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by : **Ahmed HALLACI**

Entitled

On the study of some types of differential equations of fractional orders

Jury Members

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Option : Analyse non linéaire et modélisation

Par : **Ahmed HALLACI**

Intitulée

Sur l'étude de quelques types des équations différentielles d'ordre fractionnaires

Membres de Jury

Mr. Med Zine AISSAOUI	*Prof. Univ-Guelma*	*Président*
Mr. Abderrezak CHAOUI	*Prof. Univ-Guelma*	*Encadreur*
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List of symbols

.

$FODE$: Fractional ordinary differential equation

$FPDE$: Fractional Partial differential equation

IVP : Initial value problem

BVP : Boundary value problem

$\Gamma(.)$: Gamma function

$B(.,.)$: Beta function

$E_\alpha(.)$: Mittag-Leffler function

I_{0+}^{α} : Right-fractional Riemann-Liouville integral

${}^{GL}D_{0+}^{\alpha}$: Grunwald-Letnikov fractional derivative

${}^{RL}D_{0+}^{\alpha}$: Right-fractional Riemann-Liouville derivative

${}^{C}D_{0+}^{\alpha}$: Right-fractional Caputo derivative

$[.]$: Integer part of a real number

$\triangleq$: Denoted by

$C(I,\mathbb{R})$: Space of continuous functions on I

$C^n(I,\mathbb{R})$: Space of $n-$time continuously differentiable functions on I

$AC(I,\mathbb{R})$: Space of absolutely continuous functions on I

$BC(\mathbb{R}_+,\mathbb{R})$: Space of bounded continuous functions on I

$L^1(I,\mathbb{R})$: space of Lebesgue integrable functions on I

$L^{p}(I,\mathbb{R}):$ space of measurable functions u with $|u|^{p}$ belongs to $L^{1}(I,\mathbb{R})$

$L^{p,\sigma}(I,\mathbb{R}):$ Weighted $L^{p}-$space with weighted function σ

$L^{\infty}(I,\mathbb{R}):$ space of measurable functions essentially bounded on I

$W^{m,p}(I,\mathbb{R}):$ $(m,p)-$Sobolev space

$W_{RL}^{s,p}(I,\mathbb{R}):$ $(s,p)-$Riemann-Liouville fractional Sobolev space

$\mathfrak{D}'(I):$ Space of distributions

$\nabla u:$ Gradiant of u

$\triangle u:$ Laplacian of u

$\partial u:$ Boundary of u

$resp.:$ respectively

$R-L:$ Riemann-Liouville

$a.e.:$ almost everywhere

$\rightarrow:$ strongly converge to

$\rightharpoonup:$ weakly converge to

الملخص

اكتسب موضوع المعادلات التفاضلية الكسرية شعبية كبيرة وأهمية بالغة خلال العقود الثلاثة الماضية، ويرجع ذلك أساسا إلى تطبيقاته التي تظهر في العديد من المجالات المتنوعة و المنتشرة على نطاق واسع في العلوم والهندسة. فهو يوفر بالفعل عدة أساليب وطرق يمكن أن تكون مفيدة لحل المعادلات التفاضلية والتكاملية وكذلك أهميتها في نمذجة الكثير من الظواهر الفيزيائية المرتبطة بالتغيرات السريعة جدا والقصيرة جدا.

من ناحية أخرى، وجود، تفرد واستقرار الحلول، تمثل جزءا كبيرا من الدراسة النظرية للمعادلات التفاضلية العادية وذات الإشتقاق الجزئي غير الخطية من رتب كسرية، حيث سنهتم في هذه الرسالة بمناقشة جانب التحليل النوعي لبعض أنواع المعادلات التفاضلية العادية الكسرية والمعادلات التفاضلية الجزئية الكسرية. تحقيقا لهذه الغاية، نستخدم نظريات النقطة الثابتة لباناخ، شودر و كراسنوسلسكي في فضاءات باناخ للمعادلات التفاضلية العادية الكسرية، و كذلك نستخدم طريقة روث لإثبات وجود وتفرد الحل لمعادلة الإنتشار برتبة كسرية في فضاء هيلبرت. ولضمان فعالية وفائدة نتائجنا التي تم الحصول عليها نظريا، يتم إعطاء بعض الأمثلة التوضيحية.

<u>الكلمات المفتاحية</u>: المعادلات التفاضلية الكسرية، معادلة الانتشار الكسري، مسائل القيم الأولية، مسائل القيم الحدية ، المجال الزمني غير المحدود، المشتقات المختلطة، فضاء باناخ المثقل، فضاء سوبوليف الكسري، الوجود، التفرد، الإستقرار، الحل الضعيف، طريقة روث، مبدأ التقلص لباناخ ، نظرية النقطة الثابتة لشودر ، نظرية النقطة الثابتة لكراسنوسلسكي.

Abstract

The subject of fractional differential equations has gained considerable popularity and importance during the past three decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. It does indeed provide several potentially useful tools for solving differential and integral equations as well as their importance in the modeling of a lot of physical phenomena associated to very rapid and very short changes.

On the other hand, existence, uniqueness and stability of solutions, represent a large part of the qualitative theory of nonlinear ordinary and partial differential equations of non-integer order. Where we are interested in this thesis on the discussion of qualitative analysis of some kinds of fractional ordinary differential equations and fractinal partial differential equations. To this end, we utilize the fixed point theorems of Banach, Schauder and Krasnoselskii in Banach spaces for fractional ordinary differential equations as well as Rothe discretization method is used to show the existence and uniqueness of weak solution for fractional diffusion equation of the second-order differential Volterra operator in Hilbert space. To guarantee the effectiveness and usefulness of our obtained results theoretically, some illustrative examples are given.

Keywords: Fractional differential equations, fractional diffusion equation, initial value problems, boundary value problems, unbounded interval, mixed derivatives,

weighted Banach spaces, fractional Sobolev spaces, existence, uniqueness, stability, weak solutions, Rothe method, discretisation scheme, Banach contraction principle, Schauder fixed point theorem, Krasnoselskii fixed point theorem.

Résumé

Le sujet des équations différentielles fractionnaires a acquis une popularité et une importance considérables au cours des trois dernières décennies, principalement en raison de ses applications démontrées dans de nombreux domaines de la science et de l'ingénierie apparemment diversifiés. Il fournit en effet plusieurs outils potentiellement utiles pour résoudre les équations différentielles et intégrales ainsi que leur importance dans la modélisation d'un grand nombre de phénomènes physiques associés à des changements très rapides et très courts.

D'autre part, l'existence, l'unicité et la stabilité des solutions représentent une grande partie de la théorie qualitative des équations différentielles ordinaires et partielles non linéaires d'ordre non entier. Où nous sommes intéressés dans cette thèse sur la discussion de l'analyse qualitative de quelques types des équations différentielles ordinaires et partielles fractionnaires. A cette fin, nous utilisons les théorèmes des points fixes de Banach, Schauder et Krasnoselskii dans des espaces de Banach pour les équations différentielles ordinaires fractionnaires ainsi que la méthode de discrétisation de Rothe pour montrer l'existence et l'unicité de la solution faible dans un espace de Sobolev pour l'équation différentielle partielle de diffusion d'ordre fractionnaire. Pour garantir l'efficacité et l'utilité des résultats obtenus théoriquement , quelques exemples illustratifs seront donnés.

Mots clés: Equations différentielles fractionnaires, équation de diffusion fraction-

naire, problèmes de valeurs initiales, problèmes de valeur aux limite, intervalle non borné, dérivations mixtes, espaces de Banach pondérés, espaces de Sobolev fractionnaires, existence, unicité, stabilité, solution faible, méthode de Rothe, schéma de discrétisation, principe de contraction de Banach, théorème du point fixe de Schauder, théorème du point fixe de Krasnoselskii.

Introduction

The theory of derivatives of non-integer order goes back to Leibniz's note in his list to L'Hospital, dated 30 september 1695, in which the meaning of the derivative of order one half is discussed. Leibniz's note led to the appearance of the theory of derivatives and integrals of arbitrary order, which by the end of 19 century took more or less finished form due primarily to liouville, Grunwald, Letnikov, and Riemann. For more than two centuries, the theory of fractional derivative developed mainly as a pure theoretical field of mathematics useful only for mathematics. However, in the last few decades many authors pointed out that derivatives and integrals of non-integer order are very suitable for the description of properties of various real materials. It has been shown that new fractional order models are more adequate than previously used integer-order models. For more details of fundamental works on various aspects of the fractional calculus and fundamental physical considerations in favour of the use of models based on derivatives of non-integer order we refer the monograph of Bagley [11], Engeita [33], Hilfer [44], Khare [48], Kilbas [49], Magin [60], Mainardi [61], Miller and Ross [64], Nishitomo [67], Oldham [69], Oldham and Spanier [70], Petras [71], Podlubny [72], Sabatier et al. [78], and the references therein.

Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional derivatives in comparison with classical integer order models, in which such effects are in fact neglected. The advantages of fractional derivatives become apparent in modelling mechanical and electrcal properties of real materials, as well as in the description rheological of rocks, and in many other fields us chemistry, biology, engineering, viscoelasticity, signal processing, electrotechnical, electrochemistry and controllability, see the above mentioned references.

The main objective of the present thesis is to proved some existence and uniqueness results for some kinds of ordinary differential equations of fractional order and the partial differential equations which contain a fractional derivative term. We need to this end to use various fundamental concepts of fractional calculus and fractional derivative and its properties in order to use it in fractional differential equations and in partial fractional differential equations. Also, some concepts of functional analysis are presented to show these purposee. Beside, The subject of fixed point theory become an important fild of mathematics given its great importance in the other domains of mathematics especially in ordinary differential equations, partial differential equations, integral equations, operator theory, numerical analysis and other mathematic areas. In our work, we will use fixed points theorems on a large scale to show the existence, uniqueness and stability of solutions of some problems that we will given later. For existence of solutions, we employ the Shauder fixed point theorem, the Krasnoselskii fixed point theorem, as well as Banach contraction principle is used for uniqueness, and too, by utilizing Krasnoselskii fixed point theorem we discuss the stability of solutions. Some contributions around applications of fixed point theorems in fractional differential equations to show the existence, uniqueness and stability of solution found in [1, 5, 6, 15, 21, 23, 37, 46, 47, 51, 68] and the references cited therin.

On the other hand, Fractional diffusion equations include the mathematical model of large class of problems. They describe anomalous diffusion on fractal (physical objects of fractional dimension), fractional random walk. For details, see [13, 34, 38] and the references therein. Let us cite some interesting papers dealing with this kind of problems. The first of them is that of Oldham et al. [70] whose studied the relation between usual diffusion equation and a fractional diffusion equation. In [62], F. Mainardi et al. estabilished the model of diffusion waves in viscioelasticity based on fractional calculus. Agarwal [3] discussed the solutions for fractional diffusion wave equation defined in a bounded domain. El-Borai [32] investigated the fundamental solutions of fractional evolution equations. Recently, Mophou et al. [38] considred fractional evolution equation with fractional integral condition in Sobolev space, where the authors assumed that the operational coefficient is the generator of a semigroup of contractions.

Among other methods, the Rothe's method is one of the more popular that is commonly used in the time discretization of evolution equations where the derivatives with respect to one variable are replaced by difference quotients that finally leads to systems of differential equations for functions of the remaining variables. Rothe's method as an approximative approach is well suited not only to prove the existence results, but also for various applications. This method was introduced by Rothe in 1930 for solving second order linear parabolic equations with one space variable(see [77]). Later, this method was adopted by Ladyzenskaja [53, 54] to solve linear and quasilinear parabolic problems of second order and linear equations of higher orders. Further development is connected with Rektorys (see[74, 75]) who obtained more smooth solutions. Recently, Rothe's method has been devloped to cover other types of equations as we can see in [12, 27, 28, 31, 39, 40, 52].

The scheme of the Rothe method is given as follows;

We divide the time interval into n subintervals $(t_{i-1}, t_i)\,, i = 1, ..., n$, where $t_i = ih$, $h = \frac{T}{n}$. We donote by $u_i = u(t_i, x) = u_i(x) = u(ih, x)$ the approximants of u.

We replace the derivative (of the function u) $\frac{\partial u}{\partial t}$ by $\delta u_i = \frac{u_i - u_{i-1}}{h}$, for all $t = t_i, i = 1, ..., n$.

We obtain a system consisting of n equations in x where the unknown is $u_i(x)$, so we approach the problem posed in every point by a new discrete problem.

We determine the functions u_n solutions of the obtained system.

We build the Rothe functions defined by

$$u^{(n)}(t) = u_{i-1} - \delta u_i (t - t_i)\,, \quad t \in [t_{i-1}, t_i]\,, \quad i = 1, ..., n.$$

and the corresponding test functions

$$\overline{u}^n(t) = \begin{cases} u_i & t \in (t_{i-1}, t_i] \\ u_0 & t = 0 \end{cases}, i = 1, ..., n.$$

Motivated and inspired by the works mentioned above on qualitative theorie of fractional differential equations and as a contribution to enrich the works previously conducted in this orientation, the main goal of this thesis is to show new results about existence, uniqueness and stability of solutions for initial value problems and boundary value problem for some kinds of partial and ordinary differential equations of fractional orders on bounded and on unbounded interval involving Riemann-Liouville and Caputo fractional derivatives. The discussion of solutions will be in some Banach spaces and Hilbert spaces that we will present it in later.

This thesis consists of four chapters.

Chapter 1 devoted to give a preface on the theory of functional spaces, special functions, fractional derivative and fractional integral, some tools of functional analysis and fixed point theorems.

Chapter 2 is based on the submitted paper [42] and new other results on going redaction. We give some results about uniqueness, existence, and stability of solutions on unbounded interval using Krasnoselskii fixed point theorem and Banach contraction principle in weighted Banach spaces of the following fractional initial value problem

$$\begin{cases} {}^C D^\alpha u(t) = f(t, u(t)) \ \ t \geq 0, \\ u(0) = u_0, \ u'(0) = u_1, \end{cases}$$

and the delay fractional initial value problem

$$\begin{cases} D^\alpha[{}^C D^\beta u(t) - g(t, u(t-r))] = f(t, u(t-r)), t \geq 0, \\ u(t) = \Phi(t), t \in [-r, 0], \\ \lim\limits_{t \to 0} t^{1-\alpha}\, {}^C D^\beta u(t) = 0, u'(0) = u_0. \end{cases}$$

Chapter 3 is based on the submitted paper [41]. We will interested with the study of the following boundary value problem of fractional differential equations of Riemann-Liouville type

$$\begin{cases} D^{q}u\left(t\right)=g\left(t,u\left(t\right),D^{s}u\left(t\right)\right),\ t\in I, \\ D^{q-i}u|_{t=0}=0,\ i=1,...,n,\ i\neq n-1 \text{ and } u\left(T\right)=0, \end{cases}$$

we prove the existence and uniqueness of weak solution in weighted fractional Sobolev spaces (which we will construct later) using Shauder fixed point theorem and Banach contraction principle.

Chapter 4 is based on the published paper [26] which deals with the following fractional diffusion equation of the second-order differential Volterra operator and fractional integral condition

$$D^{\alpha}u\left(t,x\right)-\triangle u\left(t,x\right)=\int\limits_{0}^{t}a\left(t-s\right)\triangle u\left(s,x\right)ds+f\left(t,x\right)\quad\text{in } I\times\Omega,$$

$$u\left(0,x\right)=u_{0}\left(x\right)\quad\text{in}\ \ \Omega,$$

$$I^{1-\alpha}u\left(0^{+}\right)=U_{1}\left(x\right),\text{ in }\Omega,$$

$$u\left(t,x\right)=0\text{ on }I\times\partial\Omega.$$

The existence and uniqueness of a weak solution in an appropriate sense as well as some regularity results are obtained by the use of Rothe's discretization method. Here, $\triangle$ is the differential operator defined by the application of the gradient operator followed by the application of the divergence operator, that is

$$\begin{aligned} \triangle\Phi &= \overrightarrow{\nabla}.\left(\overrightarrow{\nabla}\Phi\right)=\text{div}\left(\overrightarrow{\text{grad}}\Phi\right) \\ &= \left(\frac{\partial}{\partial x_{1}}\ \frac{\partial}{\partial x_{2}}\cdots\frac{\partial}{\partial x_{n}}\right)\left(\frac{\partial\Phi}{\partial x_{1}}\ \frac{\partial\Phi}{\partial x_{2}}\cdots\frac{\partial\Phi}{\partial x_{n}}\right)^{T}=\sum_{i=1}^{n}\frac{\partial^{2}\Phi}{\partial x_{i}^{2}}. \end{aligned}$$

CHAPTER 1

Preliminaries

In this chapter, we present some basic notations, definitions and properties from such topics of analysis which are used in the other chapters as special functions, functional spaces, fractional integral, fractional derivative, fixed point theorems etc. For more details we refer to the monographs of Adams[2], Besov [17], Brezis [20] , Hilfer [44] , Kilbas [49], Kolmogorov [50], Podlubny [72], Precup [73], Renardy and Rogers [76], Smart [79] and the book of Zeidler [82].

1.1 Functional spaces

Let $\mathbb{R}_+ = [0, +\infty)$ and let $J := [0, T]$ the compact interval of $\mathbb{R}_+$. we present the following functional spaces:

Definition 1.1 Let $C(J, \mathbb{R})$ is the Banach space of continuous functions $u : J \to \mathbb{R}$ have the valued in $\mathbb{R}$, equipped with the norm

$$\|u\| = \sup_{t \in J} |u(t)|.$$

Analogoustly, $C^n(J, \mathbb{R})$ the Banach space of functions $u : J \to \mathbb{R}$ where u is n time continuously differentiable on J.

Denote by $L^1(J,\mathbb{R})$ the Banach space of functions u Lebesgues integrables with the norm

$$\|u\|_{L^1} = \int\limits_J |u(t)|\, dt,$$

and we denote $L^p(J,\mathbb{R})$ ($L^{p,\sigma}(J,\mathbb{R})$ resp.) the space of Lebesgue integrable functions on J where $|u|^p$ ($\sigma |u|^p$ resp.) belongs to $L^1(J,\mathbb{R})$, endowed with the norm

$$\|u\|_{L^p} = \int\limits_J |u(t)|^p\, dt,$$

$$\|u\|_{L^{p,\sigma}} = \int\limits_J \sigma(t) |u(t)|^p\, dt, \text{ resp.}$$

In particular, if $p = \infty$, $L^\infty(J,\mathbb{R})$ is the space of all functions u that are essentially bounded on J with essential supremum

$$\|u\|_{L^\infty} = ess \sup_{t\in J} |u(t)| = \inf\{C \geq 0 : |u(t)| \leq C \text{ for a.e. } t\}.$$

Definition 1.3 Let Ω be a open set of $\mathbb{R}^n$, we difine the Sobolev space $W^{m,p}(\Omega,\mathbb{R})$ by

$$W^{m,p}(\Omega,\mathbb{R}) = \{u \in L^p(\Omega,\mathbb{R}) : D^\alpha u \in L^p(\Omega,\mathbb{R}) \text{ for } 0 \leq |\alpha| \leq m\}$$

where D^α is the weak (or distributional) partial derivative. $W^{m,p}(\Omega,\mathbb{R})$ is a Banach space equipped with the norm

$$\|u\|_{m,p} = \left(\sum_{0\leq|\alpha|\leq m} \|D^\alpha u\|_p^p \right)^{\frac{1}{p}} \text{ if } 0 \leq p < \infty,$$

$$\|u\|_{m,\infty} = \max_{0\leq|\alpha|\leq m} \|D^\alpha u\|_{L^\infty}.$$

In particular, if $p = 2$, we denote by $H^m(\Omega,\mathbb{R})$ to $W^{m,2}(\Omega,\mathbb{R})$ (the space of square-integrable functions) which is a Hilbert space with inner product

$$(u,v)_m = \sum_{0\le|\alpha|\le m} (D^\alpha u, D^\alpha v),$$

where

$$(u,v) = \int_J u(t)\, v(t)\, dt$$

is the inner product in $L^2(\Omega)$.

If $u = 0$ on $\partial J = \{0, T\}$, for all $u \in H^m(\Omega, \mathbb{R})$, then we write $H_0^m(\Omega, \mathbb{R})$ instead of $H^m(\Omega, \mathbb{R})$.

Some inequalities associated with these concepts are found in section 1.4.

Definition 1.4 A function $u : J \to \mathbb{R}$ is said absolutly continuous on J if for all $\epsilon > 0$, thre exists a number $\delta_\epsilon > 0$ such that; for all finite partition $[a_i, b_i]_{i=1}^p$ in J, then $\sum_{i=1}^p (b_i - a_i) < \delta_\epsilon$ implies that $\sum_{i=1}^p |u(b_i) - u(a_i)| < \epsilon$.

