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Dedications

I dedicate this Manuscript to:

My Mother

Father

My brothers

My sisters

My colleagues.

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Table of notations

These notations permit the reader to clearly understand the content of this work.

- A, B, C n by n matrices, $\alpha, \beta, \lambda, \mu$ are scalars and X, Y, Z, x, y, z... are vectors.
- A^{-1} The inverse of the matrix A.
- e^A The exponential matrix of A, e^{At} is fundamental matrix solution to the homogeneous system $X' = AX$.
- T, U Upper triangular matrices
- f, g, f denote for Endomorphisms over a vector space E.
- $f \circ g$ or fg Means $f(g(.))$ and we write $(f \circ g)(v) = f(g(v))$.
- f^k Means $f \circ f \circ \ldots \circ f$, i.e., the composition k-times.
- $Vect\{u_1, u_2, ..., u_n\}$ The vector space of all linear combinations of the vectors u_i ($1 \leq 1 \leq n$).
- P The passage matrix.
- E a vector space over $\mathbb{K}$.
- $\mathbb{K}^n$ The field of n-tuples of real or complex numbers.
- $(x_1, x_2, ..., x_n)$ An element of $\mathbb{K}^n$ (vector).
- $\mathbb{K}[x]$, $\mathbb{K}_n[x]$ The vector space of all polynomial of degree not exceeding n with real or complex coefficients.
- $C([a, b], \mathbb{R})$ The vector space of all continuous functions on $[a, b]$.
- $C^{\infty}([a, b], \mathbb{R})$ The v. space of all infinitely differentiable functions on $[a, b]$.
- $\mathcal{M}_n(\mathbb{K})$ The vector space of all n by n real (or complex) matrices.
- $\mathbb{GL}_n(\mathbb{K})$ The vector space of all n by n invertible matrices.

- $\{e_1, e_2, ..., e_n\}$ In general denotes for the canonical basis.
- $F \oplus G$ Direct sum between F and G.
- $diag\,\{a_1, a_2, ..., a_n\}$ Diagonal matrix whose diagonal entries are $a_1, a_2, ..., a_n$.
- $p_A\,(x)\,,\, p_f\,(x)$ Characteristic polynomials A and f, respectively.
- $Sp\,(A)$ The spectral set of A = The set of eigenvalues of A.
- λ, μ eigenvalues
- i The imaginary pure number ($i^2 = -1$).
- I or I_n The identity matrix.
- $Re\,(z)$ The real part of a complex number z and Im for the imaginary part.
- A^t The transpose of a matrix A.
- $\det\,(A)$ Determinant of a square matrix A.
- $\|v\|$ The norm of the vector v.
- $\|A\|$ The norm of the matrix A.
- $X'\,(t)$, $X''\,(t)$ The first derivative of the vector function X
- $X^{(m)}\,(t)$ The m-th derivative of the vector function X
- $\ker\,(f)$ The kernel of f : $\ker f = \{v \in E : f\,(v) = 0_E\}$.
- $\ker\,(A - \lambda I)$ The eigenspace corresponding to λ
- $\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$ Matrix formed by n vectors $x_1, ..., x_n$.
- J Jordan matrix by blocks
- $J_k\,(\lambda)$ Simple $k \times k$ Jordan matrix corresponding to λ.
- N Nilpotent matrix ($N^k = 0$ for some $k \geq 1$).
- E_λ The corresponding eigenspace of f for the eigenvalue λ.
- $\mathcal{N}_\lambda$ The corresponding characteristic subspace of f for the eigenvalue λ.
- id_E identical Endomorphism of E.
- $com\,(A)$ The comatrix of A.

General Introduction

A differential equation is an equation whose unknown is a function (generally denoted $x(t)$ or simply x); in which some of the derivatives of the function appear (first derivative x' or derivatives of higher orders: $x'', x^{(3)}, x^{(4)}$ and so on.

We consider the following differential equation; called of *order* 1.

$$x'(t) = a \cdot x(t) + f(t), t \in [0, a] \tag{E}$$

The corresponding *homogeneous* differential equation is that where the *second member* $f(t)$ is zero, i.e.

$$x'(t) = a \cdot x(t), t \in [0, a] \tag{1}$$

The solution verifies $x(t) = x(0) e^{at}$. We found that any solution of the homogeneous system (1) is written in the form $x(t) = ce^{at}$, where c is a constant. Then if we denote by $\mathcal{S}$ the set of solutions of (1), then $\mathcal{S} = Vect\{e^{at}\}$. We return to the general problem (E), it is immediate to prove that $x(t)$ is a solution of (E) if and only if

$$X(t) = x(0) e^{at} + \int_0^t e^{a(t-r)} f(r) \, dr. \tag{2}$$

Thus, the problem (E) has only one solution. For more information, one can see [1].

There are many types of such equations as well as the first order differential equations when we meet with the famous Cauchy's problem, the second and higher order differential equations. There are also homogeneous and nonhomogeneous equations and some of which are linear. To find all the solutions of the differential equation (E), it suffices to find a particular solution and add to it the general solution of the homogeneous equation. To find this particular solution we will use the method of *variation of the constant*. There is no a priori method for finding a particular solution to a second order linear equation except for sine, cosine or exponential functions. But there are many applications that provide to some differential equations sharing a common set of solutions [6]. Typically, these solutions include arbitrary real or complex constants. I propose to understand and study a system of simultaneous linear differential equations with constant coefficients with or without a second member, and to establish a mode of discussion and resolution of this system, analogous to that developed for differential equations. Therefore, the study of differential equations was generalized by considering that the variable function is a composite vector of several functions that are generally connected to each other.

More generally, in view of (E) and (1) we consider $X(t)$ as a vector function includes n components, that is $X(t) = (x_1(t), ..., x_n(t))$ and if we have an $n \times n$ matrix with real or complex entries, then it is would be important to generalize (E) as follows:

$$X'(t) = AX(t) + F(t), t \in [0, a], \tag{3}$$

where F is a continuous vector function from $[0, a]$ to $\mathbb{K}^n$ and $A \in \mathcal{M}_n(\mathbb{K})$. The above equation known as a nonhomogeneous ODE system with constant coefficients. Therefore, to determine solutions of the system (3) we must have somehow to find particulars solutions to the nonhomogeneous corresponding system and use the technique from the eigenanalysis study to obtain solution to the homogeneous system. Recall that for the ODE we know three approaches to solve nonhomogeneous equations: variating of the constant (or variating of the parameters) method, the method of an educated guess, and the Laplace transforing method. Similarly, with ODE system the same methods are used.

We present proofs for some results dealing with trigonalizations of any matrix from which we conclude that if A is upper triangular, then (3) becomes few easy. Note that in the diagonalization and trigonalization problems, we know that all matrices with complex entriesare are trigonalizable, so they are similar with to an upper triangular matrix. There are some important consequences of these results. Many applications immediate from diagonalization and trigonalization of matrices with the calculus of powers and solving systems of differential equations. More precisely, solving systems as in (3) required us to compute the exponential matrix e^{At}. We can use the triangulation of any matrix A to prove that the problem (3) has always a solution, where the system must be only treated when A is upper triangular and we use next the fact that every matrix with complex entries can written as PTP^{-1}, where P is invertible and T is triangular. We can also calculate the exponential using the eigenanalysis method when A is diagonalizable, that is, A can represented as PDP^{-1} with D diagonal or by Dunford decomposition. In fact, if $A = \mathcal{D} + N$ is the Dunford decomposition, where $\mathcal{D}$ is diagonalizable and N is nilpotent with $\mathcal{D}$ and N commute, then $e^A = e^{\mathcal{D}} e^N$. Generally we decompose the matrix of the sentence so that we can calculate its exponential. Computing the exponential matrix means solving systems of finite set of linear differential equations in which the eigenvalues are real or complex numbers. Concerning systems including higher order differential equations [4], for example if we wish to solve the one of the systems

$$X^{(m)}(t) = AX(t) + F(t), t \in [0, a], \tag{4}$$

and

$$AX^{(m)}(t) = A_{m-1}X^{(m-1)}(t) + ... + A_1X'(t) + A_0X(t) + F(t), t \in [0, a], \tag{5}$$

then we must convert such systems to (3).

An important factorization of any square matrix was made by Jordan and known as the Jordan decomposition theorem which states that every matrix A can be written as PJP^{-1}, where P is invertible and J is the Jordan matrix by blocks. Since there is a formula to compute e^{tJ} with real, we can therefore deduce the general solution of (3), (4) or (5) and many others.

Workplan.

We try to undestand the most important method to solve a system of higher-order differential equations. The most useful references are [1], [2], [5], [4] and [6]. In Chapter 1, we present the proof of Cayley-Hamilton's Theorem and some results on the matrix exponential calculus as well as the computing e^{At} for special cases. In Chapter 2, we focus on the matrix decomposition Theorems: Trigonalization, Dunford Decomposition Theorem and Jordan Decomposition Theorem. In Chapter 3, we study the eigenanalysis method for systems of differential equations of the first order, where we present important theoretically methods involving eigenpairs by applying the matrix decomposition theorems studied in Chapter 2. Moreover, in Chapter 4, we finish this manuscript by studying systems of higher order differential equations. At the end, there is a conclusion and some open problems for further research.

CHAPTER 1

MATRIX EXPONENTIAL CALCULUS

In this chapter we present some definitions and basic tools. At first, we present the proof of Cayley-Hamilton's Theorem. For more details, see [5].

1.1 Proof of Cayley-Hamilton's Theorem

Theorem 1.1 (Cayley-Hamilton Theorem)**.** *Let $A \in \mathcal{M}_n(\mathbb{R})$ and let $p_A(x)$ be its characteristic polynomial. Then $p_A(A) = 0$.*

In the proof, we need to use the following lemma.

Lemma 1.1. *For each $A \in \mathcal{M}_n(\mathbb{R})$, we have*

$$(com\,(A))^t \cdot A = A \cdot (com\,(A))^t = \det A \cdot I_n. \tag{1.1}$$

When the matrix A is invertible, then its inverse is given by

$$A^{-1} = \frac{1}{\det(A)} (com\,(A))^t .$$

Proof of Cayley-Hamilton Theorem. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \in \mathcal{M}_n(\mathbb{R}).$$

Assume further that $p_A(x) = x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + ... + c_1x + c_0$. Applying Lemma 1.1 using the matrix $xI_n - A$, we obtain

$$(xI_n - A)\, com\,(xI - A)^t = \det(xI_n - A)\, I_n,$$

where

$$xI - A = \begin{pmatrix} x-a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & x-a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & x-a_{nn} \end{pmatrix}.$$

Hence,

$$com\,(xI - A) = \begin{pmatrix} p_{n-1}^{(1,1)}(x) & p_{n-1}^{(1,2)}(x) & \dots & p_{n-1}^{(1,n)}(x) \\ p_{n-1}^{(2,1)}(x) & p_{n-1}^{(2,2)}(x) & \dots & p_{n-1}^{(2,n)}(x) \\ \vdots & \vdots & \vdots & \vdots \\ p_{n-1}^{(n,1)}(x) & p_{n-1}^{(n,2)}(x) & \dots & p_{n-1}^{(n,n)}(x) \end{pmatrix},$$

where $p_{n-1}^{(i,j)}$ are polynomials of degree $n-1$. Setting

$$com\,(xI - A)^t = B_0 + xB_1 + x^2B_2 + \dots + x^{n-1}B_{n-1}, \text{ where } (B_i)_{i=0,1,\dots,n-1} \in M_n(\mathbb{R}).$$

We deduce that

$$\begin{aligned} (xI - A)\left(B_0 + xB_1 + x^2B_2 + \dots + x^{n-1}B_{n-1}\right) &= \det(xI_n - A).I_n \\ &= x^nI_n + c_{n-1}x^{n-1}I_n + \dots + c_1xI_n + c_0I_n. \end{aligned}$$

It follows that

$$\begin{aligned} & x^nB_{n-1} + x^{n-1}(B_{n-2} - AB_{n-1}) + \dots + x(B_0 - AB_1) - AB_0 \\ = \; & x^nI_n + c_{n-1}x^{n-1}I_n + \dots + c_1xI_n + c_0I_n. \end{aligned}$$

Then

$$\begin{cases} -AB_0 = c_0I_n \\ \vdots \\ B_0 - AB_1 = c_1I_n \\ B_{n-2} - AB_{n-1} = c_{n-1}x^{n-1}I_n \\ B_{n-1} = I_n \end{cases}$$

This gives

$$\begin{aligned} p_A(A) &= c_0I_n + c_1A + \dots + c_{n-1}A^{n-1} + A^n \\ &= -AB_0 + A(B_0 - AB_1) + \dots + A^{n-1}(B_{n-2} - AB_{n-1}) + A^nB_{n-1} \\ &= 0. \end{aligned}$$

This completes the proof of Theorem 1.1. □

1.2 Fundamental matrix

$\mathbb{K}$ will be the field $\mathbb{R}$ or $\mathbb{C}$, E a $\mathbb{K}$-vector space of finite dimension. Let $x \in \mathbb{K}$. It is well-known that the power series

$$1+\frac{x}{1!}+\frac{x^2}{2!}+...+\frac{x^k}{k!}+... \tag{1.2}$$

is denoted by e^x. Let $A \in \mathcal{M}_n(\mathbb{K})$. If we replace the matrix A into (1.2), then we obtain

$$e^A = I+\frac{A}{1!}+\frac{A^2}{2!}+...+\frac{A^k}{k!}+... = \sum_{k=0}^{+\infty}\frac{A^k}{k!}.$$

Moreover, the corresponding *fundamental matrix* of any matrix A is defined as follows:

Definition 1.1 (Fundamental matrix)**.** Let A be a square matrix and let $t \in \mathbb{R}$. Then define

$$e^{At} = I_n + At + \frac{A^2t^2}{2!}+\ldots+\frac{A^kt^k}{k!}+\ldots \tag{1.3}$$

The above formula is known as the series expansion of Maclaurin for the function $t \mapsto e^{At}$. It is not obvious that this sum converges, so a natural question: Is the above series convergent?

Let $\mathbb{K}$ be the field of real numbers or complexes. We define the mapping

$$\begin{aligned} e^A \quad &: \quad \mathcal{M}_n(\mathbb{K}) \to \mathcal{M}_n(\mathbb{K}) \\ A \quad &\mapsto \quad \sum_{k=0}^{+\infty}\frac{A^k}{k!} \end{aligned}$$

If we set $n=1$, we find the well-known exponential function on $\mathbb{R}$ and on $\mathbb{C}$. Let $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{K})$ and let M be a positive real number such that $|a_{ij}| \leq M$ for $1 \leq i,j \leq n$. Since $A^2 = \left(a_{ij}^{(2)}\right)$ with $a_{ij}^{(2)} = \sum_{s=1}^n a_{is}\cdot a_{sj}$, we conclude that $\left|a_{ij}^{(2)}\right| \leq M^2n$. More generally, the entries of A^k are bounded by M^kn^{k-1}. We can prove this by induction on k. Indeed, for $k=2$, the statement is true. Assume that the statement is true for k, so if $A^k = \left(a_{ij}^{(k)}\right)$, then $A^{k+1} = \left(a_{ij}^{(k+1)}\right)$ with $a_{ij}^{(k+1)} = \sum_{s=1}^n a_{is}\cdot a_{sj}^{(k)}$, where

$$\left|a_{ij}^{(m+1)}\right| \leq \sum_{s=0}^{n}|a_{is}|\left|a_{sj}^{(k)}\right| \leq \sum_{s=0}^{n} M^{k+1}n^{k-1} = M^{k+1}n^k.$$

As required.

