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Ghellab Amira

Title

Non-Periodic Boundary Value Problem for a fractional differential equation of the Jerk type

Defended on :

Before the jury composed of:

Full name	Rank	University	
Mr Ellaggoune Fateh	**PROF**	Univ. of Guelma	President
Mm Berhail Amel	**MCA**	Univ. of Guelma	Supervisor
Mm Frioui Assia	**MCA**	Univ. of Guelma	Examiner
Mr Ali Ahmed	**MCB**	Univ. of Guelma	Examiner

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Abstract

In this work, we consider a model of the Jerk problem of fractional order in the G-Caputo sense with non periodic conditions. Firstly, we establish the existence and uniqueness of the solution, which is achieved via the Schauder fixed point theorem and Banach contraction principle. Moreover, we explore the stability of the solution to our problem in Ulam-Hyers and Ulam-Hyers–Rassias sense. Finally, we provide a numerical example in order to illustrate the obtained results.

Key words : Jerk problem, G-Caputo fractional derivative, Schauder fixed point theorem, Banach contraction principle, Ulam-Hyers stability, Ulam-Hyers–Rassias stability.

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Résumé

Dans notre travail, nous considérons un problème de Jerk d'ordre fractionnaire au sens G-Caputo detivative avec des conditions non périodiques. Tout d'abord, nous établissons l'existence et l'unicité de la solution, ce qui est réalisé via le théorème du point fixe de Schauder et le principe de contraction de Banach. De plus, nous explorons la stabilité de la solution de notre problème au sens Ulam-Hyers et d'Ulam-Hyers–Rassias. Enfin, nous fournissons un exemple numérique afin d'illustrer les résultats obtenus.

Mots clés: Problème de Jerk, dérivée fractionnaire de G-Caputo, théorème du point fixe de Schauder, principe de contraction de Banach, Stabilité d'Ulam-Hyers, Stabilité d'Ulam-Hyers–Rassias.

Contents

Notations and Abbreviations

- $\mathbb{N}$ Set of integers numbers.
- $\mathbb{R}$ Set of real numbers.
- $\mathbb{C}$ Set of complex numbers.
- $\Gamma(.)$ Gamma function.
- $\beta(.,.)$ Beta function.
- $Re(z)$ The real part of the complex number z.
- $I_a^\alpha f$ Riemann-Liouville fractional integral of order α of function f.
- ${}^cD_a^\alpha f$ Caputo fractional derivative of order α of function f.
- ${}^{RL}D_a^\alpha f$ Riemann-Liouville fractional derivative of order α of a function f.
- $\mathscr{I}_{a^+}^{\alpha;G} f$ The G-Caputo integral of order α for a function f.
- ${}^c\mathscr{D}_{a^+}^{\alpha;G} f$ The G-Caputo derivative of order α of a function f.
- X, Y Normed vectoriel space.
- $C(X, Y)$ Space of continuous functions from X to Y.

Introduction

Fractional calculus explores integrals and derivatives of functions, however, in this branch of mathematics we are not looking at the usual integer order but at the non-integer order integrals and derivatives. These are called fractional derivatives and fractional integrals, which can be of real or complex orders and therefore also include integer orders.
What if we would like to take the $\frac{1}{2}$-th order derivative?
This question was already mentioned in a letter from the mathematician Leibniz to L'Hopital in 1695. Leibniz denoted the aforementioned concept as a paradox containing potential utilities within practical contexts. However, it was not until 1990 that noteworthy advancements in this domain were realized. Since then several famous mathematicians, such as Grunwald, Letnikov, Riemann, Liouville and many more, have dealt with this problem ([7, 11, 14]).

In physics, jerk is the third derivative of an object's position with respect to time, or in other words, it is the rate of change of acceleration. The Jerk equations are extensively utilized in engineering and applied sciences. More recently, the investigation of jerk equations have concentrated the attention of scholars on itself and several new findings have been obtained in this regard. Some limited publications in the context of Jerk equations are ([5, 6, 8, 9, 10, 12]).

In this work we focus our attention on the problem of the existence, uniqueness and stability of solutions for a nonlinear fractional jerk system with non-periodic boundary conditions.

This manuscrit is organized as follow
In the first Chapter, we provide some preliminary concepts regarding fractional calculus, the notion of compactness, and some fixed point theorems, which will be beneficial for the upcoming chapters.
In the seconde chapter, we consider a nonlinear fractional jerk system with respect to G-Caputo derivative subject to non-periodic boundary conditions. Some qualitative analysis of

the solution such as existence and uniqueness are investigated by using the Banach contraction and Shauder fixed point theorems.

In the last Chapter Hyers-Ulam and Hyers-Ulam-Rassias stability criterions are considered and investigated.

Finally, we present an example illustrating the obtained results.

Chapter 1

Preliminaries

In this chapter, we present some fundamental theories related to fractional calculus. In this context, the focus will intentionally be on different approaches to generalizing the concepts of differentiation and integration for a fractional order.

1.1 Special functions

1.1.1 Gamma Function

One of the fundamental functions for fractional calculus is the Gamma function, which extends the factorial function to the set of complex numbers.

Definition 1.1 [7] The Gamma function $\Gamma(z)$ is defined by the following integral

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} \mathrm{d}t, \;\; Re(z) > 0, \tag{1.1}$$

where the improper integral converges absolutely in the complex half-plane where the real part is strictly positive. obviously, $\Gamma(1) = 1$ and $\Gamma(n+1) = n!$.

Remark 1.1 :
1) The Gamma function is strictly decreasing for $0 < z < 1$, and moreover, we have

$$\Gamma(z+1) = z\Gamma(z), \quad \forall z \in \mathbb{C}.$$

In general, we have

$$\Gamma(z+n) = z(z+1)(z+2)......(z+n-1)\Gamma(z), \;\; n \geq 1.$$

2) The Gamma function is verified

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{sin(\pi z)}.$$

We will obtain this formula under the condition $0 < Re(z) < 1$ and then show that it holds for $z \neq 0,\ \pm 1,\ \pm 2...$

Special values

The value of $\Gamma(1/2) = \sqrt{\pi}$ allows, through recurrence, the determination of other values of the gamma function for positive half-plan.

$$\Gamma(3/2) = \frac{\sqrt{\pi}}{2},\ \Gamma(5/2) = \frac{3\sqrt{\pi}}{4}, ...,$$

in general, we have

$$\Gamma(n + \frac{1}{2}) = \frac{(2n)!}{2^{2n} n!}\sqrt{\pi},\ \forall n \in \mathbb{N}.$$

1.1.2 Beta Function

The so-called Beta function, which is an Euler type integral, insteacl of a certain combination of values of the Gamma function.

Definition 1.2 [7] Let $p, q \in \mathbb{C}$ such that $Re(p) > 0$ and $Re(q) > 0$, the function beta is defined by

$$\beta(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1} dt. \tag{1.2}$$

Remark 1.2 [7] By the change of variable $s = 1 - t$, we have

$$\beta(p,q) = \beta(q,p). \tag{1.3}$$

Proposition 1.1 *[7][Relation between the Gamma and Beta functions] Let $p, q \in \mathbb{C}$ such that $Re(p) > 0$ and $Re(q) > 0$ we have*

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \tag{1.4}$$

This relation allows the continuation of the beta function over the entire complex plane, provided that the gamma function is also continued.

1.2 Fractional Calculus

This section will be devoted to basic definitions of Riemann-Liouville and Caputo fractional integrals and derivative.

1.2.1 Riemann-Liouville fractional derivative

Definition 1.3 [7] Let $\Omega = [a, b]$ be a finite interval on the real axis $\mathbb{R}$ and f is a continuous function in Ω. The Riemann-Liouville fractional integral $I_a^\alpha f(t)$ of order $\alpha > 0$ of the function f is defined by

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad t > a, \ n < \alpha < n+1. \tag{1.5}$$

Definition 1.4 [7] The Riemann-Liouville fractional derivative (noted by ${}^{RL}D^\alpha$) of order $\alpha > 0$ of the function $f \in L^1(\Omega)$ is defined by

$$\begin{aligned} {}^{RL}D_a^\alpha f(t) &= \left(\frac{d}{dt}\right)^n (I_a^{n-\alpha} f)(t) && (1.6) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s) ds, && (1.7) \end{aligned}$$

with $n - 1 < \alpha < n, \ n \in \mathbb{N}^*$.
When $\alpha = n$, then

$${}^{RL}D_a^\alpha f(t) = f^{(n)}(t),$$

where $f^{(n)}$ is the usuel derivative of y of order n.

Remark 1.3 :
The Riemann-Liouville fractional integral is a simple generalization of the following Cauchy formula

$$\begin{aligned} I_a^n f(t) &= \int_a^t dt_1 \int_a^{t_1} dt_2 \ldots.. \int_a^{t_{n-1}} f(t_n) dt_n \\ &= \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s) ds, \quad n \in \mathbb{N}^*. \end{aligned}$$

Example 1.1 :
We consider the function f defined by

$$f(t) = (t-a)^\beta, \quad \beta \in \mathbb{R}.$$

We have

$$\begin{aligned} I_a^\alpha f(t) &= I_a^\alpha (t-a)^\beta = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}(s-a)^\beta ds \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(t-a)^{\alpha+\beta}. \end{aligned}$$

Moreover, we have

$${}^{RL}D_a^\alpha (t-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}.$$

1.2.2 Caputo fractional derivative

The Riemann-Liouville approach has initial conditions that include the boundary values of the Riemann-Liouville fractional derivatives at the lower limit. Despite the fact that initial value problems with such initial conditions can be solved mathematically, their solutions are practically useless, as there is no physical interpretation for this type of initial condition. A potential solution to this problem was proposed by M. Caputo.

Definition 1.5 [7] Let $[a, b]$ be a finite interval of the real line $\mathbb{R}$ and let f be a function of class $C^n([a, b])$. The fractional Caputo derivative ${}^cD_a^\alpha$ of order $\alpha > 0$ of the function f is defined through the Riemann-Liouville fractional derivative, that is to say

$${}^cD_a^\alpha f(t) =^{RL} D_a^\alpha \left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k \right], \tag{1.8}$$

with $n-1 < \alpha < n, \quad n \in \mathbb{N}^*$.
We deduce that if $f^k(a) = 0$ for $k = 0, 1, 2, ..., n-1$, we get

$${}^cD_a^\alpha f(t) =^{RL} D_a^\alpha f(t).$$

Definition 1.6 [7] The Caputo derivative of order $\alpha > 0$ of the function f of class $C^n([a, b])$ is defined by

$${}^cD_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > a,$$

with $n-1 < \alpha < n, \ n \in \mathbb{N}^*$.
When $\alpha = n \in \mathbb{N}$, then

$${}^cD_a^\alpha f(t) = f^{(n)}(t),$$

where $f^{(n)}$ is the usuel derivative of y of order n.