We denote by $AC(J,\mathbb{R})$ (or $AC^1(J,\mathbb{R})$) the space of all absolutely continuous functions defined on J. It is known that $AC(J,\mathbb{R})$ coincides with the space of primitives of Lebesgue summable functions:

$$u \in AC(J,\mathbb{R}) \Longleftrightarrow u(t) = c + \int_0^t \Phi(s)\, ds,\ \ \Phi \in L^1(J,\mathbb{R}), \tag{1.1}$$

and therefore an absolutely continuous function u has a summable derivative $u'(t) = \Phi(t)$ almost everywhere on J. Thus (1.1) yields

$$u'(t) = \Phi(t) \text{ and } c = u(0).$$

Definition 1.5 For $n \in \mathbb{N}$, we denote by $AC^n(J,\mathbb{R})$ the space of functions $u : J \to \mathbb{R}$ which have continuous derivatives up to order $n-1$ on J such that $u^{(n-1)}$

belongs to $AC(J,\mathbb{R})$:

$$\begin{aligned} AC^n(J,\mathbb{R}) &= \left\{u \in C^{n-1}(J,\mathbb{R}) : u^{(n-1)} \in AC(J,\mathbb{R})\right\} \\ &= \left\{u \in C^{n-1}(J,\mathbb{R}) : u^{(n)} \in L^1(J,\mathbb{R})\right\}. \end{aligned}$$

The space $AC^n(J,\mathbb{R})$ consists of those and only those functions u which can be represented in the form

$$u(t) = \sum_{k=0}^{n-1} c_k t^k + \left(I_{0+}^n \Phi\right)(t), \tag{1.2}$$

where $\Phi \in L^1(J,\mathbb{R})$, $c_k\,(k = 0, 1, ..., n-1) \in \mathbb{R}$, and

$$\left(I_{0+}^n \Phi\right)(t) = \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}\,\Phi(s)\,d\tau.$$

It follows from (1.2) that

$$\Phi(t) = u^{(n)}(t), \ \ c_k = \frac{u^{(k)}(0)}{k!} \ \ (k = 0, 1, ..., n-1).$$

For more details about $AC(J,\mathbb{R})$ and $AC^n(J,\mathbb{R})$, see eg. the book of Kolmogorov and Fomin ([50], p.338).

1.2 Special functions

We give here some information on the gamma function, beta function and Mittag-Leffler function, which play moste important role in the theory of differentiation of arbitrary order and in the theory of fractional differential equations. They represent a generalizations of some usual functions.

Definition 1.6 (Gamma function):

The gamma function $\Gamma(.)$ is defined by the integral

$$\forall z \in \mathbb{R}_+^* : \; \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt,$$

which possesses the following basic properties

$$z\Gamma(z) = \Gamma(z+1),$$

and for every integer $n \geq 0$, we have

$$n! = \Gamma(n+1).$$

Farthermore

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)(z+2)\ldots(z+n-1)}, z > -n, n = 1, ..., z \neq 0, -1, -2, ...$$

Clearly, Gamma function is analytic except for $z = 0, 1, 2, ...$ which are represent simple poles.

Definition 1.7 (Beta function)

The beta function $B(.,.)$ is defined for all $p, q \geq 0$ by:

$$B(p,q) = \int_0^1 s^{p-1} (1-s)^{q-1} ds.$$

The functions $\Gamma(.)$ and $B(.,.)$ are related by the formula

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

To prove this relationship we use the Laplace transform, see [49].

Definition 1.8 (Mittag-Leffler function)

For $\alpha > 0$ and $z \in \mathbb{R}$, the one parameter Mittag-Leffler function is defined by

$$E_\alpha\left(z\right) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma\left(\alpha k + 1\right)},$$

where it was introduced by Mittag-Leffler [65].

For $\alpha > 0$, $\beta > 0$, we define the two parameter Mittag-Leffler function by

$$E_{\alpha,\beta}\left(z\right) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma\left(\alpha k + \beta\right)},$$

in particular,

$$E_{1,1}\left(z\right) = e^z \text{ and } E_{2,1}\left(z\right) = \cosh\left(\sqrt{z}\right),$$

and

$$E_{\alpha,1}\left(z\right) = E_\alpha\left(z\right).$$

1.3 Fractional integrales and derivatives

In this section, some approaches to the generalization of the notion of differentiation and integration are considered (see for instence [44, 49, 64, 72]).

Definition 1.9 (Grunwald-Letnikov fractional derivative)

The Grunwald-Letnikov fractional derivative of the function u of order $\alpha \geq 0$ is defined by

$${}^{GL}D_a^\alpha u\left(t\right) = \lim_{h \to 0+} \frac{1}{h^\alpha} \sum_{k=0}^{[(t-a)/h]} \left(-1\right)^k \left(C_k^\alpha f\left(t - kh\right)\right).$$

Definition 1.10 (Cauchy formula)

The Cauchy formula of nth integral of a locally integrable function u on $\mathbb{R}_+$ is given by

$$I^n u(t) = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} u(s)\, ds,$$

Definition 1.11 (Riemann-Liouville fractional integral)

The right-side(left-side resp.) Riemann-Liouville fractional integral of the function $u \in L^1[0,T]$ of order $\alpha \geq 0$ is defined by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds,$$

$$I_{T-}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} u(s)\, ds,$$

resp., where $t \in [0,T]$.

Riemann-Liouville fractional derivative are defined depending on their fractional integral and integer order derivative as follows.

Definition 1.12 (Riemann-Liouville fractional derivative)

The right-side(left-side resp.) Riemann-Liouville fractional derivative of the function u of order $\alpha \in (n-1, n]$ is given by

$${}^{RL}D_{0+}^{\alpha} u(t) = \frac{d^n}{dt^n}\left[I_{0+}^{n-\alpha} u(t)\right] = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} u(s)\, ds,$$

$$^{RL}D^{\alpha}_{T-}u(t) = \left(-\frac{d}{dt}\right)^{n}\left[I^{n-\alpha}_{T-}u(t)\right] = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dt}\right)^{n}\int_{t}^{T}(s-t)^{n-\alpha-1}u(s)\,ds,$$

resp.

Definition 1.13 (Caputo fractional derivative)

The left-side(right-side resp.) Caputo fractional derivatives of the function $u \in AC^{n}[0,T]$ of order $\alpha \in (n-1,n]$ is defined by

$$^{C}D^{\alpha}_{0+}u(t) = I^{n-\alpha}_{0+}\left[u^{(n)}(t)\right] = \frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}(t-s)^{n-\alpha-1}u^{(n)}(s)\,d\tau,$$

$$^{C}D^{\alpha}_{T-}u(t) = I^{n-\alpha}_{T-}\left[u^{(n)}(t)\right] = \frac{1}{\Gamma(n-\alpha)}\int_{t}^{T}(s-t)^{n-\alpha-1}u^{(n)}(s)\,d\tau,$$

resp.

Remark 1.1 Fractional integrals and fractional derivatives can themselves be extended from the case of a finite interval to the case of half-axes or axes.

Let's now consider some properties of the Riemann-Liouville and Caputo fractional integral and derivatives. In particular, we are interested by the left-side fractional derivatives and integrals. Farthermore, we denote in the rest of this thesis only by $I^{\alpha}, D^{\alpha}, ^{C}D^{\alpha}$ instead $I^{\alpha}_{0+}, ^{RL}D^{\alpha}_{0+}$ and $^{C}D^{\alpha}_{0+}$ resp.

Lemma 1.1 (Relation between R-L and Caputo derivatives)

Let $n-1 < \alpha \leq n$. If the function $u \in C^{n}[0,T]$, then

$$^{C}D^{\alpha}u(t) = D^{\alpha}u(t) - \sum_{k=0}^{n}\frac{u^{(k)}(0)}{\Gamma(k-\alpha+1)}t^{k-\alpha}.$$

Lemma 1.2 (Linearity and monotony)

- Fractional operators I^{α}, D^{α} and ${}^{C}D^{\alpha}$ are linears.
- Operator I^{α} is monotone.

Lemma 1.3 (Boundness of fractional integral)

Fractional operator I^{α} is bounded on $L^{p}(0,T)$, that is

$$\|I^{\alpha}u\|_{p} \leq K\|u\|_{p}, K = \frac{T^{\alpha}}{\Gamma(\alpha+1)}.$$

Lemma 1.4 For $\alpha, \beta \geq 0$ and $u \in L^{1}([a,b])$, we have

$$\begin{aligned} I^{\alpha}I^{\beta}u(t) &= I^{\beta}I^{\alpha}u(t) = I^{\alpha+\beta}u(t), \\ D^{\alpha}I^{\alpha}u(t) &= u(t), \\ {}^{C}D^{\alpha}I^{\alpha}u(t) &= u(t), \end{aligned}$$

also, for $\alpha > \beta > 0$ and $u \in L^{1}([a,b])$, we have

$$D^{\beta}I^{\alpha}u(t) = I^{\alpha-\beta}u(t).$$

Lemma 1.5 (examples of fractional integral and derivative for power functions, see eg.[49])

If $\alpha \geq 0, \beta > 0$, then

$$I^{\alpha}t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}t^{\beta+\alpha-1}, \alpha > 0,$$

$$D^{\alpha}t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}t^{\beta-\alpha-1}, \alpha \geq 0,$$

$${}^{C}D^{\alpha}t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}t^{\beta-1}, \beta \geq [\alpha],$$

Lemma 1.6 Let $u \in L^1(a,b)$ and $n-1 \leq \alpha < n$ with $D^\alpha u \in L^1(a,b)$, then

$$I^\alpha D^\alpha u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + ... + c_n t^{\alpha-n}, \text{ for } c_i \in \mathbb{R}, i = 1, 2, ..., n,$$

Lemma 1.7 Let $u \in AC^{n-1}([a,b])$ and $n-1 \leq \alpha < n$ with ${}^C D^\alpha u \in L^1(a,b)$, then

$$I^{\alpha C} D^\alpha u(t) = u(t) + c_0 + c_1 t + ... + c_{n-1} t^{n-1}, \quad c_i \in \mathbb{R}, i = 0, 2, ..., n-1,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of α.

Definition 1.14 The Laplace transform of a function Φ of a real variable $t \in \mathbb{R}_+$ is defined by

$$(\mathcal{L}\Phi)(s) = \mathcal{L}[\Phi(t)](s) = \int_0^\infty e^{-st} \Phi(t)\, dt, \quad (s \in \mathbb{C}).$$

Definition 1.15 The inverse Laplace transform is given for $x \in \mathbb{R}_+$ by the formula

$$\left(\mathcal{L}^{-1} g\right)(x) = \mathcal{L}^{-1}[g(s)](x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{sx} g(s)\, ds.$$

The direct and inverse Laplace transforms are inverse to each other

$$\mathcal{L}^{-1}\mathcal{L}\Phi = \Phi \text{ and } \mathcal{L}\mathcal{L}^{-1} g = g.$$

Lemma 1.8 Let $\alpha, \beta > 0$. The Laplace transform of the Riemann-Liouville fractional derivative $D^\alpha u(t)$ and the power function $t \mapsto t^\beta$ are given respectively by

(i) $\mathcal{L}\{D^\alpha u(t), z\} = z^\alpha U(z) - \sum_{i=0}^{n-1} z^i \left[D^{\alpha-i-1} u(t)\right]_{t=0},$

(*ii*) $\mathcal{L}\left\{t^{\beta}, z\right\} = \Gamma\left(\beta+1\right) z^{-(\beta+1)}$,

where $U\left(z\right)$ denotes the Laplace transforme of $u\left(t\right)$. $n = [\alpha] + 1$.

More details for Laplace transform and its applications in fractional calculus theory can be obtained in [72].

1.4 Functional tools

Some useful concepts of functional analysis are presented below:

Definition 1.16 $\Phi : J \times E \longrightarrow E$ is called a Carathéodory function if

i $t \longmapsto \Phi\left(t, u\right)$ is measurable for every $u \in E$,

ii $u \longmapsto \Phi\left(t, u\right)$ is continuous for almost everywhere $t \in J$.

Definition 1.17 Let I be a measurable subset of $\mathbb{R}$, $g : J \times \mathbb{R} \to \mathbb{R}$ be a function satisfies the condition of Carathéodory. By a Nemytskii operator we mean the mapping N_g taking a function u to the function

$$N_g u(t) = g(t, u(t)), t \in J, u \in \mathbb{R}.$$

The continuity of the operator N_g is concerned in the following Lemma.

Lemma 1.9 Let J be a measurable subset of $\mathbb{R}$, $g : J \times \mathbb{R} \to \mathbb{R}$ to be a Carathéodory function. Let $u \in L^{p}\left(J\right)$, $p \in [1, \infty)$. If there exist a function $b \in L^{r}\left(J\right), r \in [1, \infty)$, and a constant $c > 0$ such that

$$\left|g\left(t, u\right)\right| \leq b\left(t\right) + c\left|u\right|^{\frac{p}{r}}, a.e.t \in J, u \in \mathbb{R},$$

then the Nemytskii operator

$$N_g u(t) = f\left(t, u(t)\right),$$

is continuous and bounded from $L^p(J)$ to $L^r(J)$, that is

$$\|N_g u\|_{L^r(J)} \leq \|b\|_{L^r(J)} + c\|u\|_{L^p(J)}^{\frac{p}{r}}.$$

Here, we say that the g is $(p,r)-$Carathéodory. For more details about Nemytski operator and its properties, we refer, eg. to [73].

This result can be easily carried over to vector functions $u = (u_1, ..., u_d)$ with components $u_j \in L_{p_j}$ and to functions $f : J \times R^d \to R$. In this case, we have(see [17])

$$|g(t,u)| \leq b(t) + c\sum_{j=1}^{d} |u_j|^{\frac{p_j}{r}}, a.e.t \in J, u \in \mathbb{R}^d.$$

Notation : We set $(\tau_h f)(x) = f(x+h)$.

Lemma 1.10 (Compactness Criteria in L^p)

Let Ω be a bounded domain in $\mathbb{R}^n$. A set $\mathcal{F}$ of functions $f \in L^p(\Omega)$ $(1 \leq p < \infty)$ has a compact closure in $L^p(\Omega)$ if and only if there is equicontinuous, i.e.

$$\lim_{|h|\to 0} \|\tau_h f - f\|_p = 0,$$

uniformly in $f \in \mathcal{F}$.

For more details see [2, 20, 66].

Definition 1.18 (Weak convergence)

Let E be a Banach space, a sequence u_n in E converges weakly to u if $f(u_n)$ converges to $f(u)$ for every $f \in E^*$. A sequence f in E^* converges weakly$-*$ to f if $f_n(u)$ converges to $f(u)$ for every $u \in E$.

To distinguish notations, one writes $u_n \to u$ for convergence in norm, $u_n \rightharpoonup u$ for weak convergence, and $u_n \overset{*}{\rightharpoonup} u$ for weak$-*$ convergence.

Theorem 1.1 (Weak compactness, [76]) Let E be a separable Banach space and let f_n be a bounded sequence in E^*. Then f_n has a weakly$-*$ convergent subsequence.

Theorem 1.2 (Minty-Browder, [35])

Let Ω be a bounded domain in $\mathbb{R}^N$ and $f : (0,T) \times \Omega \to \mathbb{R}$ be monotone in the second variable, i.e.

$$(f(t,u) - f(t,v), u - v) \geq 0 \text{ for } u, v \in \Omega,$$

and

$$u_n \to u \text{ weakly in } L^p(\Omega),$$

$$d(t,x,u_n) \to \chi \text{ weakly in } L^q(\Omega),$$

$$\limsup_{n\to\infty} \int_\Omega d(t,x,u_n)\, u_n dx \leq \int_\Omega \chi u dx.$$

Then

$$\chi = d(t,x,u).$$

Lemma 1.11 (Holder's inequality)

Let $1 \leq p \leq \infty$ and let p' denote the conjugate exponent defined by

$$p' = \frac{p}{p-1}, \quad \text{that is} \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

If $u \in L^p(J,\mathbb{R})$ and $v \in L^{p'}(J,\mathbb{R})$, then $uv \in L^1(J,\mathbb{R})$, and

$$\int_J |u(t)\, v(t)|\, dt \leq \|u\|_p \|v\|_{p'}.$$

Lemma 1.12 (Minkowski's Inequality)

If $1 \leq p < \infty$, then

$$\|u+v\|_p \leq \|u\|_p + \|v\|_p .$$

Lemma 1.13 (Cauchy-Schwartz inequality)

Let u, v belong to $L^2(I)$ equipped with the inner product $(.,.)$ and the norm $\|.\|_2$, then

$$(u,v) \leq \|u\|_2 \|v\|_2 .$$

Lemma 1.14 ($\epsilon-$Young's inequality)

Let $a, b \geq 0$, then for all $\epsilon > 0$, we have

$$2ab \leq \epsilon a^2 + \frac{1}{\epsilon} b^2.$$

Lemma 1.15 (Poincaré's Inequality)

For all $u \in H_0^1(J, \mathbb{R})$, there exists a constant C_J (depending on J) such that

$$\|u\|_2 \leq C_J \|\nabla u\|_2 .$$

Lemma 1.16 (Green formula)

If $u \in H^2(J)$ and $v \in H^1(J)$, we have

$$\int\limits_J (\triangle f)\, g dx = -\int\limits_J \nabla f \nabla g dx + \int\limits_{\partial J} \nabla (f\eta)\, g.$$

Lemma 1.17 (Discrete Gronwall Lemma)

Let $(u_n), (f_n)$ and (g_n) are nonnegative sequences and

$$u_n \leq f_n + \sum_{0 \leq k \leq n} g_k u_k \quad \text{for } n \geq 0,$$

then

$$u_n \leq f_n + \sum_{0\leq k\leq n} f_k g_k \exp\left(\sum_{k<j<n} g_j\right) \quad \text{for } n \geq 0.$$

1.5 Fixed point theorems

In the following, we are interested by giving some fixed point theorems with related notions:

Definition 1.19 For a mapping A from a set E into itself, an element u of E is a fixed point of A if $A(u) = u$.

Definition 1.20 Let E be a Banach space with a norm $\|.\|$. A mapping $A : E \to E$ is called $\Phi-$Lipschitzian, if there exists a continuous nondecreasing function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\|Au - Av\| \leq \Phi\left(\|u - v\|\right),$$

for all $u, v \in E$ with $\Phi(0) = 0$. The function Φ is sometimes called a $\Phi-$function of A on E.

In particular:

- ◦ If $\Phi(r) = kr$ for some $k > 0$, A is a Lipschitz mapping with a Lipschitzian constant k(k-Lipschitzian). In this case if $k < 1$ then A is called a contraction mapping with a contraction constant k.
- ◦ If $\Phi(r) < r$, A is called a nonlinear contraction mapping ($\Phi-$contraction).

Definition 1.21 [30]A function $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is said to be generalized Lipshitz, if there exists a positive function Φ, such that

$$|f(t, u) - f(t, v)| \leq \Phi(t)\,|u - v|,$$

for all $t \in [a, b], u, v \in \mathbb{R}$. Φ is called the Lipschitz function of f.

Definition 1.22 Let E and F two Banach spaces and A be an application defined on E in F. We say that A is completely continuous if it is continuous and transforms any bounded of E into a relatively compact set in F. A is called compact if $A(E)$ is relatively compact in F.

Lemma 1.17 [82](Ascoli-Arzelà Theorem)

Let Ω be a bounded subset of $\mathbb{R}^n$. Let M a subset of $C\left(\overline{\Omega}\right)$ the space of continuous functions on $\overline{\Omega}$, Then M is relatively compact if and only if

- ∘ M is unifomely bounded, i.e. $\|u\| \leq c$, for all $u \in M$ and $c > 0$ is a fixed number.
- ∘ M is equicontinuous, i.e.

$$\forall \epsilon > 0, \exists \delta > 0, \text{ if } \left|t - t^{'}\right| < \delta \text{ and } u \in M \Rightarrow \left|u\left(t\right) - u\left(t^{'}\right)\right| < \epsilon.$$

We present now, a more generally version of Ascoli-Arzelà Theorem in the case when the set Ω is unbounded.

Definition 1.23 Let $h : \mathbb{R}_+ \to [1, +\infty)$ be a strictly increasing continuous function with

$$h(0) = 1,\ h(t) \to \infty \text{ as } t \to \infty,\ h(s)h(t-s) \leq h(t),$$

for all $0 \leq s \leq t \leq \infty$. We introduce the space

$$E = \left\{ u \in C\left[0, +\infty\right) : \sup_{t \geq 0} \frac{|u(t)|}{h(t)} < \infty \right\},$$

which is a Banach space equipped with the norm $\|u\| = \sup\limits_{t\geq 0} \frac{|u(t)|}{h(t)}$. For more properties of this Banach space, see [23, 51].

In order to prove the compactness in E, we give the following modified compactness criterion.