Now, if we put $e^A = (c_{ij})_{1\leq i,j\leq n}$, then

$$|c_{ij}| \leq 1 + M + \frac{nM^2}{2!} + \ldots + \frac{n^{k-1}M^k}{k!} + \ldots \leq e^{nM}.$$

This proves the existence of the entries of e^A. So for any matrix A, the exponential matrix e^A exists.

Proposition 1.1. *The series defined in* (1.3) *converges.*

It suffices to present the proof of the following result:

Proposition 1.2. *For any $A \in \mathcal{M}_n(\mathbb{C})$, the series $\sum_{k=0}^{+\infty} \frac{A^k}{k!}$ is absolutely convergent.*

Proof. Let $\|.\|$ be any matrix norm. For every $k \geq 0$, we have $\left\|\frac{A^k}{k!}\right\| \leq \frac{\|A\|^k}{k!}$. By d'Alembert[1] rule, we obtain

$$\left|\frac{\frac{\|A\|^{k+1}}{(k+1)!}}{\frac{\|A\|^k}{k!}}\right| = \frac{\|A\|}{k+1} \to 0 < 1.$$

Hence, $\sum_{k=0}^{\infty} \frac{A^k}{k!}$ converges. Since

$$\left\|\sum_{k=0}^{+\infty} \frac{A^k}{k!}\right\| \leq \sum_{k=0}^{+\infty} \frac{\|A\|^k}{k!}.$$

we deduce that the series $\sum_{k=0}^{+\infty} \dfrac{A^k}{k!}$ converges absolutely. □

Next, we present the exponential of a diagonal matrix by blocks. In fact, if $A = diag\left\{A_1, A_2, ..., A_k\right\}$, then

$$e^{At} = diag\left\{e^{A_1 t}, e^{A_2 t}, ..., e^{A_k t}\right\}. \tag{1.4}$$

We can compute the exponential matrix when A is diagonalizable, which gives the corresponding fundamental matrix. For example, if $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then the corresponding fundamental matrix is given by

$$\mathcal{F}_{\mathcal{M}}(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix},$$

[1] Let $\sum u_n$ such that u_n is positive. If the limit $l = \lim \frac{u_{n+1}}{u_n}$ (finite or not) exists, then

- The series $\sum u_n$ is convergent when $l < 1$,
- The series $\sum u_n$ is divergent when $l > 1$.

where $\mathcal{F}_{\mathcal{M}}(0) = I$. Note that the matrix A is diagonalizable which has the eigenvalues $\lambda_1 = -i$ and $\lambda_2 = i$ with the corresponding eigenvectors $v_1 = (i, 1)$ and $v_2 = (-i, 1)$. So we can easily compute e^{At}.

Some well-known facts on the exponential matrix:

1.2.1 Some facts on exponential matrix

Proposition 1.3. *We have the following facts:*

1. *Let* 0 *be the zero matrix. Then* $e^{o} = I$.
2. *If* $AB = BA$, *then* $e^{A+B} = e^{A} \cdot e^{B} = e^{B} \cdot e^{A}$. *In particular,* $\left(e^{A}\right)^{-1} = e^{-A}$.
3. *If* $A, P \in \mathcal{M}_n(\mathbb{C})$ *with* P *is invertible, then* $e^{PAP^{-1}} = Pe^{A}P^{-1}$.
4. *If* $D = diag\{\lambda_1, \lambda_2, ..., \lambda_n\}$, *then* $e^{D} = diag\left\{e^{\lambda_1}, e^{\lambda_2}, ..., e^{\lambda_n}\right\}$.
5. *If* T *is an upper triangular matrix whose diagonal entries are* (t_{ii}), *then* e^{T} *is also upper triangular matrix whose diagonal entries are* $(e^{t_{ii}})$.
6. $\det\left(e^{A}\right) = e^{tr(A)}$.
7. $\frac{\partial}{\partial t}\left(e^{tA}\right) = A \cdot e^{tA}$.
8. *Let* $A \in \mathcal{M}_n(\mathbb{C})$. *Then*
$$\lim_{k \to +\infty}\left(I_n + \frac{A}{k}\right)^{k} = e^{A}.$$
9. *If* $AB = BA$, *then* $Be^{At} = e^{At}A$ *and* $e^{At}e^{Bt} = e^{(A+B)t}$.
10. *Since* At *and* As *commute,* $e^{At}e^{As} = e^{A(t+s)}$.
11. $\left(e^{At}\right)^{-1} = e^{-At}$. *Thus,* $e^{At}e^{-At} = I$.
12. *If* $A \in \mathcal{M}_2(\mathbb{R})$ *with* $Sp(A) = \{\lambda_1, \lambda_2\}$, *then* $e^{At} = e^{\lambda_1 t}I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(A - \lambda_1 I)$.
13. *If* $A \in \mathcal{M}_2(\mathbb{R})$ *with* $Sp(A) = \{\lambda\}$, *then* $e^{At} = e^{\lambda t}I + te^{At}(A - \lambda I)$.
14. *If* $A \in \mathcal{M}_2(\mathbb{C})$ *with* $Sp(A) = \{\lambda_1, \lambda_2\}$ *and* $\lambda_1 = \overline{\lambda_2} = a + ib$, $b > 0$, *then*
$$e^{At} = e^{at}\cos bt I + \frac{e^{at}\sin bt}{b}(A - aI).$$
15. *For any matrix* $A \in \mathcal{M}_2(\mathbb{K})$ *and* $t \in \mathbb{R}$,*we have* $\left\|e^{At}\right\| \leq e^{\|A\| \cdot |t|}$.

Recall that if A, B are two $n \times n$ matrices. We say that A is similar to B if there exists an invertible matrix P such that $A = PBP^{-1}$ or $B = PAP^{-1}$.

Remark 1.1. On $\mathbb{R}$ the matrices which are similar to symmetric matrices are only diagonalizable matrices. However, on $\mathbb{C}$, every matrix A is similar to a symmetric matrix.

1.2.2 Computing e^{At} for some cases

If A is an upper triangular matrix (resp. lower triangular matrix), then a column $v(t)$ of e^{At} can be easily found by solving the system $v'(t) = Av(t)$, where $v(0) = v$ is the corresponding column of the unit matrix. Note that such problem can always be solved. In particular, when $A = diag\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, then the columns $v(t) = (v_1(t), ..., v_k(t))$ of e^{At} can be easily found by solving the equation $v_i'(t) = \lambda_i v_i(t)$ with $v_i(0) = 1$, which gives $v_i(t) = e^{\lambda_i t}$. So, $e^{At} = diag\left\{e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_k t}\right\}$.

Theorem 1.2. *If $A = diag\{A_1, A_2, \ldots, A_k\}$ where each of $A_1, A_2, \ldots, A_k$ is square, then*

$$e^{At} = diag\left\{e^{A_1 t}, e^{A_2 t}, \ldots, e^{A_k t}\right\}.$$

Proof. The proof holds immediately since $A^s = diag\{A_1^s, A_2^s, \ldots, A_k^s\}$ whenever $A = diag\{A_1, A_2, \ldots, A_k\}$. □

Corollary 1.1. Let $A \in \mathcal{M}_n(\mathbb{R})$ be a square matrix having a unique eigenvalue, say λ. Then

$$e^{tA} = e^{\lambda t} \sum_{k=0}^{n-1} (A - \lambda I_n)^k \frac{t^k}{k!}.$$

Proof. We first have $p_A(x) = (x - \lambda)^n$ since A has a unique eigenvalue λ. We have

$$\begin{aligned}
e^{tA} &= e^{\lambda t I_n + t(A - \lambda I_n)} && (1.5)\\
&= e^{\lambda t I_n} e^{t(A - \lambda I_n)} \text{ (because } \lambda t I_n \text{ and } t(A - \lambda I_n) \text{ commute)}\\
&= e^{\lambda t} e^{t(A - \lambda I_n)} \text{ (because } e^{\alpha I_n} B = e^{\alpha} B \text{ for any } B \in \mathcal{M}_n(\mathbb{R}) \text{ and } \alpha \in \mathbb{R})\\
&= e^{\lambda t} \sum_{k=0}^{+\infty} (A - \lambda I_n)^k \frac{t^k}{k!} && (1.6)\\
&= e^{\lambda t} \sum_{k=0}^{n-1} (A - \lambda I_n)^k \frac{t^k}{k!},
\end{aligned}$$

where $\sum_{k=n}^{+\infty} (A - \lambda I_n)^k = 0$; this is obtained by Cayley-Hamilton theorem since $p_A(A) = (A - \lambda I_n)^n = 0$. □

Theorem 1.3. *Let $A \in \mathcal{M}_3(\mathbb{R})$. If A has two distinct eigenvalues λ and μ (where λ has multiplicity 2), then*

$$e^{tA} = e^{\lambda t}\left(I + t\left(A - \lambda I\right)\right) + \frac{e^{\mu t} - e^{\lambda t}}{\left(\mu - \lambda\right)^2}\left(A - \lambda I\right)^2 - \frac{te^{\lambda t}}{\mu - \lambda}\left(A - \lambda I\right)^2. \tag{1.7}$$

Proof. From (1.5) and (1.6), we have

$$\begin{aligned} e^{tA} &= e^{\lambda t}\sum_{k=0}^{+\infty}\left(A - \lambda I\right)^k \frac{t^k}{k!} \\ &= e^{\lambda t}\left(I + \left(A - \lambda I\right)\right) + e^{\lambda t}\sum_{r=2}^{+\infty}\left(A - \lambda I\right)^k \frac{t^k}{k!} \\ &= e^{\lambda t}\left(I + \left(A - \lambda I\right)\right) + e^{\lambda t}\sum_{k=0}^{+\infty}\left(A - \lambda I\right)^{2+r} \frac{t^{2+r}}{(2+r)!} \end{aligned} \tag{1.8}$$

Now, let $p_A(x) = (x - \lambda)^2 (x - \mu)$ be the characteristic polynomial of A. First, we note that

$$A - \mu I = \left(A - \lambda I_n\right) - \left(\mu - \lambda\right) I.$$

By Cayley-Hamilton theorem, we get

$$0 = \left(A - \lambda I\right)^2\left(A - \mu I\right) = \left(A - \lambda I\right)^3 - \left(\mu - \lambda\right)\left(A - \lambda I\right)^2,$$

from which is follows that

$$\left(A - \lambda I\right)^3 = \left(\mu - \lambda\right)\left(A - \lambda I\right)^2.$$

By induction, for every $r \geq 1$,

$$\left(A - \lambda I\right)^{2+r} = \left(\mu - \lambda\right)^r\left(A - \lambda I\right)^2.$$

It follows from (1.8) that

$$\begin{aligned} \sum_{k=0}^{+\infty}\left(A - \lambda I\right)^{2+r}\frac{t^{2+r}}{(2+r)!} &= \sum_{k=0}^{+\infty}\left(\mu - \lambda\right)^r \frac{t^{2+r}}{(2+r)!}^{r}\left(A - \lambda I\right)^2 \\ &= \frac{1}{\left(\mu - \lambda\right)^2}\sum_{k=0}^{+\infty}\frac{t^k}{k!}\left(\mu - \lambda\right)^k\left(A - \lambda I\right)^2. \end{aligned}$$

Finally, we obtain

$$\begin{aligned} e^{tA} &= e^{\lambda t}\left(I+(A-\lambda I)\right)+\frac{e^{\lambda t}}{(\mu-\lambda)^2}\left\{e^{(\mu-\lambda)t}-1-(\mu-\lambda)t\right\}(A-\lambda I)^2 \\ &= e^{\lambda t}\left(I+t(A-\lambda I)\right)+\frac{e^{\mu t}-e^{\lambda t}}{(\mu-\lambda)^2}(A-\lambda I)^2-\frac{te^{\lambda t}}{\mu-\lambda}(A-\lambda I)^2. \end{aligned}$$

This completes the proof. □

Example 1.1. Consider the matrix

$$A=\begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$

We have

$$e^A=e^x\begin{pmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{pmatrix}. \tag{1.9}$$

In fact, if we put $z=x+yi$, where $i^2=-1$, then we can easily prove by induction on n that

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix}^n=\begin{pmatrix} Re\left(z^n\right) & Im\left(z^n\right) \\ -Im\left(z^n\right) & Re\left(z^n\right) \end{pmatrix}. \tag{1.10}$$

Indeed, for $n=1$ the statement is true by definition. Assume that (1.10) holds for $n=1$. Then

$$\begin{aligned} \begin{pmatrix} x & y \\ -y & x \end{pmatrix}^{n+1} &= \begin{pmatrix} x & y \\ -y & x \end{pmatrix}\begin{pmatrix} Re\left(z^n\right) & Im\left(z^n\right) \\ -Im\left(z^n\right) & Re\left(z^n\right) \end{pmatrix} \\ &= \begin{pmatrix} xRe\left(z^n\right)-yIm\left(z^n\right) & xIm\left(z^n\right)+yRe\left(z^n\right) \\ -xIm\left(z^n\right)-yRe\left(z^n\right) & xRe\left(z^n\right)-yIm\left(z^n\right) \end{pmatrix} \end{aligned}$$

On the other hand, if we put $z=r\cos\theta+r\sin\theta$, then we have $z^n=r^n\left(\cos(n\theta)+\sin(n\theta)\right)$. Hence,

$$\begin{aligned} \begin{pmatrix} x & y \\ -y & x \end{pmatrix}^{n+1} &= r^n\begin{pmatrix} x\cos(n\theta)-y\sin(n\theta) & x\sin(n\theta)+y\cos(n\theta) \\ -x\sin(n\theta)-y\cos(n\theta) & x\cos(n\theta)-y\sin(n\theta) \end{pmatrix} \\ &= r^{n+1}\begin{pmatrix} \cos\theta\cos(n\theta)-\sin\theta\sin(n\theta) & \cos\theta\sin(n\theta)+\sin\theta\cos(n\theta) \\ -\cos\theta\sin(n\theta)-\sin\theta\cos(n\theta) & \cos\theta\cos(n\theta)-\sin\theta\sin(n\theta) \end{pmatrix} \\ &= \begin{pmatrix} r^{n+1}\cos((n+1)\theta) & r^{n+1}\sin((n+1)\theta) \\ -r^{n+1}\sin((n+1)\theta) & r^{n+1}\cos((n+1)\theta) \end{pmatrix} \\ &= \begin{pmatrix} Re\left(z^{n+1}\right) & Im\left(z^{n+1}\right) \\ -Im\left(z^{n+1}\right) & Re\left(z^{n+1}\right) \end{pmatrix} \end{aligned}$$

Therefore, by definition, we get

$$e^A = \sum_{n=0}^{+\infty} \begin{pmatrix} Re\left(\frac{z^n}{n!}\right) & Im\left(\frac{z^n}{n!}\right) \\ -Im\left(\frac{z^n}{n!}\right) & Re\left(\frac{z^n}{n!}\right) \end{pmatrix} = \begin{pmatrix} Re\,(e^z) & Im\,(e^z) \\ -Im\,(e^z) & Re\,(e^z) \end{pmatrix},$$

where $e^z = e^{x+yi} = e^x\left[\cos y + i \sin y\right]$. Thus, we obtain (1.9).

