People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research
University of 8 Mai 1945 Guelma

جامعة 8 ماي 1945 قالمة
UNIVERSITE 8 MAI 1945 GUELMA

Faculty: Mathematics, Computer Science and Science of Matter
Department of Mathematics

Master thesis

Submitted in Candidacy for the Degree of Academic Master in Mathematics

Specialty : Partial Differential Equations and Numerical Analysis

Presented by:
Sifeddine AOUISSI

Title

Novel Existence and Uniquenss results for Seqential Fractional Neutral Functional Differential Equations

Defended on: 24/06/2024

Before the jury composed of:

Full name	Rank	University	
Mr Hamid BOULARES	PROF	Univ. of 08 MAI 1945 GUELMA	President
Mr Rabah DEBBAR	MCA	Univ. of 08 MAI 1945 GUELMA	Supervisor
Mr Tarek BAHLOUL	MCA	Univ. of 08 MAI 1945 GUELMA	Examiner
Mr Ghania REBAI	MCB	Univ. of 08 MAI 1945 GUELMA	Examiner

Academic year: 2023/2024

Dedications

To my dear mother Nadia

Whatever I do or say, I will not be able to thank you properly, your affection covers me, your kindness guides me and your presence at my side has always been my source of strength to face different obstacles.

To my dear father Abd Rezzak

You have always been by my side to support and encourage me. May this work reflect my gratitude and affection

To my brother Ahmed Karim and my sisters, Marwa and Ruqaiya May God protect them and give them luck and happiness.

Thanks

First of all, I would like to thank God the Almighty and Merciful, who gave me the strength, courage and patience to accomplish this modest work.

I would like to thank my dissertation supervisor. Mr. RABAH DEBBAR. Head of department mathematics at university May 8, 1945 - Guelma, for all the support she gave me throughout the conception and writing of this dissertation. He knew how to lead me in an exemplary way. Without his ideas and expertise, the realization of this memory would not have been possible.

We would also like to thank all the members of the jury Mr Hamid BOULARES for having agree to chair and judge this work Mr Tarek BAHLOUL and Ms Ghania REBAI for having agreed to examine

and judge this work. We extend our sincere thanks to all Our teachers and all people who, through their words, advice and criticism, have guided our thoughts while throughout our studies during our university career. We would like to thank our dear colleagues who gave us their moral support and for the pleasant moments throughout our paths and big thanks to all the members of my family. Once again a big thank you to all those who supported us directly or indirectly in carrying out this modest work.

Abstract.

This work was devoted to studying the existence, uniqueness of solutions for two classes of sequential fractional neutral functional differential equations.

The first category is the Caputo-Hadamard type, while the second is the ψ-Caputo operator type.

The method used to study this type of equation depends on converting the equation into an integral equation before using the appropriate fixed point theory, The Banach fixed point theorem, a nonlinear alternative of Leray-Schauder type and Krasnoselski fixed point theorem are used to obtain the desired results.

Finally, examples illustrating the main results are presented.

ملخص

كرس هذا العمل لدراسة وجود، وحدانية حلول فئتين لمتتابعة المعادلات التفاضلية الكسرية

الفئة الاولى من **النوع كابوتو هادمارد** بينما الثانية من **النوع المؤثر بسي كابوتو**

ان الطريقة المستعملة لدراسة هذا النوع من المعادلات يعتمد على تحويل المعادلة الى معادلة تكاملية قبل استعمال نظرية النقطة الثابتة المناسبة، نظرية بناخ للنقطة الثابتة، البديل غير الخطي لنوع لراي شودر ونظرية النقطة الثابتة كراسنوسلسكي يتم استخدامها للحصول على النتائج المرجوة .

وأخيرا يتم عرض أمثلة توضح النتائج الرئيسية.

Contents

Introduction

Functional differential equations are found to be of central importance in many disciplines such as control theory, neural networks, epidemiology, etc. [19]. In analyzing the behavior of real populations, delay differential equations are regarded as effective tools. Since the delay terms can be finite as well as infinite in nature, one needs to study these two cases independently. Moreover, the delay terms may appear in the derivatives involved in the given equation. As it is difficult to formulate such a problem, an alternative approach is followed by considering neutral functional differential equations. On the other hand, fractional derivatives are capable to describe hereditary and memory effects in many processes and materials. So the study of neutral functional differential equations in presence of fractional derivatives constitutes an important area of research. For more details, see the text [33].

In recent years, there has been a significant development in fractional calculus, and initial and boundary value problems of fractional differential equations, see the monographs of Kilbas *et al.* [21], Lakshmikantham *et al.* [24], Miller and Ross [25], Podlubny [26], Samko *et al.* [27], Diethelm [13] and a series of papers [1, 2, 3, 4, 12, 14, 15, 20, 30, 32] and the references therein. One can notice that much of the work on the topic involves Riemann-Liouville and Caputo type fractional derivatives. Besides these derivatives, there is an other fractional derivative introduced by Hadamard in 1892 [18], which is known as

Hadamard derivative and differs from aforementioned derivatives in the sense that the kernel of the integral in its definition contains logarithmic function of arbitrary exponent. A detailed description of Hadamard fractional derivative and integral can be found in [9, 10, 11] and references cited therein.

In [7], the authors studied an initial value problem (IVP) for Riemman-Liouville type fractional functional and neutral functional differential equations with infinite delay. Recently, initial value problems for fractional order Hadamard-type functional and neutral functional differential equations and inclusions were respectively investigated in [3, 5], while an IVP for retarded functional Caputo type fractional impulsive differential equations with variable moments was discussed in [16].

In this memory, we investigate a new class of Hadamard-type sequential fractional neutral functional differential equations. Our study is based on fixed point theorems due to Banach and Krasnoselskii [23], and nonlinear alternative of Leray-Schauder type [17].

Chapter 1

Preliminaries

1.1 Functional spaces

Definition 1.1.1. ***(Norm [28]).***
Let E be a vector space on $\mathbb{R}$. We call a norm on E any application $\|.\| : E \to \mathbb{R}_+$ checked

1. $\forall x \in E : \|x\| = 0 \iff x = 0.$
2. $\forall \lambda \in R, \forall x \in E : \|\lambda x\| = |\lambda| \|x\|$ *(homogeneity).*
3. $\forall x, y \in E : \|x + y\| \leq \|x\| + \|y\|$ *(triangular inequality).*

we say that then (E,$\|.\|$) is a normalized vector space.

Exemple 1.1.1. *C(J;$\mathbb{R}$) This space with norm given by*

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\}$$

Definition 1.1.2. ***(Cauchy sequence)***
Let $(X, \|.\|_X)$ a normalized vector space $(x_n)_{n\in\mathbb{N}}$ a sequence of X. We say that $(x_n)_{n\in\mathbb{N}}$ is said to be-Cauchy (or fundamental) if
$\forall \epsilon > 0, \exists N_\epsilon \geq 0, \exists n > N_\epsilon, \forall m \geq N_\epsilon, \|x_{n+m} - x_n\|_X \leq \epsilon.$

Definition 1.1.3. ***(Complete metric spaces [8]).***
The space $(X, \|.\|)$ *is said to be complete if every Cauchy sequence in* X *converges (that is, has a limit which is an element of* X.

Exemple 1.1.2. *The real line* $(\boldsymbol{R}, |.|)$ *and the complex plane* $(\boldsymbol{C}, |.|)$ *are complete metric spaces.*

Definition 1.1.4. ***(Banach Space [28]).***
A normed space X *is a vector space with a norm defined on it, A Banach space is a complete normed space (complete in the metric defined by the norm) Here a nonn on a (real or complex) vector space* X *is a real-valued function on* X *whose value at an* $x \in X$

Exemple 1.1.3. $C(J;\mathbb{R})$ *This space is a Banach space.*

Definition 1.1.5. ***(Continnous mapping [8]).***
A mapping A *is said to be continuous, if for every* $\epsilon > 0$ *there is a* $\delta > 0$ *such that*

$$(x', x'' \in D_A) : \|x' - x''\| < \delta \Longrightarrow \|Ax' - Ax''\| < \epsilon$$

Definition 1.1.6. ***(Bounded linear operator [8]).***
Let X *and* Y *be normed spaces and* $T : D(T) \to Y$ *a linear operator, where* $D(T) \subset X$. *The operator* T *is said to be bounded if there is a real number* c *such that for all* $x \in D(T)$,

$$\|Tx\| \leq c\|x\|$$

Hence the answer to our question is that the smallest possible c *is that supremum. This quantity is denoted by*

$$\|T\| = sup_{\|x\| \leq 1} \|Tx\| = sup_{x \in D_T} \frac{\|Tx\|}{\|x\|} \quad (x \neq 0)$$

Definition 1.1.7. ***(Space*** $C^{'}[a,b]$ ***).*** *The space* $C^{1}[a,b]$ *or* $C^{'}[a,b]$ *is the normed space of all continuously differentiable functions on* $J=[a,b]$ *with norm defined by*

$$x=\max_{t\in J}|x(t)|+\max_{t\in J}|x^{'}(t)|.$$

1.2 Integrals and fractional derivatives

In this section, we introduce notation, definitions, and preliminary facts that we need in the sequel.

By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_{\infty}:=\sup\{|y(t)|:t\in J\}.$$

Also C_r is endowed with norm

$$\|\phi\|_{C}:=\sup\{|\phi(\theta)|:-r\leq\theta\leq 0\}.$$

Let $\psi:[a,\Im]\to\mathbb{R}$ be increasing and $\psi'(\varsigma)\neq 0$, $\forall\varsigma$.

Definition 1.2.1 ([21])**.** The Hadamard derivative of fractional order q for a function $g:[1,\infty)\to\mathbb{R}$ is defined as

$$D^{q}g(t)=\frac{1}{\Gamma(n-q)}\left(t\frac{d}{dt}\right)^{n}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{n-q-1}\frac{g(s)}{s}ds,\quad n-1<q<n,\ n=[q]+1,$$

where $[q]$ denotes the integer part of the real number q and $\log(\cdot)=\log_{e}(\cdot)$.

Definition 1.2.2 ([21])**.** The Hadamard fractional integral of order q for a function g is defined as

$$I^{q}g(t)=\frac{1}{\Gamma(q)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{q-1}\frac{g(s)}{s}ds,\quad q>0,$$

provided the integral exists.