Lemma 1.18 [51]Let $\mathcal{M}$ be a subset of the Banach space E. Then $\mathcal{M}$ is relatively compact in E if the following conditions are satisfied:

i) $\left\{\frac{u}{h} : u \in \mathcal{M}\right\}$ is uniformly bounded;

ii) $\left\{\frac{u}{h} : u \in \mathcal{M}\right\}$ is equicontinuous on any compact interval of $\mathbb{R}_+$;

iii) $\left\{\frac{u}{h} : u \in \mathcal{M}\right\}$ is equiconvergent at infinity. i.e. for any given $\epsilon > 0$, there exists a $T_0 > 0$ such that for all $u \in \mathcal{M}$ and $t_1, t_2 > T_0$, it holds

$$\left|\frac{u(t_1)}{h(t_1)} - \frac{u(t_2)}{h(t_2)}\right| < \epsilon.$$

Theorem 1.2 (Banach contraction principle [79, 82])

Let E be a Banach space. If $A : E \to E$ is a contraction, then A has a unique fixed point in E.

Theorem 1.3 (Schauder's fixed point theorem [79, 82])

Let $\mathcal{M}$ be a closed convex subset of a Banach space E. If $A : \mathcal{M} \to \mathcal{M}$ is continuous and the set $\overline{A(\mathcal{M})}$ is compact, then A has a fixed point in $\mathcal{M}$.

Theorem 1.4 (Krasnoselskii fixed point theorem [79, 82])

If $\mathcal{M}$ is a nonempty bounded; closed and convex subset of a Banach space E, A and B two operators defined on $\mathcal{M}$ with values in E such as:

i) $Au + Bv \in \mathcal{M}$, for all $u, v \in \mathcal{M}$.

ii) A is continuous and compact,

iii) B is a contraction,

then there exists $w \in \mathcal{M}$ such as: $w = Aw + Bw$.

CHAPTER 2

FODE on unbounded interval

2.1 FODE with initial conditions

2.1.1 Position of problem

Recently, the study of fractional differential equations on infinite domain has begun to emerge and evolve extentively. The interest of authors in this study is mainly due to the excitement inspired by the adoption of unusual Banach spaces as well as some other concepts which have been extended to the case of unbounded interval as compactness criteria and others, we refer for example to the papers [18, 36, 58, 83] and the references therein.

Based on these papers and others, we are interested in this subsection in the study of the existence and uniqueness for the following inital value problem of nonlinear fractional differential equations

$$\begin{aligned} {}^{C}D^{\alpha}u(t) &= f\left(t, u(t)\right), \ t \geq 0, \qquad (2.1) \\ u(0) &= u_0, \ u'(0) = u_1, \qquad (2.2) \end{aligned}$$

where $1 < \alpha < 2$, $u_0, u_1 \in \mathbb{R}$, $\mathbb{R}_+ = [0, +\infty)$, $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function, ${}^{C}D^{\alpha}$ is the standard Caputo fractional derivative.

We will mainly discuss the existence and uniqueness of the nonlinear FDE of order $\alpha(1 < \alpha < 2)$ given by equations (2.1) and (2.2). For satisfy this aim, we first transform the fractional differential equation into a first-order ordinary differential equation with a fractional integral perturbation, then by using the means of the variation of constants formula and some analytical skills, we obtain the equivalent integral equations of (2.1)-(2.2). Furthermore, we investigate the existence and uniqueness of nonlinear FDEs (2.1)-(2.2) by using the contraction mapping principle.

2.1.2 Corresponding integral equation

In the following, we give the integral equation coresponding to the FODE (2.1)-(2.2) using the concepts of fractional calculus given in Chapter one. We start by defining the following Banach space:

Let $h : \mathbb{R}_+ \to [1, +\infty)$ be a strictly increasing continuous function with

$$h(0) = 1,\ h(t) \to \infty \text{ as } t \to \infty,\ h(s)h(t-s) \leq h(t),$$

for all $0 \leq s \leq t \leq \infty$. Let

$$E = \left\{ u(t) \in C\,[0, +\infty) : \sup_{t \geq 0} \frac{|u(t)|}{h(t)} < \infty \right\}.$$

Then E is a Banach space equipped with the norm $\|u\| = \sup\limits_{t\geq 0} \frac{|u(t)|}{h(t)}$. For more properties of this Banach space, see [23].

Lemma 2.1 Let $y \in C\,[0, +\infty)$. Then u is a solution of the Cauchy type problem

$$\begin{cases} {}^{C}D^{\alpha}u(t) = y(t),\ t \in \mathbb{R}_+,\ 1 < \alpha < 2, \\ u(0) = u_0,\ u'(0) = u_1, \end{cases} \tag{2.3}$$

if and only if u is a solution of the Cauchy type problem

$$\begin{cases} u^{'}(t) = I^{\alpha-1}y(t) + u_1, \\ u(0) = u_0. \end{cases} \tag{2.4}$$

Proof. (i) Let $u \in C\,[0, +\infty)$ be a solution of the problem (2.3).
For any $t \in \mathbb{R}_+$, we have

$${}^{C}D^{\alpha}u(t) = \left({}^{C}D^{\alpha-1}D^{1}u\right)(t) = y(t).$$

According to Lemma 1.7, we have

$$u'(t) = c + I^{\alpha-1}y(t) = I^{\alpha-1}y(t) + u_1,$$

which means that u is a solution of the problem (2.4).
(ii) Let u be a solution of the problem (2.4).
For any $t \in \mathbb{R}_+$, it is easy to see that

$$^{C}D^{\alpha}u(t) =^{C} D^{\alpha-1}u'(t) = \left(^{C}D^{\alpha-1}I^{\alpha-1}y\right)(t) +^{C} D^{\alpha-1}u_1 = y(t).$$

Besides, note that $y \in C\,[0,+\infty)$, we have $u'(0) = I^{\alpha-1}y(0) + u_1 = u_1$ ■

Lemma 2.1.2 shows that the system (2.1)-(2.2) is equivalent to the system

$$\begin{aligned} u'(t) &= \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2} f(s,u(s))ds + u_1, \\ u(0) &= u_0. \end{aligned} \tag{2.5}$$

Lemma 2.2 Let $k \in \mathbb{R}$ satisfies that

$$e^{-kt}/h(t) \in C\,[0,+\infty) \cap L^1\,[0,+\infty). \tag{2.6}$$

Then (2.5) can be equivalently written as

$$\begin{aligned} u(t) &= u_0 e^{-kt} + \frac{1-e^{-kt}}{k}u_1 + k\int_0^t e^{-k(t-\tau)}u(\tau)d\tau \\ &\quad + \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_\tau^t e^{-k(t-s)}(s-\tau)^{\alpha-2}ds f(\tau,u(\tau))d\tau. \end{aligned} \tag{2.7}$$

Proof. It is clear that (2.5) can be written as follow

$$\begin{aligned} u'(t) + ku(t) &= ku(t) + \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2} f(s,u(s))ds + u_1, \\ u(0) &= u_0. \end{aligned} \tag{2.8}$$

By the variation of constants formula, we have

$$u(t) = u_0 e^{-kt}$$

$$
\begin{aligned}
&+e^{-kt}\int\limits_0^t \left[ku(s)+\frac{1}{\Gamma(\alpha-1)}\int\limits_0^s (s-\tau)^{\alpha-2} f(\tau,u(\tau))d\tau+u_1\right]e^{ks}ds \\
=\;& u_0e^{-kt}+e^{-kt}\int\limits_0^t ku(s)e^{ks}ds \\
&+\frac{1}{\Gamma(\alpha-1)}\int\limits_0^t\int\limits_0^s e^{-k(t-s)}(s-\tau)^{\alpha-2} f(\tau,u(\tau))dsd\tau+u_1e^{-kt}\int\limits_0^t e^{ks}ds \\
=\;& u_0e^{-kt}+k\int\limits_0^t e^{-k(t-s)}u(s)ds \\
&+\frac{1}{\Gamma(\alpha-1)}\int\limits_0^t\int\limits_0^s e^{-k(t-s)}(s-\tau)^{\alpha-2} f(\tau,u(\tau))dsd\tau+u_1e^{-kt}\int\limits_0^t e^{ks}ds
\end{aligned}
$$

so

$$
\begin{aligned}
u(t) \;=\;& u_0e^{-kt}+\frac{1-e^{-kt}}{k}u_1+k\int\limits_0^t e^{-k(t-\tau)}u(\tau)d\tau \\
&+\frac{1}{\Gamma(\alpha-1)}\int\limits_0^t\int\limits_\tau^t e^{-k(t-s)}(s-\tau)^{\alpha-2}dsf(\tau,u(\tau))d\tau.
\end{aligned}
$$

Conversely, it is clear that

$$
\left(e^{kt}u(t)\right)'=\left(u'(t)+ku(t)\right)e^{kt},
$$

using this fact we get

$$
\left(u'(t)+ku(t)\right)e^{kt}=\Bigg[u_0+\frac{e^{kt}-1}{k}u_1+k\int\limits_0^t e^{k\tau}u(\tau)d\tau
$$

$$+\frac{1}{\Gamma(\alpha-1)}\int\limits_0^t\int\limits_\tau^t e^{ks}(s-\tau)^{\alpha-2}dsf(\tau,u(\tau))d\tau\Bigg]'$$

$$= e^{kt}u_1+ke^{kt}u(t)+\left[\int\limits_0^t e^{k\tau}I^{\alpha-1}f(\tau,u(\tau))d\tau\right]'$$

$$= e^{kt}\left(u_1+I^{\alpha-1}f(t,u(t))+ku(t)\right),$$

Farthermore, if (2.7) holds, we have $u(0)=u_0$.

From the argument above, we get that the system (2.1)-(2.2) can be equivalently written as (2.7). Then our following study will focus on the integral equation (2.7) ■

2.1.3 Uniqueness and existence result

Our result based on the Banach contraction principle (Theorem 1.2). We define the nonlinear operator $A:E\longrightarrow E$ by

$$Au(t)=u_0e^{-kt}+\frac{1-e^{-kt}}{k}u_1+k\int\limits_0^t e^{-k(t-\tau)}u(\tau)d\tau$$

$$+\frac{1}{\Gamma(\alpha-1)}\int\limits_0^t\int\limits_\tau^t e^{-k(t-s)}(s-\tau)^{\alpha-2}dsf(\tau,u(\tau))d\tau, \tag{2.9}$$

for all $t\in\mathbb{R}_+$ and $k\in\mathbb{R}$.

We shall investigate the existence and uniqueness of fixed point of the operator A.

Theorem 2.1 Assume that (2.6) holds and

$(H1)$ There exists a constant $l>0$ and a bounded function $\Phi:[0,\infty)\to[0,\infty)$ so that if $|u|,|v|\leq l$ then

(i)

$$|f(t,u)-f(t,v)| \leq \Phi(t)\,|u-v|, \tag{2.10}$$

for all $t \in \mathbb{R}_+$, and

(ii) There exists a constant $c \in (0,1)$ which satisfies

$$|k| \int_0^\infty \frac{e^{-kt}}{h(t)} dt + \sup_{t\geq 0} \int_0^t \frac{K(t-\tau)}{h(t-\tau)} \Phi(\tau)\, d\tau \leq c \tag{2.11}$$

where

$$K(t-\tau) = \begin{cases} \frac{1}{\Gamma(\alpha-1)} \int_\tau^t e^{-k(t-s)} (s-\tau)^{\alpha-2} ds, & t \geq \tau, \\ 0, & t \leq \tau. \end{cases} \tag{2.12}$$

Then the system (2.1)-(2.2) has a unique solution.

Proof. We claim that $A : E \to E$ is a contraction mapping.
Let $u, v \in E$, $t \geq 0$ and from (2.10), on gets

$$\begin{aligned} & \left| \frac{Au(t)}{h(t)} - \frac{Av(t)}{h(t)} \right| \\ = & \left| k \int_0^t \frac{e^{-k(t-\tau)}}{h(t)} \left[u(\tau) - v(\tau)\right] d\tau \right. \\ & \left. + \int_0^t \frac{K(t-\tau)}{h(t)} \left[f(\tau, u(\tau)) - f(\tau, v(\tau))\right] d\tau \right| \\ \leq & \ |k| \int_0^t \frac{e^{-k(t-\tau)}}{h(t-\tau)} \frac{|u(\tau) - v(\tau)|}{h(\tau)} d\tau \\ & + \int_0^t \frac{K(t-\tau)}{h(t-\tau)} \frac{|f(\tau, u(\tau)) - f(\tau, v(\tau))|}{h(\tau)} d\tau \end{aligned}$$

$$
\begin{aligned}
&\leq \ |k| \int_0^t \frac{e^{-k(t-\tau)}}{h(t-\tau)} \frac{|u(\tau)-v(\tau)|}{h(\tau)} d\tau + \int_0^t \frac{K(t-\tau)}{h(t-\tau)} \Phi(\tau) \frac{|u(\tau)-v(\tau)|}{h(\tau)} d\tau \\
&\leq \ |k| \sup_{t\geq 0} \left[\frac{|u(\tau)-v(\tau)|}{h(\tau)} \right] \int_0^t \frac{e^{-k(t-\tau)}}{h(t-\tau)} d\tau \\
&\quad + \sup_{t\geq 0} \left[\frac{|u(\tau)-v(\tau)|}{h(\tau)} \right] \int_0^t \frac{K(t-\tau)}{h(t-\tau)} \Phi(\tau) d\tau \\
&\leq \ |k| \, \|u-v\| \int_0^t \frac{e^{-k(t-\tau)}}{h(t-\tau)} d\tau + \|u-v\| \int_0^t \frac{K(\tau)}{h(u)} \Phi(u) du \\
&\leq \ \left[|k| \int_0^t \frac{e^{-k(t-u)}}{h(t-u)} du + \int_0^t \frac{K(t-u)}{h(t-u)} \Phi(u) du \right] \|u-v\|,
\end{aligned}
$$

then

$$\|Au - Av\| \leq c \, \|u - v\|,$$

follows from (2.6) and (2.11). So, A is a contraction mapping from E into E.

Hence, using the contraction principle mapping and from Theorem 2.1, the operator A given by (2.9) has a unique fixed point. Then, the system (2.1)-(2.2) has a unique solution ■

2.1.4 An example

Let us consider the following nonlinear fractional initial value problem

$$
\begin{cases}
{}^{C}D^{\frac{3}{2}} u(t) = \omega(t) \left(\frac{1+\alpha u+\sin u}{e^{\rho t}} \right), t \geq 0, \\
u(0) = u_0 \in \mathbb{R}, u'(0) = u_1 \in \mathbb{R},
\end{cases}
\tag{2.13}
$$

$f(t) = \omega(t) \left(\frac{1+\alpha u+\sin u}{e^{\rho t}} \right), \omega(t) = \epsilon^{\frac{-1}{2}} \frac{1}{\rho+t^2}, \rho > 0, \alpha > -1$. Let $\lambda > 1, k \in \mathbb{R}, h(t) = e^{\lambda t}$ and suppose that $0 < |k| \leq \frac{\lambda-1}{2}$, clearly that (2.6) holds and $|k| \int_0^\infty \frac{e^{-kt}}{h(t)} dt \leq \frac{|k|}{\lambda+k}$, then the Banach space is

$$E_\lambda = \left\{ u(t) \in C\left[0,+\infty\right) : \sup_{t\geq 0} \left|u(t)\right| / e^{\lambda t} < \infty \right\},$$

equipped with the norm $\|u\| = \sup_{t\geq 0} \frac{|u(t)|}{e^{\lambda t}}$. Clearly

$$\left|f\left(t,u\right) - f\left(t,v\right)\right| \leq \Phi\left(t\right)\left|u-v\right|,$$

where $\Phi\left(t\right) = \left(1+\alpha\right)\omega\left(t\right)$ for all $t \geq 0$. Morover

$$\begin{aligned} \frac{K(t-\tau)}{e^{\lambda(t-\tau)}} &= \frac{1}{\Gamma(1/2)} \int_\tau^t \frac{1}{e^{(\lambda+k)(t-s)}} \frac{(s-\tau)^{-1/2}}{e^{\lambda(s-\tau)}} ds \\ &\leq \frac{\int_\tau^t \frac{(s-\tau)^{-1/2}}{e^{\lambda(s-\tau)}} ds}{\Gamma(1/2)} = \frac{\int_0^{t-\tau} \frac{\tau^{-1/2}}{e^{\lambda\tau}} d\tau}{\Gamma(1/2)} \leq \lambda^{1/2}, \end{aligned}$$

for all $t \geq 0$. Also, if we choose $\rho \geq \left(1+\alpha\right)\lambda^{1/2}\pi\left(\lambda+k\right)$ then for all $t \geq 0$ we get

$$\begin{aligned} & \int_0^t \frac{K(t-\tau)}{h(t-\tau)} \Phi\left(\tau\right) d\tau \\ = \ & \left(1+\alpha\right)\lambda^{1/2}\rho^{-1/2} \int_0^t \frac{1}{\epsilon+\tau^2} d\tau = \left(1+\alpha\right)\lambda^{1/2}\rho^{-3/2} \int_0^t \frac{1}{1+\left(\tau\epsilon^{\frac{-1}{2}}\right)^2} d\tau \\ = \ & \left(1+\alpha\right)\lambda^{1/2}\rho^{-1} \int_0^{t\epsilon^{\frac{-1}{2}}} \frac{dz}{1+z^2} = \left(1+\alpha\right)\lambda^{1/2}\rho^{-1} \arctan\left(t\epsilon^{\frac{-1}{2}}\right) \\ \leq \ & \left(1+\alpha\right)\lambda^{1/2}\rho^{-1}\frac{\pi}{2} \leq \frac{1}{2\left(\lambda+k\right)} < 1 - \frac{|k|}{\lambda+k}. \end{aligned}$$

Then there exists $c = \frac{|k|}{\lambda+k} + \frac{1}{2(\lambda+k)} < 1$ which satisfies $\|Au - Av\| \leq c\,\|u-v\|$. So, the system 2.13 has a unique solution follows from Theorem 2.1.

2.2 Delay FODE with mixed derivatives

2.2.1 Previous works, Position of problem

To the best of our knowledge, the use of mixed fractional derivative in fractional differential equations with delay is still not sufficiently generalized as the other important kinds of fractional differential equations, where we will interest in this subsection to study this type of fractional differential equations. Beside, delay fractional differential equations have been studied extensively in the last decades and by different methods as fixed point theorems, upper and lower solution method, spectral theory and others. For some recent contributions in fractional boundary value problems of fractional differential equations with delay, we can see the papers of Benchohra et al. [15], Agarwal [4], Nouri [68], Bachir et al. [6] and the references therein.

In 2008, Benchohra et al. [15], investigated the existence of solutions for the following Riemann-Liouville fractional order functional differential equations with infinite delay using the Leray-Schauder fixed point theorem.

$$\begin{cases} D^{\alpha}[u(t)-g(t,u\left(t-r\right))]=f(t,u\left(t-r\right)), t\in J=[0,T], 0<\alpha\leq 1, \\ u\left(t\right)=\Phi\left(t\right);\ t\in\left[-\infty,0\right]. \end{cases}$$

Agarwal et al. [4], studied the initial value problem of fractional neutral Caputo fractional derivative

$$\begin{cases} {}^{C}D^{\alpha}[u(t)-g(t,u\left(t-r\right))]=f(t,u\left(t-r\right)), t\in J=[0,T], 0<\alpha\leq 1, \\ u\left(t\right)=\Phi\in B, \end{cases}$$

and established the existence results of solution of this problem by using Krasnoselskii's fixed point theorem.

Nouri et al. [68], by utilizing the Banach fixed point theorem and Krasnoselskii's fixed point theorem, discussed the existence and uniqueness of solutions to the following semilinear Caputo type neutral fractional differential equations

$$\begin{cases} {}^{C}D^{\alpha}[u(t)-g(t,u\left(t-r\right))]=f(t,u\left(t-r\right),Ku\left(t\right)),t\in J=[0,T], \\ u\left(t\right)=\Phi\in B, \end{cases}$$

where $0<\alpha\leq 1$.

Farthermore, there are some works about delay fractional differential equations with sequential fractional derivative. In [6], Bashir et al. studied the qualitative theory of the following boundary value problem

$$\begin{cases} D^{\alpha}\left(D^{\beta}u(t)-g\left(t,u\left(t-r\right)\right)\right)=f(t,u\left(t-r\right)),\ t\in[1,b], \\ u(t)=\phi(t), \quad t\in[1-r,1], \\ D^{\beta}u(1)=\eta\in\mathbb{R}, \end{cases}$$

where D^{α}, D^{β} are the Caputo-Hadamard fractional derivatives, $0<\alpha,\beta<1$.

On the other hand, in 2017, Guezane-Lakoud et al. [55], studied the following mixed fractional boundary value problem

$$\begin{cases} -{}^{C}D_{1^{-}}^{\alpha}D_{0^{+}}^{\beta}u(t)+f(t,u\left(t\right))=0, \quad t\in[0,1], \\ u(0)=u^{'}(0)=u\left(1\right)=0, \end{cases}$$

where $0<\alpha\leq 1, 1<\beta\leq 2, {}^{C}D_{1^{-}}^{\alpha}$ denotes the right Caputo derivative and $D_{0^{+}}^{\beta}$ denotes the left Riemann-Liouville.