Remark 1.2. As above, we can compute e^A for the matrix

$$A = \begin{pmatrix} x & y & & & & & \\ -y & x & & & & & \\ & & x & y & & & \\ & & -y & x & & & \\ & & & & \ddots & & \\ & & & & & x & y \\ & & & & & -y & x \end{pmatrix}.$$

Thus, by Lemma, we have

$$e^A = e^x \begin{pmatrix} \cos y & \sin y & & & & & \\ -\sin y & \cos y & & & & & \\ & & \cos y & \sin y & & & \\ & & -\sin y & \cos y & & & \\ & & & & \ddots & & \\ & & & & & \cos y & \sin y \\ & & & & & -\sin y & \cos y \end{pmatrix}.$$

1.3 Nilpotent matrices and Binomial formula

At first, consider the following definition:

Definition 1.2. An n by n matrix N (resp. an Endomorphism f) is said to be nilpotent if there exists a nonnegative integer k such that $N^k = 0$ (resp. $f^k = 0$). The smallest positive integer k such that $N^{k-1} \neq 0$ and $N^k = 0$ is called the index[2] of N.

In other words, A nilpotent matrix is a square matrix N such that $N^k = 0$ for some

[2]We say that N is *nilpotent* with index k.

positive integer k. Consider the n by n matrix

$$N = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \mathbf{1} & 0 & 0 & \ddots & 0 \\ \vdots & \mathbf{1} & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & \mathbf{1} & 0 \end{pmatrix}.$$

This matrix is nilpotent with index n, since $N^{n-1} \neq 0$ and $N^n = 0$. Therefore,

$$e^N = I_n + \frac{N}{1!} + \frac{N^2}{2!} + ... + \frac{N^{n-1}}{(n-1)!}.$$

Proposition 1.4. *Let N be a nilpotent matrix. Then*

- $Sp\,(N) = \{0\}$,
- *$I - N$ is invertible.*

Proof. Assume that N is nilpotent with index k, and let (λ, x) be an eigenpair of N. Since $Nx = \lambda x$, we have $\lambda^k x = s$. But, since x is nonzero, we deduce that $\lambda = 0$. □

Theorem 1.4. *Let A be a nonzero nilpotent matrix. Then A is nondiagonalizable.*

Theorem 1.5. *Any strictly triangular matrix is nilpotent.*

Proof. Setting

$$A = \begin{pmatrix} \mathbf{0} & 0 & \cdots & 0 \\ a_{21} & \mathbf{0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \mathbf{0} \end{pmatrix}.$$

Since $p_A\,(x) = x^n$. By Cayley-Hamilton theorem, $A^n =$. That is, $\exists\, k \leq n$ such that $A^k = 0$, and hence A is nilpotent. □

Theorem 1.6. *Let N be nilpotent with index k and let $v \in \mathbb{R}^n$ be a nonzero vector such that $N^{k-1}v \neq 0$. The family*

$$\left\{Iv, Nv, N^2v, ..., N^{k-1}v\right\}$$

is free.

Proof. Let $(\alpha_i)_{0\leq i\leq k-1} \in \mathbb{R}$ such that

$$\sum_{i=0}^{k-1} \alpha_i N^i v = 0,$$

from which it follows that

$$\begin{cases} \alpha_0 N^{k-1}v + \alpha_1 N^k v + ... + \alpha_{k-1}N^{2k-2}v = 0 \\ \alpha_0 N^{k-2}v + \alpha_1 N^{k-1} v + ... + \alpha_{k-1}N^{2k-3}v = 0 \\ \alpha_0 N v + \alpha_1 N^2 v + ... + \alpha_{k-1}N^{k}v = 0 \\ \vdots \\ \alpha_0 I v + \alpha_1 N v + ... + \alpha_{k-1}N^{k-1}v = 0 \end{cases} \Rightarrow \begin{cases} \alpha_0 N^{k-1}x = 0 \\ \alpha_1 N^{k-1}x \\ \vdots \\ \alpha_{k-2} N^{k-1}x = 0 \\ \alpha_{k-1} N^{k-1}x = 0 \end{cases}$$

Since $N^{k-1}v \neq 0$, then $\alpha_0 = \alpha_1 = ... = \alpha_{k-1} = 0$. This completes the proof. $\square$

Let us consider the following example:

Example 1.2. Consider the 3 by 3 matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We can easily calculate e^{At} since A has many zeros. Indeed, by Definition we have $A^s = 0$ for $s \geq 3$, and so

$$e^{At} = I_n + At + \frac{A^2t^2}{2!} = \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix},$$

where I_n is the identity matrix.

Remark 1.3. Let N be nilpotent with index k. By Definition , it is clear to see that

$$e^{At} = I_n + At + \frac{A^2t^2}{2!} + \ldots + \frac{A^{k-1}t^{k-1}}{(k-1)!}. \tag{1.11}$$

Since multiplication is not commutative for matrices, the usual binomial identities are false. In particular, $(A+B)^2$ is not generally worth $A^2 + 2AB + B^2$, but we only know that $(A+B)^2 = A^2 + AB + BA + B^2$.

On the calculus of $(A+B)^k$ whenever A and B commute.

Proposition 1.5. *Let A and B be two elements of $\mathcal{M}_n(\mathbb{R})$ which commute, that is to say such that*

$AB = BA$. Then, for any integer $k \geq 1$, we have the formula:

$$(A+B)^k = \sum_{i=0}^{k} C_k^i A^i B^{k-i} = \sum_{i=0}^{k} C_k^{k-i} B^i,$$

where C_k^i denotes the coefficient of the binomial.

The proof is (by induction on k) similar to that of the binomial formula for $(a+b)^k$ with $k \geq 1$.

As application of Proposition, we consider the upper triangular matrix

$$A = \begin{pmatrix} \lambda & \times & \times & \times & \times \\ & \lambda & \times & \times & \times \\ & & \ddots & \times & \times \\ & & & \lambda & \times \\ & & & & \lambda \end{pmatrix}.$$

We put $N = A - \lambda I$ which is a nilpotent matrix[3], i.e, there exists a positive integer m such that $N^{m-1} \neq 0$ and $N^m = 0$. Since $A = \lambda I + N$ with λI and N commute, we conclude that u

$$A^k = \sum_{i=0}^{k} C_k^i A^i B^{k-i} = \sum_{i=0}^{m-1} C_k^i A^i B^{k-i}.$$

Thus, when we need to apply the binomial formula with matrices, we must have two matrices A and B such that $AB = BA$. Otherwise, this formula is not useful.

[3]We recall that a matrix A is said to be *unipotent* if $A - I$ is nilpotent. Or, equivalently, if its spectrum is reduced to $\{1\}$.

CHAPTER 2

SOME MATRIX DECOMPOSITION THEOREMS

Recall that to solve a system of differential equations of any order we first write the system in the matrix form and then we need to compute the corresponding fundamental matrix e^{At}. To do this, we must represent the matrix A in some special forms so that the powers can be deduced easily. So any matrix $A \in \mathcal{M}_n(\mathbb{C})$ is either diagonalizable or trigonalizable. When A is diagonalizable, the corresponding fundamental matrix can be derived from the fact that e^{At} is the product of the passage matrix P, its inverse and the diagonal matrix which includes all eigenvalues of A. However, when A is not diagonalizable, so by a Theorem we can show that is of the form $A = PTP^{-1}$, where P is invertible and T is upper triangular. This means that A is similar to an upper triangular matrix. This factorization is not enough to compute an explicit formula of the fundamental matrix. For this purpose, we will show that any matrix, whose characteristic polynomial is split, can be written as the sum of a diagonalizable matrix and a nilpotent matrix. In other words, this matrix is similar to the sum of a diagonal matrix and a nilpotent matrix. Other suitable representation is to show that any matrix, whose characteristic polynomial is split, can be written as PJP^{-1}, where P is invertible and J is the matrix by blocks of the Jordan simple matrices corresponding to the eigenvalues of A whose entries are equal to λ_i on the diagonal, 1 right above the diagonal and 0 elsewhere.

2.1 Trigonalization

We have seen that not all matrices are diagonalizable. However, we can break down some of them into the simplest form possible. We will see three decompositions.

- Digonalization: transforming a matrix into a diagonal matrix.
- Trigonalization: transforming a matrix into a triangular matrix.
- The Dunford decomposition: writing a matrix as the sum of a diagonalizable matrix and a nilpotent matrix.

- Jordan reduction: transforming a matrix into a block-diagonal matrix.

The goal of these factorization of any matrix A is to find an explicit formula for e^{At} with $t \in \mathbb{K}$. The later matrix gives us the general solution of the system of differential equations given by its matrix expression $X' = AX$. First, recall the necessary and sufficient condition of the diagonalization.

Theorem 2.1 (Necessary and sufficient condition of the diagonalization)**.** *Let $A \in \mathcal{M}_n(\mathbb{R})$ be square. Then A is diagonalizable if and only if there exists a base B of $\mathbb{R}^n$ formed by n eigenvectors of A.*

Definition 2.1. A matrix $A = (a_{ij})_{1\leq i,j\leq n}$ is said to be upper triangular if $a_{ij} = 0$ whenever $i > j$, that is the coefficients below the diagonal are all zero. Here, we can write

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}. \tag{2.1}$$

In particular, if $a_{ij} = 0$ whenever $i \geq j$, then A is said to be strictly upper triangular.

We will show that any matrix, whose characteristic polynomial is split, is similar to a triangular matrix.

Definition 2.2. A matrix $A \in \mathcal{M}_n(\mathbb{K})$ is trigonalizable on $\mathbb{K}$ if there exists an invertible matrix $P \in \mathcal{M}_n(\mathbb{K})$ invertible such that $P^{-1}AP$ is upper triangular. Moreover, an Endomorphism f of E is trigonalizable if there exists a basis of E in which the matrix of f is upper triangular.

Of course, a diagonalizable matrix is in particular trigonalizable.

Theorem 2.2. *A matrix $A \in \mathcal{M}_n(\mathbb{K})$ (resp. an endomorphism f) is trigonalizable on $\mathbb{K}$ if and only if its characteristic polynomial is split over $\mathbb{K}$.*

Recall that a polynomial is *split* on $\mathbb{K}$ if it decomposes into the product of linear factors in $\mathbb{K}[x]$ or the product of simple factors in $\mathbb{K}[x]$. Clearly, if f is trigonalizable, then there exists a base of E in which the matrix of f is written as in (2.1). It follows that $p_{\mathcal{M}_f}(x) = (a_{11} - x)(a_{22} - x)\ldots(a_{nn} - x)$, which proves that $p_{\mathcal{M}_f}(x)$ decomposes into the product of linear factors in $\mathbb{K}[x]$. We will see the converse later.

Example 2.1. Define the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$. Then $p_A(x) = x^2 + 1$. This polynomial is not split on $\mathbb{R}$, therefore A is not trigonalizable on $\mathbb{R}$. If we consider the same matrix A as an element of $\mathcal{M}_2(\mathbb{C})$, then it is trigonalizable (and here even diagonalizable) on $\mathbb{C}$.

Remark 2.1. Note that if $\mathbb{K} = \mathbb{C}$, by the well-known d'Alembert-Gauss theorem, we have:

Theorem 2.3. *Any matrix with complex entries is trigonalizable over $\mathcal{M}_n(\mathbb{C})$.*

Proof. Let $A \in \mathcal{M}_n(\mathbb{C})$. We will show that A is trigonalizable over $\mathcal{M}_n(\mathbb{C})$. We use induction on n. Indeed, for $n = 1$ we have

$$A = (a_{11}), \text{ where } a_{11} \in \mathbb{C}.$$

In this case, we write

$$A = I(a_{11})I^{-1} = PTP^{-1} \text{ with } P = I = (1) \text{ and } T = (a_{11}) = A.$$

Assume that every matrix $A_1 \in \mathcal{M}_n(\mathbb{C})$ is trigonalizable. Let (λ, x) be an eigenpair of A, and let $\{x, u_2, ..., u_n\}$ be a basis of $\mathbb{C}^n$. We put $U = (x, u_2, ..., u_n)$, it follows that

$$AU = \begin{pmatrix} Ax & Au_2 & \dots & Au_n \end{pmatrix} = \begin{pmatrix} \lambda x & Au_2 & \dots & Au_n \end{pmatrix}.$$

Now, calculate $U^{-1}AU$. In fact, we have

$$U^{-1} = U^{-1}Ue_1 = e_1,$$

where $e_1 = (1, 0, ..., 0)$. Therefore,

$$U^{-1}AU = U^{-1}\begin{pmatrix} \lambda x & Au_2 & \dots & Au_n \end{pmatrix} = \begin{pmatrix} \lambda e_1 & U^{-1}Au_2 & \dots & U^{-1}Au_n \end{pmatrix}.$$

Also we obtain

$$U^{-1}AU = \begin{pmatrix} \lambda & \times & \dots & \times \\ 0 & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \dots & * \end{pmatrix} = \begin{pmatrix} \lambda & C \\ 0 & A_1 \end{pmatrix} = T_1,$$

where $C \in \mathcal{M}_{1,n-1}(\mathbb{C})$ and $A_1 \in \mathcal{M}_{n-1}(\mathbb{C})$. From the hypothesis, there exists an invertible matrix W such that

$$\begin{pmatrix} 1 & C \\ 0 & W^{-1} \end{pmatrix}\begin{pmatrix} \lambda & C \\ 0 & A_1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & W \end{pmatrix} = \begin{pmatrix} \lambda & CW \\ 0 & W^{-1}A_1W \end{pmatrix} = \begin{pmatrix} \lambda & CW \\ 0 & T' \end{pmatrix}.$$

Hence,

$$A \sim T_1 \sim \begin{pmatrix} \lambda & CW \\ 0 & T' \end{pmatrix} = T,$$

where T is upper triangular. That is, $A \sim T_1 \sim T$ which gives $A \sim T$. The proof is finished. □

Let us take

$$A = \begin{pmatrix} 1 & 4 & -2 \\ 0 & 6 & -3 \\ -1 & 4 & 0 \end{pmatrix}$$

Let us prove that A is trigonalizable on $\mathbb{R}$ and find a matrix P such that $P^{-1}AP$ is upper triangular. Since $p_A(x) = (x-2)^2(x-3)$, we see that this polynomial is split on $\mathbb{R}$. So, the matrix is trigonalizable on $\mathbb{R}$. The roots of the characteristic polynomial are the real numbers $\lambda_1 = 3$ (with multiplicity 1), and $\lambda_2 = 2$ (with multiplicity 2). Let us determine the associated eigensubspaces. Let E_3 be the eigensubspace associated with the simple eigenvalue 3, i.e., $E_3 = Vect\{(1,1,1)\}$. Let E_2 be the eigensubspace associated with the eigenvalue 2, i.e., $E_2 = Vect\{(4,3,4)\}$. The dimension of E_2 is equal to 1 while the multiplicity of the corresponding eigenvalue λ_2 is equal to 2. Consequently, we know that the matrix A will not be diagonalizable. Let $v_3 = (0,0,1)$. The vectors (v_1, v_2, v_3) form a basis of $\mathbb{R}^3$. The passage matrix (consisting of the v_i written in column) is

$$P = \begin{pmatrix} 1 & 4 & 0 \\ 1 & 3 & 0 \\ 1 & 4 & 1 \end{pmatrix}, \text{ so } P^{-1} = \begin{pmatrix} -3 & 4 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

We have $A \cdot v_1 = 3v_1$ and $A \cdot v_2 = 2v_2$. It remains to express $A \cdot v_3$ in the basis (v_1, v_2, v_3). After simple computation, we see that

$$A \cdot v_3 = (-2, -3, 0) = -6v_1 + v_2 + 2v_3.$$

Thus, the endomorphism which has matrix A in the canonical basis of $\mathbb{R}^3$ has matrix T in the basis (v_1, v_2, v_3), where

$$T = \begin{pmatrix} \mathbf{3} & 0 & -6 \\ 0 & \mathbf{2} & 1 \\ 0 & 0 & \mathbf{2} \end{pmatrix}.$$

This is an upper triangular matrix whose diagonal entries are the eigenvalues of A. We could also have calculated T by the formula: $T = P^{-1} \cdot A \cdot P$.