Remark 1.4 One difference between the Riemann-Liouville definition and the Caputo definition is that the Caputo derivative of a constant is zero, where as the Riemann-Liouville fractional derivative of a constant C is

$$^{RL}D_a^\alpha C = \frac{Ct^{-\alpha}}{\Gamma(1-\alpha)} \neq 0.$$

Lemma 1.1 *[7]:*

1) Let $f \in C([a,b])$, $\forall t \in [a,b]$, $\forall \alpha \in]n-1, n[$, we have the following propertie

$$^{c}D_a^\alpha I_a^\alpha f(t) = f(t).$$

2) Let $f \in C^n([a,b])$, $\forall t \in [a,b]$, $\forall \alpha \in]n-1, n[$, we have the following propertie

$$I^\alpha({}^cD^\alpha) f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k. \tag{1.9}$$

Theorem 1.1 *Let $Re(\alpha) > 0$. If $f(t) \in C^n[a,b]$ then the Caputo fractional derivative ${}^cD_a^\alpha f$ exist almost every where on $[a,b]$ and we have*

$$^{c}D_a^\alpha f(t) = I^{n-\alpha} D^n f(t), \quad t > a. \tag{1.10}$$

1.2.3 G-Caputo fractional derivative

Some primitive notions, definitions and notations about G-Caputo derivative, which will be utilized throughout the manuscript, are recalled here.

Definition 1.7 [7] Let $[a,b]$ be a finite interval of the real line $\mathbb{R}$ and $\alpha > 0$. Let $G \in C^1([a,b], \mathbb{R})$ be an increasing function having a continuous derivative such $G'(t) \neq 0$ on $[a,b] \subset \mathbb{R}$. The α^{th} G- integral for an integrable function $\mathrm{x} : [a,b] \to \mathbb{R}$ with respect to G is given as

$$\mathscr{I}_{a^+}^{\alpha;G} \mathrm{x}(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (G(t) - G(s))^{\alpha-1} G'(s) \mathrm{x}(s)\, \mathrm{d}s,\ t > a, \tag{1.11}$$

in which $n < \alpha < n+1, n \in \mathbb{N}$.

Remark 1.5 This integral gives the Caputo integral when $G(t) = t$. Also, in the case $G(t) = \ln t$, it yields the Hadamard derivative given by

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (\ln t - \ln s)^{\alpha-1} \frac{f(s)}{s} ds, \quad t > a > 0,\ n < \alpha < n+1. \tag{1.12}$$

Definition 1.8 [7] Let $n \in \mathbb{N}$, $\mathrm{x} \in C^n([a,b],\mathbb{R})$, and $G \in C^n([a,b],\mathbb{R})$. The α^{th} G-Caputo derivative of x is defined by

$$ {}^c\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(t) = \mathscr{I}_{a^+}^{n-\alpha;G}\ \partial_G^n \mathrm{x}(t), $$

in which $n = [\alpha]+1$ for $\alpha \notin \mathbb{N}$, $n = \alpha$ for $\alpha \in \mathbb{N}$ and $\partial_G = \left(\frac{1}{G'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)$.
In other words,

$$ {}^c\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(t) = \begin{cases} \int_a^t \frac{G'(s)(G(t)-G(s))^{n-\alpha-1}}{\Gamma(n-\alpha)}\partial_G^n \mathrm{x}(s)\,\mathrm{d}s, & \alpha \notin \mathbb{N}, \\ \\ \partial_G^n \mathrm{x}(t), & \alpha = n \in \mathbb{N}. \end{cases} $$

Remark 1.6 :

If $\mathrm{x} \in C^{n-1}([a,b],\mathbb{R})$, the α^{th} G-Caputo derivative of x is specified as

$$ {}^c\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(t) = \mathscr{D}_{a^+}^{\alpha;G}\left(\mathrm{x}(t) - \sum_{k=0}^{n-1}\frac{\partial_G^k \mathrm{x}(a)}{k!}(G(t)-G(a))^k\right). $$

The composition rules for above G-operators are recalled in this lemma.

Lemma 1.2 *[2] Let $n-1<\alpha<n$, $G \in C^n([a,b],\mathbb{R})$, and $\mathrm{x} \in C^{n-1}([a,b],\mathbb{R})$. Then the following holds*

$$ \mathscr{I}_{a^+}^{\alpha;G}\ {}^c\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(t) = \mathrm{x}(t) - \sum_{k=0}^{n-1}\frac{\partial_G^k \mathrm{x}(a)}{k!}\left[G(t)-G(a)\right]^k, $$

for all $t \in [a,b]$. Moreover, if $m \in \mathbb{N}$ and $\mathrm{x} \in C^{n+m-1}([a,b],\mathbb{R})$, then, the following holds:

$$ \partial_G^m\left({}^c\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}\right)(t) = {}^c\mathscr{D}_{a^+}^{\alpha+m;G}\mathrm{x}(t) + \sum_{k=0}^{n-1}\frac{[G(t)-G(a)]^{k+n-\alpha-m}}{\Gamma(k+n-\alpha-m+1)}\partial_G^{k+m}\mathrm{x}(a). $$

Observe that if $\partial_G^k \mathrm{x}(a) = 0$, $\forall\, k = n, n+1, \ldots, n+m-1$, we can get the following relation

$$ \partial_G^m\left({}^c\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}\right)(t) = {}^c\mathscr{D}_{a^+}^{\alpha+m;G}\mathrm{x}(t), \qquad t \in [a,b]. $$

Lemma 1.3 *[2] Let $\alpha, l > 0$, and $x \in C([a,b],\mathbb{R})$. Then $\forall t \in [a,b]$ and we suppose that $F_a(t) = G(t)-G(a)$, we have*

1. $\mathscr{I}_a^{\alpha;G}\mathscr{I}_a^{l;G}x(t) = \mathscr{I}_a^{\alpha+l;G}x(t)$.

2. ${}^c\mathscr{D}_{a^+}^{\alpha;G}\mathscr{I}_a^{\alpha;G}x(t) = x(t)$.

3. $\mathscr{I}_a^{\alpha;G}(F_a(t))^{l-1} = \dfrac{\Gamma(l)}{\Gamma(l+\alpha)}(F_a(t))^{l+\alpha-1}$.

4. ${}^c\mathscr{D}_{a^+}^{\alpha;G}(F_a(t))^{l-1} = \dfrac{\Gamma(l)}{\Gamma(l-\alpha)}(F_a(t))^{l-\alpha-1}$.

5. ${}^c\mathscr{D}_{a^+}^{\alpha;G}(F_a(t))^k = 0,\ k \in \{0,...,n-1\},\ n \in \mathbb{N},\ \alpha \in (n-1,n)$.

1.3 Compact Operator

Definition 1.9 [17] [Relatively compact sets]A set U of a normed space X is relatively compact if the closure $\overline{U}$ is compact it means if every sequence of points in U has a cluster point in X.

Remark 1.7 Relatively compact sets are granting some compactness properties. They are commonly used to study the convergence and properties of sequences and continuous functions.

Definition 1.10 [17] [Compact space] Let (E, d) be a metric space. We say that (E, d) is a compact space if and only if, for every open covering of E, we can extract a finite open subcovering.

A space is compact if it is relatively compact in itself.

Definition 1.11 [18][Completely continuous Operator] A bounded linear operator T acting from a Banach space X into another Banach space Y is completely continuous if it transforms weakly-convergent sequences in X to norm-convergent sequences in Y.
Equivalently, the operator $T$is completely continuous if it maps every relatively weakly compact subset of X into a relatively compact subset of Y.

Remark 1.8 :
1) It is easy to see that every completely continuous operator is compact, however the converse is false.
2) If X is reflexive, the two classes of operators (completely continuous operator and compact) do coincide.

Definition 1.12 [22][Equicontinuity] Let X, Y be two Banach spaces and let $A \in C(X, Y)$ be a family of continuous operator defined from X into Y. The family A is equicontinuous at a point $a \in X$ if and only if

$$\forall \epsilon > 0, \exists \delta > 0,\ \forall x \in X, \|x - a\|_X < \delta \Rightarrow \forall T \in A, \|T(x) - T(a)\|_Y < \epsilon,$$

The family is equicontinuous if it is equicontinuous at each point of X.

1.3.1 Ascoli-Arzela Theorem

Arzela-Ascoli theorem, demonstrated by Italian mathematicians Giulio Ascoli and Cesare Arzela, characterizes, using the notion of equicontinuity, the relatively compact subsets of the space of continuous functions from a compact space into a metric space.

Theorem 1.2 *[19] Let (X, d_X) be a compact metric space and (Y, d_Y) a complete metric space, a subset A of $C(X, Y)$ is relatively compact if and only if:*
1) A is equicontinuous on X i.e., for all $\epsilon > 0$, $\exists \delta > 0$ such that

$$|t_1 - t_2| < \delta \Rightarrow ||f(t_1) - f(t_2)|| \leqslant \epsilon, \ \forall \ t_1 > t_2 > 0 \text{ and } \forall f \in A.$$

2) The set A is uniformly bounded, i.e., there exists a constant $K > 0$ such that

$$||f(x)|| \leqslant K, \ \forall \ x \in X \text{ and } f \in A.$$

1.3.2 Point fixed Theorems

Theorem 1.3 *[21][Banach fixed-point theorem] Let $X = (X, \|.\|_X)$ be a Banach space, and let $T : X \to X$ be a contraction mapping on X i.e. such that*

$$\exists \ 0 < k < 1, \ \|Tu - Tv\| \leq k\|u - v\|, \ \forall \ u, v \in X.$$

Then, T admits a unique fixed point in X, i.e Tu=u.

