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Lecture-Note

By

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Algebra 4
(summary of lessons, examples, solved and suggested exercises)

Presented to students of the second year Maths.

2023 /2024

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Theme

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Thanks

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Dedications

I dedicate this Lecture-Note of Algebra 4 to:

My Students

Mathematicians

Readers

Les paroles s'envolent mais les écrits restent... á cet effet ce travail.

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Notations

The following notations allow the reader to clearly understand the content of this manuscript.

- $\alpha, \beta, \lambda, \mu$ are scalars and $x, y, u, v, w, u', v', w'$ are vectors.
- $\oplus$ This is an internal composition law.
- $\otimes$ This is an external composition law.
- E a vector space over $\mathbb{K}$.
- $\mathbb{K}^n$ The field of n-tuples of real or complex numbers.
- $(x_1, x_2, ..., x_n)$ An element of $\mathbb{K}^n$ (vector).
- $\mathbb{K}_n[x]$ The vector space of all polynomial of degree not exceeding n with real or complex coefficients.
- $C([a,b], \mathbb{R})$ The vector space of all continuous functions on $[a,b]$.
- $C^\infty([a,b], \mathbb{R})$ The v. space of all infinitely differentiable functions on $[a,b]$.
- $\mathcal{M}_n(\mathbb{K})$ The vector space of all n by n real (or complex) matrices.
- $\mathcal{S}_n(\mathbb{K})$ The vector space of all n by n real (or complex) symmetric matrices.
- $\mathcal{A}_n(\mathbb{K})$ The vector space of all n by n real (or complex) skew-symmetric matrices.
- $\mathbb{GL}_n(\mathbb{K})$ The vector space of all n by n invertible matrices.
- $\mathcal{M}_f(B)$ The matrix of the mapping f with respect to the basis B.
- P The passage matrix.
- $\{e_1, e_2, ..., e_n\}$ In general denotes for the canonical basis.
- $Vect\{u_1, u_2, ..., u_n\}$ The vector space of all linear combinations of the vectors u_i ($1 \leq 1 \leq n$).

- $\ker f$ The kernel of the linear mapping f or the kernel of the bilinear symmetric form f.
- $Im\ f$ The vector subspace $\{f(v) : v \in E\}$.
- $F \oplus G$ Direct sum between F and G.
- $\mathcal{L}_2(E)$ The v. space[1] of all bilinear forms on E.
- $\mathcal{L}(E, F)$ The v. space of all linear mappings from E to F.
- $\mathcal{L}(E, \mathbb{K})$ The v. space of all linear mappings from E to $\mathbb{K}$.
- q or Q Quadratic forms.
- C The isotropic cone; $C = \{v \in E : f(v, v) = 0\}$.
- E^* dual v. space of a vector space E.
- Φ^* The dual mapping of Φ.
- $\mathcal{S}_2(E)$ The v. subspace of all symmetric bilinear forms on E.
- $\mathcal{A}_2(E)$ The v. subspace of all skew-symmetric bilinear forms on E.
- $\mathcal{Q}_2(E)$ The set of all quadratic forms on E.
- $diag\{a_1, a_2, ..., a_n\}$ Diagonal matrix whose diagonal entries are $a_1, a_2, ..., a_n$.
- $tr(A)$ The trace of an n by n matrix A.
- $Sp(A)$ The spectral set of A = The set of eigenvalues of A.
- $\overline{z}$ The conjugate of the vector $z \in \mathbb{C}^n$.
- i The imaginary pure number ($i^2 = -1$).
- I The identity matrix.
- $Re(z)$ The real part of a complex number z.
- A^t The transpose of a matrix A.
- $\det(A)$ Determinant of a square matrix A.
- A^* The transpose conjugate of a complex n by n matrix A.

[1] v. space means vector space.

- $\|v\|$ the norm of the vector v.
- $\langle u, v \rangle$ The scalar product (or inner product) between the vectors u and v.

General Introduction

This work is the fruit of teaching of this subject at the University of 8 Mai 45 Guelma. It is intended for students of the **2nd** year mathematics. This volume is devoted to a part of the program of Algebra 4 (bilinear forms, quadratic forms, sesquilinear forms and hermitian forms. One can see [1], [2], [3], [4], [5], [6]).

Each chapter begins with a clear presentation of definitions, lemmas and theorems, illustrated with numerous examples. This is followed by a graduated number of a set of solved exercises.

The course summary is sufficiently developed so that everyone will find the results they need to solve proposed problems. Although the large number of additional problems makes their solution difficult, special importance should nevertheless be given to those presented in the first two chapters. After engaging in it, the student will feel more confident.

I had been teaching this material by French from *2012* to *2016*. Then I have taught it to students a second time, but by English, starting *2020* to now. Being the first subject presented to students at the beginning of their education, they gladly accepted presenting it in English language. Indeed, this course, which is based on bilinear forms (linearity from the right and those from the left), is a continuation study of Algebra II taught in the first year M.I. It is composed of five chapters. In Chapter 1, we recall some definitions and give without proof some classical results on vector spaces and linear mappings, that is, we list in the this chapter the basic notions on a vector space and its dual space. In chapter 2 we deal with bilinear forms over a real vector space. It is not possible to understand such properties without examining the related concepts of linear forms. More precisely, this chapter describes the most properties of bilinear forms on a vector space and gives examples of the three most common types of such forms as well as symmetric, skew-symmetric and alternating bilinear forms. Chapter 3 deals with the spectral decomposition of self-adjoint linear mappings. The important condition of nondegeneracy for a bilinear form, Gauss decomposition theorem and the orthogonal basis for a symmetric bilinear form are the subject of Chapter 4. An introduction to Hermitian space is given in Chapter 5. At the end of this lecture-note, the reader will find a conclusion and a bibliography.

CHAPTER 1

LINEAR FORMS, DUALITY (VECTOR SPACE AND ITS DUAL SPACE

As a continuation of Algebra 2, we present in this chapter many relationships between scalars, vectors and linear mappings having many variables defined on a finite-dimensional v. space. Recall that the v. space is a basic object in the study of linear algebra. It is a set of several vectors which are objects that can be added together and multiplied by numbers, which are called scalars in this context. This chapter deals with mappings defined on some special v. spaces that display one or two variables.

1.1 Vector space (a summary of lessons)

Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ and let E be a non-empty set equipped with two operations $\oplus$ and $\otimes$, where

1. $\oplus$ is an *internal composition law*; i.e., $\forall\, u, v \in E : u \oplus v \in E$.
2. $\otimes$ is an *external composition law*; i.e., $\forall\, \lambda \in \mathbb{K}, \forall\, v \in E : \lambda \otimes v \in E$.

We say that $(E, \oplus, \otimes)$ is a v. space on the field $\mathbb{K}$ if the following conditions hold:

1. $(E, \oplus)$ is a commutative (Abelian) group.
2. $\forall\, \lambda \in \mathbb{K}, \forall\, u, v \in E : \lambda \otimes (u \oplus v) = (\lambda \otimes u) \oplus (\lambda \otimes v)\,,$
3. $\forall\, \lambda, \mu \in \mathbb{K}, \forall\, v \in E : (\lambda + \mu) \otimes v = (\lambda \otimes v) \oplus (\mu \otimes v)\,,$
4. $\forall\, \lambda, \mu \in \mathbb{K}, \forall\, v \in E : \lambda \otimes (\mu \otimes v) = (\lambda.\mu) \otimes v,$
5. $\forall\, v \in E : 1_{\mathbb{K}} \otimes v = v.$ (if $\mathbb{K}=\mathbb{R}$ or $\mathbb{C} \Rightarrow 1_{\mathbb{K}} = 1$).

To make statements (things) easier; in a v. space $(E, \oplus, \otimes)$ over $\mathbb{K}$, the internal law $\oplus$ we designate it $+$ and the external law $\otimes$ we designate it $\cdot$ or nothing. The definition of a v. space becomes:

We say that $(E, +, .)$ is a vector space (or just **v.s.**) over the field $\mathbb{K}$ if:

i. $+$ is an internal composition law on E; i.e., $\forall\, u, v \in E : u + v \in E$.

ii. $\cdot$ is an external composition law on E; i.e., $\forall\, \lambda \in \mathbb{K}, \forall\, v \in E : \lambda v \in E$.

1. $(E, +)$ is a commutative (abelian) group.
2. $\forall\, \lambda \in \mathbb{K}, \forall\, u, v \in E : \lambda\,(u + v) = \lambda u + \lambda v$,
3. $\forall\, \lambda, \mu \in \mathbb{K}, \forall\, v \in E : (\lambda + \mu)\, v = \lambda v + \mu v$,
4. $\forall\, \lambda, \mu \in \mathbb{K}, \forall\, v \in E : \lambda\,(\mu v) = (\lambda\mu)\, v$,
5. $\forall\, v \in E : 1_{\mathbb{K}} v = v$.

We must know the following facts:

- The elements of the vector space E are called *vectors* and the elements of the field $\mathbb{K}$ are called *scalars* ($\Rightarrow$ the sum of two vectors is a vector and the multiplication of a vector by a scalar is a vector).
- The neutral element with respect to $+$ in the vector space E we designate it 0_E; and we call it the *zero vector*.
- In the v. space E over $\mathbb{K}$; we have $\forall\, v \in E : -v = (-1)\, v$; where $-v$ is the symmetric element of v with respect to $+$, and $(-1)\, v$ is the multiplication of the vector v by the scalar -1.
- For two vectors u and v of the vector space E, we write by convention $u - v$ instead of $u + (-v)$ and $u + (-1)\, v$:

Let $(E, +, .)$ a vector space over the field $\mathbb{K}$ and let F be a subset of E.

Definition 1.1. We say that F is a vector subspace **(**or subspave**)** of E if $(F, +, .)$ is a vector space over $\mathbb{K}$, where $0_F = 0_E$.

Remark 1.1. From the above definition we deduce that every vector space is a vector subspace of itself.

Let E be a v. space, and let F and G be two subspaces of E.

- We have $F \cap G = \{v \in E \;/\; v \in F \text{ and } v \in G\}$ and $F + G = \{u + v \in E \;/\; u \in F \text{ and } v \in G\}$ are vector subspaces of E.
- Note that $F + G = G + F,\ F + F = F,\ F \cap G \subset F \subset F + G$ and $F \cap G \subset G \subset F + G$.
- Note that if $v \in F + G$, then $\exists\, a \in F, \exists\, b \in G : v = a + b$; where a and b are not unique.

- The v. space E is a direct sum of G and F; and write $E = G \oplus F$, if $E = G + F$ and $G \cap F = \{0_E\}$.
- We say that G is supplementary of F in E (or the opposite) if $E = G \oplus F$.
- We have $E = G \oplus F \Leftrightarrow$ Every vector v of E is written in a unique way $a + b$, where $a \in G$ and $b \in F$.

Proposition 1.1. *Let F be a subset of E. We have*

$$F \text{ is a vector subspace of } E \overset{\text{iff}}{\Leftrightarrow} \begin{array}{l} \text{(i) } \forall\, u, v \in F : u + v \in F \\ \text{(ii) } \forall\, \lambda \in \mathbb{K}, \forall\, v \in F : \lambda \cdot v \in F \\ \text{(iii) } F \neq \emptyset \ \ (0_E \in F) \end{array}$$

Or equivalently,

$$F \text{ is a v. subspace of } E \overset{\text{iff}}{\Leftrightarrow} \begin{array}{l} \text{(i) } F \neq \emptyset \ \ (0_E \in F) \\ \text{(ii) } \forall\, \lambda, \mu \in \mathbb{K}, \forall\, u, v \in F : \lambda \cdot u + \mu \cdot v \in F. \end{array}$$

Example 1.1. Suppose that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then

$$\mathbb{K}^n = \{(x_1, x_2, ..., x_n) : x_i \in \mathbb{K}\}$$

is a vector space on $\mathbb{K}$ with the laws $+$ and $\cdot$ defined by

1. $\forall\, (x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n) \in \mathbb{K}^n$:

$$(x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n) = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n),$$

2. $\forall\, \lambda \in \mathbb{K}, \forall\, (x_1, x_2, ..., x_n) \in \mathbb{K}^n$:

$$\lambda (x_1, x_2, ..., x_n) = (\lambda x_1, \lambda x_2, ..., \lambda x_n),$$

where $0_{\mathbb{K}^n} = \underbrace{(0, 0, ..., 0)}_{n\text{-times}}$ is the zero vector of this space.

For these laws, we have

- $\mathbb{R}^n$ is a v. space over $\mathbb{R}$,
- $\mathbb{R}^n$ is not a v. space over $\mathbb{C}$,
- $\mathbb{C}^n$ is a v. space over $\mathbb{C}$,

- $\mathbb{C}^n$ is a v. space over $\mathbb{R}$.

Let E be a v. space on $\mathbb{K}$, and let $v, v_1, v_2, ..., v_n \in E$.

- We have: v is a *linear combination* of $v_1, v_2, ..., v_n \overset{\text{def}}{\Leftrightarrow} \exists\, \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K}$:

$$v = \lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_n v_n.$$

- We always have $0_E = 0.v_1 + 0.v_2 + ... + 0.v_n$ (where 0_E is the zero vector of space E).
- The sum of two linear combinations is a linear combination.
- Multiplying a linear combination by a scalar is a linear combination.

Let E be a v. space on $\mathbb{K}$ and let $v_1, v_2, ..., v_n \in E$. The set of all linear combinations of vectors $v_1, v_2, ..., v_n$ we note it $Vect\,(v_1, v_2, ..., v_n)$ or $\langle v_1, v_2, ..., v_n \rangle$ and we call it the subspace *generated* by the vectors $v_1, v_2, ..., v_n$. We have then

$$Vect\,(v_1, v_2, ..., v_n) = \{\lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_n v_n \;:\; \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K}\}\,.$$

Moreover, we have

- $Vect\,(0_E) = \{0_E\}\,.$
- $Vect\,(v_1, v_2, ..., v_n)$ is a vector subspace of E (with $v_1, v_2, ..., v_n \in Vect\,(v_1, v_2, ..., v_n)$). Therefore, the subspace generated by vectors of a space is a vector subspace. of this space.
- If F is a vector subspace of E, then we have $v_1, v_2, ..., v_n \in F \Leftrightarrow Vect\,(v_1, v_2, ..., v_n) \subset F$. Therefore, the subspace generated by vectors is the smallest v. subspace contains these vectors.
- If $v = \lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_n v_n$, then $Vect\,(v_1, v_2, ..., v_n, v) = Vect\,(v_1, v_2, ..., v_n)\,.$
- $Vect\,(v_1, v_2, ..., v_n, 0_E) = Vect\,(v_1, v_2, \dots, v_n)\,.$
- If $F = Vect\,(v_1, v_2, ..., v_n)$ and $G = Vect\,(u_1, u_2, ..., u_m)$, then

$$F + G = Vect\,(v_1, v_2, ..., v_n, u_1, u_2, ..., u_m)\,.$$

Let E be a v. space over $\mathbb{K}$, and let $v_1, v_2, ..., v_n \in E$.
We call a *linear relationship* between the vectors $v_1, v_2, ..., v_n$; any relation of the form

$$\lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_n v_n = 0_E, \text{ where } \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K}.$$

1. If $\lambda_1, \lambda_2, ..., \lambda_n$ are all zero, we say that this linear relation is trivial.

2. If $\lambda_1, \lambda_2, ..., \lambda_n$ are not all zero, we say that this linear relation is non-trivial.

We say that the vectors $v_1, v_2, ..., v_n$ are *linearly independent* (or *free*) if there is no non-trivial linear relationship between the vectors $v_1, v_2, ..., v_n$, in other words; any linear relationship between vectors $v_1, v_2, ..., v_n$ is trivial; i.e.,
$v_1, v_2, ..., v_n$ are free $\overset{\text{def}}{\Leftrightarrow}$

$$\forall\ \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K} : \lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_n v_n = 0_E \Rightarrow \lambda_1 = \lambda_2 = ... = \lambda_n = 0.$$

- We say that the vectors $v_1, v_2, ..., v_n$ are *linearly dependent* (or linked) if they are not free, in other words; if there is at least one non-trivial linear relationship between the vectors $v_1, v_2, ..., v_n$; i.e.,

$v_1, v_2, ..., v_n$ are linked $\overset{\text{def}}{\Leftrightarrow}$

$$\exists\ \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K}\ \text{(are not all null)} : \lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_n v_n = 0_E.$$

- The family of vectors $\{v_1, v_2, ..., v_n\}$ are said to be *free* if the vectors $v_1, v_2, ..., v_n$ are free.

- The family of vectors $\{v_1, v_2, ..., v_n\}$ are said to be *linked* if the vectors $v_1, v_2, ..., v_n$ are linked.

- Note that if a family contains a linked part, then this family is linked.

- If $v \in E$, then $v \neq 0_E \Leftrightarrow v$ is free; since we have

$$v \neq 0_E \Leftrightarrow (\forall \lambda \in \mathbb{K} : \lambda v = 0_E \Rightarrow \lambda = 0).$$

- The null vector or the zero vector 0_E is linked; since we have $1.0_E = 0_E$, which is a non-trivial linear relationship.

- If a family of vectors contains the zero vector, then that family is related; i.e., family $\{v_1, v_2, ..., v_n, 0_E\}$ is linked, since $\{0_E\}$ is linked; or because

$$0.v_1 + 0.v_2 + ... + 0.v_n + 1.0_E = 0_E.$$

Let E be a vector space over $\mathbb{K}$, and let $v_1, v_2, ..., v_n \in E$.

- The family $\{v_1, v_2, ..., v_n\}$ is a *base* **(**or *basis* **)** of $E \Leftrightarrow \begin{cases} \textbf{1)}\ E = Vect(v_1, v_2, ..., v_n) \\ \textbf{2)}\ v_1, v_2, ..., v_n \text{ are free.} \end{cases}$

Note that $E = Vect(v_1, v_2, \ldots, v_n)$, i.e., E is spanned by $v_1, v_2, ..., v_n$; or, we say that $\{v_1, v_2, ..., v_n\}$ is a generated part of E.

- Please note, a basis of E is not always exists or unique.

- If $E = Vect(v_1, v_2, ..., v_n)$, then E admits at least one basis $\{u_1, u_2, ..., u_m\}$; with $m \leq n$, and all the bases of E have the same number of vectors m. This unique number m; denoted by $\dim E$, is called the dimension of E.

- If$v_1, v_2, ..., v_n$ are free, then by definition $\{v_1, v_2, ..., v_n\}$ is a basis of E, and so $\dim E = n$. Notice, in this case, that every other basis E contains exactly n vectors.

- If $v_1, v_2, ..., v_n$ are linked, then a vector of them is a linear combination of the other vectors. For example, $v_1 = \lambda_2 v_2 + ... + \lambda_n v_n$. Therefore,

$$Vect(v_1, v_2, ..., v_n) = Vect(v_2, v_3, ..., v_n).$$

Now, if $v_2, v_3, ..., v_n$ are free, then by definition $\{v_2, v_3, ..., v_n\}$ is a basis of E. Hence, $\dim E = n - 1$. But, if $v_2, v_3, ..., v_n$ are linked, then, a vector of them is a linear combination of the others; For example $v_n = \alpha_2 v_2 + \alpha_3 v_3 ... + \alpha_{n-1} v_{n-1}$. Hence,

$$E = Vect(v_2, v_3, ..., v_n) = Vect(v_2, v_3, ..., v_{n-1}) \quad \text{...and so on.}$$

- Note that the vector subspace $\{0_E\}$ has no basis; but by convention we put $\dim \{0_E\} = 0$ ($\{0_E\} = Vect(\{0_E\})$, where $\{0_E\}$ is linked).

- Note that if $E = Vect(v)$, where $v \neq 0_E$ (i.e., v is free), then $\{v\}$ is a base of E. In this case, $\dim E = 1$.

- For the vector space $\mathbb{K}^n$ over $\mathbb{K}$, we have $\dim \mathbb{K}^n = n$; since the family of vectors $\{e_1, e_2, ..., e_n\}$ form a basis of $\mathbb{K}^n$; which is called the *canonical basis* of $\mathbb{K}^n$, where

$$e_1 = (1, 0, ..., 0), e_2 = (0, 1, ..., 0), ..., e_n = (0, 0, ..., 1).$$

- For the vector space $\mathbb{C}^n$ on the field $\mathbb{R}$, we have $\dim \mathbb{C}^n = 2n$; since the family of vectors

$$\{e_1, ie_1, e_2, ie_2, ..., e_n, ie_n\}, \text{ where } i^2 = -1$$

form a basis of $\mathbb{C}^n$ over $\mathbb{R}$; which is called the *canonical basis* of $\mathbb{C}^n$ over $\mathbb{R}$.

- For the vector space $\mathbb{R}_n[x]$, we have $\dim \mathbb{R}_n[x] = n+1$; because the family of vectors $\{1, x, x^2, ..., x^n\}$ form a basis of $\mathbb{R}_n[x]$; which is called the *canonical basis* of $\mathbb{R}_n[x]$.

- If $\dim E = n$, then

$$\{v_2, v_3, ..., v_n\} \text{ is a basis of } E \Leftrightarrow E = Vect(v_2, v_3, ..., v_n) \Leftrightarrow v_2, v_3, ..., v_n \text{ are free.}$$

- If F is a vector subspace of E, then we have $\dim F \leq \dim E$.

- If F is a vector subspace of E, then we have $\dim F = \dim E \Leftrightarrow F = E$.

- *Dimension theorem*. If F and G are two vector subspaces E, then we have

$$\dim(F+G) + \dim(F \cap G) = \dim F + \dim G. \tag{1.1}$$

- Assume that $\mathcal{B} = \{v_1, v_2, ..., v_n\}$ is a basis of E and let $v \in E$. Then

$$\exists\, \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K} : v = \lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_n v_n,$$

since $E = Vect(v_1, v_2, ..., v_n)$. But; since the vectors $v_1, v_2, ..., v_n$ are free, then the scalars $\lambda_1, \lambda_2, ..., \lambda_n$ are unique. In this case the scalars $(\lambda_1, \lambda_2, ..., \lambda_n)$ we call them the coordinates of v in the basis $\mathcal{B}$.

- In the vector space $\mathbb{K}^n$ over $\mathbb{K}$, we have $\forall\, (x_1, x_2, ..., x_n) \in \mathbb{K}^n$:

$$(x_1, x_2, ..., x_n) = x_1 e_1 + x_2 x_2 + ... + x_n e_n,$$

where $\{e_1, e_2, ..., e_n\}$ is the canonical basis of $\mathbb{K}^n$. Therefore, $(x_1, x_2, ..., x_n)$ are the coordinates of the vector $(x_1, x_2, ..., x_n)$ in the canonical basis $\{e_1, e_2, ..., e_n\}$.

- In the vector space $\mathbb{C}^n$ on $\mathbb{R}$, we have $\forall\, (z_1, z_2, ..., z_n) \in \mathbb{C}^n$:

$$(z_1, z_2, ..., z_n) = x_1 e_1 + y_1 (ie_1) + x_2 x_2 + y_2 (ie_2) + ... + x_n e_n + y_n (ie_n),$$

where $z_k = x_k + iy_k$ $(1 \leq k \leq n)$ and $\{e_1, ie_1, e_2, ie_2 ..., e_n, ie_n\}$ is the canonical basis of $\mathbb{C}^n$ over $\mathbb{R}$. Hence, $(x_1, y_1, x_2, x_2, ..., x_n, y_n)$ are the coordinates of the vector $(z_1, z_2, ..., z_n)$ in the canonical basis.

- In the v. space $\mathbb{R}_n[X]$, we have

$$\forall P \in \mathbb{R}_n[x] : P = a_0 + a_1 \cdot x + a_2 \cdot x^2 + ... + a_n \cdot x^n,$$

where $\{1, x, x^2, ..., x^n\}$ is the canonical basis of $\mathbb{R}_n[x]$. Hence, $(a_0, a_1, a_2, ..., a_n)$ are the coordinates of $P = a_0 + a_1 \cdot x + a_2 \cdot x^2 + ... + a_n \cdot x^n$ in the canonical basis.

- In the vector space $\mathcal{M}_2(\mathbb{R})$, we have: $\forall A = (a_{ij}) \in \mathcal{M}_2(\mathbb{R})$:

$$A = a_{11} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_{12} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + a_{21} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + a_{22} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

where

$$\left\{ e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is the canonical basis of $\mathcal{M}_2(\mathbb{R})$. Hence, $\dim \mathcal{M}_2(\mathbb{R}) = 4$. More generally, $\dim \mathcal{M}_n(\mathbb{R}) = n^2$.

1.2 Linear mappings and linear forms

Let E and F be two vector spaces over the same field $\mathbb{K}$, and let $f : E \to F$ be a mapping[1] from E to F.

- f is a linear mapping[2] $\overset{\text{def}}{\Leftrightarrow} \forall\, u, v \in E, \forall\, \alpha \in \mathbb{K}$:

$$\begin{cases} f(u+v) = f(u) + f(v) \\ f(\alpha \cdot v) = \alpha \cdot f(v). \end{cases} \tag{1.2}$$

- f is a linear mapping $\overset{\text{prop}}{\Leftrightarrow} \forall\, \alpha, \beta \in \mathbb{K}, \forall\, u, v \in E$:

$$f(\alpha \cdot u + \beta \cdot v) = \alpha \cdot f(u) + \beta \cdot f(v). \tag{1.3}$$

- We denote by $\mathcal{L}(E, F)$ the set of all linear mappings from E to F.
- If $E = F$, then we denote by $\mathcal{L}(E)$ instead of $\mathcal{L}(E, E)$.

Definition 1.2. Let E be a vector space over $\mathbb{K}$. A linear form over $\mathbb{K}$ is a linear mapping from E to $\mathbb{K}$. The vector space of all linear forms on E, denoted by E^*, is called the *dual* vector space of E.

Example 1.2. Using (1.2) or (1.3), we can easily prove that the following mappings are linear forms on E

[1]Sometimes we say a *map* instead of mapping.
[2]– – we say a *linear functional* instead of a linear mapping.

1. The mapping $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x, y) = 2x - y$ is a linear form on $\mathbb{R}^2$.

2. The mapping $f : \mathbb{K}^n \to \mathbb{R}$ such that

$$f(x_1, x_2, ..., x_n) = a_1 x_1 + a_2 x_2 + ... + a_n x_n$$

   is a linear form on $\mathbb{R}^n$, where $a_i \in \mathbb{R}$, for $i = 1, 2, ..., n$.

3. The mapping $f : \mathbb{R}_n[x] \to \mathbb{R}$ such that

$$f(p) = \int_a^b p(t)\, dt$$

   is a linear form on $\mathbb{R}_n[x]$.

4. The mapping $f : \mathcal{M}_n(\mathbb{K}) \to \mathbb{R}$ such that $f(A) = tr(A)$ is a linear form on $\mathcal{M}_n(\mathbb{K})$.

5. Let E be vector space of finite dimension (or a finite-dimensional vector space), say $\dim E = n$ and let $\mathcal{B} = \{u_1, u_2, ..., u_n\}$ be a basis of E. Note that every vector $v \in E$ can be written (uniquely) as $u = \alpha_1 u_1 + ... + \alpha_n u_n$. For each $i \in \overline{1, n}$, the mapping

$$\begin{aligned} u_i^* &: E \to \mathbb{K} \\ u &\mapsto u_i^*(u) = \alpha_i \end{aligned} \tag{1.4}$$

   is a linear form on E.

The dual space of E, denoted E^*, is the v. space of all linear mappings on E. In other words, $E^* = \mathcal{L}(E, \mathbb{K})$.

We have the following facts:

- If E has finite dimension, then $\dim E = \dim E^*$.

- If $u_1, u_2, ..., u_n$ is a basis of E, then the dual basis of $u_1, u_2, ..., u_n$ is the list $\Phi_1, \Phi_2, ..., \Phi_n$ of elements of E^*, where each $\Phi_i : E \to \mathbb{K}$ is a linear mapping such that

$$\Phi_i(u_j) = \begin{cases} 1, \text{ if } i = j \\ 0, \text{ otherwise.} \end{cases} \tag{1.5}$$

  In the case when $E = \mathbb{K}^n$, we can easily find the corresponding dual basis of the canonical basis of $\mathbb{K}^n$, namely $(e) = \{e_1, e_2, ..., e_n\}$. Define the mappings:

$$\begin{aligned} \Phi_i &: \mathbb{K}^n \to \mathbb{K} \\ (x_1, x_2, ..., x_n) &\mapsto x_i \end{aligned}$$

We see that $\Phi_i(e_j)$ satisfies (1.5). Hence, $\{\Phi_1, \Phi_2, ..., \Phi_n\}$ is the corresponding dual basis of the canonical basis (e) of $\mathbb{K}^n$.