Remark 2.2. Note that other choices for v_3 are possible. Here, any vector v_3' completing (v_1, v_2) in a basis of $\mathbb{R}^3$ would be suitable. On the other hand, another choice would lead to a different triangular matrix T' (for the last column).

2.2 Dunford Decomposition Theorem

We will present the proof of Dunford's Theorem which allow us to compute the powers of any matrix A and so e^{At} for any $t \in \mathbb{R}$. Let us start by proving the following lemma:

Lemma 2.1 (Kernel Lemma)**.** *Let f be an endomorphism of E. Let P and Q be polynomials of $\mathbb{K}[x]$, relatively prime. Then*

$$\ker (PQ)(f) = \ker P(f) \oplus \ker Q(f).$$

More general, let $P_1, P_2, \ldots, P_n$ be polynomilas pairwise relatively prime. Then

$$\ker (P_1 P_2 \ldots P_n)(f) = \ker P_1(f) \oplus \ker P_2(f) \oplus \ldots \oplus \ker P_n(f). \tag{2.2}$$

Of course we have similar statements with matrices.

Let P and Q be polynomials of $\mathbb{K}[x]$. We say that $P(X)$ and $Q(X)$ are coprime in $K[X]$ if the only polynomials which divide both P and Q are the constant polynomials. In particular, on $\mathbb{C}$, two polynomials are relatively prime if and only if they have no common root. Also we need to Bézout's theorem, which is stated as follows: P and Q are coprime if and only if there exist U and V in $K[X]$ such that $P \cdot U + Q \cdot V = 1$.

Proof of Lemma ... Let P and Q be two mutually prime polynomials. Then, according to Bézout's theorem, there exist polynomials U and V such that $P \cdot U + Q \cdot V = 1$. We therefore have, for any endomorphism f,

$$P(f) \circ U(f) + Q(f) \circ V(f) = id_E.$$

In other words, for all $x \in E$

$$P(f)(x) \circ U(f)(x) + Q(f)(x) \circ V(f)(x) = x.$$

We will show that $\ker P(f) \cap \ker Q(f) = \{0\}$. Let $x \in P(f) \cap \ker Q(f)$, we have

$$\underbrace{P(f)(x)}_{=0} \circ U(f)(x) + \underbrace{Q(f)(x)}_{=0} \circ V(f)(x) = x.$$

therefore $x = 0$, which proves $P(f) \cap \ker Q(f) \subset \{0\}$. On the other hand, since $\{0\} \subset P(f) \cap \ker Q(f)$ we have the equality. By double inclusion, we prove that $\ker (PQ)(f) = \ker P(f) + \ker Q(f)$. Let $x \in \ker (PQ)(f)$. We have, again due to Bézout's theorem

$$P(f)(x) \circ U(f)(x) + Q(f)(x) \circ V(f)(x) = x.$$

We show that $P(f)(x) \circ U(f)(x) \in \ker Q(f)$. Indeed,

$$Q(f) \circ P(f)(x) \circ U(f)(x) = U(f) \circ ((PQ(f)))(x) = 0.$$

We used that the endomorphism polynomials in f commute and that $PQ(f)(x) = 0$. Similarly, we can easily prove that and $Q(f)(x) \circ V(f)(x) \in \ker P(f)$. Thus,

$$\underbrace{P(f)(x) \circ U(f)(x)}_{\ker Q(f)} + \underbrace{Q(f)(x) \circ V(f)(x)}_{\in \ker P(f)} = x.$$

and so $x \in \ker P(f) + \ker Q(f)$.

Now, we will show that $\ker P(f) + \ker Q(f) \subset \ker (PQ)(f)$. Let $u \in \ker P(f)$ and $v \in \ker Q(f)$. Then

$$PQ(f)(u+v) = P(f) \circ \underbrace{Q(f)(u)}_{=0} + P(f) \circ \underbrace{Q(f)(v)}_{=0}$$

and thus $u + v \in \ker (PQ)(f)$. As a conclusion, $\ker (PQ)(f) = \ker P(f) \oplus \ker Q(f)$. The proof of Lemma Lemma 2.1 is finished. □

2.2.1 Characteristic subspaces

We have seen that, when f is diagonalizable, we have $E = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \ldots \oplus E_{\lambda_k}$, where $E_{\lambda_i} = \ker (f - \lambda_i id_E)$ is the corresponding eigenspace of the eigenvalue λ_i. We will demonstrate that even if f is not diagonalizable, but if its characteristic polynomial is split on $\mathbb{K}$, we can write

$$E = \ker ((f - \lambda_1 id_E)^{m_1}) \oplus \ker ((f - \lambda_2 id_E)^{m_2}) \oplus \ldots \oplus \ker ((f - \lambda_k id_E)^{m_k}),$$

where m_i is the multiplicity of the eigenvalue λ_i, as the root of the characteristic polynomial of f.

Definition 2.3. Let f be an endomorphism of E. Let λ be an eigenvalue of f and let m be its multiplicity as the root of $\mathcal{M}_f$. The characteristic subspace of f for the eigenvalue λ is given by

$$\mathcal{N}_\lambda = \ker ((f - \lambda id_E)^m). \tag{2.3}$$

When λ is an eigenvalue of A, we have $E_\lambda \subset \mathcal{N}_\lambda$ since $\ker (f - \lambda id_E) \subset \ker ((f - \lambda id_E)^s)$ for any $s \geq 1$.

Example 2.2. Let us take the matrix formed by the vectors $v_1 = (-2, -3, 1, 9)$, $v_2 = (3, 4, 1, -5)$, $v_3 = (0, 0, 1, -1)$ and $v_4 = (0, 0, 0, 3)$. We calculate the characteristic subspaces of A. First, we have $p_A(x) = (x-1)^3(x-3)$. The eigenvalue 3 has multiplicity 1 and eigenvalue 1 has multiplicity 3. Next, we find the characteristic subspace associated with $\lambda_1 = 3$. As the multiplicity of this eigenvalue is 1 then the characteristic subspace is also the eigensubspace

$$\mathcal{N}_{\lambda_1} = \ker(A - \lambda_1 I) = E_{\lambda_1} = Vect\{(0,0,0,1)\}.$$

Since $\mathcal{N}_{\lambda_1} = E_{\lambda_1}$ is of dimension 1 and $v_1 = (0,0,0,1) \in \mathcal{N}_{\lambda_1}$, we deduce that $\mathcal{N}_{\lambda_1} = \mathbb{R}\cdot v_1$. We also find the characteristic subspace associated with $\lambda_2 = 1$. The multiplicity of this eigenvalue is 3, so

$$\mathcal{N}_{\lambda_2} = \ker\left((A - \lambda_2 I)^3\right),$$

where $(A-\lambda_2 I)^3$ is formed by the vectors $(0,0,0,16)$, $(0,0,0,-4)$, $(0,0,0,-4)$ and $(0,0,0,8)$. We are looking for a basis of $\mathcal{N}_{\lambda_2}$, it is a vector space of dimension 3, of which for example

$$\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 1 & -2 \end{pmatrix}.$$

Theorem 2.4. *Let f be an Endomorphism of E such that P_f is split. Setting*

$$P_f(x) = (x-\lambda_1)^{m_1}\cdots(x-\lambda_k)^{m_k}$$

For $1 \le i \le k$, let $\mathcal{N}_{\lambda_1}$,let $\mathcal{N}_{\lambda_i}$be the characteristic subspace associated to the eigenvalue λ_i. We have:

1. *Every $\mathcal{N}_{\lambda_i}$ is stable by f.*
2. *$E = \mathcal{N}_{\lambda_1} \oplus \mathcal{N}_{\lambda_2} \oplus \cdots \oplus \mathcal{N}_{\lambda_k}$*
3. *$\dim \mathcal{N}_{\lambda_i} = m_i$.*

In other words, the vector space E is the direct sum of the characteristic subspaces $\mathcal{N}_{\lambda_i}$ for $i = 1, 2, \ldots, k$. In addition, the dimension of the characteristic subspace associated with eigenvalue λ_i is the multiplicity of λ_i as the roots of the characteristic polynomial.

Let us return to Example 2.2. We have $\lambda_1 = 3$ is an eigenvalue of multiplicity 1, where $\dim \mathcal{N}_{\lambda_1} = 1$ and $\lambda_2 = 1$ is an eigenvalue of multiplicity 3, where $\dim \mathcal{N}_{\lambda_2} = 3$. Thus, we have $\mathbb{R}^n = \mathcal{N}_{\lambda_1} \oplus \mathcal{N}_{\lambda_2}$.

Proof of Theorem 2.4. The proof is presented as follows:

1. We will prove that $f(\mathcal{N}_{\lambda_i}) \subset \mathcal{N}_{\lambda_i}$. Let $v \in \mathcal{N}_{\lambda_i}$. Then $f(v) \in f(\mathcal{N}_{\lambda_i})$. Thus, $v \in \ker(f - \lambda_i id_E)^{m_i}$. This means that $(f - \lambda_i id_E)^m(v) = 0$. Or, equivalently,

$$(f - \lambda_i id_E)^m \circ f(x) = f \circ (f - \lambda_i id_E)^m(v) = 0.$$

   Hence, $f(v) \in \ker(f - \lambda_i id_E)^m = \mathcal{N}_{\lambda_i}$.

2. This is an application of the Kernel lemma (see Lemma 2.1). Recall that

$$P_f(x) = \prod_{i=1}^{k} (x - \lambda_i)^{m_i}.$$

   The polynomials $(x - \lambda_i)^{m_i}$ are coprime since the eigenvalues are distinct. By Lemma 2.1 (see Equation (2.2)), we get

$$\begin{aligned} \ker P_f(f) &= \ker(f - \lambda_1 id_E)^{m_1} \oplus \cdots \oplus \ker(f - \lambda_k id_E)^{m_k} \\ &= \mathcal{N}_{\lambda_1} \oplus \mathcal{N}_{\lambda_2} \oplus \cdots \oplus \mathcal{N}_{\lambda_k}. \end{aligned}$$

   Now, according to Cayley-Hamilton Theorem (see Theorem 1.1), we have $P_f(f) = 0$. Hence, $\ker P_f(f) = E$. The result is proved.

3. We have $\dim \mathcal{N}_{\lambda_i} = m_i$ since $\mathcal{N}_{\lambda_i} \cap \mathcal{N}_{\lambda_j} = \{0\}$ for $i \neq j$.

□

Theorem 2.5 (Dunford Decomposion Theorem with matrices). *For any matrix $A \in \mathcal{M}_n(\mathbb{K})$ having a split characteristic polynomial on $\mathbb{K}$, there exist a unique nilpotent matrix N and a unique $\mathcal{D}$ diagonalizable matrix such that $A = \mathcal{D} + N$ with $\mathcal{D}N = N\mathcal{D}$ (that is $\mathcal{D}$ and N commute).*

From this theorem, we can immediately deduce the following corollary:

Corollary 2.1. Let $A \in \mathcal{M}_n(\mathbb{K})$ and assume that $A = \mathcal{D} + N$ with $\mathcal{D}$ diagonalizable, N nilpotent and $\mathcal{D}N = N\mathcal{D}$. Then we have:

1. A is diagonalizable if and only if $A = \mathcal{D}$ and $N = 0$.

2. A is nilpotent if and only if $A = N$ and $\mathcal{D} = 0$.

Theorem 2.6 (Dunford Decomposion Theorem with Endomorphisms). *Let f be an Endomorphism of E with $P_f(x)$ is split. There exist unique two Endomorphisms g and h. Such that*

1. *g is diagonalizable and h is nilpotent.*

2. $f = g + h$.

3. $g \circ h = h \circ g$.

For the proof of Theorem 2.5 or Theorem 2.6, we need to some lemmas.