Theorem 1.4 *[20][Schauder fixed point theorem] Let $X = (X, \|.\|_X)$ be a Banach space and let U be a closed convex subset of X. Let $T : U \to U$ be a continuous and compact mapping. Then T admits a fixed point belonging to U.*

Chapter 2

Existence and uniqueness of the solution of a non-periodic FDE of the Jerk type

2.1 Introduction

In physics, jerk is the third derivative of an object's position with respect to time, or in other words, it is the rate of change of acceleration. Mathematically, if $a(t)$ represents an object's acceleration at time t, then the jerk $j(t)$ is given by the following expression

$$j(t) = \frac{da}{dt}.$$

Jerk is a vector quantity because acceleration is a vector (having both magnitude and direction). The unit of jerk is meters per second cubed (m/s^3), or, in some contexts, it may be expressed in terms of standard gravity per second (g/s).

Understanding jerk is important in the study of motion because it provides information on how an object's acceleration changes over time. Jerk becomes particularly relevant in situations where smooth and controlled motion is important, such as in robotic systems, transportation systems, or any application where sudden changes in acceleration can have undesirable effects.

As a vector, jerk can be expressed as the first time derivative of acceleration, the second time derivative of velocity, and the third time derivative of position

$$\begin{cases} \frac{\mathrm{dx}}{\mathrm{dt}} = v(t), \\ \frac{dv}{\mathrm{dt}} = a(t), \\ \frac{da}{\mathrm{dt}} = j(t), \end{cases}$$

where

-a is the acceleration,

- v is the velocity,

- x is the position, and t is the time.

Third-order differential equations of the form

$$F(x''', x'', x', x) = 0,$$

are sometimes called jerk equations.

Example 2.1 Consider the effects of acceleration and jerk when traveling by car

Skilled and experienced drivers can accelerate smoothly, but beginners often cause jerky driving. When shifting gears in a car with a foot-operated clutch, the acceleration force is limited by the engine power, but an inexperienced driver may cause severe jerk due to the intermittent closure of force on the clutch.

The sensation of being pressed against the seats in a powerful sports car is due to acceleration. When the car starts from a stop, there is a large positive jerk as its acceleration increases rapidly. After starting, there is a small sustained negative jerk as the air resistance force increases with the car's speed, gradually reducing acceleration and decreasing the force pressing the passenger against the seat. When the car reaches its maximum speed, acceleration reaches zero and remains constant, after which there is no more jerk until the driver decelerates or changes direction.

In sudden braking or collisions, passengers are thrown forward with an initial acceleration greater than during the rest of the braking process, as muscle tension quickly regains control of the body after the beginning of the braking or impact. These effects are not modeled in vehicle tests because crash test dummies and cadavers do not have active muscle control.

To minimize jerk, curves along roads are designed to be clothoids, as are railroad curves and roller coaster loops.

2.2 Statement of the problem

We consider the following fractional Jerk problem

$$\begin{cases} {}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(t) = \mathrm{x}_1(t), \quad a \leq t \leq b, \quad \mathrm{x}(a) = \lambda \mathrm{x}(b), \\ {}^{c}\mathscr{D}_{a^+}^{\beta;G}\mathrm{x}_1(t) = \mathrm{x}_2(t), \qquad \mathrm{x}_1(a) = \mu \mathrm{x}_1(b), \\ {}^{c}\mathscr{D}_{a^+}^{\gamma;G}\mathrm{x}_2(t) = f(t, \mathrm{x}, \mathrm{x}_1, \mathrm{x}_2), \qquad \mathrm{x}_2(a) = \nu \mathrm{x}_2(b), \end{cases} \tag{2.1}$$

where ${}^{c}\mathscr{D}_{a^+}^{\omega;G}$, $\omega \in \{\alpha, \beta, \gamma\}$ are the G-Caputo derivative such that $0 < \alpha, \beta, \gamma \leq 1$, the increasing function $G \in C^1([a,b])$ such that $G'(t) \neq 0$, $\forall\, t \in [a,b]$, $f \in C([a,b] \times \mathbb{R}^3, \mathbb{R})$ et $\lambda, \mu, \nu \in \mathbb{R}/\{1\}$. We can write this system in the form of

$$\begin{cases} {}^{c}\mathscr{D}_{a^+}^{\gamma;G}\left({}^{c}\mathscr{D}_{a^+}^{\beta;G}\left({}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(t)\right)\right) = f\left(t, \mathrm{x}(t), {}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(t), {}^{c}\mathscr{D}_{a^+}^{\beta;G}\left({}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(t)\right)\right), \\ \mathrm{x}(a) = \lambda \mathrm{x}(b), \\ {}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(a) = \mu {}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(b), \\ {}^{c}\mathscr{D}_{a^+}^{\beta;G}\left({}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(a)\right) = \nu {}^{c}\mathscr{D}_{a^+}^{\beta;G}\left({}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(b)\right), \end{cases} \tag{2.2}$$

In this study, we analyze the existence and uniqueness of solutions for the proposed fractional non-periodic Jerk problem.

2.3 Integral Equation

We need the following lemma, which defines the corresponding integral equation.

Lemma 2.1 *Let $\alpha, \beta, \gamma \in (0,1]$, $\lambda, \mu, \nu \neq 1$, and ${}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}, {}^{c}\mathscr{D}_{a^+}^{\beta;G}({}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}) \in C([a,b],\mathbb{R})$, for a given function $g \in C([a,b],\mathbb{R})$, the integral equation of the following problem*

$$\begin{cases} {}^{c}\mathscr{D}_{a^+}^{\gamma;G}\left({}^{c}\mathscr{D}_{a^+}^{\beta;G}\left({}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(t)\right)\right) = g\left(t\right), \\ \mathrm{x}(a) = \lambda \mathrm{x}(b), \\ {}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(a) = \mu {}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(b), \\ {}^{c}\mathscr{D}_{a^+}^{\beta;G}\left({}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(a)\right) = \nu {}^{c}\mathscr{D}_{a^+}^{\beta;G}\left({}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(b)\right), \end{cases} \tag{2.3}$$

is given by

$$\begin{aligned} x(t) &= \frac{\nu}{(1-\nu)}\left(\frac{\mu(G(b)-G(a))^{\beta}}{(1-\mu)\Gamma(\alpha+1)\Gamma(\beta+1)}\left(\frac{\lambda(G(b)-G(a))^{\alpha}}{(1-\lambda)}+(G(t)-G(a))^{\alpha}\right)\right. \quad (2.4)\\ &+ \left.\frac{1}{\Gamma(\beta+\alpha+1)}\left((G(t)-G(a))^{\beta+\alpha}+\frac{\lambda(G(b)-G(a))^{\alpha+\beta}}{(1-\lambda)}\right)\right)\mathscr{I}_a^{\gamma;G}g\,(b)\\ &+ \frac{\mu}{(1-\mu)\Gamma(\alpha+1)}\left(\frac{\lambda(G(b)-G(a))^{\alpha}}{(1-\lambda)}+(G(t)-G(a))^{\alpha}\right)\mathscr{I}_a^{\beta+\gamma;G}g\,(b)\\ &+ \mathscr{I}_a^{\alpha+\beta+\gamma;G}\left(g(t)+\frac{\lambda g(b)}{1-\lambda}\right). \end{aligned}$$

Proof. Consider $x(t)$ satisfying the jerk problem (2.3). By using the γ-th integral operator $\mathscr{I}_a^{\gamma;G}$ to both sides of equation (2.3), we obtain

$$ {}^{c}\mathscr{D}_{a^+}^{\beta;G}\left({}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(t)\right)=c_0+\mathscr{I}_a^{\gamma;G}g(t), \quad (2.5)$$

and

$$ {}^{c}\mathscr{D}_{a^+}^{\beta;G}\left({}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(b)\right)=c_0+\mathscr{I}_a^{\gamma;G}g\,(b)\,,$$

then

$$c_0={}^{c}\mathscr{D}_{a^+}^{\beta;G}\left({}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(b)\right)-\mathscr{I}_a^{\gamma;G}g\,(b)={}^{c}\mathscr{D}_{a^+}^{\beta;G}\left({}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(a)\right)-\mathscr{I}_a^{\gamma;G}g\,(a)\,.$$

By using the boundary condition, we obtain

$$c_0=\nu\,{}^{c}\mathscr{D}_{a^+}^{\beta;G}\left({}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(b)\right)=\frac{\nu}{1-\nu}\mathscr{I}_a^{\gamma;G}g\,(b)\,.$$

Similarly, applying the $\mathscr{I}_a^{\beta;G}$ to both sides of equation (2.6), we get

$$\mathscr{I}_a^{\beta;G}\left({}^{c}\mathscr{D}_{a^+}^{\beta;G}\left({}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(t)\right)\right)=\mathscr{I}_a^{\beta;G}\left(c_0+\mathscr{I}_a^{\gamma;G}g(t)\right).$$

Then

$$\begin{aligned} {}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(t) &= c_1+\mathscr{I}_a^{\beta;G}c_0+\mathscr{I}_a^{\beta+\gamma;G}g(t) \quad (2.6)\\ &= c_1+\frac{\nu(G(t)-G(a))^{\beta}}{(1-\nu)\Gamma(\beta+1)}\mathscr{I}_a^{\gamma;G}g\,(b)+\mathscr{I}_a^{\beta+\gamma;G}g(t). \end{aligned}$$

By using the seconde boundary condition, we obtain

$$ {}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(a)=\mu\,{}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(b),$$

then

$$c_1=\mu\,{}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(b)=\frac{\mu\nu(G(b)-G(a))^{\beta}}{(1-\mu)(1-\nu)\Gamma(\beta+1)}\mathscr{I}_a^{\gamma;G}g(b)+\frac{\mu}{1-\mu}\mathscr{I}_a^{\beta+\gamma;G}g(b).$$

Next, we apply the α-th integral operator to both side of equation (2.5), we obtain

$$\mathscr{I}_a^{\alpha;G}\left({}^c\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(t)\right) = \mathscr{I}_a^{\alpha;G}\left(c_1 + \mathscr{I}_a^{\beta;G}c_0 + \mathscr{I}_a^{\beta+\gamma;G}g(t)\right),$$

$$\mathrm{x}(t) = c_2 + \mathscr{I}_a^{\alpha;G}c_1 + \mathscr{I}_a^{\alpha+\beta;G}c_0 + \mathscr{I}_a^{\alpha+\beta+\gamma;G}g(t),$$

it means that

$$\begin{aligned}\mathrm{x}(t) &= c_2 + \frac{(G(t)-G(a))^\alpha}{\Gamma(\alpha+1)}\left(\frac{\mu\nu(G(b)-G(a))^\beta}{(1-\mu)(1-\nu)\Gamma(\beta+1)}\mathscr{I}_a^{\gamma;G}g\,(b) + \frac{\mu}{1-\mu}\mathscr{I}_a^{\beta+\gamma;G}g\,(b)\right)\\ &+ \frac{\nu(G(t)-G(a))^{\beta+\alpha}}{(1-\nu)\Gamma(\beta+\alpha+1)}\mathscr{I}_a^{\gamma;G}g\,(b) + \mathscr{I}_a^{\alpha+\beta+\gamma;G}g\,(t)\,.\end{aligned}$$