- Every basis of $E^*$is the dual basis of a unique basis of E, it is called the *predual basis*.
- Let f be a nonzero linear form over E. Then there exists a nonzero vector v such that $f(v) = 1$. In fact, since $f \neq 0$, there exists a nonzero vector x such that $f(x_0) \neq 0$. The results holds for $v = \dfrac{x_0}{f(x_0)}$.
- Let E be a finite-dimensional v. space, namely $\dim E = n$. If $v \in E$ is a nonzero vector, then there exits a linear form $f \in E^*$ such that $f(v) = 1$. Indeed, let $v = \alpha_1 u_1 + ... + \alpha_n u_n$ be a a nonzero vector. Then there exists $i_0 \in \overline{1,n}$ such that $\alpha_{i_0} \neq 0$. Define $u_{i_0}^*$ as in (1.4). That is, $u_{i_0}^*(v) = \alpha_{i_0} \neq 0$. Hence, the result holds if we put $f = \dfrac{u_{i_0}^*}{u_{i_0}^*(v)}$.

Proposition 1.2 (Changing dual basis)**.** *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two basses of E and let P be the passage matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$. Then $(P^{-1})^t$ is passage matrix from $\mathcal{B}_1^*$ to $\mathcal{B}_2^*$.*

Definition 1.3 (dual mapping)**.** If $\Phi \in \mathcal{L}(E, F)$, then the dual mapping of f is the linear mapping $\Phi^* \in \mathcal{L}(E^*, F^*)$ defined by $\Phi^*(f) = f \circ \Phi$, for $f \in E^*$.

Example 1.3. Define the mapping

$$\begin{array}{rcl} \Phi & : & \mathbb{R}_n[x] \to \mathbb{R} \\ p & \mapsto & \Phi(p) = p', \end{array}$$

where p' denotes the derivative of p. Let us take, for example $f : \mathbb{R}_n[x] \to \mathbb{R}$ such that $f(p) = p(n)$ (here n is a positive integer). Then $\Phi^*(f)$ is the linear mapping on $\mathbb{R}_n[x]$ given by

$$(\Phi^*(f))(p) = (f \circ \Phi)(p) = f[\Phi(p)] = f(p') = p'(n).$$

Hence, $\Phi^*(f)$ is the linear map on $\mathbb{R}_n[x]$ that takes p to $p'(n)$.

Suppose further that $f : \mathbb{R}_n[x] \to \mathbb{R}$ such that $f(p) = \int_a^b p(t)\,dt$. Then $\Phi^*(f)$ is the linear mapping on $\mathbb{R}_n[x]$ given by

$$(\Phi^*(f))(p) = (f \circ \Phi)(p) = f[\Phi(p)] = f(p') = \int_a^b p'(t)\,dt = p(b) - p(a).$$

Hence, $\Phi^*(f)$ is the linear map on $\mathbb{R}_n[x]$ that takes p to $p(b) - p(a)$.

Let us state some algebraic properties of dual maps:

1. $(\Phi_1 + \Phi_2)^* = \Phi_1^* + \Phi_2^*$ for every $\Phi_1 + \Phi_2 \in \mathcal{L}(E, F)$.

2. $(\alpha \cdot \Phi_2)^* = \alpha \cdot \Phi^*$ for all $\Phi \in \mathcal{L}(E, F)$ and $\alpha \in \mathbb{K}$.

3. $(\Phi_1 \circ \Phi_2)^* = \Phi_2^* \circ \Phi_1^*$ for all $\Phi_1 \in \mathcal{L}(E, F)$ and $\Phi_2 \in \mathcal{L}(F, G)$.

- Any linear mapping is a homomorphism (we can talk about the *kernel*, the *image* and so on).

- If $f : E \to F$ is linear mapping, then $f(0_E) = 0_F$ (the converse is false).

- If $f(0_E) \neq 0_F$, then $f : E \to F$ is not a linear mapping.

- Be careful, if $f : E \to F$ is linear and $v \in E$, then : $f(v) = 0_F \not\Rightarrow v = 0_E$ (in general). But, if f is injective, then $f(v) = 0_F \Rightarrow v = 0_E$ (since $f(v) = 0_F \Leftrightarrow f(v) = f(0_E)$).

- Every linear mapping $f : E \to E$ is called *Endomorphism* of E.

- Every linear mapping $f : E \to F$ bijective is called *Isomorphism.*

- Every bijective Endomorphism of E is called *Automorphism* of E.

- Every linear mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is uniquely defined as follows:

$$f(x_1, x_2, ..., x_n) = (a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n, ..., a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n),$$

  where $(a_{ij}) \in \mathbb{R}$ for all i, j ($i = 1, 2, ..., m$ and $j = 1, 2, ..., n$).

- The *kernel* of a linear mapping $f : E \to F$ is the set defined by

$$\ker f = \{v \in E \ : \ f(v) = 0_F\}. \tag{1.6}$$

  We can easily prove that $\ker f$ is a vector subspace of E.

- The *image* of a linear mapping $f : E \to F$ is the set defined by

$$Im\ f = \{f(v) \ : \ v \in E\}.$$

  We can easily prove that $Im\ f$ is a vector subspace of F.

- If $f : E \to F$ is the zero linear mapping (i.e., $f(v) = 0, \forall\, v \in E$), then $\ker f = E$ and $Im\ f = \{0_F\}$.

- The identical mapping of E, i.e., id_E is linear, where $\ker(id_E) = \{0_E\}$ and $Im(id_E) = E$.

- If $f : E \to F$ is linear, then

$$E = Vect\left(v_1, v_2, ..., v_n\right) \Rightarrow Im\ f = Vect\left(f\left(v_1\right), f\left(v_2\right), ..., f\left(v_n\right)\right).$$

In practice, we use the canonical basis of E. So, we have in particular:

- If $f : \mathbb{R}^n \longrightarrow F$ is linear and $\{e_1, e_2, ..., e_n\}$ is the canonical basis of $\mathbb{R}^n$, then we have

$$Im\ f = Vect\left(f\left(e_1\right), f\left(e_2\right), ..., f\left(e_n\right)\right),$$

  where $e_1 = (1, 0, ..., 0)\,, e_2 = (0, 1, ..., 0)\,, e_n = (0, 0, ..., 1)\,.$

- If $f : \mathbb{R}_n[x] \longrightarrow F$ is linear and $\{1, x, x^2, ..., x^n\}$ is the canonical basis of $\mathbb{R}_n[x]$, then we have

$$Im\ f = Vect\left(f\left(1\right), f\left(x\right), ..., f\left(x^n\right)\right),$$

  where $e_1 = (1, 0, ..., 0)\,, e_2 = (0, 1, ..., 0)\,, e_n = (0, 0, ..., 1)\,.$

- If f is linear, then the number $\dim\left(Im\ f\right)$ is called the *rank* of f and we note it by $rank\left(f\right)$, i.e.,

$$rank\left(f\right) = \dim\left(Im\ f\right).$$

- If $f : E \to F$ is linear, then

$$f \text{ is injective } \Leftrightarrow \ker f = \{0_E\} \Leftrightarrow \dim\left(\ker f\right) = 0,$$

  and also, we have

$$f \text{ is surjective } \Leftrightarrow Im\ f = F \Leftrightarrow \dim\left(Im\ f\right) = \dim F.$$

- If $f : E \to F$ is linear, then

$$\dim E = \dim \ker f + \dim Im\ f. \quad (\textit{Rank Theorem})$$

- If $f : E \to F$ is linear with $\dim E = \dim F$, then

$$f \text{ is surjective} \Leftrightarrow f \text{ is bijective} \Leftrightarrow f \text{ is injective.}$$

  In practice, we use this result if $E = F$, i.e., if f is an Endomorphism on E.

1.3 Proposed Problems on linear forms

Exercise 1. Determine the linear form f defined by

$$f(1,1,1)=0,\ f(2,0,1)=1 \text{ and } f(1,2,3)=4,$$

then, determine $\ker f$. The same question for g, where $g(1,0,1)=-1$, $g(0,1,1)=0$ and $g(-1,1,1)=2$.

Exercise 2. Let $E=\mathbb{R}^2$ and $f_1,f_2\in E^*$ such that

$$f_1(x,y)=x+y,\ \ f_2(x,y)=x-y.$$

1. Show that $\{f_1,f_2\}$ is a base of E^*.
2. Express g and h, in this base, where $g(x,y)=x$ and $h(x,y)=2x-6y$.
3. Determine the predual base of $\{f_1,f_2\}$.
4. Note that $\{(1,2),(-1,1)\}$ is a base of E, find its dual base.

Exercise 3. Let $\{e_1,e_2,e_3\}$ be the canonical basis of $E=\mathbb{R}^3$ and let $f_1,f_2,f_3\in E^*$ defined by

$$\begin{cases} f_1=2e_1^*+e_2^*+e_3^* \\ f_2=-e_1^*+2e_3^* \\ f_3=e_1^*+3e_2^*. \end{cases}$$

1. Prove that $\{f_1,f_2,f_3\}$ is a basis of E^*.
2. Determine the predual basis of $\{f_1,f_2,f_3\}$.
3. Prove that $\mathcal{A}=\{(1,1,1),(-1,2,1),(0,1,3)\}$ is a basis of E, and find its dual basis, say $\mathcal{A}^*$.
4. Calculate φ the passage matrix from $\{f_1,f_2,f_3\}$ to $\mathcal{A}^*$.

Exercise 4. Consider the vector space of real polynomials of degree not exceeding 2, i.e., $E=\mathbb{R}_2[x]$. Define the mappings $\varphi_0,\varphi_1,\varphi_2$ from E to $\mathbb{R}$ by

$$\forall\, p\in E,\ \varphi_0(p)=p(0),\ \varphi_1(p)=p(1) \text{ et } \varphi_2(p)=\int_0^1 p(t)\,dt.$$

1. Prove that $\varphi_i\in E^*$ for $i=0,1,2$.
2. Show that $\{\varphi_0,\varphi_1,\varphi_2\}$ is a basis of E^*.

3. Determine the predual basis of $\{\varphi_0, \varphi_1, \varphi_2\}$.

4. Prove that $\{1, 1+x, 1+x+x^2\}$ is a basis of E, and find its dual.

CHAPTER 2

BILINEAR FORMS OVER A VECTOR SPACE

In this chapter we present a basic introduction on Bilinear forms over a vector space including rank, kernel, Orthogonalization of Gram-Schmidt, Orthogonal matrices and diagonalization of real symmetric matrices.

2.1 Bilinear forms (Definitions)

In this section, $\mathbb{R}$ is the *field* of real numbers and E is a *vector space* over $\mathbb{R}$. For example, $E = \mathbb{R}^n$, $\mathbb{R}_n[x]$ or $\mathbb{P}_n[x]$, $C([a,b], \mathbb{R})$, $C^\infty([a,b], \mathbb{R})$ and $\mathcal{M}_n(\mathbb{R})$ with $n \geq 1$, and so on.

Let E be a vector space on $\mathbb{R}$. As above, a linear form[1] is a mapping f from E to $\mathbb{R}$ such that for every $(x, y) \in E^2$ and $\lambda \in \mathbb{R}$, we have

(i) $f(x+y) = f(x) + f(y)$,

(ii) $f(\lambda x) = \lambda f(x)$.

Similarly, we have the following definition:

Definition 2.1. Let E be a vector space on $\mathbb{R}$. A bilinear form is a mapping f from E^2 to $\mathbb{R}$ such that for every $(x, x', y, y') \in E^4$ and $\lambda \in \mathbb{R}$, one has

(i) $f(\lambda x + x', y) = \lambda f(x, y) + f(x', y)$,

(ii) $f(x, \lambda y + y') = \lambda f(x, y) + f(x, y')$.

As in (1.2) and (1.3), note that a bilinear form is a mapping f from E^2 to $\mathbb{R}$ such that f is linear from the *left* and linear from the *right*. For details, we present the following remark.

Remark 2.1. Let $f : E \times E \to \mathbb{R}$ be a bilinear form on E. This means that for all $x, x', y, y' \in E$ and $\lambda \in \mathbb{R}$ we have

- $f(x + x', y) = f(x, y) + f(x', y)$,

[1] If $f : E \times E \to F$ is bilinear, then f is called a bilinear mapping. However, if $f : E \times E \to \mathbb{K}$ is bilinear, then f is called a bilinear form.

- $f(x, y + y') = f(x, y) + f(x, y')$,
- $f(\lambda x, y) = \lambda f(x, y)$,
- $f(x, \lambda y) = \lambda f(x, y)$.

Definition 2.2. Let $f : E^2 \to \mathbb{R}$ be a bilinear form.

1. f is said to be **symmetric** if for each $(x, y) \in E^2$, $f(x, y) = f(y, x)$.
2. f is said to be **skew-symmetric** if for each $(x, y) \in E^2$, $f(x, y) = -f(y, x)$.
3. f is said to be **alternating** if for each $x \in E$, $f(x, x) = 0$.

Example 2.1. We can easily check that the following mappings are symmetric bilinear forms.

1. $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f(x, y) = xy$.
2. $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $((x, y), (x', y')) \mapsto xx' + yy'$.
3. $\varphi : \mathbb{P}[x] \times \mathbb{P}[x] \to \mathbb{R}$ with $\varphi(p, q) = \int_a^b p(t)\, q(t)\, dt$.
4. Let $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ with $f(x, y) = x_1y_1 + x_2y_2 - x_3y_3$.

Example 2.2. Let $u = (x, y), v = (x', y') \in \mathbb{R}^2$ with $f(u, v) = xy' - x'y$. Then f is a skew-symmetric alternating bilinear form.

Notation 2.1. Let $\mathcal{L}_2(E)$ denote the vector space of all bilinear forms over E, $\mathcal{S}_2(E)$ denote the vector space of all symmetric bilinear forms over E and $\mathcal{A}_2(E)$ denote the vector space of all skew-symmetric bilinear forms over E.

We can prove the following fact: $\mathcal{L}_2(E) = \mathcal{S}_2(E) \oplus \mathcal{A}_2(E)$. Indeed, we have $f_0 = 0 \in \mathcal{S}_2(E) \cap \mathcal{A}_2(E)$. Also, if $f \in \mathcal{S}_2(E) \cap \mathcal{A}_2(E)$, then by Definition 2.2 $f(x, y) = f(y, x) = -f(y, x)$ for every $x, y \in E$. Hence, $f(x, y) = 0$ for every $x, y \in E$. So, $f = f_0$. Thus, we have proved that $\mathcal{S}_2(E) \cap \mathcal{A}_2(E) = \{f_0\}$. Now, let $f \in \mathcal{L}_2(E)$. For any $x, y \in E$, we see that

$$f(x, y) = \frac{f(x, y) - f(y, x)}{2} + \frac{f(x, y) + f(y, x)}{2} = h_1(x, y) + h_2(x, y),$$

where h_1 is skew-symmetric and h_2 is symmetric.

Theorem 2.1. *Let*

$$x=\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix},\ y=\begin{pmatrix}y_1\\y_2\\\vdots\\y_n\end{pmatrix}\in\mathbb{R}^n.$$

Let $A\in\mathcal{M}_n(\mathbb{R})$ be a square matrix. Define $f:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$, $(x,y)\mapsto x^t\cdot A\cdot y$. Then f is a bilinear form over $\mathbb{R}^n$. Moreover, if A is symmetric, then f is also symmetric.

Proof. For all $x,x',y\in\mathbb{R}^n$ and for all $\lambda\in\mathbb{R}$ we have

$$\begin{aligned}f(\lambda x+x',y) &= (\lambda x+x')^t\cdot A\cdot y\\ &= \lambda x^t\cdot A\cdot y+(x')^t\cdot A\cdot y\\ &= \lambda f(x,y)+f(x',y).\end{aligned}$$

Thus, f is linear from the left. We use the same manner to show that f is linear from the right. For every $x,y,y'\in\mathbb{R}^n$ and $\lambda\in\mathbb{R}$ we have

$$\begin{aligned}f(x,\lambda y+y') &= x^tA(\lambda y+y')\\ &= \lambda x^tAy+x^tAy'\\ &= \lambda f(x,y)+f(x,y').\end{aligned}$$

Next, assume that A is symmetric. We show that f is also symmetric. In fact, we have

$$\begin{aligned}f(x,y) &= x^tAy\\ &= \left(x^tAy\right)^t \quad \text{(since } x^tAy\in\mathbb{R}\text{)}\\ &= y^tA^t\left(x^t\right)^t \quad \text{(well-known result)}\\ &= y^tAx \quad \text{(since } A \text{ is symmetric)}\\ &= f(y,x).\end{aligned}$$

Hence, $f(x,y)=f(y,x)$.

The proof is finished. □

We conclude from Theorem 2.1 the following corollary.

Corollary 2.1. Every matrix $A\in\mathcal{M}_n(\mathbb{R})$ produces a bilinear form over $\mathbb{R}^n$ and every symmetric matrix $A\in\mathcal{M}_n(\mathbb{R})$ produces a symmetric bilinear form over $\mathbb{R}^n$.

Theorem 2.2. *Let B and B' be two bases of E. Let P be the passage matrix from B to B' and let $f:E\times E\to\mathbb{R}$ be a bilinear form over E. If $A=\mathcal{M}_f(B)$ and $A'=\mathcal{M}_f(B')$, then $A'=P^t\cdot A\cdot P$.*

Proof. Assume that $B = \{e_1, e_2, ..., e_n\}$ and $B' = \{e'_1, e'_2, ..., e'_n\}$. For every $x, y \in E$, we see that

$$\begin{cases} x = \sum_{i=1}^{n} x_i \cdot e_i \\ \\ x = \sum_{i=1}^{n} x'_i \cdot e'_i \end{cases} \text{and} \begin{cases} y = \sum_{i=1}^{n} y_i \cdot e_i \\ \\ y = \sum_{i=1}^{n} y'_i \cdot e'_i \end{cases}$$

That is, $x = P \cdot x'$ and $y = P \cdot y'$. Therefore,

$$f(x, y) = x^t \cdot A \cdot y = (P \cdot x')^t \cdot A \cdot (P \cdot y') = (x')^t \cdot \underbrace{P^t A P} \cdot y'.$$

Thus, the matrix of f with respect to the basis B' is given by: $A' = P^t \cdot A \cdot P$, where A is the matrix of f with respect to the basis B. The proof is finished. □

Theorem 2.3. *If* $\dim E = n$, *then* $\dim \mathcal{L}_2(E) = n^2$.

Proof. Let $\{u_1, u_2, ..., u_n\}$ be a basis of E. Define the bilinear forms $f_{i,j}$ by

$$f_{i,j}(e_r, e_s) = \begin{cases} 1, \text{ for } (i, j) = (r, s) \\ 0, \text{ for } (i, j) \neq (r, s) \end{cases}$$

Let $x = \sum_{i=1}^{n} x_i u_i$ and $y = \sum_{j=1}^{n} y_j u_j$ be two vectors of E. It is clear that

$$f_{i,j}(x, y) = x_i y_j, \text{ for } i = 1, 2, ..., n.$$

Now, let $f \in \mathcal{L}_2(E)$ and put $f(e_r, e_s) = a_{rs}$. It follows that

$$\begin{aligned} f(x, y) &= f\left(\sum_{i=1}^{n} x_i u_i, \sum_{j=1}^{n} y_j u_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i y_j f(e_i, e_j) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i y_j a_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i y_j f_{i,j}(x, y). \end{aligned}$$

Then these n^2 bilinear forms $f_{i,j}$ generated the vector space $f_{i,j}$. Since $(f_{i,j})_{1 \leq i,j \leq n}$ form a free family, we conclude that $(f_{i,j})_{1 \leq i,j \leq n}$ is a basis of $\mathcal{L}_2(E)$. The proof is finished. □

2.2 Orthogonal matrices

Definition 2.3. An invertible square matrix A is said to be **orthogonal** if $A^t = A^{-1}$.

Clearly, a sufficient and necessary condition for A to be orthogonal is that $AA^t = A^t A = I$, , where I is the identity matrix.

Example 2.3. By the above definition, the following matrices:

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \text{ with } \alpha \in \mathbb{R}, \tag{2.1}$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \text{ and } \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

are orthogonal.

Proposition 2.1. *Let* $A \in \mathcal{M}_n(\mathbb{R})$ *be an orthogonal matrix. Then* $\det(A) = \pm 1$.

Proof. Since $A^t = A^{-1}$, we conclude that $A^t A = I_n$. This gives

$$\det\left(A^t A\right) = \det\left(A^t\right)\det(A) = (\det(A))^2 = \det(I_n) = 1.$$

Hence, $\det(A) = \pm 1$. □

We need to define matrix norms and scalar product over a vector space E.

2.2.1 Matrix norms

Definition 2.4. Let E be a vector space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). The norm over E, denoted by $\|.\|$, is a mapping

$$\begin{array}{ccl} \|.\| & : & E \to \mathbb{R}_+ \\ x & \mapsto & \|x\| \text{ (we say: the norm of } x) \end{array}$$

which satisfy the following properties:

1. For every $x \in E : \|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0_E$;
2. For every $x \in E$ and scalar $\alpha \in \mathbb{K} : \|\alpha x\| = |\alpha| . \|x\|$;
3. For every $x, y \in E : \|x + y\| \leq \|x\| + \|y\|$.

In this case, the couple $(E, \|.\|)$ is called *normed vector space* or *normed space.* So, a normed space E is a v. space endowed by a norm.

Example 2.4. Here, we only use the two vector spaces, $\mathbb{K}^n$ and $\mathcal{M}_n(\mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

1. Define over $\mathbb{K}^n$ the following norms:

$$\begin{aligned}\|x\|_1 &= \sum_{i=1}^{n} |x_i|\,,\ \|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}},\\ \|x\|_\infty &= \max_{1\leq i\leq n} (|x_i|)\,.\end{aligned}$$

2. Define over $\mathcal{M}_n(\mathbb{K})$ the following norms:

$$\begin{aligned}\|A\|_1 &= \max_j \sum_{i=1}^{n} |a_{ij}| \text{ and } \|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|\\ \|A\|_2 &= \left(\sum_{i,j}^{n} |a_{ij}|^2\right)^{\frac{1}{2}} \text{ and } \|A\|_p = \left(\sum_{i,j}^{n} |a_{ij}|^p\right)^{\frac{1}{p}}\end{aligned}$$

As an application, for $x = \begin{pmatrix} -1 & 1 & -2 \end{pmatrix}^t$, we have

$$\|x\|_1 = 4,\ \ \|x\|_2 = \sqrt{6} \text{ and } \|x\|_\infty = 2.$$

and for $A = \begin{pmatrix} -1 & -2 \\ 7 & 3 \end{pmatrix} \in \mathcal{M}_n(\mathbb{R})$, we also have

$$\|A\|_1 = \max(8,5) = 8, \|A\|_2 = 3\sqrt{7} \text{ and } \|A\|_\infty = \max(3,10) = 10.$$

Lemma 2.1. *For each matrix $A \in \mathcal{M}_n(\mathbb{K})$ and for each $x \in \mathbb{K}^n$, we have the following inequality:*

$$\|Ax\| \leq \|A\| \, \|x\|\,.$$

The above lemma remains interesting for future study.

2.2.2 Scalar Product (Inner product) over a real vector space

Definition 2.5. Let E be real vector space. The inner product over E is a mapping $\langle .,. \rangle$ defined by

$$\begin{aligned}\langle .,. \rangle &: E \times E \to \mathbb{R}\\ (x,y) &\mapsto \langle x,y \rangle\end{aligned}$$

which satisfy the following properties:

1. For all $x \in E$, $\langle x,x \rangle \geq 0$ and $\langle x,x \rangle = 0 \Leftrightarrow x = 0$.

2. For all $x, y \in E$, we have $\langle x, y\rangle = \langle y, x\rangle$.

3. For all $x \in E$ and scalar $\alpha \in \mathbb{R}$, we have $\langle \lambda x, y\rangle = \lambda \langle x, y\rangle$

4. For all $x, y, z \in E$, we have $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$.

We say, the scalar product between x and y, or the inner product between x and y.

Example 2.5. Define over $\mathbb{R}^n$ the scalar product $\langle ., .\rangle$ by

$$\forall\, x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},\ y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n : \langle x, y\rangle = \sum_{i=1}^{n} x_i y_i. \tag{2.2}$$

We can write (2.2) as[2]: $\langle x, y\rangle = x^t \cdot y$. In particular, for $x = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^t$ and $y = \begin{pmatrix} y_1 & y_2 \end{pmatrix}^t$, we have

$$\langle x, y\rangle = \langle (x_1, x_2), (y_1, y_2)\rangle = x_1 y_1 + x_2 y_2.$$

Example 2.6. Define over the vector space $C([a, b])$ the inner product:

$$\forall\, f, g \in C([a, b]) : \langle f, g\rangle = \int_a^b f(x) \cdot g(x)\, dx.$$

Theorem 2.4. *Let $A \in \mathcal{M}_n(\mathbb{R})$. The following properties are equivalent:*

(i) A is orthogonal.

(ii) For every $x \in \mathbb{R}^n$, $\|Ax\| = \|x\|$.

(iii) For every $x, y \in \mathbb{R}^n$, $\langle Ax, Ay\rangle = \langle x, y\rangle$.

Proof. 1)⇒2). Assume that A is orthogonal. Let $x \in \mathbb{R}^n$ we have

$$\|Ax\|^2 = \langle Ax, Ax\rangle = \left\langle x, A^t A x\right\rangle = \langle x, I_n x\rangle = \langle x, x\rangle = \|x\|^2 . \tag{2.3}$$

Therefore, $\|Ax\| = \|x\|$.

2)⇒3). Suppose that $\forall\, x \in \mathbb{R}^n : \|Ax\| = \|x\|$. Let $x, y \in \mathbb{R}^n$ we see that

$$\|A(x + y)\|^2 = \|x + y\|^2 .$$

[2]Sometimes we use the notation ${}^t x \cdot y$ instead of $x^t \cdot y$. We also write ${}^t A \cdot A$ instead of $A^t \cdot A$ when A is a square matrix.

This means that $\langle Ax+Ay, Ax+Ay\rangle = \langle x+y, x+y\rangle$, or, equivalently,

$$\langle Ax, Ax\rangle + \langle Ay, Ay\rangle + 2\langle Ax, Ay\rangle = \langle x, x\rangle + \langle y, y\rangle + 2\langle x, y\rangle .$$

Thus, $\langle Ax, Ay\rangle = \langle x, y\rangle$.

3)⇒1). Assume that $\forall\, x, y \in \mathbb{R}^n : \langle Ax, Ay\rangle = \langle x, y\rangle$. Then

$$\left\langle x, A^t Ay\right\rangle = \langle x, y\rangle ,$$

i.e., $\langle x, A^t Ay - y\rangle = 0$. In particular, for $x = x^t Ay - y$, we obtain

$$\left\|A^t Ay - y\right\|^2 = 0.$$

Hence, $A^t Ay = y$. Consequently, $A^t A = I_n$. □

Example 2.7 (Homework)**.** Consider the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

For any θ real, show that $e^{\theta A}$ is orthogonal[3].

2.3 Gram-Schmidt Orthonormalization Theorem

Let E be a Euclidean space and let $B = \{u_1, u_2, ..., u_n\}$ be a basis of E. There exists a unique orthonormal basis $\{e_1, e_2, ..., e_n\}$ of E satisfying the following conditions:

1. $Vect\,\{e_1, e_2, ..., e_n\} = Vect\,\{u_1, u_2, ..., u_n\}$.

2. $\langle e_i, u_i\rangle = 0$ for $i = 1, 2, ..., n$.