Lemma 2.2. *If f is nilpotent, then 0 is its unique eigenvalue. Moreover, we have*

$$P_f(x) = (-1)^n x^n.$$

Proof. Let (λ, v) be an eigenpair of f. Then $f(v) = \lambda v$. Since f is nilpotent, we deduce that there exists $k \geq 1$ such that $0 = f^k v = \lambda^k v$. Hence, $\lambda = 0$. Consequently, $P_f(x) = (-1)^n x^n$ since eigenvalues are the roots of $P_f(x)$, where $(-1)^n$ is the leading coefficients of P_f. □

Lemma 2.3. *Let f be an Endomorphisim on E which is diagonalizable. Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be the eigenvalues of f with the corresponding eigenspace $E_1, E_2, \cdots, E_k$. If F is a vector subspace of E stable by f $(f(F) \subset F)$, then we have*

$$F = \left(F \cap E_{\lambda_1}\right) \oplus \left(F \cap E_{\lambda_2}\right) \oplus \cdots \oplus \left(F \cap E_{\lambda_K}\right).$$

Proof. Let $v \in E$. Since $x \in E = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$ there exist unique $v_1, v_2, \cdots, v_k$. with $v_i \in E_{\lambda_i}$ $(1 \leq i \leq k)$ such that $v = v_1 + v_2 + \cdots + v_k$. The vector subspace F is stable by f, it is also stable by $P(f)$ for any polynomial $F \in \mathbb{K}[x]$. Since $v_i \in E_{\lambda_i}$ we have $f(v_i) = \lambda_i v_i$. More generally,

$$P(f)(v_i) = P(\lambda_i) v_i \text{ for } 1 \leq i \leq k.$$

Define

$$P_i(x) = \prod_{\substack{i \neq r \\ r=1}}^{k} (x - \lambda_r)$$

We have also $P_i(\lambda_j) = 0$ if $i \neq j$ and $P_i(\lambda_i) = 0.$ if $i \neq j$ and $P_i(\lambda_i) \neq 0$. We can also write

$$\begin{aligned} P_i(f)(v) &= P_i(f)(v_1 + \cdots + v_k) \\ &= P_i(\lambda_i) v_1 + \cdots + P_i(\lambda_i) v_k \\ &= P_i(\lambda_i) v_i \end{aligned}$$

Since $P_i(f)(v) \in F$ by stability of f, Then $v_i \in F$. Thus, for every $1 \leq i \leq k$, $v_i \in F \cap E_{\lambda_i}$. This prover the result. □

Lemma 2.4. *If f is diagonilizable and F is a vector subspace of E, stable by f. Then the restriction of f to F is also diagonalizable.*

Proof. Let $g = f_F$ be denote the restriction of f to the space F so g is well-defined since F is stable by f . Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the eigenvalues of f. Since f is assumed diagonaizable, we deduce that

$$E = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k},$$

where each E_{λ_i} is the eigenspace corresponding to the eigenvalue λ_i. By Lemma , we obtain

$$F = (F \cap E_{\lambda_1}) \oplus \cdots \oplus (F \cap E_{\lambda_k}).$$

Moreover, for every scalar $\alpha \in \mathbb{K}$, we see that $\ker(g - \alpha id_F) = F \cap \ker(f - \alpha id_E)$ and then the eigenvalues of g, denoted by $\alpha_1, \alpha_2, \ldots, \alpha_s$ are in $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$. The numbers $\alpha_1, \alpha_2, \ldots, \alpha_s$ form exactly the set of values of $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ for which $F \cap E_{\lambda_i} \neq \{0\}$. Thus, we have

$$F = \ker(g - \alpha_1 id_F) \oplus \cdots \oplus \ker(g - \alpha_s id_F).$$

This proves g is diagonalizable. □

Lemma 2.5. *Let f and g be two diagonalizable Endomorphisms. Assume that $f \circ g = g \circ f$, then there exists a common basis of eigenvectors of f and g. Similarly, if $A, B \in \mathcal{M}_n(\mathbb{R})$ with A and B commute and diagonalizable, then we can diagonlize then in a common basis, that is to say that there exists a matrix $P \in \mathcal{M}_n(\mathbb{K})$ invertible such that $P^{-1}AP$ and $P^{-1}BP$ are both diagonal.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the eigenvalues of f . Note that

$$E_{\lambda_i} = \{x \in E : f(x) = \lambda_i x\}$$

There for every $x \in E_{\lambda_i}$, we have $f[g(x)] = g[f(x)] = g(\lambda_i x) = \lambda_i g(x)$. This proves that $g(x) \in E_{\lambda_i}$. Hence, E_{λ_i} is stable by g. By lemma, their restriction of g to E_{λ_i} is diagonalizable. Let us consider, in the space E_{λ_i}, a basis B_i of eigenvectors of g, which are also eigenvectors of f (be cause they are in E_{λ_i}). Since f is diagonalizable, we have

$$E = \underbrace{E_{\lambda_1}}_{\beta_1} \oplus \cdots \oplus \underbrace{E_{\lambda_k}}_{\beta_k},$$

So, the basis $\beta = \beta_1 \cup \ldots \cup \beta_k$ is therefore a basis of E formed of vectors which are eigenvectors for both f and g. □

Let us present the proof of Dunford's decomposition theorem.

Proof of Theorem 2.6. Here is the general idea:

- We decompose the vector space E (resp $\mathbb{K}^n$) into the sum of the characteristic subspaces $\mathcal{N}_{\lambda_i}$, where λ_i are the eigenvalues of E (resp $\mathbb{K}^n$).

- On each of these subspaces $\mathcal{N}_{\lambda_i}$, we decompose the restriction of f into $g_i + h_i$ with $g_i = \lambda_i id_{\mathcal{N}_{\lambda_i}}$ which is of course diagonalizable[1].
- We show that h_i, which is $f - g_i$ restricted to $\mathcal{N}_{\lambda_i}$ is nilpotent.
- Since g_i is λ_i times the identity, we deduce that g_i commutes in particular with h_i.

Let x_f be the characteristic polynomial of f which, by hypothesis, is split on $\mathbb{K}$. Let us denote by λ_i an eigenvalue of f and m_i its multiplicity as the root of the characteristic polynomial

$$p_f(x) = \pm \prod_{i=1}^{k} (x - \lambda_i)^{m_i}.$$

Let $\mathcal{N}_{\lambda_1}, \ldots, \mathcal{N}_{\lambda_k}$ be the characteristic subspaces of f. For $1 \leq i \leq r$, we have $\mathcal{N}_{\lambda_i} = \ker(f - \lambda_i id_E)^{m_i}$ and $E = \mathcal{N}_{\lambda_1} \oplus \ldots \oplus \mathcal{N}_{\lambda_k}$. We will define the endomorphism g on each $\mathcal{N}_{\lambda_i}$, in the following way: for all $v \in \mathcal{N}_{\lambda_i}$, we set $g(v) = \lambda_i v$. The vector space E being a direct sum of the $\mathcal{N}_{\lambda_i}$, g is defined on the set E. Indeed, if $v \in \mathcal{N}_{\lambda_i}$ is decomposed into $v = v_1 + \cdots + v_k$, with $v_i \in \mathcal{N}_{\lambda_i}$, (for $1 \leq i \leq k$), then

$$g(v) = g(v_1 + \cdots + v_k) = g(v_1) + \cdots + v(v_k) = \lambda_1 v_1 + \cdots + \lambda_k v_k.$$

For $1 \leq i \leq k$, we have $g_i = g_{|\mathcal{N}_{\lambda_i}} = \lambda_i id_{\mathcal{N}_{\lambda_i}}$. We finally set

$$h(x) = f(x) - g(x).$$

We still have to check that h is nilpotent.

1. By construction, g is diagonalizable. Indeed, let us set a base for each subspace $\mathcal{N}_{\lambda_i}$. For each vector v of this base, $g(v) = \lambda_i v$. As E is the direct sum of the $\mathcal{N}_{\lambda_i}$ then, in the base of E formed from the union of the bases of the $\mathcal{N}_{\lambda_i}(1 \leq i \leq k)$, the matrix of g is diagonal.

2. We defined $h = f - g \cdot \mathcal{N}_{\lambda_i}$ which is stable by h (because it is true for f and g). We set $h_i = h_{|\mathcal{N}_{\lambda_i}} = f_{|\mathcal{N}_{\lambda_i}} - \lambda_i id_{\mathcal{N}_{\lambda_i}}$. Then, by definition, $\mathcal{N}_{\lambda_i} = \ker(h_i^{m_i})$, and therefore $h_i^{m_i} = 0$. Thus, by setting $m = \max m_i(1 < i < k)$, since h^m is zero on each $\mathcal{N}_{\lambda_i}$, then $h^m = 0$, which proves that h is nilpotent.

3. We will check that $g \circ h = h \circ g$. If $v \in E$, it decomposes into $v = v_1 + \cdots + v_k$ with $v_i \in \mathcal{N}_{\lambda_i}$, for $1 \leq i \leq k$. On each $\mathcal{N}_{\lambda_i}$, we have $g_{|\mathcal{N}_{\lambda_i}} = \lambda_i id_{\mathcal{N}_{\lambda_i}}$, therefore commutes

[1] Every square matrix of the form λI_n is diagonalizable.

with any endomorphism. In particular, $g \circ h(x) = h \circ g(x)$ since $\mathcal{N}_{\lambda_i}$ is stable by h. We therefore have

$$\begin{aligned} g \circ h(x) &= g \circ h(v_1 + \cdots + v_k) = g \circ h(v_1) + \cdots + g \circ h(v_k) \\ &= h \circ h(v_1) + \cdots + h \circ g(v_k) = h \circ g(v). \end{aligned}$$

So, g and h commute.

4. It remains to prove uniqueness. Suppose that (h, g), is the couple constructed above and is (h', g') another couple satisfying the properties of Dunford decomposition Theorem. **4.1.** Let us show that g and g' commute, as well as h and h'. We have $f = g + h = g' + h'$, hence

$$g \circ f = g \circ (g + h) = g \circ g + g \circ h = g \circ g + h \circ g = (g + h) \circ g = f \circ g.$$

Thus, f and g commute and we also show that f and g' commute. Let us show that $\mathcal{N}_{\lambda_i}$ is stable by g'. Let $v \in \mathcal{N}_{\lambda_i} = \ker(f - \lambda_i id_E)^{m_i}$. So,

$$(f - \lambda_i id_E)^{m_i} \circ g'(x) = g' \circ (f - \lambda_i id_E)^{m_i} = 0.$$

So $g' \in \mathcal{N}_{\lambda_i}$. By construction, $g_{|\mathcal{N}_{\lambda_i}} = \lambda_i id_E$, therefore g and g' commute on each $\mathcal{N}_{\lambda_i}$, therefore on E as a whole. Or $h = f - g$ and $h' = f - g'$ therefore, as g and g' commute and f commutes with g and g', then h and h' also commute.

4.2. Since d and d' commute, according to Lemma 2.5, there exists a common basis of eigenvectors. In particular, $g - g'$ is diagonalizable.

4.3. Since the endomorphisms h and h' are nilpotent and commute, $h - h'$ is also nilpotent. Indeed, if r and s are integers such that $h^r = 0$ and $(h')^s = 0$, then $(h - h')^{r+s} = 0$ (by Newton's binomial formula and see that, in each term $h^k \left(h'\right)^{r+s} = 0$, we have $k \geq r$ or $r + s - k \geq s$).

4.4. Thus $g - g' = h - h'$ is an endomorphism which is both diagonalizable and nilpotent. As it is nilpotent, its only eigenvalue is 0 (this is Lemma 2.2). And since it is diagonalizable, it is necessarily zero endomorphism. We therefore have $g - g' = h - h'$.

4.5. As a conclusion, $g = g'$ and $h = h'$.

This completes the proof of uniqueness. □

Remark 2.3. Note that a triangular matrix can always be written as the sum of a diagonal matrix and a nilpotent matrix, but, in general, these do not commute. Remember this counterexample carefully to avoid this trivial decomposition. Let

$$A = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}, \; \mathcal{D} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \; N = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}.$$

Then we have $A = \mathcal{D} + N$, $\mathcal{D}$ is diagonal, N is nilpotent $(N^2 = 0)$. However, it is not the Dunford decomposition because the matrices do not commute: $\mathcal{D} \cdot N \neq N \cdot \mathcal{D}$.

Remark 2.4. The Dunford decomposition is simply $D = N$, and N is the zero matrix. D is clearly diagonalizable (its characteristic polynomial is split with simple roots) but is not diagonal; N is nilpotent and $DN = ND$.

2.2.2 How to apply Dundord's Theorem in the practice

The method to find the Dunford decomposition of a matrix $A \in \mathcal{M}_n(\mathbb{K})$ consists the following steps.

1. We calculate the characteristic polynomial $p_A(x)$ of A. so, it must be split. We calculate its roots, which are the eigenvalues of A. This is a we have done with the diagonalization or trigonalization.

2. For each eigenvalue λ of multiplicity m as a root of $p_A(x)$, we note $N_\lambda = \ker(A - \lambda I_n)^m$. It is a vector space of dimension m. We determine m vectors forming a basis of N_λ. The union of all bases $\mathfrak{B}_\lambda$, of $\mathcal{N}_\lambda$ forms a basis $\mathfrak{B} = (v_1, \ldots, v_n)$ of $\mathbb{K}^n$.

3. We define the endomorphism f by $f(v_i) = \lambda v_i$, for each $v_i \in N_\lambda$. (In the basis $\mathfrak{B}$, the matrix of f is diagonal.) Let $\mathfrak{B}_0 = (e_1, \ldots, e_n)$ be denote the canonical basis of $\mathbb{K}^n$. (A is the matrix of the endomorphism f in the basis $\mathfrak{B}_0$), $\mathfrak{D}$ will be the matrix of f in the basis $\mathfrak{B}_0$, that is to say that the columns of $\mathfrak{D}$ are the coordinates of $f(e_i)$ expressed in the basis $(e_1, \ldots, e_n)$.

4. We set $N = A - \mathfrak{D}$. By the proof of Theorem 2.5, $\mathfrak{D}$ is diagonalizable, N is nilpotent and $\mathfrak{D}N = N\mathfrak{D}$ The transition matrix P from the basis $\mathfrak{B}$ to the canonical basis $\mathfrak{B}_0$ transforms $\mathfrak{D}$ into a diagonal matrix $D = P^{-1}\mathfrak{D}P$.

Let us calculate the Dunford decomposition of the matrix formed by the vectors $v_1 = (1,0,0)$, $v_2 = (1,1,0)$ and $v_3 = (1,1,2)$. Clearly, if we consider the matrix $\mathfrak{D}$ formed by the vectors $v_1' = (1,0,0)$, $v_2' = (0,1,0)$ and $v_3' = (0,0,2)$ which is diagonalizable and N the nilpotent matrix formed by $v_1'' = (0,0,0)$, $v_2'' = (1,0,0)$ and $v_3'' = (1,1,0)$ which is strictly upper triangular, then after computation $\mathfrak{D}N \neq N\mathfrak{D}$. So, in general the Dunford decomposition of an upper triangular matrix cannot be deduced easily.

Let us now calculate the Dunford decomposition of A. The characteristic polynomial $p_A(\lambda)$ is equal to $(\lambda-1)^2(\lambda-2)$. We therefore have two eigenvalues which are $\lambda_1=1$ and $\lambda_2=2$. The eigenvalue λ_1 has a multiplicity $m_1=2$, while, for the eigenvalue λ_2, $m_2=1$. We note that $\mathcal{N}_1=\ker(A-I_3)^2$ and $\mathcal{N}_2=\ker(A-2I_3)$. The vector space $\mathbb{R}^3$ is written as a direct sum: $\mathbb{R}^3=\ker(A-I_3)^2\oplus\ker(A-2I_3)$. Let us determine these characteristic subspaces. We know that it is a vector space of dimension $m_1=2$. We obtain

$$\mathcal{N}_1=\ker(A-I_3)^2=Vect\{(1,0,0),(0,1,0)\}$$

The matrix A is not diagonalizable: in fact, the eigenvalue 1 has multiplicity 2, but E_1 has dimension 1.