Lemma 1.2.1. *The function* $y \in C^2([1-r,b],\mathbb{R})$ *is a solution of the problem*

$$\begin{aligned} D^{\alpha}[D^{\beta}y(t)-g(t,y_t)] &= f(t,y_t), \quad t\in J:=[1,b],\\ y(t)&=\phi(t), \quad t\in[1-r,1], \\ D^{\beta}y(1)&=\eta\in\mathbb{R}, \end{aligned} \tag{1.1}$$

if and only if

$$y(t)=\begin{cases} \phi(t), & \text{if } t\in[1-r,1],\\ \phi(1)+(\eta-g(1,\phi(1)))\dfrac{(\log t)^{\beta}}{\Gamma(\beta+1)} & \\ +\dfrac{1}{\Gamma(\beta)}\displaystyle\int_1^t\Big(\log\frac{t}{s}\Big)^{\beta-1}\frac{g(s,y_s)}{s}ds & \\ +\dfrac{1}{\Gamma(\alpha+\beta)}\displaystyle\int_1^t\Big(\log\frac{t}{s}\Big)^{\alpha+\beta-1}\frac{f(s,y_s)}{s}ds, & \text{if } t\in[1,b]. \end{cases} \tag{1.2}$$

where D^{α}, D^{β} *are the Caputo-Hadamard fractional derivatives,* $0<\alpha,\beta<1$, $f,g: J\times C([-r,0],\mathbb{R})\to\mathbb{R}$ *are given functions and* $\phi\in C([1-r,1],\mathbb{R})$.

Proof 1.2.1. *The solution of Hadamard differential equation in* (1.1) *can be written as*

$$D^{\beta}y(t)-g(t,y_t)=\frac{1}{\Gamma(\alpha)}\int_1^t\Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{f(s,y_s)}{s}ds+c_1, \tag{1.3}$$

where $c_1\in\mathbb{R}$ *is arbitrary constant. Using the condition* $D^{\beta}y(1)=\eta$ *we find that* $c_1=\eta-g(1,\phi(1))$. *Then we obtain*

$$\begin{aligned} y(t)=(\eta-g(1,\phi))\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}+\frac{1}{\Gamma(\beta)}\int_1^t\Big(\log\frac{t}{s}\Big)^{\beta-1}\frac{g(s,y_s)}{s}ds\\ +\frac{1}{\Gamma(\alpha+\beta)}\int_1^t\Big(\log\frac{t}{s}\Big)^{\alpha+\beta-1}\frac{f(s,y_s)}{s}ds+c_2. \end{aligned}$$

From the above equation we find $c_2=\phi(1)$ *and* (1.2) *is proved. The converse follows by direct computation.*

Definition 1.2.3 ([6, 21])**.** The ψ-Riemann-Liouville fractional integral (ψ-RLFI) of order $\alpha > 0$ for a CF $\varphi : [a, \Im] \to \mathbb{R}$ is referred to as

$$\mathscr{I}_a^{\alpha;\psi}\varphi(\varsigma) = \frac{1}{\Gamma(\alpha)}\int_a^{\varsigma}\left(\psi(\varsigma)-\psi(s)\right)^{\alpha-1}\psi'(s)\varphi(s)ds.$$

Definition 1.2.4 ([6, 21])**.** The ψ-Caputo fractional derivative (ψ-CFD) of order $\alpha > 0$ for a CF $\varphi : [a, \Im] \to \mathbb{R}$ is the aim of

$$\mathscr{D}_a^{\alpha;\psi}\varphi(\varsigma) = \frac{1}{\Gamma(n-\alpha)}\int_a^{\varsigma}\left(\psi(\varsigma)-\psi(s)\right)^{n-\alpha-1}\psi'(s)\partial_\psi^n\varphi(s)ds, \quad \varsigma > a, \quad n-1 < \alpha < n,$$

where $\partial_\psi^n = (\frac{1}{\psi'(\varsigma)}\frac{d}{d\varsigma})^n, \quad n \in \mathbb{N}$

Lemma 1.2.2 ([6, 21])**.** *Let $q, \ell > 0$, and $\varphi \in \mathscr{C}([a,b],\mathbb{R})$. Then,$\forall \varsigma \in [a,b]$, and by assuming $F_a(\varsigma) = \psi(\varsigma) - \psi(a)$, we have*

1. $\mathscr{I}_a^{q;\psi}\mathscr{I}_a^{\ell;\psi}\varphi(\varsigma) = \mathscr{I}_a^{q+\ell;\psi}\varphi(\varsigma)$,
2. $\mathscr{D}_a^{q;\psi}\mathscr{I}_a^{q;\psi}\varphi(\varsigma) = \varphi(\varsigma)$,
3. $\mathscr{I}_a^{q;\psi}(F_a(\varsigma))^{\ell-1} = \dfrac{\Gamma(\ell)}{\Gamma(\ell+q)}(F_a(\varsigma))^{\ell+q-1}$,
4. $\mathscr{D}_a^{q;\psi}(F_a(\varsigma))^{\ell-1} = \dfrac{\Gamma(\ell)}{\Gamma(\ell-q)}(F_a(\varsigma))^{\ell-q-1}$,
5. $\mathscr{D}_a^{q;\psi}(F_a(\varsigma))^k = 0$, *for* $k \in \{0, ..., n-1\}$, $n \in \mathbb{N}$, $q \in (n-1, n]$.

Lemma 1.2.3 ([6, 21])**.** *Let $n-1 < \alpha < n$, $\beta > 0$, $a > 0$, $\varphi \in L(a,\mathscr{T})$,$\mathscr{D}_a^{\alpha;\psi}\varphi \in L(a,\mathscr{T})$. Then, the differential equation*

$$\mathscr{D}_a^{\alpha;\psi}\varphi = 0$$

has the unique solution

$\varphi(\varsigma) = c_0 + c_1(\psi(\varsigma) - \psi(a)) + c_2(\psi(\varsigma) - \psi(a))^2 + \cdots + c_{n-1}(\psi(\varsigma) - \psi(a))^{n-1},$

and

$\mathscr{I}_a^{\alpha;\psi}\mathscr{D}_a^{\alpha;\psi}\varphi(\varsigma) = \varphi(\varsigma) + c_0 + c_1(\psi(\varsigma) - \psi(a)) + c_2(\psi(\varsigma) - \psi(a))^2 + \cdots + c_{n-1}(\psi(\varsigma) - \psi(a))^{n-1},$

with $c_\ell \in \mathbb{R}$, $\ell = 0, 1, ..., n-1$.

Furthermore,

$$\mathscr{D}_a^{\alpha;\psi}\mathscr{I}_a^{\alpha;\psi}\varphi(\varsigma) = \varphi(\varsigma),$$

and

$$\mathscr{I}_a^{\alpha;\psi}\mathscr{I}_a^{\beta;\psi}\varphi(\varsigma) = \mathscr{I}_a^{\beta;\psi}\mathscr{I}_a^{\alpha;\psi}\varphi(\varsigma) = \mathscr{I}_a^{\alpha+\beta;\psi}\varphi(\varsigma).$$

Lemma 1.2.4. *The function* $y \in C^2([a-r, b], \mathbb{R})$ *is a solution of the problem*

$$\begin{aligned} \mathscr{D}_a^{\alpha_1;\psi}[\mathscr{D}_a^{\alpha_2;\psi}y(t) - g(t, y_t)] &= f(t, y_t), \quad t \in J := [a, b], \\ y(t) &= \phi(t), \quad t \in [a-r, a], \\ \mathscr{D}_a^{\alpha_2;\psi}y(a) &= \eta \in \mathbb{R} \end{aligned} \tag{1.4}$$

if and only if

$$y(t) = \begin{cases} \phi(t), & \text{if } t \in [a-r, a], \\ \phi(a) + (\eta - g(a, \phi(a))\dfrac{(\psi(t) - \psi(a))^{\alpha_2}}{\Gamma(\alpha_2 + 1)} & \\ +\dfrac{1}{\Gamma(\alpha_2)}\displaystyle\int_a^t \big(\psi(t) - \psi(s)\big)^{\alpha_2 - 1}\psi'(s)g(s, y_s)\,ds & \\ +\dfrac{1}{\Gamma(\alpha_1 + \alpha_2)}\displaystyle\int_a^t \big(\psi(t) - \psi(s)\big)^{\alpha_1 + \alpha_2 - 1}\psi'(s)f(s, y_s)\,ds, & \text{if } t \in [a, b]. \end{cases} \tag{1.5}$$

where $\mathscr{D}_a^{\alpha_1;\psi}, \mathscr{D}_a^{\alpha_2;\psi}$ *are the* ψ*-Caputo fractional derivatives,* $0 < \alpha_1, \alpha_2 < 1$, $f, g : J \times C([-r, 0], \mathbb{R}) \to \mathbb{R}$ *are given functions and* $\phi \in C([a-r, a], \mathbb{R})$.

Proof 1.2.2. *The solution of ψ-Caputo differential equation in* (1.4) *can be written as*

$$\mathscr{D}_a^{\alpha_2;\psi} y(t) - g(t, y_t) = \frac{1}{\Gamma(\alpha_1)} \int_a^t \big(\psi(t) - \psi(s)\big)^{\alpha_1 - 1} \psi'(s) f(s, y_s) ds + c_1, \tag{1.6}$$

where $c_1 \in \mathbb{R}$ is arbitrary constant. Using the condition $\mathscr{D}_a^{\alpha_2;\psi} y(a) = \eta$ we find that $c_1 = \eta - g(a, \phi(a))$. Then we obtain

$$\begin{aligned} y(t) = (\eta - g(a, \phi(a))) \frac{(\psi(t) - \psi(a))^{\alpha_2}}{\Gamma(\alpha_2 + 1)} \\ + \frac{1}{\Gamma(\alpha_2)} \int_a^t \big(\psi(t) - \psi(s)\big)^{\alpha_2 - 1} \psi'(s) g(s, y_s) ds \\ + \frac{1}{\Gamma(\alpha_1 + \alpha_2)} \int_a^t \big(\psi(t) - \psi(s)\big)^{\alpha_1 + \alpha_2 - 1} \psi'(s) f(s, y_s) ds + c_2. \end{aligned}$$

From the above equation we find $c_2 = \phi(a)$ and (1.5) *is proved. The converse follows by direct computation.*

1.3 Some fixed point theorems

Fixed point theorems allow us to transform a fractional differential problem into a problem of the following form $Tx = x$. These theorems provide sufficient conditions for our fractional problem to admit a solution.

Definition 1.3.1. ***(Bounded set).*** *Show that a subset M in a normed space X is bounded if and only if there is a positive nwnber c such that$\|x\| \leq c$for every $x \in M$.*

Definition 1.3.2. ***(Compactness).*** *A metric space X is said to be compact if every sequence in X has a convergent subsequence. A subset M of X is said to be compact if M is compact considered as a subspace of X, that is, if every sequence in M has a convergent subsequence whose limit is an element of M.*

Definition 1.3.3. ***(Relatively compact).*** *A subset M of X is said to be relatively compact , that is, the closure $\overline{M}$ is compact.*

Definition 1.3.4. ***(Equicontinuous).*** *A sequence (x_n) in $C[a,b]$ is said to be equicontinuous if for every $\epsilon > 0$ there is a $\delta > 0$, depending only on ϵ, such that for all x_n and all $s_1, s_2 \in [a,b]$ satisfying $|s_1 - s_2| < \delta$ we have*

$$|x_n(s_1) - x_n(s_2)| < \epsilon.$$

Definition 1.3.5. ***(Contraction)****. Let $X = (X,d)$ be a metric space. A mapping $T : X \to X$ is called a contraction on X if there is a positive real number $a < 1$ such that for all $x, y \in X$*

$$d(Tx, Ty) \leqq a d(x,y)$$

Definition 1.3.6. ***(Fixed point)*** *A fixed point of a mapping $T : X \to X$ of a set X into itself is an $x \in X$ which is mapped onto itself (is "kept fixed" by T), that is,*

$$Tx = x$$

the image Tx coincides with x .