The following formulas permit us to find such orthonormal basis recursively as follows:

$$\begin{cases} e_1 = \dfrac{u_1}{\|u_1\|}, \\ v_k = u_k - \sum_{i=1}^{k-1} \langle e_i, u_k\rangle \cdot e_i, \\ e_k = \dfrac{v_k}{\|v_k\|}. \end{cases} \tag{2.4}$$

Example 2.8. Let us take $E = \mathbb{R}^2$ and $B = \{(1,-1), (1,1)\} = \{u_1, u_2\}$. Clearly, B is a basis of $\mathbb{R}^2$. Now, we construct the corresponding orthonormal basis using Gram-Schmith

[3]We can prove that A is diagonalizable, where $e^{\theta A}$ is given by (2.1).

method. First, we have

$$e_1 = \frac{(1,-1)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$$

and

$$\begin{aligned} v_2 &= u_2 - \langle e_1, u_2\rangle \cdot e_1 \\ &= (1,1) - \left\langle \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right), (1,1) \right\rangle \cdot \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = (1,1). \end{aligned}$$

Hence,

$$e_2 = \frac{v_2}{\|v_2\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

We deduce that $\mathcal{B} = \{e_1, e_2\}$ is an orthonormal basis of $\mathbb{R}^2$. Similarly, let

$$B = \{(1,1,1,), (0,1,1,), (0,0,1)\} = \{u_1, u_2, u_3\}.$$

We have $e_1 = \frac{u_1}{\|u_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. Next, by (2.4) we have

$$\begin{aligned} v_2 &= u_2 - \langle e_1, u_2\rangle \cdot e_1 \\ &= (0,1,1) - \left\langle \left(\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), (0,1,1) \right) \right\rangle \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \\ &= (0,1,1) - \frac{2}{\sqrt{3}} \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \left(\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}\right). \end{aligned}$$

Hence, $e_2 = \frac{v_2}{\|v_2\|} = \left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$. Also, by by (2.4),

$$\begin{aligned} v_3 &= u_3 - \langle e_1, u_3\rangle e_1 - \langle e_2, u_3\rangle e_2 \\ &= (0,0,1) - \left\langle \left(\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), (0,0,1) \right) \right\rangle \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) - \\ &\quad \left\langle \left(\left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), (0,1,1) \right) \right\rangle \cdot \left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \\ &= \left(0, \frac{-1}{2}, \frac{1}{2}\right). \end{aligned}$$

Thus, $e_3 = \left(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. We deduce that $\mathcal{B} = \{e_1, e_2, e_3\}$ is an orthonormal basis of $\mathbb{R}^3$.

Example 2.9 (Homework)**.** Let $B = \{(1,2), (3,4)\}$. Prove that B is a basis of $\mathbb{R}^2$. Transform B to an orthonormal basis. **Ans**. $\mathcal{B} = \left\{ \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right), \left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right) \right\}$.

2.4 Diagonalization of real symmetric matrices

Recall that a matrix $A \in \mathcal{M}_n(\mathbb{C})$ is said to be *symmetric* if $A^t = A$. It is well known that for any matrix $A \in \mathcal{M}_n(\mathbb{R})$, the matrices A^tA and AA^t are symmetric. If A is skew-symmetric, then A^2 is symmetric. Moreover, $A + A^t$ is always symmetric and $A - A^t$ is always skew-symmetric. So we can easily prove that

$$\mathcal{M}_n(\mathbb{R}) = \mathcal{S}_n(\mathbb{R}) \oplus \mathcal{A}_n(\mathbb{R}).$$

We can easily show that if A is symmetric, then e^A is also symmetric. Indeed, by definition, if A is symmetric then we have

$$\left(e^A\right)^t = \left(\sum_{i=0}^{+\infty} \frac{A^i}{i!}\right)^t = \sum_{i=0}^{+\infty} \left(\frac{A^i}{i!}\right)^t = \sum_{i=0}^{+\infty} \frac{\left(A^t\right)^i}{i!} = \sum_{i=0}^{+\infty} \frac{A^i}{i!} = e^A.$$

The result holds. Another important result is given by:

Lemma 2.2. *Every symmetric matrix $A \in \mathcal{M}_n(\mathbb{R})$ is diagonalizable. Moreover, every symmetric matrix $A \in \mathcal{M}_n(\mathbb{R})$ can be represented in the form:*

$$A = P \cdot D \cdot P^t, \tag{2.5}$$

where P is orthogonal and D is diagonal whose diagonal entries are the eigenvalues of A.

Proof. The proof is found in the course of Algebra III. □

Definition 2.6. Let $A \in \mathcal{M}_n(\mathbb{R})$ be a symmetric matrix. We have

- A is said to be **positive** if $x^tAx \geq 0$ for every $x \in \mathbb{R}^n$.
- A is said to be **definite positive** if $x^tAx > 0$ for every $x \in \mathbb{R}^n - \{0_{\mathbb{R}^n}\}$.

Next, we present the following theorem.

Theorem 2.5. *Let $A \in \mathcal{M}_n(\mathbb{R})$. Then A is symmetric definite positive if and only if there exists an invertible matrix M such that*

$$A = M^tM. \tag{2.6}$$

Proof. Assume that $A = M^tM$ with M is invertible. Then A is symmetric, since

$$A^t = \left(M^tM\right)^t = M^t\left(M^t\right)^t = M^tM = A.$$

On the other hand, let $x \neq 0$ be a column vector. We put

$$(Mx)^t = \begin{pmatrix} y_1 & y_2 & \dots & y_n \end{pmatrix}.$$

Since $M \in \mathbb{GL}_n(\mathbb{R})$, then $Mx = y \neq 0$. Therefore,

$$x^t Ax = x^t \left(M^t M\right) x = (Mx)^t (Mx) = y^t y = \sum_{i=1}^{n} y_i^2 > 0.$$

Thus, A is definite positive.

Conversely, assume that A is symmetric definite positive. By Lemma 2.5, we write

$$A \text{ is symmetric} \Rightarrow \exists\, P \in \mathbb{GL}_n(\mathbb{R}) \text{ such that } A = PDP^t,$$

where $D = (\lambda_{ii})$ is diagonal whose diagonal elements are the eigenvalues of A. However, since A est definite positive, the matrix D is also definite positive, that is, its diagonal entries are strictly positive. Thus, we can define the diagonal matrix:

$$\sqrt{D} = diag\left\{\sqrt{\lambda_1}, \sqrt{\lambda_2}, ..., \sqrt{\lambda_n}\right\},$$

and rewriting, we get

$$A = PDP^t = P\sqrt{D}\sqrt{D}P^t = P\sqrt{D}\left(\sqrt{D}\right)^t P^t = \left(P\sqrt{D}\right)\left(P\sqrt{D}\right)^t = M^t M,$$

where $M = \left(P\sqrt{D}\right)^t \in \mathbb{GL}_n(\mathbb{R})$; since $P, \sqrt{D} \in \mathbb{GL}_n(\mathbb{R})$. The proof of Theorem 2.2 is finished. □

Corollary 2.2. Let A be a symmetric definite positive matrix. Then $\det(A) > 0$.

Proof. *First method.* Since A is a symmetric definite positive then by Theorem 2.5, $A = M^t M$, where M is invertible. Therefore,

$$\det(A) = \det\left(M^t M\right) = \det\left(M^t\right)\det(M) = (\det(M))^2 > 0.$$

Second method. Since A is a symmetric definite positive then $Sp(A) \subset \mathbb{R}_+^*$. On the other hand, it is well-known that

$$\det(A) = \prod \lambda_i,$$

from which it follows that $\det(A) > 0$. □

Another interesting property of symmetric matrices is the following result:

Proposition 2.2. *Let A be a symmetric matrix and let $(\lambda_1, x), (\lambda_2, y)$ be two eigenpairs of A with $\alpha \neq \beta$. Then $\langle x, y\rangle = 0$.*

Proof. Indeed, we see that

$$\lambda_1 \langle x, y\rangle = \langle \lambda_1 x, y\rangle = \langle Ax, y\rangle = \left\langle x, A^t y\right\rangle = \langle x, Ay\rangle = \langle x, \lambda_2 y\rangle = \lambda_2 \langle x, y\rangle ,$$

and since $\alpha \neq \beta$, we deduce that $\langle x, y\rangle = 0$. □

2.5 Proposed Problems on bilinear forms

Exercise 1. Let $v_1 = (2, 1)$ and $v_2 = (1, -2)$ two elements of th real vector space $\mathbb{R}^2$ reported by its canonical basis. Show that $\{v_1, v_2\}$ is a basis of $\mathbb{R}^2$. Consider the linear form φ over $\mathbb{R}^2$ defined by $\varphi(v_1) = 15$ and $\varphi(v_2) = -10$. Find $\varphi(x)$ for any$x = (x_1, x_2)$ in $\mathbb{R}^2$. Give the dual basis $\{v_1, v_2\}$.

Exercise 2. Let E be real vector space $\mathbb{R}^2$ related to its canonical basis $\{e_1, e_2\}$ and let f be the bilinear form defined on E setting for every $x = (x_1, x_2)$ and (y_1, y_2) in E,

$$f(x, y) = 33x_1y_1 - 14(x_1y_2 + x_2y_1) + 6x_2y_2.$$

1. Find the matrix of f relative to the basis $\{e_1, e_2\}$.

2. Prove that the vectors $v_1 = e_1 + 2e_2$, $v_2 = 2e_1 + 5e_2$ form a basis of E.

3. Write the matrix of f with respect to the basis $\{v_1, v_2\}$.

4. What is the rank of f?

Exercise 3. Let f be the bilinear form defined on the vector space $\mathbb{R}^2$ by

$$\begin{cases} f(e_1, e_1) = 1, f(e_1, e_2) = 1 \\ f(e_2, e_1) = -1, f(e_1, e_2) = 3, \end{cases}$$

where $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$. Specify the value $f(x, y)$ for every x, y in $\mathbb{R}^2$.

Exercise 4. Let f be the bilinear form on $\mathbb{R}^2$ setting $x = (x_1, x_2)$ and (y_1, y_2) in $\mathbb{R}^2$,

$$f(x, y) = 2x_1y_1 - 3x_1y_2 + x_2y_2.$$

1. Find the matrix A' of f related to the basis $\{u_1 = (1, 0), u_2 = (1, 1)\}$.

2. Find the matrix B of f related to the basis $\{v_1 = (2, 1), u_2 = (1, -1)\}$.

3. Find the passage matrix P from the basis $\{u_1, u_2\}$ to the basis $\{v_1, v_2\}$ and verify that $B = P^t A' P$.

4. What is the rank of f?

Exercise 5. Let E be a vector space over a commutative field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). We denote by S the set of symmetric bilinear forms on E and by A the set of antisymmetric bilinear forms on E.

1. Show that S and A are two vector subspaces of E.

2. Show that the vector space of bilinear forms $B(E)$ over E is the direct sum of S and A.

3. We assume that E is of finite dimension n. What are the dimensions of S and A.

Exercise 6. Are the following functions $E \times E \to \mathbb{R}$ bilinear forms over the vector space E? If yes, write their matrix in the canonical basis. Are they symmetric? When $E = \mathbb{R}^3$, give their matrix in the basis $B = \{v_1, v_2, v_3\}$, where $v_1 = (1, 0, 1)$, $v_2 = (1, 1, 0)$ and $v_3 = (-1, 0, 1)$.

- $f(x, y) = x_1 y_1 + x_2 y_2 + x_3 y_3$, $E = \mathbb{R}^3$
- $f(x, y) = y_1 y_2 + x_1 y_1 + x_3 y_3$, $E = \mathbb{R}^2$
- $f(x, y) = x_2^2 y_1 + 3 x_2 y_2$, $E = \mathbb{R}^3$
- $f(x, y) = x_1 y_2 - 2 x_3 (y_2 + 2 y_1) + 4 x_3 y_2 - y_1 x_2$, $E = \mathbb{R}^3$

Exercise 7. Let f_1, f_2 be bilinear forms on $\mathbb{R}^3$ whose matrices in the canonical basis are

$$A_1 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & -3 & 2 \\ 0 & -2 & -1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 1 & -2 & \frac{-1}{2} \\ \frac{1}{2} & \frac{-1}{2} & 0 \end{pmatrix}.$$

Write the matrices B_1 and B_2 of f_1 and f_2 with respect to the basis $\{v_1, v_2, v_3\}$, where $v_1 = (1, 0, 0)$, $v_2 = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$, $v_3 = \left(\frac{-1}{2}, \frac{-1}{2}, 1\right)$. Deduce the rank of each of the linear forms $f_1$and f_2.

Exercise 8. Prove that the vectors $e_1 = (1, 0, 2)$, $e_2 = (0, 1, 1)$ and $e_3 = (-1, 0, 1)$ form a basis of $\mathbb{R}^3$.

Determine the matrix with respect to this basis of the bilinear form $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ defined, for every $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{R}^3$ by

$$f(x, y) = 2 x_1 y_2 + x_2 y_2 - x_2 y_3 - 2 x_3 y_1 + x_3 y_2 - x_3 y_3.$$

Exercise 9. Let f be a bilinear form on E and A its matrix representation in a given basis. f is said to be symmetrical or symmetric (resp., skew-symmetric, alternating) if $f(x,y) = f(y,x)$ (resp., $f(x,y) = -f(y,x)$, $f(x,x) = 0$) for every x, y belong to E.

1. Prove that f is symmetric (resp., skew-symmetric) if and only if $A^t = A$ (resp., $A^t = -A$).

2. Prove that if f is alternating, then f is skew-symmetric.

3. Recall that the basic field $\mathbb{K}$ of E is infinite. Show that if f is antisymmetric, then f is alternating.

4. Define $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ such that $f(x,y) = xy$ and

$$g((x_1, x_2), (y_1, y_2)) = x_1 y_2 - x_2 y_1,$$

where $x, y \in \mathbb{R}$ and $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$. Study whether f and g are symmetric, skew-symmetric or alternating.

Exercise 10. Let $(e) = \{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$ and let f the bilinear symmetric form over $\mathbb{R}^3$ given by

$$f(x,y) = x_1y_1 + 6x_2y_2 + 56x_3y_3 - 2(x_1y_2 + x_2y_1) + 7(x_1y_3 + x_3y_1) - 18(x_2y_3 + x_3y_2),$$

for $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{R}^3$.

1. Find the matrix of f with respect to the basis (e).

2. Prove that the vectors $e'_1 = e_1$, $e'_2 = 2e_1 + e_2$ and $e'_3 = -3e_1 + 2e_2 + e_3$ form a basis of $\mathbb{R}^3$.

3. Write the matrix of f with respect to the basis $(e') = \{e'_1, e'_2, e'_3\}$.

CHAPTER 3

SYMMETRIC BILINEAR FORMS AND QUADRATIC FORMS

In this chapter we focus on the goal of symmetric bilinear forms which define quadratic forms, where every bilinear form is uniquely represented as the sum of a symmetric bilinear form and a skew-symmetric bilinear form. Let us start by the following definition:

Definition 3.1. Let E be a vector space over $\mathbb{R}$. A mapping $q : E \to \mathbb{R}$ is said to be **quadratic form** if there exists a symmetric bilinear form $f : E \times E \to \mathbb{R}$ such that $f(x, x) = q(x)$ for any $x \in E$. In this case, f is said to be the **polar form** of q. Thus, f is the polar for of q if and only if f is bilinear, symmetric and $f(x, x) = q(x)$ for any $x \in E$.

Example 3.1. Using the above definition, we can easily show that the following mappings are quadratic forms over E.

1. $E = \mathbb{R}$ and $q : E \to \mathbb{R}$, $x \mapsto x^2$.
2. $E = \mathbb{R}^2$ and $q : E \to \mathbb{R}$, $(x, y) \mapsto x^2 + y^2$.
3. $E = \mathbb{P}[x]$ and $q : E \to \mathbb{R}$, $p \mapsto = \int_a^b p^2(t)\, dt$.
4. $E = \mathbb{R}^3$ and $q : E \to \mathbb{R}$, $(x_1, x_2, x_3) \mapsto x_1^2 + x_2^2 - x_3^2$.

   The corresponding polar forms are given in Example 2.1.

Notation 3.1. Let $q : E \to \mathbb{R}$ be a quadratic form over E. We denote by $\mathcal{Q}_2(E)$ the set of all quadratic forms over E.

3.1 Relation between a quadratic form and its polar form

Let $q : E \to \mathbb{R}$ be a quadratic form and let $u, v \in E$. Then the *polar* form of q, namely f satisfies:

$$f(u, v) = \frac{1}{4}\left[q(u + v) - q(u - v)\right] = \frac{1}{2}\left[q(u + v) - q(u) - q(v)\right]. \tag{3.1}$$

In general, if we would like to prove that a mapping $q : E \to \mathbb{R}$ is a quadratic form, we first define the mapping f from E^2 to $\mathbb{R}$ by $f : E \times E \to \mathbb{R}$,

$$(u, v) \mapsto \frac{1}{2}\left[q(u+v) - q(u) - q(v)\right].$$

and by Definition 3.1 we must prove the following facts:

1. f is bilinear,
2. f is symmetric,
3. $f(x, x) = q(x)$ for any $x \in E$.

Example 3.2. Define the mapping $Q : \mathbb{P}_2[x] \to \mathbb{R}$, $p \mapsto p(0)\,p(1)$. Show that Q is a quadratic form over $\mathbb{P}_2[x]$. In deed, by (3.1) we obtain $\varphi : \mathbb{P}_2[x] \times \mathbb{P}_2[x] \to \mathbb{R}$, where

$$(p, q) \mapsto \varphi(p, q) = \frac{1}{2}p(0)\,q(1) + \frac{1}{2}q(0)\,p(1).$$

We can easily check that φ is bilinear, symmetric and $\varphi(p, p) = Q(p)$.

3.2 Quadratic forms over $\mathbb{R}^n$

First, the *analytic expression* of q is given by:

$$q = \sum_{i,j}^{n} a_{ij} \cdot x_i x_j = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \cdot x_i x_j, \tag{3.2}$$

where (a_{ij}) are real numbers. There are two cases to consider:

Case 1. $a_{ij} = a_{ji}$ for $1 \le i, j \le n$. The quadratic form q is given by the following *matrix form*:

$$\begin{aligned} q(x_1, x_2, ..., x_n) &= \sum_{i=1}^{n} a_{ii}x_i^2 + \sum_{i\neq j}^{n} a_{ij} \cdot x_i x_j \\ &= \sum_{i=1}^{n} a_{ii}x_i^2 + 2\sum_{i<j}^{n} a_{ij} \cdot x_i x_j \\ &= \begin{pmatrix} x_1 & x_2 & ... & x_n \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{12} & a_{22} & ... & a_{2n} \\ & & \ddots & \\ a_{1n} & a_{2n} & ... & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \\ &= x^t \cdot A \cdot x, \end{aligned}$$

Case 2. $a_{ij} \neq a_{ji}$ for some $1 \leq i, j \leq n$. Here, we see that

$$\begin{aligned} q &= \sum_{i=1}^{n} a_{ii} \cdot x_i^2 + \sum_{i \neq j}^{n} a_{ij} \cdot x_i x_j = \sum_{i=1}^{n} a_{ii} \cdot x_i^2 + \sum_{i<j}^{n} (a_{ij} + a_{ji}) \cdot x_i x_j \\ &= \sum_{i=1}^{n} a_{ii} \cdot x_i^2 + 2\sum_{i<j}^{n} \frac{a_{ij} + a_{ji}}{2} \cdot x_i x_j \\ &= \sum_{i=1}^{n} a_{ii} \cdot x_i^2 + 2\sum_{i<j}^{n} b_{ij} \cdot x_i x_j, \end{aligned}$$

where $b_{ij} = b_{ji}$ ($1 \leq i, j \leq n$). It follows that

$$\begin{aligned} q &= \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & \frac{a_{12}+a_{21}}{2} & \dots & \frac{a_{1n}+a_{n1}}{2} \\ \frac{a_{12}+a_{21}}{2} & a_{22} & \dots & \frac{a_{2n}+a_{n2}}{2} \\ & & \ddots & \\ \frac{a_{1n}+a_{n1}}{2} & \frac{a_{2n}+a_{n2}}{2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= x^t \cdot A \cdot x. \end{aligned}$$

In both cases, q can be written in the form $q = x^t \cdot A \cdot x$ with A *symmetric*.

Corollary 3.1. Every symmetric matrix $A \in \mathcal{M}_n(\mathbb{R})$ produces a quadratic form over $\mathbb{R}^n$.

Example 3.3. For $q = x_1^2 + 5x_1x_2 + 7x_2^2$, we have

$$q = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & \frac{5}{2} \\ \frac{5}{2} & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Here $E = \mathbb{R}^2$. But, if $E = \mathbb{R}^3$ we also have

$$q = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & \frac{5}{2} & 0 \\ \frac{5}{2} & 7 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Similarly, for $q = -x_1^2 + 5x_1x_2 + x_1x_3 + 2x_2^2 + 2x_2x_3 - x_3^2$, we also have

$$q = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} -1 & \frac{5}{2} & \frac{1}{2} \\ \frac{5}{2} & 2 & 1 \\ \frac{1}{2} & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

3.3 Quadratic forms over an arbitrary vector space

Let E be a vector space of dimension n (finite-dimensional space) and let $(e) = \{e_1, e_2, ..., e_n\}$ be a basis of E. Let $f \in \mathcal{S}_2(E)$ and let $q \in \mathcal{Q}_2(E)$ be the corresponding quadratic form. For each $u, v \in E$ we have

$$u = \sum_{i=1}^{n} x_i e_i \text{ and } v = \sum_{i=1}^{n} y_i e_i, \text{ where } x_i, y_i \in \mathbb{R} \text{ for } 1 \leq i, j \leq n.$$

Then

$$f(u, v) = f\left(\sum_{i=1}^{n} x_i e_i, \sum_{i=1}^{n} y_i e_i\right) = \sum_{i=1}^{n} x_i y_i f(e_i, e_i) + \sum_{i<j}^{n} (x_i x_j + x_j x_i) f(e_i, e_j).$$

In the matrix form (for $E = \mathbb{R}^n$), we obtain

$$\begin{aligned} f(x, y) &= \begin{pmatrix} x_1 & x_2 & ... & x_n \end{pmatrix} \begin{pmatrix} f(e_1, e_1) & f(e_1, e_2) & ... & f(e_1, e_n) \\ f(e_2, e_1) & f(e_2, e_2) & ... & f(e_2, e_n) \\ & & \ddots & \\ f(e_n, e_1) & f(e_n, e_2) & ... & f(e_n, e_n) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \\ &= x^t \cdot A \cdot y. \end{aligned}$$

By definition, the following matrix

$$\mathcal{M}_f((e)) = \begin{pmatrix} f(e_1, e_1) & f(e_1, e_2) & ... & f(e_1, e_n) \\ f(e_2, e_1) & f(e_2, e_2) & ... & f(e_2, e_n) \\ & & \ddots & \\ f(e_n, e_1) & f(e_n, e_2) & ... & f(e_n, e_n) \end{pmatrix}.$$

is called the matrix of f in the basis (e).

Example 3.4 (Homework)**.** Show that the mappings:

$$\begin{aligned} q_1 &: A \mapsto q_1(A) = tr\left(A^t A\right) \\ q_2 &: A \mapsto q_2(A) = tr\left(A^2\right) \end{aligned}$$

are quadratic forms, where $tr(M)$ denotes the *trace*[1] of M.

[1] Recall that the trace of an n by n matrix $M = (a_{ij})$ is defined by $tr(M) = a_{11} + a_{22} + ... + a_{nn}$.

3.4 Orthogonalization method

Using some properties of symmetric matrices, we prove the following theorem:

Theorem 3.1. *Every quadratic form over $\mathbb{R}^n$ is diagonalizable. That is, if $q = x^t Ax$ for some A symmetric and $x \in \mathbb{R}^n$, then $q = v^t Dv$ for some D diagonal and $v \in \mathbb{R}^n$. In other word, we have! Every quadratic form over $\mathbb{R}^n$ is of the form:*

$$q = \lambda_1 \cdot v_1^2 + \lambda_2 \cdot v_2^2 + ... + \lambda_n \cdot v_n^2, \tag{3.3}$$

where the scalars $\lambda_1, ..., \lambda_n$ and the vectors $(v_1, v_2, ..., v_n) \in \mathbb{R}^n$ satisfy $Av_i = \lambda_i v_i$. That is, by (3.3) *we get*

$$q = \begin{pmatrix} v_1 & v_2 & ... & v_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

Proof. We know that

$$q = x^t \cdot A \cdot x,$$

where A is symmetric. By Lemma 2.2, we have

$$q = x^t \cdot \left(PDP^t\right) \cdot x = x^t P \cdot D \cdot P^t x = \left(P^t x\right)^t \cdot D \cdot P^t x.$$

Now, if we put $v = P^t x$, then we obtain $q = v^t \cdot D \cdot v$. Since

$$D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

is diagonal and $v^t = \begin{pmatrix} v_1 & v_2 & ... & v_n \end{pmatrix}$, the proof of (3.3) finished. □

Example 3.5. Let $q = 2x_1^2 - 4x_1x_2 + 5x_2^2$.

1) Write q in the form $x^t Ax$, where $A \in \mathcal{M}_2(\mathbb{R})$.

2) Using the Orthogonalization method, write q in the form $\lambda_1 v_1^2 + \lambda_2 v_2^2$, where λ_1, λ_2 are the eigenvalues of A. **Solution. 1)** In fact, we have

$$q = 2x_1^2 - 4x_1x_2 + 5x_2^2 = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

2) We put $A = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$. After few computation, the eigenpairs of A are:

$$\begin{aligned} \lambda_1 &= 1 \to u_1 = (2,1), \\ \lambda_2 &= 6 \to u_2 = (1,-2). \end{aligned}$$

We see that $\|u_1\|_2 = \|u_2\|_2 = \sqrt{5}$. Setting

$$P = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}.$$

Clearly, P is orthogonal ($PP^t = I_2$). Moreover, we have

$$PDP^t = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} = A.$$

It follows that

$$q = x^t A x = x^t \left(PDP^t\right) x = x^t P \cdot D \cdot P^t x = \left(P^t x\right)^t \cdot D \cdot P^t x.$$

Now, we put $v = P^t x$. That is,

$$v = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{5}\sqrt{5}x_1 + \frac{1}{5}\sqrt{5}x_2 \\ \frac{1}{5}\sqrt{5}x_1 - \frac{2}{5}\sqrt{5}x_2 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} q &= v^t \cdot D \cdot v = \lambda_1 v_1^2 + \lambda_2 v_2^2 \\ &= \left(\frac{2}{5}\sqrt{5}x_1 + \frac{1}{5}\sqrt{5}x_2\right)^2 + 6\left(\frac{1}{5}\sqrt{5}x_1 - \frac{2}{5}\sqrt{5}x_2\right)^2 \\ &= 1 \cdot \frac{(2x_1 + x_2)^2}{5} + 6 \cdot \frac{(x_1 - 2x_2)^2}{5}. \end{aligned}$$

Thus, we have written q as in (3.3).

Example 3.6 (Homework 1)**.** Let $q = 2x_1x_2$.

1. Write q in the form $x^t A x$, where $A \in \mathcal{M}_2(\mathbb{R})$.

2. Using the Orthogonalization method, write q in the form $\lambda_1 v_1^2 + \lambda_2 v_2^2$, where λ_1, λ_2 are the eigenvalues of A.

Example 3.7 (Homework 2). Define the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Write A as $A = P \cdot D \cdot P^t$, where P is orthogonal.

3.5 Definitions and results

Let $f \in S_2(E)$ (that is, f is a bilinear symmetric form over E).

Definition 3.2. A bilinear form f is called nondegenerate[2] if it satisfies the condition: $f(x, y) = 0$ for all $x \in E$ implies that $y = 0$.

Thus, f is called nondegenerate if $\ker f = \{0_E\}$. In the case when $\ker f \neq \{0_E\}$, f is said to be **degenerate.**

Definition 3.3. A vector $v \in E$ is said to be isotropic if $f(v, v) = 0$. A subset $A \subset E$ is called isotropic if $f(v, v) = 0$ for any $v \in E$.

We denote by C the set of all *isotropic vectors*. That is ,

$$C = \{v \in E : \ f(v, v) = 0\}. \tag{3.4}$$

The set C is called the *isotropic cone* of E. Note that if $v \in C$, then $\alpha \cdot v \in C$. In fact, for every $v \in C$ and $\alpha \in \mathbb{R}$ we have

$$q(\alpha v) = f(\alpha v, \alpha v) = \alpha^2 f(v, v) = 0. \tag{3.5}$$

Proposition 3.1. *If f is skew-symmetric, then every vector is isotropic.*

Proof. If f is skew-symmetric, i.e., $f(x, y) = -f(y, x)$ for any $x, y \in E$, then $f(x, x) = -f(x, x)$ for any $x \in E$, so $f(x, x) = 0$ for any $x \in E$. □

Proposition 3.2. *Let $f \in \mathcal{L}_2(E)$. Then f is alternating if and only if $C = E$.*

3.5.1 Is the isotropic cone a vector space?

In general, the isotropic cone of E is not a vector subspace of E. Thus, we have the following theorem.

[2] In some references we find the word "*nonsingular*" instead of nondegenerate.

Theorem 3.2. *Let f be a nonzero bilinear symmetric form defined on a vector space E. Then*

$$C = \ker f \overset{iff}{\Leftrightarrow} C \text{ is a vector subspace of } E.$$

For the proof we need to the following lemma.