For $\mathcal{N}_2$. We know that it is a vector space of dimension $m_2=1$. To determine the kernel $\ker(A-2I_3)=\{v\in\mathbb{R}^3 : Av=2v\}$, if $v=(x,y,z)$, we have $\mathcal{N}_2=Vect\{(2,1,1)\}$. The family $\mathfrak{B}=(u_1,u_2,u_3)$ is a basis of $\mathbb{R}^3$, where $u_1=(1,0,0)$, $u_2=(0,1,0)$ and $u_3=(2,1,1)$. Therefore,

$$\mathbb{R}^3=\underbrace{\mathbb{R}v_1\oplus\mathbb{R}v_2}_{\mathcal{N}_1}\oplus\underbrace{\mathbb{R}v_3}_{\mathcal{N}_2}.$$

We define the endomorphism f by $f(v_1)=v_1, f(v_2)=v_2$ (because $v_1,v_2\in\mathcal{N}_1$) and $d(v_3)=2v_3$ (because $v_3\in\mathcal{N}_2$). In the basis $\mathfrak{B}$, the matrix of f is therefore the diagonal matrix D formed by the vectors $(1,0,0)$, $(0,1,0)$ and $(0,0,2)$. Now we need to the matrix of f in the canonical basis $\mathfrak{B}_0=(e_1,e_2,e_3)$. The computations will be quite simple because $e_1=v_1$ and $e_2=v_2$.

- $f(e_1)=f(1,0,0)=(1,0,0)=e_1$.
- $f(e_2)=f(0,1,0)=(0,1,0)=e_2$.
- We also have $v_3=(2,1,1)=2e_1+e_2+e_3$, where $e_3=(0,0,1)=2v_1+v_2+v_3$. So

$$\begin{aligned}f(e_3) &= f(-2v_1+v_2+v_3)=-2f(v_1)-f(v_2)+f(v_3)=-2v_1-v_2+2v_3\\ &= -2e_1-e_2+2(2e_1+e_2+e_3)=2e_1+e_2+2e_3.\end{aligned}$$

Thus, $\mathfrak{D}$ is the matrix of f with respect the basis $\mathfrak{B}_0$, that is, the matrix formed by the vectors $(1,0,0)$, $(0,1,0)$ and $(2,1,2)$. Next, we put $N=A-\mathfrak{D}$, that is N is the nilpotent matrix formed by the vectors $(0,0,0)$, $(1,0,0)$ and $(-1,0,0)$. The Dunford decomposition is $A=\mathfrak{D}+N$. The proof of the decomposition theorem states that A is diagonalizable, N is nilpotent and $\mathfrak{D}N=N\mathfrak{D}$. We note P the passage matrix from the basis $\mathfrak{B}_0$ to the basis $\mathfrak{B}$. Therefore, P contains the vectors of the new basis $\mathfrak{B}=(v_1,v_2,v_3)$ expressed in the old basis $\mathfrak{B}_0=(e_1,e_2,e_3)$. Since $v_1=e_1$, $v_2=e_2$ and $v_3=2e_1+e_2+e_3$, then P is the matrix formed by the vectors $(1,0,0)$, $(0,1,0)$ and $(2,1,1)$, and so P^{-1} is the matrix formed by the vectors $(1,0,0)$, $(0,1,0)$ and $(-2,-1,0)$. If necessary, we can diagonalize $\mathfrak{D}$ as follows:

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}_{\mathfrak{D}} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}_{P} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}_{D} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}_{P^{-1}}.$$

Remark 2.5. D and N are unique, but there are several possible choices for the vectors v_i and therefore for the matrix P.

Let us calculate the Dunford decomposition of A formed by the vectors $(2,3,3)$, $(1,3,1)$ and $(-1,-4,-2)$.

1. We find $p_A(\lambda) = (\lambda-1)(\lambda-2)^2$. The eigenvalue $\lambda_1 = -1$ has multiplicity 1, and the value $\lambda_2 = 2$ has multiplicity 2.

2. We calculate $\mathcal{N}_{-1} = \ker(A + I_3) = \mathbb{R}v_1$, where $v_1 = (0,1,1)$ (it is indeed an eigenspace of dimension $m_{-1} = 1$).

3. We calculate $\mathcal{N}_2 = \ker(A - 2I_3)^2$ which will be of dimension $m_2 = 2$: Since $A - 2I_3$ is formed by the vectors $(0,3,3)$, $(1,1,1)$ and $(-1,-4,-4)$ and $(A-2I_3)^2$ is formed by the vectors $(0,-9,-9)$, $(0,0,0)$ and $(0,9,9)$. For a base of $\mathcal{N}_2$, we first choose $v_2 \in E_2 = \ker(A - 2I_3) \subset \mathcal{N}_2$, for example $v_2 = (1,1,1)$. We look for $v_3 \in \mathcal{N}_2$, linearly independent with v_2. For example, $v_3 = (1,0,1)$. The family $\mathfrak{B} = (v_1, v_2, v_3)$ is a base of $\mathbb{R}^3$. Thus, we have
$$\mathbb{R}^3 = \underbrace{\mathbb{R}v_1}_{\mathcal{N}_{\lambda_1}} \oplus \underbrace{\mathbb{R}v_2 \oplus \mathbb{R}v_3}_{\mathcal{N}_{\lambda_2}}.$$

4. We denote by $\mathfrak{B}_0 = (e_1, e_2, e_3)$ the canonical base of $\mathbb{R}^3$. The passage matrix P from the basis $\mathfrak{B}_0$ to the basis $\mathfrak{B}$ is obtained by writing the vectors v_i in columns. So P is formed by $(0,1,1)$, $(0,1,-1)$ and $(1,-1,1)$. Also P^{-1} is formed by the vectors $(-1,1,0)$, $(0,1,-1)$ and $(1,-1,1)$.

5. We define the endomorphism f by $f(v_1) = -v_1$ (because $v_1 \in \mathcal{N}_{-1}$), and $f(v_2) = 2v_2$, $f(v_3) = 2v_3$ (because $v_2, v_3 \in \mathcal{N}_2$). In the basis $\mathfrak{B} = (v_1, v_2, v_3)$, the matrix of f (denoted by D) is the diagonal matrix forme by the vectors $(1,0,0)$, $(0,2,0)$ and $(0,0,2)$. The matrix of f in the basis $\mathfrak{B}_0$ is obtained by expressing $f(e_i)$ in the basis (e_1, e_2, e_3). Therefore, $\mathfrak{D} = PDP^{-1}$ is formed by the vectors $(0,1,1)$, $(1,1,1)$ and $(1,0,1)$.

6. We set $N = A - \mathfrak{D}$ which is the matrix formed by $(0,0,0)$, $(1,1,1)$ and $(-1,-1,-1)$. Thus, the Dunford decomposition is $A = \mathfrak{D} + N$. We have $\mathfrak{D}$ is diagonalizable since $D = P^{-1}\mathfrak{D}P$. Also we can easily see that $N^2 = 0$ and that $\mathfrak{D}N = N\mathfrak{D}$.

As a conclusion, if A est diagonal or nilpotent, there is no problem. Otherwise we use the Dunford decomposition: $A = \mathcal{D} + N$ with $\mathcal{D}$ diagonalizable, N nilpotent and $N \cdot \mathcal{D} = \mathcal{D} \cdot N$ which allows us to write $e^A = e^{\mathcal{D}} \cdot e^N$. The matrix $\mathcal{D}$ is diagonalizable, which means that there exists an invertible matrix P and a diagonal matrix D such that $\mathcal{D} = PDP^{-1}$. Thus, $e^{\mathcal{D}} = e^{PDP^{-1}} = pe^D P^{-1}$. We can therefore always calculate the exponential of any matrix with real or complex coefficients.

2.3 Jordan Decomposition Theorem

We start with the definition of simple Jordan (resp. by block) matrix corresponding to the value λ, which is denoted by $J_k(\lambda)$ or simply $J(\lambda)$. Also we show its importance and how to find the associated fundamental matrix $e^{tJ_k(\lambda)}$.

Definition 2.4. An n by n simple Jordan matrix corresponding to the value λ (denoted by J or $J_n(\lambda)$) is given by

$$J_n(\lambda) = \begin{pmatrix} \lambda & \mathbf{1} & & & \\ & \lambda & \mathbf{1} & & \\ & & \ddots & \ddots & \\ & & & \lambda & \mathbf{1} \\ & & & & \lambda \end{pmatrix}. \tag{2.4}$$

In other words, a Jordan simple matrix corresponding to λ is an n by n matrix (denoted by $J(\lambda)$) whose entries are equal to λ on the diagonal, 1 right above the diagonal and 0 elsewhere.

Remark 2.6. In some references, the simple Jordan matrix is given by the following formula:

$$J_n(\lambda) = \begin{pmatrix} \lambda & & & & \\ \mathbf{1} & \lambda & & & \\ & \ddots & \ddots & & \\ & & \mathbf{1} & \lambda & \\ & & & \mathbf{1} & \lambda \end{pmatrix}. \tag{2.5}$$

Definition 2.5. Let A be a square matrix and let $\lambda \in Sp(A)$. If a nonzero vector satisfies the equation $(A - \lambda I)^s v = 0$, then we say that v is generalized eigenvector of λ.

Note that if $(A - \lambda I)^m v = 0$, then

$$e^{At} v = e^{\lambda I t + (A - \lambda I)t} v = e^{\lambda t} \left(v + t(A - \lambda I) v + \ldots + \frac{t^{s-1}}{(s-1)!} (A - \lambda I)^{s-1} v \right).$$

Definition 2.6 (Jordan chain)**.** Let $A \in \mathcal{M}_n(\mathbb{R})$ be a square matrix and let $\lambda \in \mathbb{R}$ be an eigenvalue of A. The family of nonzero vectors $\{v_1, v_2, ..., v_s\}$ forms a Jordan chain corresponding to λ if it satisfies the following condition:

$$(A - \lambda I_n) v_i = \begin{cases} v_{i+1}, \text{ if } i < s \\ 0, \text{ if } i = s \end{cases} \tag{2.6}$$

Remark 2.7. From the above definition, we remark that the last vector v_s is an eigenvector of A Corresponding to λ since $A \cdot v_s = \lambda \cdot v_s$.

Remark 2.8. Given a matrix $A \in \mathcal{M}_n(\mathbb{K})$ we know that there exists a sequence of inclusions of vector subspaces:

$$\{0\} \subset \ker A^0 \subset \ker A \subset \ker A^2 \subset ... \subset \ker A^k \subset ... \subset \mathbb{K}^n,$$

where $(\dim \ker A^i)$ form an increasing sequence. In fact, if $v \in \ker A^k$ then $A^k v = 0$ and so $A^{k+i}v = 0$ for $i \geq 0$.

Definition 2.7. Let $n \geq 1$. We say a block of the simple Jordan matrix any matrix $n \times n$ matrix of the form

$$J = \begin{pmatrix} J_\alpha & & & \\ & J_\beta & & \\ & & \ddots & \\ & & & J_\gamma \end{pmatrix},$$

where all the matrices $J_\alpha, J_\beta, \ldots, J_\gamma$ are of the form (2.4) or (2.5).

In other words, a Jordan matrix by Blocks is a square matrix formed by Jordan simple matrices. In particular, the possible cases of the simple 3 by 3 Jordan matrix are the following:

$$\begin{pmatrix} \boxed{\lambda_1} & & \\ & \boxed{\lambda_2} & \\ & & \boxed{\lambda_3} \end{pmatrix}, \begin{pmatrix} \boxed{\begin{matrix} \lambda & \mathbf{1} \\ & \lambda \end{matrix}} & \\ & \boxed{\lambda'} \end{pmatrix}, \begin{pmatrix} \lambda & \mathbf{1} & \\ & \lambda & \mathbf{1} \\ & & \lambda \end{pmatrix}$$

An example on Jordan matrix by Blocks:

$$J = \begin{pmatrix} \boxed{\begin{matrix} \alpha & 1 & \\ & \alpha & 1 \\ & & \alpha \end{matrix}} & & & \\ & \boxed{\begin{matrix} \beta & 1 \\ & \beta \end{matrix}} & & \\ & & \boxed{\gamma} & \\ & & & \boxed{\begin{matrix} \mu & 1 & \\ & \mu & 1 \\ & & \mu \end{matrix}} \end{pmatrix},$$

which is formed by four simple Jordan matrices.

We state without proof (the proof is few long) the following important theorem.

Theorem 2.7 (Jordan Decomposition Theorem)**.** *Every spuare matrix A can be written as:*

$$A = P \cdot J \cdot P^{-1}, \tag{2.7}$$

where P is invertible and J is the Jordan matrix by blocks.