Definition 1.3.7. ***(Compact linear operator).*** *Let X and Y be normed spaces. An operator $T : X \longrightarrow Y$ is called a compact linear operator (or completely continuous linear operator) if T is linear and if for every bounded subset M of X, the image $T(M)$ is relatively compact, that is, the closure $\overline{T(M)}$ is compact.*

Theorem 1.3.1. ***(Banach Fixed Point Theorem (Contraction Theorem [17])).***
Consider a metric space $X = (X,d)$, where $X \neq \emptyset$. Suppose that X is complete and let $T : X \to X$ be a contraction on X. Then T has precisely one fixed point.

Theorem 1.3.2. ***(Arzelà-Ascoli theorem [31]).***
Let Ω be a set of X. Then Ω is relatively compact in X if and only if the following conditions are verified
1. Ω is uniformly bounded.
2. Ω is equicontinuous.

Theorem 1.3.3. ***(Schauder [29]).***
Let C be a nonempty closed convex subset of a Banach space X and $\Phi : C \longrightarrow C$ be a continuous compact application. Then Φ has a fixed point in C.

Chapter 2

Existence and uniqueness of solutions for Caputo-Hadamard

This work is concerned with the existence and uniqueness of solutions to the following initial value problem (IVP) of Caputo-Hadamard sequential fractional order neutral functional differential equations

$$D^{\alpha}[D^{\beta}y(t) - g(t, y_t)] = f(t, y_t), \quad t \in J := [1, b], \tag{2.1}$$

$$y(t) = \phi(t), \quad t \in [1-r, 1], \tag{2.2}$$

$$D^{\beta}y(1) = \eta \in \mathbb{R}, \tag{2.3}$$

where D^{α}, D^{β} are the Caputo-Hadamard fractional derivatives, $0 < \alpha, \beta < 1$, $f, g : J \times C([-r, 0], \mathbb{R}) \to \mathbb{R}$ are given functions and $\phi \in C([1-r, 1], \mathbb{R})$. For any function y defined on $[1-r, b]$ and any $t \in J$, we denote by y_t the element of $C_r := C([-r, 0], \mathbb{R})$ defined by

$$y_t(\theta) = y(t+\theta), \quad \theta \in [-r, 0].$$

2.1 Existence and uniqueness result

In this section, we establish the existence and uniqueness of a solution for the IVP (2.1)–(2.3).

Definition 2.1.1. A function $y \in C^2([1-r, b], \mathbb{R})$, is said to be a solution of (2.1)–(2.3) if y satisfies the equation $D^\alpha[D^\beta y(t) - g(t, y_t)] = f(t, y_t)$ on J, the condition $y(t) = \phi(t)$ on $[1-r, 1]$ and $D^\beta y(1) = \eta$.

The next theorem gives us a uniqueness result using the assumptions

(A1) there exists $\ell > 0$ such that

$$|f(t,u) - f(t,v)| \le \ell \|u - v\|_C, \quad \text{for } t \in J \text{ and every } u, v \in C_r;$$

(A2) there exists a nonnegative constant k such that

$$|g(t,u) - g(t,v)| \le k \|u - v\|_C, \quad \text{for } t \in J \text{ and every } u, v \in C_r.$$

Theorem 2.1.1. *Assume that* (A1), (A2) *hold. If*

$$\frac{k(\log b)^\beta}{\Gamma(\beta+1)} + \frac{\ell(\log b)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} < 1, \tag{2.4}$$

then there exists a unique solution for IVP (2.1)–(2.3) *on the interval* $[1-r, b]$.

Proof 2.1.1. *Consider the operator* $N : C([1-r, b], \mathbb{R}) \to C([1-r, b], \mathbb{R})$ *defined by*

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [1-r, 1], \\ \phi(1) + (\eta - g(1,\phi))\dfrac{(\log t)^\beta}{\Gamma(\beta+1)} & \\ +\dfrac{1}{\Gamma(\beta)}\displaystyle\int_1^t \left(\log\frac{t}{s}\right)^{\beta-1}\frac{g(s, y_s)}{s}ds & \\ +\dfrac{1}{\Gamma(\alpha+\beta)}\displaystyle\int_1^t \left(\log\frac{t}{s}\right)^{\alpha+\beta-1}\frac{f(s, y_s)}{s}ds, & \text{if } t \in J. \end{cases} \tag{2.5}$$

To show that the operator N is a contraction, let $y, z \in C([1-r,b],\mathbb{R})$. Then we have

$$\begin{aligned}
|N(y)(t)-N(z)(t)| &\leq \frac{1}{\Gamma(\beta)}\int_1^t \big(\log\frac{t}{s}\big)^{\beta-1}\frac{|g(s,y_s)-g(s,z_s)|}{s}ds \\
&\quad + \frac{1}{\Gamma(\alpha+\beta)}\int_1^t \big(\log\frac{t}{s}\big)^{\alpha+\beta-1}\frac{|f(s,y_s)-f(s,z_s)|}{s}ds \\
&\leq \frac{k}{\Gamma(\beta)}\int_1^t \big(\log\frac{t}{s}\big)^{\beta-1}\frac{\|y_s-z_s\|_C}{s}ds \\
&\quad + \frac{\ell}{\Gamma(\alpha+\beta)}\int_1^t \big(\log\frac{t}{s}\big)^{\alpha+\beta-1}\|y_s-z_s\|_C\,ds \\
&\leq \frac{k(\log t)^{\beta}}{\Gamma(\beta+1)}\|y-z\|_{[1-r,b]} + \frac{\ell(\log t)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|y-z\|_{[1-r,b]}.
\end{aligned}$$

Consequently we obtain

$$\|N(y)-N(z)\|_{[1-r,b]} \leq \Big[\frac{k(\log b)^{\beta}}{\Gamma(\beta+1)} + \frac{\ell(\log b)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big]\|y-z\|_{[1-r,b]},$$

which, in view of (2.4)*, implies that N is a contraction. Hence N has a unique fixed point by Banach's contraction principle. This, in turn, shows that problem* (2.1)–(2.3) *has a unique solution on* $[1-r,b]$.

2.2 Existence results

In this section, we establish our existence results for the IVP (2.1)–(2.3). The first result is based on Leray-Schauder nonlinear alternative.

Lemma 2.2.1 (Nonlinear alternative for single valued maps [17])**.** *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) F has a fixed point in $\overline{U}$, or

(ii) there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.

For the next theorem we need the following assumptions:

(A3) $f, g : J \times C_r \to \mathbb{R}$ are continuous functions;

(A4) there exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in C(J, \mathbb{R}^+)$ such that

$$|f(t,u)| \leq p(t)\psi(\|u\|_C) \text{for each } (t,u) \in J \times C_r;$$

(A5) there exist constants $d_1 < \Gamma(\beta+1)(\log b)^{-\beta}$ and $d_2 \geq 0$ such that

$$|g(t,u)| \leq d_1\|u\|_C + d_2, \quad t \in J, \ u \in C_r.$$

(A6) there exists a constant $M > 0$ such that

$$\frac{\left(1 - \frac{d_1(\log b)^\beta}{\Gamma(\beta+1)}\right)M}{M_0 + \psi(M)\|p\|_\infty \frac{1}{\Gamma(\alpha+\beta+1)}(\log b)^{\alpha+\beta}} > 1,$$

where

$$M_0 = \|\phi\|_C + [|\eta| + d_1\|\phi\|_C + 2d_2]\frac{(\log b)^\beta}{\Gamma(\beta+1)}.$$

Theorem 2.2.1. *Under assumptions* (A3)–(A6) *hold, IVP* (2.1)–(2.3) *has at least one solution on* $[1-r, b]$.

Proof 2.2.1. *We shall show that the operator* $N : C([1-r,b],\mathbb{R}) \to C([1-r,b],\mathbb{R})$ *defined by* (2.5) *is continuous and completely continuous.*

Step 1: *N is continuous. Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C([1-r,b],\mathbb{R})$. Then*

$$\begin{aligned}
&|N(y_n)(t) - N(y)(t)| \\
&\leq \frac{1}{\Gamma(\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta-1}\frac{|g(s,y_{ns}) - g(s,y_s)|}{s}ds \\
&\quad + \frac{1}{\Gamma(\alpha+\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha+\beta-1}|f(s,y_{ns}) - f(s,y_s)|\frac{ds}{s} \\
&\leq \frac{1}{\Gamma(\beta)}\int_1^b \left(\log\frac{t}{s}\right)^{\beta-1}\sup_{s\in[1,b]}|g(s,y_{ns}) - g(s,y_s)|\frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(\alpha+\beta)}\int_1^b \left(\log\frac{t}{s}\right)^{\alpha+\beta-1}\sup_{s\in[1,b]}|f(s,y_{ns}) - f(s,y_s)|\frac{ds}{s} \\
&\leq \frac{\|g(\cdot,y_{n.}) - g(\cdot,y_.)\|_\infty}{\Gamma(\beta)}\int_1^b \left(\log\frac{t}{s}\right)^{\beta-1}\frac{ds}{s} \\
&\quad + \frac{\|f(\cdot,y_{n.}) - f(\cdot,y_.)\|_\infty}{\Gamma(\alpha+\beta)}\int_1^b \left(\log\frac{t}{s}\right)^{\alpha+\beta-1}\frac{ds}{s} \\
&\leq \frac{(\log b)^\beta\|g(\cdot,y_{n.}) - g(\cdot,y_.)\|_\infty}{\Gamma(\beta+1)} \\
&\quad + \frac{(\log b)^{\alpha+\beta}\|f(\cdot,y_{n.}) - f(\cdot,y_.)\|_\infty}{\Gamma(\alpha+\beta+1)}.
\end{aligned}$$

Since f,g are continuous functions, we have

$$\begin{aligned}
&\|N(y_n) - N(y)\|_\infty \\
&\leq \frac{(\log b)^\beta\|g(\cdot,y_{n.}) - g(\cdot,y_.)\|_\infty}{\Gamma(\beta+1)} + \frac{(\log b)^{\alpha+\beta}\|f(\cdot,y_{n.}) - f(\cdot,y_.)\|_\infty}{\Gamma(\alpha+\beta+1)} \to 0
\end{aligned}$$

as $n \to \infty$.