For stability of fractional differential equations, Ge and Kou [36], by utilizing the Krasnoselskii's fixed point theorem, discussed the stability and assymptotic stability of zero solution to the following Caputo type fractional differential equations

$$\begin{cases} {}^{C}D^{\alpha}u(t)=f(t,u\left(t\right)),t\geq 0,1<\alpha\leq 2, \\ u\left(0\right)=u_{0},u^{'}\left(0\right)=u_{1}. \end{cases}$$

Farthermore, In [18], Boulares et al. discussed the stability and assymptotic stability of of zero solution of the following boundary value problem with delay

$$\begin{cases} {}^{C}D^{\alpha}u(t) = f(t, u\left(t\right), (t - r\left(t\right))) +^{C} D^{\alpha-1}g(t, u\left(t - r\left(t\right)\right)), t \geq 0, \\ u(t) = \Phi(t), \quad t \in [m_0, 0], u^{'}\left(0\right) = u_1. \quad 1 < \alpha < 2. \end{cases}$$

Motivated and inspired by the works above and the papers ([37], [57] , [59]) and the references therein, we give sufficient conditions to investigate the stability of trivial solution for the following IVP of mixed Riemann-Liouville and Caputo fractional differential equation with delay on undounded interval

$$\begin{cases} D^{\alpha}[{}^{C}D^{\beta}u(t) - g(t, u\left(t - r\right))] = f(t, u\left(t - r\right)), t \geq 0, \\ u\left(t\right) = \Phi\left(t\right), t \in [-r, 0] \\ \lim\limits_{t\to 0} t^{1-\alpha\, C}D^{\beta}u(t) = 0, u'(0) = u_0, \end{cases} \tag{2.14}$$

where $D^{\alpha},^{C} D^{\beta}$ are the left Riemann Liouville and left Caputo fractional derivatives respectively, $0 < \alpha \leq 1$, $1 < \beta \leq 2$, $f, g : \mathbb{R}_{+} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions with $f(t, 0) = g(t, 0) = 0$, $\Phi \in C([-r, 0], \mathbb{R})$ is continuous function.

2.2.2 Study space, integral equation

In this subsection, we present a suitable Banach space for investigate in which the qualitative theory of problem (2.14), then we transforme it in a fixed point problem to show the required end.

Let C_{λ} be the Banach space of all continuous functions defined on $[-r, +\infty)$ with the norm

$$\|u\|_{\lambda} = \sup_{t \geq -r} \left\{ e^{-\lambda t} \left|u\left(t\right)\right| \right\},$$

for all positive real number $\lambda > 1$.

Lemma 2.3 Problem (2.14) is equivalent to the following Caputo type fractional differential equation with delay

$$\begin{cases} {}^{C}D^{\beta}u(t) = I^{\alpha}f(t, u\,(t-r)) + g(t, u\,(t-r)), t \geq 0, \\ u\,(t) = \Phi\,(t)\,, t \in [-r, 0]\,, \\ u^{'}\,(0) = u_0 \end{cases} \tag{2.15}$$

Proof. The first equation of (2.14) can be written as

$${}^{C}D^{\beta}u(t) = I^{\alpha}f(t, u\,(t-r)) + g(t, u\,(t-r)) + c_0 t^{\alpha-1},$$

using condition $\lim\limits_{t\to 0} t^{1-\alpha}\,{}^{C}D^{\beta}u(t) = 0$, we get $c_0 = 0$. Then we obtain the required result ■

Lemma 2.4 Let f, g are continuous functions. Then u is a solution of the problem (2.15) if and only if u is a solution of the delay Cauchy type problem

$$\begin{cases} u^{'}\,(t) = I^{\alpha+\beta-1}f(t, u\,(t-r)) + I^{\beta-1}g(t, u\,(t-r)) + u_0, t \geq 0, \\ u\,(t) = \Phi\,(t)\,, t \in [-r, 0]\,. \end{cases} \tag{2.16}$$

Proof. Let $u \in C\,[-r, +\infty)$ be a solution of the problem (2.15), for any $t \geq 0$ and for $\beta \in (1, 2]$, we have

$${}^{C}D^{\beta}u(t) = \left({}^{C}D^{\beta-1}D^{1}u\right)(t) = I^{\alpha}f(t, u\,(t-r)) + g(t, u\,(t-r)).$$

It is obvious that

$$u'(t) = I^{\beta-1}\left[I^{\alpha}f(t, u\,(t-r)) + g(t, u\,(t-r))\right] + u_0,$$

due to the condition $u^{'}\,(0) = u_0$,

which means that u is a solution of the problem (2.16).

Conversly, let u be a solution of the problem (2.16).

Also, for any $t \geq 0$, it is easy to see that

$$\begin{aligned} {}^{C}D^{\beta}u(t) &= {}^{C}D^{\beta-1}u'(t) \\ &= {}^{C}D^{\beta-1}\left(I^{\alpha+\beta-1}f(t,u\left(t-r\right)) + I^{\beta-1}g(t,u\left(t-r\right))\right) +^{C} D^{\beta-1}u_0 \\ &= I^{\alpha}f(t,u\left(t-r\right)) + g(t,u\left(t-r\right)), \end{aligned}$$

Also, we have $u'(0) = u_0$ ■

Lemma 2.5 Let $k \in \mathbb{R} \backslash \{0\}$ satisfies that $|k| \leq \frac{\lambda-1}{2}$, clearly: $\lambda + k > 0$. Then (2.16) can be equivalently written as

$$\begin{aligned} u\left(t\right) &= \Phi\left(0\right)e^{-kt} + \frac{1-e^{-kt}}{k}u_0 + k\int\limits_0^t e^{-k(t-s)}u(s)ds \\ &\quad + \frac{1}{\Gamma(\alpha+\beta-1)}\int\limits_0^t\int\limits_\tau^t e^{-k(t-s)}\left(s-\tau\right)^{\alpha+\beta-2}dsf(\tau,u(\tau-r))d\tau \\ &\quad + \frac{1}{\Gamma(\beta-1)}\int\limits_0^t\int\limits_\tau^t e^{-k(t-s)}\left(s-\tau\right)^{\beta-2}dsg(\tau,u(\tau-r))d\tau. \end{aligned} \tag{2.17}$$

Proof. It is clear that (2.16) can be written as follow

$$\begin{cases} u'(t) + ku(t) = ku(t) + \frac{1}{\Gamma(\alpha+\beta-1)}\int\limits_0^t (t-s)^{\alpha+\beta-2}f(s,u(s-r))ds \\ +\frac{1}{\Gamma(\beta-1)}\int\limits_0^t (t-s)^{\beta-2}g(s,u(s-r))ds + u_0, \\ u\left(t\right) = \Phi\left(t\right), t \in \left[-\tau, 0\right]. \end{cases}$$

By the variation of constants formula, we have

$$\begin{aligned} u(t) &= \Phi(0)\, e^{-kt} + e^{-kt} \\ &\quad \times \int_0^t \left[ku(s) + \frac{1}{\Gamma(\alpha+\beta-1)} \int_0^s (s-\tau)^{\alpha+\beta-2} f(\tau, u(\tau-r))d\tau \right. \\ &\quad \left. + \frac{1}{\Gamma(\beta-1)} \int_0^s (s-\tau)^{\beta-2} g(\tau, u(\tau-r))d\tau + u_0 \right] e^{ks} ds, \\ &= \Phi(0)\, e^{-kt} + k \int_0^t e^{-k(t-s)} u(s) ds \end{aligned}$$

$$+\frac{1}{\Gamma(\alpha+\beta-1)} \int_0^t \int_\tau^t e^{-k(t-s)} (s-\tau)^{\alpha+\beta-2}\, ds f(\tau, u(\tau-r))d\tau$$
$$+\frac{1}{\Gamma(\beta-1)} \int_0^t \int_\tau^t e^{-k(t-s)} (s-\tau)^{\beta-2}\, ds g(\tau, u(\tau-r))d\tau + \frac{1-e^{-kt}}{k} u_0.$$

Farthermore, it is clear that

$$\left(e^{kt} u(t)\right)' = \left(u'(t) + ku(t)\right) e^{kt},$$

using this fact, we get

$$\left(u'(t) + ku(t)\right) e^{kt}$$

$$
\begin{aligned}
&= \left[\Phi(0) + k\int_0^t e^{ks}u(s)ds + \frac{1}{\Gamma(\alpha+\beta-1)}\int_0^t\int_\tau^t e^{ks}(s-\tau)^{\alpha-2}dsf(\tau,u(\tau-r))d\tau\right. \\
&\quad \left. + \frac{1}{\Gamma(\beta-1)}\int_0^t\int_\tau^t e^{ks}(s-\tau)^{\beta-2}dsg(\tau,u(\tau-r))d\tau + \frac{e^{kt}-1}{k}u_0\right]' \\
&= e^{kt}u_0 + ke^{kt}u(t) + \left[\int_0^t e^{k\tau}I^{\alpha+\beta-1}f(\tau,u(\tau-r))d\tau + \int_0^t e^{k\tau}I^{\beta-1}g(\tau,u(\tau))d\tau\right]' \\
&= e^{kt}\left(u_0 + I^{\alpha+\beta-1}f(t,u(t-r)) + I^{\beta-1}g(t,u(t-r)) + ku(t)\right),
\end{aligned}
$$

this means that

$$u'(t) = I^{\alpha+\beta-1}f(t,u(t-r)) + I^{\beta-1}g(t,u(t-r)) + u_0.$$

On the other hand, if (2.17) holds, we have $u(0) = \Phi(0)$.

From the argument above, we get that the system (2.16) can be equivalently written as (2.17) ■

2.2.3 Stability of solutions

The following definition is needed.

Definition 2.1 The trivial solution $u = 0$ of (2.14) is said to be stable in Banach space C_λ, if for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that $|\Phi(t)|+|u_0| \leq \delta$ implies that the solution $u(t) = u(t, \Phi, u_0)$ exists for all $t \in [-r, +\infty)$ and satisfies $\|x\| \leq \epsilon$.

Let us assume the following hypotheses:

$(H2)$ $f, g : I \times C_r \to \mathbb{R}$ are continuous functions.

$(H3)$ There exists a constant $l > 0$ and a bounded continuous function $\eta(t) > 0$ so that if $|u|, |v| \leq l$ then

$$|g(t,u) - g(t,v)| \leq \eta(t)\,|u - v|, \text{ for } t \in \mathbb{R}_+.$$

$(H4)$ There exist a constant $\gamma > 0$ and tow continuous functions $\zeta : \mathbb{R}_+ \to \mathbb{R}_+, \Psi : (0,\gamma] \to \mathbb{R}_+$ such that

$$|f(t, e^{\lambda(t-r)}u)| \leq e^{\lambda t}\zeta(t)\,\Psi(|u|),$$

holds for all $t \geq 0$, $0 < |u| \leq \gamma$, where Ψ is bounded nondecreasing function and $\zeta \in L^1([0,\infty))$.

Now, we present the stability of trivial solution of system (2.14).

Theorem 2.2 Assume that $(H2) - (H4)$ hold. Then the trivial solution $u = 0$ of (2.14) is stable in Banach space C_λ, provided that there exists constants $M_1, M_2 > 0$ such that

$$\Psi(z) \sup_{t\geq 0} \int_0^t e^{-\lambda(t-\tau)}\mathcal{K}(t-\tau)\,\zeta(\tau)\,d\tau \leq zM_2, \text{ for all } z \in (0,\gamma], \text{ and} \tag{2.18}$$

$$\sup_{t\geq 0} \int_0^t e^{-\lambda(t-\tau)}\mathcal{H}(t-\tau)\,\eta(\tau)\,d\tau \leq M_1 < 1 - \frac{|k|}{\lambda + k} - M_2 < 1, \tag{2.19}$$

where

$$\mathcal{K}(t-\tau) = \begin{cases} \frac{1}{\Gamma(\alpha+\beta-1)} \int_\tau^t e^{-k(t-s)}(s-\tau)^{\alpha+\beta-2}ds, & \text{if } t - \tau \geq 0, \\ 0, & \text{if } t - \tau \leq 0, \end{cases}$$

and

$$\mathcal{H}(t-\tau)=\begin{cases}\frac{1}{\Gamma(\beta-1)}\int\limits_{\tau}^{t} e^{-k(t-s)}(s-\tau)^{\beta-2}ds, & \text{if } t-\tau\geq 0,\\ 0, & \text{if } t-\tau\leq 0,\end{cases}$$

Proof. For any given $\epsilon>0$, we first prove the existence of $\delta>0$ such that

$$|\Phi(t)|+|u_0|<\delta \text{ implies } \|u\|\leq\epsilon.$$

Let $0<\delta\leq\frac{|k|}{|k|+2}\left[\left(1-M_1-M_2-\frac{|k|}{\lambda+k}\right)\epsilon\right]$. Consider the non-empty closed convex subset $B_\epsilon=\{u\in C_\lambda([-r,+\infty),\mathbb{R}):\sup\limits_{t\geq -r}|u(t)|\leq\epsilon$ for $t\geq -r$ and $u(t)=\Phi(t)$ if $t\in[-r,0]\}$ for any $\epsilon>0$. We define two mapping $\mathcal{A},\mathcal{B}:B_\epsilon\to C_\lambda([-r,+\infty],\mathbb{R})$ by:

$$\mathcal{A}u(t)=\begin{cases}0, & \text{if } t\in[-r,0],\\ k\int\limits_0^t e^{-k(t-s)}u(s)ds+\int\limits_0^t\mathcal{K}(t-\tau)f(\tau,u(\tau-r))d\tau & \text{if } t\in I,\end{cases}\tag{2.20}$$

$$\mathcal{B}u(t)=\begin{cases}\Phi(t), & \text{if } t\in[-r,0],\\ \Phi(0)e^{-kt}+\frac{1-e^{-kt}}{k}u_0+\int\limits_0^t\mathcal{H}(t-u)g(\tau,u(\tau-r))d\tau & \text{if } t\in I.\end{cases}\tag{2.21}$$

Clearly, for $u\in B_\epsilon$, both $\mathcal{A}u$ and $\mathcal{B}u$ are continuous functions on $[-r,+\infty)$. Also, for $u\in B_\epsilon$, for any $t\geq 0$, we have

$$\begin{aligned}&e^{-\lambda t}|\mathcal{A}u(t)|\\ \leq\ & |k|e^{-\lambda t}\int\limits_0^t e^{-k(t-s)}|u(s)|ds+\int\limits_0^t e^{-\lambda t}|\mathcal{K}(t-\tau)||f(\tau,u(\tau-r))|d\tau\end{aligned}$$

$$
\begin{aligned}
&\leq \; |k| \int_{0}^{t} e^{-(\lambda+k)(t-s)} \left| e^{-\lambda s} u(s) \right| ds \\
&\quad + \int_{0}^{t} e^{-\lambda(t-\tau)} \left| \mathcal{K}\left(t-\tau\right) \right| \zeta\left(\tau\right) \Psi\left(e^{-\lambda(\tau-r)} \left| u(\tau-r) \right| \right) d\tau \\
&\leq \; |k| \left\| u \right\|_{\lambda} \int_{0}^{\infty} e^{-(\lambda+k)s} ds + \int_{0}^{t} e^{-\lambda(t-\tau)} \left| \mathcal{K}\left(t-\tau\right) \right| \zeta\left(\tau\right) \Psi\left(e^{-\lambda(\tau-r)} \left| u(\tau-r) \right| \right) d\tau \\
&\leq \; \left(\frac{|k|}{\lambda+k} + M_2 \right) \epsilon < \infty, \qquad (2.22)
\end{aligned}
$$

$$
\begin{aligned}
& e^{-\lambda t} \left| \mathcal{B} u(t) \right| \\
&\leq \; \left| \Phi\left(0\right) \right| e^{-(\lambda+k)t} + \frac{e^{-\lambda t} + e^{-(\lambda+k)t}}{|k|} \left| u_0 \right| + \int_{0}^{t} e^{-\lambda t} \mathcal{H}\left(t-u\right) g(\tau, u(\tau-r)) d\tau \\
&\leq \; \left| \Phi\left(0\right) \right| + 2 \frac{|u_0|}{|k|} + \int_{0}^{t} e^{-\lambda(t-\tau)} \mathcal{H}\left(t-u\right) \eta\left(\tau\right) \left| e^{-\lambda \tau} u\left(\tau\right) \right| d\tau \\
&\leq \; \left| \Phi\left(0\right) \right| + 2 \frac{|u_0|}{|k|} + \left\{ \int_{0}^{t} e^{-\lambda(t-\tau)} \mathcal{H}\left(t-u\right) \eta\left(\tau\right) d\tau \right\} \left\| u \right\|_{\lambda} \\
&\leq \left| \Phi\left(0\right) \right| + 2 \frac{|u_0|}{|k|} + M_1 \epsilon < \infty. \qquad (2.23)
\end{aligned}
$$

Then $\mathcal{A}B_\epsilon \subset C_\lambda$ and $\mathcal{B}B_\epsilon \subset C_\lambda$. Now we shall to prove that there exists at least one fixed point of the operator $\mathcal{A}+\mathcal{B}$. To this end, we divide the proof into three claims.

Claim 1: we show that $\mathcal{A}u + \mathcal{B}v \in B_\epsilon$ for all $u, v \in B_\epsilon$, from (2.22) and (2.23), we get

$$\|\mathcal{A}u+\mathcal{B}v\|_{\lambda} \leq \frac{|k|+2}{|k|}\delta + \left(M_1+M_2+\frac{|k|}{\lambda+k}\right)\epsilon \leq \epsilon, \tag{2.24}$$

this means that $\mathcal{A}u+\mathcal{B}v \in B_\epsilon$, for all $u, v \in B_\epsilon$.