Lemma 3.1. *Let q be a quadratic form defined over E and let f be its polar form, where E is vector space over $\mathbb{R}$. Assume that C is a vector subspace of E and there exists an element $x_0 \in C/\ker f$. Then*

$$\forall\, y \in E; \text{ if } f(x_0, y) \neq 0, \text{ then } y \in C,$$

where C denotes the isotropic cone.

Proof. Let $y \in E$ such that $f(x_0, y) \neq 0$. We have

$$\begin{aligned} \forall\, \lambda &\in \mathbb{R} : q(\lambda x_0 + y) = f(\lambda x_0 + y, \lambda x_0 + y) \\ &= 2\lambda f(x_0, y) + q(y) \quad \text{(since } q(x_0) = 0\text{).} \end{aligned}$$

If we let $\lambda_0 = \dfrac{-q(y)}{2f(x_0, y)} \in \mathbb{R}$, then clearly $q(\lambda_0 x_0 + y) = 0$, from which we deduce that $\lambda_0 x_0 + y \in C$. But, C is given as a subspace containing x_0. Thus, we deduce that y belongs to C, as claimed. □

Proof of Theorem 3.2. ($\Rightarrow$) Let q be the quadratic form of f. For every $(x, y) \in C^2$ and $\lambda \in \mathbb{R}$, we get that

- $q(x + y) = q(x) + q(y) + 2f(x, y) = 0$. Implies $x + y \in C$.
- $q(\lambda x) = \lambda^2 q(x) = 0$; i.e., $\lambda x \in C$.

Thus, C is a subspace of E.

($\Leftarrow$) Suppose that C is a subspace of E. Note that the inclusion $\ker f \subset C$ is always true; since

$$f(x, y) = 0,\ \forall\, y \in E \ \Rightarrow f(x, x) = 0 \ \text{ (by taking the case } y = x\text{).}$$

We would like to prove that if C is a subspace of E, then $C \subset \ker f$. Assume by the way of contradiction that $C \nsubseteq \ker f$. There exists a nonzero vector x_0 with $x_0 \in C/\ker f$. Define

$$H = \{y \in E \,;\ f(x_0, y) = 0\}.$$

It suffices to verify that $E \subset C$. In fact, let $z \in E$ and $y \notin H$. We have

$$z = y + z - y.$$

From Lemma 3.1, we have $y \in C$. We distinguish two cases:

Case 1. If z is in H, then $y + z \notin H$, since

$$f(x_0, y+z) = \underbrace{f(x_0, y)}_{\neq 0} + \underbrace{f(x_0, z)}_{=0} \neq 0$$

Likewise, from Lemma 3.1, we have $y + z \in C$. In this case, we have

$$z = \underbrace{y+z}_{\in C} - \underbrace{y}_{\in C} \in C \text{ (since } C \text{ is a subspace of } E\text{)}.$$

Case 2. If $z \notin H$, by Lemma 3.1 once again, we have $z \in C$. Thus, $E = C$, and so $f = 0$; since $q = 0$. But this is a contradiction with our hypothesis that f is a nonzero bilinear form. Our proof of Theorem 3.2 is finished. □

Definition 3.4. Two vectors x and y are said to be orthogonal by f if $f(x, y) = 0$. We denote by $x \perp y$.

We deduce from the above definition that C consists all vectors x such that $x \perp x$. Also, $\ker f$ consists vectors which are orthogonal with all the vectors of E.

Definition 3.5 (Orthogonal set)**.** Let $A \subset E$. The orthogonal[3] of A with respect to f is usually denoted by $A^{\perp}$ and defined by

$$A^{\perp} = \{x \in E, f(x, y) = 0 \text{ for every } y \in A\}. \tag{3.6}$$

Example 3.8 (Homework)**.** When does $A \subset A^{\perp}$?

Remark 3.1. In the case when a nondegenerate bilinear form on E is not symmetric, there are two different orthogonals of A :

1. $A^{\perp,R} = \{x \in E, f(x, y) = 0 \text{ for every } y \in A\}$.
2. $A^{\perp,L} = \{x \in E, f(y, x) = 0 \text{ for every } y \in A\}$.

Here, we can prove that

$$\left(A^{\perp,L}\right)^{\perp,R} = \left(A^{\perp,R}\right)^{\perp,L} = A.$$

Definition 3.6 (Kernel of a bilinear symmetric form)**.** Let $f \in S_2(E)$. The kernel of f is defined by

$$\ker f = \{x \in E, \ f(x, y) = 0 \text{ for every } y \in E\}. \tag{3.7}$$

[3]In some references we say "*perp space to* A" instead of the orthogonal of A.

From Definition 3.5, we deduce that $\ker f = E^{\perp}$. Note that $x_0 \in \ker f$ iff $f(x,y)=0$ for every $y \in E$. Similarly, $x_0 \notin \ker f$ if and only if there exists $\overline{y} \in E$ such that $f(x_0, \overline{y}) \neq 0$, or equivalently $f(x_0, \overline{y}) \neq 0$ for some $\overline{y} \in E$.

Theorem 3.3. *Let $f \in \mathcal{S}_2(E)$ and let $A, B \subset E$. Then*

1. $\left(A^{\perp}\right)^{\perp} \supset A$.
2. $(A \cap B)^{\perp} \supset A^{\perp} + B^{\perp}$.
3. $(A \cup B)^{\perp} \supset A^{\perp} \cap B^{\perp}$

Proof. **1.** Let $v \in A$. For any $u \in A^{\perp}$, we have $f(u,x)=0$ for any $x \in A$. In particular, for $x = v$ we have $f(u,v)=0$. This means that $\left(A^{\perp}\right)^{\perp}$ contains v. As required.

2. Let $v = a + b \in A^{\perp} + B^{\perp}$, where $A^{\perp}$ contains a and $B^{\perp}$ contains b. We will prove that $(A \cap B)^{\perp}$ contains v. For every $x \in A \cap B$ we have $f(a,x) = f(b,x) = 0$ and so $f(a+b,x)=0$ since $f \in S_2(E)$. Thus, $f(v,x)=0$.

3. Let $v \in A^{\perp} \cap B^{\perp}$. For every $x \in A \cup B$ we have

- If $x \in A$, then $f(v,x)=0$ since $v \in A^{\perp}$.
- If $x \in B$, then $f(v,x)=0$ since $v \in B^{\perp}$.

In both cases we have $f(v,x)=0$ for any $x \in A \cup B$. Thus, $v \in (A \cup B)^{\perp}$, as asked. □

Proposition 3.3. *Let f be a bilinear form over E. Two subsets A and B of E are called orthogonal with respect to f if $f(x,y)=0$ for any x in A and y in B. The following conditions are equivalent:*

1. *A and B are orthogonal,*
2. $A \subset B^{\perp}$,
3. $B \subset A^{\perp}$.

Proof. We prove $(a) \Rightarrow (b)$. Let $a_0 \in A$. For each vector $v \in B$, $f(a_0, v) = 0$ since A and B are orthogonal. Hence, $a_0 \in B^{\perp}$. Next, $(b) \Rightarrow (c)$. Let $b_0 \in B$. For each vector $v \in A$, we have $v \in B^{\perp}$, and so $f(b_0, v) = 0$ since $b_0 \in B$. Hence, $b_0 \in A^{\perp}$. Finally, $(c) \Rightarrow (a)$. Let $u \in A$ and $v \in B$. Since $v \in A^{\perp}$, then $f(u,v)=0$. □

Definition 3.7. Let E be a v. space on $\mathbb{R}$ and let $\{e_1, e_2, ..., e_n\}$ be a family of n vectors of E. We have

- $\{e_1, e_2, ..., e_n\}$ is **orthogonal** by f if $f(e_i, e_j) = 0$ for $i \neq j$.

- $\{e_1, e_2, ..., e_n\}$ is **orthonormal** by f if $f(e_i, e_j) = 0$ for $i \neq j$ and $f(e_i, e_i) = 1$ for $i = 1, 2, ..., n$.

Definition 3.8. Let $q \in \mathcal{Q}_2(E)$ and f its polar form. Then

- q or f is said to be **positive** if $q(x) \geq 0$ for every $x \in E$.
- q or f is said to be **definite positive** if $q(x) > 0$ for every $x \in E - \{0_E\}$.

Example 3.9. $q = x_1^2 - 4x_1x_2 + 4x_2^2$ is positive. In fact, we see that

$$q = (x_1 - 2x_2)^2 \geq 0 \text{ for every } (x_1, x_2) \in \mathbb{R}^2.$$

But, $q = x_1^2 - 2x_1x_2 + 2x_2^2$ is definite positive. In fact, we have

$$q = (x_1 - x_2)^2 + x_2^2 > 0 \text{ for every } (x_1, x_2) \in \mathbb{R}^2 - \{(0, 0)\}.$$

Theorem 3.4. *Let $f \in \mathcal{S}_2(E)$. If f is definite positive, then f is nondegenerate.*

Proof. Let $x \in \ker f$. Then by (3.7), $f(x, y) = 0$ for every $y \in E$. In the case when $y = x$, we get $f(x, x) = 0$. But, since f is definite positive, $f(x, x) = 0$ implies $x = 0$. □

Theorem 3.5 (Cauch-Schwarz inequality). *Let $q \in \mathcal{Q}_2(E)$ and $f \in \mathcal{S}_2(E)$. If q is positive, then*

$$(f(x, y))^2 \leq q(x)\, q(y) \text{ for every } x, y \in E.$$

3.5.2 When is a quadratic form surjective?

Let $q \in \mathcal{Q}_2(E)$. Here we ask if any real number is represented by this quadratic form. We present the following result:

Theorem 3.6. *Let q be a quadratic form on a real vector space E. The following three properties are statements:*

1. *q is surjective.*
2. *q is neither positive nor negative.*
3. *There exists an isotropic vector which is not in the kernel.*

Proof. $(1) \overset{?}{\Rightarrow} (2)$. Since q is surjective, then there exists $x_0 \in E$ (resp. $x_1 \in E$) such that

$$\begin{cases} q(x_0) > 0, \\ q(x_1) < 0. \end{cases}$$

Then q is not negative (resp. positive).

(2) $\overset{?}{\Rightarrow}$ (3). Let x_0 and x_1 in E such that $q(x_0) > 0$ and $q(x_1) < 0$. Consider then a vector of the form $\lambda x_0 + x_1$, where $\lambda \in \mathbb{R}$. Let f be the polar form of q, we have

$$\begin{aligned} q(\lambda x_0 + x_1) &= f(\lambda x_0 + x_1, \lambda x_0 + x_1) \\ &= \lambda^2 q(x_0) + 2\lambda f(x_0, x_1) + q(x_1) = p(\lambda). \end{aligned}$$

Assume that $p(\lambda) = 0$. By computation, we find

$$\Delta = f^2(x_0, x_1) - q(x_0) q(x_1) > 0.$$

Then the equation $p(\lambda) = 0$ has two roots. Let λ_0 be one of them. Then the vector $y_0 = \lambda_0 x_0 + x_1$ is by construction, isotropic. We prove by the way of contradiction that y_0 is not in the kernel of f, that is, assume that $y_0 \in \ker f$. Hence, $f(y_0, x) = 0$ for each $x \in E$. In particular, for $x = x_0$ and for $x = x_1$ we have

$$\begin{cases} 0 = f(y_0, x_0) = \lambda_0 q(x_0) + f(x_0, x_1), \\ 0 = f(y_0, x_1) = \lambda_0 f(x_0, x_1) + q(x_1). \end{cases}$$

That is,

$$\begin{cases} \lambda_0^2 q(x_0) + \lambda_0 f(x_0, x_1) = 0, \\ \lambda_0 f(x_0, x_1) + q(x_1) = 0. \end{cases}$$

We deduce that $\lambda_0^2 q(x_0) = q(x_1) < 0$. A contradiction.

(3) $\overset{?}{\Rightarrow}$ (1). Let y_0 be an isotropic vector which is not in the kernel of f. There exists $y_1 \in E$ such that $f(y_0, y_1) \neq 0$. Then for each $\gamma \in \mathbb{R}$, we put

$$\lambda = \frac{\gamma - q(y_1)}{2f(y_0, y_1)} \in \mathbb{R},$$

from which it follows that

$$\begin{aligned} q(\lambda y_0 + y_1) &= q\left(\frac{\gamma - q(y_1)}{2f(y_0, y_1)} y_0 + y_1\right) \\ &= f\left(\frac{\gamma - q(y_1)}{2f(y_0, y_1)} y_0 + y_1, \frac{\gamma - q(y_1)}{2f(y_0, y_1)} y_0 + y_1\right) \\ &= \gamma. \end{aligned}$$

Thus, q is surjective. The proof of Theorem 3.6 is finished. $\square$

3.6 Gauss Decomposition (Silvester's Theorem)

First, we need to the following definition:

Definition 3.9 (signature of a quadratic form)**.** Assume that

$$q = +f_1^2 + f_2^2 + ... + f_r^2 - f_{r+1}^2 - .... - f_{r+s}^2, \tag{3.8}$$

where $f_1, f_2, ..., f_{r+s}$ are linearly independent forms over $\mathbb{R}^n$. The couple (r, s) is called the signature of q.

Recall that if $f_i(x_1, x_2, ..., x_n) = a_1^{(i)}x_1 + a_2^{(i)}x_2 + ... + a_n^{(i)}x_n$ with $a_j^{(i)} \in \mathbb{R}$ for $1 \leq i \leq r+s$ and $1 \leq j \leq n$, then $f_1, f_2, ..., f_{r+s}$ are linearly independent if and only if

$$\begin{vmatrix} a_1^{(1)} & a_1^{(2)} & \dots & a_1^{(r+s)} \\ a_2^{(1)} & a_1^{(2)} & \dots & a_1^{(r+s)} \\ \vdots & \vdots & \dots & \vdots \\ a_n^{(1)} & a_1^{(2)} & \dots & a_1^{(r+s)} \end{vmatrix} \neq 0.$$

In the rest of this section we show how to write any quadratic form over $\mathbb{R}^n$ as in (3.8). To make this, we use the well-known identity:

$$(a + b + c + ...)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + ... + 2bc + 2bd + ... \tag{3.9}$$

Let $q(x_1, x_2, ..., x_n) = \sum_{i,j}^n a_{ij} \cdot x_i x_j$ be a quadratic form over $\mathbb{R}^n$, where $a_{ij} = a_{ji}$ for $1 \leq i, j \leq n$. We distinguish two cases:

Case 1. When $a_{11} \neq 0$, we put

$$\begin{cases} x_1 = y_1 - \dfrac{1}{a_{11}}(a_{12}y_2 + ... + a_{1n}y_n) \\ x_2 = y_2 \\ \vdots \\ x_n = y_n \end{cases} \tag{3.10}$$

It follows that

$$q(x_1, x_2, ..., x_n) = a_{11}y_1^2 + q'(y_2, y_3, ..., y_n),$$

where q' is also a quadratic form; but over $\mathbb{R}^{n-1}$. Then we repeat the same argument with q'.

Case 2. When $a_{11} = 0$, but $a_{12} \neq 0$. Here, we put

$$\begin{cases} x_1 = y_1 + y_2 \\ x_2 = y_1 - y_2 \\ x_3 = y_3 \\ \vdots \\ x_n = y_n. \end{cases} \tag{3.11}$$

It follows that

$$q(x_1, x_2, ..., x_n) = \sum_{i,j}^{n} b_{ij} \cdot y_i y_j,$$

where $b_{11} \neq 0$. This is the first case (we have transformed q so that we can apply the first case). By this method we can write q in the following form:

$$q = \pm f_1^2 \pm f_2^2 \pm ... \pm f_m^2,$$

where $m \leq n$ and $f_1, f_2, ..., f_m$ are linearly independent forms over $\mathbb{R}^n$.

Example 3.10. Using Gauss' Method, diagonalize the following two quadratic forms and deduce their signatures :

- $q_1 = x_1^2 + x_2^2 + 2x_3^2 - 4x_1x_2 + 6x_2x_3$
- $q_2 = 2x_1x_2 + 2x_2x_3 + 2x_1x_3.$

Solution: For q_1, since $a_{11} = 1$ it follows from (3.10) that

$$\begin{cases} x_1 = y_1 + 2y_2 \\ x_2 = y_2 \\ x_3 = y_3. \end{cases}$$

This implies

$$\begin{aligned} q_1 &= x_1^2 + x_2^2 + 2x_3^2 - 4x_1x_2 + 6x_2x_3 \\ &= (y_1 + 2y_2)^2 + y_2^2 + 2y_3^2 - 4(y_1 + 2y_2) y_2 + 6y_2y_3 \\ &= y_1^2 - 3y_2^2 + 6y_2y_3 + 2y_3^2 \\ &= y_1^2 + q_1'(y_2, y_3). \end{aligned}$$

Likewise by (3.10), let us take

$$\begin{cases} y_2 = z_2 + z_3 \\ y_3 = z_3. \end{cases}$$

It follows that

$$\begin{aligned} q_1' &= -3y_2^2 + 6y_2y_3 + 2y_3^2 \\ &= -3\left(z_2+z_3\right)^2 + 6\left(z_2+z_3\right)z_3 + 2z_3^2 \\ &= -3z_2^2 + 5z_3^2. \end{aligned}$$

Finally, we obtain

$$q_1 = \left(x_1 - 2x_2\right)^2 + 5x_3^2 - 3\left(x_2 - x_3\right)^2 = f_1^2 + f_2^2 - f_3^2,$$

where f_1, f_2 and f_3 are linearly independent forms over $\mathbb{R}^3$ since

$$\begin{vmatrix} 1 & -2 & 0 \\ 0 & 0 & 5 \\ 0 & 1 & -1 \end{vmatrix} = -5 \neq 0.$$

Thus, the signature of q_1 is $(2,1)$. For the quadratic form q_2, by (3.11) we put

$$\begin{cases} x_1 = y_1 + y_2 \\ x_2 = y_1 - y_2 \\ x_3 = y_3. \end{cases}$$

We obtain

$$\begin{aligned} q_2 &= 2\left(y_1+y_2\right)\left(y_1-y_2\right) + 2\left(y_1-y_2\right)y_3 + 2x_1y_3 \\ &= 2y_1^2 + 4y_3y_1 - 2y_2^2 \\ &= q_2'. \end{aligned}$$

Setting once again

$$\begin{cases} y_1 = z_1 - z_3 \\ y_2 = z_2 \\ y_3 = z_3. \end{cases}$$

It follows that

$$\begin{aligned} q_2' &= 2y_1^2 + 4y_3y_1 - 2y_2^2 \\ &= 2\left(z_1-z_3\right)^2 + 4z_3\left(z_1-z_3\right) - 2z_2^2 \\ &= 2z_1^2 - 2z_2^2 - 2z_3^2. \end{aligned}$$

Hence,

$$\begin{aligned} q_2 &= 2\left(y_1+y_3\right)^2-2y_2^2-2y_3^2 \\ &= 2\left(\frac{x_1+x_2}{2}+x_3\right)^2-2\left(\frac{x_1-x_2}{2}\right)^2-2x_3^2 \\ &= f_1^2-f_2^2-f_3^2, \end{aligned}$$

where f_1, f_2 and f_3 are linearly independent forms over $\mathbb{R}^3$; since

$$\begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{vmatrix} = 2 \neq 0$$

The signature of q_2 is $(1,2)$ and the rank is 3.

Example 3.11. Consider the quadratic form $q = x_1x_3 + x_2x_3$, where $E = \mathbb{R}^3$. Find the signature of q.

Solution: We put

$$\begin{cases} x_1 = y_1 + y_2, \\ x_2 = y_1 - y_2, \\ x_3 = y_3. \end{cases}$$

We obtain

$$q = \left(y_1+y_2\right)y_3 + \left(y_1-y_2\right)y_3 = 2y_1y_3$$

We put once again

$$\begin{cases} y_1 = z_1 + z_2 \\ y_3 = z_1 - z_2 \end{cases}$$

Then

$$\begin{aligned} q &= 2\left(z_1+z_2\right)\left(z_1-z_2\right) = 2z_1^2-2z_2^2 \text{ (the signature is } (1,1)\,) \\ &= 2\left(\frac{y_1+y_3}{2}\right)^2-2\left(\frac{y_1-y_3}{2}\right)^2 \\ &= 2\left[\frac{\frac{x_1+x_2}{2}+x_3}{2}\right]^2-2\left[\frac{\frac{x_1+x_2}{2}-x_3}{2}\right]^2. \end{aligned}$$

Remark 3.2. The *inner product* is a bilinear form, symmetric and definite positive. For each

$(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, we have

$$\langle x, y \rangle = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = x^t y.$$

Corollary 3.2. Let $A \in \mathcal{M}_n(\mathbb{R})$. Then there exists a symmetric matrix $B \in \mathcal{S}_n(\mathbb{R})$ such that

$$x^t Ax = x^t Bx \text{ for every } x \in \mathbb{R}^n.$$

Proof. Since $x^t Ax = a \in \mathbb{R}$, for every $x \in \mathbb{R}^n$ we have

$$x^t Ax = \left(x^t Ax\right)^t = x^t A^t x.$$

It follows that

$$x^t Ax = \frac{1}{2} x^t Ax + \frac{1}{2} x^t A^t x = x^t \left(\frac{A + A^t}{2}\right) x.$$

Note that the matrix $B = \dfrac{A + A^t}{2}$ is always symmetric. □

Proposition 3.4. *Let $A \in \mathcal{M}_n(\mathbb{R})$ be a symmetric and let $(\alpha, x), (\beta, y)$ be two eigenpairs of A with $\alpha \neq \beta$. Then x and y are orthogonal, i.e., $x \perp y$. Or, equivalently, $\langle x, y \rangle = 0$.*

Proof. Indeed, we have

$$\alpha \langle x, y \rangle = \langle \alpha x, y \rangle = \langle Ax, y \rangle = \left\langle x, A^t y \right\rangle = \langle x, Ay \rangle = \langle x, \beta y \rangle = \beta \langle x, y \rangle ,$$

and since $\alpha \neq \beta$, it follows that $\langle x, y \rangle = 0$. □

Example 3.12 (Homework).

1. Consider the equation

$$ax^2 + 2hxy + by^2 = 0. \tag{3.12}$$

   Write (3.12) in the form $v^t Av = 0$, where $A \in \mathcal{M}_2(\mathbb{R})$ and $v = \begin{pmatrix} x & y \end{pmatrix}^t$.

2. Write the equation $\lambda_1 x_1^2 + \lambda_2 x_2^2 = 0$ in the matrix form.

3. Let $A \in \mathcal{M}_n(\mathbb{R})$. We ask if $v^t Av = 0 \; \forall \, v \in \mathbb{R}^n$, implies $A = 0$?

   Ans. No, take the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Definition 3.10. Let E be a real vector space equipped with an inner product $\langle .,. \rangle$. The couple $(E, \langle .,. \rangle)$ is said to be a real **pre-Hilbert space**. A real pre-Hilbert space of **finite dimension** is said to be **Euclidian space**.

Let $(E, \langle .,. \rangle)$ be a pre-Hilbert space. The *related norm* is defined by

$$\forall\, x \in E : \|x\| = \sqrt{\langle x, x \rangle}. \tag{3.13}$$

Note that (3.13) a well-known identity which is used in (2.3).

3.7 Proposed problems (quadratic forms)

Exercise 1. Determine in the canonical basis of $\mathbb{R}^3$ the matrix of the symmetric bilinear form f such that for $v_1 = (1, 2, 1)$, $v_2 = (-1, 2, 0)$, $v_3 = (1, 0, 1)$, one has $f(v_1, v_2) = 0$, $f(v_2, v_3) = 4$, $f(v_1, v_3) = -1$, $f(v_1, v_1) = 5$, $f(v_2, v_2) = 1$, $f(v_3, v_3) = 0$. Find the quadratic form associated with f.

Exercise 2. Let f be a symmetric bilinear form on E and q the quadratic form associated with f. Show that for all x, y in E, one has

$$f(x, y) = \frac{1}{4}\left(q(x+y) - q(x-y)\right). \tag{3.14}$$

Consider the mapping $q : E \to \mathbb{K}$ such that for all $x \in E$, and $\lambda \in \mathbb{K}$ we have $q(\lambda x) = \lambda^2 q(x)$. The map $f : E \times E \to \mathbb{K}$ given by (3.14) is bilinear. Show that q is the quadratic form associated with f.

Exercise 3. In the vector space $\mathbb{R}^2$ define the quadratic form:

$$q(x) = 33x_1^2 - 28x_1x_2 + 6x_2^2,$$

where $\begin{pmatrix} x_1 & x_2 \end{pmatrix}^t$ are the coordinates of x in the canonical basis $\{e_1, e_2\}$ of $\mathbb{R}^2$. Determine the expression of q when we take as basis $\{e_1', e_2'\} = \{e_1 + 2e_2, 2e_1 + 5e_2\}$. Write the polar form of q in both bases.

Exercise 4. Let f be a symmetric bilinear form on E and let A and B be two parts of E. Prove that $A \subset \left(A^\perp\right)^\perp$ and if $A \subset B$, then $B^\perp \subset A^\perp$.

In the vector space $\mathbb{R}^3$ related to its canonical base the symmetrical bilinear form defined by

$$f(x, y) = x_1y_1 + x_2y_2,$$

where $\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^t$ and $\begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix}^t$ are the coordinates of x and y. Find $e_1^\perp$ and $\left(e_1^\perp\right)^\perp$.

What can we deduce from this?

Exercise 5. Let f be a symmetric bilinear form on E and let F and G be two subspaces of E. Show that

$$(F+G)^{\perp} = (F \cup G)^{\perp} = F^{\perp} \cap G^{\perp} \text{ and } F^{\perp} + G^{\perp} \subset (F \cap G)^{\perp}.$$

In the vector space $\mathbb{R}^2$ related to its canonical base the symmetric bilinear form defined by

$$f(x,y) = x_1 y_1,$$

where $\begin{pmatrix} x_1 & x_2 \end{pmatrix}^t$ and $\begin{pmatrix} y_1 & y_2 \end{pmatrix}^t$ are the coordinates of x and y. Calculate $(Vect\{e_1\})^{\perp}$, $(Vect\{e_1+e_2\})^{\perp}$, $(Vect\{e_1\})^{\perp} + (Vect\{e_1+e_2\})^{\perp}$and $(Vect\{e_1\} \cap Vect\{e_1+e_2\})^{\perp}$. What can we deduce from this?

Exercise 6. In the vector space $\mathbb{R}^3$related to its canonical base the quadratic form defined by

$$q(x) = x_1^2 + x_2^2 + x_3^2 - 4(x_1x_2 + x_1x_3 + x_2x_3),$$

where $\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^t$ are the coordinates of x. Without using the Gauss method, find a basis of $\mathbb{R}^3$ which is orthogonal by f, where f is the polar form of q.

Exercise 7. In the vector space $E = \mathbb{R}^3$ define to its canonical basis the quadratic form

$$q(x) = x_1^2 + x_2^2 + x_3^2 - (x_1x_2 + x_1x_3 + x_2x_3),$$

where $\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^t$ are the coordinates of x,and let f be the polar form of q.

1. Decompose q into sum of squares using the Gaussian method.
2. Find a base of E which is orthogonal to f.
3. Find the matrix A and B of f respectively in the canonical and orthogonal basis of E.
4. Verify by calculation that $P^tAP = B$, where P is the passage matrix from the canonical basis to the orthogonal basis.

Exercise 8. Let $A \in \mathcal{M}_n(\mathbb{C})$. Show that if A is symmetric, then there exists $B \in \mathcal{M}_n(\mathbb{C})$ such that $A = B^t \cdot B$.

Exercise 9. Let $(e) = \{e_1, e_2, e_3\}$ a basis of a real vector space E of dimension 3 and let

$$q(x) = x_1^2 + 4x_2^2 + x_3^2 + 4x_1x_2 - 2x_1x_3 - 12x_2x_3,$$

be a quadratic form over E, where $x = x_1e_1 + x_2e_2 + x_3e$ is any vector of E. Reduce $q(x)$ in sum of squares using the Gaussian method and deduce its rank and signature.

Exercise 10. Let $(e) = \{e_1, e_2, e_3\}$ a basis of a real vector space E of dimension 3. Define

$$q(x) = x_1^2 + 4x_2^2 + x_3^2 + 4x_1x_2 - 2x_1x_3 - 12x_2x_3,$$

which is a quadratic form on E, where $x = x_1e_1 + x_2e_2 + x_3e$ is any vector of E. Construct, without using the Gauss method, a basis (e') of E which is orthogonal with respect to q.