Step 2: *N maps bounded sets into bounded sets in $C([1-r,b],\mathbb{R})$. Indeed, it is sufficient to show that for any $\theta > 0$ there exists a positive constant $\tilde{\ell}$ such that for each $y \in B_\theta = \{y \in$*

$C([1-r,b],\mathbb{R}) : \|y\|_\infty \leq \theta\}$, we have $\|N(y)\|_\infty \leq \tilde{\ell}$. By (A4) and (A5), for each $t \in J$, we have

$$\begin{aligned}
|N(y)(t)| &\leq \|\phi\|_C + [|\eta| + d_1\|\phi\|_C + d_2]\frac{(\log b)^\beta}{\Gamma(\beta+1)} \\
&\quad + \frac{1}{\Gamma(\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta-1}|g(s,y_s)|\frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(\alpha+\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha+\beta-1}|f(s,y_s)|\frac{ds}{s} \\
&\leq \|\phi\|_C + [|\eta| + d_1\|\phi\|_C + d_2]\frac{(\log b)^\beta}{\Gamma(\beta+1)} \\
&\quad + \frac{d_1\|y\|_{[1-r,b]} + d_2}{\Gamma(\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta-1}\frac{ds}{s} \\
&\quad + \frac{\psi(\|y\|_{[1-r,b]})\|p\|_\infty}{\Gamma(\alpha+\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha+\beta-1}\frac{ds}{s} \\
&\leq \|\phi\|_C + [|\eta| + d_1\|\phi\|_C + d_2]\frac{(\log b)^\beta}{\Gamma(\beta+1)} \\
&\quad + \frac{d_1\|y\|_{[1-r,b]} + d_2}{\Gamma(\beta+1)}(\log b)^\beta + \frac{\psi(\|y\|_{[1-r,b]})\|p\|_\infty}{\Gamma(\alpha+\beta+1)}(\log b)^{\alpha+\beta}.
\end{aligned}$$

Thus

$$\begin{aligned}
\|N(y)\|_\infty &\leq \|\phi\|_C + [|\eta| + d_1(\theta + \|\phi\|_C) + 2d_2]\frac{(\log b)^\beta}{\Gamma(\beta+1)} \\
&\quad + \frac{\psi(\theta)\|p\|_\infty}{\Gamma(\alpha+\beta+1)}(\log b)^{\alpha+\beta} := \tilde{\ell}.
\end{aligned}$$

Step 3: *N maps bounded sets into equicontinuous sets of $C([1-r,b],\mathbb{R})$. Let $t_1, t_2 \in J$, $t_1 < t_2$,*

B_θ be a bounded set of $C([1-r,b],\mathbb{R})$ as in Step 2, and let $y \in B_\theta$. Then

$$\begin{aligned}
&|N(y)(t_2)-N(y)(t_1)| \\
&\le \frac{|\eta|+d_1\|\phi\|_C+d_2}{\Gamma(\beta+1)}[(\log t_2)^\beta-(\log t_1)^\beta] \\
&\quad + \Big|\frac{1}{\Gamma(\beta)}\int_1^{t_1}\Big[\Big(\log\frac{t_2}{s}\Big)^{\beta-1}-\Big(\log\frac{t_1}{s}\Big)^{\beta-1}\Big]g(s,y_s)\frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(\beta)}\int_{t_1}^{t_2}\Big(\log\frac{t_2}{s}\Big)^{\beta-1}g(s,y_s)\frac{ds}{s}\Big| \\
&\quad + \Big|\frac{1}{\Gamma(\alpha+\beta)}\int_1^{t_1}\Big[\Big(\log\frac{t_2}{s}\Big)^{\alpha+\beta-1}-\Big(\log\frac{t_1}{s}\Big)^{\alpha+\beta-1}\Big]f(s,y_s)\frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(\alpha+\beta)}\int_{t_1}^{t_2}\Big(\log\frac{t_2}{s}\Big)^{\alpha+\beta-1}f(s,y_s)\frac{ds}{s}\Big| \\
&\le \frac{|\eta|+d_1\|\phi\|_C+d_2}{\Gamma(\beta+1)}[(\log t_2)^\beta-(\log t_1)^\beta] \\
&\quad + \frac{d_1\theta+d_2}{\Gamma(\beta)}\int_1^{t_1}\Big[\Big(\log\frac{t_2}{s}\Big)^{\beta-1}-\Big(\log\frac{t_1}{s}\Big)^{\beta-1}\Big]\frac{ds}{s} \\
&\quad + \frac{d_1\theta+d_2}{\Gamma(\beta)}\int_{t_1}^{t_2}\Big(\log\frac{t_2}{s}\Big)^{\beta-1}\frac{ds}{s} \\
&\quad + \frac{\psi(\theta)\|p\|_\infty}{\Gamma(\alpha+\beta)}\int_1^{t_1}\Big[\Big(\log\frac{t_2}{s}\Big)^{\alpha+\beta-1}-\Big(\log\frac{t_1}{s}\Big)^{\alpha+\beta-1}\Big]\frac{ds}{s} \\
&\quad + \frac{\psi(\theta)\|p\|_\infty}{\Gamma(\alpha+\beta)}\int_{t_1}^{t_2}\Big(\log\frac{t_2}{s}\Big)^{\alpha+\beta-1}\frac{ds}{s} \\
&\le \frac{|\eta|+d_1\|\phi\|_C+d_2}{\Gamma(\beta+1)}[(\log t_2)^\beta-(\log t_1)^\beta] \\
&\quad + \frac{d_1\theta+d_2}{\Gamma(\beta+1)}[\big|(\log t_2)^\beta-(\log t_1)^\beta\big|+|\log t_2/t_1|^\beta] \\
&\quad + \frac{\psi(\theta)\|p\|_\infty}{\Gamma(\alpha+\beta+1)}[\big|(\log t_2)^{\alpha+\beta}-(\log t_1)^{\alpha+\beta}\big|+|\log t_2/t_1|^{\alpha+\beta}].
\end{aligned}$$

As $t_1 \to t_2$ the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $t_1 < t_2 \le 0$ and $t_1 \le 0 \le t_2$ is obvious.

As a consequence of Steps 1 to 3, it follows by the Arzelá-Ascoli theorem that $N : C([1-r,b],\mathbb{R}) \to C([1-r,b],\mathbb{R})$ is continuous and completely continuous.

Step 4: *We show that there exists an open set* $U \subseteq C([1-r,b],\mathbb{R})$ *with* $y \neq \lambda N(y)$ *for* $\lambda \in (0,1)$ *and* $y \in \partial U$. *Let* $y \in C([1-r,b],\mathbb{R})$ *and* $y = \lambda N(y)$ *for some* $0 < \lambda < 1$. *Then, for each* $t \in J$, *we have*

$$y(t) = \lambda\Big(\phi(1) + (\eta - g(1,\phi(1)))\frac{(\log t)^{\beta}}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta)}\int_1^t \big(\log\frac{t}{s}\big)^{\beta-1}\frac{g(s,y_s)}{s}ds$$
$$+ \frac{1}{\Gamma(\alpha+\beta)}\int_1^t \big(\log\frac{t}{s}\big)^{\alpha+\beta-1}\frac{f(s,y_s)}{s}ds\Big).$$

By our assumptions, for each $t \in J$, *we obtain*

$$\begin{aligned}|y(t)| \leq{}& \|\phi\|_C + [|\eta| + d_1\|\phi\|_C + d_2]\frac{(\log b)^{\beta}}{\Gamma(\beta+1)}\\ &+ \frac{d_1\|y\|_{[1-r,b]} + d_2}{\Gamma(\beta)}\int_1^t \big(\log\frac{t}{s}\big)^{\beta-1}\frac{ds}{s}\\ &+ \frac{1}{\Gamma(\alpha+\beta)}\int_1^t \big(\log\frac{t}{s}\big)^{\alpha+\beta-1}p(s)\psi(\|y_s\|_C)\frac{ds}{s}\\ \leq{}& \|\phi\|_C + [|\eta| + d_1\|\phi\|_C + d_2]\frac{(\log b)^{\beta}}{\Gamma(\beta+1)} + \frac{d_1\|y\|_{[1-r,b]} + d_2}{\Gamma(\beta+1)}(\log b)^{\beta}\\ &+ \frac{\|p\|_{\infty}\psi(\|y\|_{[1-r,b]})}{\Gamma(\alpha+\beta+1)}(\log b)^{\alpha+\beta},\end{aligned}$$

which can be expressed as

$$\frac{\Big(1 - \frac{d_1(\log b)^{\beta}}{\Gamma(\beta+1)}\Big)\|y\|_{[1-r,b]}}{M_0 + \psi(\|y\|_{[1-r,b]})\|p\|_{\infty}\frac{1}{\Gamma(\alpha+\beta+1)}(\log b)^{\alpha+\beta}} \leq 1.$$

In view of (A6), there exists M *such that* $\|y\|_{[1-r,b]} \neq M$. *Let us set*

$$U = \{y \in C([1-r,b],\mathbb{R}) : \|y\|_{[1-r,b]} < M\}.$$

Note that the operator $N : \overline{U} \to C([1-r,b],\mathbb{R})$ *is continuous and completely continuous. From the choice of* U, *there is no* $y \in \partial U$ *such that* $y = \lambda N y$ *for some* $\lambda \in (0,1)$. *Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 2.2.1), we deduce that* N *has a fixed point* $y \in \overline{U}$ *which is a solution of the problem* (2.1)-(2.3). *This completes the proof.*

The second existence result is based on Krasnoselskii's fixed point theorem.

Lemma 2.2.2 (Krasnoselskii's fixed point theorem [23])**.** *Let S be a closed, bounded, convex and nonempty subset of a Banach space X. Let A, B be the operators such that (a) $Ax + Bx \in S$ whenever $x, y \in S$; (b) A is compact and continuous; (c) B is a contraction mapping. Then there exists $z \in S$ such that $z = Az + Bz$.*

Theorem 2.2.2. *Assume that* (A2) *and* (A3) *hold. In addition we assume that*

(A7) $|f(t,x)| \le \mu(t)$, $|g(t,x)| \le \nu(t)$, *for all* $(t,x) \in J \times \mathbb{R}$, *and* $\mu, \nu \in C(J, \mathbb{R}^+)$.