Claim 2: Obviously, $\mathcal{A}$ is continuous operator on C_λ, it remains to prove that $\mathcal{A}B_\epsilon$ is relatively compact in C_λ. In fact, from (2.24), we get that $\left\{e^{-\lambda t}u(t) : u \in B_\epsilon\right\}$ is uniformly bounded in C_λ. Moreover, for $t \geq \tau$, we have

$$\begin{aligned} 0 &\leq \lim_{t\to\infty} e^{-\lambda(t-\tau)}\mathcal{K}(t-\tau) \\ &\leq \lim_{t\to\infty}\frac{1}{\Gamma(\alpha+\beta-1)}\int_{\tau}^{t}\left[e^{-(\lambda+k)(t-s)}\right]\left[e^{-\lambda(s-\tau)}(s-\tau)^{\alpha+\beta-2}\right]ds \\ &= \lim_{t\to\infty}\frac{1}{\Gamma(\alpha+\beta-1)}\int_{0}^{t-\tau}\left[e^{-(\lambda+k)(t-\tau-s)}\right]\left[e^{-\lambda s}s^{\alpha+\beta-2}\right]ds = 0 \end{aligned} \tag{2.25}$$

Together with the continuity of functions $\mathcal{K}$ and $t \longmapsto e^{-\lambda t}$, we get that there exists a constant $M_3 > 0$ such that

$$e^{-\lambda(t-\tau)}\left|\mathcal{K}(t-\tau)\right| \leq M_3.$$

Also, for any fixed $T_0 > 0$ and any $t_1, t_2 \in [0, T_0]$, $t_1 < t_2$, we have

$$\begin{aligned} &\left|e^{-\lambda t_2}\mathcal{A}u(t_2) - e^{-\lambda t_1}\mathcal{A}u(t_1)\right| \\ = &\left|k\int_0^{t_2} e^{-\lambda t_2}e^{-k(t_2-s)}u(s)ds - k\int_0^{t_1} e^{-\lambda t_1}e^{-k(t_1-s)}u(s)ds\right. \\ &+ \int_0^{t_2} e^{-\lambda t_2}\mathcal{K}(t_2-\tau)f(\tau, u(\tau-r))\,d\tau \end{aligned}$$

$$
\begin{aligned}
& \left. - \int_0^{t_1} e^{-\lambda t_1} \mathcal{K}\left(t_1-\tau\right) f\left(\tau, u(\tau-r)\right) d\tau \right| \\
\leq\ & |k| \int_0^{t_1} \left| e^{-\lambda t_2} e^{-k(t_2-s)} - e^{-\lambda t_1} e^{-k(t_1-s)} \right| |u(s)|\, ds \\
& + \int_0^{t_1} \left| e^{-\lambda t_2} \mathcal{K}\left(t_2-\tau\right) - e^{-\lambda t_1} \mathcal{K}\left(t_1-\tau\right) \right| \left| f\left(\tau, u(\tau-r)\right) \right| d\tau \\
& + |k| \int_{t_1}^{t_2} e^{-\lambda t_2} e^{-k(t_2-s)} |u(s)|\, ds + \int_{t_1}^{t_2} e^{-\lambda t_2} \mathcal{K}\left(t_2-\tau\right) \left| f\left(\tau, u(\tau-r)\right) \right| d\tau \\
\leq\ & |k| \int_0^{t_1} \left| e^{-(\lambda+k)(t_2-s)} - e^{-(\lambda+k)(t_1-s)} \right| \left| e^{-\lambda s} u(s) \right| ds \\
& + \int_0^{t_1} \left| e^{-\lambda t_2} \mathcal{K}\left(t_2-\tau\right) - e^{-\lambda t_1} \mathcal{K}\left(t_1-\tau\right) \right| e^{\lambda\tau} \zeta\left(\tau\right) \Psi\left(e^{-\lambda(\tau-r)} \left| u(\tau-r) \right| \right) d\tau \\
& + \int_{t_1}^{t_2} e^{-\lambda t_2} \mathcal{K}\left(t_2-\tau\right) e^{\lambda\tau} \zeta\left(\tau\right) \Psi\left(e^{-\lambda(\tau-r)} \left| u(\tau-r) \right| \right) d\tau \\
& + |k| \int_{t_1}^{t_2} e^{-(\lambda+k)(t_2-s)} \left| e^{-\lambda s} u(s) \right| ds \\
\leq\ & \frac{R\,|k|}{\lambda+k} \left(e^{-(\lambda+k)t_1} - e^{-(\lambda+k)t_2} \right) \\
& + \left\{ \int_0^{t_1} \left| e^{-\lambda(t_2-\tau)} \mathcal{K}\left(t_2-\tau\right) - e^{-\lambda(t_1-\tau)} \mathcal{K}\left(t_1-\tau\right) \right| \zeta\left(\tau\right) d\tau \right.
\end{aligned}
$$

$$\left. + M_3 \int_{t_1}^{t_2} \zeta(\tau)\, d\tau \right\} \Psi(\epsilon)$$
$$\to \quad 0 \text{ as } t_2 \to t_1,$$

this means that $\left\{ e^{-\lambda t} u(t) : u \in B_\epsilon \right\}$ is equicontinuous on any compact interval of $\mathbb{R}_+$, it remains to show that the set $\left\{ e^{-\lambda t} u(t) : u \in B_\epsilon \right\}$ is equiconvergent at infinity. In fact, for any $\epsilon_1 > 0$ such that $\epsilon \leq \frac{\lambda + k}{6|k|} \epsilon_1$, there exists a $L > 0$ such that

$$M_3 \Psi(\epsilon) \int_L^{\infty} \zeta(\tau)\, d\tau \leq \frac{\epsilon_1}{6}.$$

According to (2.25), we get

$$\lim_{t \to \infty} \sup_{\tau \in [0,L]} e^{-\lambda(t-\tau)} \mathcal{K}(t-\tau) = 0.$$

Then, there exists $T > L$ such that for every $t_1, t_2 \geq T$, we have

$$\begin{aligned} & \sup_{\tau \in [0,L]} \left| e^{-\lambda t_2} \mathcal{K}(t_2 - \tau) e^{\lambda \tau} - e^{-\lambda t_1} \mathcal{K}(t_1 - \tau) e^{\lambda \tau} \right| \\ \leq & \sup_{\tau \in [0,L]} \left| e^{-\lambda(t_2 - \tau)} \mathcal{K}(t_2 - \tau) \right| + \sup_{\tau \in [0,L]} \left| e^{-\lambda(t_1 - \tau)} \mathcal{K}(t_1 - \tau) \right| \end{aligned}$$

$$\leq \frac{\epsilon_1}{6} \left(\Psi(\epsilon) \int_0^{\infty} \zeta(\tau)\, d\tau \right)^{-1}.$$

Farthermore, for $t \geq s$, one gets

$$\lim_{t \to \infty} e^{-(\lambda + k)(t-s)} = 0,$$

then for $t_1, t_2 \geq T$, we have

$$\begin{aligned} & \sup_{s\in[0,L]} \left| e^{-(\lambda+k)(t_2-s)} - e^{-(\lambda+k)(t_1-s)} \right| \\ \leq & \sup_{s\in[0,L]} \left| e^{-(\lambda+k)(t_2-s)} \right| + \sup_{s\in[0,L]} \left| e^{-(\lambda+k)(t_1-s)} \right| \leq \frac{\epsilon_1}{6} \left(\epsilon \left| k \right| L \right)^{-1}. \end{aligned}$$

Therefore, for $t_1, t_2 \geq T$, we have

$$\begin{aligned} & \left| e^{-\lambda t_2} \mathcal{A}u\left(t_2\right) - e^{-\lambda t_1} \mathcal{A}u\left(t_1\right) \right| \\ = & \left| k \int_0^{t_2} e^{-\lambda t_2} e^{-k(t_2-s)} u(s) ds - k \int_0^{t_1} e^{-\lambda t_1} e^{-k(t_1-s)} u(s) ds \right. \\ & \left. + \int_0^{t_2} e^{-\lambda t_2} \mathcal{K}\left(t_2-\tau\right) f\left(\tau, u(\tau-r)\right) d\tau - \int_0^{t_1} e^{-\lambda t_1} \mathcal{K}\left(t_1-\tau\right) f\left(\tau, u(\tau-r)\right) d\tau \right| \\ \leq & \ \epsilon \left| k \right| \int_0^L \left| e^{-(\lambda+k)(t_2-s)} - e^{-(\lambda+k)(t_1-s)} \right| ds + 2\epsilon \left| k \right| \int_0^\infty e^{-(\lambda+k)s} ds \\ & + \Psi\left(\epsilon\right) M_3 \int_L^{t_2} \zeta\left(\tau\right) d\tau + \Psi\left(\epsilon\right) M_3 \int_L^{t_1} \zeta\left(\tau\right) d\tau \\ & + \Psi\left(\epsilon\right) \int_0^L \left| e^{-\lambda(t_2-\tau)} \mathcal{K}\left(t_2-\tau\right) - e^{-\lambda(t_1-\tau)} \mathcal{K}\left(t_1-\tau\right) \right| \zeta\left(\tau\right) d\tau \\ \leq & \ \frac{\epsilon_1}{6} + \frac{2\epsilon \left| k \right|}{\lambda + k} + 2\Psi\left(\epsilon\right) M_3 \int_L^\infty \zeta\left(\tau\right) d\tau + \frac{\epsilon_1}{6} \leq \epsilon_1, \end{aligned}$$

this achieves the proof.

Claim 3: We show that $\mathcal{B} : B_\epsilon \to C_\lambda$ is a contraction mapping.

In fact, for any $u, v \in B_\epsilon$, using $(H2)$, we have

$$
\begin{aligned}
& \sup_{t\geq 0} e^{-\lambda t} \left|\mathcal{B}u(t) - \mathcal{B}v(t)\right| \\
= & \sup_{t\geq 0} \left\{ \int_0^t e^{-\lambda t} \mathcal{H}\left(t-\tau\right) \left|g(\tau, u(\tau - r)) - g\left(\tau, v(\tau - r)\right)\right| d\tau \right\} \\
\leq & \sup_{t\geq 0} \left\{ \int_0^t e^{-\lambda t} \mathcal{H}\left(t-\tau\right) \eta\left(\tau\right) \left|u\left(\tau\right) - v\left(\tau\right)\right| d\tau \right\} \\
\leq & \sup_{t\geq 0} \left\{ \int_0^t e^{-\lambda(t-\tau)} \mathcal{H}\left(t-\tau\right) \eta\left(\tau\right) \left[e^{-\lambda\tau} \left|u(\tau) - v\left(\tau\right)\right|\right] d\tau \right\} \\
\leq & \left\{ \sup_{t\geq 0} \int_0^t e^{-\lambda(t-\tau)} \mathcal{H}\left(t-\tau\right) \eta\left(\tau\right) d\tau \right\} \left\|u - v\right\|_\lambda \leq M_1 \left\|u - v\right\|_\lambda,
\end{aligned}
$$

from (2.19), $\mathcal{A}$ is a contraction mapping.

By Krasnoselskii fixed point theorem, we know that there exists at least one fixed point of the operator $\mathcal{A} + \mathcal{B}$.

Finally, let $t \geq 0$, for any $\epsilon_2 > 0$, if $0 < \delta_1 \leq \frac{|k|}{|k|+2} \left[\left(1 - M_1 - M_2 - \frac{|k|}{\lambda + k}\right) \epsilon_2\right]$, then $|\phi(t)| + |u_0| < \delta_1$ implies that

$$
\begin{aligned}
& e^{-\lambda t} \left|u\left(t\right)\right| \\
\leq & \left|k\right| e^{-\lambda t} \int_0^t e^{-k(t-s)} \left|u(s)\right| ds + \int_0^t e^{-\lambda t} \left|\mathcal{K}\left(t-\tau\right)\right| \left|f(\tau, u(\tau - r))\right| d\tau
\end{aligned}
$$

$$+\left|\Phi\left(0\right)\right|e^{-(\lambda+k)t}+\frac{e^{-\lambda t}+e^{-(\lambda+k)t}}{|k|}+\int\limits_{0}^{t}e^{-\lambda t}\mathcal{H}\left(t-u\right)g(\tau,u(\tau-r))d\tau$$

$$\leq\ |k|\int\limits_{0}^{t}e^{-(\lambda+k)(t-s)}\left|e^{-\lambda s}u(s)\right|ds+\int\limits_{0}^{t}e^{-\lambda(t-\tau)}\left|\mathcal{K}\left(t-\tau\right)\right|\zeta\left(\tau\right)\Psi\left(e^{-\lambda(\tau-r)}\left|u(\tau-r)\right|\right)d\tau$$

$$+\left|\Phi\left(0\right)\right|+2\frac{|u_{0}|}{|k|}+\int\limits_{0}^{t}e^{-\lambda(t-\tau)}\mathcal{H}\left(t-u\right)\eta\left(\tau\right)\left|e^{-\lambda\tau}u\left(\tau\right)\right|d\tau$$

$$\leq\ \left(M_{1}+M_{2}+\frac{|k|}{\lambda+k}\right)\|u\|_{\lambda}+\frac{|k|+2}{|k|}\delta_{1},$$

this means that

$$\|u\|_{\lambda}\left(1-\left(M_{1}+M_{2}+\frac{|k|}{\lambda+k}\right)\right)\leq\frac{|k|+2}{|k|}\delta_{1},$$

so

$$\|u\|_{\lambda}\leq\frac{|k|+2}{|k|\left(1-M_{1}-M_{2}-\frac{|k|}{\lambda+k}\right)}\delta_{1}\leq\epsilon_{2}.$$

Thus, we know that the trivial solution of (2.14) is stable in Banach space C_{λ} ■

2.2.4 An example

Let us consider the following nonlinear fractional initial value problem with delay:

$$\begin{cases}D^{\frac{1}{2}}[{}^{C}D^{\frac{3}{2}}u(t)-\frac{1}{\theta^{2}+t^{2}}\sin\left(u\left(t-r\right)\right)]=\sigma\frac{e^{\lambda\left(\left(1-\lambda^{-1}\right)t+2e^{-\lambda(t-r)}u(t-r)\right)}\arctan\left(tu^{3}(t-r)\right)}{1+e^{2\lambda e^{-\lambda(t-r)}u(t-r)}},t\geq0,\\ u\left(t\right)=\Phi\left(t\right),t\in\left[-r,0\right]\\ \lim\limits_{t\to0}t^{1-\alpha}\,{}^{C}D^{\beta}u(t)=0,u'(0)=u_{0}\in\mathbb{R},\end{cases}\tag{2.26}$$

$\alpha=\frac{1}{2},\beta=\frac{3}{2}$, $g\left(t,x\right)=\frac{1}{\theta^{2}+t^{2}}\sin\left(x\right)$, $\theta>0$, $g\left(t,0\right)=0$,

$f(t,x) = f_{\lambda,r}(t,x) = \sigma \frac{e^{\lambda\left(\left(1-\lambda^{-1}\right)t+2xe^{-\lambda(t-r)}\right)}\arctan\left(tx^3\right)}{1+e^{2\lambda xe^{-\lambda(t-r)}}}$, then we have:

$|g(t,x) - g(t,y)| \le \frac{1}{\theta^2+t^2}|x-y|$ *i.e.* $\eta(t) = \frac{1}{\theta^2+t^2}$, $\left|f(t, e^{\lambda(t-r)}x)\right| \le e^{\lambda t}\frac{\sigma\pi}{2e^t}\frac{e^{2\lambda x}}{1+e^{2\lambda x}}$ *i.e.* $\zeta(t) = \sigma\frac{\pi}{2e^t}$ and $\Psi(x) = \frac{e^{2\lambda x}}{1+e^{2\lambda x}}$ positive nondecreasing function. ζ is a positive continuous integrable function on $[0,\infty)$ and $\int\limits_0^\infty \zeta(t)\,dt = \frac{\sigma\pi}{2}$.

Farthermore, if there exists $\eta > 0$ such that for all $z \in (0,\eta]$ and $\sigma \le \frac{z}{2\pi M_3(\lambda+k)\Psi(z)}$, then we have

$$\frac{\Psi(z)}{z}\int\limits_0^t e^{-\lambda(t-\tau)}\mathcal{K}(t-\tau)\zeta(\tau)\,d\tau \le \frac{\Psi(z)}{z}M_3\sigma\frac{\pi}{2} \le \frac{1}{4(\lambda+k)} = M_2,$$

$$\begin{aligned} e^{\lambda(t-u)}\mathcal{H}(t-u) &= \frac{1}{\Gamma(\frac{1}{2})}\int\limits_u^t \frac{1}{e^{(\lambda+k)(t-s)}}\frac{(s-u)^{-\frac{1}{2}}}{e^{\lambda(s-u)}}ds, \\ &\le \frac{1}{\Gamma(\frac{1}{2})}\int\limits_u^t \frac{(s-u)^{-\frac{1}{2}}}{e^{\lambda(s-u)}}ds = \frac{1}{\Gamma(\frac{1}{2})}\int\limits_0^{t-u}\frac{\tau^{-\frac{1}{2}}}{e^{\lambda\tau}}d\tau \le \lambda^{\frac{1}{2}}, \text{ for all} t \ge 0. \end{aligned}$$

Also, if we choose $\theta \ge 2\lambda^{\frac{1}{2}}\pi(\lambda+k)$ then for all $t \ge 0$ we get

$$\begin{aligned} \int\limits_0^t e^{-\lambda(t-\tau)}\mathcal{H}(t-\tau)\eta(\tau)\,d\tau &= \theta^{-1}\lambda^{\frac{1}{2}}\int\limits_0^{t\theta^{-1}}\frac{d\tau}{1+\tau^2} \\ &\le \frac{1}{4(\lambda+k)} = M_1 < 1 - \frac{|k|}{\lambda+k} - M_2. \end{aligned}$$

All conditions of theorem 2.2 are satisfied, the trivial solution of (2.26) is stable.

CHAPTER 3

FODE in fractional sobolev spaces

3.1 Introduction

In this chapter, we are interested in the existence and uniqueness of weak solution of a boundary value problem of fractional differential equations in fractional sobolev spaces using fixed point theory.

Let $T > 0$ be a real number and $I = [0, T]$ be a closed and bounded interval of the set of real numbers $\mathbb{R}$. Consider the following nonlinear functional boundary value problem of the higher-order fractional differential equations with Riemann-Liouville derivative $q \in (n-1, n]$

$$\begin{aligned} D^q u(t) &= g(t, u(t), D^s u(t)), \ t \in I, & (3.1) \\ D^{q-i} u|_{t=0} &= 0, \ i = 1, ..., n, \ i \neq n-1 \text{ and } u(T) = 0, & (3.2) \end{aligned}$$

where $1 \leq n-1 < q \leq n, \ 0 < s < 1, \ g : I \times \mathbb{R}^2 \to \mathbb{R}$ is given function, D^q denotes the Riemann-Liouville's fractional derivative.

At recent decades, majority of published papers has been devoted to give the existence and uniqueness of solution of various classes of fractional differential and integral equations in the space of continuous functions $C([a, b])$ or $C(\mathbb{R}_+)$. But the discussion on measurable solutions of differential and integral equations remains relatively few compared to continuous solutions of differential and integral equations, we refer to some papers about this side as [24, 46, 47]. Where the L^p-solutions of fractional differential equations are discussed by Burton and Zhang in [24] using some techniques to show the belonging of solutions to $L^p(\mathbb{R}_+)$. In [46], Schauder's and Darbo's fixed point theorems are employed to study the existence of $L^p(\mathbb{R}_+)$ −solutions of nonlinear quadratic integral equations. Also in [47], the authors give different existence results for $L^p[a, b]$ and $C([a, b])$ −solutions of some nonlinear integral equations of the Hammerstein and Volterra types using some fixed point theorems combined with a general version of Gronwall's inequality.

Motivated by those valuable contributions mentioned above, we mainly discuss the existence and uniqueness of solution for nonlinear FDE given by (3.1)-(3.2) in an apropriate weighted fractional Sobolev space. To achieve our mentioned purpose, we first transform the fractional differential equation (3.1) with conditions (3.2) into an equivalent integral equation with Green continuous function using Laplace transform technic of the Riemann-Liouville fractional derivative and some analytical skills, then we present a suitable study space which is based essentialy on the classical concepts of weighted L^p-spaces and Sobolev spaces.

3.2 Study space

We start by introducing the Riemann-Liouville fractional Sobolev space. Let

$$W_{RL}^{s,p}(I) = \left\{ u \in L^p(I) \text{ and } I^{1-s}u \in W^{1,p}(I) \right\}.$$

Now, we present the completeness of $W_{RL}^{s,p}(I)$.

Lemma 3.1 The Riemann-Liouville fractional Sobolev space $W_{RL}^{s,p}(I)$ is a Banach space endowed with the norm

$$\|u\|_{W_{RL}^{s,p}(I)} = \left(\|u\|_p^p + \left\| I^{1-s}u \right\|_{W^{1,p}(I)}^p \right)^{\frac{1}{p}}.$$

Proof. It is easy to verify that $\|.\|_{W_{RL}^{s,p}(I)}$ defines a norm so we pass to prove the completeness. Let $(u_n) \in W_{RL}^{s,p}(I)$ be a Cauchy sequence, this implies that (u_n) and $(I^{1-s}u_n)$ are Cauchy sequences in $L^p(I)$ and $W^{1,p}(I)$ respectively, since $L^p(I)$ and $W^{1,p}(I)$ are completes, there exist functions u and u_s such that $u_n \to u$ in $L^p(I)$ and $I^{1-s}u_n \to u_s$ in $W^{1,p}(I)$ [*i.e.* $I^{1-s}u_n \to u_s$ in $L^p(I)$ and $(I^{1-s}u_n)' \to u_s'$ in $L^p(I)$].

We have $(I^{1-s}u_n)$ is a Cauchy sequence in $W^{1,p}(I)$, then $(I^{1-s}u_n)$ is a Cauchy sequence in $L^p(I)$, therefore, there exist $v \in L^p(I)$ such that $I^{1-s}u_n \to v$ in $L^p(I)$. Beside, we have $u_n \to u$ in $L^p(I)$, then by using the fact $I^{1-s} : L^p(I) \to L^p(I), s \in (0,1)$, we get $I^{1-s}u_n \to I^{1-s}u$ in $L^p(I)$, so, $I^{1-s}u = v$ and $I^{1-s}u = u_s$.

It remains to show that $(I^{1-s}u)' = u_s'$, where u_s' denotes the first derivatives in distributions sens of u_s. In other term we prove that $(I^{1-s}u_n)' \to (I^{1-s}u)'$ in $L^p(I)$. Clearly, $L^p(I) \subset L_{loc}^1(I)$, then $I^{1-s}u_n$ determines a distribution $\check{T}_{I^{1-s}u_n} \in \mathfrak{D}'(I)$.

For $\Phi \in C_c^1(I)$ and we use Holder inequality we get

$$\begin{aligned}\left|\check{T}_{I^{1-s}u_n}(\Phi) - \check{T}_{I^{1-s}u}(\Phi)\right| &\leq \int\limits_I \left|I^{1-s}u_n(t) - I^{1-s}u(t)\right| |\Phi(t)|\, dt \\ &\leq \|\Phi\|_{p'} \left\|I^{1-s}u_n - I^{1-s}u\right\|_p .\end{aligned}$$

where p' is the exponent conjugate to p. therefore: $\check{T}_{I^{1-s}u_n}(\Phi) \to \check{T}_{I^{1-s}u}(\Phi)$ as $n \to \infty$.