Exercise 11. Let $(e) = \{e_1, e_2, e_3\}$ a basis of a real vector space E of dimension 3. Define the quadratic form on E,

$$q(x) = x_1^2 + ax_2^2 + 5x_3^2 + 2x_1x_2 - 6x_1x_3 + 2x_2x_3,$$

where $a \in \mathbb{R}$ and $x = x_1e_1 + x_2e_2 + x_3e_3$ is any vector of E.

1. Give the polar form f of q as well as the matrix A associated with q relative to the base (e).
2. Reduce q to sum of squares using the Gaussian method.
3. Construct, without using the Gauss method, a basis (e') of E which is orthogonal for f.
4. Give the matrix B associated with f in the basis (e').
5. Deduce the rank and signature of q.

Exercise 12. Let $(e) = \{e_1, e_2, e_3\}$ be a basis of a real vector space E of dimension 3. Define

$$q(x) = 4x_1^2 + 25x_2^2 + ax_3^2 - 12x_1x_2 + 4x_1x_3 + 2x_2x_3$$

a quadratic form on E, where a is a real number and $x = x_1e_1 + x_2e_2 + x_3e_3$ is any vector of E, and let f be the polar form of q.

1. Give the matrix A associated with f relative to the base (e).
2. Reduce q to sum of squares using the Gaussian method.
3. Deduce the rank and signature of q.
4. Study if f is degenerate, positive, definite.
5. Construct, without using the Gauss method, a basis (e') of E which is orthogonal by f.

6. Give the quadratic form associated with f in the basis (e').

Exercise 13. Let q be a real quadratic form of signature (s,t). Show that

1. q is non-degenerate if and only if $s+t=n$,
2. q is positive if and only if $t=0$,
3. q is negative if and only if $s=0$,
4. q is definite positive if and only if $s=n$,
5. q is definite negative if and only if $t=n$.

Exercise 14. Show that $q_1 : A \longmapsto tr\left(A^t A\right)$ and $q_2 : A \longmapsto tr\left(A^2\right)$ are quadratic forms.
Exercise 15. Find the signature of the quadratic form related to the polar form

$$\begin{array}{rcl} f & : & \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \\ (x,y) & \mapsto & (x_1 + x_2 + \ldots + x_n)(y_1 + y_2 + \ldots + y_n) - (x_1 y_1 + x_2 y_2 + \ldots + x_n y_n) . \end{array}$$

CHAPTER 4

INTRODUCTION TO HERMITIAN SPACE

Throughout this chapter, the field used here is the field of complex numbers and E is a **vector space** over $\mathbb{C}$. For example, $E = \mathbb{C}^n$ with $n \geq 2$, $\mathbb{C}_n[x]$, $\mathcal{M}_n(\mathbb{C})$, and so on. The basic goal of this chapter is to define quadratic forms over a complex pre-Hilbert space of finite dimension, namely, *hermitian space*.

4.1 Sesquilinear forms and hermitian quadratic forms

In this section, we deal with a sesquilinear form defined over a complex vector space E, which is a mapping from $E \times E$ to $\mathbb{C}$, linear according to one of the variables and semi-linear with respect to the other variable.

4.1.1 Definitions and examples

Definition 4.1. Let E be a vector space on $\mathbb{C}$. A **semi-linear form** is a mapping f from E to $\mathbb{C}$ such that for every $(u, v) \in E^2$ and $\alpha \in \mathbb{C}$, one has

1. $f(u+v) = f(u) + f(v)$,
2. $f(\alpha v) = \overline{\alpha} f(v)$.

Example 4.1. The mapping

$$\begin{array}{rcl} f & : & \mathbb{C} \to \mathbb{C} \\ z & \mapsto & f(z) = \overline{z} \end{array}$$

is a semi-linear form over $\mathbb{C}$. In fact, we see that

- For every $z_1, z_2 \in \mathbb{C}$,

$$f(z_1 + z_2) = \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} = f(z_1) + f(z_2).$$

- For every $z \in \mathbb{C}$ and $\alpha \in \mathbb{K} = \mathbb{C}$ we have

$$f(\alpha v) = \overline{\alpha z} = \overline{\alpha} \cdot \overline{z} = \overline{\alpha} f(z).$$

Definition 4.2. A **sesquilinear** form is a mapping f from E^2 to $\mathbb{C}$ such that f is **linear** from the left and **semi-linear** from the right. Thai is, for every $(x, x', y, y') \in E^4$ and $\lambda \in \mathbb{C}$, one has

1. $f(\lambda x + x', y) = \lambda f(x, y) + f(x', y)$,
2. $f(x, \lambda y + y') = \overline{\lambda} f(x, y) + f(x, y')$

Example 4.2. Let $f : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, $(z_1, z_2) \mapsto z_1 \cdot \overline{z_2}$ and we prove that f is a sesquilinear form. In fact, for every $(z_1, z_2, z_1', z_2') \in \mathbb{C}^4$ and $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} f(\lambda z_1 + z_2, z_1') &= (\lambda z_1 + z_2) \cdot \overline{z_1'} = \lambda z_1 \overline{z_1'} + z_2 \overline{z_1'} \\ &= \lambda f(z_1, z_1') + f(z_2, z_1'). \end{aligned}$$

That is, f is linear from the left.

$$\begin{aligned} f(z_1, \lambda z_1' + z_2') &= z_1 \cdot \overline{(\lambda z_1' + z)_2'} = z_1 \cdot \left(\overline{\lambda z_1'} + \overline{z_2'}\right) \\ &= \overline{\lambda} \cdot z_1 z_1' + z_1 \overline{z_2'} = \overline{\lambda} f(z_1, z_1') + f(z_1, z_2'). \end{aligned}$$

That is, f is semi-linear from the right.

As we have done above, we deduce:

Theorem 4.1. *Let B and B' be two bases of E. Let P be the passage matrix from B to B' and let $f : E \times E \to \mathbb{R}$ be a sesquilinear form over E. If $A = \mathcal{M}_f(B)$ and $A' = \mathcal{M}_f(B')$, then $A' = P^t \cdot A \cdot \overline{P}$.*

Definition 4.3. A hermitian sesquilinear form is a sesquilinear form f over E satisfying

$$f(x, y) = \overline{f(y, x)}, \text{ for each } (x, y) \in E^2.$$

Example 4.3. The sesquilinear form defined over $\mathbb{C}$ by

$$\begin{aligned} f \quad &: \quad \mathbb{C} \times \mathbb{C} \to \mathbb{C} \\ (z_1, z_2) &\mapsto z_1 \overline{z_2} \end{aligned}$$

is hermitian. In fact, for each $(z_1, z_2) \in \mathbb{C}^2$, one has

$$f(z_1, z_2) = z_1 \overline{z_2} = \overline{\overline{z_1}} \overline{z_2} = \overline{\overline{z_1} z_2} = \overline{z_2 \overline{z_1}} = \overline{f(z_2, z_1)}.$$

That is, f is hermitian.

Theorem 4.2. *Let A be a hermitian matrix, and let $f : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$, $f(x,y) \mapsto x^t A \overline{y}$. Then f is a hermitian sesquilinear form over $\mathbb{C}^n$.*

Proof. We use the same argument as in the proof of Theorem 2.1. □

Theorem 4.3. *Let f be a sesquilinear form over E. Then f is hermitian if and only if $f(x,x) \in \mathbb{R}$ for every $x \in E$.*

Proof. Assume that f is Hermitian. Then by definition, $f(x,y) = \overline{f(y,x)}$ for every $x, y \in E$. In particular, when $x = y$ we have $f(x,x) = \overline{f(x,x)}$ for every $x \in E$. Thus, $f(x,x) \in \mathbb{R}$ for every $x \in E$.

Conversely. Assume that $f(x,x) \in \mathbb{R}$ for every $x \in E$. Then for every $x, y \in E$ we also have

$$\begin{cases} f(x+y, x+y) \in \mathbb{R}, \\ f(ix+y, ix+y) \in \mathbb{R}. \end{cases}$$

It follows that

$$\begin{cases} \underbrace{f(x,x)}_{\in\mathbb{R}} + \underbrace{f(y,y)}_{\in\mathbb{R}} + f(x,y) + f(y,x) \in \mathbb{R}, \\ \underbrace{f(x,x)}_{\in\mathbb{R}} + \underbrace{f(y,y)}_{\in\mathbb{R}} + i\left[f(x,y) - f(y,x)\right] \in \mathbb{R}. \end{cases}$$

We put

$$\begin{cases} \alpha = f(x,y) + f(y,x) \in \mathbb{R}, \\ \beta = i\left[f(x,y) - f(y,x)\right] \in \mathbb{R}. \end{cases}$$

It is clear that

$$\frac{\alpha + i\beta}{2} = f(y,x) \text{ and } \frac{\alpha - i\beta}{2} = f(x,y),$$

and so $\overline{f(y,x)} = f(x,y)$. This completes the proof. □

4.1.2 Hermitian matrices

At first, define hermitian matrices:

Definition 4.4. Let $A = (a_{ij})_{1 \le i,j \le n} \in \mathcal{M}_n(\mathbb{C})$. The matrix $(\overline{a_{ij}})_{1 \le i,j \le n}$ is called **conjugate** of A, denoted by $\overline{A}$. The **transpose conjugate** matrix of A is called the **adjoint** of A, and denoted by A^*.

Note that for any matrix $A \in \mathcal{M}_n(\mathbb{C})$, we have $A^* = \overline{A^t} = \left(\overline{A}\right)^t$. That is, the conjugate transpose is the same with the transpose conjugate.

Definition 4.5. A matrix $A \in \mathcal{M}_n(\mathbb{C})$ is said to be **hermitian** if $A^* = A$. That is, if $\overline{A^t} = A$. Thus,

$$A \text{ is hermitian} \overset{\text{def}}{\Leftrightarrow} a_{ij} = \overline{a_{ji}} \text{ for } 1 \leq i, j \leq n.$$

Example 4.4. The matrices

$$A = \begin{pmatrix} 3 & 2+i \\ 2-i & 7 \end{pmatrix}, B = \begin{pmatrix} 1 & 1+i & 2+3i \\ 1-i & -2 & -i \\ 2-3i & i & 0 \end{pmatrix}$$

are hermitian.

We also state the following elementary properties:

1. $I^* = I$,
2. $(A^*)^* = A$,
3. $(A+B)^* = A^* + B^*$,
4. $(\alpha A)^* = \overline{\alpha} \cdot A^*$,
5. $(AB)^* = B^* A^*$.

Remark 4.1. Let $A \in \mathcal{M}_n(\mathbb{C})$. We can easily prove that the matrices $A + A^*$, AA^* and A^*A are hermitian.

Proposition 4.1. *The diagonal entries of a hermitian matrix A are real numbers.*

Proof. Let $A = (a_{ij})_{1 \leq i,j \leq n} \in \mathcal{M}_n(\mathbb{C})$ be a hermitian matrix. Since $a_{ij} = \overline{a_{ji}}$ for each $1 \leq i, j \leq n$, then

$$a_{ii} = \overline{a_{ii}}, \forall\, i = 1, 2, ..., n.$$

It follows that $a_{ii} \in \mathbb{R}$ for $i = 1, 2, ..., n$. □

Proposition 4.2. *Let A and B be two hermitian matrices. Then AB is hermitian if and only if $AB = BA$.*

Proof. We see that $(AB)^* = AB$ iff $B^*A^* = AB$ iff $BA = AB$, as desired. □

Definition 4.6. Let $A \in \mathcal{M}_n(\mathbb{C})$.

1. A is said to be skew-hermitian if $A^* = -A$. That is, if $\overline{A^t} = -A$.

Proposition 4.3. *Let $A \in \mathcal{M}_n(\mathbb{C})$. The diagonal entries of a skew-hermitian matrix A are zero or imaginary pure.*

Proof. Let $A = (a_{ij})_{1\le i,j\le n} \in \mathcal{M}_n(\mathbb{C})$ be a skew-hermitian matrix. Since $-a_{ij} = -\overline{a_{ji}}$ for each $1 \le i, j \le n$, then

$$-a_{ii} = \overline{a_{ii}}, \forall\, i = 1, 2, ..., n.$$

It follows that $Re\,(a_{ii}) = 0$, so $a_{ii} = 0$ or $\alpha_i \cdot i$ with $\alpha_i \in \mathbb{R}^*$ for $i = 1, 2, ..., n$. □

Proposition 4.4. *Let $A \in \mathcal{M}_n(\mathbb{C})$. Then A is skew-hermitian if and only if iA is hermitian.*

Proof. We have

$$(iA)^* = iA \Leftrightarrow -iA^* = iA \Leftrightarrow A^* = -A.$$

The proof is finished. □

Example 4.5 (Homework)**. 1.** Find the complex number b for which the matrix

$$A = \begin{pmatrix} 0 & b & 0 \\ \overline{b} & 0 & 1-b \\ 0 & b-1 & 0 \end{pmatrix}, b \in \mathbb{C}$$

is hermitian.

2. Let

$$A = \begin{pmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{pmatrix}, x, y, z \in \mathbb{C}$$

Find the complex numbers x, y, z such that (i) $A^* = A$, (ii) $A^* = -A$, (i) A is unitary.

4.2 Hermitian quadratic forms over $\mathbb{C}^n$

Let E be a v. space over $\mathbb{C}$. Recall that a map $q : E \to \mathbb{C}$ is said to be **hermitian quadratic form** if there exists a hermitian sesquilinear form $f : E \times E \to \mathbb{R}$ such that $f(x, x) = q(x)$ for any $x \in E$.

Remark 4.2. Every hermitian matrix $A \in \mathcal{M}_n(\mathbb{C})$ produces a hermitian quadratic form over $\mathbb{C}^n$.

Next, the *analytic expression* of q is given by:

$$q = \sum_{i,j}^{n} a_{ij} \cdot x_i\overline{x_j} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \cdot x_i\overline{x_j}.$$

Thus, every hermitian quadratic form over $\mathbb{C}^n$ is given by the following *matrix form*[1].

$$\begin{aligned} q(x_1, x_2, ..., x_n) &= \begin{pmatrix} x_1 & x_2 & ... & x_n \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{12} & a_{22} & ... & a_{2n} \\ & & \ddots & \\ a_{1n} & a_{2n} & ... & a_{nn} \end{pmatrix} \begin{pmatrix} \overline{x_1} \\ \overline{x_2} \\ \vdots \\ \overline{x_n} \end{pmatrix} \\ &= x^t \cdot A \cdot \overline{x}, \end{aligned}$$

where $a_{ij} = \overline{a_{ji}}$ for $1 \leq i, j \leq n$.

Definition 4.7. 1. A hermitian sesquilinear form $f : E \times E \to \mathbb{C}$ is said to be **positive** if for any $v \in E$,

$$f(v, v) \in \mathbb{R}_+.$$

2. A hermitian sesquilinear form $f : E \times E \to \mathbb{C}$ is said to be **definite positive** if for any $v \in E$,

$$f(v, v) \in \mathbb{R}_+^*.$$

Theorem 4.4. *Let $A \in \mathcal{M}_n(\mathbb{C})$. Then A is hermitian definite positive iff there exists an invertible matrix M such that*

$$A = M^t \cdot \overline{M}. \tag{4.1}$$

Definition 4.8. Let E be a vector space over $\mathbb{C}$. An inner product over E is a sesquilinear form, hermitian and definite positive.

Thus, a vector space E over $\mathbb{C}$ equipped with a sesquilinaer form which is hermitian and definite positive is called *pre-Hilbert space*. If a pre-Hilbert space E has finite dimension, it is called *Euclidean space*.

4.3 Gauss decomposition for hermitian forms

Here, we have not a direct method as in Section 3.6; but we usually use the following well-known facts:

- For every $z \in \mathbb{C}$: $z \cdot \overline{z} = |z|^2$.
- For every $z \in \mathbb{C}$: $z + \overline{z} = 2Re(z)$.
- For every $z_1, z_2 \in \mathbb{C}$:

$$z_1 \cdot \overline{z_2} + \overline{z_1} \cdot z_2 = \frac{1}{2}|z_1 + z_2|^2 - \frac{1}{2}|z_1 - z_2|^2.$$

[1] In some references $\overline{x}^t \cdot A \cdot x$ is the matrix representation of a quadratic hermitian form over $\mathbb{C}^n$, where $\overline{x}^t \cdot A \cdot x = x^t \cdot A \cdot \overline{x}$.

Example 4.6. Diagonalize the following hermitian quadratic forms:

1. $q_1 = ix_1\overline{x_2} - ix_2\overline{x_1}$, $E = \mathbb{C}^2$.
2. $q_2 = x_1\overline{x_1} + ix_1\overline{x_2} - ix_2\overline{x_1} + x_2\overline{x_2}$, $E = \mathbb{C}^2$.
3. $q_3 = x_1\overline{x_1} + a_{12}x_1\overline{x_2} + a_{21}x_2\overline{x_1} + a_{22}x_2\overline{x_2}$.
4. Deduce the signature of the quadratic form given by:

$$q_2' = \alpha x_1\overline{x_1} + ix_1\overline{x_2} - ix_2\overline{x_1} + x_2\overline{x_2},\ \alpha \in \mathbb{R}.$$

Solution. We can write

$$\begin{aligned} q_1 &= ix_1\overline{x_2} - ix_2\overline{x_1} \\ &= x_1\left(i\overline{x_2}\right) + \overline{x_1}\left(-ix_2\right) \\ &= x_1\left(\overline{-ix_2}\right) + \overline{x_1}\left(-ix_2\right) \quad \text{(which is of the form } z_1\overline{z_2} + \overline{z_1}z_2\text{)} \\ &= \frac{1}{2}\left|x_1 - ix_2\right|^2 - \frac{1}{2}\left|x_1 + ix_2\right|^2 \quad \text{(since } z_1\overline{z_2} + \overline{z_1}z_2 = \frac{1}{2}\left|z_1 + z_2\right|^2 - \frac{1}{2}\left|z_1 - z_2\right|^2\text{)} \\ &= \left|f_1\right|^2 - \left|f_2\right|^2, \end{aligned}$$

where f_1 et f_2 are linearly independent forms over $\mathbb{C}^2$, since

$$\begin{vmatrix} 1 & -i \\ 1 & i \end{vmatrix} \neq 0.$$

The signature of q_1 is $(1,1)$. Likewise, we have

$$\begin{aligned} q_2 &= x_1\overline{x_1} + ix_1\overline{x_2} - ix_2\overline{x_1} + x_2\overline{x_2} \\ &= \left(x_1 - ix_2\right)\left(\overline{x_1} + i\overline{x_2}\right) \\ &= \left(x_1 - ix_2\right)\overline{\left(x_1 - ix_2\right)} \\ &= \left|x_1 - ix_2\right|^2 \\ &= \left|f_1\right|^2. \end{aligned}$$

The signature of q_2 is $(1,0)$.

For the quadratic form $q_2' = \alpha x_1\overline{x_1} + ix_1\overline{x_2} - ix_2\overline{x_1} + x_2\overline{x_2}$, $\alpha \in \mathbb{R}$. We see that

$$q_2' = (\alpha - 1)\, x_1\overline{x_1} + q_2 = (\alpha - 1)\left|x_1\right|^2 + \left|x_1 - ix_2\right|^2.$$

We deduce that

$$\begin{cases} \alpha = 1, \text{ the signature is } (1,0). \\ \alpha > 1, \text{ the signature is } (2,0). \\ \alpha < 1, \text{ the signature is } (1,1). \end{cases}$$

Finally, we have

$$\begin{aligned} q_3 &= x_1\overline{x_1} + a_{12}x_1\overline{x_2} + a_{21}x_2\overline{x_1} + a_{22}x_2\overline{x_2} \\ &= (x_1 + a_{21}x_2)(\overline{x_1} + a_{12}\overline{x_2}) + (a_{22} - a_{12}a_{21})\, x_2\overline{x_2} \\ &= (x_1 + a_{21}x_2)\overline{(x_1 + a_{21}x_2)} + (a_{22} - a_{12}a_{21})\, x_2\overline{x_2} \\ &= |x_1 + a_{21}x_2|^2 + \underbrace{(a_{22} - a_{12}a_{21})}_{\in\mathbb{R}} |x_2|^2 . \end{aligned}$$

Example 4.7. Let $E = \mathbb{C}$, and let q be the Hermitian quadratic form over E given by

$$q = a_{12}x_1\overline{x_2} + a_{13}x_1\overline{x_3} + \overline{a_{12}}x_2\overline{x_1} + a_{23}x_2\overline{x_3} + \overline{a_{13}}x_3\overline{x_1} + \overline{a_{23}}x_3\overline{x_2}.$$

Give the diagonal form of Gauss.

Solution. We have

$$\begin{aligned} q &= a_{12}x_1\overline{x_2} + a_{13}x_1\overline{x_3} + \overline{a_{12}}x_2\overline{x_1} + a_{23}x_2\overline{x_3} + \overline{a_{13}}x_3\overline{x_1} + \overline{a_{23}}x_3\overline{x_2} \\ &= x_1(a_{12}\overline{x_2} + a_{13}\overline{x_3}) + \overline{x_1}(\overline{a_{12}}x_2 + \overline{a_{13}}x_3) + a_{23}x_2\overline{x_3} + \overline{a_{23}}x_3\overline{x_2} \\ &= x_1(a_{12}\overline{x_2} + a_{13}\overline{x_3}) + a_{23}x_2\overline{x_3} + \overline{x_1}(\overline{a_{12}}x_2 + \overline{a_{13}}x_3) + \overline{a_{23}}x_3\overline{x_2} \\ &= x_1(a_{12}\overline{x_2} + a_{13}\overline{x_3}) + \frac{a_{23}}{a_{13}}x_2(a_{12}\overline{x_2} + a_{13}\overline{x_3}) - \frac{a_{23}a_{12}}{a_{13}}x_2\overline{x_2} + \\ &\quad \underbrace{\overline{x_1}(\overline{a_{12}}x_2 + \overline{a_{13}}x_3) + \frac{\overline{a_{23}}}{\overline{a_{13}}}\overline{x_2}(\overline{a_{12}}x_2 + \overline{a_{13}}x_3) - \frac{\overline{a_{23}a_{12}}}{\overline{a_{13}}}x_2\overline{x_2}} \\ &= \left(x_1 + \frac{a_{23}}{a_{13}}x_2\right)(a_{12}\overline{x_2} + a_{13}\overline{x_3}) + \left(\overline{x_1} + \frac{\overline{a_{23}}}{\overline{a_{13}}}\overline{x_2}\right)(\overline{a_{12}}x_2 + \overline{a_{13}}x_3) - \left(\frac{a_{23}a_{12}}{a_{13}} + \frac{\overline{a_{23}a_{12}}}{\overline{a_{13}}}\right)x_2\overline{x_2} \\ &= \frac{1}{2}\left|x_1 + \left(\frac{a_{23}}{a_{13}} + \overline{a_{12}}\right)x_2 + \overline{a_{13}}x_3\right|^2 - \frac{1}{2}\left|x_1 + \left(\frac{a_{23}}{a_{13}} - \overline{a_{12}}\right)x_2 - \overline{a_{13}}x_3\right|^2 - 2Re\left(\frac{a_{23}a_{12}}{a_{13}}\right)|x_2|^2 . \end{aligned}$$

Example 4.8. Diagonalize the Hermitian quadratic form given by its matrix:

$$M_q = \begin{pmatrix} 0 & 1-i & 0 \\ 1+i & 0 & i \\ 0 & -i & 0 \end{pmatrix}.$$

Here, M_q is the matrix of the hermitian quadratic form q with respect to the standard basis of $\mathbb{C}^3$.

Solution. We have

$$\begin{aligned} q &= (1-i)\,x_1\overline{x_2} + (1+i)\,x_2\overline{x_1} + ix_2\overline{x_3} - ix_3\overline{x_2} \\ &= x_2\left[(1+i)\,\overline{x_1} + i\overline{x_3}\right] + \overline{x_2}\left[(1-i)\,x_1 - ix_3\right] \\ &= x_2\overline{\left[(1-i)\,x_1 - ix_3\right]} + \overline{x_2}\left[(1-i)\,x_1 - ix_3\right] \quad \text{(which is of the form } z_1\overline{z_2} + \overline{z_1}z_2\text{)} \\ &= \frac{1}{2}\left|x_2 + (1-i)\,x_1 - ix_3\right|^2 - \frac{1}{2}\left|x_2 - (1-i)\,x_1 + ix_3\right|^2 \\ &= \left|f_1\right|^2 - \left|f_2\right|^2. \end{aligned}$$

The signature is $(1,1)$.

CHAPTER 5

SPECTRAL DECOMPOSITION OF SELF-ADJOINT LINEAR MAPPINGS

In this chapter we present a sufficiently and necessary condition for a linear form to be normal in a complex pre-Hilbert space of finite dimension. But first, define the inner product on a complex vector space and then we state, without proof, the spectral decomposition theorem of self-adjoint linear mappings.

5.1 Scalar Product over a complex vector space

Definition 5.1. Let E be complex v. space. The inner product of E (over E) is a function $\langle ., . \rangle$ defined by

$$\begin{aligned} \langle ., . \rangle \quad &: \quad E \times E \to \mathbb{C} \\ (x, y) \quad &\mapsto \quad \langle x, y \rangle \end{aligned}$$

satisfying the following properties:

1. For all $x \in E$, $\langle x, x \rangle \in \mathbb{R}_+$ and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$.
2. For all $x, y \in E$, we have $\langle x, y \rangle = \overline{\langle y, x \rangle}$.
3. For all $x \in E$ and scalar $\alpha \in \mathbb{R}$, we have $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$
4. For all $x, y, z \in E$, we have $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.

We say, the scalar product between x and y, or the inner product between x and y.

Definition 5.2. Let E be a complex vector space equipped with an inner product $\langle ., . \rangle$. The couple $(E, \langle ., . \rangle)$ is said to be a **complex pre-Hilbert space**. A complex pre-Hilbert space of finite dimension is said to be **hermitian space**.

Example 5.1. Define over $\mathbb{C}^n$ the scalar product $\langle .,. \rangle$ by

$$\forall\, x,\ y \in \mathbb{C}^n : \langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}. \tag{5.1}$$

We can write (5.1) as[1]: $\langle x, y \rangle = x^t \cdot \overline{y}$. In particular, for $x = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^t$ and $y = \begin{pmatrix} y_1 & y_2 \end{pmatrix}^t$, we have

$$\langle x, y \rangle = \langle (x_1, x_2), (y_1, y_2) \rangle = x_1 \overline{y_1} + x_2 \overline{y_2}.$$

We will accept the following lemma without proof.

Lemma 5.1. *For every* $x, y \in \mathbb{C}^n$:

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|. \tag{5.2}$$

5.2 Spectral decomposition of self-adjoint linear mapping

At first, define unitary and normal matrices or linear mapping.

Definition 5.3. Let $A \in \mathcal{M}_n(\mathbb{C})$.

1. A is said to be **unitary** if $A^* = A^{-1}$.

2. A is said to be **normal** if $A^*A = AA^*$. This means that A commutes with its transpose conjugate.

Example 5.2. We can check that the matrix

$$U = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

is unitary; however for the matrix:

$$N = \begin{bmatrix} -i & -i \\ -i & i \end{bmatrix}$$

we can easily check that $N^*N = NN^*$, so N is normal.

Proposition 5.1. *Every* n *by* n *complex invertible matrix* A *can be represented as* $A = U \cdot T$, *where* U *is unitary and* $T = (t_{ij})$ *is upper triangular with* $t_{ij} \geq 0$.

Proof. The proof is similar to the real case. □

[1]Sometimes we use the notation ${}^t x \cdot \overline{y}$ instead of $x^t \cdot \overline{y}$.

Lemma 5.2. *Every hermitian matrix $A \in \mathcal{M}_n(\mathbb{R})$ can be represented in the form:*

$$A = P^t \cdot D \cdot \overline{P}, \tag{5.3}$$

where P is orthogonal and D is diagonal whose diagonal entries ($\in \mathbb{R}$) are the eigenvalues of A.

From the above lemma, we deduce that every hermitian definite positive matrix A can written as $A = M^t \cdot \overline{M}$, where $M = \sqrt{D}P$ is invertible.

Definition 5.4. Let $f \in \mathcal{L}(E)$, the adjoint (or the hermitian conjugate) of f is the mapping $f^* \in E^*$ satisfying

$$\langle f(u), v\rangle = \langle u, f^*(v)\rangle,$$

for any $u, v \in E$. Further, f is said to be **self-adjoint** or **hermitian** if $f = f^*$.