Then problem (2.1)-(2.3) *has at least one solution on* $[1-r, b]$, *provided*

$$\frac{k(\log b)^{\beta}}{\Gamma(\beta+1)} < 1. \tag{2.6}$$

Proof 2.2.2. *We define the operators* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *by*

$$\mathcal{G}_1 y(t) = \begin{cases} 0, & \text{if } t \in [1-r, 1], \\ (\eta - g(1,\phi))\dfrac{(\log t)^{\beta}}{\Gamma(\beta+1)} & \\ +\dfrac{1}{\Gamma(\beta)}\displaystyle\int_1^t \left(\log\frac{t}{s}\right)^{\beta-1}\frac{g(s,y_s)}{s}\,ds, & \text{if } t \in J. \end{cases} \tag{2.7}$$

$$\mathcal{G}_2 y(t) = \begin{cases} \phi(t), & \text{if } t \in [1-r, 1], \\ \phi(1) + \dfrac{1}{\Gamma(\alpha+\beta)}\displaystyle\int_1^t \left(\log\frac{t}{s}\right)^{\alpha+\beta-1}\frac{f(s,y_s)}{s}\,ds, & \text{if } t \in J. \end{cases} \tag{2.8}$$

Setting $\sup_{t\in[1,b]} \mu(t) = \|\mu\|_\infty$, $\sup_{t\in[1,b]} \nu(t) = \|\nu\|_\infty$ *and choosing*

$$\rho \ge \|\phi\|_C + [|\eta| + \|\nu\|_\infty]\frac{(\log b)^{\beta}}{\Gamma(\beta+1)} + \frac{\|\nu\|(\log b)^{\beta}}{\Gamma(\beta+1)} + \|\mu\|_\infty\frac{(\log b)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}, \tag{2.9}$$

we consider $B_\rho = \{y \in C([1-r,b],\mathbb{R}) : \|y\|_\infty \le \rho\}$. *For any* $y, z \in B_\rho$, *we have*

$$\begin{aligned}
&|\mathscr{G}_1 y(t) + \mathscr{G}_2 z(t)| \\
&\le \sup_{t\in[1,b]} \Big\{(\eta - g(1,\phi))\frac{(\log t)^\beta}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta)}\int_1^t \big(\log\frac{t}{s}\big)^{\beta-1}\frac{g(s,y_s)}{s}ds \\
&\quad + \phi(1) + \frac{1}{\Gamma(\alpha+\beta)}\int_1^t \big(\log\frac{t}{s}\big)^{\alpha+\beta-1}\frac{f(s,z_s)}{s}ds\Big\} \\
&\le \|\phi\|_C + [|\eta| + \|\nu\|_\infty]\frac{(\log b)^\beta}{\Gamma(\beta+1)} + \frac{\|\nu\|(\log b)^\beta}{\Gamma(\beta+1)} + \|\mu\|_\infty\frac{(\log b)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \\
&\le \rho.
\end{aligned}$$

This shows that $\mathscr{G}_1 y + \mathscr{G}_2 z \in B_\rho$. *Using* (2.6) *it is easy to see that* $\mathscr{G}_1$ *is a contraction mapping.*

Continuity of f *implies that the operator* $\mathscr{G}_2$ *is continuous. Also,* $\mathscr{G}_2$ *is uniformly bounded on* B_ρ *as*

$$\|\mathscr{G}_2 y\| \le \|\phi\|_C + \|\mu\|_\infty\frac{(\log b)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}.$$

Now we prove the compactness of the operator $\mathscr{G}_2$. *We define*

$$\bar{f} = \sup_{(t,y)\in[1,b]\times B_\rho} |f(t,y)| < \infty,$$

and consequently, for $t_1, t_2 \in [1,b]$, $t_1 < t_2$, *we have*

$$\begin{aligned}
&|\mathscr{G}_2 y(t_2) - \mathscr{G}_2 y(t_1)| \\
&\le \frac{\bar{f}}{\Gamma(\alpha+\beta)}\int_1^{t_1}\left|\Big(\log\frac{t_2}{s}\Big)^{\alpha+\beta-1} - \Big(\log\frac{t_1}{s}\Big)^{\alpha+\beta-1}\right|\frac{ds}{s} \\
&\quad + \frac{\bar{f}}{\Gamma(\alpha+\beta)}\int_{t_1}^{t_2}\Big(\log\frac{t_2}{s}\Big)^{\alpha+\beta-1}\frac{ds}{s} \\
&\le \frac{\bar{f}}{\Gamma(\alpha+\beta+1)}[\left|(\log t_2)^{\alpha+\beta} - (\log t_1)^{\alpha+\beta}\right| + |\log t_2/t_1|^{\alpha+\beta}],
\end{aligned}$$

which is independent of y *and tends to zero as* $t_2 - t_1 \to 0$. *Thus,* $\mathscr{G}_2$ *is equicontinuous. So* $\mathscr{G}_2$ *is relatively compact on* B_ρ. *Hence, by the Arzelá-Ascoli theorem,* $\mathscr{G}_2$ *is compact on* B_ρ. *Thus* *all the assumptions of Lemma* 2.2.2 *are satisfied. So the conclusion of Lemma* 2.2.2 *implies that the problem* (2.1)-(2.3) *has at least one solution on* $[1-r,b]$

2.3 Examples

In this section we give an example to illustrate the usefulness of our main results. Let us consider the fractional functional differential equation,

$$D^{1/2}\Big[D^{3/4}y(t)-\frac{1+e^{-t}}{8+e^{t}}\frac{\|y_t\|_C}{(1+\|y_t\|_C)}\Big]=\frac{\|y_t\|_C}{2(1+\|y_t\|_C)}+e^{-t},\quad t\in J:=[1,e], \tag{2.10}$$

$$y(t)=\phi(t),\quad t\in[1-r,1], \tag{2.11}$$

$$D^{3/4}y(1)=1/2. \tag{2.12}$$

Let

$$f(t,x)=\frac{x}{2(1+x)},\quad g(t,x)=\frac{1+e^{-t}}{8+e^{t}}\Big(\frac{x}{1+x}\Big),\quad (t,x)\in[1,e]\times[0,\infty).$$

For $x,y\in[0,\infty)$ and $t\in J$, we have

$$|f(t,x)-f(t,y)|=\frac{1}{2}\Big|\frac{x}{1+x}-\frac{y}{1+y}\Big|=\frac{|x-y|}{2(1+x)(1+y)}\le\frac{1}{2}|x-y|,$$

and

$$|g(t,x)-g(t,y)|=\frac{1+e^{-t}}{8+e^{t}}\Big|\frac{x}{1+x}-\frac{y}{1+y}\Big|=\frac{1+e^{-t}}{8+e^{t}}\frac{|x-y|}{(1+x)(1+y)}\le\frac{e+1}{e(e+8)}|x-y|.$$

Hence conditions (A1) and (A2) hold with $\ell=1/2$ and $k=\dfrac{e+1}{e(e+8)}$ respectively. Since $\dfrac{k(\log b)^{\alpha}}{\Gamma(\alpha+1)}+\dfrac{\ell(\log b)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\approx 0.5853088<1$, therefore, by Theorem 2.2.2, problem (2.10)-(2.12) has a unique solution on $[1-r,e]$.

Also $|f(t,x)| \leq (1+2e^{-t})/2 = \mu(t)$, $|g(t,x)| \leq (1+e^{-t})/(8+e^{t}) = \nu(t)$ and $k(\log b)^{\alpha}/\Gamma(\alpha+1) = 2(e+1)/\sqrt{\pi}e(e+8) \approx 0.144005 < 1$. Clearly the assumptions of Theorem 2.2.2 are satisfied. Consequently, by the conclusion of Theorem 2.2.2, there exists a solution of the problem (2.10)-(2.12) on $[1-r,e]$.

2.4 Initial value integral condition case

The results of this paper can be extended to the case of an initial value integral condition of the form

$$D^{\beta}y(1) = \int_{1}^{b} h(s,y_s)ds, \tag{2.13}$$

where $h : J \times C([-r,0],\mathbb{R}) \to \mathbb{R}$ is a given function. In this case η will be replaced with $\int_{1}^{b} h(s,y_s)ds$ in (2.5) and the statement of the existence and uniqueness result for the problem (2.1)–(2.2)–(2.13) can be formulated as follows.

Theorem 2.4.1. *Assume that the conditions* (A1) *and* (A2) *hold. Further, we suppose that*

(A8) there exists a nonnegative constant m such that

$$|h(t,u) - h(t,v)| \leq m\|u-v\|_C, \quad \textit{for } t \in J \textit{ and every } u,v \in C_r.$$

Then the problem (2.1)–(2.2)–(2.13) *has a unique solution on* $[1-r,b]$ *if*

$$\frac{(m(b-1)+k)(\log b)^{\beta}}{\Gamma(\beta+1)} + \frac{\ell(\log b)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} < 1.$$

We do not provide the proof of the above theorem as it is similar to that of Theorem 2.1.1.

The analog form of the existence results: Theorems 2.2.1 and 2.2.2 for the problem (2.1)–(2.2)–(2.13) can be constructed in a similar manner.

Chapter 3

Existence and uniqueness of solutions for the ψ-Caputo operator

This work is concerned with the existence and uniqueness of solutions to the following initial value problem (IVP) of ψ-Caputo sequential fractional order neutral functional differential equations

$$\mathscr{D}_a^{\alpha_1;\psi}[\mathscr{D}_a^{\alpha_2;\psi}y(t)-g(t,y_t)]=f(t,y_t),\quad t\in J:=[a,b], \tag{3.1}$$

$$y(t)=\phi(t),\quad t\in[a-r,a], \tag{3.2}$$

$$\mathscr{D}_a^{\alpha_2;\psi}y(a)=\eta\in\mathbb{R}, \tag{3.3}$$

where $\mathscr{D}_a^{\alpha_1;\psi}, \mathscr{D}_a^{\alpha_2;\psi}$ are the ψ-Caputo fractional derivatives, $0<\alpha_1,\alpha_2<1$, $f,g:J\times C([-r,0],\mathbb{R})\to\mathbb{R}$ are given functions and $\phi\in C([a-r,a],\mathbb{R})$. For any function y defined on $[a-r,b]$ and any $t\in J$, we denote by y_t the element of $C_r:=C([-r,0],\mathbb{R})$ defined by

$$y_t(\theta)=y(t+\theta),\quad \theta\in[-r,0].$$

3.1 Existence and uniqueness result

In this section, we establish the existence and uniqueness of a solution for the IVP (3.1)–(3.3).

Definition 3.1.1. A function $y \in C^2([a-r,b],\mathbb{R})$, is said to be a solution of (3.1)–(3.3) if y satisfies the equation $\mathscr{D}_a^{\alpha_1;\psi}[\mathscr{D}_a^{\alpha_2;\psi} y(t) - g(t,y_t)] = f(t,y_t)$ on J, the condition $y(t) = \phi(t)$ on $[a-r,a]$ and $\mathscr{D}_a^{\alpha_2;\psi}(a) = \eta$.

The next theorem gives us a uniqueness result using the assumptions

(A1) there exists $\ell > 0$ such that

$$|f(t,u) - f(t,v)| \leq \ell \|u-v\|_C, \quad \text{for } t \in J \text{ and every } u,v \in C_r;$$

(A2) there exists a nonnegative constant k such that

$$|g(t,u) - g(t,v)| \leq k \|u-v\|_C, \quad \text{for } t \in J \text{ and every } u,v \in C_r.$$

Theorem 3.1.1. *Assume that* (A1), (A2) *hold. If*

$$\frac{k(F_a(b))^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{\ell(F_a(b))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} < 1, \tag{3.4}$$

then there exists a unique solution for IVP (3.1)–(3.3) *on the interval* $[a-r,b]$.