Also, $(I^{1-s}u_n)'$ determine a distribution $\widehat{T}$, then for $\Phi \in C_c^1(I)$ we have

$$\widehat{T}_{(I^{1-s}u_n)'}(\Phi) = \int\limits_I \left(I^{1-s}u_n\right)'(t)\, \Phi(t)\, dt = -\int\limits_I \left(I^{1-s}u_n\right)(t)\, \Phi'(t)\, dt = -\widehat{T}_{I^{1-s}u_n}(\Phi'),$$

we pass to the limit when $n \to \infty$, we obtain

$$\widehat{T}_{u_s'}(\Phi) = -\widehat{T}_{I^{1-s}u}(\Phi') = \widehat{T}_{(I^{1-s}u)'}(\Phi)$$

for every $\Phi \in C_c^1(I)$. Thus $u_s' = (I^{1-s}u)'$ in the distributional sense on I for $s \in (0,1)$.

Consequently, $I^{1-s}u \in W^{1,p}(I)$ and $(I^{1-s}u)' = u_s'$ in distributional sens. Therefore $I^{1-s}u_n \to I^{1-s}u$ in $W^{1,p}(I)$. Accordingly, $u_n \to u$ in $W_{RL}^{s,p}(I)$. Whence $\left(W_{RL}^{s,p}(I), \|.\|_{W_{RL}^{s,p}(I)}\right)$ is a Banach space ■

Remark 3.1 In [16], the authors discussed more broadly about fractional Sobolev space $W_{RL}^{s,p}(I)$ in the case where $p = 1$ to make the relation between this spaces and the classical spaces of functions of bounded variation BV. The authors shown also the completeness of the fractional Sobolev spaces $W_{RL}^{s,1}(I)$.

However, we note that we can not show the existence and uniqueness of solution with using the fixed point theorems in $W_{RL}^{s,p}(I)$. To overcome these problem, we can use a more suitable weighted norm.

We define the weighted L^p-space

$$L^{p,\sigma}(I)=\left\{u\in L^{p}(I),\|u\|_{p,\sigma}<+\infty\right\},$$

where, $\|u\|_{p,\sigma}$ is the positive real valued function defined on $L^{p}(I)$ by

$$\|u\|_{p,\sigma}=\left(\int_{I}\sigma(t)|u(t)|^{p}\,dt\right)^{\frac{1}{p}}\text{ for all }u\in L^{p}(I).$$

Also, we define the weighted fractional sobolev space with Riemann-Liouville fractional derivative by

$$E_{\sigma}(I)=\left\{u\in L^{p,\sigma}(I):I^{1-s}u\in W_{1}^{p,\sigma}(I)\right\},$$

equiped with the norm

$$\|u\|_{\sigma}=\left(\|u\|_{p,\sigma}^{p}+\left\|I^{1-s}u\right\|_{W_{1}^{p,\sigma}}^{p}\right)^{\frac{1}{p}},$$

where

$$W_{1}^{p,\sigma}(I)=\left\{v\in L^{p,\sigma}(I):v'\in L^{p,\sigma}(I)\right\},$$

σ is a given function defined on I and such that there exists a real number $\sigma_{*}>1$ satisfies $1\leq\sigma(t)\leq\sigma_{*}$, for all $t\in I$, and

$$K'(t)\in L^{p,\sigma}(I),\text{ for a.e. }t\in I,\tag{3.3}$$

where

$$K(t)=\begin{cases}\int_{0}^{t}\frac{(\sigma(t-\tau))^{\frac{1}{p}}}{(t-\tau)^{s}}d\tau, & t\geq\tau,\\ 0, & t\leq\tau.\end{cases}\tag{3.4}$$

Clearly

$$\sigma(t-\tau)\geq1,\text{ for all }t,\tau\in I\text{ with }t\geq\tau,$$

3.3 Integral equation

Definition 3.1 A function u is a solution of the system (3.1)-(3.2) if $u \in E_\sigma(I)$ and u satisfies (3.1)-(3.2).

Lemma 3.2 System (3.1)-(3.2) is equivalent to the following integro-differential equation

$$u(t) = \int_0^T G(t,\tau)\, g(\tau, u(\tau), D^s u(\tau))\, d\tau, \quad t \in I,$$

where $G(.,.)$ denotes the Green's function defined on I^2 by

$$G(t,\tau) = \begin{cases} \frac{1}{\Gamma(q)}\left[(t-\tau)^{q-1} - \left(\frac{t}{T}\right)^{q-n+1}(T-\tau)^{q-1}\right]; & 0 \le \tau \le t \le T, \\ \frac{1}{\Gamma(q)}\left[-\left(\frac{t}{T}\right)^{q-n+1}(T-\tau)^{q-1}\right]; & 0 \le t \le \tau \le T. \end{cases} \tag{3.5}$$

Proof. For conveniece we take $\left[D^{q-i}u(t)\right]_{t=0}$ instead b_i. Applying Laplace transform on both side of (3.1) with putting $g(t) = g(t, u(t), D^s u(t))$ and using Lemma 1.8, we get

$$z^q U(z) - \sum_{i=0}^{n-1} z^i \left[D^{q-i-1}u(t)\right]_{t=0} = G(z),$$

where $U(z)$ and $G(z)$ denote the Laplace transformes of $u(t)$ and $g(t)$ respectively.

In other words, we can write

$$U(z) = z^{-q}G(z) + \sum_{i=0}^{n-1} b_{i+1} z^{i-q}.$$

Inverse Laplace transform give us

$$u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1}\, g(\tau, u(\tau), D^s u(\tau))\, d\tau + \sum_{i=0}^{n-1} \frac{b_{i+1}}{\Gamma(q-i)} t^{q-i-1}$$

$$= \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} g(\tau, u(\tau), D^s u(\tau))\, d\tau + \sum_{i=1}^{n} \frac{b_i}{\Gamma(q-i+1)} t^{q-i},$$

we have $b_i = 0$, $i = 1, ..., n$ for $i \neq n-1$ then

$$u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} g(\tau, u(\tau), D^s u(\tau))\, d\tau + \frac{b_{n-1}}{\Gamma(q-n+2)} t^{q-n+1}. \tag{3.6}$$

But condition $u(T) = 0$, then we obtain

$$\frac{b_{n-1}}{\Gamma(q-n+2)} = \frac{-T^{n-q-1}}{\Gamma(q)} \int_0^T (T-\tau)^{q-1} g(\tau, u(\tau), D^s u(\tau))\, d\tau,$$

substuting in (3.6), we get

$$u(t) = \int_0^T G(t,\tau)\, g(\tau, u(\tau), D^s u(\tau))\, d\tau,$$

where $G(.,.)$ is the Green's kernel defined by (3.5). The proof of Lemma 3.2 is complete ■

Define the operator $\mathcal{T} : E_\sigma(I) \to E_\sigma(I)$ by

$$\mathcal{T}u(t) = \int_0^T G(t,\tau)\, g(\tau, u(\tau), D^s u(\tau))\, d\tau. \tag{3.7}$$

In the following, we present some existence results for the boundary value problem (3.1)-(3.2).

3.4 Existence results

In this section, we present two existence results for BVP of FODE (3.1)-(3.2) using fixed point theorems. First result is based on Shauder fixed point theorem in order to investigate the existence of the weak solutions of the system (3.1)-(3.2).

Assume the following hypotheses:

$(H5)$ The function $g : I \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory's condition.

$(H6)$ There exist a real constant $c > 0$ and a function $b : I \to \mathbb{R}_+$ belongs to $L^{1,\sigma}(I)$ and such that

$$|g(\tau, u, v)| \leq b(\tau) + c(|u|^p + |v|^p),$$

for any $\tau \in I$ and any $u, v \in \mathbb{R}$.

$(H7)$ There exists a real number $R > 0$ satisfies

$$G_* \left[T\sigma_* + \frac{\sigma_* T^{1+p(1-s)}}{(\Gamma(2-s))^p} + \frac{\|K'\|_{p,\sigma}}{(\Gamma(1-s))^p} \right]^{\frac{1}{p}} \left[\|b\|_{1,\sigma} + cR^p \right] \leq R,$$

where $G_* = \sup\limits_{(t,\tau)\in I^2} |G(t,\tau)|$.

Theorem 3.1 If $(H5) - (H7)$ hodl, then problem (3.1)-(3.2) has at least one solution.

Proof. Consider the operator $\mathcal{T}$ given by (3.7) and we define the set

$$B_R = \{u \in E_\sigma, \|u\|_\sigma \leq R\},$$

where R is the same constant defined in $(H7)$. It is clear that B_R is convex, closed and bounded subset of E_σ.

Firstly, we show that $\mathcal{T}B_R \subset B_R$. Let $u \in B_R$, then by using $(H5), (H6)$, we get

$$\sigma(t)^{\frac{1}{p}} |\mathcal{T}u(t)| \leq \sigma(t)^{\frac{1}{p}} \int_0^T |G(t,\tau)| \, |g(\tau, u(\tau), D^s u(\tau))| \, d\tau$$

$$
\begin{aligned}
&\leq \ \sigma(t)^{\frac{1}{p}} G_* \int_0^T \left[\sigma(\tau)\left(b(\tau) + c\left(|u(\tau)|^p + |D^s u(\tau)|^p\right)\right)\right] d\tau \\
&\leq \ \sigma_*^{\frac{1}{p}} G_* \left[\|b\|_{1,\sigma} + c\left(\|u\|_{p,\sigma}^p + \left\|\left(I^{1-s}u\right)'\right\|_{p,\sigma}^p\right)\right] \\
&\leq \ \sigma_*^{\frac{1}{p}} G_* \left[\|b\|_{1,\sigma} + c\,\|u\|_{\sigma}^p\right].
\end{aligned}
$$

$$
\|\mathcal{T}u\|_{p,\sigma}^p \leq T\sigma_* G_*^p \left[\|b\|_{1,\sigma} + cR^p\right]^p. \tag{3.8}
$$

Similarly, we obtain the following estimates

$$
\left\|I^{1-s}\mathcal{T}u\right\|_{p,\sigma}^p \leq \frac{\sigma_* T^{1+p(1-s)}}{(\Gamma(2-s))^p} G_*^p \left[\|b\|_{1,\sigma} + cR^p\right]^p, \tag{3.9}
$$

and

$$
\begin{aligned}
&\sigma(t)^{\frac{1}{p}} \left|\left(I^{1-s}\mathcal{T}u\right)'(t)\right| \\
\leq \ & \frac{\sigma(t)^{\frac{1}{p}}}{\Gamma(1-s)} \frac{d}{dt}\int_0^t (t-\tau)^{-s} \int_0^T \frac{|G(\tau,\theta)|}{\sigma(\theta)} \sigma(\theta) \left|g\left(\theta, u(\theta), D^s u(\theta)\right)\right| d\theta d\tau \\
\leq \ & \frac{G_*}{\Gamma(1-s)} \left[\sigma^{\frac{1}{p}}(t) \frac{d}{dt}\int_0^t \frac{(\sigma(t-\tau))^{\frac{1}{p}}}{(t-\tau)^s} d\tau\right] \left[\|b\|_{1,\sigma} + cR^p\right],
\end{aligned}
$$

$$
\left\|\left(I^{1-s}\mathcal{T}u\right)'\right\|_{p,\sigma}^p \leq \frac{\|K'\|_{p,\sigma}}{(\Gamma(1-s))^p} G_*^p \left[\|b\|_{1,\sigma} + cR^p\right]^p. \tag{3.10}
$$

We combine (3.8)-(3.10), it yilds

$$
\|\mathcal{T}u\|_{\sigma} \leq G_* \left[T\sigma_* + \frac{\sigma_* T^{1+p(1-s)}}{(\Gamma(2-s))^p} + \frac{\|K'\|_{p,\sigma}}{(\Gamma(1-s))^p}\right] \left[\|b\|_{1,\sigma} + cR^p\right] \leq R. \tag{3.11}
$$

Hence $\mathcal{T}B_R \subset B_R$.

Secondly, We prove that A is continuous operator. Let u_n, u in E_σ such that: $u_n \to u$ in E_σ, then for all $t \in I$ we have

$$\begin{aligned}
& \sigma(t)^{\frac{1}{p}} \left|\mathcal{T}u_n(t) - \mathcal{T}u(t)\right| \\
\leq\ & \sigma(t)^{\frac{1}{p}} \int_0^T \frac{|G(t,\tau)|}{\sigma(\tau)} \left[\sigma(\tau)\left|g(\tau, u_n(\tau), D^s u_n(\tau)) - g(\tau, u(\tau), D^s u(\tau))\right|\right] d\tau \\
\leq\ & \sigma_*^{\frac{1}{p}} G_* \left\|N_g u_n - N_g u\right\|_{1,\sigma} \\
\leq\ & \sigma_*^{\frac{1}{p}} G_* \left\|N_g u_n - N_g u\right\|_{1,\sigma},
\end{aligned}$$

applying L^p-norm, one gets

$$\left\|\mathcal{T}u_n - \mathcal{T}u\right\|_{p,\sigma} \leq (T\sigma_*)^{\frac{1}{p}} G_* \left\|N_g u_n - N_g u\right\|_{1,\sigma}. \tag{3.12}$$

Also

$$\begin{aligned}
& \sigma(t)^{\frac{1}{p}} \left|I^{1-s}Au_n(t) - I^{1-s}Au(t)\right| \\
\leq\ & \frac{\sigma(t)^{\frac{1}{p}}}{\Gamma(1-s)} \int_0^t (t-\tau)^{-s} \int_0^T \frac{|G(\tau,\theta)|}{\sigma(\theta)} \left(\sigma(\theta)\left|g(\theta, u_n(\theta), D^s u_n(\theta))\right.\right. \\
& - \left.\left. g(\theta, u(\theta), D^s u(\theta))\right|\right) d\theta d\tau \\
\leq\ & \frac{\sigma(t)^{\frac{1}{p}} G_*}{\Gamma(1-s)} \int_0^t (t-\tau)^{-s} \left\|N_g u_n - N_g u\right\|_{1,\sigma} d\tau \\
\leq\ & \frac{\sigma_*^{\frac{1}{p}} T^{1-s}}{\Gamma(2-s)} G_* \left\|N_g u_n - N_g u\right\|_{1,\sigma},
\end{aligned}$$

then

$$\left\|I^{1-s}\mathcal{T}u_n - I^{1-s}\mathcal{T}u\right\|_{p,\sigma}^p \leq \frac{\sigma_*^{\frac{1}{p}} T^{\frac{1}{p}+1-s}}{\Gamma(2-s)} G_* \left\|N_g u_n - N_g u\right\|_{1,\sigma}. \tag{3.13}$$

Moreover

$$\begin{aligned}
&\sigma(t)^{\frac{1}{p}}\left|\left(I^{1-s}\mathcal{T}u_n\right)'(t)-\left(I^{1-s}\mathcal{T}u\right)'(t)\right| \\
\leq\ & \frac{G_*}{\Gamma(1-s)}\left[\sigma^{\frac{1}{p}}(t)\frac{d}{dt}\int_0^t\frac{(\sigma(t-\tau))^{\frac{1}{p}}}{(t-\tau)^s}d\tau\right]\|N_g u_n-N_g u\|_{1,\sigma},
\end{aligned}$$

i.e.

$$\left\|\left(I^{1-s}\mathcal{T}u_n\right)'-\left(I^{1-s}\mathcal{T}u\right)'\right\|_{p,\sigma}\leq\frac{\|K'\|_{p,\sigma}G_*}{\Gamma(1-s)}\|N_g u_n-N_g u\|_{1,\sigma}. \tag{3.14}$$

Combining (3.12)-(3.14), one finds

$$\begin{aligned}
\|\mathcal{T}u_n-\mathcal{T}u\|_\sigma \ \leq\ & G_*\left[(T\sigma_*)^{\frac{1}{p}}+\frac{\sigma_*^{\frac{1}{p}}T^{\frac{1}{p}+1-s}}{\Gamma(2-s)}+\frac{\|K'\|_{p,\sigma}}{\Gamma(1-s)}\right] \\
&\times\|N_g u_n-N_g u\|_{1,\sigma}.
\end{aligned}\tag{3.15}$$

Taking $(H5)$,$(H6)$ and Lemma 1.9 into acount, we deduce that the Nemytskii operator N_g is continuous from $L^{p,\sigma}$ to $L^{1,\sigma}$, then the right side term of (3.15) tends to zero when n tends to infinity. This show that the operator $\mathcal{T}$ is continuous.

Thirdly, we prove that the set $\mathcal{T}B_R=\{\mathcal{T}u : u\in B_R\}$ is relatively compact in E_σ using Kolmogorov theorem.

For any $u\in B_R$ and any $\delta\geq 0$, we have

$$\begin{aligned}
&\sigma(t)^{\frac{1}{p}}\left|\mathcal{T}u(t+\delta)-\mathcal{T}u(t)\right| \\
\leq\ & \sigma(t)^{\frac{1}{p}}\int_0^T|G(t+\delta,\tau)-G(t,\tau)|\,|g(\tau,u(\tau),D^s u(\tau))|\,d\tau \\
\leq\ & \sigma(t)^{\frac{1}{p}}\int_0^T\frac{|G(t+\delta,\tau)-G(t,\tau)|}{\sigma(\tau)}\left[\sigma(\tau)\left(b(\tau)+c\left(|u(\tau)|^p+|D^s u(\tau)|^p\right)\right)\right]d\tau
\end{aligned}$$

$$
\begin{aligned}
&\leq \sigma(t)^{\frac{1}{p}} \sup_{\tau\in I} |G(t+\delta,\tau) - G(t,\tau)| \left[\|b\|_{1,\sigma} + c\left(\|u^p\|_{1,\sigma} + \|(D^s u)^p\|_{1,\sigma}\right)\right] \\
&= \sigma_*^{\frac{1}{p}} \sup_{t\in I} \left[\sup_{\tau\in I} |G(t+\delta,\tau) - G(t,\tau)|\right] \left[\|b\|_{1,\sigma} + c\left(\|u\|_{p,\sigma}^p + \left\|\left(I^{1-s}u\right)'\right\|_{p,\sigma}^p\right)\right] \\
&\leq \sigma_*^{\frac{1}{p}} \sup_{t\in I} \left[\sup_{\tau\in I} |G(t+\delta,\tau) - G(t,\tau)|\right] \left[\|b\|_{1,\sigma} + c\|u\|_{\sigma}^p\right],
\end{aligned}
$$

$$
\frac{\|\mathcal{T}u_n(.+\delta) - \mathcal{T}u(.)\|_{p,\sigma}}{(T\sigma_*)^{\frac{1}{p}}\left[\|b\|_{1,\sigma} + cR^p\right]} \leq \sup_{t\in I}\left[\sup_{\tau\in I} |G(t+\delta,\tau) - G(t,\tau)|\right]. \tag{3.16}
$$

Similarly

$$
\begin{aligned}
&\sigma(t)^{\frac{1}{p}} \left|I^{1-s}\mathcal{T}u(t+\delta) - I^{1-s}\mathcal{T}u(t)\right| \\
\leq\; & \frac{\sigma(t)^{\frac{1}{p}}}{\Gamma(1-s)} \int_0^t (t-\tau)^{-s} \int_0^T |G(\tau+\delta,\theta) - G(\tau,\theta)| \, |g(\theta,u(\theta),D^s u(\theta))| \, d\theta d\tau \\
\leq\; & \frac{\sigma(t)^{\frac{1}{p}}}{\Gamma(1-s)} \int_0^t (t-\tau)^{-s} \int_0^T \frac{|G(\tau+\delta,\theta) - G(\tau,\theta)|}{\sigma(\theta)} \sigma(\theta) \left[b(\theta)\right. \\
&\left. + c\left(|u(\theta)|^p + |D^s u(\theta)|^p\right)\right] d\theta d\tau \\
\leq\; & \frac{\sigma(t)^{\frac{1}{p}}}{\Gamma(1-s)} \int_0^t (t-\tau)^{-s} \sup_{\theta\in I} |G(\tau+\delta,\theta) - G(\tau,\theta)| \left[\|b\|_{1,\sigma}\right. \\
&\left. + c\left(\|u^p\|_{1,\sigma} + \|(D^s u)^p\|_{1,\sigma}\right)\right] d\tau \\
\leq\; & \frac{T^{1-s}\sigma_*^{\frac{1}{p}}}{\Gamma(2-s)} \sup_{\tau\in I}\left[\sup_{\theta\in I} |G(\tau+\delta,\theta) - G(\tau,\theta)|\right] \left[\|b\|_{1,\sigma} + c\|u\|_{\sigma}^p\right],
\end{aligned}
$$

$$
\frac{\Gamma(2-s)\left\|\left(I^{1-s}\mathcal{T}u_n\right)(.+\delta) - \left(I^{1-s}\mathcal{T}u\right)(.)\right\|_{p,\sigma}}{T^{1-s}(T\sigma_*)^{\frac{1}{p}}\left[\|b\|_{1,\sigma} + cR^p\right]} \leq \sup_{\tau\in I}\left[\sup_{\theta\in I} |G(\tau+\delta,\theta) - G(\tau,\theta)|\right]. \tag{3.17}
$$

Using same method, one finds

$$
\begin{aligned}
& \sigma(t)^{\frac{1}{p}}\left|\left(I^{1-s}\mathcal{T}u\right)'(t+\delta)-\left(I^{1-s}\mathcal{T}u\right)'(t)\right| \\
\leq\; & \frac{\sigma(t)^{\frac{1}{p}}}{\Gamma(1-s)}\frac{d}{dt}\int_{0}^{t}(t-\tau)^{-s}\int_{0}^{T}\left|G(\tau+\delta,\theta)-G(\tau,\theta)\right|\left|g\left(\theta,u(\theta),D^{s}u(\theta)\right)\right|d\theta d\tau \\
\leq\; & \frac{\sigma(t)^{\frac{1}{p}}}{\Gamma(1-s)}\frac{d}{dt}\int_{0}^{t}(t-\tau)^{-s}\int_{0}^{T}\frac{\left|G(\tau+\delta,\theta)-G(\tau,\theta)\right|}{\sigma(\theta)} \\
& \times\left[\sigma(\theta)\left(b(\theta)+c\left(|u(\theta)|^{p}+|D^{s}u(\theta)|^{p}\right)\right)\right]d\theta d\tau \\
\leq\; & \frac{\sigma(t)^{\frac{1}{p}}}{\Gamma(1-s)}\frac{d}{dt}\int_{0}^{t}(t-\tau)^{-s}\sup_{\tau\in I}\left|G(t+\delta,\tau)-G(t,\tau)\right|d\tau \\
& \times\left[\|b\|_{1,\sigma}+c\left(\|u^{p}\|_{1,\sigma}+\|(D^{s}u)^{p}\|_{1,\sigma}\right)\right] \\
\leq\; & \frac{\left[\sigma(t)^{\frac{1}{p}}\,\frac{d}{dt}\int_{0}^{t}\frac{(\sigma(t-\tau))^{\frac{1}{p}}}{(t-\tau)^{s}}d\tau\right]}{\Gamma(1-s)}\sup_{t\in I}\left[\sup_{\tau\in I}\left|G(t+\delta,\tau)-G(t,\tau)\right|\right] \\
& \times\left[\|b\|_{1,\sigma}+c\left(\|u\|_{p,\sigma}^{p}+\left\|\left(I^{1-s}u\right)'\right\|_{p,\sigma}^{p}\right)\right],
\end{aligned}
$$

$$
\frac{\Gamma(1-s)\left\|\left(I^{1-s}\mathcal{T}u\right)'(.+\delta)-\left(I^{1-s}\mathcal{T}u\right)'(.)\right\|_{p,\sigma}}{\|K'\|_{p,\sigma}\left[\|b\|_{1,\sigma}+cR^{p}\right]}\leq\sup_{t\in I}\left[\sup_{\tau\in I}\left|G(t+\delta,\tau)-G(t,\tau)\right|\right]. \tag{3.18}
$$

From the continuity of the function $G(.,.)$ on I^2, we conclude that the second members of (3.16)-(3.18) tend to zero when δ tends to zero, these prove the condition (i) of Lemma 1.10.