Theorem 5.1. *Let $A \in \mathcal{M}_n(\mathbb{C})$ be a hermitian matrix (resp. self-adjoint mapping). Then $x^t A\overline{x} \in \mathbb{R}$ for each $x \in \mathbb{C}^n$.*

Proof. We have

$$\begin{aligned} x^t A\overline{x} &= \left(x^t A\overline{x}\right)^t \quad \text{(since } x^t A\overline{x} = a \in \mathbb{C}\text{)} \\ &= (\overline{x})^t A^t x \quad \text{(known result)} \\ &= (\overline{x})^t \overline{A^*} x \\ &= \overline{x^t A^* \overline{x}} \\ &= \overline{x^t A\overline{x}} \quad \text{(since } A^* = A\text{)}. \end{aligned}$$

This implies that $x^t A\overline{x} = \overline{x^t A\overline{x}}$. Hence, $x^t A\overline{x} \in \mathbb{R}$. □

Second proof. We know that

$$x^t A\overline{x} = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{12} & a_{22} & \dots & a_{2n} \\ & & \ddots & \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} \overline{x_1} \\ \overline{x_2} \\ \vdots \\ \overline{x_n} \end{pmatrix},$$

where $a_{ii} \in \mathbb{R}$ for $1 \leq i \leq n$ and $a_{ij} = \overline{a_{ji}}$ for $i \neq j$ because the matrix A is Hermitian.

Therefore,

$$\begin{aligned} x^t A\overline{x} &= \sum_{i,j} a_{ij} x_i \overline{x_j} \\ &= \sum_{i=1}^{n} a_{ii} x_i \overline{x_i} + \sum_{i \neq j} a_{ij} x_i \overline{x_j} \\ &= \underbrace{\sum_{i=1}^{n} a_{ii} \left| x_i \right|^2}_{\in \mathbb{R}} + \sum_{i<j} \left(a_{ij} x_i \overline{x_j} + a_{ji} x_j \overline{x_i} \right) \\ &= \underbrace{\sum_{i=1}^{n} a_{ii} \left| x_i \right|^2}_{\in \mathbb{R}} + \sum_{i<j} \left(a_{ij} x_i \overline{x_j} + \overline{a_{ij} x_i \overline{x_j}} \right) \\ &= \underbrace{\sum_{i=1}^{n} a_{ii} \left| x_i \right|^2}_{\in \mathbb{R}} + \underbrace{2Re \sum_{i<j} a_{ij} x_i \overline{x_j}}_{\in \mathbb{R}} \end{aligned}$$

The proof is finished. □

Remark 5.1. By a second method we prove that the eigenvalues of a hermitian matrix A (resp. self-adjoint mapping) are real numbers. Let f_A be the corresponding hermitian sesquilinear form of A and let (λ, x) be an eigenpair of A. Applying Theorem 5.1, we obtain

$$\begin{aligned} \underbrace{f_A\left(\overline{x}, \overline{x}\right)}_{\in \mathbb{R}} &= \left(\overline{x}\right)^t A \overline{\overline{x}} = \left(\overline{x}\right)^t Ax = \left(\overline{x}\right)^t \lambda x = \lambda \left(\overline{x}\right)^t x \\ &= \lambda \cdot \underbrace{\sum_{i=1}^{n} \left| x_i \right|^2}_{\in \mathbb{R}} \in \mathbb{R}. \end{aligned}$$

Hence, $\lambda \in \mathbb{R}$.

Theorem 5.2. *The eigenvalues of a hermitian matrix (resp. self-adjoint mapping) are real numbers.*

Proof. Let (λ, x) be an eigenpair of a hermitian matrix A (note that $x \neq 0$)[2]. We can write

$$\begin{aligned} \lambda \left\langle x, x \right\rangle &= \left\langle \lambda x, x \right\rangle = \left\langle Ax, x \right\rangle = (Ax)^t \,\overline{x} = x^t A^t \overline{x} \\ &= x^t \left(\left(\overline{A}\right)^t \right)^t \overline{x} \quad \text{(since } \left(\overline{A}\right)^t = A\text{)} = x^t \overline{A}\overline{x} \\ &= x^t \overline{Ax} = \left\langle x, Ax \right\rangle = \left\langle x, \lambda x \right\rangle = \overline{\lambda} \left\langle x, x \right\rangle. \end{aligned}$$

[2]The eigenvectors are always nonzero.

Thus, $\lambda = \overline{\lambda}$ and so $\lambda \in \mathbb{R}$. □

Corollary 5.1. The eigenvalues of a real skew-symmetric matrix are imaginary pure.

Proof. **First Method.** It suffices to show that iA is hermitian. In fact, $(iA)^* = \left(\overline{(iA)}\right)^t = -i \cdot A^t = -i(-A) = iA$. By Theorem 5.2, the eigenvalues of iA are real, and so the eigenvalues of A are imaginary pure.

Second Method. Proceeding along the same manner as in the proof of Theorem 5.1. Let (λ, x) be an eigenpair of A. Then

$$\begin{aligned}
\lambda \langle x, x \rangle &= \langle \lambda x, x \rangle = \langle Ax, x \rangle = (Ax)^t \, \overline{x} = x^t A^t \overline{x} \\
&= x^t \left(-\overline{A}\right) \overline{x} \quad (\text{since } \overline{A} = A \text{ and } A^t = -A) \\
&= -x^t \cdot \overline{Ax} = -\langle x, Ax \rangle = -\langle x, \lambda x \rangle = -\overline{\lambda} \langle x, x \rangle .
\end{aligned}$$

Therefore, $\left(\lambda + \overline{\lambda}\right) \langle x, x \rangle = 0$. Since $x \neq 0$, we deduce that $2Re(\lambda) = 0$ and hence λ is imaginary pure. The proof is finished. □

Theorem 5.3 (**Spectral decomposition of self-adjoint linear mapping**). *Let E be a pre-Hilbert space over $\mathbb{C}$ with $\dim E = n$ and let $f \in \mathcal{L}(E)$. Then f is normal iff there exists an orthonormal basis for E formed by the eigenvectors of f.*

Proof. For the proof, one can see [1]. □

We finish this subsection by a simple comparison between linear algebra and sesquilinear algebra.

Linear Algebra	**Sesquilinear Algebra**
Linear	Semi-linear
f is bilinear	f is sesquilinear
f is bilinear symmetric	f is sesquilinear hermitian
q is a quadratic form	q is a hermitian quadratic form
Euclidian space	Hermitian space
Symmetric matrix	Hermitian matrix
Anti-symmetric (skew-symmetric) matrix	Anti-hermitian (skew-hermitian) matrix
Orthogonal matrix	Unitary matrix
Pre-Hilbert space over $\mathbb{R}$	Complex Pre-Hilbert space
...	...

5.3 Proposed problems

Exercise 1. Let f be a sesquilinear form on E. Show that f is hermitian sesquilinear form on E if, and only if, for every x in E, $f(x, x)$ is real.

Exercise 2. Show that

i. The set of all sesquilinear forms over E, equipped with the usual sum of functions and multiplication by a scalar, is a vector space over $\mathbb{C}$.

ii. The set of all hermitian forms over E, equipped with the usual sum of functions and multiplication by a scalar, is a vector space over $\mathbb{C}$.

Exercise 3. Let f be a hermitian sesquilinear form on E. Two parts A and B of E are said to be orthogonal with respect to f if $f(x, y) = 0$ for any x in A and y in B. Prove that the following conditions are equivalent:

(*i*) A and B are orthogonal.

(*ii*) $A \subset B^{\perp}$

(*iii*) $B \subset A^{\perp}$.

Exercise 4. Let f be a hermitian sesquilinear form on E and q its associated Hermitian quadratic form. Prove that for all x, y in E and α, β in $\mathbb{C}$ we have

- $q(x+y) + q(x-y) = 2q(x) + 2q(y)$,
- $q(\alpha x + \beta y) = |\alpha|^2 q(x) + 2Re\left(\alpha\overline{\beta} f(x, y)\right) + |\beta|^2 q(y)$.

Exercise 5. Let $(e) = \{e_1, e_2, e_3\}$ a basis for a vector space E of dimension 3.

1. Let f be the sesquilinear form defined by

$$f(x, y) = 3x_1\overline{y_1} + 2ix_1\overline{y_2} - 5ix_1\overline{y_3} + (2+i)\, x_2\overline{y_1} - 7x_2\overline{y_2} + x_2\overline{y_3} + ix_3\overline{y_1} - x_3\overline{y_2} + (1-i)\, x_3\overline{y_3}.$$

1.1. Determine the matrix of f with respect to the basis (e).

1.2. Is f hermitian?

2. Explain the hermitian form g whose matrix in the base (e) is given by

$$\begin{pmatrix} -2 & i & 5 \\ -i & -1 & 3-2i \\ 5 & 3+2i & 4 \end{pmatrix}.$$

Give the hermitian quadratic form associated with g.

3. Determine, in the basis (e), the matrix of the hermitian form h whose associated hermitian quadratic form is

$$q(x) = 3x_1\overline{x_1} - 5ix_1\overline{x_3} + (2-i)\,x_2\overline{x_3} - 7x_2\overline{x_2} + 5ix_3\overline{x_1} + (2+i)\,x_3\overline{x_2} + x_3\overline{x_3}.$$

Exercise 6. Show that the product of two hermitian matrices A and B is a Hermitian matrix if and only if $AB = BA$.

Exercise 7. Let A be the hermitian matrix:

$$A = \begin{pmatrix} 1 & 1+i & 2i \\ 1-i & 4 & 2-3i \\ -2i & 2+3i & 7 \end{pmatrix}.$$

Find an invertible matrix P such that $P^t \cdot A \cdot \overline{P}$ is diagonal. Deduce the rank and signature of A.

Exercise 8. Let A be an invertible complex matrix. Show that the matrix $\left(\overline{A}\right)^t A$ is hermitian definite positive.

Exercise 9. Let q be a hermitian quadratic form on E with polar form f and let x be an isotropic vector for q.

1. Show that if q is defined then f is non-degenerate.

2. Show that for all $y \in E$ and $\lambda \in \mathbb{C}$, we have

$$q(y + \lambda x) = q(y) + 2Re\left(\lambda f(x, y)\right).$$

3. Deduce that if q is positive then for all $y \in E$ and $\mu \in \mathbb{R}$, we get

$$0 \leq q(y) + 2\mu \left|f(x, y)\right|^2.$$

4. Using the previous questions, show that if q is positive and f is nondegenerate, then q is definite.

Exercise 10. Let A be an invertible complex matrix. Show that if A is hermitian then A^{-1} is also hermitian.

Exercise 11. A complex matrix A is said to be anti-hermitian if $\left(\overline{A}\right)^t = -A$. Show that the matrix A is anti-hermitian if and only if iA is hermitian.

Exercise 12. Give a Gaussian decomposition of the Hermitian quadratic forms of $\mathbb{C}^3$

whose matrices in the canonical basis are

$$A = \begin{pmatrix} 1 & 1-i & 0 \\ 1+i & 3 & i \\ 0 & -i & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & -i & i \\ i & 0 & -i \\ -i & i & 0 \end{pmatrix}.$$

Deduce their core, rank and signature.

Exercise 13. Show that a hermitian quadratic form on a vector space E is non-degenerate if, and only if, the matrix A which represents it in a basis of E is invertible.

Exercise 14. We consider the hermitian quadratic form on $\mathbb{C}^3$ given by:

$$q(x) = x_1\overline{x_1} + (1+a)\,x_2\overline{x_2} + \left(1+a+a^2\right)x_3\overline{x_3} + ix_1\overline{x_2} - ix_2\overline{x_1} - iax_2\overline{x_3} + iax_3\overline{x_2},$$

where a is a real number and $\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^t$ are the coordinates of x in the canonical basis of $\mathbb{C}^3$.

1. Give the matrix of q in the canonical basis as well as its polar form f.
2. Using the Gauss method, decompose q into the sum of squares of modules of independent linear forms.
3. Deduce an orthogonal basis of $\mathbb{C}^3$ relative to f and give the matrix of q in the new basis.
4. Discuss according to the values of a the rank, signature and kernel of q.

Chapter 6

Solutions to some exercises and problems

The present chapter consists a detailed solution to some exercises and problems related to symmetric bilinear forms and quadratic forms. These problems were the subject of some previous TD's at department of mathematics.

Exercise 01. Find the corresponding symmetric matrix of each of the following quadratic forms:

1. $q(x,y) = 4x^2 - 6xy - 7y^2$, where $E = \mathbb{R}^2$.
2. $q(x,y) = xy + y^2$, where $E = \mathbb{R}^2$.
3. $q(x,y,z) = x^2 + y^2 - 2z^2 + xy + yz$, where $E = \mathbb{R}^3$.
4. $q(x,y,z) = 2x^2 + 2y^2 + 2z^2 + 2xy + 2yz + 2xz$, where $E = \mathbb{R}^3$.
5. $q(x,y,z,t) = 2x^2 + 2y^2 + 2z^2 + 2xy + 2yz + 2xz$, where $E = \mathbb{R}^4$.

Solution. We can easily write

1) $A = \begin{pmatrix} 4 & -3 \\ -3 & -7 \end{pmatrix}$; since $q(x,y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 4 & -3 \\ -3 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. In fact, we have

$$\begin{aligned} \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 4 & -3 \\ -3 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 4x - 3y & -3x - 7y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= 4x^2 - 6xy - 7y^2 = q(x,y). \end{aligned}$$

Using the same manner, we obtain

$$2)\ A = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}, 3)\ A = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & -2 \end{pmatrix}$$

$$4)\ A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, 5)\ A = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Exercise 02. Consider the quadratic form

$$\begin{array}{rcl} q & : & \mathbb{R}^2 \to \mathbb{R} \\ (x, y) & \mapsto & x^2 - y^2. \end{array}$$

1. Calculate the polar form of q, say f.
2. Write f in the matrix form.
3. Calculate the isotropic cone C.
4. Verify that q is nondegenerate.

Solution.

1. We know that
$$f(u, v) = \frac{1}{4}(q(u+v) - q(u-v)),$$
where $u = (x, y)$ and $v = (x', y') \in \mathbb{R}^2$; i.e.,
$$\begin{array}{rcl} f & : & \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \\ (u, v) & \mapsto & f(u, v). \end{array}$$
Then
$$\begin{aligned} f(u, v) &= f((x, y), (x', y')) \\ &= \frac{1}{4}(q(x + x', y + y') - q(x - x', y - y')) \\ &= \frac{1}{4}\left((x + x')^2 - (y + y')^2 - (x - x')^2 + (y - y')^2\right) \\ &= xx' - yy'. \end{aligned}$$

2. We see that the matrix form of f is given by

$$\begin{aligned} f(u,v) &= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \\ &= u^t A v, \text{ where } u = (x,y) \text{ and } v = (x', y'). \end{aligned}$$

3. We calculate the isotropic cone C. By (3.4), we have

$$\begin{aligned} C &= \left\{ (x,y) \in \mathbb{R}^2 \,;\, q(x,y) = 0 \right\} \\ &= \left\{ (x,y) \in \mathbb{R}^2 \,;\, x^2 - y^2 = 0 \right\} \\ &= \left\{ (x,y) \in \mathbb{R}^2 \,;\, (x-y)(x+y) = 0 \right\} \\ &= \left\{ (x,y) \in \mathbb{R}^2 \,;\, y = x \text{ or } y = -x \right\}. \end{aligned}$$

4. We verify that q is nondegenerate. Indeed, we have

$$\begin{aligned} \ker f &= \left\{ (x,y) \in \mathbb{R}^2 \,;\, f((x,y),(x',y')) = 0, \ \forall \ (x',y') \in \mathbb{R}^2 \right\} \\ &= \left\{ (x,y) \in \mathbb{R}^2 \,;\, xx' - yy' = 0, \ \forall \ (x',y') \in \mathbb{R}^2 \right\} \\ &= \{(0,0)\}. \end{aligned}$$

   Thus, f or q is nondegenerate.

Exercise 03. Let $f \in \mathcal{S}_2(E)$, and let q be the associated quadratic form. Let $x_0 \in E$ with $q(x_0) \neq 0$. Setting

$$\begin{cases} F: \text{ is the subspace generated (spanned) by } x_0, \\ \quad G = \{ y \in E; f(x_0, y) = 0 \}. \end{cases}$$

Prove that $E = F \oplus G$.

Solution. At first, we can check that $F \cap G = \{0_E\}$.

Let $u \in F \cap G$. Since $u \in F$, $u = kx_0$ for some scalar $k \in \mathbb{K}$. Since $u \in G$, then $f(x_0, kx_0) = kf(x_0, x_0) = 0$. But, $f(x_0, x_0) \neq 0$, then $k = 0$. This gives $u = 0$. Thus, $F \cap G \subset \{0_E\}$.

Second, we prove that $E = F + G$. Let $x \in E$ and let

$$x = \underbrace{\frac{f(x_0, x)}{f(x_0, x_0)} \cdot x_0}_{u} + \underbrace{x - \frac{f(x_0, x)}{f(x_0, x_0)} \cdot x_0}_{v},$$

where $u \in F$ (since u is of the form λx_0 with $\lambda = \dfrac{f(x_0, x)}{f(x_0, x_0)} \in \mathbb{R}$). Likewise, since

$$f(x_0, v) = f\left(x_0, x - \frac{f(x_0, x)}{f(x_0, x_0)} \cdot x_0\right) = f(x_0, x) - f(x_0, x) = 0,$$

then $v \in G$. Thus, we have shown that $F \cap G = \{0_E\}$ and $F + G = E$, and hence $E = F \oplus G$.

Exercise 04. Let $A \in \mathcal{M}_n(\mathbb{R})$ and $x \in \mathbb{R}^n$. Prove that

$$x^t A x = x^t \left(\frac{A + A^t}{2}\right) x.$$

Solution. For each $A \in \mathcal{M}_n(\mathbb{R})$ and $x \in \mathbb{R}^n$, we have

$$\begin{aligned} x^t A x &= x^t A x = \left(x^t A x\right)^t \quad \text{(since } x^t A x \in \mathbb{R}) \\ &= x^t A^t \left(x^t\right)^t \quad \text{(well-known result)} \\ &= x^t A^t x. \end{aligned}$$

Then we can write

$$x^t A x = \frac{1}{2} x^t A x + \frac{1}{2} x^t A x = \frac{1}{2} x^t A x + \frac{1}{2} x^t A^t x = x^t \left(\frac{A + A^t}{2}\right) x.$$

This completes the proof.

Exercise 05. Define the quadratic form

$$q = x_1^2 + 4x_1x_2 + 3x_2^2.$$

Calculate the polar form associated with q, denoted by f.

Solution. The polar form f of q is given by

$$\begin{aligned} f \quad &: \quad \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \\ (u, v) \quad &\mapsto \quad f(u, v) = u^t A v, \end{aligned}$$

where $u = (x_1, x_2)$, $v = (y_1, y_2) \in \mathbb{R}^2$ and $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$. Hence,

$$\begin{aligned} f &= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ &= x_1y_1 + 2x_1y_2 + 2x_2y_1 + 3x_2y_2. \end{aligned}$$

Exercise 06. Let $x = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^t \in \mathbb{R}^2$. Show that there are infinitely many matrices $A \in \mathcal{M}_2(\mathbb{R})$ such that

$$x^t A x = x^t \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix} x, \tag{6.1}$$

where $x \in \mathbb{R}^2$.

Solution. Let $n \in \mathbb{N}$. From Exercise 06, we have

$$\begin{aligned} X^t \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix} X &= X^t \frac{\begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 4 & 0 \end{pmatrix}}{2} X \\ &= X^t \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} X \quad \text{(in this case, the matrix is symmetric)} \\ &= X^t \begin{pmatrix} 1 & n \\ 4-n & 0 \end{pmatrix} X\,. \end{aligned}$$

Then (6.1) is true for infinitely many matrices A.

Exercise 7. Let $f \in \mathcal{S}_2(E)$ and let F be a subspace of E. Prove that

$$F \subset F^{\perp} \Leftrightarrow f(x,x) = 0, \text{ for every } x \in F. \tag{6.2}$$

Assume that $E = \mathbb{R}^3$, and let

$$\begin{aligned} f \quad &: \quad \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \\ ((x_1, x_2, x_3), (y_1, y_2, y_3)) \quad &\mapsto \quad x_1 y_1 + x_2 y_2 - x_3 y_3. \end{aligned}$$

Define $F = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \; ; \; x_1 = x_3 \text{ and } x_2 = 0\}$. Prove by two methods that $F \subset F^{\perp}$.

Solution.

1. Suppose that $f(x,x) = 0$ for all $x \in F$, and we prove that $F \subset F^{\perp}$. Let $y \in F$, we have

$$\begin{aligned} f(x+y, x+y) &= 0 \text{ for each } x \in F \\ &= f(x,x) + f(y,y) + 2f(x,y)\,. \end{aligned}$$

   Then for each $x \in F$, $f(x,y) = 0$. Hence, $y \in F^{\perp}$.

Assume that $F \subset F^{\perp}$ and we show that $f(x,x) = 0$ for each $x \in F$. In fact, let $x \in F$. For all $y \in F$, we have

$$f(y,x) = 0. \quad \text{(since } x \in F^{\perp}\text{)}.$$

In particular, $f(x,x)=0$ for each $x\in F$.

2. **1**st **Method.** For every $x=(\lambda,0,\lambda)\in F$, we have by (6.2) that

$$f(x,x)=f((\lambda,0,\lambda),(\lambda,0,\lambda))=\lambda^2+0^2-\lambda^2=0.$$

Hence, $F\subset F^{\perp}$.

2nd **Method.** By Definition 3.5, we can compute $F^{\perp}$ as follows:

$$\begin{aligned} F^{\perp} &= \left\{y\in\mathbb{R}^3;\, f(x,y)=0;\ \forall\, x\in F\right\} \\ &= \left\{(y_1,y_2,y_3)\in\mathbb{R}^3;\, f((\lambda,0,\lambda),(y_1,y_2,y_3))=0;\ \forall\, x=(\lambda,0,\lambda)\in F\right\} \\ &= \left\{(y_1,y_2,y_3)\in\mathbb{R}^3;\, \lambda y_1-\lambda y_3=0;\ \forall\, x=(\lambda,0,\lambda)\in F\right\} \\ &= \left\{(y_1,y_2,y_3)\in\mathbb{R}^3;\, \lambda(y_1-y_3)=0;\ \forall\, x=(\lambda,0,\lambda)\in F\right\} \\ &= \left\{(y_1,y_2,y_3)\in\mathbb{R}^3;\, y_1=y_3\right\} \\ &= Vect\left\{(1,0,1),(0,1,0)\right\}. \end{aligned}$$

Since

$$\begin{aligned} F &= \left\{(x_1,x_2,x_3)\in\mathbb{R}^3\,;\ x_1=x_3 \text{ and } x_2=0\right\} \\ &= Vect\left\{(1,0,1)\right\}, \end{aligned}$$

then clearly, $F\subset F^{\perp}$.

Exercise 8.

1. Using Gauss' Method, diagonalize the following two quadratic forms:

a. $q_1=x_1^2+x_2^2+2x_3^2-4x_1x_2+6x_2x_3$

b. $q_2=2x_1x_2+2x_2x_3+2x_1x_3$.

Then, determine their associated signatures.

2. Diagonalize the following quadratic form (use two methods).

$$q(x_1,x_2)=-12x_1x_2+5x_2^2.$$

Solution.

a. Using Gauss's method, we put

$$\begin{cases} x_1 = y_1 - \dfrac{1}{a_{11}}\left(a_{12}y_2 + \ldots + a_{1n}y_n\right) \\ x_2 = y_2 \\ \vdots \\ x_n = y_n \end{cases}$$

That is,

$$\begin{cases} x_1 = y_1 + 2y_2 \\ x_2 = y_2 \\ x_3 = y_3. \end{cases}$$

This implies

$$\begin{aligned} q_1 &= x_1^2 + x_2^2 + 2x_3^2 - 4x_1x_2 + 6x_2x_3 \\ &= \left(y_1 + 2y_2\right)^2 + y_2^2 + 2y_3^2 - 4\left(y_1 + 2y_2\right)y_2 + 6y_2y_3 \\ &= y_1^2 - 3y_2^2 + 6y_2y_3 + 2y_3^2 \\ &= y_1^2 + q_1'\left(y_2, y_3\right). \end{aligned}$$

Likewise, let us take

$$\begin{cases} y_2 = z_2 + z_3 \\ y_3 = z_3. \end{cases}$$

It follows that

$$\begin{aligned} q_1' &= -3y_2^2 + 6y_2y_3 + 2y_3^2 \\ &= -3\left(z_2 + z_3\right)^2 + 6\left(z_2 + z_3\right)z_3 + 2z_3^2 \\ &= -3z_2^2 + 5z_3^2. \end{aligned}$$

Finally, we obtain

$$q_1 = \left(x_1 - 2x_2\right)^2 + 5x_3^2 - 3\left(x_2 - x_3\right)^2 = f_1^2 + f_2^2 - f_3^2,$$

where f_1, f_2 and f_3 are linearly independent forms over $\mathbb{R}^3$ since

$$\begin{vmatrix} 1 & -2 & 0 \\ 0 & 0 & 5 \\ 0 & 1 & -1 \end{vmatrix} = -5 \neq 0.$$

The signature of q_1 is $(2, 1)$.

b. Consider the quadratic form

$$q_2 = 2x_1x_2 + 2x_2x_3 + 2x_1x_3.$$

In this case, we put

$$\begin{cases} x_1 = y_1 + y_2 \\ x_2 = y_1 - y_2 \\ x_3 = y_3. \end{cases}$$

We obtain

$$\begin{aligned} q_2 &= 2(y_1 + y_2)(y_1 - y_2) + 2(y_1 - y_2)y_3 + 2x_1y_3 \\ &= 2y_1^2 + 4y_3y_1 - 2y_2^2 \\ &= q_2'. \end{aligned}$$

Setting once again

$$\begin{cases} y_1 = z_1 - z_3 \\ y_2 = z_2 \\ y_3 = z_3. \end{cases}$$

It follows that

$$\begin{aligned} q_2' &= 2y_1^2 + 4y_3y_1 - 2y_2^2 \\ &= 2(z_1 - z_3)^2 + 4z_3(z_1 - z_3) - 2z_2^2 \\ &= 2z_1^2 - 2z_2^2 - 2z_3^2. \end{aligned}$$

Hence,

$$\begin{aligned} q_2 &= 2(y_1 + y_3)^2 - 2y_2^2 - 2y_3^2 \\ &= 2\left(\frac{x_1 + x_2}{2} + x_3\right)^2 - 2\left(\frac{x_1 - x_2}{2}\right)^2 - 2x_3^2 \\ &= f_1^2 - f_2^2 - f_3^2, \end{aligned}$$

where f_1, f_2 and f_3 are linearly independent forms over $\mathbb{R}^3$; since

$$\begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{vmatrix} = 2 \neq 0$$

The signature of q_2 is $(1, 2)$. The rank is 3.

2. Using two methods, we diagonalize the following quadratic form

$$q(x_1, x_2) = -12x_1x_2 + 5x_2^2.$$

1st method. Setting

$$x_2 = y_2 - \left(\frac{-6}{5}y_1\right) = y_2 + \frac{6}{5}y_1,\ x_1 = y_1$$

We obtain

$$\begin{aligned} q(x_1, x_2) &= -12x_1x_2 + 5x_2^2 \\ &= -12y_1\left(y_2 + \frac{6}{5}y_1\right) + 5\left(y_2 + \frac{6}{5}y_1\right)^2 \\ &= 5y_2^2 - \frac{36}{5}y_1^2 \\ &= 5\left(x_2 - \frac{6}{5}x_1\right)^2 - \frac{36}{5}x_1^2 \\ &= |f_1|^2 - |f_2|^2. \end{aligned}$$

where f_1, f_2 and f_3 are linearly independent forms over $\mathbb{R}^2$, since

$$\begin{vmatrix} 1 & -\frac{6}{5} \\ 1 & 0 \end{vmatrix} \neq 0.$$

The signature of q is $(1, 1)$.