Proof 3.1.1. *Consider the operator* $N : C([a-r,b],\mathbb{R}) \to C([a-r,b],\mathbb{R})$ *defined by*

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [a-r,a], \\ \phi(a) + (\eta - g(a,\phi(a)))\dfrac{(\psi(t)-\psi(a))^{\alpha_2}}{\Gamma(\alpha_2+1)} & \\ +\dfrac{1}{\Gamma(\alpha_2)}\displaystyle\int_a^t \big(\psi(t)-\psi(s)\big)^{\alpha_2-1}\psi'(s)g(s,y_s)ds & \\ +\dfrac{1}{\Gamma(\alpha_1+\alpha_2)}\displaystyle\int_a^t \big(\psi(t)-\psi(s)\big)^{\alpha_1+\alpha_2-1}\psi'(s)f(s,y_s)ds, & \text{if } t \in J. \end{cases} \tag{3.5}$$

To show that the operator N is a contraction, let $y,z \in C([a-r,b],\mathbb{R})$. Then we have

$$\begin{aligned}|N(y)(t)-N(z)(t)| &\leq \frac{1}{\Gamma(\alpha_2)}\int_a^t \big(F_s(t)\big)^{\alpha_2-1}\psi'(s)|g(s,y_s)-g(s,z_s)|ds \\ &\quad + \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t \big(F_s(t)\big)^{\alpha_1+\alpha_2-1}\psi'(s)|f(s,y_s)-f(s,z_s)|ds \\ &\leq \frac{k}{\Gamma(\alpha_2)}\int_a^t \big(F_s(t)\big)^{\alpha_2-1}\psi'(s)\|y_s-z_s\|_C ds \\ &\quad + \frac{\ell}{\Gamma(\alpha_1+\alpha_2)}\int_a^t \big(F_s(t)\big)^{\alpha_1+\alpha_2-1}\psi'(s)\|y_s-z_s\|_C ds \\ &\leq \frac{k(F_a(t))^{\alpha_2}}{\Gamma(\alpha_2+1)}\|y-z\|_{[a-r,b]} + \frac{\ell(F_a(t))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}\|y-z\|_{[a-r,b]}.\end{aligned}$$

Consequently we obtain

$$\|N(y)-N(z)\|_{[a-r,b]} \leq \left[\frac{k(F_a(b))^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{\ell(F_a(b))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}\right]\|y-z\|_{[a-r,b]},$$

which, in view of (3.4)*, implies that N is a contraction. Hence N has a unique fixed point by Banach's contraction principle. This, in turn, shows that problem* (3.1)–(3.3) *has a unique solution on* $[a-r,b]$.

3.2 Existence results

In this section, we establish our existence results for the IVP (3.1)–(3.3). The first result is based on Leray-Schauder nonlinear alternative.

Lemma 3.2.1 (Nonlinear alternative for single valued maps [17])**.** *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $R : \overline{U} \to C$ is a continuous, compact (that is, $R(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) R has a fixed point in $\overline{U}$, or

(ii) there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda R(u)$.

For the next theorem we need the following assumptions:

(A3) $f, g : J \times C_r \to \mathbb{R}$ are continuous functions;

(A4) there exist a continuous nondecreasing function $H : [0,\infty) \to (0,\infty)$ and a function $p \in C(J, \mathbb{R}^+)$ such that

$$|f(t,u)| \leq p(t) H(\|u\|_C) \text{for each } (t,u) \in J \times C_r;$$

(A5) there exist constants $d_1 < \Gamma(\alpha_2+1)(F_a(b))^{-\alpha_2}$ and $d_2 \geq 0$ such that

$$|g(t,u)| \leq d_1 \|u\|_C + d_2, \ \ t \in J, \ u \in C_r.$$

(A6) there exists a constant $M > 0$ such that

$$\frac{\left(1 - \frac{d_1 (F_a(b))^{\alpha_2}}{\Gamma(\alpha_2+1)}\right) M}{M_0 + H(M)\|p\|_\infty \frac{1}{\Gamma(\alpha_1+\alpha_2+1)} (F_a(b))^{\alpha_1+\alpha_2}} > 1,$$

where

$$M_0 = \|\phi\|_C + \left[|\eta| + d_1 \|\phi\|_C + 2d_2 \right] \frac{(F_a(b))^{\alpha_2}}{\Gamma(\alpha_2+1)}.$$

Theorem 3.2.1. *Under assumptions* (A3)–(A6) *hold, IVP* (3.1)–(3.3) *has at least one solution on* $[a-r, b]$.

Proof 3.2.1. *We shall show that the operator* $N : C([a-r,b],\mathbb{R}) \to C([a-r,b],\mathbb{R})$ *defined by* (3.5) *is continuous and completely continuous.*

Step 1: *N is continuous. Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C([a-r,b],\mathbb{R})$. Then*

$$\begin{aligned}
&|N(y_n)(t) - N(y)(t)| \\
&\leq \frac{1}{\Gamma(\alpha_2)} \int_a^t \big(F_s(t)\big)^{\alpha_2 - 1} \psi'(s) |g(s, y_{ns}) - g(s, y_s)| ds \\
&\quad + \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_a^t \big(F_s(t)\big)^{\alpha_1+\alpha_2 - 1} \psi'(s) |f(s, y_{ns}) - f(s, y_s)| \frac{ds}{s} \\
&\leq \frac{1}{\Gamma(\alpha_2)} \int_a^b \big(F_s(t)\big)^{\alpha_2 - 1} \psi'(s) \sup_{s\in[a,b]} |g(s, y_{ns}) - g(s, y_s)| ds \\
&\quad + \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_a^b \big(F_s(t)\big)^{\alpha_1+\alpha_2 - 1} \psi'(s) \sup_{s\in[a,b]} |f(s, y_{ns}) - f(s, y_s)| ds \\
&\leq \frac{\|g(\cdot, y_{n.}) - g(\cdot, y_.)\|_\infty}{\Gamma(\alpha)} \int_a^b \big(F_s(t)\big)^{\alpha_2 - 1} \psi'(s) ds \\
&\quad + \frac{\|f(\cdot, y_{n.}) - f(\cdot, y_.)\|_\infty}{\Gamma(\alpha_1+\alpha_2)} \int_a^b \big(F_s(t)\big)^{\alpha+\beta - 1} \psi'(s) ds \\
&\leq \frac{(F_a(b))^{\alpha_2} \|g(\cdot, y_{n.}) - g(\cdot, y_.)\|_\infty}{\Gamma(\alpha_2+1)} \\
&\quad + \frac{(F_a(b))^{\alpha_1+\alpha_2} \|f(\cdot, y_{n.}) - f(\cdot, y_.)\|_\infty}{\Gamma(\alpha_1+\alpha_2+1)}.
\end{aligned}$$

Since f, g are continuous functions, we have

$$\begin{aligned}
&\|N(y_n) - N(y)\|_\infty \\
&\leq \frac{(F_a(b))^{\alpha_2} \|g(\cdot, y_{n.}) - g(\cdot, y_.)\|_\infty}{\Gamma(\alpha_2+1)} + \frac{(F_a(b))^{\alpha_1+\alpha_2} \|f(\cdot, y_{n.}) - f(\cdot, y_.)\|_\infty}{\Gamma(\alpha_1+\alpha_2+1)} \to 0
\end{aligned}$$

as $n \to \infty$.

Step 2: *N maps bounded sets into bounded sets in $C([a-r,b],\mathbb{R})$. Indeed, it is sufficient to show that for any $\theta > 0$ there exists a positive constant $\tilde{\ell}$ such that for each $y \in B_\theta = \{y \in$*

$C([a-r,b],\mathbb{R}):\|y\|_\infty \le \theta\}$, we have $\|N(y)\|_\infty \le \tilde{\ell}$. By (A4) and (A5), for each $t \in J$, we have

$$\begin{aligned}
|N(y)(t)| \le \|\phi\|_C &+ \Big[|\eta| + d_1\|\phi\|_C + d_2\Big]\frac{(F_a(b))^{\alpha_2}}{\Gamma(\alpha_2+1)} \\
&+ \frac{1}{\Gamma(\alpha_2)}\int_a^t \big(F_s(t)\big)^{\alpha_2-1}\psi'(s)|g(s,y_s)|ds \\
&+ \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t \big((F_s(t)\big)^{\alpha_1+\alpha_2-1}\psi'(s)|f(s,y_s)|ds \\
&\le \|\phi\|_C + \Big[|\eta| + d_1\|\phi\|_C + d_2\Big]\frac{(F_a(b))^{\alpha_2}}{\Gamma(\alpha_2+1)} \\
&+ \frac{d_1\|y\|_{[a-r,b]} + d_2}{\Gamma(\alpha_2)}\int_a^t \big(F_s(t)\big)^{\alpha_2-1}\psi'(s)ds \\
&+ \frac{H(\|y\|_{[a-r,b]})\|p\|_\infty}{\Gamma(\alpha_1+\alpha_2)}\int_a^t \big(F_s(t)\big)^{\alpha_1+\alpha_2-1}\psi'(s)ds \\
&\le \|\phi\|_C + \Big[|\eta| + d_1\|\phi\|_C + d_2\Big]\frac{(F_a(b))^{\alpha_2}}{\Gamma(\alpha_2+1)} \\
&+ \frac{d_1\|y\|_{[a-r,b]} + d_2}{\Gamma(\alpha_2+1)}(F_a(b))^{\alpha_2} + \frac{H(\|y\|_{[a-r,b]})\|p\|_\infty}{\Gamma(\alpha_1+\alpha_2+1)}(F_a(b))^{\alpha_1+\alpha_2}.
\end{aligned}$$

Thus

$$\begin{aligned}
\|N(y)\|_\infty \le \|\phi\|_C &+ \Big[|\eta| + d_1(\|\phi\|_C + \theta) + 2d_2\Big]\frac{(F_a(b))^{\alpha_2}}{\Gamma(\alpha_2+1)} \\
&+ \frac{H(\theta)\|p\|_\infty}{\Gamma(\alpha_1+\alpha_2+1)}(F_a(b))^{\alpha_1+\alpha_2} := \tilde{\ell}.
\end{aligned}$$

Step 3: *N maps bounded sets into equicontinuous sets of $C([a-r,b],\mathbb{R})$. Let $t_1, t_2 \in J$, $t_1 < t_2$,*