By using the third boundary conditions $\mathrm{x}(a) = \lambda\mathrm{x}(b)$, we obtain

$$\begin{aligned}c_2 &= \lambda\mathrm{x}(b) = \frac{\lambda(G(b)-G(a))^\alpha}{(1-\lambda)\,\Gamma(\alpha+1)}\\ &\times\left(\frac{\mu\nu(G(b)-G(a))^\beta}{(1-\mu)\,(1-\nu)\,\Gamma(\beta+1)}\mathscr{I}_{a^+}^{\gamma;G}g(b) + \frac{\mu}{1-\mu}\mathscr{I}_{a^+}^{\beta+\gamma;G}g(b)\right)\\ &+\frac{\lambda\nu(G(b)-G(a))^{\alpha+\beta}}{(1-\lambda)\,(1-\nu)\,\Gamma(\beta+\alpha+1)}\mathscr{I}_{a^+}^{\gamma;G}g(b) + \frac{\lambda}{1-\lambda}\mathscr{I}_{a^+}^{\alpha+\beta+\gamma;G}g(b).\end{aligned}$$

The proof is ended. ■

Remark 2.1 By sing lemma (2.1), the integral equation associed to the Jerk problem (2.2) is given by

$$\begin{aligned}\mathrm{x}(t) &= \frac{\nu}{(1-\nu)}\left(\frac{\mu(G(b)-G(a))^\beta}{(1-\mu)\Gamma(\alpha+1)\Gamma(\beta+1)}\left(\frac{\lambda(G(b)-G(a))^\alpha}{(1-\lambda)} + (G(t)-G(a))^\alpha\right)\right.\\ &+ \left.\frac{1}{\Gamma(\beta+\alpha+1)}\left((G(t)-G(a))^{\beta+\alpha} + \frac{\lambda(G(b)-G(a))^{\alpha+\beta}}{(1-\lambda)}\right)\right)\mathscr{I}_a^{\gamma;G}f_x\,(b)\\ &+ \frac{\mu}{(1-\mu)\Gamma(\alpha+1)}\left(\frac{\lambda(G(b)-G(a))^\alpha}{(1-\lambda)} + (G(t)-G(a))^\alpha\right)\mathscr{I}_a^{\beta+\gamma;G}f_x\,(b)\\ &+ \mathscr{I}_a^{\alpha+\beta+\gamma;G}\left(f_x(t) + \frac{\lambda f_x(b)}{1-\lambda}\right),\end{aligned}$$

where

$$f_x(t) = f\left(t, x(t), {}^c\mathscr{D}_{a^+}^{\alpha;G}x(t), {}^c\mathscr{D}_{a^+}^{\beta;G}\left({}^c\mathscr{D}_{a^+}^{\alpha;G}x(t)\right)\right).$$

2.4 Existence and uniqueness

In this section, we focus on the existence and uniqueness of the solution of the Jerk problem (2.2). We define the space X as follows

$$X = \{\mathrm{x} \in C([a;b],\mathbb{R}) : {}^{c}\mathscr{D}_{a^+}^{\alpha;G}x, {}^{c}\mathscr{D}_{a^+}^{\beta;G}({}^{c}\mathscr{D}_{a^+}^{\alpha;G}x) \in C([a,b],\mathbb{R})\}.$$

Then X is a Banach space equipped by the norme

$$||\mathrm{x}|| = \sup_{t\in[a,b]} |\mathrm{x}(t)| + \sup_{t\in[a,b]} \left| {}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(t) \right| + \sup_{t\in[a,b]} \left| {}^{c}\mathscr{D}_{a^+}^{\beta;G}\left({}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}(t)\right)\right|.$$

We definie the operator $\Psi : X \to X$ by

$$\begin{aligned}
\Psi\mathrm{x}(t) &= \frac{\nu}{(1-\nu)}\left(\frac{\mu(G(b)-G(a))^{\beta}}{(1-\mu)\Gamma(\alpha+1)\Gamma(\beta+1)}\left(\frac{\lambda(G(b)-G(a))^{\alpha}}{(1-\lambda)} + (G(t)-G(a))^{\alpha}\right)\right.\\
&+ \left.\frac{1}{\Gamma(\beta+\alpha+1)}\left((G(t)-G(a))^{\beta+\alpha} + \frac{\lambda(G(b)-G(a))^{\alpha+\beta}}{(1-\lambda)}\right)\right)\mathscr{I}_{a}^{\gamma;G} f_x\,(b)\\
&+ \frac{\mu}{(1-\mu)\Gamma(\alpha+1)}\left(\frac{\lambda(G(b)-G(a))^{\alpha}}{(1-\lambda)} + (G(t)-G(a))^{\alpha}\right)\mathscr{I}_{a}^{\beta+\gamma;G} f_x\,(b)\\
&+ \mathscr{I}_{a}^{\alpha+\beta+\gamma;G}\left(f_x(t) + \frac{\lambda f_x(b)}{1-\lambda}\right).
\end{aligned}$$

Notations

The following notations will be helpful

$$E = \frac{|\nu|}{\Gamma(\gamma+1)|1-\lambda||1-\nu|}\left(\frac{|\mu|}{|1-\mu|\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta+\alpha+1)}\right).$$

$$F = \frac{|\mu|}{|1-\mu||1-\lambda|\Gamma(\alpha+1)\Gamma(\beta+\gamma+1)}.$$

$$G = \left(\frac{\mu\nu\Gamma(\alpha+\beta+1)}{(1-\mu)(1-\nu)(1-\lambda)\Gamma(\alpha+1)(\Gamma(\beta+1))^2\Gamma(\gamma+1)} + \frac{\nu}{(1-\nu)(1-\lambda)\Gamma(\beta+1)\Gamma(\gamma+1)}\right).$$

$$H = \frac{\mu}{(1-\lambda)^2(1-\mu)\Gamma(\beta+\gamma+1)}.$$

$$\begin{aligned}
R &= \left(\frac{\mu\nu\Gamma(\alpha+\beta+1)}{(1-\mu)(1-\nu)(1-\lambda)\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)} + \frac{\nu}{(1-\nu)(1-\lambda)\Gamma(\gamma+1)}\right.\\
&+ \left.\frac{\mu}{(1-\lambda)(1-\mu)\Gamma(\gamma+1)}\right).
\end{aligned}$$

$$\begin{aligned}
\Omega &= \left[\left(E + F + \frac{1}{(1-\lambda)\Gamma(\alpha+\beta+\gamma+1)}\right)\times|G(b)-G(a)|^{\alpha+\beta+\gamma}\right.\\
&+ \left.(G+H)\times|G(b)-G(a)|^{\beta+\gamma} + R\times|G(b)-G(a)|^{\gamma}\right].
\end{aligned}$$

Now, we show the existence of the solution to problem (2.2) by applying the Schauder fixed point theorem.

Theorem 2.1 *Let the following Hypotheses*

(H1) The functions $\mathrm{x}, {}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x}, {}^{c}\mathscr{D}_{a^+}^{\beta;G}({}^{c}\mathscr{D}_{a^+}^{\alpha;G}\mathrm{x})$ *are continuous.*

(H2) $\exists k > 0$ *such that* $\forall t \in [a, b]$ *and* $x_j, y_j \in C([a, b], \mathbb{R}), j = 1, 2, 3,$ *we have*

$$\begin{aligned} |f_x(t) - f_y(t)| &= |f(t, x_1(t), x_2(t), x_3(t)) - f(t, y_1(t), y_2(t), y_3(t))| \\ &\leq k\sum_{j=1}^{3}|x_j(t) - y_j(t)|. \end{aligned}$$

(H3) The function $f \in C([a, b] \times \mathbb{R}^3, \mathbb{R})$ *and* $\exists N > 0$ *such that* $\forall t \in [a, b]$,

$$|f_x(t)| = |f(t, x_1(t), x_2(t), x_3(t))| \leq N.$$

Then there exists at least one solution for the fractional non-periodic jerk problem (2.2) on the interval $[a, b]$.

Proof. To prove the existence of the solution, we transform Jerk problem (2.2) into a fixed point problem. Since this problem is equivalent to the integral equation (2.4), the fixed points of the operator Ψ are the solutions to problem (2.2).
We demonstrate that the operator $\Psi : X \to X$ is a completely continuous operator. Therefore, the proof consists of several steps.
Step 1 : The continuity of the operator Ψ is established by the continuity of the function f .
Step 2 : We show that $\Psi(A)$ is uniformly bounded. Define a subset $A \subset X$ such that

$$A = \{x \in X : ||x|| \leq m\}.$$

For a satisfying constant $m > 0$ and

$$m \geqslant \frac{s\Omega}{1 - \Omega k}.$$

We denoted

$$s = \sup_{t\in[a,b]} |f(t, 0, 0, 0)| = \sup_{t\in[a,b]} |f_0(t)|.$$

To apply the schauder fixed-point theorem, we verify that $\Psi A \subset A$.