Fartheremore, from 3.11, we have $\|\mathcal{T}u\|_{\sigma}\leq R$ for all $u\in B_R$, this proves that $\mathcal{T}B_R$ is uniformly bounded. Consiquently, $\mathcal{T}B_R$ is relatively compact in E_σ. Finally, using Schauder's fixed point theorem, we conclude that $\mathcal{T}$ has at least one fixed point in B_R and the proof of Theorem 3.1 is complete ■

Second result devoted to show the existence of unique solution of (3.1)-(3.2) using Banach contraction principle.

Consider the following hypotheses on g:

$(H8)$ There exist a positive real number $p^{'} \geq 1$ and a function $\varphi : I \to \mathbb{R}_+$ and such that

(i) $\varphi \in L^{p^{'}}(I)$ a.e. $t \in I$, where $\frac{1}{p} + \frac{1}{p^{'}} = 1$.

(ii) For any $t \in I$ and any $u, v, \overline{u}, \overline{v} \in \mathbb{R}$, we have

$$\left|g\left(t,u,v\right) - g\left(\tau,\overline{u},\overline{v}\right)\right| \leq \varphi\left(t\right)\left[\left|u-\overline{u}\right| + \left|v-\overline{v}\right|\right].$$

Theorem 3.2 Assume that $(H5), (H8)$ hold. Then the boundary value problem (3.1)-(3.2) has a unique solution provided

$$G_* \left\|\varphi\right\|_{p'} \left[\left(T\sigma_*\right)^{\frac{1}{p}} + \frac{T^{1-\beta}\left(T\sigma_*\right)^{\frac{1}{p}}}{\Gamma\left(2-\beta\right)} + \frac{\left\|K^{'}\right\|_{p,\sigma}}{\Gamma\left(1-\beta\right)}\right] < 1,$$

where $G_* = \max\limits_{(t,\tau)\in I^2} \left|G\left(t,\tau\right)\right|$.

Proof. Consider the operator $\mathcal{T}$ given by (3.7), we want to show that $\mathcal{T}$ is a contraction mapping on $E_\sigma(I)$. To this purpose, let u, v in E_σ, using $(H5)$ and $(H8)$, then for a.e. $t \in I$ we have

$$\begin{aligned}
& \sigma\left(t\right)^{\frac{1}{p}}\left|\mathcal{T}u\left(t\right) - \mathcal{T}v\left(t\right)\right| \\
\leq\ & \sigma\left(t\right)^{\frac{1}{p}} \int\limits_0^T \frac{\left|G\left(t,\tau\right)\right|}{\left(\sigma\left(\tau\right)\right)^{\frac{1}{p}}}\left[\left(\sigma\left(\tau\right)\right)^{\frac{1}{p}}\left|g\left(\tau,u\left(\tau\right),D^\beta u\left(\tau\right)\right) - g\left(\tau,v\left(\tau\right),D^\beta v\left(\tau\right)\right)\right|\right] d\tau \\
\leq\ & G_*\sigma\left(t\right)^{\frac{1}{p}} \int\limits_0^T \varphi\left(\tau\right)\left[\left(\sigma\left(\tau\right)\right)^{\frac{1}{p}}\left(\left|u\left(\tau\right)-v\left(\tau\right)\right| + \left|D^\beta u\left(\tau\right) - D^\beta v\left(\tau\right)\right|\right)\right] d\tau
\end{aligned}$$

$$
\begin{aligned}
&\leq G_*\sigma_*^{\frac{1}{p}}\|\varphi\|_{p'}\left\|\sigma^{\frac{1}{p}}\left(|u-v|+\left|D^\beta u-D^\beta v\right|\right)\right\|_p \\
&\leq G_*\sigma_*^{\frac{1}{p}}\|\varphi\|_{p'}\left[\|u-v\|_{p,\sigma}+\left\|D^\beta u-D^\beta v\right\|_{p,\sigma}\right] \\
&\leq G_*\sigma_*^{\frac{1}{p}}\|\varphi\|_{p'}\left[\|u-v\|_{p,\sigma}+\left\|\left(I^{1-\beta}u\right)'-\left(I^{1-\beta}v\right)'\right\|_{p,\sigma}\right] \\
&\leq G_*\sigma_*^{\frac{1}{p}}\|\varphi\|_{p'}\|u-v\|_\sigma
\end{aligned}
$$

applying L^p-norm, we get

$$
\|\mathcal{T}u-\mathcal{T}v\|_{p,\sigma}\leq G_*\left(T\sigma_*\right)^{\frac{1}{p}}\|\varphi\|_{p'}\|u-v\|_\sigma \tag{3.19}
$$

Also

$$
\begin{aligned}
&\sigma(t)^{\frac{1}{p}}\left|I^{1-\beta}\mathcal{T}u(t)-I^{1-\beta}\mathcal{T}v(t)\right| \\
\leq\;& \frac{\sigma(t)^{\frac{1}{p}}}{\Gamma(1-\beta)}\int_0^t(t-\tau)^{-\beta}\int_0^T\frac{|G(\tau,\theta)|}{\sigma(\theta)^{\frac{1}{p}}}\left[\sigma(\theta)^{\frac{1}{p}}\left|g\left(\theta,u(\theta),D^\beta u(\theta)\right)\right.\right. \\
&-\left.\left.g\left(\theta,v(\theta),D^\beta v(\theta)\right)\right|\right]d\theta d\tau \\
\leq\;& \frac{G_*\sigma(t)^{\frac{1}{p}}}{\Gamma(1-\beta)}\int_0^t(t-\tau)^{-\beta}\int_0^T\varphi(\theta)\left[(\sigma(\theta))^{\frac{1}{p}}\left(|u(\theta)-v(\theta)|\right.\right. \\
&+\left.\left.\left|D^\beta u(\theta)-D^\beta v(\theta)\right|\right)\right]d\theta d\tau
\end{aligned}
$$

$$
\begin{aligned}
\leq\;& \frac{G_*\sigma_*^{\frac{1}{p}}}{\Gamma(1-\beta)}\left(\int_0^t(t-\tau)^{-\beta}d\tau\right)\|\varphi\|_{p'}\left\|\sigma^{\frac{1}{p}}\left(|u-v|+\left|D^\beta u-D^\beta v\right|\right)\right\|_p \\
\leq\;& \frac{G_*\sigma_*^{\frac{1}{p}}}{\Gamma(2-\beta)}T^{1-\beta}\|\varphi\|_{p'}\left[\|u-v\|_{p,\sigma}+\left\|\left(I^{1-\beta}u\right)'-\left(I^{1-\beta}v\right)'\right\|_{p,\sigma}\right],
\end{aligned}
$$

therefore

$$\left\| I^{1-\beta}\mathcal{T}u - I^{1-\beta}\mathcal{T}v \right\|_{p,\sigma} \leq \frac{T^{1-\beta}G_*\left(T\sigma_*\right)^{\frac{1}{p}}}{\Gamma\left(2-\beta\right)} \left\|\varphi\right\|_{p'} \left\|u-v\right\|_{\sigma}. \tag{3.20}$$

Moreover

$$\begin{aligned}
& \sigma\left(t\right)^{\frac{1}{p}} \left| \left(I^{1-\beta}\mathcal{T}u\right)'\left(t\right) - \left(I^{1-\beta}\mathcal{T}v\right)'\left(t\right) \right| \\
\leq\ & \frac{G_*\sigma\left(t\right)^{\frac{1}{p}}}{\Gamma\left(1-\beta\right)} \frac{d}{dt}\int_0^t \left(t-\tau\right)^{-\beta} \int_0^T \varphi\left(\theta\right) \left[\left(\sigma\left(\theta\right)\right)^{\frac{1}{p}} \left(\left|u\left(\theta\right) - v\left(\theta\right)\right| \right.\right. \\
& \left.\left. + \left| D^{\beta}u\left(\theta\right) - D^{\beta}v\left(\theta\right) \right| \right) \right] d\theta d\tau \\
\leq\ & \frac{G_*}{\Gamma\left(1-\beta\right)} \left[\sigma\left(t\right)^{\frac{1}{p}} \frac{d}{dt}\int_0^t \frac{\left(\sigma\left(t-\tau\right)\right)^{\frac{1}{p}}}{\left(t-\tau\right)^{\beta}} d\tau \right] \left\|\varphi\right\|_{p'} \\
& \times \left\| \sigma^{\frac{1}{p}} \left(\left|u-v\right| + \left|D^{\beta}u - D^{\beta}v\right| \right) \right\|_{p},
\end{aligned}$$

using some precedent method and applying L^p-norm on both sides of previous inequatity, we get

$$\left\| \left(I^{1-\beta}\mathcal{T}u\right)' - \left(I^{1-\beta}\mathcal{T}v\right)' \right\|_{p,\sigma} \leq \frac{G_*}{\Gamma\left(1-\beta\right)} \left\| K' \right\|_{p,\sigma} \left\|\varphi\right\|_{p'} \left\|u-v\right\|_{\sigma} \tag{3.21}$$

Combining inequalities (3.19)-(3.21) then we obtain

$$\begin{aligned}
\left\|\mathcal{T}u - \mathcal{T}v\right\|_{\sigma} &\leq G_* \left\|\varphi\right\|_{p'} \left[\left(T\sigma_*\right)^{\frac{1}{p}} + \frac{T^{1-\beta}\left(T\sigma_*\right)^{\frac{1}{p}}}{\Gamma\left(2-\beta\right)} + \frac{\left\|K\right\|_{p,\sigma}}{\Gamma\left(1-\beta\right)} \right] \left\|u-v\right\|_{\sigma} \\
&< \left\|u-v\right\|_{\sigma},
\end{aligned}$$

this means that the operator is a contraction. Hence, by using Banach contraction principle and according to the theorem 3.2, we conclude that $\mathcal{T}$ has a unique fixed point in E_{σ}. Then (3.1)-(3.2) has a unique fixed point ■

3.5 Examples

Example 3.1 Consider the following boundary value problem of fractional differential equations for $p=4$:

$$\begin{gathered} D^{q}u(t)=\frac{t^{\frac{-1}{2}}e^{t}+\left[(tu(t))^{2}-(tD^{s}u(t))^{2}\right]^{2}}{(1+t)^{3}e^{t+|u(t)|}},\ t\in I=[0,1], \\ D^{(q-i)}u|_{t=0}=0,\ i=1,2,3,5,\ u(1)=0 \end{gathered} \tag{3.22}$$

$q=\frac{9}{2}$, $s=\frac{1}{6}$, $g(t,u,v)=\frac{t^{\frac{-1}{2}}e^{t}+\left[(tu)^{2}-(tv)^{2}\right]^{2}}{(1+t)^{3}e^{t+|u|}}$, then

$$\begin{aligned} |g(t,u,v)| &\leq \frac{t^{\frac{-1}{2}}e^{t}+T^{4}(u^{4}+v^{4})}{(1+t)^{3}e^{t+|u|}}\leq\frac{t^{\frac{-1}{2}}e^{t}}{(1+t)^{3}e^{t}}+\frac{T^{4}(u^{4}+v^{4})}{(1+t)^{3}e^{t+|u|}} \\ &\leq \frac{1}{t^{\frac{1}{2}}(1+t)^{3}}+\frac{T^{4}(u^{4}+v^{4})}{(1+t)^{3}}\leq b(t)+c\left(u^{4}+v^{4}\right), \end{aligned}$$

where $b(t)=\frac{1}{t^{\frac{1}{2}}(1+t)^{3}}$ and $c=1$. $\sigma(t)=(1+t)^{4}$, it is clear that $\sigma(t)\leq\sigma(t)\sigma(t-\tau)$ for $t\geq\tau$, and

$$K(t)=\int_{0}^{t}\frac{(\sigma(t-\tau))^{\frac{1}{p}}}{(t-\tau)^{s}}d\tau=\int_{0}^{t}\frac{1+z}{z^{\frac{1}{6}}}dz=\frac{6t^{\frac{5}{6}}(5t+11)}{55},$$

and

$$K'(t)=(t^{\frac{1}{6}}+t^{-\frac{1}{6}}),$$

some computations give us

$$\|K'\|_{4,\sigma}\simeq 3.187991075720807,\quad \|b\|_{1,\sigma}=\frac{8}{3},$$

then

$$R^{4}-4.672639065946051R+2.666666666666665\leq 0,$$

so $R\in[0.598080985027521,1.405251623483919]$.

Using theorem 3.1, we deduce that the nonlinear functional boundary value problem (3.22) has at least one solution for any

$$R \in [0.598080985027521, 1.405251623483919].$$

Example 3.2 Consider the following boundary value problem of fractional differential equations with $p = 4$.

$$\begin{cases} D^{\alpha} u(t) = \overline{\varphi}(t) \left[e^{-1} \sin(tu) + t^2 h\left(D^{\beta} u\right)\right], & t \in I = [0,1], \\ D^{(\alpha - i)} u|_{t=0} = 0, \ i = 1,2,3,5, \quad u(1) = 0, \end{cases} \tag{3.23}$$

$\alpha = \frac{9}{2}$, $\beta = \frac{1}{6}$, $g(t,x,y) = \overline{\varphi}(t) \left[e^{-1} \sin(tu) + t^2 h\left(D^{\beta} u\right)\right]$ where: $h(x) = e^{-e^{-x}}$, $\overline{\varphi}(t) = \frac{1}{(9t)^2(1+t)}$. By the finite increments theorem we get

$$|h(x) - h(y)| \leq e^{-1} |x - y|,$$

for $x, y \in \mathbb{R}$ (since $z + e^{-z} \geq 1$ for all real z), also

$$|\sin(tx) - \sin(ty)| \leq t^2 |x - y|,$$

then

$$\begin{aligned} |g(t,x,y) - g(t,\overline{x},\overline{y})| &\leq \overline{\varphi}(t) \left[e^{-1} |\sin(tx) - \sin(t\overline{x})| + t^2 |h(y) - h(\overline{y})| \right] \\ &\leq \varphi(t) [|x - \overline{x}| + |y - \overline{y}|], \end{aligned}$$

so, condition $(H8)$ holds with $\varphi(t) = \frac{e^{-1}}{9^2(1+t)}$, obviously $\varphi \in L^{\frac{3}{4}}([0,1])$ and

$$\|\varphi\|_{3/4} = 0.0363.$$

$\sigma(t) = (1+t)^4$, it is clear that $\sigma(t) \geq 1$ for $t \in [0,1]$, and the Banach space is

$$E_{\sigma}^{*}(I) = \left\{ u \in L^{4,\sigma}(I) : I^{\frac{5}{6}} u \in W_1^{4,\sigma}(I) \right\},$$

also

$$K(t)=\int_{0}^{t}\frac{(\sigma(t-\tau))^{\frac{1}{p}}}{(t-\tau)^{\beta}}d\tau=\int_{0}^{t}\frac{1+z}{z^{\frac{1}{6}}}dz=\frac{6t^{\frac{5}{6}}(5t+11)}{55},$$

and

$$K^{'}(t)=(t^{\frac{1}{6}}+t^{-\frac{1}{6}}),$$

then, some computations give us

$$\left\|K^{'}\right\|_{4,\sigma}\simeq 3.187991075720807,$$

and

$$G_{*}\left\|\varphi\right\|_{3/4}\left[(T\sigma_{*})^{\frac{1}{p}}+\frac{T^{1-\beta}(T\sigma_{*})^{\frac{1}{p}}}{\Gamma(2-\beta)}+\frac{\left\|K^{'}\right\|_{p,\sigma}}{\Gamma(1-\beta)}\right]\simeq 0.5046<1.$$

So, using theorem 3.2, we deduce that the nonlinear functional boundary value problem 3.23 has a unique solution.

CHAPTER 4

Fractional diffusion integrodifferential equation

4.1 Position of problem

The main goal of this chapter is to prove the existence of the unique solution and some regularity results for the following FPDE with fractional integral condition by the use of Rothe time discretization method:

$$D^{\alpha}u\left(t,x\right)-\triangle u\left(t,x\right)=\int_{0}^{t}a\left(t-s\right)\triangle u\left(s,x\right)ds+f\left(t,x\right)\ \text{ in } I\times\Omega, \tag{4.1}$$

With initial condition

$$u\left(0,x\right)=u_{0}\left(x\right),\ \text{ in }\ \Omega, \tag{4.2}$$

Fractional integral condition

$$I^{1-\alpha}u\left(0^{+}\right)=U_{1}\left(x\right),\text{ in }\ \Omega, \tag{4.3}$$

And boundary condition

$$u\left(t,x\right)=0,\text{ on } I\times\partial\Omega, \tag{4.4}$$

where $\alpha\in\left]0,1\right[$, $I=\left[0,T\right]$ and Ω is an open bounded domain of $\mathbb{R}^{n}$, with a smooth boundary $\partial\Omega$. The fractional integral $I^{1-\alpha}$ and the derevative D^{α} are understood here in Riemann-Liouville sense.

we start by the discretization formula of the integro-differential fractional diffusion equation (4.1), using an implicit scheme, then we construct a discrete numerical solution of the discretized problem, then, we derive some a priori estimates for the approximations. The convergence of the method and the well posedness of the problem under study are also established. At last, we discuss the uniqueness of the weak solution.

Clearly, if $g\in L^{p}\left(0,T\right)$, $1\leq p\leq\infty$, and $\varphi:\left]0,T\right]\longrightarrow\mathbb{R}^{+}$ is a function defined by

$$\varphi\left(t\right)=\frac{t^{-\alpha}}{\Gamma\left(1-\alpha\right)},$$

then(see [49])

$$\varphi * g \in L^p(0,T), \text{ where } \varphi * g(t) = \int_0^t \varphi(t-s)\, g(s)\, ds$$

and $\varphi \times g$ is absolutly continuous, since

$$\varphi(t-s)\, g(s) \in L^1(0,T).$$

Moreover, if h is a function such that $g \geq h$, then

$$I^{\alpha} g(t) \geq I^{\alpha} h(t)$$

which means that I^{α} is increasing.