B_θ be a bounded set of $C([a-r,b],\mathbb{R})$ as in Step 2, and let $y \in B_\theta$. Then

$$
\begin{aligned}
&|N(y)(t_2) - N(y)(t_1)| \\
&\le \frac{|\eta| + d_1\|\phi\|_C + d_2}{\Gamma(\alpha_2+1)}\Big[(F_a(t_2))^{\alpha_2} - (F_a(t_1))^{\alpha_2}\Big] \\
&\quad + \Big|\frac{1}{\Gamma(\alpha_2)}\int_a^{t_1}\Big[\Big(F_s(t_2)\Big)^{\alpha_2-1} - \Big(F_s(t_1)\Big)^{\alpha_2-1}\Big]\psi'(s)g(s,y_s)ds \\
&\quad + \frac{1}{\Gamma(\alpha_2)}\int_{t_1}^{t_2}\Big(F_s(t_2)\Big)^{\alpha_2-1}\psi'(s)g(s,y_s)ds\Big| \\
&\quad + \Big|\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^{t_1}\Big[\Big(F_s(t_2)\Big)^{\alpha_1+\alpha_2-1} - \Big(F_s(t_1)\Big)^{\alpha_1+\alpha_2-1}\Big]\psi'(s)f(s,y_s)ds \\
&\quad + \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_{t_1}^{t_2}\Big(F_s(t_2)\Big)^{\alpha_1+\alpha_2-1}\psi'(s)f(s,y_s)ds\Big| \\
&\le \frac{|\eta| + d_1\|\phi\|_C + d_2}{\Gamma(\alpha_2+1)}\Big[(F_a(t_2))^{\alpha_2} - (F_a(t_1))^{\alpha_2}\Big] \\
&\quad + \frac{d_1\theta + d_2}{\Gamma(\alpha_2)}\int_a^{t_1}\Big[\Big(F_s(t_2)\Big)^{\alpha_2-1} - \Big(F_s(t_1)\Big)^{\alpha_2-1}\Big]\psi'(s)ds \\
&\quad + \frac{d_1\theta + d_2}{\Gamma(\alpha_2)}\int_{t_1}^{t_2}\Big(F_s(t_2)\Big)^{\alpha_2-1}\psi'(s)ds \\
&\quad + \frac{H(\theta)\|p\|_\infty}{\Gamma(\alpha_1+\alpha_2)}\int_a^{t_1}\Big[\Big(F_s(t_2)\Big)^{\alpha_1+\alpha_2-1} - \Big(F_s(t_1)\Big)^{\alpha_1+\alpha_2-1}\Big]\psi'(s)ds \\
&\quad + \frac{H(\theta)\|p\|_\infty}{\Gamma(\alpha_1+\alpha_2)}\int_{t_1}^{t_2}\Big(F_s(t_2)\Big)^{\alpha_1+\alpha_2-1}\psi'(s)ds \\
&\le \frac{|\eta| + d_1\|\phi\|_C + d_2}{\Gamma(\alpha_2+1)}\Big[(F_a(t_2))^{\alpha_2} - (F_a(t_1))^{\alpha_2}\Big] \\
&\quad + \frac{d_1\theta + d_2}{\Gamma(\alpha_2+1)}\Big[\Big|(F_a(t_2))^{\alpha_2} - (F_a(t_1))^{\alpha_2}\Big| + |F_{t_1}(t_2)|^{\alpha_2}\Big] \\
&\quad + \frac{H(\theta)\|p\|_\infty}{\Gamma(\alpha_1+\alpha_2+1)}\Big[\Big|(F_a(t_2))^{\alpha_1+\alpha_2} - (F_a(t_1))^{\alpha_1+\alpha_2}\Big| + |F_{t_1}(t_2)|^{\alpha_1+\alpha_2}\Big].
\end{aligned}
$$

As $t_1 \to t_2$ the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $t_1 < t_2 \le 0$ and $t_1 \le 0 \le t_2$ is obvious.

As a consequence of Steps 1 to 3, it follows by the Arzelá-Ascoli theorem that $N : C([a-r,b],\mathbb{R}) \to C([a-r,b],\mathbb{R})$ is continuous and completely continuous.

Step 4: *We show that there exists an open set* $U \subseteq C([a-r,b],\mathbb{R})$ *with* $y \neq \lambda N(y)$ *for* $\lambda \in (0,1)$ *and* $y \in \partial U$. *Let* $y \in C([a-r,b],\mathbb{R})$ *and* $y = \lambda N(y)$ *for some* $0 < \lambda < 1$. *Then, for each* $t \in J$, *we have*

$$y(t) = \lambda\Big(\phi(a) + (\eta - g(a,\phi(a)))\frac{(F_a(b))^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{1}{\Gamma(\alpha_2)}\int_a^t \big(F_s(t)\big)^{\alpha_2-1}\psi'(s)g(s,y_s)ds$$
$$+ \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t \big(F_s(t)\big)^{\alpha_1+\alpha_2-1} f(s,y_s)ds\Big).$$

By our assumptions, for each $t \in J$, *we obtain*

$$\begin{aligned}
|y(t)| &\leq \|\phi\|_C + \Big[|\eta| + d_1\|\phi\|_C + d_2\Big]\frac{(F_a(b))^{\alpha_2}}{\Gamma(\alpha_2+1)} \\
&\quad + \frac{d_1\|y\|_{[a-r,b]} + d_2}{\Gamma(\alpha_2)}\int_a^t \big(F_s(t)\big)^{\alpha_2-1}\psi'(s)ds \\
&\quad + \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t \big(F_s(t)\big)^{\alpha_1+\alpha_2-1} p(s)H(\|y_s\|_C)\psi'(s)ds \\
&\leq \|\phi\|_C + \Big[|\eta| + d_1\|\phi\|_C + d_2\Big]\frac{(F_a(b))^{\alpha_2}}{\Gamma(\alpha_2+1)} + \Big[d_1\|y\|_{[a-r,b]} + d_2\Big](\frac{(F_a(b))^{\alpha_2}}{\Gamma(\alpha_2+1)} \\
&\quad + \frac{\|p\|_\infty H(\|y\|_{[a-r,b]})}{\Gamma(\alpha_1+\alpha_2+1)}(F_a(b))^{\alpha_1+\alpha_2},
\end{aligned}$$

which can be expressed as

$$\frac{\Big(1 - \frac{d_1(F_a(b))^{\alpha_2}}{\Gamma(\alpha_2+1)}\Big)\|y\|_{[a-r,b]}}{M_0 + H(\|y\|_{[a-r,b]})\|p\|_\infty \frac{1}{\Gamma(\alpha_1+\alpha_2+1)}(F_a(b))^{\alpha_1+\alpha_2}} \leq 1.$$

In view of (A6), there exists M *such that* $\|y\|_{[a-r,b]} \neq M$. *Let us set*

$$U = \{y \in C([a-r,b],\mathbb{R}) : \|y\|_{[a-r,b]} < M\}.$$

Note that the operator $N : \overline{U} \to C([a-r,b],\mathbb{R})$ *is continuous and completely continuous. From the choice of* U, *there is no* $y \in \partial U$ *such that* $y = \lambda Ny$ *for some* $\lambda \in (0,1)$. *Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.2.1), we deduce that* N *has a fixed point* $y \in \overline{U}$ *which is a solution of the problem* (3.1)-(3.3). *This completes the proof.*

The second existence result is based on Krasnoselskii's fixed point theorem.

Lemma 3.2.2 (Krasnoselskii's fixed point theorem [23])**.** *Let S be a closed, bounded, convex and nonempty subset of a Banach space X. Let A, B be the operators such that*

- *$Ax + Bx \in S$ whenever $x, y \in S$;*
- *A is compact and continuous;*
- *B is a contraction mapping.*

Then there exists $z \in S$ such that $z = Az + Bz$.

Theorem 3.2.2. *Assume that* (A2) *and* (A3) *hold. In addition we assume that*

(A7) $|f(t,x)| \le \mu(t)$, $|g(t,x)| \le \nu(t)$, *for all* $(t,x) \in J \times \mathbb{R}$, *and* $\mu, \nu \in C(J, \mathbb{R}^+)$.

Then problem (3.1)-(3.3) *has at least one solution on* $[a-r, b]$, *provided*

$$\frac{k(F_a(b))^{\alpha_2}}{\Gamma(\alpha_2+1)} < 1. \tag{3.6}$$

Proof 3.2.2. *We define the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ by*

$$\mathcal{G}_1 y(t) = \begin{cases} 0, & \text{if } t \in [a-r, a], \\ (\eta - g(a,\phi(a)))\dfrac{(F_a(b))^{\alpha_2}}{\Gamma(\alpha_2+1)} & \\ +\dfrac{1}{\Gamma(\alpha_2)}\displaystyle\int_a^t \left(F_s(t)\right)^{\alpha_2-1}\psi'(s) g(s, y_s)\,ds, & \text{if } t \in J. \end{cases} \tag{3.7}$$

$$\mathcal{G}_2 y(t) = \begin{cases} \phi(t), & \text{if } t \in [a-r, a], \\ \phi(a) + \dfrac{1}{\Gamma(\alpha_1+\alpha_2)}\displaystyle\int_a^t \left(F_s(t)\right)^{\alpha_1+\alpha_2-1}\psi'(s) f(s, y_s)\,ds, & \text{if } t \in J. \end{cases} \tag{3.8}$$

Setting $\sup_{t\in[a,b]} \mu(t) = \|\mu\|_\infty$, $\sup_{t\in[a,b]} \nu(t) = \|\nu\|_\infty$ *and choosing*

$$\rho \ge \|\phi\|_C + \left[|\eta| + 2\|\nu\|_\infty\right]\frac{(F_a(b))^{\alpha_2}}{\Gamma(\alpha_2+1)} + \|\mu\|_\infty \frac{(F_a(b))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}, \tag{3.9}$$

we consider $B_\rho = \{y \in C([a-r,b],\mathbb{R}) : \|y\|_\infty \le \rho\}$. *For any* $y, z \in B_\rho$, *we have*

$$\begin{aligned}
&|\mathscr{G}_1 y(t) + \mathscr{G}_2 z(t)| \\
&\le \sup_{t\in[a,b]} \Big\{ (\eta - g(a,\phi)) \frac{(F_a(t))^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{1}{\Gamma(\alpha_2)} \int_a^t \big(F_s(t)\big)^{\alpha_2-1} \psi'(s) g(s,y_s)\,ds \\
&\quad + \phi(a) + \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_a^t \big(F_s(t)\big)^{\alpha_1+\alpha_2-1} \psi'(s) f(s,y_s)\,ds \Big\} \\
&\le \|\phi\|_C + \Big[|\eta| + 2\|\nu\|_\infty \Big] \frac{(F_a(b))^{\alpha_2}}{\Gamma(\alpha_2+1)} + \|\mu\|_\infty \frac{(F_a(b))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} \\
&\le \rho.
\end{aligned}$$

This shows that $\mathscr{G}_1 y + \mathscr{G}_2 z \in B_\rho$. *Using* (3.6) *it is easy to see that* $\mathscr{G}_1$ *is a contraction mapping.*

Continuity of f *implies that the operator* $\mathscr{G}_2$ *is continuous. Also,* $\mathscr{G}_2$ *is uniformly bounded on* B_ρ *as*

$$\|\mathscr{G}_2 y\| \le \|\phi\|_C + \|\mu\|_\infty \frac{(F_a(b))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}.$$