2nd method. We have

$$\begin{aligned} q(x_1, x_2) &= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 0 & -6 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= x^t Ax, \text{ where } A = \begin{pmatrix} 0 & -6 \\ -6 & 5 \end{pmatrix} \in S_2(\mathbb{R}). \end{aligned}$$

Then we can write A in the form PDP^t, where P is orthogonal and D is diagonal whose diagonal entries are the eigenvalues of A. The eigenpairs of A are

$$\begin{cases} \lambda_1 = -4,\ v_1 = (3, 2) \\ \lambda_2 = 9,\ v_2 = (-2, 3). \end{cases}$$

Therefore,

$$P=\begin{pmatrix} \frac{v_1}{\|v_1\|_2} & \frac{v_2}{\|v_2\|_2} \end{pmatrix}=\begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix}.$$

which gives

$$\begin{aligned} q(x_1,x_2) &= x^tAx \\ &= x^tPDP^tx \quad (\text{since } A=PDP^t) \\ &= \left(P^tx\right)^t D\left(P^tx\right) \\ &= v^tDv, \text{ where } v=P^tx. \end{aligned}$$

It follows that

$$v=\begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \\ \frac{-2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} \frac{3}{13}\sqrt{13}x_1+\frac{2}{13}\sqrt{13}x_2 \\ \frac{3}{13}\sqrt{13}x_2-\frac{2}{13}\sqrt{13}x_1 \end{pmatrix}=\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

That is,

$$\begin{aligned} q &= \begin{pmatrix} v_1 & v_2 \end{pmatrix}\begin{pmatrix} -4 & 0 \\ 0 & 9 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ &= 9v_2^2-4v_1^2 \\ &= 9\left(\frac{3}{13}\sqrt{13}x_2-\frac{2}{13}\sqrt{13}x_1\right)^2-4\left(\frac{3}{13}\sqrt{13}x_1+\frac{2}{13}\sqrt{13}x_2\right)^2 \\ &= |f_1|^2-|f_2|^2. \end{aligned}$$

where f_1 are f_2 linearly independent form over $\mathbb{R}^2$, since

$$\begin{vmatrix} 3\sqrt{13} & -2\sqrt{13} \\ 3\sqrt{13} & 2\sqrt{13} \end{vmatrix}\neq 0.$$

The signature is $(1,1)$.

Exercise 9. Let $E=\mathcal{M}_2(\mathbb{R})$ be the vector space of 2×2 square matrices on $\mathbb{R}$. Let

$$M=\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix},$$

and let $f(A,B)=tr\left(A^tMB\right)$, where $A,B\in E$.

1. Prove that f is a bilinear form on the space E.

2. Find the matrix of f with respect to the canonical (standard) basis of E:

$$\mathcal{B}=\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}.$$

Solution.

1. We first prove that f is a bilinear form on the space E. Indeed, $\forall\ A, A', B \in \mathcal{M}_2(\mathbb{R})$, $\forall\ \lambda \in \mathbb{R}$ we have

$$\begin{aligned} f\left(\lambda A + A', B\right) &= tr\left(\left(\lambda A + A'\right)^t MB\right) \\ &= tr\left(\lambda A^t MB + \left(A'\right)^t MB\right) \\ &= \lambda tr\left(A^t MB\right) + tr\left(\left(A'\right)^t MB\right) \\ &= \lambda f\left(A, B\right) + f\left(A', B\right). \end{aligned}$$

2. We compute $M_f\left(\mathcal{B}\right)$, where

$$\mathcal{B}=\left\{\underbrace{\begin{pmatrix}1&0\\0&0\end{pmatrix}}_{e_1},\underbrace{\begin{pmatrix}0&1\\0&0\end{pmatrix}}_{e_2},\underbrace{\begin{pmatrix}0&0\\1&0\end{pmatrix}}_{e_3},\underbrace{\begin{pmatrix}0&0\\0&1\end{pmatrix}}_{e_4}\right\}$$

From a simple calculation, we obtain

$$\begin{aligned} f\left(e_1, e_1\right) &= tr\left(e_1^t M e_1\right) \\ &= tr\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1&2\\3&5\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}\right\} \\ &= tr\begin{pmatrix}1&0\\0&0\end{pmatrix} \\ &= 1. \end{aligned}$$

Similarly, we have $f\left(e_1, e_2\right) = 0$, $f\left(e_1, e_3\right) = 2$,...

It follows that

$$\mathcal{M}_f\left(\mathcal{B}\right)=\begin{pmatrix} f\left(e_1,e_1\right) & f\left(e_1,e_2\right) & f\left(e_1,e_3\right) & f\left(e_1,e_4\right) \\ f\left(e_2,e_1\right) & f\left(e_2,e_2\right) & f\left(e_2,e_3\right) & f\left(e_2,e_4\right) \\ f\left(e_3,e_1\right) & f\left(e_3,e_2\right) & f\left(e_3,e_3\right) & f\left(e_3,e_4\right) \\ f\left(e_4,e_1\right) & f\left(e_4,e_2\right) & f\left(e_4,e_3\right) & f\left(e_4,e_4\right) \end{pmatrix}=\begin{pmatrix}1&0&2&0\\0&1&0&2\\3&0&4&0\\0&3&0&4\end{pmatrix}.$$

Exercise 10. Recall that a bilinear form on a vector space E is called alternating form if and only if

$$\forall\, x \in E,\ f(x, x) = 0.$$

1. Let f be an alternating bilinear form on a vector space E. Prove that f is skew-symmetric.

2. Assume that $f \neq 0$ and $2 \leq \dim E < \infty$. Prove that there exist two vectors $u_1, u_2 \in E$ such that

$$f(u_1, u_2) = 1.$$

   Calculate $f(u_2, u_1)$.

3. Let U be the v. subspace spanned by u_1 and u_2. Verify that $\{u_1, u_2\}$ is a base of U. Write the associated matrix of f in this basis.

4. Setting

$$W = \{w \in E;\ f(w, u) = 0, \forall\, u \in U\} = U^{\perp}.$$

   Prove that $E = U \oplus W$ and deduce that there exists a basis $\mathcal{B}$ of the vector space E for which

$$\mathcal{M}_f(\mathcal{B}) = \begin{pmatrix} \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & & & \\ & \ddots & & & \\ & & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} \in \mathcal{M}_n(\mathbb{R}).$$

Solution.

1. For each $(x, y) \in E^2$ we have

$$\begin{aligned} f(x+y, x+y) &= 0 \quad \text{(since } f \text{ is alternating)} \\ &= \underbrace{f(x, x)}_{=0} + \underbrace{f(y, y)}_{=0} + f(x, y) + f(y, x). \end{aligned}$$

   Hence, $f(x, y) = -f(y, x)$. Then f is skew-symmetric.

2. Since $f \neq 0$, there exist two vectors $x, y \in E$ such that $f(x, y) = \alpha \neq 0$, and so

$$f\left(\frac{x}{\alpha}, y\right) = f(u_1, u_2) = 1.$$

Since $f(u_1, u_2) = -f(u_2, u_1)$, then $f(u_2, u_1) = -1$.

3. Let U be the vector subspace generated by u_1 and u_2. We prove that u_1 and u_2 are linearly independent. By the way of contradiction, if we put $u_2 = ku_1$, then

$$f(u_1, u_2) = f(u_1, ku_1) = kf(u_1, u_1) = 0.$$

A contradiction. Then $\{u_1, u_2\}$ is a base of U.

The matrix of f associated of $\{u_1, u_2\}$ is

$$\mathcal{M}_f(\{u_1, u_2\}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

4. Setting $W = \{w \in E;\ f(w, v) = 0, \forall\, v \in U\}$. We prove that $E = U \oplus W$.

It is clear that $\{0_E\} \subset U \cap W$. Further, if $x \in U \cap W$ implies

$$\begin{cases} x = \alpha u_1 + \beta u_2, \\ f(x, u_1) = 0, \\ f(x, u_2) = 0. \end{cases}$$

where $\alpha, \beta \in \mathbb{R}$. Hence,

$$\begin{cases} \alpha f(u_1, u_2) = 0 \Rightarrow \alpha = 0 \\ \beta f(u_2, u_1) = 0 \Rightarrow \beta = 0. \end{cases}$$

Then $x = 0$. Therefore, $U \cap W = \{0_E\}$.

It remains to be shown that $E = U + W$. For each $x \in E$, setting

$$u = f(x, u_2)\, u_1 - f(x, u_1)\, u_2$$

We see that $x = u + x - u$. Let u is a linear combination of u_1 and u_2, then $u \in U$.

It suffices to prove that $x - u \in W$. In fact, we see

$$\begin{aligned} f(x - u, u_1) &= f(x - f(x, u_2)\, u_1 + f(x, u_1)\, u_2, u_1) \\ &= f(x, u_1) - f(x, u_1) \\ &= 0. \end{aligned}$$

Similarly, we also see

$$\begin{aligned} f(x-u,u_2) &= f(x-f(x,u_2)u_1+f(x,u_1)u_2,u_2) \\ &= f(x,u_2)-f(x,u_2) \\ &= 0. \end{aligned}$$

Hence $f(x-u,v)=0, \forall\, v\in U$. Then $x-u\in W$, which gives the result.

Now, the restriction of f on the set W is an alternating bilinear form. By induction, there exists a basis $\mathcal{B}=\{u_3,u_4,...,u_n\}$ of W with

$$\mathcal{M}_f(\mathcal{B})=\begin{pmatrix} \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & & & \\ & \ddots & & & \\ & & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} \in \mathcal{M}_{n-2}(\mathbb{R}).$$

Thus, $u_1,u_2,...,u_n$ is a basis of E for which the matrix representing f has the desired form.

Exercise 11. Let E be a vector space over $\mathbb{R}$ with dimension 2. Let $f\in\mathcal{S}_2(E)$, and let q be the associated quadratic form. Prove that the following three statements are equivalent:

a. f is nondegenerate and there is a nonzero vector e_1 such that $q(e_1)=0$.

b. There exists a basis of E for which the matrix of f is given by

$$A=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

c. There exists a basis of E for which the matrix of f is given by

$$D=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Solution. $(a)\overset{?}{\Rightarrow}(b)$. Since f is nondegenerate and $e_1\neq 0$, there exists a vector $y\in E$ such that

$$f(e_1,y)\neq 0.$$

We put

$$z = \frac{1}{f(e_1, y)} y,$$

so we get

$$f(e_1, z) = f\left(e_1, \frac{1}{f(e_1, y)} y\right) = 1.$$

For the vector $e_2 = z - \frac{1}{2} q(z) e_1$, we find

$$\begin{cases} f(e_1, e_2) = f\left(e_1, z - \frac{1}{2} q(z) e_1\right) = 1 = f(e_2, e_1), \\ f(e_2, e_2) = 0. \end{cases}$$

The family $\{e_1, e_2\}$ is a basis of E. Otherwise, $e_2 = k e_1$ and $f(e_1, e_2) = 0$. Here, the matrix of f is given by

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$(b) \overset{?}{\Rightarrow} (c)$. Conserving the previous notations. The vectors

$$\begin{cases} e_1' = \frac{1}{2} e_1 + e_2, \\ e_2' = \frac{1}{2} e_1 - e_2 \end{cases}$$

satisfy the following equations

$$\begin{cases} q(e_1') = f\left(\frac{1}{2} e_1 + e_2, \frac{1}{2} e_1 + e_2\right) = 1, \\ q(e_2') = f\left(\frac{1}{2} e_1 - e_2, \frac{1}{2} e_1 - e_2\right) = -1, \\ f(e_2', e_1') = f\left(\frac{1}{2} e_1 + e_2, \frac{1}{2} e_1 - e_2\right) = 0. \end{cases}$$

The family $\{e_1', e_2'\}$ is a basis of E. Otherwise, we get $e_2' = \alpha e_1'$, where $\alpha \in \mathbb{R}$. Then

$$-1 = q(e_2') = q(\alpha e_1') = \alpha^2 q(e_1') = \alpha^2.$$

A contradiction. In this basis the matrix of f is given by

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$(c) \overset{?}{\Rightarrow} (a)$. Conserving the previous notations. The quadratic form is nondegenerate since the matrix D is invertible. For the nonzero vector $v' = e'_1 + e'_2$, we have

$$q(v') = f(v', v') = f(e'_1 + e'_2, e'_1 + e'_2) = 0.$$

Exercise 12. Let E be a real vector space and let $a \in E$. Let q be a quadratic form over E with the polar form f. Define the mapping q' from E to $\mathbb{R}$, by setting:

$$\forall\, x \in E,\ q'(x) = q(a)\, q(x) - (f(a, x))^2 .$$

1. Prove that q' is a quadratic form whose polar form f will be specified.
2. Verify that $a \in \ker f'$ and that $\ker f \subset \ker f'$. Deduce the following inclusion set: $\mathbb{R}.a \subset \ker f'$.
3. If a is nonisotropic, i.e., $q(a) \neq 0$, then prove that $\ker f \oplus \mathbb{R} \cdot a = \ker f'$.

Solution.

1. We see that q' is a quadratic form because the mapping

$$\begin{aligned} f' \quad &: \quad E \times E \to \mathbb{R} \\ (x, y) \quad &\mapsto \quad q(a)\, f(x, y) - f(a, x)\, f(a, y) \end{aligned}$$

   is a symmetric bilinear form (since f is also a symmetric bilinear form). Further, $f'(x, x) = q'(x)$ for every $x \in E$.

2. We Verify that $a \in \ker f'$ and $\ker f \subset \ker f'$.

For each $y \in E$, we have

$$f'(a, y) = q(a)\, f(a, y) - f(a, a)\, f(a, y) = 0.$$

Hence, $a \in \ker f'$.

We show that $\ker f \subset \ker f'$. In fact, if $x \in \ker f$, then for each $y \in E$ we have

$$f'(x, y) = q(a) \underbrace{f(x, y)}_{=0} - \underbrace{f(a, x)}_{=0} f(a, y) = 0.$$

Thus, $x \in \ker f'$.

We show that $\mathbb{R}a \subset \ker f'$. Let $\lambda \in \mathbb{R}$. For each $y \in E$, we have

$$\begin{aligned} f'(\lambda a, y) &= f(\lambda a, y)\, q(a) - f(a, \lambda a)\, f(a, y) \\ &= \lambda q(a)\, f(a, y) - \lambda q(a)\, f(a, y) \\ &= 0. \end{aligned}$$

Therefore, $\mathbb{R}a \subset \ker f'$.

If a is nonisotropic, we prove that $\ker f \oplus \mathbb{R}a = \ker f'$.

Since $\ker f$ and $\mathbb{R}.a$ are two subspace of E, then $\{0_E\} \subset \ker f \cap \mathbb{R}a$. If $x \in \ker f \cap \mathbb{R}a$, then $x = \lambda a$ and $f(\lambda a, y) = 0$ for each $y \in E$. That is,

$$f(\lambda a, a) = \lambda q(a) = 0.$$

Hence $\lambda = 0$ (since $q(a) \neq 0$). Which implies $x = 0$. Consequently, $\ker f \cap \mathbb{R}a \subset \{0_E\}$. Finaly, we obtain $\ker f \cap \mathbb{R}a = \{0_E\}$.

For each $x \in \ker f'$ and $y \in E$, we write

$$q(a)\, x = f(a, x)\, a + \underbrace{q(a)\, x - f(a, x)\, a}, \tag{6.3}$$

where $f(a, x)\, a \in \mathbb{R}a$. It suffices to prove that $q(a)\, x - f(a, x)\, a \in \ker f$. In fact, for each $y \in E$, we have

$$\begin{aligned} f(q(a)\, x - f(a, x)\, a, y) &= q(a)\, f(x, y) - f(a, x)\, f(a, y) \\ &= 0 \text{ (since } x \in \ker f'\text{).} \end{aligned}$$

From (6.3), we have

$$x = \underbrace{\frac{f(a, x)}{q(a)} a}_{\in \mathbb{R}a} + \underbrace{\frac{q(a)\, x - f(a, x)\, a}{q(a)}}_{\in \ker f},$$

since $u \in \ker f \Leftrightarrow \alpha u \in \ker f$.

Exercise 13.

1. Let q be a quadratic form on a vector space E and let $\{e_1, e_2, ..., e_n\}$ be a finite orthogonal set for q. Prove the following equality:

$$q(e_1 + e_2 + ... + e_r) = q(e_1) + q(e_2) + ... + q(e_r).$$

2. Let $(E, \langle ., . \rangle)$ be a Hilbert space and let $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of E.

Prove that

$$\forall\, x \in E : x = \sum_{i=1}^{n} \langle x, e_i \rangle\, e_i.$$

3. Let $A = \{u_1, u_2, ..., u_n\}$ be a finite orthonormal set. Show that A is free. Further, for each $x \in E$ prove that the vector

$$y = x - \langle x, u_1 \rangle\, u_1 - \langle x, u_2 \rangle\, u_2 - ... - \langle x, u_n \rangle\, u_n$$

is orthogonal with u_i, for $i = 1, 2, ..., n$.

Solution.

1. Let q be a quadratic form on a vector space E and let $\{e_1, e_2, ..., e_n\}$ be a finite orthogonal set for q. We have

$$\begin{aligned} q\,(e_1 + e_2 + ... + e_r) &= f\,(e_1 + e_2 + ... + e_r, e_1 + e_2 + ... + e_r) \\ &= q\,(e_1) + q\,(e_2) + ... + q\,(e_r) \quad \text{(since } f\,(e_i, e_j) = 0\text{) for } i \neq j\text{)} \end{aligned}$$

2. Let $\{e_1, e_2, ..., e_n\}$ be an arbitrarily orthonormal basis of E. We prove that

$$\forall\, x \in E : x = \sum_{i=1}^{n} \langle x, e_i \rangle\, e_i.$$

For each $x \in E$, we have $x = \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n$. Further, we have

$$\langle x, e_i \rangle = \langle \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n, e_i \rangle = \alpha_i \langle e_i, e_i \rangle = \alpha_i, \tag{6.4}$$

for $i = 1, 2, ..., n$. We replace α_i by $\langle x, e_i \rangle$ in the equation (6.4), we obtain for the desired result.

3. Let $A = \{u_1, u_2, ..., u_n\}$ be a finite orthonormal set. We show that A is free.

For each $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R}$, we have

$$\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n = 0,$$

implies

$$0 = \langle 0, u_i \rangle = \langle \alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n, u_i \rangle = \alpha_i, \forall\, i = 1, 2, ..., n.$$

Further, for each $x \in E$, the vector

$$y = x - \langle x, u_1 \rangle u_1 - \langle x, u_2 \rangle u_2 - ... - \langle x, u_n \rangle u_n$$

is orthogonal with u_i, $i = 1, 2, ..., n$; since

$$\begin{aligned} \langle y, u_i \rangle &= \langle x - \langle x, u_1 \rangle u_1 - \langle x, u_2 \rangle u_2 - ... - \langle x, u_n \rangle u_n, u_i \rangle \\ &= \langle x, u_i \rangle - \langle x, u_i \rangle \underbrace{\langle u_i, u_i \rangle}_{=1} \\ &= 0. \end{aligned}$$

Exercise 14. Let q be a quadratic form over $\mathbb{R}^n$ which has the matrix A in the standard basis, and let $\lambda_{\max}$ be the greatest eigenvalue of A. Prove the following inequality:

$$q(x_1, x_2, ..., x_n) \leq \lambda_{\max} \left(x_1^2 + x_2^2 + ... + x_n^2 \right).$$

Solution. Let $x \in \mathbb{R}^n$ with

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + ... + x_n^2} = 1,$$

and let $\{u_1, u_2, ..., u_n\}$ an orthonormal basis formed by the eigenvectors of A. We have

$$x = \alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n$$

with $\alpha_1^2 + \alpha_2^2 + ... + \alpha_n^2 = 1$, since $\|x\|_2^2 = \langle x, x \rangle = 1$. In this case, we can write

$$\begin{aligned} q(x) &= x^t A x \\ &= (\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n)^t A (\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n) \\ &= \alpha_1^2 u_1^t A u_1 + \alpha_2^2 u_2^t A u_2 + ... + \alpha_n^2 u_n^t A u_n \\ &= \lambda_1 \alpha_1^2 u_1^t u_1 + \lambda_2 \alpha_2^2 u_2^t u_2 + ... + \lambda_n \alpha_n^2 u_n^t u_n \quad (\text{since } A u_i = \lambda_i u_i,\ i = 1, 2, ..., n) \\ &\leq \lambda_{\max} \left(\alpha_1^2 u_1^t u_1 + \alpha_2^2 u_2^t u_2 + ... + \alpha_n^2 u_n^t u_n \right) \\ &= \lambda_{\max} \left(\alpha_1^2 + \alpha_2^2 + ... + \alpha_n^2 \right) \quad (\text{since } u_i^t u_i = 1,\ i = 1, 2, ..., n) \\ &= \lambda_{\max} \quad (\text{since } \alpha_1^2 + \alpha_2^2 + ... + \alpha_n^2 = 1). \end{aligned}$$

Hence, $q(x) \leq \lambda_{\max}$.

Now, for each $x \in \mathbb{R}^n$ we put

$$u = \frac{x}{\|x\|_2} \text{; i.e., } \|u\|_2 = 1.$$

Since $q(u) \leq \lambda_{\max}$, it follows that

$$q\left(\frac{x}{\|x\|_2}\right) = \frac{1}{\|x\|_2} q(x) \leq \lambda_{\max} .$$

Therefore, $q(x) \leq \lambda_{\max} \|x\|_2 = \lambda_{\max} (x_1^2 + x_2^2 + ... + x_n^2)$. This completes the proof.

Exercise 15.

1. Let A be a hermitian matrix, and let

$$\begin{array}{rcl} f & : & \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C} \\ (x, y) & \mapsto & x^t A \overline{y}. \end{array}$$

   Prove that f is a hermitian form.

2. Let f be a sesquilinear Hermitian form over a vector space E. Show that

$$\forall\, x, y, y' \in E, \forall\, \alpha, \beta \in \mathbb{C} : f(x, \alpha y + \beta y') = \overline{\alpha} f(x, y) + \overline{\beta} f(x, y').$$

Notice that if $f : E \times E \to \mathbb{C}$ is linear on the left and semilinear on the right, then f is called "**sesquilinear form**", that is, $\forall\, x, x', y, y' \in E$, $\forall\, \lambda \in \mathbb{C}$:

$$\begin{array}{rcl} \bullet\ f(\lambda x + x', y) & = & \lambda f(x, y) + f(x', y) \\ \bullet\ f(x, \lambda y + y') & = & \overline{\lambda} f(x, y) + f(x, y'). \end{array}$$

A hermitian sesquilinear form is a sesquilinear form f over E satisfying

$$f(x, y) = \overline{f(y, x)} \text{ for all } x, y \in E.$$

1. Let A be a hermitian matrix. We prove that the mapping

$$\begin{array}{rcl} f & : & \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C} \\ (x, y) & \mapsto & x^t A \overline{y} \end{array}$$

   is a hermitian form. (i.e., f is hermitian sesquilinear form). In fact, for every $x, x', y \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$, we have

$$\begin{array}{rcl} f(\lambda x + x', y) & = & (\lambda x + x')^t A \overline{y} \\ & = & \lambda x^t A \overline{y} + (x')^t A \overline{y} \\ & = & \lambda f(x, y) + f(x', y). \end{array}$$

Thus, f is linear from the left. Similarly, for every $x, y, y' \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} f(x, \lambda y + y') &= x^t A \overline{(\lambda y + y')} \\ &= \lambda x^t A \overline{y} + x^t A \overline{(y)'} \\ &= \lambda f(x, y) + f(x, y'). \end{aligned}$$

Thus, f is semi-linear from the right.

Moreover, for each $x, y \in \mathbb{C}^n$, we have

$$\begin{aligned} \overline{f(x, y)} &= \overline{x^t A \overline{y}} \\ &= \overline{(x^t A \overline{y})^t} \quad \text{(since } x^t A \overline{y} \in \mathbb{C}\text{)} \\ &= \overline{(\overline{y})^t A^t x} \\ &= y^t \overline{(A^t)} \overline{x} \\ &= y^t A^* \overline{x} \\ &= y^t A \overline{x} \quad \text{(since } A \text{ is hermitian)} \\ &= f(y, x). \end{aligned}$$

2. Let f be a sesquilinear hermitian form over a vector space E. We show that

$$\forall x, y, y' \in E, \forall \alpha, \beta \in \mathbb{C} : f(x, \alpha y + \beta y') = \overline{\alpha} f(x, y) + \overline{\beta} f(x, y').$$

In fact, we have

$$\begin{aligned} \forall\, x, y, y' &\in E, \forall\, \alpha, \beta \in \mathbb{C} : f(x, \alpha y + \beta y') = \overline{f(\alpha y + \beta y', x)} \\ &= \overline{\alpha f(y, x)} + \overline{\beta f(y', x)} \\ &= \overline{\alpha}\, \overline{f(y, x)} + \overline{\beta}\, \overline{f(y', x)} \\ &= \overline{\alpha} f(x, y) + \overline{\beta} f(x, y'). \end{aligned}$$

Exercise 16. Let

$$\begin{aligned} f &: \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C} \\ (x, y) &\mapsto 4x_1\overline{y_1} + (2 - i)\, x_1\overline{y_2} + (2 + i)\, x_2\overline{y_1} - 5x_2\overline{y_2}. \end{aligned}$$

Show that f is a hermitian sesquilinear form. Calculate $f(x, x)$, where $x \in \mathbb{C}^2$.

Solution. We write f in the form

$$\begin{aligned} f(x,y) &= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 4 & 2-i \\ 2+i & -5 \end{pmatrix} \begin{pmatrix} \overline{y_1} \\ \overline{y_2} \end{pmatrix} \\ &= X^t A \overline{Y}, \text{ where } X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \end{aligned}$$

Since $A^* = A$, then A is Hermitian. Then f is a Hermitian sesquilinear form.

Calculate $f(x,x)$, where $x \in \mathbb{C}^2$. In fact, we have

$$f(x,x) = 4|x_1|^2 + (2-i)\, x_1\overline{x_2} + (2+i)\, x_2\overline{x_1} - 5|x_2|^2.$$

Exercise 17.

1. Diagonalize the following Hermitian quadratic forms:

i) $q_1 = ix_1\overline{x_2} - ix_2\overline{x_1}$, $E = \mathbb{C}^2$.

ii) $q_2 = x_1\overline{x_1} + ix_1\overline{x_2} - ix_2\overline{x_1} + x_2\overline{x_2}$, $E = \mathbb{C}^2$.

iii) $q_3 = x_1\overline{x_1} + a_{12}x_1\overline{x_2} + a_{21}x_2\overline{x_1} + a_{22}x_2\overline{x_2}$.

2. Deduce the signature of the quadratic form given by:

$$q_2' = \alpha x_1\overline{x_1} + ix_1\overline{x_2} - ix_2\overline{x_1} + x_2\overline{x_2},\ \alpha \in \mathbb{R}.$$

Solution.

- We can write

$$\begin{aligned} q_1 &= ix_1\overline{x_2} - ix_2\overline{x_1} \\ &= x_1\left(i\overline{x_2}\right) + \overline{x_1}\left(-ix_2\right) \\ &= x_1\left(\overline{-ix_2}\right) + \overline{x_1}\left(-ix_2\right) \quad \text{(which is of the form } z_1\overline{z_2} + \overline{z_1}z_2\text{)} \\ &= \frac{1}{2}|x_1 - ix_2|^2 - \frac{1}{2}|x_1 + ix_2|^2 \quad \left(\text{since } z_1\overline{z_2} + \overline{z_1}z_2 = \frac{1}{2}|z_1+z_2|^2 - \frac{1}{2}|z_1 - z_2|^2\right) \\ &= |f_1|^2 - |f_2|^2, \end{aligned}$$

where f_1 et f_2 are linearly independent forms over $\mathbb{C}^2$; since

$$\begin{vmatrix} 1 & -i \\ 1 & i \end{vmatrix} \neq 0.$$

The signature of q_1 is $(1,1)$.

- Likewise, we have

$$\begin{aligned} q_2 &= x_1\overline{x_1} + ix_1\overline{x_2} - ix_2\overline{x_1} + x_2\overline{x_2} \\ &= (x_1 - ix_2)(\overline{x_1} + i\overline{x_2}) \\ &= (x_1 - ix_2)\overline{(x_1 - ix_2)} \\ &= |x_1 - ix_2|^2 \\ &= |f_1|^2 . \end{aligned}$$

The signature of q_2 is $(1,0)$.