Let Ψ be a function from A to itself and $x \in A$, we have

$$\begin{aligned}
|(\Psi x)(t)| &\leq \left|\frac{\nu}{(1-\nu)(1-\lambda)}\right|\left|\frac{\mu}{(1-\mu)\Gamma(\alpha+1)\Gamma(\beta+1)}+\frac{1}{\Gamma(\alpha+\beta+1)}\right| \\
&\times |G(b)-G(a)|^{\alpha+\beta}|\mathscr{I}_a^{\gamma;G}f_x(b)| + \left|\frac{\mu}{(1-\lambda)(1-\mu)\Gamma(\alpha+1)}\right||G(b)-G(a)|^{\alpha}|\mathscr{I}_a^{\beta+\gamma;G}f_x(b)| \\
&+ \left|\mathscr{I}_a^{\alpha+\beta+\gamma;G}\left(f_x(t)+\frac{\lambda f_x(b)}{1-\lambda}\right)\right| \\
&\leqslant \left[E+F+\frac{1}{|1-\lambda|\Gamma(\alpha+\beta+\gamma+1)}\right]\times|G(b)-G(a)|^{\alpha+\beta+\gamma}|\sup_{t\in[a,b]}(|f_x(t)-f_0(t)|+|f_0(t)|) \\
&\leqslant \left[E+F+\frac{1}{|1-\lambda|\Gamma(\alpha+\beta+\gamma+1)}\right]\times|G(b)-G(a)|^{\alpha+\beta+\gamma}|k||x||_X+s|,
\end{aligned}$$

and

$$\begin{aligned}
|{}^c\mathscr{D}_{a^+}^{\alpha;G}\Psi x(t)| &\leq \left(\frac{\mu\nu\Gamma(\alpha+\beta+1)}{(1-\mu)(1-\nu)(1-\lambda)\Gamma(\alpha+1)(\Gamma(\beta+1))^2}+\frac{\nu}{(1-\nu)(1-\lambda)\Gamma(\beta+1)}\right) \\
&\times |G(b)-G(a)|^{\beta}|\mathscr{I}_a^{\gamma;G}f_x(b)|+|\frac{\mu}{(1-\lambda)(1-\mu)}||\mathscr{I}_a^{\beta+\gamma;G}(f_x(t)+\frac{f_x(b)}{1-\lambda})|.
\end{aligned}$$

Then,

$$\begin{aligned}
|{}^c\mathscr{D}_{a^+}^{\alpha;G}\Psi x(t)| &\leqslant \left(\frac{\mu\nu\Gamma(\alpha+\beta+1)}{(1-\mu)(1-\nu)(1-\lambda)\Gamma(\alpha+1)(\Gamma(\beta+1))^2\Gamma(\gamma+1)}\right. \\
&+ \left.\frac{\nu}{(1-\nu)(1-\lambda)\Gamma(\beta+1)\Gamma(\gamma+1)}\right)|G(b)-G(a)|^{\beta}\int_a^b(G(b)-G(a))^{\gamma-1}G'(s)f_x(b)ds \\
&+ |\frac{\mu}{(1-\lambda)(1-\mu)\Gamma(\beta+\gamma+1)}|\int_a^b(G(b)-G(a))^{\beta+\gamma-1}G'(s)\left(f_x(t)+\frac{f_x(b)}{1-\lambda}\right)ds \\
&\leqslant (G+H)\times|G(b)-G(a)|^{\beta+\gamma}.|k||x||_X+s|.
\end{aligned}$$

However, we have

$$\begin{aligned}
|{}^c\mathscr{D}_{a^+}^{\beta;G}({}^c\mathscr{D}_{a^+}^{\alpha;G}(\Psi x))(t)| &\leqslant \left(\frac{\mu\nu\Gamma(\alpha+\beta+1)}{(1-\mu)(1-\nu)(1-\lambda)\Gamma(\alpha+1)\Gamma(\beta+1)}+\frac{\nu}{(1-\nu)(1-\lambda)}\right)|\mathscr{I}_a^{\gamma;G}f_x(b)| \\
&+ |\frac{\mu}{(1-\lambda)(1-\mu)}||\mathscr{I}_a^{\gamma;G}(f_x(t)+\frac{f_x(b)}{1-\lambda})|.
\end{aligned}$$

Then

$$\begin{aligned}
|{}^c\mathscr{D}_{a^+}^{\beta;G}({}^c\mathscr{D}_{a^+}^{\alpha;G}(\Psi x))(t)| &\leqslant \left(\frac{\mu\nu\Gamma(\alpha+\beta+1)}{(1-\mu)(1-\nu)(1-\lambda)^2\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}+\frac{\nu}{(1-\nu)(1-\lambda)^2\Gamma(\gamma+1)}\right. \\
&+ \left.\frac{\mu}{(1-\lambda)^2(1-\mu)\Gamma(\gamma+1)}\right)|G(b)-G(a)|^{\gamma}\sup_{t\in[a,b]}(|f_x(t)-f_0(t)|+|f_0(t)|). \\
&\leqslant R\times|G(b)-G(a)|^{\gamma}|k||x||_X+s|.
\end{aligned}$$

From the above inequalities, we obtain $||\Psi x||\leqslant \Omega|k.m+s|$.

Then, $\Psi(A)$ is uniformly bounded.

2.4. Existence and uniqueness

Step 3 : Now, we demontrate that $\Psi(A)$ is equicontinue.

For all $t_1, t_2 \in [0,T], 0 \leqslant t_1 \leqslant t_2 \leqslant T$ et $x \in A$, we deduce that

$$\begin{aligned}(\Psi x)(t_2) - (\Psi x)(t_1) &\leqslant \frac{\nu}{(1-\nu)}\Bigg(\frac{\mu(G(b)-G(a))^{\beta}}{(1-\mu)\Gamma(\alpha+1)\Gamma(\beta+1)(1-\lambda)} + \Big((G(t_2)-G(a))^{\alpha} \\ &- (G(t_1)-G(a))^{\alpha}\Big) \\ &+ \frac{1}{\Gamma(\beta+\alpha+1)(1-\lambda)}\Big((G(t_2)-G(a))^{\beta+\alpha} - (G(t_1)-G(a))^{\beta+\alpha}\Big)\Bigg)\mathscr{I}_a^{\gamma;G} f_x(b) \\ &+ \frac{\mu}{(1-\mu)\Gamma(\alpha+1)(1-\lambda)}\Big((G(t_2)-G(a))^{\alpha} - (G(t_1)-G(a))^{\alpha}\Big)\mathscr{I}_a^{\beta+\gamma;G} f_x(b) \\ &+ \frac{\mathscr{I}_a^{\alpha+\beta+\gamma;G}}{1-\lambda}\Big(f_x(t_2) - f_x(t_1)\Big).\end{aligned}$$

Then

$$\begin{aligned}|(\Psi x)(t_2) - (\Psi x)(t_1)| &\leqslant \frac{N\nu}{(1-\nu)\Gamma(\gamma+1)}\Bigg(\frac{\mu|G(b)-G(a)|^{\beta+\gamma}}{(1-\mu)\Gamma(\alpha+1)\Gamma(\beta+1)(1-\lambda)} \\ &\times \Big(|G(t_2)-G(a)|^{\alpha} - |G(t_1)-G(a)|^{\alpha}\Big) \\ &+ \frac{|G(b)-G(a)|^{\gamma}}{\Gamma(\beta+\alpha+1)(1-\lambda)}\Big(|G(t_2)-G(a)|^{\beta+\alpha} - |G(t_1)-G(a)|^{\beta+\alpha}\Big)\Bigg) \\ &+ \frac{N\mu|G(b)-G(a)|^{\beta+\gamma}}{(1-\mu)\Gamma(\alpha+1)(1-\lambda)\Gamma(\beta+\gamma+1)}\Big(|G(t_2)-G(a)|^{\alpha} - |G(t_1)-G(a)|^{\alpha}\Big) \\ &+ \frac{1}{(1-\lambda)\Gamma(\alpha+\beta+\gamma+1)}\Big(|G(t_2)-G(a)|^{\alpha+\beta+\gamma} - |G(t_1)-G(a)|^{\alpha+\beta+\gamma} \\ &+ 2|G(t_2)-G(t_1)|^{\alpha+\beta+\gamma}\Big),\end{aligned}$$

and

$$\begin{aligned}|{}^{c}\mathscr{D}_{a^+}^{\alpha;G}\Psi x(t_2) - {}^{c}\mathscr{D}_{a^+}^{\alpha;G}\Psi x(t_1)| &\leqslant \Bigg(\frac{\mu\nu\Gamma(\alpha+\beta+1)}{(1-\mu)(1-\nu)(1-\lambda)\Gamma(\alpha+1)(\Gamma(\beta+1))^2} \\ &+ \frac{\nu}{(1-\nu)(1-\lambda)\Gamma(\beta+1)}\Bigg)\Big(|G(t_2)-G(a)|^{\beta} - |G(t_1)-G(a)|^{\beta}\Big)|\mathscr{I}_a^{\gamma;G} f_x(b)| \\ &+ \left|\frac{\mu}{(1-\lambda)(1-\mu)}\right|\left|\frac{\mathscr{I}_a^{\beta+\gamma;G}}{1-\lambda}(f_x(t_2) - f_x(t_1))\right| \\ &\leqslant \Bigg(\frac{N\mu\nu\Gamma(\alpha+\beta+1)|G(b)-G(a)|^{\gamma}}{(1-\mu)(1-\nu)(1-\lambda)\Gamma(\alpha+1)(\Gamma(\beta+1))^2\Gamma(\gamma+1)} \\ &+ \frac{N\nu|G(b)-G(a)|^{\gamma}}{(1-\nu)(1-\lambda)\Gamma(\beta+1)\Gamma(\gamma+1)}\Bigg)\Big(|G(t_2)-G(a)|^{\beta} - |G(t_1)-G(a)|^{\beta}\Big) \\ &+ \left|\frac{N\mu}{(1-\lambda)^2(1-\mu)\Gamma(\beta+\gamma+1)}\right|\Big(|G(t_2)-G(a)|^{\beta+\gamma} - |G(t_1)-G(a)|^{\beta+\gamma} \\ &+ 2|G(t_2)-G(t_1)|^{\beta+\gamma}\Big).\end{aligned}$$

Similary, we get

$$\begin{aligned}|{}^c\mathscr{D}_{a^+}^{\beta;G}({}^c\mathscr{D}_{a^+}^{\alpha;G}(\Psi x))(t_2) - {}^c\mathscr{D}_{a^+}^{\beta;G}({}^c\mathscr{D}_{a^+}^{\alpha;G}(\Psi x))(t_1)| &\leqslant \Bigg(\frac{N\mu\nu\Gamma(\alpha+\beta+1)}{(1-\mu)(1-\nu)(1-\lambda)^2\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)} \\ &+ \frac{N\nu}{(1-\nu)(1-\lambda)^2\Gamma(\gamma+1)} + \frac{N\mu}{(1-\lambda)^2(1-\mu)\Gamma(\gamma+1)}\Bigg) \\ &\times \Big(|G(t_2)-G(a)|^\gamma - |G(t_1)-G(a)|^\gamma \\ &+ 2|G(t_2)-G(t_1)|^\gamma\Big).\end{aligned}$$

From the above inequalities, we obtain

$$|(\Psi x)(t_2) - (\Psi x)(t_1)| \to 0, \ as\ t_2 \to t_1,$$

$$|{}^c\mathscr{D}_{a^+}^{\alpha;G}(\Psi x)(t_2) - {}^c\mathscr{D}_{a^+}^{\alpha;G}(\Psi x)(t_1)| \to 0, \ as\ t_2 \to t_1,$$

$$|{}^c\mathscr{D}_{a^+}^{\beta;G}({}^c\mathscr{D}_{a^+}^{\alpha;G}(\Psi x))(t_2) - {}^c\mathscr{D}_{a^+}^{\beta;G}({}^c\mathscr{D}_{a^+}^{\alpha;G}(\Psi x))(t_1)| \to 0, \ as\ t_2 \to t_1.$$

This implise that $\Psi(A)$ is equicontinuous.