Now we prove the compactness of the operator $\mathscr{G}_2$. *We define*

$$\bar{f} = \sup_{(t,y)\in[a,b]\times B_\rho} |f(t,y)| < \infty,$$

and consequently, for $t_1, t_2 \in [a,b]$, $t_1 < t_2$, *we have*

$$\begin{aligned}
&|\mathscr{G}_2 y(t_2) - \mathscr{G}_2 y(t_1)| \\
&\le \frac{\bar{f}}{\Gamma(\alpha_1+\alpha_2)} \int_a^{t_1} \Big| \Big(F_s(t_2)\Big)^{\alpha_1+\alpha_2-1} - \Big(F_s(t_1)\Big)^{\alpha_1+\alpha_2-1} \Big| ds \\
&\quad + \frac{\bar{f}}{\Gamma(\alpha_1+\alpha_2)} \int_{t_1}^{t_2} \Big(F_s(t_2)\Big)^{\alpha_1+\alpha_2-1} ds \\
&\le \frac{\bar{f}}{\Gamma(\alpha_1+\alpha_2+1)} \Big[\big| (F_a(t_2))^{\alpha_1+\alpha_2} - (F_a(t_1))^{\alpha_1+\alpha_2} \big| + |F_{t_1}(t_2)|^{\alpha_1+\alpha_2} \Big],
\end{aligned}$$

which is independent of y *and tends to zero as* $t_2 - t_1 \to 0$. *Thus,* $\mathscr{G}_2$ *is equicontinuous. So* $\mathscr{G}_2$ *is relatively compact on* B_ρ. *Hence, by the Arzelá-Ascoli theorem,* $\mathscr{G}_2$ *is compact on* B_ρ. *Thus all the assumptions of Lemma* 3.2.2 *are satisfied. So the conclusion of Lemma* 3.2.2 *implies that the problem* (3.1)-(3.3) *has at least one solution on* $[a-r,b]$

3.3 Examples

In this section we give an example to illustrate the usefulness of our main results. Let us consider the fractional functional differential equation,

$$\mathscr{D}_1^{\frac{2}{5};\log_e}\Big[\mathscr{D}_1^{\frac{1}{5};\log_e} y(t) - \frac{e^{-t}}{\sqrt{5}}\frac{\|y_t\|_C}{(1+\|y_t\|_C)}\Big] = \frac{\|y_t\|_C}{\sqrt{7}(1+\|y_t\|_C)} + t, \quad t \in J := [1,e], \tag{3.10}$$

$$y(t) = \phi(t), \quad t \in [1-r, 1], \tag{3.11}$$

$$\mathscr{D}_1^{\frac{1}{5};\log_e} y(1) = 1/2. \tag{3.12}$$

Let

$$f(t,x) = \frac{x}{\sqrt{7}(1+x)} + t, \quad g(t,x) = \frac{e^{-t}}{\sqrt{5}}\Big(\frac{x}{1+x}\Big), \quad (t,x) \in [1,e]\times[0,\infty).$$

For $x, y \in [0,\infty)$ and $t \in J$, we have

$$|f(t,x) - f(t,y)| = \frac{1}{\sqrt{7}}\Big|\frac{x}{1+x} - \frac{y}{1+y}\Big| = \frac{|x-y|}{\sqrt{7}(1+x)(1+y)} \le \frac{1}{\sqrt{7}}|x-y|,$$

and

$$|g(t,x) - g(t,y)| = \frac{e^{-t}}{\sqrt{5}}\Big|\frac{x}{1+x} - \frac{y}{1+y}\Big| = \frac{e^{-t}}{\sqrt{5}}\frac{|x-y|}{(1+x)(1+y)} \le \frac{1}{\sqrt{5}}|x-y|.$$

Hence conditions (A1) and (A2) hold with $\ell = \frac{1}{\sqrt{7}}$ and $k = \frac{1}{\sqrt{5}}$ respectively. Since $\frac{k(F_1(b))^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{\ell(F_1(b))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} \approx 0.910079666 < 1$, therefore, by Theorem 3.1.1, problem (3.10)-(3.12) has a unique solution on $[1-r,e]$.

Also $|f(t,x)| \le \frac{1}{\sqrt{7}} + t = \mu(t)$, $|g(t,x)| \le \frac{1}{\sqrt{5}}e^{-t} = \nu(t)$ and $k\frac{(F_1(b))^{\alpha_2}}{\Gamma(\alpha_2+1)} \approx 0.487071271 < 1$. Clearly the assumptions of Theorem 3.2.2 are satisfied. Consequently, by the conclusion of Theorem 3.2.2, there exists a solution of the problem (3.10)-(3.12) on $[1-r,e]$.

3.4 Initial value integral condition case

The results of this paper can be extended to the case of an initial value integral condition of the form

$$D_a^{\alpha_2,\psi} y(a) = \int_a^b W(s, y_s)\,ds, \tag{3.13}$$

where $W : J \times C([-r,0],\mathbb{R}) \to \mathbb{R}$ is a given function. In this case η will be replaced with $\int_a^b W(s, y_s)\,ds$ in (3.5) and the statement of the existence and uniqueness result for the problem (3.1)–(3.2)–(3.13) can be formulated as follows.

Theorem 3.4.1. *Assume that the conditions* (A1) *and* (A2) *hold. Further, we suppose that*

(A8) there exists a nonnegative constant m such that

$$|W(t,u) - W(t,v)| \le m\|u - v\|_C, \quad \text{for } t \in J \text{ and every } u, v \in C_r.$$

Then the problem (3.1)-(3.2)-(3.13) *has a unique solution on* $[a-r, b]$ *if*

$$\frac{\Big[m(b-1)+k\Big](F_a(b))^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{\ell(F_a(b))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} < 1.$$

We do not provide the proof of the above theorem as it is similar to that of Theorem 3.1.1.

The analog form of the existence results: Theorems 3.2.1 and 3.2.2 for the problem (3.1)-(3.2)-(3.13) can be constructed in a similar manner.

Bibliography

[1] R. P. Agarwal, S. Hristova, D. O'Regan; A survey of Lyapunov functions, stability and impulsive Caputo fractional differential equations, *Fract. Calc. Appl. Anal*, **19** (2016), 290-318.

[2] B. Ahmad, S. K. Ntouyas; Nonlocal fractional boundary value problems with slit-strips boundary conditions, *Fract. Calc. Appl. Anal*, **18** (2015), 261-280.

[3] B. Ahmad, S. K. Ntouyas; Initial value problems of fractional order Hadamard-type functional differential equations, *Electron. J. Differential Equ*, Vol. 2015 (2015), No. 77, 9 pp.

[4] B. Ahmad, S. K. Ntouyas, J. Tariboon; Fractional differential equations with nonlocal integral and integer-fractional-order Neumann type boundary conditions, *Mediterr. J. Math*, **13** (2016), 2365-2381.

[5] B. Ahmad, S. K. Ntouyas; Initial value problems for functional and neutral functional Hadamard type fractional differential inclusions, Miskolc Math. Notes, **17** (2016), 15-27.

[6] R. Almeida; Malinowska, A.B.; Monteiro, M.T.T. Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications. Math. Methods Appl. Sci. 2018, 41, 336?352.

[7] M. Benchohra, J. Henderson, S. K. Ntouyas, A. Ouahab; Existence results for fractional order functional differential equations with infinite delay, *J. Math. Anal. Appl.* **338** (2008), 1340-1350.

[8] N. Boccara, Analyse Fonctionnelle une introduction pour physiciens, 1984.

[9] P. L. Butzer, A. A. Kilbas, J. J. Trujillo; Compositions of Hadamard-type fractional integration operators and the semigroup property, *J. Math. Anal. Appl.* **269** (2002), 387-400.

[10] P. L. Butzer, A. A. Kilbas, J. J. Trujillo; Fractional calculus in the Mellin setting and Hadamard-type fractional integrals, *J. Math. Anal. Appl*, **269** (2002), 1-27.

[11] P. L. Butzer, A. A. Kilbas, J. J. Trujillo; Mellin transform analysis and integration by parts for Hadamard-type fractional integrals, *J. Math. Anal. Appl*, **270** (2002), 1-15.

[12] B.C. Dhage, S. K. Ntouyas; Existence results for boundary value problems for fractional hybrid differential inclucions, *Topol. Methods Nonlinar Anal.* **44** (2014), 229-238.

[13] K. Diethelm; *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics, Springer-verlag Berlin Heidelberg, 2010.

[14] Y. Ding, Z. Wei, J. Xu, D. O'Regan; Extremal solutions for nonlinear fractional boundary value problems with p-Laplacian, *J. Comput. Appl. Math*, 288 (2015), 151-158.

[15] A. M. A. El-Sayed; Nonlinear functional differential equations of arbitrary orders, *Nonlinear Anal*, **33** (1998), 181-186.

[16] H. Ergören; Impulsive retarded fractional differential equations with variable moments, *Contemporary Analysis and Applied Mathematics*, **4** (2016), 156-170.

[17] A. Granas, J. Dugundji; *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[18] J. Hadamard; Essai sur l'etude des fonctions donnees par leur developpment de Taylor, *J. Mat. Pure Appl. Ser.* **8** (1892) 101-186.

[19] J. Hale, S. M. Verduyn Lunel; *Introduction to Functional Differential Equations*, Applied Mathematical Sciences, 99, Springer-Verlag, New York, 1993.

[20] J. Henderson, R. Luca; On a system of Riemann-Liouville fractional boundary value problems, *Comm. Appl. Nonlinear Anal.* **23** (2016), 1-19.

[21] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo; *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, 204. Elsevier Science B.V, Amsterdam, 2006.

[22] A. Kolmogorov, S. Fomine, Eléments de la théorie des fonctions et de l?analyse fonctionnelle., 2ème éditions Mir-Moscou. 1974.

[23] M. A. Krasnoselskii; Two remarks on the method of successive approximations, *Uspekhi Mat. Nauk*, **10** (1955), 123-127.

[24] V. Lakshmikantham, S. Leela, J. V. Devi; *Theory of Fractional Dynamic Systems*, Cambridge Academic Publishers, Cambridge, 2009.

[25] K. S. Miller, B. Ross; *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

[26] I. Podlubny; *Fractional Differential Equations*. Academic Press, San Diego, 1999.

[27] S. G. Samko, A. A. Kilbas, O. I. Marichev; *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[28] L. Schwartz, Analyse Topologie générale et analyse fonctionnelle, Édition Corrigée Paris. 2008.

[29] D.R. Smart : Fixed point theorems. Cambridge University Press, Cambridge (1980).

[30] J. R. Wang, M. Feckan, Y. Zhou; A survey on impulsive fractional differential equations, *Fract. Calc. Appl. Anal*, **19** (2016), 806-831.

[31] W. Yang, Positive solutions for singular coupled integral boundary value problems of nonlinear Hadamard fractional differential equations. J. Nonlinear Sci. Appl. 8 (2015), 110-129.

[32] L. Zhang, B. Ahmad, G. Wang; Successive iterations for positive extremal solutions of nonlinear fractional differential equations on a half line, *Bull. Aust. Math. Soc*, 91 (2015), 116-128.

[33] Y. Zhou; *Basic theory of fractional differential equations*, World Scientific Publishing Co. Pte. Ltd, Hackensack, NJ, 2014.