For the quadratic form $q_2' = \alpha x_1\overline{x_1} + ix_1\overline{x_2} - ix_2\overline{x_1} + x_2\overline{x_2}$, $\alpha \in \mathbb{R}$. We see that

$$\begin{aligned} q_2' &= (\alpha - 1)\, x_1\overline{x_1} + q_2 \\ &= (\alpha - 1)|x_1|^2 + |x_1 - ix_2|^2 . \end{aligned}$$

We deduce that

$$\begin{cases} \alpha = 1, \text{ the signature is } (1,0). \\ \alpha > 1, \text{ the signature is } (2,0). \\ \alpha < 1, \text{ the signature is } (1,1). \end{cases}$$

- We have

$$\begin{aligned} q_3 &= x_1\overline{x_1} + a_{12}x_1\overline{x_2} + a_{21}x_2\overline{x_1} + a_{22}x_2\overline{x_2} \\ &= (x_1 + a_{21}x_2)(\overline{x_1} + a_{12}\overline{x_2}) + (a_{22} - a_{12}a_{21})\, x_2\overline{x_2} \\ &= (x_1 + a_{21}x_2)\overline{(x_1 + a_{21}x_2)} + (a_{22} - a_{12}a_{21})\, x_2\overline{x_2} \\ &= |x_1 + a_{21}x_2|^2 + \underbrace{(a_{22} - a_{12}a_{21})}_{\in\mathbb{R}} |x_2|^2 . \end{aligned}$$

Exercise 18. Let E be a vector space with $\dim E = 3$, and let q be the hermitian quadratic form over E given by

$$q = a_{12}x_1\overline{x_2} + a_{13}x_1\overline{x_3} + \overline{a_{12}}x_2\overline{x_1} + a_{23}x_2\overline{x_3} + \overline{a_{13}}x_3\overline{x_1} + \overline{a_{23}}x_3\overline{x_2}.$$

Give the diagonal form of q using Gauss method.

Solution. We have

$$\begin{aligned}
q &= a_{12}x_1\overline{x_2}+a_{13}x_1\overline{x_3}+\overline{a_{12}}x_2\overline{x_1}+a_{23}x_2\overline{x_3}+\overline{a_{13}}x_3\overline{x_1}+\overline{a_{23}}x_3\overline{x_2}\\
&= x_1\left(a_{12}\overline{x_2}+a_{13}\overline{x_3}\right)+\overline{x_1}\left(\overline{a_{12}}x_2+\overline{a_{13}}x_3\right)+a_{23}x_2\overline{x_3}+\overline{a_{23}}x_3\overline{x_2}\\
&= x_1\left(a_{12}\overline{x_2}+a_{13}\overline{x_3}\right)+a_{23}x_2\overline{x_3}+\overline{x_1}\left(\overline{a_{12}}x_2+\overline{a_{13}}x_3\right)+\overline{a_{23}}x_3\overline{x_2}\\
&= x_1\left(a_{12}\overline{x_2}+a_{13}\overline{x_3}\right)+\frac{a_{23}}{a_{13}}x_2\left(a_{12}\overline{x_2}+a_{13}\overline{x_3}\right)-\frac{a_{23}a_{12}}{a_{13}}x_2\overline{x_2}+\\
&\quad \underbrace{\overline{x_1}\left(\overline{a_{12}}x_2+\overline{a_{13}}x_3\right)+\frac{\overline{a_{23}}}{\overline{a_{13}}}\overline{x_2}\left(\overline{a_{12}}x_2+\overline{a_{13}}x_3\right)-\frac{\overline{a_{23}a_{12}}}{\overline{a_{13}}}x_2\overline{x_2}}\\
&= \left(x_1+\frac{a_{23}}{a_{13}}x_2\right)\left(a_{12}\overline{x_2}+a_{13}\overline{x_3}\right)+\left(\overline{x_1}+\frac{\overline{a_{23}}}{\overline{a_{13}}}\overline{x_2}\right)\left(\overline{a_{12}}x_2+\overline{a_{13}}x_3\right)-\left(\frac{a_{23}a_{12}}{a_{13}}+\frac{\overline{a_{23}a_{12}}}{\overline{a_{13}}}\right)x_2\overline{x_2}\\
&= \frac{1}{2}\left|x_1+\left(\frac{a_{23}}{a_{13}}+\overline{a_{12}}\right)x_2+\overline{a_{13}}x_3\right|^2-\frac{1}{2}\left|x_1+\left(\frac{a_{23}}{a_{13}}-\overline{a_{12}}\right)x_2-\overline{a_{13}}x_3\right|^2-2Re\left(\frac{a_{23}a_{12}}{a_{13}}\right)|x_2|^2.
\end{aligned}$$

Exercise 19. Diagonalize the Hermitian quadratic form given by its matrix:

$$M_q=\begin{pmatrix}0 & 1-i & 0\\ 1+i & 0 & i\\ 0 & -i & 0\end{pmatrix}.$$

Here, M_q is the matrix of the Hermitian quadratic form q with respect to the standard basis of $\mathbb{C}^3$.

Solution. We have

$$\begin{aligned}
q &= (1-i)\,x_1\overline{x_2}+(1+i)\,x_2\overline{x_1}+ix_2\overline{x_3}-ix_3\overline{x_2}\\
&= x_2\left[(1+i)\,\overline{x_1}+i\overline{x_3}\right]+\overline{x_2}\left[(1-i)\,x_1-ix_3\right]\\
&= x_2\overline{\left[(1-i)\,x_1-ix_3\right]}+\overline{x_2}\left[(1-i)\,x_1-ix_3\right]\quad \text{(which is of the form } z_1\overline{z_2}+\overline{z_1}z_2\text{)}\\
&= \frac{1}{2}\left|x_2+(1-i)\,x_1-ix_3\right|^2-\frac{1}{2}\left|x_2-(1-i)\,x_1+ix_3\right|^2\\
&= |f_1|^2-|f_2|^2.
\end{aligned}$$

The signature is $(1,1)$.

Exercise 20. Let

$$B=\begin{pmatrix}1 & 0 & b\\ 0 & a+i & \overline{a}\\ \overline{b} & b+1 & b-ai\end{pmatrix},\ a,b\in\mathbb{C}.$$

For which values of the parameters a and b is the matrix B Hermitian? In the case when B is Hermitian, find its Hermitian quadratic form.

Solution. The matrix B is Hermitian if and only if

$$B = \left(\overline{B}\right)^t \Leftrightarrow \begin{pmatrix} 1 & 0 & b \\ 0 & a+i & \overline{a} \\ \overline{b} & b+1 & b-ai \end{pmatrix} = \begin{pmatrix} 1 & 0 & b \\ 0 & \overline{a}-i & \overline{b}+1 \\ \overline{b} & a & \overline{b}+\overline{a}i \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} a+i \in \mathbb{R} \\ b+1 = a \\ b-ai \in \mathbb{R} \end{cases}$$

$$\Leftrightarrow \begin{cases} a = \alpha - i, \text{ where } \alpha \in \mathbb{R} \\ b = \alpha - 1 - i \\ \alpha - 1 - i - (\alpha - i)\, i \in \mathbb{R} \end{cases}$$

$$\Leftrightarrow \begin{cases} a = \alpha - i, \text{ where } \alpha \in \mathbb{R} \\ b = (\alpha - 1) - i \\ \alpha - 2 - (1+\alpha)\, i \in \mathbb{R} \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha = -1, \\ a = -1 - i \\ b = -2 - i. \end{cases}$$

Therefore,

$$B = \begin{pmatrix} 1 & 0 & -2-i \\ 0 & -1 & -1+i \\ -2+i & -1-i & -3 \end{pmatrix}.$$

Now, we give the analytic expression of the corresponding Hermitian quadratic form of B. (see (**??**)):

$$q_B = x_1\overline{x_1} + (-2-i)\, x_1\overline{x_3} - x_2\overline{x_2} + (-1+i)\, x_2\overline{x_3} + (-2+i)\, x_3\overline{x_1} + (-1-i)\, x_3\overline{x_2} - 3x_3\overline{x_3}.$$

Exercise 21 Prove that every real quadratic form $q = x^t A x$ is diagonalizable.

Further, prove that if q is definite positive, then the integral

$$I = \int\int ... \int_{\mathbb{R}^n} e^{-q(x_1,x_2,...,x_n)} dx_1 dx_2 ... dx_n$$

converges and calculate its value[1].

[1]Use the following well-known formula: $\int_{-\infty}^{+\infty} e^{-t^2} = \sqrt{\pi}$.

Solution. Let

$$\begin{aligned} q &= x^t A x \\ &= \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ & & & \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \end{aligned}$$

be quadratic form over $\mathbb{R}^n$. We prove that q is diagonalizable. However, since A is symmetric, there exists an orthogonal matrix P such that $A = PDP^t$, where

$$D = diag\{\lambda_1, \lambda_2, ..., \lambda_n\}.$$

It follows that

$$q = x^t A x = x^t \left(PDP^t\right) x = \left(x^t P\right) D \left(P^t x\right) = \left(P^t x\right)^t DP^t x.$$

Setting

$$P^t x = v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix},$$

Implies

$$\begin{aligned} q &= v^t D v \\ &= \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \\ &= \lambda_1 v_1^2 + \lambda_2 v_2^2 + ... + \lambda_n v_n^2, \end{aligned}$$

where $(\lambda_i)_{i=1,2,...,n}$ are the eigenvalues of A. Further, suppose that q is definite positive, i.e.,

$\lambda_i > 0$ for every $i = 1, 2, ..., n$. Then

$$\begin{aligned} I &= \int\int ... \int_{\mathbb{R}^n} e^{-q(x_1,x_2,...,x_n)} dx_1 dx_2 ... dx_n \\ &= \alpha_J \int\int ... \int_{\mathbb{R}^n} e^{-\left(\lambda_1 v_1^2 + \lambda_2 v_2^2 + ... + \lambda_n v_n^2\right)} dv_1 dv_2 ... dv_n, \text{ where } \alpha_J \in \mathbb{R}^* \\ &= \frac{\alpha_J}{\sqrt{\lambda_1 \lambda_2 ... \lambda_n}} \left(\int_{-\infty}^{+\infty} e^{-t^2} dt \right)^n \\ &= \frac{\alpha_J}{\sqrt{\lambda_1 \lambda_2 ... \lambda_n}} \left(\sqrt{\pi}\right)^n. \end{aligned}$$

Note that

$$\boxed{dv_1 dv_2 ... dv_n = \frac{1}{\alpha_J} dx_1 dx_2 ... dx_n}$$

Exercise 22. Let $q = x^t A x$ be a quadratic form over the vector space $\mathbb{R}^n$. Prove that

$$q \text{ is nondegenerate} \Leftrightarrow \det(A) \neq 0 \ (\textit{i.e.}, A \text{ is invertible}).$$

Solution. By Definition 3.6, recall that $\ker f = \{x \in E;\ x^t A y = 0 \text{ for each } y \in E\}$. Then

$$\begin{aligned} \ker f &= \{0\} \Leftrightarrow \forall\, y \in \mathbb{R}^n : x^t A y = 0 \Rightarrow x = 0 \\ &\Leftrightarrow \forall\, y \in \mathbb{R}^n : y^t A^t x = 0 \Rightarrow x = 0 \\ &\Leftrightarrow A^t x = 0 \Rightarrow x = 0 \quad ; \text{since } \left(\forall\, y \in \mathbb{R}^n : y^t A^t x = 0\right) \Leftrightarrow A^t x = 0 \\ &\Leftrightarrow A^t \in \mathbb{GL}_n(\mathbb{R}) \\ &\Leftrightarrow A \in \mathbb{GL}_n(\mathbb{R}). \end{aligned}$$

Exercise 23. Let $E = \mathbb{R}_2[x]$ be the vector space of polynomials having degree ≤ 2, and let

$$\begin{aligned} Q &: E \to \mathbb{R} \\ p &\mapsto p(0)\, p(1). \end{aligned}$$

1. Prove that Q is a quadratic form, and then give its polar form f.
2. Determine $\mathcal{M}_Q(\mathcal{B})$, where $\mathcal{B}$ is the canonical basis of E.
3. Prove that f is degenerate. Is it positive ?, definite positive?, negative?, definite negative?

Solution.

1. From simple computation, the polar form of Q is given by

$$\begin{array}{rcl} f & : & \mathbb{R}_2[x]\ \mathbb{R}_2[x] \to \mathbb{R} \\ (p,q) & \mapsto & f(p,q) = \dfrac{1}{4}\left(Q(p+q) - Q(p-q)\right) \\ & = & \dfrac{1}{2}p(0)\,q(1) + \dfrac{1}{2}p(1)\,q(0)\,. \end{array}$$

2. Calculate the matrix $\mathcal{M}_Q(\mathcal{B})$, where $\mathcal{B}$ is the canonical basis of $\mathbb{R}_2[x]$. We have

$$f(1,1) = 1, f(1,x) = \frac{1}{2}, f\left(1,x^2\right) = \frac{1}{2}, f\left(x,x^2\right) = 0, f\left(x^2,x^2\right) = 0.$$

Therefore,

$$\mathcal{M}_Q(\mathcal{B}) = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

3. Since $\det(\mathcal{M}_Q(\mathcal{B})) = 0$, then f is degenerate.

Further, Q neither positive nor negative; since

$$\begin{cases} Q(2x-1) = (-1)\times 1 = -1 < 0, \\ Q(-x-2) = (-2)\times(-3) = 6 > 0. \end{cases}$$

Remark 6.1. The eigenvectors of $\mathcal{M}_Q(\mathcal{B})$ are $\frac{1}{2}\sqrt{3}+\frac{1}{2}, \frac{1}{2}-\frac{1}{2}\sqrt{3}, 0$. Then $\mathcal{M}_Q(\mathcal{B})$ is neither positive nor negative.

Exercise 24. Let $(E, \langle .,. \rangle)$ an inner product space (a pre-Hilbert space) and let F be a subspace of E. Prove that $F \subset \left(F^{\perp}\right)^{\perp}$ and so $F = \left(F^{\perp}\right)^{\perp}$ whenever E has finite dimension.

Solution. We have

$$\begin{cases} F^{\perp} = \{x \in E\ ;\ \langle x,y \rangle = 0 \text{ for each } y \in F\}, \\ \left(F^{\perp}\right)^{\perp} = \left\{x \in E\ ;\ \langle x,y \rangle = 0 \text{ for each } y \in F^{\perp}\right\}. \end{cases}$$

We prove that $F \subset \left(F^{\perp}\right)^{\perp}$. Let $x_0 \in F$. Assume that $x_0 \notin \left(F^{\perp}\right)^{\perp}$, there exists $y_0 \in F^{\perp}$ such that $\langle x_0, y_0 \rangle \neq 0$. But, $\langle x, y_0 \rangle = 0$ for every $x \in F$. A contradiction.

Next, assume that E is a finite dimension space. Since $E = F \oplus F^{\perp} = F^{\perp} \oplus \left(F^{\perp}\right)^{\perp}$, by (1.1) we get

$$\dim F + \dim F^{\perp} = \dim E \text{ et } \dim\left(F^{\perp}\right)^{\perp} + \dim F^{\perp} = \dim E.$$

which gives $\dim F = \dim\left(F^{\perp}\right)^{\perp}$. Moreover, since $F \subset \left(F^{\perp}\right)^{\perp}$, we have $F = \left(F^{\perp}\right)^{\perp}$.

Exercise 25. Let φ be the mapping defined on the vector space $E = \mathbb{R}_n[x]$ by

$$\varphi(P,Q) = \int_a^b P(t)\,Q(t)\,dt, \text{ where } a < b.$$

1. Prove that φ is an inner product (a scalar product).
2. For $n = 2$, calculate $\mathcal{M}_\varphi(\mathcal{B})$ "this is the matrix of φ in the standard basis of $\mathbb{R}_2[x]$".
3. Apply Cauchy-Schwarz's inequality.

Solution. We prove that φ is an inner product. That is, φ is a symmetric bilinear form definite positive.

For each $(P, Q, P_1, Q_1) \in E^4$ and for each $\lambda \in \mathbb{R}$, we have

$$\begin{aligned}
\varphi(\lambda P + P_1, Q) &= \int_a^b (\lambda P + P_1)(t)\,Q(t)\,dt \\
&= \int_a^b (\lambda P(t)\,Q(t) + P_1(t)\,Q(t))\,dt \\
&= \lambda \int_a^b P(t)\,Q(t)\,dt + \int_a^b P_1(t)\,Q(t)\,dt \\
&= \lambda\varphi(P,Q) + \varphi(P_1,Q),
\end{aligned}$$

and also, we have

$$\begin{aligned}
\varphi(P, \lambda Q + Q_1) &= \int_a^b P(t)\,(\lambda Q + Q_1)(t)\,dt \\
&= \int_a^b P(t)\,(\lambda Q(t) + Q_1(t))\,dt \\
&= \lambda \int_a^b P(t)\,Q(t)\,dt + \int_a^b P(t)\,Q_1(t)\,dt \\
&= \lambda\varphi(P,Q) + \varphi(P,Q_1).
\end{aligned}$$

Then φ is a bilinear form. Further, φ is symmetric since for each $(P,Q) \in E^2$, one has

$$\varphi(P,Q) = \int_a^b P(t)\,Q(t)\,dt = \int_a^b Q(t)\,P(t)\,dt = \varphi(Q,P).$$

For all $P \in E - \{0\}$, we have $\varphi(P,P) = \int_a^b P^2(t)\,dt > 0$. Then φ is definite positive. Hence φ is an inner product.

Now, we calculate $M_\varphi(\mathcal{B})$:

$$\begin{aligned} M_\varphi(\mathcal{B}) &= \begin{pmatrix} \varphi(1,1) & \varphi(1,t) & \varphi(1,t^2) \\ \varphi(1,t) & \varphi(t,t) & \varphi(t,t^2) \\ \varphi(1,t^2) & \varphi(t,t^2) & \varphi(t^2,t^2) \end{pmatrix} \\ &= \begin{pmatrix} \int_a^b dt & \int_a^b tdt & \int_a^b t^2dt \\ \int_a^b tdt & \int_a^b t^2dt & \int_a^b t^3dt \\ \int_a^b t^2dt & \int_a^b t^3dt & \int_a^b t^4dt \end{pmatrix} \\ &= \begin{pmatrix} \frac{b-1}{1} & \frac{b^2-a^2}{2} & \frac{b^3-a^3}{3} \\ \frac{b^2-a^2}{2} & \frac{b^3-a^3}{3} & \frac{b^4-a^4}{4} \\ \frac{b^3-a^3}{3} & \frac{b^4-a^4}{4} & \frac{b^5-a^5}{5} \end{pmatrix}. \end{aligned}$$

- From Cauchy-Schwarz inequality, for each $(P,Q) \in E^2$, we have

$$|\varphi(P,Q)|^2 = |\langle P,Q\rangle|^2 \leq \langle P,P\rangle \langle Q,Q\rangle$$

That is,

$$\left|\int_a^b P(t)\,Q(t)\,dt\right|^2 \leq \int_a^b P^2(t)\,dt \int_a^b Q^2(t)\,dt.$$

Exercise 26. Let A be a symmetric matrix with real entries. Prove that the quadratic form $q = x^tAx$ is definite positive if and only, if the eigenvalues of A are strictly positive.

Solution. Let $q = x^tAx$ be quadratic form definite positive, where $A \in \mathcal{S}_n(\mathbb{R})$ and let (λ, x) be eigenpair of A. Since $x \neq 0$, it follows that

$$0 < x^tAx = \langle x, Ax\rangle = \langle x, \lambda x\rangle = \lambda \underbrace{\langle x, x\rangle}_{>0;\text{ since } x\neq 0.} \Leftrightarrow \lambda > 0; \text{because } \lambda \in \mathbb{R}\ (A \text{ is symmetric}).$$

Exercise 27. Let f be a bilinear form on a vector space E. Show that the mapping:

$$\begin{aligned} q &: E \to \mathbb{R} \\ x &\mapsto f(x,x) \end{aligned}$$

is a quadratic form.

Solution. Let f be a bilinear form over E. Clearly, the mapping

$$\begin{aligned} \varphi &: E \times E \to \mathbb{R} \\ (x,y) &\mapsto \varphi(x,y) = \frac{f(x,y) + f(y,x)}{2} \end{aligned}$$

is symmetric bilinear form. Further, for each $x \in E$ we have $\varphi(x,x) = f(x,x) = q(x)$. Then q is a quadratic form over E.

Exercise 28. Let q be a quadratic form over E. Prove that two vectors x and y satisfying $q(x)q(y) < 0$ are independent.

Solution. Assume, by the way of contradiction that x, y are dependent. Since x and y are nonzero (otherwise, if x or y is zero then $q(x)q(y) = 0$), there exists $\lambda \in \mathbb{R}^*$ such that $y = \lambda x$. By (3.5), $q(x)q(y) = \lambda^2 (q(x))^2 > 0$, this contradicts our assumption.

Exercise 29. Diagonalize the quadratic form

$$q(x,y) = ax^2 + 2bxy + cy^2.$$

Deduce its signature.

Solution. Note that

$$q(x,y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

1. Assume that $a \neq 0$. By (3.11), we put

$$x = x' - \frac{1}{a}(by') \text{ and } y = y'.$$

It follows that

$$\begin{aligned} q(x,y) &= ax^2 + bxy + cy^2 \\ &= a\left(x' - \frac{b}{a}y'\right)^2 + 2b\left(x' - \frac{b}{a}y'\right)y' + c(y')^2 \\ &= a(x')^2 + \left(c - \frac{b^2}{a}\right)(y')^2 \\ &= a(x + \frac{b}{a}y)^2 + \left(c - \frac{b^2}{a}\right)y^2 \\ &= a \cdot |f_1|^2 + \left(c - \frac{b^2}{a}\right) \cdot |f_2|^2, \end{aligned}$$

where $f_1 = x + \frac{b}{a}y$ and $f_2 = y$ are two independent linear forms over $\mathbb{R}^2$, since

$$\begin{vmatrix} 1 & 0 \\ \frac{b}{a} & 1 \end{vmatrix} \neq 0.$$

- If $a, c - \frac{b^2}{a} > 0$, then the signature of q is $(2, 0)$.

- If $a, c - \frac{b^2}{a} < 0$, then the signature of q is $(0, 2)$.
- If $a > 0$ and $c - \frac{b^2}{a} < 0$ or $a < 0$ and $c - \frac{b^2}{a} > 0$, then the signature of q is $(1, 1)$.

2. Assume that $a = 0$ and $b \neq 0$. There are two cases.

2.1. For $c = 0$, we let

$$x = x' + y' \text{ and } y = x' - y',$$

which implies

$$\begin{aligned} q(x, y) &= 2bxy = 2b(x' + y')(x' - y') \\ &= 2b(x')^2 - 2b(y')^2 = 2b(\frac{x+y}{2})^2 - 2b(\frac{x-y}{2})^2. \end{aligned}$$

2.2. For $c \neq 0$, we let

$$y = u - \frac{1}{c}(bv) \text{ and } x = v.$$

It follows that

$$\begin{aligned} q &= 2bxy + cy^2 \\ &= 2bv\left(u - \frac{b}{c}v\right) + c\left(u - \frac{b}{c}v\right)^2 \\ &= cu^2 - \frac{b^2}{c}v^2 \\ &= c\left(y + \frac{b}{c}x\right)^2 - \frac{b^2}{c}x^2. \end{aligned}$$

Exercise 30. Let $A \in \mathcal{M}_n(\mathbb{R})$ be a symmetric definite positive matrix. Using two methods, prove that $\det(A)$ is strictly positive.

Solution. 1st method. We show that A is definite positive $\Leftrightarrow \forall\, \lambda \in Sp(A) : \lambda > 0$. In fact, if A is definite positive, for each eigenpair (λ, x) of A we have

$$0 < x^t Ax = \langle x, Ax\rangle = \langle x, \lambda x\rangle = \lambda \underbrace{\langle x, x\rangle}_{>0,\text{ since } x\neq 0.} \Leftrightarrow \lambda > 0, \text{ since } \lambda \in \mathbb{R} \ (A \text{ is symmetric}).$$

It follows that

$$\det(A) = \prod_{\lambda\in Sp(A)} \lambda > 0.$$

2nd method. In the case when A is symmetric definite positive, we deduce from Theorem 2.5 that $A = M^t M$, where $M \in \mathbb{GL}_n(\mathbb{R})$. Hence, $\det(A) = \det(M^t M) = (\det(M))^2 > 0$ (note that $\det(M) = \det(M^t)$ and $\det M \neq 0$ since M is invertible).

Remark 6.2. Let $A \in \mathcal{M}_n(\mathbb{C})$ be a hermitian definite positive matrix. Then $\det(A) \in \mathbb{R}_+^*$.

Exercise 31. Let E be a real vector space and let q be a nondegenerate quadratic form over E of the polar form f. Let $a \in E$ be a nonisotropic vector. Define the mapping:

$$\begin{array}{rcl} S_a & : & E \to E \\ x & \mapsto & S_a(x) = x - 2\dfrac{f(x,a)}{q(a)}a. \end{array}$$

1. Verify the equality

$$f(S_a(x), S_a(y)) = f(x,y) \text{ for any } (x,y) \in E^2.$$

2. Let x_1 and x_2 be two vectors of E such that $q(x_1) = q(x_2) \neq 0$. Prove that at least one of the vectors $x_1 + x_2$ and $x_1 - x_2$ is nonisotropic (use the way of contradiction).
3. Deduce that there exists a nonisotropic vector $a' \in E$ such that

$$S_{a'}(x_1) = -x_2 \text{ or } S_{a'}(x_1) = x_2.$$

Solution.

1. For any $(x,y) \in E^2$, we see that

$$\begin{aligned} & f(S_a(x), S_a(y))\, f\left(x - 2\frac{f(x,a)}{q(a)}a, y - 2\frac{f(x,a)}{q(a)}a\right) \\ = \;& f(x,y) - \frac{2f(y,a)\,f(x,a)}{q(a)} - \frac{2f(x,a)\,f(a,y)}{q(a)} + 4\frac{f(x,a)\,f(y,a)}{q(a)} = f(x,y). \end{aligned}$$

2. Assume that both $x_1 + x_2$ and $x_1 + x_2$ are isotropic. Therefore,

$$\begin{cases} f(x_1 + x_2, x_1 + x_2) = 0, \\ f(x_1 - x_2, x_1 - x_2) = 0. \end{cases}$$

Implies

$$\begin{cases} q(x_1) + q(x_2) + 2f(x_1, x_2) = 0, \\ q(x_1) + q(x_2) - 2f(x_1, x_2) = 0. \end{cases}$$

So, $4q(x_1) = 4q(x_2) = 0$. This is a contradiction.

3. In the case when $x_1 + x_2 = a'$ is nonisotropic, we get

$$\begin{aligned} S_{a'}(x_1) &= S_{x_1+x_2}(x_1) = x_1 - 2\frac{f(x_1, x_1+x_2)}{q(x_1+x_2)}(x_1+x_2) \\ &= x_1 - \frac{2(q(x_1)+f(x_1,x_2))}{q(x_1)+q(x_2)+2f(x_1,x_2)}(x_1+x_2) \\ &= x_1 - \frac{q(x_1)+f(x_1,x_2)}{q(x_1)+f(x_1,x_2)}(x_1+x_2) \\ &= -x_2. \end{aligned}$$

Similarly, in the case when $x_1 - x_2 = a'$ is nonisotropic, we can prove that $S_a(x_1) = x_1$. Indeed, we have

$$\begin{aligned} S_{a'}(x_1) &= S_{x_1-x_2}(x_1) = x_1 - 2\frac{f(x_1, x_1-x_2)}{q(x_1-x_2)}(x_1-x_2) \\ &= x_1 - \frac{2(q(x_1)-f(x_1,x_2))}{q(x_1)+q(x_2)-2f(x_1,x_2)}(x_1+x_2) \\ &= x_1 - \frac{q(x_1)-f(x_1,x_2)}{q(x_1)-f(x_1,x_2)}(x_1-x_2) \\ &= x_2. \end{aligned}$$

As required.

Exercise 32. Compute the signature of each of the following quadratic forms:

1. (i) $q = 2x_1x_3 + 2x_2x_3$.

   (ii) $q = \sum_{i,j}^{n} ij \cdot x_ix_j$, $n = 1, 2, ...$

Solution. (i) See Example 3.11.
(ii) We see that

$$q = \sum_{i,j}^{n} ij \cdot x_ix_j = (x_1 + 2x_2 + 3x_3 + \ldots + nx_n)^2,$$

and so, the signature of this quadratic form is $(1, 0)$.

Conclusion

Quadratic forms have many applications in cryptography. In the context, it is very interesting to know large prime numbers which are represented by some special quadratic forms. For example, it is well known that every prime number of the form $4k + 1$ can be represented by the quadratic form $q = x^2 + y^2$. Currently, there are many open problems on the distribution of values of quadratic forms, some others include quadratic forms involving systems of forms having k-tuple of variables. For more information, see the paper *Ten problems on quadratic forms* stated in [7]. In addition, in sesquilinear algebra, the study of Hermitian spaces is the basic of Hermitian Geometry.

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