Therefore, according to the Arzela-Ascoli theorem the operator Ψ is completely continuous and by Schauder fixed point theorem, Ψ has a fixed point, which is considered a solution to the non-periodic jerk problem (2.2) on the interval $[a, b]$. ■

Now, we show the uniqueness of the solution to Jerk problem (2.2).

Theorem 2.2 *Let (H1)-(H3) be holds. Then the non-periodic jerk system (2.2) admits a unique solution on* $[a, b]$ *if*

$$k\Omega < 1.$$

Proof. The contractive property of the operator Ψ is investigated. Let $x, y \in X$, we estimate

$$\begin{aligned}|(\Psi x)(t) - (\Psi y)(t)| &\leqslant \frac{\nu}{(1-\nu)}\Bigg(\frac{\mu(G(b)-G(a))^\beta}{(1-\mu)\Gamma(\alpha+1)\Gamma(\beta+1)}\Bigg(\frac{\lambda(G(b)-G(a))^\alpha}{(1-\lambda)} + (G(t)-G(a))^\alpha\Bigg) \\ &+ \frac{1}{\Gamma(\beta+\alpha+1)}\Bigg((G(t)-G(a))^{\beta+\alpha} + \frac{\lambda(G(b)-G(a))^{\alpha+\beta}}{(1-\lambda)}\Bigg)\Bigg)\mathscr{I}_a^{\gamma;G}|f_x(b) - f_y(b)| \\ &+ \frac{\mu}{(1-\mu)\Gamma(\alpha+1)}\Bigg(\frac{\lambda(G(b)-G(a))^\alpha}{(1-\lambda)} + (G(t)-G(a))^\alpha\Bigg)\mathscr{I}_a^{\beta+\gamma;G}|f_x(b) - f_y(b)| \\ &+ \mathscr{I}_a^{\alpha+\beta+\gamma;G}\Bigg|\Bigg(f_x(t) + \frac{\lambda f_x(b)}{1-\lambda}\Bigg) - \Bigg(f_y(t) + \frac{\lambda f_y(b)}{1-\lambda}\Bigg)\Bigg| \\ &\leqslant \Bigg(E + F + \frac{1}{|1-\lambda|\Gamma(\alpha+\beta+\gamma+1)}\Bigg)\times|G(b)-G(a)|^{\alpha+\beta+\gamma}k||x-y||_X.\end{aligned}$$

Additionally, we have

$$\begin{aligned}|{}^{c}\mathscr{D}^{\alpha;G}_{a^+}(\Psi x)(t) - {}^{c}\mathscr{D}^{\alpha;G}_{a^+}(\Psi y)(t)| &\leqslant \Bigg(\frac{\mu\nu\Gamma(\alpha+\beta+1)}{(1-\mu)(1-\nu)(1-\lambda)\Gamma(\alpha+1)(\Gamma(\beta+1))^2} \\ &+ \frac{\nu}{(1-\nu)(1-\lambda)\Gamma(\beta+1)}\Bigg)|G(b)-G(a)|^{\beta}|\mathscr{I}^{\gamma;G}_{a}(f_x(b)-f_y(b))| \\ &+ |\frac{\mu}{(1-\lambda)(1-\mu)}||\mathscr{I}^{\beta+\gamma;G}_{a}((f_x(t)-f_y(t)) + \frac{(f_x(b)-f_y(b)}{1-\lambda})|. \\ &\leqslant (G+H)\times|G(b)-G(a)|^{\beta+\gamma}.k||x-y||_X.\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\left|{}^{c}\mathscr{D}^{\beta;G}_{a^+}\left({}^{c}\mathscr{D}^{\alpha;G}_{a^+}(\Psi x)\right)(t) - {}^{c}\mathscr{D}^{\beta;G}_{a^+}\left({}^{c}\mathscr{D}^{\alpha;G}_{a^+}(\Psi y)\right)(t)\right| &\leqslant \Bigg(\frac{\mu\nu\Gamma(\alpha+\beta+1)}{(1-\mu)(1-\nu)(1-\lambda)\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)} \\ &+ \frac{\nu}{(1-\nu)(1-\lambda)\Gamma(\gamma+1)} + \frac{\mu}{(1-\lambda)(1-\mu)\Gamma(\gamma+1)}\Bigg) \\ &\times \int_a^b (G(b)-G(a))^{\gamma-1}G'(s) \\ &\times \left((f_x(t)-f_y(t)) + \frac{(f_x(b)-f_y(b))}{1-\lambda}\right)ds \\ &\leqslant R\times|G(b)-G(a)|^{\gamma}k||x-y||_X.\end{aligned}$$

From the above inequalities, we obtain

$$\begin{aligned}||\Psi x - \Psi y|| &\leqslant \Bigg[\left(E+F+\frac{1}{(1-\lambda)\Gamma(\alpha+\beta+\gamma+1)}\right)\times|G(b)-G(a)|^{\alpha+\beta+\gamma} \\ &+ (G+H)\times|G(b)-G(a)|^{\beta+\gamma} + R\times|G(b)-G(a)|^{\gamma}\Bigg]k||x-y||_X \\ &\leqslant k\Omega||x-y||_X.\end{aligned}$$

Then

$$||\Psi x - \Psi y|| \leqslant k\Omega||x-y||_X.$$

We have $k\Omega < 1$, then Ψ is a contraction on X. These guarantee the existence of a unique fixed point for Ψ and accordingly the existence of a unique solution for the non-periodic jerk (2.2) and the proof is ended. ∎

Chapter 3

Stability of the fractional system of the non-periodic Jerk

In this section, we examine the stability criteria based on the Ulam-Hyers theorem and its extended version, as well as the Ulam-Hyers-Rassias theorem and its extended version, for the solutions of the non-periodic jerk problem (2.2) on the interval $[a, b]$.

3.1 Ulam-Hyers-Stability

Ulam-Hyers Stability is a concept used to study the stability of functional equations and differential equations. It originated from a question posed by Stanislaw Ulam in 1940 concerning the stability of functional equations. This concept is concerned with the behavior of solutions to equations when they are subject to small perturbations. If a small perturbation in the initial conditions or the function itself leads to only a small deviation in the solution, the equation is considered stable in the Ulam-Hyers sense.

Definition 3.1 The fractional non-periodic Jerk problem (2.2) is Ulam-Hyers stable if $\exists 0 < c_f^* \in \mathbb{R}$, such as $\forall \epsilon > 0$, and $\forall x^* \in X$

$$\left| {}^c\mathscr{D}_{a^+}^{\gamma;G}\left({}^c\mathscr{D}_{a^+}^{\beta;G}\left({}^c\mathscr{D}_{a^+}^{\alpha;G}x^*(t)\right)\right) - f\left(t, x^*(t), {}^c\mathscr{D}_{a^+}^{\alpha;G}x^*(t), {}^c\mathscr{D}_{a^+}^{\beta;G}\left({}^c\mathscr{D}_{a^+}^{\alpha;G}x^*(t)\right)\right)\right| < \epsilon, \quad (3.1)$$

$\exists x \in X$ satisfying the fractional non-periodic Jerk problem (2.2) with

$$||x^* - x||_X \leqslant \epsilon c_f^*.$$

Definition 3.2 The fractional non-periodic Jerk problem (2.2) is generalized Ulam-Hyers stable if $\exists c_f^* \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $c_f^*(0) = 0$ such as $\forall \epsilon > 0$, and $\forall x^* \in X$ satisfying the inequality

$$\left| {}^c\mathscr{D}_{a^+}^{\gamma;G} \left({}^c\mathscr{D}_{a^+}^{\beta;G} \left({}^c\mathscr{D}_{a^+}^{\alpha;G} x^*(t) \right) \right) - f\left(t, x^*(t), {}^c\mathscr{D}_{a^+}^{\alpha;G} x^*(t), {}^c\mathscr{D}_{a^+}^{\beta;G} \left({}^c\mathscr{D}_{a^+}^{\alpha;G} x^*(t) \right) \right) \right| < \epsilon, \tag{3.2}$$

$\exists x \in X$ as a solution of the fractional non-periodic Jerk problem (2.2) with

$$||x^* - x||_X \leqslant c_f^*(\epsilon).$$

Remark 3.1 Noted that $x^* \in X$ is the solution of the inequality (3.1) if and only if $\exists h \in C([a,b], \mathbb{R})$ and x^* such that $\forall t \in [a,b]$,

(i)$|h(t)| < \epsilon$.

(ii)${}^c\mathscr{D}_{a^+}^{\gamma;G} \left({}^c\mathscr{D}_{a^+}^{\beta;G} \left({}^c\mathscr{D}_{a^+}^{\alpha;G} x^*(t) \right) \right) = f\left(t, x^*(t), {}^c\mathscr{D}_{a^+}^{\alpha;G} x^*(t), {}^c\mathscr{D}_{a^+}^{\beta;G} \left({}^c\mathscr{D}_{a^+}^{\alpha;G} x^*(t) \right) \right) + h(t)$.