Let λ be real or complex number. By induction on the type of a simple Jordan matrix $J_k(\lambda)$, we can prove that

$$J_k(\lambda)^n = \begin{pmatrix} \lambda^n & C_n^1\lambda^{n-1} & C_n^2\lambda^{n-2} & \dots & & \dots & C_n^{k-1}\lambda^{n-k+1} \\ & \lambda^n & & \dots & & \dots & C_n^{k-2}\lambda^{n-k+2} \\ & & \ddots & & & & \\ & & & \ddots & & & \\ & & & & \lambda^n & \dots & C_n^2\lambda^{n-2} \\ & & & & & \lambda^n & C_n^1\lambda^{n-1} \\ & & & & & & \lambda^n \end{pmatrix},$$

where $C_n^s = 0$ for $n < s$. To do this, in view of (2.4) or (2.5) we see that

$$J_k(\lambda) = \begin{pmatrix} \lambda & \mathbf{1} & & & \\ & \lambda & \mathbf{1} & & \\ & & \ddots & \ddots & \\ & & & \lambda & \mathbf{1} \\ & & & & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & & & & \\ & \lambda & & & \\ & & \ddots & & \\ & & & \lambda & \\ & & & & \lambda \end{pmatrix}_D + \begin{pmatrix} \mathbf{0} & \mathbf{1} & & & \\ & 0 & \mathbf{1} & & \\ & & \ddots & \ddots & \\ & & & \mathbf{0} & \mathbf{1} \\ & & & & \mathbf{0} \end{pmatrix}_N,$$

where $DN = ND$. By Theorem 1.5, N is nilpotent. Thus, by Binomial formula (see Section 1.5) we obtain an explicit formula of $J_k(\lambda)^n$. Moreover, we have

$$e^{J_k(\lambda)} = e^{\lambda} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \dots & & \dots & \frac{t^{k-1}}{(k-1)!} \\ & 1 & t & \dots & & & \frac{t^{k-2}}{(k-2)!} \\ & & \ddots & & & & \\ & & & \ddots & & & \\ & & & & 1 & t & \frac{t^2}{2!} \\ & & & & & 1 & t \\ & & & & & & 1 \end{pmatrix}.$$

2.3.1 Examples

Consider the following examples:

Example 2.3. Let us take the matrix formed by the vectors $v_1 = (2,0,0,0)$, $v_2 = (-1,3,1,-1)$, $v_3 = (0,-1,1,0)$ and $v_3 = (1,0,0,3)$. We obtain, $p_A(x) = (x-2)^3(x-3)$, so this matrix has the eigenvalues $\lambda_1 = 2$ with multiplicity $m_1 = 3$ and $\lambda_2 = 3$ with multiplicity $m_2 = 1$. Also we have $E_{\lambda_1} = Vect\{(1,0,0,0),(0,1,1,1)\}$ and $E_{\lambda_2} = Vect\{(1,0,0,1)\}$. Since

$$\ker(A-2I)^2 = Vect\{(1,0,0,0),(0,1,2,0),(0,1,0,2)\},$$

let us choose $v_3 = (0,1,2,0)$. Put $B = \{v_1, (A-2I)v_3, (A-2I)^2 v_3, , v_2\}$, where $v_1 = (1,0,0,0)$ is an eigenvector of λ_1, $v_3 = (0,1,2,0)$ is a generalized eigenvector of λ_1 and $v_2 = (1,0,0,1)$ is the eigenvector of λ_2. After computation, we get

$$\begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}_P \begin{pmatrix} \boxed{2} & & & \\ & \boxed{\begin{matrix} \mathbf{2} & \mathbf{1} \\ 0 & \mathbf{2} \end{matrix}} & \\ & & & \boxed{3} \end{pmatrix}_J \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -2 & 1 & 1 \end{pmatrix}_{P^{-1}}$$

Example 2.4. Let $A \in \mathcal{M}_n(\mathbb{R})$ the matrix formed by the vectors $(1,1,...,1)$. Then by induction on n we can prove that

$$J = \begin{pmatrix} 0 & 0 & 0 & & 0 \\ 0 & n & 0 & & 0 \\ 0 & 0 & 0 & & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which is formed by n simple Jordan matrices. However, if

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 \\ & & \ddots & \ddots & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}$$

then

$$J = \begin{pmatrix} \mathbf{1} & & & & \\ \mathbf{1} & \mathbf{1} & & & \\ & \ddots & \ddots & & \\ & & \mathbf{1} & \mathbf{1} & \\ & & & \mathbf{1} & \mathbf{1} \end{pmatrix}$$

which includes one simple Jordan matrix.

CHAPTER 3

EIGENANALYSIS METHOD FOR SYSTEMS OF DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

We present some general notions on matrices and derivative. Let $A(t) = (a_{ij}(t))$ be a square matrix. If the functions $a_{ij}(t)$ are differentiable, then we introduce the derivative of the matrix $A'(t)$ as

$$A'(t) = \frac{dA(t)}{dt} = \begin{bmatrix} a'_{11}(t) & a'_{12}(t) \\ a'_{21}(t) & a'_{22}(t) \end{bmatrix}.$$

We can immediately verify the following properties:

1) If G is a constant matrix, then $\frac{dG}{dt} = 0$ (0 = zero matrix)

2) If $A(t)$ and $B(t)$ have the same dimension,then

$$\frac{d}{dt}[A(t) + B(t)] = A'(t) + B'(t)$$

3) If the product of the matrices $A(t)$ and $B(t)$ has sense, then

$$\frac{d}{dt}[A(t)B(t)] = A'(t)B(t) + A(t)B'(t)$$

4) Let $A^{-1}(t)$ be the inverse of $A(t)$. Then

$$\begin{cases} A'(t) \cdot A^{-1}(t) - A(t) \cdot \frac{d}{dt}A^{-1}(t) = 0 \\ A(t) \cdot \frac{d}{dt}A^{-1}(t) = A'(t) \cdot A^{-1}(t) \end{cases}$$

We can introduce the notion of the integral of a matrix

$$\int_{t_0}^{t} A(t)\,dt = \left(\int_{t_0}^{t} a_{ij}(t)\,dt\right), t_0, t \in [a, b].$$

Now, we present the general form of a system of differential equations (linear, with constant coefficients and the second member):

$$\begin{cases} \dfrac{dx_1}{dt} = a_{11}x_1 + \cdots + a_{1n}x_n + f_1(t) \\ \dfrac{dx_2}{dt} = a_{21}x_1 + \cdots + a_{2n}x_n + f_2(t) \\ \vdots \\ \dfrac{dx_n}{dt} = a_{n1}x_1 + \cdots + a_{nn}x_n + f_n(t) \end{cases} \tag{3.1}$$

whete $f(t) = (f_1(t), \cdots, f_n(t))$ is a continuous vector function on $]a, b[$. The general solution of (3.1)is given by

$$x(t) = \sum_{k=1}^{n} C_k x^k + x^0(t), \tag{3.2}$$

where $x^0(t)$ is a particular solution of (3.1) and $\sum_{k=1}^{n} C_k x^k$ is a general solution of the homogeneous system:

$$L[x] = \frac{dx}{dt} - Ax = 0. \tag{3.3}$$

In fact, for any constant C_k, the sum (3.2) is clearly a solution of the system (3.1).

$$L\left[\sum_{k=1}^{n} C_k x^k + x^0\right] = L\left[x^0\right] = f.$$

On the other hand, if x is a solution of the system (3.1), then

$$L\left[x - x^0\right] = L[x] - L\left[x^0\right] = f - f = 0$$

and for certain constants C_k,

$$x - x^0 = \sum_{k=1}^{n} C_k x^k. \tag{3.4}$$

If we have fond the general solution of the system (3.3), we can find a particular solution of the the system with the second member (3.1) by the method of arbitrarily constants (this is the Lagrange method).

Theorem 3.1 (Picard approximation). *Let $A \in \mathcal{M}_n(\mathbb{R})$ be independent on t. Assume that $X'(t) = A \cdot X(t)$ with $X(0) = X_0$. Then $X(t) = e^{At} \cdot X_0$.*

Proof. From the hypothesis, we see that

$$X(t) = X_0 + \int_0^t AX(r)\,dr. \tag{3.5}$$

It follows from (3.5) (we apply it many times) that

$$\begin{aligned}
X(t) &= X_0 + \int_0^t A\left(X_0 + \int_0^t AX(r)\,dr\right)dr \\
&= X_0 + A\int_0^t dr + A^2\int_0^t\left(\int_0^t X(r)\,dr\right)dr \\
&= X_0 + A\int_0^t dr + A^2\int_0^t\left(\int_0^t\left(X_0 + \int_0^t AX(r)\,dr\right)dr\right)dr \\
&= \left[I + A\int_0^t dx_1 + A^2\int_0^t\left(\int_0^{x_1} dx_2\right)dx_1 + \ldots\right]X_0 \\
&= \left[I + At + \frac{A^2t^2}{2!} + \ldots\right]X_0 \\
&= \sum_{n=0}^{+\infty}\frac{(At)^n}{n!}X_0 = e^{At}\cdot X_0.
\end{aligned}$$

Hence, $X(t) = e^{At}\cdot X_0$. The proof is finished. □

Proposition 3.1. *Let $A \in \mathcal{M}_n(\mathbb{R})$ and let $\lambda \in Sp(A)$ with a corresponding nonzero eigenvector v. Then the mapping $X : \mathbb{R} \to \mathbb{R}^n$ with $X(t) = e^{\lambda t}\cdot v$ is a solution of the system $X'(t) = A\cdot X(t)$.*

Proof. Let $X(t) = e^{\lambda t}\cdot v$. Then we have

$$X'(t) = \lambda e^{\lambda t}v = e^{\lambda t}(\lambda v) = e^{\lambda t}(Av) = A\left(e^{\lambda t}v\right) = A\cdot X(t).$$

This proves that $X(t)$ is indeed a solution of the homogeneous system $X'(t) = A\cdot X(t)$. □

Definition 3.1. Let $F_S = \{x_1(t), x_2(t), \ldots, x_n(t)\}$ be a set of n linearly independent solutions of the system $X'(t) = A\cdot X(t)$. The set F_S is called fundamental solution set of the system. The corresponding matrix $\mathcal{M} = \begin{bmatrix} x_1(t) & x_2(t) & \ldots & x_n(t)\end{bmatrix}$ is called fundamental matrix of the system. In this case,

$$X(t) = c_1x_1(t) + c_2x_2(t) + \ldots + c_nx_n(t)$$

is called the general solution of the system.

3.1 Eigenanalysis Method for $X' = AX$

In this chapter we first systems include differential equations of the first-order. Consider the system of ODES: $x'(t) = Ax(t)$, where $X = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix}^t \in \mathbb{R}^n$ and

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \in \mathbb{M}_n(\mathbb{R}).$$

If we put $x(t) = ve^{\lambda t}$. After derivation, we obtain $A\left(ve^{\lambda t}\right) = \lambda ve^{\lambda t}$. Hence, $Av = \lambda v$, from which we deduce that (λ, v) is an eigenpair of A.Therefore, $(A - \lambda I)v = 0$. Or, equivalently, $\det(A - \lambda I) = 0$ (this is the characteristic equation of A and denoted by $p_A(\lambda)$). In particular, for the matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

we get $p_A(\lambda) = \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}$. By Section 1.2.1, e^{At} is computed according to the following three cases:

1. Distinct real roots,
2. Complex conjugate roots,
3. repeated roots.

Proposition 3.2. *Let $A \in \mathcal{M}_n(\mathbb{R})$ be diagonalizable and let $P = \begin{bmatrix} X_1 & X_n & \dots & X_n \end{bmatrix}$ the invertible matrix formed by n eigenvectors of A. Then the system $X' = AX$ with $X(0) = C_0$ has a unique solution given by the following formula:*

$$X(t) = c_1 e^{\lambda_1 t} X_1 + c_2 e^{\lambda_2 t} X_2 + \dots + c_n e^{\lambda_n t} X_n, \tag{3.6}$$

where $c_1, c_2, \dots, c_n \in \mathbb{R}$.

Proof. From the equality $X' = AX$ we must have $X(t) = e^{At}.\xi$, where $\xi \in \mathcal{M}_{n,1}(\mathbb{R})$. On the other hand, since A is diagonalizable, we obtain

$$X(t) = Pe^{Dt}P^{-1} = P \begin{pmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{pmatrix} P^{-1}\xi \tag{3.7}$$

If we put $P^{-1}\xi = C$, then by (3.7) we get

$$\begin{aligned} X(t) &= \begin{bmatrix} X_1 & X_n & \ldots & X_n \end{bmatrix} \begin{pmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \\ &= \begin{bmatrix} e^{\lambda_1 t} X_1 & e^{\lambda_2 t} X_n & \ldots & e^{\lambda_n t} X_n \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \\ &= c_1 e^{\lambda_1 t} X_1 + c_2 e^{\lambda_2 t} X_2 + \ldots + c_n e^{\lambda_n t} X_n. \end{aligned}$$

Since $X(0) = C_0$, we conclude that $C = P^{-1}C_0$. The proof is finished □

Homogenous systems including n first-order differential equations having constant coefficients. The general form is given by (3.1). Or, equivalently, (3.1) can be written in the matrix form: $X'(t) = A \cdot X(t)$ with $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{R})$. For $n = 2$, we have the result:

Theorem 3.2. *Consider the system $X'(t) = A \cdot X(t)$, where $A \in \mathcal{M}_2(\mathbb{R})$ with $Sp(A) = \{\lambda_1, \lambda_2\}$. Then there exist two differentiable functions $a_1(t)$ and $a_2(t)$ and two square matrices A_1 and $A_2 \in \mathcal{M}_2(\mathbb{R})$ such that*

$$X(t) = (a_1(t) \cdot A_1 + a_2(t) \cdot A_2) X(0).$$

Proof. Define the matrices $A_1 = I_2$ and $A_2 = A - \lambda_1 I_2$. Let $a_1(t)$ and $a_2(t)$ be two differentiable functions such that

$$\begin{cases} a_1'(t) = \lambda_1 a_1(t) \\ a_2'(t) = \lambda_2 a_2(t) + a_1(t) \end{cases} \quad \begin{cases} a_1(0) = 1, \\ a_2(0) = 0. \end{cases}$$

Setting $X(t) = (a_1(t) \cdot A_1 + a_2(t) \cdot A_2) X(0)$. Applying Cayley-Hamilton Theorem, we get $p_A(x) = x^2 - tr(A)x + \det A$ and so $A^2 - tr(A)A + \det A$ from which we also get

$$A^2 - (\lambda_1 + \lambda_2) A + \lambda_1 \lambda_2 I_2 = (A - \lambda_1 I_2)(A - \lambda_2 I_2) = 0,$$

and so $(A-\lambda_1 I_2)\,A_2 = 0$ from which we have $AA_2 = \lambda_2 A_2$. It follows that

$$\begin{aligned} X'(t) &= (\lambda_1 a_1(t)\,A_1 + \lambda_2 a_2(t)\,A_2 + a_1(t)\,A_2)\,X(0) \\ &= (a_1(t)\,(\lambda_1 A_1 + A_2) + \lambda_2 a_2(t)\,A_2)\,X(0) \\ &= (a_1(t)\,(\lambda_1 I_2 + A - \lambda_1 I_2) + a_2(t)\,AA_2)\,X(0) \\ &= A\,(a_1(t)\,I_2 + a_2(t)\,A_2)\,X(0) \\ &= A\cdot(a_1(t)\,A_1 + a_2(t)\,A_2)\,X(0) \\ &= A\cdot X(t)\,, \end{aligned}$$

which means that $X(t)$ is a solution. The proof is finished. □

More general, we have the following theorem:

Theorem 3.3. *Consider the system $X'(t) = A\cdot X(t)$, where $A \in \mathcal{M}_n(\mathbb{R})$ with $Sp(A) = \{\lambda_1, \lambda_2, ..., \lambda_n\}$. Then there exist differentiable functions $a_i(t)$ and square matrices $A_i \in \mathcal{M}_n(\mathbb{R})$ ($1 \le i \le n$) such that*

$$X(t) = (a_1(t)\cdot A_1 + a_2(t)\cdot A_2 + \ldots + a_n(t)\cdot A_n)\,X(0)\,.$$

Proof. Define the matrices

$$A_1 = I_n, A_m = \prod_{i=1}^{m-1}(A-\lambda_i I_n)\,,\ \text{for } m = 2,3,\ldots,n$$

and let $a_i(t)$ such that

$$\begin{cases} a_1'(t) = \lambda_1 a_1(t) \\ a_2'(t) = \lambda_2 a_2(t) + a_1(t) \\ \vdots \\ a_n'(t) = \lambda_n a_n(t) + a_{n-1}(t) \end{cases} \qquad \begin{cases} a_1(0) = 1, \\ a_2(0) = 0, \\ \vdots \\ a_n(0) = 0. \end{cases}$$

By Cayley-Hamilton Theorem 1.1, we have $(A-\lambda_1 I_n)(A-\lambda_2 I_n)\ldots(A-\lambda_n I_n) = 0$. As above, we deduce that $AA_n = \lambda_n A_n$ and that $\lambda_m A_m + A_{m+1} = AA_m$ for $1 \le m \le n-1$.