The problem (4.1)-(4.4) can be written as

$$\frac{\partial}{\partial t} I^{1-\alpha} u(t,x) - \triangle u(t,x) = \int_0^t a(t-s)\, \triangle u(s,x)\, ds + f(t,x). \tag{4.5}$$

4.2 Assumptions and discretization scheme

In this section, we give the assumptions that will ensure the existence of the unique weak solution. Let $L^2(\Omega)$ be the usual space of Lebesgue square integrable real functions on Ω whose inner product and norm will be denoted by $(.,.)$ and $\|.\|$, respectively, and $\|.\|_{-1}$ stands for the norm in $H^{-1}(\Omega)$(the dual space to $H_0^1(\Omega)$).

We make the following assumptions:

$(H9)$ $u_0 \in H_0^1(\Omega) \cap L^{\infty}(\Omega)$

$(H10)$ $f(t) \in L^2(\Omega)$ and $\|f(t) - f(t')\| \leq l\,|t-t'|$

$(H11)$ a is a continuous function such that $|a(t) - a(t')| \leq c_1\,|t-t'|$

We look for a weak solution in the following sense.

Definition 4.1 By a weak solution of problem (4.1)-(4.4) we mean a function u satisfying:

1 $u \in L^2\left(I, H_0^1(\Omega)\right)$ with $I^{1-\alpha}(u) \in C\left(I, H^{-1}(\Omega)\right).$

2 $\partial_t I^{1-\alpha}(u) \in L^2\left(I, H^{-1}(\Omega)\right).$

3 u satisfies (4.2) and (4.3).

4 For any $\phi \in H_0^1(\Omega)$, we have

$$\int_I \left(\partial_t I^{1-\alpha}(u), \phi\right) dt + \int_I (\nabla u, \nabla \phi)\, dt = \int_I (f, \phi)\, dt + \int_I \left(\int_0^t a(t-s) \nabla u(s)\, ds, \nabla \phi\right) dt.$$

We divide the interval I into n subintervals of length $h = \frac{T}{n}$ and denote $u_i = u(t_i, x)$, $t_i = ih$, $\delta u_i = \frac{u_i - u_{i-1}}{h}$, $i = 1, ..., n$. We will omit x for sake of simplicity, the associated discretized problem is

$$\left(I^{1-\alpha}(u_i) - I^{1-\alpha}(u_{i-1}), \phi\right) + h(\nabla u_i, \nabla \phi) = h(f_i, \phi) + h^2 \sum_{j=1}^{i-1} (a_{ij} \nabla u_j, \nabla \phi) \tag{4.6}$$

The existence of a weak solution $u_i \in H_0^1(\Omega)$ at each time step is ensured due to the monotony and the coercivity of the operator $\frac{I^{1-\alpha}(u_i)}{h} - \triangle u_i - a_{ij} h \triangle u_i$.

4.3 A priori estimates

In this section, we establish some useful a priori estimates.

Lemma 4.1 The following estimates hold uniformly in n, i, j and h:

$$\sum_{i=1}^{l} h \left\|\delta I^{1-\alpha}(u_i)\right\|^2 \leq C, \quad \|\nabla u_i\|^2 \leq C, \quad \sum_{i=1}^{l} \|\nabla u_i - \nabla u_{i-1}\|^2 \leq C. \tag{4.7}$$

Proof. Setting $\phi = u_i - u_{i-1}$ in (4.6) and summing up for $i = 1, ..., l$, we obtain

$$\sum_{i=1}^{l} h\left(\delta I^{1-\alpha}(u_i), \delta u_i\right) + \sum_{i=1}^{l} (\nabla u_i, \nabla u_i - \nabla u_{i-1}) =$$
$$\sum_{i=1}^{l} h(f_i, \delta u_i) + \sum_{i=1}^{l} \left(h \sum_{j=1}^{i-1} (a_{ij} \nabla u_j, \nabla u_i - \nabla u_{i-1}) \right). \tag{4.8}$$

The equality (4.8) is briefly denoted as: $J_1 + J_2 = J_3 + J_4$. Now we estimate each term, by the use of mean value theorem we get

$$J_1 \geq C \sum_{i=1}^{l} h \left\| \delta I^{1-\alpha}(u_i) \right\|^2. \tag{4.9}$$

Moreover

$$\begin{aligned} 2J_2 &= 2\sum_{i=1}^{l} (\nabla u_i - \nabla u_{i-1}, \nabla u_i - \nabla u_{i-1}) + 2\sum_{i=1}^{l} (\nabla u_{i-1} - \nabla u_i, \nabla u_i - \nabla u_{i-1}) \\ &\quad -2\sum_{i=1}^{l} \|\nabla u_{i-1}\|^2 \\ &= \sum_{i=1}^{l} \|\nabla u_i - \nabla u_{i-1}\|^2 + \sum_{i=1}^{l} \|\nabla u_i\|^2 - \sum_{i=1}^{l} \|\nabla u_{i-1}\|^2 \\ &= \sum_{i=1}^{l} \|\nabla u_i - \nabla u_{i-1}\|^2 + \|\nabla u_l\|^2 - \|\nabla u_0\|^2. \end{aligned} \tag{4.10}$$

Due to Cauchy-Schwarz, $\epsilon-$Young, Poincaré's inequalities, we obtain

$$|J_3| = \left| \sum_{i=1}^{l} h(f_i, \delta u_i) \right| \leq C \left(\varepsilon + \frac{1}{\varepsilon} \sum_{i=1}^{l} h \|\delta u_i\|^2 \right) \leq C \left(\varepsilon + \frac{1}{\varepsilon} \sum_{i=1}^{l} h \|\nabla \delta u_i\|^2 \right). \tag{4.11}$$

By the use of Cauchy-Schwarz and $\epsilon-$Young inequalities, the memory term can be estimated

$$\begin{aligned}|J_4| &= \left|\sum_{i=1}^{l} h^2 \sum_{j=1}^{i-1} (a_{ij}\nabla u_j, \nabla\delta u_i)\right| \\ &\leq C\left(\varepsilon + \frac{1}{\varepsilon}\sum_{i=1}^{l} h \|\nabla\delta u_i\|^2\right). \qquad (4.12)\end{aligned}$$

Summarizing all these consideration, collecting (4.9)-(4.12), choosing ε small, then the discrete Gronwall Lemma conclude the proof of the lemma 4.1 ■

Let $\overline{u}^n$ be a step functions defined by:

$$\overline{u}^n(t) = \begin{cases} u_i & t \in (t_{i-1}, t_i] \\ u_0 & t = 0 \end{cases}, \ i = 1, ..., n, \qquad (4.13)$$

$$\overline{I_n}(\overline{u}^n(t)) = \begin{cases} I^{1-\alpha}(u_i), & t \in (t_{i-1}, t_i] \\ U_1 & t = 0 \end{cases}, \quad i = 1, ..., n. \qquad (4.14)$$

We denote by f^n and M^n the functions

$$f^n(t) = \begin{cases} f_i & t \in [t_{i-1}, t_i] \\ f_0 & t = 0 \end{cases} \qquad i = 1, ..., n, \qquad (4.15)$$

$$M^n(t) = \begin{cases} h\sum_{j=1}^{i-1} a_{ij}\nabla u_j & t \in [t_{i-1}, t_i] \\ ha_{10}\nabla u_0 & t = 0 \end{cases} \qquad i = 1, ..., n. \qquad (4.16)$$

We define Rothe's functions on the interval I by

$$u^n(t) = u_{i-1} + (t - t_{i-1})\,\delta u_i, \quad t \in [t_{i-1}, t_i], \quad i = 1, ..., n, \qquad (4.17)$$

$$I_n(\overline{u}^n(t)) = I^{1-\alpha}(u_{i-1}) + (t - t_{i-1})\,\delta I^{1-\alpha}(u_i) \quad t \in [t_{i-1}, t_i], \quad i = 1, ..., n. \qquad (4.18)$$

Lemma 4.2 The a priori estimate

$$\int_I \|\partial_t I_n\|_{-1}^2 \leq C,$$

holds for $1 \leq i \leq n$, where $\|\partial_t I_n\|_{-1} = \sup_{\|\phi\|\leq 1,\ \phi\in H_0^1} |(\partial_t I_n, \phi)|$.

Proof. Applying Lemma 4.1 we conclude the proof ■

Lemma 4.3 There exists a positive constant C such that

$$\sum_{i=1}^{j} h \|\nabla u_i\|^2 \leq C, \quad 1 \leq j \leq n.$$

Proof. According to Lemma 4.1, we have

$$h \|\nabla u_i\|^2 \leq hC,$$

summing over i yields

$$\sum_{i=1}^{j} h \|\nabla u_i\|^2 \leq \sum_{i=1}^{j} hC = j\frac{T}{n}C \leq C,$$

since $j \leq n$ ■

4.4 Convergence and existence results

From lemmas 4.2 and 4.3, we could say

$$\max_I \left\|\overline{I_n}\right\| + \|\partial_t I_n\|_{L^2(I,H^{-1}(\Omega))} \leq C.$$

Hence, there exist (see [60] lemma.1.3.13) $w \in C\left(I, H^{-1}(\Omega)\right) \cap L^\infty\left(I, L^2(\Omega)\right)$ with $\partial_t w \in L^2\left(I, H^{-1}(\Omega)\right)$ and subsequence I_{n_k} such that

$$\begin{aligned} I_{n_k} &\longrightarrow w \text{ in } C\left(I, H^{-1}(\Omega)\right), \quad \overline{I_{n_k}}(t) \rightharpoonup w(t) \text{ in } L^2(\Omega) \\ I_{n_k}(t) &\rightharpoonup w(t) \text{ in } L^2(\Omega), \quad \partial_t I_{n_k} \rightharpoonup \partial_t w \text{ in } L^2\left(I, H^{-1}(\Omega)\right) \end{aligned} \tag{4.19}$$

In view of lemma 4.1 one can deduce that $\{\overline{u}^n\}_n$ is uniformely bounded in $L^2(I, H_0^1(\Omega))$.
Thereafter, we can extract subsequence $\{\overline{u}^{n_k}\}_{k\in\mathbb{N}}$ such that

$$\overline{u}^{n_k} \underset{k\longrightarrow\infty}{\rightharpoonup} u \quad \text{in } L^2\left(I, H_0^1(\Omega)\right). \tag{4.20}$$

Therefore, it follows from $(H10)$ that

$$\|f^n(t) - f\|_{L^2(I,L(\Omega))} \leq \frac{C}{n},$$

and so

$$f^n(t) \underset{n\longrightarrow\infty}{\longrightarrow} f \text{ in } L^2(I, L(\Omega)). \tag{4.21}$$

Proceeding as in [63], we will be able to state the following lemma.

Lemma 4.4 The sequence $\{M^n\}_n$ is uniformly bounded and possess a subsequence $\{M^{n_k}\}_k$ such that

$$M^{n_k} \underset{k\longrightarrow\infty}{\rightharpoonup} M \quad \text{in } L^2\left(I, L^2(\Omega)\right), \tag{4.22}$$

where

$$(M(u), \phi) = \left(\int_0^t a(t-s)\nabla u(s)\, ds, \nabla\phi\right).$$

Our next target is to prove the strong convergence of $\{I_n\}_n$ and $\{\overline{u}^n\}_n$ in $L^2(I, L^2(\Omega))$ and $L^2(I, H_0^1(\Omega))$ respectively.

Lemma 4.5 There exist subsequences $\{I_{n_k}\}_k$ of $\{I_n\}_n$ and $\{\overline{u}^{n_k}\}_{k\in\mathbb{N}}$ of $\{\overline{u}^n\}_n$ for which

$$\begin{aligned} I_{n_k} &\longrightarrow w \text{ in } L^2\left(I, L^2(\Omega)\right), \\ \overline{u}^{n_k} &\longrightarrow u \text{ in } L^2\left(I, H_0^1(\Omega)\right). \end{aligned} \tag{4.23}$$

Proof. By vertue of Kolmogorov compactness criterion, it is sufficient to prove that

$$\|I_n\|^2_{L^2(I,L^2(\Omega))} + \|\overline{u}^n\|^2_{L^2\left(I,H_0^1(\Omega)\right)} \leq C,$$

$$\|I_n(t+s,x+h) - I_n(t,x)\|_{L^2(I,L^2(\Omega))} \underset{s,|h|\longrightarrow 0}{\longrightarrow} 0,$$

$$\|\overline{u}^n(t+s,x+h) - \overline{u}^n(t,x)\|_{L^2\left(I,H_0^1(\Omega)\right)} \underset{s,|h|\longrightarrow 0}{\longrightarrow} 0.$$

This can be easily obtained using the above lemmas ■

The main result of this paper is given in the following theorem.

Theorem 4.1 The limit u is a weak solution of the problem (4.1)-(4.4) in sense of definition 4.1.

Proof. In view of lemma 4.5 and the well-known Minty-Browder's trick, we can conclude that $w = I^{1-\alpha}(u)$. On the other hand, from the equality

$$\overline{I}_{n_k}(\overline{u}^{n_k}) - U_1 = \int_0^t \partial_t \overline{I}_{n_k}(\overline{u}^{n_k}(s))\,ds \tag{4.24}$$

we get as $k \longrightarrow \infty$

$$I^{1-\alpha}(u) - U_1 = \int_0^t \partial_t I^{1-\alpha} u(s)\,ds \tag{4.25}$$

This implies that

$$I^{1-\alpha}\left(u\left(0^+\right)\right) = U_1.$$

Hence, the condition (4.3) is fulfilled. Obviously, in light of (4.13)-(4.18), the identity (4.6) can be rewritten as

$$\int_I (\partial_t I_n(t), \phi)\,dt + \int_I (\nabla\overline{u}^n, \nabla\phi)\,dt = \int_I (f^n, \phi)\,dt + \int_I (M^n, \nabla\phi)\,dt,\ \forall\phi \in H_0^1(\Omega). \tag{4.26}$$

Now, replacing n by $n_k \longrightarrow \infty$ in (4.26) then taking (4.19), (4.21), Lemmas 4.4 and 4.5 into account, it yields

$$\int_I \left(\partial_t I^{1-\alpha}(u), \phi\right) dt + \int_I (\nabla u, \nabla\phi)\,dt = \int_I (f, \phi)\,dt + \int_I \left(\int_0^t a(t-s)\nabla u(s)\,ds, \nabla\phi\right) dt.$$

Thus, u is a weak solution of problem (4.1)-(4.4) ■

4.5 Uniqueness of weak solution

Now, we are prepared to prove the uniqueness of the weak solution as follows.

Theorem 4.2 Under the assumptions $(H9)-(H11)$, the problem (4.1)-(4.4) has a unique weak solution.

Proof. Let us suppose that the problem (4.1)-(4.4) has two weak solutions u_1, u_2, then $u = u_1 - u_2$ satisfies

$$\int_I \left(\partial_t \left(I^{1-\alpha}u_1 - I^{1-\alpha}u_2\right), \phi\right) dt + \int_I \left(\nabla u, \nabla \phi\right) dt = \int_I \left(\int_0^t a\left(t-s\right) \nabla u\left(s\right) ds, \nabla \phi\right) dt \tag{4.27}$$

We divide the interval I into subintervals with the length p such that

$$\max_I \left|a\left(t\right)\right| .p < 1,$$

then we choose the function ϕ in (4.27) as

$$\phi\left(t\right) = \begin{cases} u\left(t\right) & t \in [0, p] \\ 0 & t \in]p, T] \end{cases}$$

we obtain

$$\begin{aligned} &\left(I^{1-\alpha}\left(u_1\right) - I^{1-\alpha}\left(u_1\right), u_1 - u_2\right) + \int_0^p \left\|\nabla u\left(t\right)\right\|^2 dt \\ = &\int_0^p \left(\int_0^t a\left(t-s\right) \nabla u\left(s\right), \nabla u\left(t\right)\right) dt \end{aligned} \tag{4.28}$$

Monotony of $I^{1-\alpha}$ yields

$$\left(I^{1-\alpha}\left(u_1\right) - I^{1-\alpha}\left(u_2\right), u_1 - u_2\right) \geq 0.$$

Hypothesis $(H11)$ and Cauchy-Schwarz inequality give us

$$\begin{aligned}
\int_0^p \|\nabla u(t)\|^2 dt &\leq \int_0^p \left| \left(\int_0^t a(t-s) \nabla u(s), \nabla u(t) \right) \right| dt \\
&\leq \int_0^p \left\| \int_0^t a(t-s) \nabla u(s) \, ds \right\| . \|\nabla u(t)\| \, dt \\
&\leq \max_I |a(t)| \left\| \int_0^p \nabla u(t) \, dt \right\| . \int_0^p \|\nabla u(t)\| \, dt. \qquad (4.29)
\end{aligned}$$

It follows by means of

$$\left\| \int_0^p \nabla u(t) \, dt \right\| \leq \int_0^p \|\nabla u(t)\| \, dt;$$

that

$$\int_0^p \|\nabla u(t)\|^2 dt \leq \max_I |a(t)| \left(\int_0^p \|\nabla u(t)\| \, dt \right)^2. \qquad (4.30)$$

Cauchy-Schwarz inequality once more implies

$$\int_0^p \|\nabla u(t)\|^2 dt \leq \max_I |a(t)| .p \int_0^p \|\nabla u(t)\|^2 dt. \qquad (4.31)$$

Since

$$\max_I |a(t)| .p < 1,$$

we get

$$\|u\|_{L^2\left([0,p],H_0^1(\Omega)\right)} = 0. \tag{4.32}$$

Which proves that

$$u(t) = 0,\ \forall t \in [0,p]. \tag{4.33}$$

Repeating the same procedure on the intervals $[ip,(i+1)p]$, we obtain

$$u(t) = 0,\ \forall t \in I.$$

Consequently, $u_1 = u_2$ ■

4.6 An example

Consider the following fractional integrodifferential equation

$${}^{RL}D^{\frac{1}{2}}u(t,x) - \partial_{xx}u(t,x) = tx + \int_0^t e^{(t-s)}\partial_{xx}u(s,x)\,ds,\ \ (t,x) \in [0,T]\times(0,1), \tag{4.34}$$

$$u(0,x) = u_0(x) = x(1-x),\ x \in (0,1), \tag{4.35}$$

$$u = 0 \text{ on } [0,T]\times\{0,1\}, \tag{4.36}$$

$$I^{\frac{1}{2}}u\left(0^+\right) = U_1(x) \in H_0^1(0,1). \tag{4.37}$$

We have

$$\|f(t)\| = \left(\int_0^1 (tx)^2\,dx\right)^{\frac{1}{2}} = \frac{t}{\sqrt{3}} \le \frac{T}{\sqrt{3}} < \infty,$$

then $f(t) \in L^2(0,1)$ and

$$\|f(t_1) - f(t_2)\| = \frac{|t_1 - t_2|}{\sqrt{3}} \le \frac{2}{3}|t_1 - t_2|,$$

which implies that $(H10)$ is satisfied. On the other hand, it is easy to see that $u_0 \in L^2(0,1)$, $u_0' \in L^2(0,1)$ and $u_0(0) = u_0(1) = 0$, hence $u_0 \in H_0^1(0,1)$. Moreover, $u_0(x) = x(1-x) \leq 1$ therefore, $u_0 \in L^\infty(0,1)$ and consequently $(H11)$ is also verified. All conditions of theorem 4.1 and theorem 4.2 are now fulfilled so we deduce that (4.34)-(4.37) has a unique weak solution.

CONCLUSION

We think that the presented results in this thesis are considerable and important contributions in the field of fractional differential equations. We proveded several new and different results for existence, uniqueness and stability of solutions for some kinds of fractional differential equations by the use of some methods and skills:

We Exploited the new Banach space discussed by Burton in 2006 [23] to provided a new uniqueness result for fractional differential equations on unbounded domain in the submitted paper [42] as well as we investigated a new stability result for a developed type of mixed fractional differential equations with finite constant delay on unbounded interval. Farthermore, we presented a generalization of the fractional Sobolev spaces presented in the paper [16] in early 2016 as shown in Chapter 3. In parallel with this, Some existence results for fractional differential equations are obtained in this fractional Sobolev spaces, see the submitted paper [41]. On the other hand, we presented a contribution in the fractional partial differential equations through the published paper [26] in 2018. Where we shown the existence of unique weak solution for fractional diffusion integrodifferential equations by utilizing the Rothe's time discretization method, all our results were supported by confirmation examples.

Otherwise, this field is very rich in different discussions ,projections, extentions, and open questions; therefore different applications can be launched as a result of the fractional calculus as the submitted paper [43] which concerned with the existence of solutions under weak topology in Banach space setting for fractional differential equations. Moreover, we intend to study the existence of weak solutions fo fractional differential equations with integrable delay, as well as we foresee the study of the uniqueness and stability of the solution of such a class of equations by employing the progressive contractions which discussed by Burton in 2017 [22]. Despite all that, numerical study of fractional differential equations arouse a lot of our interest and remains the biggest obsession for us due to their importance in the validation of theoritical study and their invaluable credibility in physical and tangible reality.

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