3.1.1 Stability Theorems

Theorem 3.1 *Let (H1)-(H3) be holds then, the fractional non-periodic Jerk problem (2.2) is Ulam-Hyers stable in $[a,b]$, and accordingly, is generalized Ulam-Hyers stable provided that $k\Omega < 1$.*

Proof. For every $\epsilon > 0$, and $\forall x^* \in X$ satisfaying

$$\left| {}^c\mathscr{D}_{a^+}^{\gamma;G} \left({}^c\mathscr{D}_{a^+}^{\beta;G} \left({}^c\mathscr{D}_{a^+}^{\alpha;G} x^*(t) \right) \right) - f\left(t, x^*(t), {}^c\mathscr{D}_{a^+}^{\alpha;G} x^*(t), {}^c\mathscr{D}_{a^+}^{\beta;G} \left({}^c\mathscr{D}_{a^+}^{\alpha;G} x^*(t) \right) \right) \right| < \epsilon,$$

We can found a function $h(t)$ satisfying

$${}^c\mathscr{D}_{a^+}^{\gamma;G} \left({}^c\mathscr{D}_{a^+}^{\beta;G} \left({}^c\mathscr{D}_{a^+}^{\alpha;G} x^*(t) \right) \right) = f\left(t, x^*(t), {}^c\mathscr{D}_{a^+}^{\alpha;G} x^*(t), {}^c\mathscr{D}_{a^+}^{\beta;G} \left({}^c\mathscr{D}_{a^+}^{\alpha;G} x^*(t) \right) \right) + h(t).$$

avec $|h(t)| < \epsilon$. Then

$$\begin{aligned} x^*(t) &= \frac{\nu}{(1-\nu)} \Bigg(\frac{\mu (G(b)-G(a))^\beta}{(1-\mu)\Gamma(\alpha+1)\Gamma(\beta+1)} \Bigg(\frac{\lambda (G(b)-G(a))^\alpha}{(1-\lambda)} + (G(t)-G(a))^\alpha \Bigg) \\ &+ \frac{1}{\Gamma(\beta+\alpha+1)} \Bigg((G(t)-G(a))^{\beta+\alpha} + \frac{\lambda (G(b)-G(a))^{\alpha+\beta}}{(1-\lambda)} \Bigg) \Bigg) \mathscr{I}_a^{\gamma;G} (f_{x^*}(b) + h(b)) \\ &+ \frac{\mu}{(1-\mu)\Gamma(\alpha+1)} \Bigg(\frac{\lambda (G(b)-G(a))^\alpha}{(1-\lambda)} + (G(t)-G(a))^\alpha \Bigg) \mathscr{I}_a^{\beta+\gamma;G} (f_{x^*}(b) + h(b)) \\ &+ \mathscr{I}_a^{\alpha+\beta+\gamma;G} \Bigg((f_{x^*}(t) + h(t)) + \frac{\lambda (f_{x^*}(b) + h(b))}{1-\lambda} \Bigg). \end{aligned}$$

Using Theorem (2.2), there exists a unique solution $x \in X$ satisfying fractional non-periodic Jerk problem (2.2). Then,

$$|x^*(t) - x(t)| \leqslant \left(|E+F| + \frac{1}{|1-\lambda|\Gamma(\alpha+\beta+\gamma+1)} \right) \times |G(b)-G(a)|^{\alpha+\beta+\gamma} (k||x^*-x||_X + \epsilon).$$

Similary, we have

$$|{}^{c}\mathscr{D}_{a^+}^{\alpha;G}x^*(t) - {}^{c}\mathscr{D}_{a^+}^{\alpha;G}x(t)| \leqslant |G+H| \times |G(b)-G(a)|^{\beta+\gamma}(k||x^*-x||_X+\epsilon),$$

and

$$|{}^{c}\mathscr{D}_{a^+}^{\beta;G}({}^{c}\mathscr{D}_{a^+}^{\alpha;G}x^*)(t) - {}^{c}\mathscr{D}_{a^+}^{\beta;G}({}^{c}\mathscr{D}_{a^+}^{\alpha;G}x)(t)| \leqslant R \times |G(b)-G(a)|^{\gamma}(k||x^*-x||_X+\epsilon).$$

From the above inequalities, we obtain

$$||x^*-x||_X \leq \Omega(k||x^*-x||_X+\epsilon).$$

Then

$$||x^*-x||_X \leq \frac{\Omega\epsilon}{1-k\Omega}.$$

Since $k\Omega < 1$, this shows the existence of a positive real

$$c_f^* = \frac{\Omega}{1-k\Omega} > 0,$$

and hence according to definition (3.1), the solution of (2.2) is d'Ulam-Hyers stable. Similarly, it shows the existence of a function $c_f^* \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $c_f^*(0) = 0$ such that

$$c_f^*(\epsilon) = \frac{\Omega}{1-k\Omega}\epsilon.$$

Hence, the solution of (2.2) is General Ulam-Hyers stable. ■

3.2 Ulam-Hyers-Rassias Stability

Ulam-Hyers-Rassias Stability is an extension of the concept of Ulam-Hyers stability, introduced to address certain types of functional equations and differential equations with more general perturbations. The idea was developed by Themistocles M. Rassias to generalize the notion of stability introduced by Ulam and Hyers, allowing for the study of equations under a broader class of perturbations.

Definition 3.3 The fractional non-periodic Jerk problem (2.2) is Ulam-Hyers-Rassias stable with respect to a function Φ , if $\exists 0 < c_f^* \in \mathbb{R}$, such that $\forall \epsilon > 0$, $\forall x^* \in X$ satisfying

$$\left|{}^{c}\mathscr{D}_{a^+}^{\gamma;G}\left({}^{c}\mathscr{D}_{a^+}^{\beta;G}\left({}^{c}\mathscr{D}_{a^+}^{\alpha;G}x^*(t)\right)\right) - f\left(t, x^*(t), {}^{c}\mathscr{D}_{a^+}^{\alpha;G}x^*(t), {}^{c}\mathscr{D}_{a^+}^{\beta;G}\left({}^{c}\mathscr{D}_{a^+}^{\alpha;G}x^*(t)\right)\right)\right| < \epsilon\Phi(t), \quad (3.3)$$

$\exists x \in X$ as a solution of the fractional non-periodic Jerk problem (2.2) with

$$||x^*-x||_X \leqslant \epsilon c_f^*\Phi(t), \ \forall t \in [a,b].$$

Definition 3.4 The fractional non-periodic Jerk problem (2.2) is generalized Ulam-Hyers-Rassias stable with respect to a function Φ if $\exists c_f^* \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $c_f^*(0) = 0$, such that $\forall x^* \in X$ satisfying

$$\left| {}^c\mathscr{D}_{a^+}^{\gamma;G}\left({}^c\mathscr{D}_{a^+}^{\beta;G}\left({}^c\mathscr{D}_{a^+}^{\alpha;G}x^*(t)\right)\right) - f\left(t, x^*(t), {}^c\mathscr{D}_{a^+}^{\alpha;G}x^*(t), {}^c\mathscr{D}_{a^+}^{\beta;G}\left({}^c\mathscr{D}_{a^+}^{\alpha;G}x^*(t)\right)\right)\right| < \epsilon\Phi(t), \quad (3.4)$$

$\exists x \in X$ as a solution of the fractional non-periodic Jerk problem (2.2) with

$$||x^* - x||_X \leqslant c_f^*(\epsilon)\Phi(t), \ \forall t \in [a, b].$$

Remark 3.2 The $x^* \in X$ is the solution of (3.3) iff $\exists \rho \in C([a, b], \mathbb{R})$ and x^* such that $\forall t \in [a, b]$,

(i)$|\rho(t)| < \epsilon\Phi(t)$.

(ii)${}^c\mathscr{D}_{a^+}^{\gamma;G}\left({}^c\mathscr{D}_{a^+}^{\beta;G}\left({}^c\mathscr{D}_{a^+}^{\alpha;G}x^*(t)\right)\right) = f\left(t, x^*(t), {}^c\mathscr{D}_{a^+}^{\alpha;G}x^*(t), {}^c\mathscr{D}_{a^+}^{\beta;G}\left({}^c\mathscr{D}_{a^+}^{\alpha;G}x^*(t)\right)\right) + \rho(t)$.

3.2.1 Stability Theorems

The Ulam-Hyers-Rassias stability for the non-periodic jerk problem (2.2) is verified below.

Theorem 3.2 *Let (H1)-(H3) be holds. Then, the fractional non-periodic Jerk problem (2.2) is Ulam-Hyers-Rassias stable, and accordingly, is generalized Ulam-Hyers-Rassias stable provided that* $k\Omega < 1$.

Proof. For every $\epsilon > 0$, and $\forall x^* \in X$ satisfying

$$\left| {}^c\mathscr{D}_{a^+}^{\gamma;G}\left({}^c\mathscr{D}_{a^+}^{\beta;G}\left({}^c\mathscr{D}_{a^+}^{\alpha;G}x^*(t)\right)\right) - f\left(t, x^*(t), {}^c\mathscr{D}_{a^+}^{\alpha;G}x^*(t), {}^c\mathscr{D}_{a^+}^{\beta;G}\left({}^c\mathscr{D}_{a^+}^{\alpha;G}x^*(t)\right)\right)\right| < \epsilon\Phi(t),$$

we can found a function $\rho(t)$ satisfying

$${}^c\mathscr{D}_{a^+}^{\gamma;G}\left({}^c\mathscr{D}_{a^+}^{\beta;G}\left({}^c\mathscr{D}_{a^+}^{\alpha;G}x^*(t)\right)\right) = f\left(t, x^*(t), {}^c\mathscr{D}_{a^+}^{\alpha;G}x^*(t), {}^c\mathscr{D}_{a^+}^{\beta;G}\left({}^c\mathscr{D}_{a^+}^{\alpha;G}x^*(t)\right)\right) + \rho(t),$$

with

$$|\rho(t)| \leqslant \epsilon\Phi(t).$$

Then

$$\begin{aligned} x^*(t) &= \frac{\nu}{(1-\nu)}\left(\frac{\mu(G(b)-G(a))^\beta}{(1-\mu)\Gamma(\alpha+1)\Gamma(\beta+1)}\left(\frac{\lambda(G(b)-G(a))^\alpha}{(1-\lambda)} + (G(t)-G(a))^\alpha\right)\right. \\ &+ \left.\frac{1}{\Gamma(\beta+\alpha+1)}\left((G(t)-G(a))^{\beta+\alpha} + \frac{\lambda(G(b)-G(a))^{\alpha+\beta}}{(1-\lambda)}\right)\right)\mathscr{I}_a^{\gamma;G}(f_{x^*}(b)+\rho(b)) \\ &+ \frac{\mu}{(1-\mu)\Gamma(\alpha+1)}\left(\frac{\lambda(G(b)-G(a))^\alpha}{(1-\lambda)} + (G(t)-G(a))^\alpha\right)\mathscr{I}_a^{\beta+\gamma;G}(f_{x^*}(b)+\rho(b)) \\ &+ \mathscr{I}_a^{\alpha+\beta+\gamma;G}\left((f_{x^*}(t)+\rho(t)) + \frac{\lambda(f_{x^*}(b)+\rho(b))}{1-\lambda}\right). \end{aligned}$$

Based on theorem (2.2), there exists a unique solution $x \in X$ satisfying the non-periodic fractional jerk problem (2.2). Then

$$|x^*(t)-x(t)| \leqslant \left(|E+F|+\frac{1}{|1-\lambda|\Gamma(\alpha+\beta+\gamma+1)}\right)\times|G(b)-G(a)|^{\alpha+\beta+\gamma}(k||x^*-x||_X+\epsilon \sup_{t\in[a,b]} \Phi(t)).$$

Similary, we have

$$|{}^c\mathscr{D}^{\alpha;G}_{a^+}x^*(t) - {}^c\mathscr{D}^{\alpha;G}_{a^+}x(t)| \leqslant |G+H|\times|G(b)-G(a)|^{\beta+\gamma}(k||x^*-x||_X+\epsilon \sup_{t\in[a,b]} \Phi(t)),$$

and

$$|{}^c\mathscr{D}^{\beta;G}_{a^+}({}^c\mathscr{D}^{\alpha;G}_{a^+}x^*)(t) - {}^c\mathscr{D}^{\beta;G}_{a^+}({}^c\mathscr{D}^{\alpha;G}_{a^+}x)(t)| \leqslant R\times|G(b)-G(a)|^{\gamma}(k||x^*-x||_X+\epsilon \sup_{t\in[a,b]} \Phi(t)).$$

From the above inequalities, we obtain

$$||x^*-x||_X \leqslant \Omega(k||x^*-x||_X+\epsilon \sup_{t\in[a,b]} \Phi(t)).$$

$$\leqslant \frac{\Omega\epsilon}{1-k\Omega} \sup_{t\in[a,b]} \Phi(t).$$

Since $k\Omega < 1$, this shows the existence of a positive real number

$$c_f^* = \frac{\Omega}{1-k\Omega} > 0,$$

and hence according to Definition (3.3),the solution of (2.2) is d'Ulam-Hyers-Rassias stable. Similarly, this demonstrates the existence of a function $c_f^* \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $c_f^*(0) = 0$ such that

$$c_f^*(\epsilon) = \frac{\Omega}{1-k\Omega}\epsilon.$$

Therefore, the solution to (2.2) is generalized Ulam-Hyers-Rassias stable ■

3.3 Example

Example 3.1 We consider a nonlinear fractional non-periodic Jerk problem as

$$\begin{cases} {}^cD^{0.65;G(t)}_{0,1^+}x(t) = x_1(t), \\ {}^cD^{0.34;G(t)}_{0,1^+}x_1(t) = x_2(t), \\ {}^cD^{0.85;G(t)}_{0,1^+}x_2(t) = \dfrac{\cos^2(x(t))^3}{20(1+\cos^2(x(t))^3)} + \dfrac{1}{24}\cos^{-1}x_1(t) + \dfrac{t}{40}\tan^{-1}\dfrac{|x_2(t)|}{(1+|x_2(t)|)} + \dfrac{11t}{240}, \end{cases} \tag{3.5}$$

for $t \in [0.1, 2]$ and

$$x(0.1) = \frac{-18}{5},\ x_1(0.1) = \frac{13}{8},\ x_2(0.1) = \frac{-19}{7},$$

with

$$a = 0.1,\ b = 2,\ \alpha = 0.65,\ \beta = 0.34,\ \gamma = 0.85,\ \lambda = \frac{1}{7},\ \mu = \frac{1}{8},\ \nu = \frac{1}{9},$$

also, we have

$$f(t, x, x_1, x_2) = \frac{\cos^2 x^3}{20(1+\cos^2 x^3)} + \frac{1}{24}\cos^{-1}x_1 + \frac{t}{40}\tan^{-1}\frac{|x_2|}{(1+|x_2|)} + \frac{11t}{240}.$$

Thus we can rewrite the above system as

$$\begin{cases} {}^cD^{0.85;G(t)}_{0,1^+}\left({}^cD^{0.34;G(t)}_{0,1^+}\left({}^cD^{0.65;G(t)}_{0,1^+}x(t)\right)\right) = \dfrac{\cos^2(x(t))^3}{20(1+\cos^2(x(t))^3)} \\ +\dfrac{1}{24}\cos^{-1}\left({}^cD^{0.65;G(t)}_{0,1^+}x(t)\right) \\ +\dfrac{t}{40}\tan^{-1}\dfrac{\left|{}^cD^{0.34;G(t)}_{0,1^+}\left({}^cD^{0.65;G(t)}_{0,1^+}x(t)\right)\right|}{(1+\left|{}^cD^{0.34;G(t)}_{0,1^+}\left({}^cD^{0.65;G(t)}_{0,1^+}x(t)\right)\right|)} + \dfrac{11t}{240}, \\ x(0.1) = \dfrac{-18}{5}, {}^cD^{0.65;G(t)}_{0,1^+}x(0.1) = \dfrac{13}{8}, {}^cD^{0.34;G(t)}_{0,1^+}\left({}^cD^{0.65;G(t)}_{0,1^+}x(0.1)\right) = \dfrac{-19}{7}. \end{cases} \tag{3.6}$$

We have

(H1) The functions cos and tan are continuous and differentiable on the interval $[0.1, 2]$.

(H2) For all $t \in [0.1, 2]$, we have

$$|f(t, x(t), x_1(t), x_2(t)) - f(t, y(t), y_1(t), y_2(t))|$$

$$\begin{aligned}
&= \left|\left(\frac{\cos^2 x^3}{20(1+\cos^2 x^3)} + \frac{1}{24}\cos^{-1} x_1 + \frac{t}{40}\tan^{-1}\frac{|x_2|}{(1+|x_2|)} + \frac{11t}{240}\right)\right. \\
&- \left.\left(\frac{\cos^2 y^3}{20(1+\cos^2 y^3)} + \frac{1}{24}\cos^{-1} y_1 + \frac{t}{40}\tan^{-1}\frac{|y_2|}{(1+|y_2|)} + \frac{11t}{240}\right)\right| \\
&+ \frac{1}{20}\left|\frac{\cos^2 x^3}{(1+\cos^2 x^3)} - \frac{\cos^2 y^3}{(1+\cos^2 y^3)}\right| + \frac{1}{24}\left|\cos^{-1} x_1 - \cos^{-1} y_1\right| \\
&+ \frac{t}{40}\left|\tan^{-1}\frac{|x_2|}{(1+|x_2|)} - \tan^{-1}\frac{|y_2|}{(1+|y_2|)}\right| \\
&\leqslant \frac{1}{20}|x-y| + \frac{1}{24}|x_1-y_1| + \frac{1}{20}|x_2-y_2|.
\end{aligned}$$

Then

$$|f(t,x(t),x_1(t),x_2(t)) - f(t,y(t),y_1(t),y_2(t))| \le \frac{1}{20}\sum_{j=1}^{3}|x_j(t)-y_j(t)|.$$

Then $k = \frac{1}{20}$.

(H3) For all $t \in [0.1, 2]$, we have

$$\begin{aligned}
|f_x(t)| &= |f(t,x(t),x_1(t),x_2(t))| \\
&\le \frac{\cos^2 x^3}{20(1+\cos^2 x^3)} + \frac{1}{24}\cos^{-1} x_1 + \frac{t}{40}\tan^{-1}\frac{|x_2|}{(1+|x_2|)} + \frac{11t}{240} \\
&\le \frac{1}{20}.
\end{aligned}$$

Then $N = \frac{1}{20}$. Furthermore,

$$s = \sup_{t\in[0.1,2]} |f(t,0,0,0)| = \frac{11}{120}.$$

Then there exists at least one solution for the non-periodic jerk problem (3.6) on the interval $[0.1, 2]$.

We consider the case: $G(t) = t$ (Caputo derivative), we get

$$\begin{aligned}
\Omega &= \Bigg[\Bigg(\frac{|\nu|}{\Gamma(\gamma+1)|1-\lambda||1-\nu|}\left(\frac{|\mu|}{|1-\mu|\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta+\alpha+1)}\right) \\
&+ \frac{|\mu|}{|1-\mu||1-\lambda|\Gamma(\alpha+1)\Gamma(\beta+\gamma+1)} + \frac{1}{|1-\lambda|\Gamma(\alpha+\beta+\gamma+1)}\Bigg)\times|G(2)-G(0.1)|^{\alpha+\beta+\gamma} \\
&+ \Bigg(\frac{\mu\nu\Gamma(\alpha+\beta+1)}{|1-\mu||1-\nu||1-\lambda|\Gamma(\alpha+1)(\Gamma(\beta+1))^2\Gamma(\gamma+1)} + \frac{\nu}{|1-\nu||1-\lambda|\Gamma(\beta+1)\Gamma(\gamma+1)} \\
&+ \frac{\mu}{|1-\lambda|^2|1-\mu|\Gamma(\beta+\gamma+1)}\Bigg)\times|G(2)-G(0.1)|^{\beta+\gamma} \\
&+ \Bigg(\frac{\mu\nu\Gamma(\alpha+\beta+1)}{|1-\mu||1-\nu||1-\lambda|\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)} + \frac{\nu}{|1-\nu||1-\lambda|\Gamma(\gamma+1)} \\
&+ \frac{\mu}{|1-\lambda||1-\mu|\Gamma(\gamma+1)}\Bigg)\times|G(2)-G(0.1)|^{\gamma}\Bigg] \\
&= 6.1170.
\end{aligned}$$

3.3. Example

Since

$$k\Omega = 0.3058 < 1,$$

then the fractional non-periodic Jerk problem (3.6) has a unique solution. On the other hand, all requirements of Theorems (3.1) and (3.2) are fulfilled, this guarantees the stability in both Ulam-Hyers and generalized Ulam-Hyers sense of the fractional non-periodic Jerk problem (3.5) admits one stable solution on $[0.1, 2]$.

General Conclusion

The contribution of our work is primarily in the following point: The extension to the study of a nonlinear fractional jerk system with respect to G-Caputo derivative subject to non-periodic boundary conditions. We were able to provide an integral representation of our problem, which allowed us to transform it into a fixed-point problem.

The fixed-point theorems of Schauder and Banach were key to the analysis of our problem. Moreover, by considering just three conditions related to the existence and uniqueness of the solution to our problem, we were able to investigate and confirm the stability of the solution in both the Ulam-Hyers and Ulam-Hyers-Rassias senses.

The numerical example confirms the theoretical results obtained.

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