Therefore,

$$\begin{aligned}
X'(t) &= \left(a_1'(t)\cdot A_1 + a_2'(t)\cdot A_2 + \ldots + a_n'(t)\cdot A_n\right)X(0)\\
&= \left(\sum_{i=1}^{n}\lambda_i a_i(t)A_i + \sum_{i=2}^{n} a_{i-1}(t)A_i\right)X(0)\\
&= \left(\sum_{i=1}^{n-1}\lambda_i a_i(t)A_i + a_n(t)\lambda_n A_n + \sum_{i=1}^{n-1} a_i(t)A_{i+1}\right)X(0)\\
&= \left(\sum_{i=1}^{n-1} a_i(t)\left(\lambda_i A_i + A_{i+1}\right) + a_n(t)\lambda_n A_n\right)X(0)\\
&= \left(\sum_{i=1}^{n-1} a_i(t)\left(AA_i\right) + a_n(t)AA_n\right)X(0)\\
&= A\left(\sum_{i=1}^{n} a_i(t)A_i\right)X(0)\\
&= A\cdot X(t).
\end{aligned}$$

So, $X(t)$ is a solution. This completes the proof. □

Proposition 3.3. *The set S of solutions of the homogeneous system $X'(t) = AX(t)$ forms a vector subspace of $C^1([0,a],\mathbb{K}^n)$ of dimension n.*

Proof. Let (e_i) be denote the canonical basis. Let Y_i denote the unique solution of the Cauchy problem $X'(t) = AX(t)$ for $t \in [0,a]$ and $X_i(0) = e_i$. Any linear combination X, of X_i, is still solution of $X'(t) = AX(t)$, therefore $Vet\{X_1, X_2, ..., X_n\} \subset S$. Conversely, let X be a solution of $X'(t) = AX(t)$. Let us denote $X_0 = X(0)$, with coordinates (X_{0i}). We consider the linear combination $Y(t) = \sum X_{0i}\cdot X_i(t)$. We note that Y is indeed solution of the homogeneous system $Y'(t) = AY(t)$ and furthermore $Y(0) = X_0$. Thus $X(t)$ and $Y(t)$ are two solutions of the Cauchy problem $X'(t) = AX(t)$ with $X(0) = X_0$, therefore by uniqueness of the solutions $X \equiv Y$. This proves that $S \subset Vet\{X_1, X_2, ..., X_n\}$, and hence the equality $S = Vet\{X_1, X_2, ..., X_n\}$.

Finally, let us verify that the family (X_i) is free. Suppose that $\sum \alpha_i X_i = 0$ in the space of functions $C^1([0,a],\mathbb{K}^n)$. Then for all $t \in [0,a]$, $\sum \alpha_i X_i(t) = 0$. In particular for $t = 0$ we obtain $\sum \alpha_i e_i = 0$, so $\alpha_i = 0$ for $1 \le i \le n$ since (e_i) is the canonical basis. □

3.2 Two examples on Jordan decomposition Theorem

We would like to solve the systems of differential equations:

$$\begin{cases} x_1' = 2x_1 - x_2 + x_4 \\ x_2' = 3x_2 - x_3 \\ x_3' = x_2 + x_3 \\ x_4' = -x_2 + 3x_4 \end{cases} \tag{3.8}$$

The corresponding matrix is given by

$$A = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{pmatrix}.$$

It follows from Example 2.3 that

$$e^{Jt} = \begin{pmatrix} e^{J_1 t} & & \\ & e^{J_2 t} & \\ & & e^{J_3 t} \end{pmatrix}.$$

Since $e^{J_2 t} = \begin{pmatrix} e^2 & te^2 \\ 0 & e^2 \end{pmatrix}$, we deduce that

$$e^{At} = P \cdot e^{Jt} \cdot P^{-1} = \begin{pmatrix} e^{2t} & 0 & 0 & 0 \\ 0 & e^2 & te^2 & 0 \\ 0 & 0 & e^2 & 0 \\ 0 & 0 & 0 & e^{3t} \end{pmatrix}$$

The general solution of (3.8) is given by $X(t) = e^{At} \cdot c$, which gives

$$\begin{cases} x_1(t) = c_1 e^{2t} + c_2 \left(e^2 + e^{2t} - 2e^{3t} + e^2 t\right) + c_3 \left(-e^{2t} + e^{3t} - e^2 t\right) + c_4 \left(e^{2t} + e^{3t}\right) \\ x_2(t) = c_2 \left(e^2 + e^2 t\right) - c_3 e^2 t \\ x_3(t) = c_2 e^2 t + c_3 \left(e^2 - e^2 t\right) \\ x_4(t) = c_2 \left(2e^2 - 2e^{3t} + e^2 t\right) + c_3 \left(-e^2 + e^{3t} - e^2 t\right) + c_4 e^{3t}, \end{cases}$$

where $c = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \end{pmatrix}^t$ is a constant. As required.

Another example, we wish to solve the system of differential equations $X' = AX$, where $A \in \mathcal{M}_7(\mathbb{R})$ is upper triangular matrix which has 2 in the diagonal-entries and 1

elsewhere. As in (2.7), we can check that the corresponding Jordan matrix form is given by

$$\begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & 0 \\ 5 & -4 & 3 & -2 & 1 & 0 & 0 \\ -10 & 6 & -3 & 1 & 0 & 0 & 0 \\ 10 & -4 & 1 & 0 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_P \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}_J \times$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 & 4 & 10 \\ 0 & 0 & 0 & 1 & 3 & 6 & 10 \\ 0 & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{P^{-1}}$$

so $e^{At} = P \cdot e^{Jt} \cdot P^{-1}$, where

$$e^{Jt} = e^{2t} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ t & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{t^2}{2!} & t & 1 & 0 & 0 & 0 & 0 \\ \frac{t^3}{3!} & \frac{t^2}{2!} & t & 1 & 0 & 0 & 0 \\ \frac{t^4}{4!} & \frac{t^3}{3!} & \frac{t^2}{2!} & t & 1 & 0 & 0 \\ \frac{t^5}{5!} & \frac{t^4}{4!} & \frac{t^3}{3!} & \frac{t^2}{2!} & t & 1 & 0 \\ \frac{t^6}{6!} & \frac{t^5}{5!} & \frac{t^4}{4!} & \frac{t^3}{3!} & \frac{t^2}{2!} & t & 1 \end{pmatrix}$$

Thus, $X(t) = e^{At} \cdot c$ with $c = \begin{pmatrix} c_1 & c_2 & \dots & c_7 \end{pmatrix}^t$ is a constant.

3.3 Nonhomogeneous Linear Systems

Let us now give our attention from the study of homogeneous systems to nonhomogeneous systems. Fortunately, the basic tools and the most important materials for solving nonhomogeneous systems quite well parallels to those used in the study for solving nonhomogeneous differential equations.

Definition 3.2. We call Cauchy problem any problem of the form

$$\begin{cases} X'(t) = AX(t) + F(t) \\ X(0) = X_0, \, m \geq 1 \end{cases}, t \in [0, a] \tag{3.9}$$

where $A \in \mathcal{M}_n(\mathbb{R})$, X_0 and $F(t)$ are given.

In this work, we will show that there exists at most one solution of the Cauchy problem (3.9). We can easily check that if $X_1(t)$ and $X_2(t)$ are two solutions then $Y(t) = X_1(t) - X_2(t)$ is a solution of $Y'(t) = AY(t)$ with $Y(0) = 0$.

Theorem 3.4. *The problem* (3.9) *has a solution.*

Proof. We start by treating the case $\mathbb{K} = \mathbb{C}$. The proof can be done by triangulation of the matrix A. Let us start by assuming that A is an upper triangular matrix. To simplify, suppose that $n = 3$. The problem is then written

$$\begin{cases} x_1'(t) = a_{11}x_1(t) + a_{12}x_2(t) + a_{13}x_3(t) + f_1(t), \\ x_2'(t) = a_{22}x_2(t) + a_{23}x_3(t) + f_2(t), \\ x_3'(t) = a_{33}x_3(t) + f_3(t). \end{cases}$$

Let $X_0 = (x_{01}, x_{02}, x_{03})^t$ denote the components of X_0. The third equation only involves $x_3(t)$ and can therefore be solved explicitly

$$x_3(t) = e^{a_{33}t}x_2(t) + \int_0^t e^{a_{33}(t-r)} f_3(r)\, dr.$$

If we put $\widetilde{f_2}(t) = a_{23}x_3(t) + f_2(t)$ which is now known we see that the second relation is written

$$x_2'(t) = a_{22}x_2(t) + \widetilde{f_2}(t).$$

The above equation hass one and only one solution verifying $x_0(0) = x_{02}$. Finally, the first relation is written as

$$x_1'(t) = a_{11}x_1(t) + \widetilde{f_1}(t),$$

where $\widetilde{f_1}(t) = a_{12}x_2(t) + a_{13}x_3(t) + f_1(t)$. So there is one and only one possible solution. Thus, the problem (3.9) is completely solved.

We now assume that $A \in \mathcal{M}_n(\mathbb{C})$ is any matrix. We know that A is trigonalizable, that is, there exists an upper triangular matrix T and an invertible matrix P such that $A = PTP^{-1}$. We put $Y(t) = P^{-1}X(t)$ and $\widetilde{F}(t) = P^{-1}F(t)$. The system of differential equations $X'(t) = AX(t) + F(t)$ after multiplication by P^{-1} becomes

$$P^{-1}X'(t) = TY(t) + \widetilde{F}(t).$$

Furthermore, we easily show that for any matrix Q (with constant coefficients $\frac{\partial}{\partial t}(QY(t)) = QY'(t)$. Thus, $Y'(t) = P^{-1}X'(t)$ and the system becomes

$$Y'(t) = TY(t) + \widetilde{F}(t).$$

The initial condition is fixed by $Y(0) = Y_0 = P^{-1}X_0$. Thus the existence and uniqueness of the solution hold because we are in the case of an upper triangular matrix. We finally obtain, necessarily, the solution X in the form $Y(t) = PY(t)$. This completes the proof in the case $\mathbb{K} = \mathbb{C}$.

Obtaining solutions in the real case: In the case where the data are real ($A \in \mathcal{M}_n(\mathbb{R})$ and $F(t) \in \mathbb{R}$ with $X_0 \in \mathbb{R}^n$), we can see the problem as having a value in $\mathbb{C}$ and obtain a unique solution $X(t)$ in this framework, therefore $X(t) \in \mathbb{C}$. We then define $Y(t) = Re(X(t))$, the real part. We see that $Y'(t) = AY(t) + F(t)$ because A and $B(t)$ are real-valued. So Y is a (real-valued) solution to the problem. On the other hand, as there is only one complex solution to the problem, we also have $Y(t) \equiv X(t)$. So in fact the solution $X(t)$ constructed at the beginning has a real value and is indeed the only solution to the problem. □

Recall that if we have $y'(t) = A \cdot y(t) + f(t)$, where y and f are n by 1 matrices and A is n by n matrix with constant entries, then

$$y(t) = \int_0^t e^{A(t-x)} f(x)\,dx + e^{At} \cdot c. \tag{3.10}$$

Moreover, if f is a continuous function over $\mathbb{R}$ with values in $\mathbb{R}^n$. The solution of the above system such that $y(t_0) = y_0$ is given by

$$y(t) = \int_{t_0}^t e^{A(t-x)} f(x)\,dx + e^{A(t-t_0)} \cdot y_0.$$

Note that a second order differential equation can represented as first order system and conversely. First, the following equation

$$x'' + ax' + x = 0,\ a \in \mathbb{R}_+^* \tag{3.11}$$

can be written as a system of differential equations. Indeed, if we set $x_1 = x$ and $x_2 = x'$, then we get the system

$$\begin{cases} x_1' = x_2 \\ x_2' = -x_1 - ax_2 \end{cases} \tag{3.12}$$

whose matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & -a \end{pmatrix}$. Thus, we can solve (3.12) and we deduce the solution of (3.11).

Second, consider the system of differential equations

$$\begin{cases} x_1' = -2x_2 \\ x_2' = \dfrac{x_1}{2} \end{cases}$$

By differentiating x_1' we have $x_1'' = -2x_2' = -x_1$. Hence, $x_1'' + x_1 = 0$ which is a second order differential equation.

Proposition 3.4. *The general solution to system* (3.9) *is the sum of the general solution to the corresponding homogeneous system and a particular solution to the nonhomogeneous system. That is,* $X(t) = X_h(t) + X_p(t)$.

Therefore, to find solutions to the system (3.9) we need to find a particular solution to the corresponding nonhomogeneous system. So, how to find a particular solution?

3.3.1 Educated guess method and applying Laplace transform

Sometimes it is not easy to find a particular solution. Consider, for example the system of the form $y' = Ay + se^{at}$, where $s \in \mathbb{R}^n$ is a constant and $a \in \mathbb{R}$ is fixed. We try to look for a particular solution as $y_p(t) = we^{at}$. We obtain that

$$aw = Aw + s \Rightarrow (A - aI)\, w = s.$$

Supposing that a is not an eigenvalue of the matrix A, we can solve the last system $w = (A - aI)^{-1} s$.

Example 3.1. Let us take the system

$$\begin{cases} y_1' = 2y_1 - y_2 + e^{3t} \\ y_2' = 3y_1 - 2y_2 + e^{3t} \end{cases}$$

After simple computations, the matrix A has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$ with the corresponding eigenvectors $u_1 = (1,1)$ and $u_2 = (3,1)$. Thus, the general solution to the homogeneous system is

$$\begin{cases} y_{1,h}(t) = c_1e^t + 3c_2e^{-t} \\ y_{2,h}(t) = c_1e^t + c_2e^{-t} \end{cases}$$

where C_1, C_2 are constants. since 3 is not an eigenvalue of A, we can look for a particular solution to the nonhomogeneous system in the form

$$y_p(t) = \begin{bmatrix} A \\ B \end{bmatrix} e^{3t},$$

Where A and B are constant to be determined. After plugging $y_p(t)$ and canceling all the exponents,we obtain the system

$$\begin{aligned} 3A &= 2A - B + 1, \\ 3B &= 3A - 2B + 1, \end{aligned}$$

which has the unique solution $A = 1/2, B = 1/2$. Therefore, the general solution to our system is given by

$$y_h(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + C_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-t} + \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}.$$

We can also apply Laplace transform to deal with nonhomogeneous systems. Suppose that we need to give solutions to the system (3.9). Let $\mathbf{X}(s)$ be the Laplace transform of $X(t)$ given as follows

$$\mathbf{X}(s) = \mathcal{L}\{X(t)\}.$$

From a well-known fact, we have $\mathcal{L}\{X'(t)\} = s\mathbf{X}(s) - X_0$. In view of (3.9), by using the Laplace transform, we find $s\mathbf{X}(s) - X_0 = AX(s) + F(s)$, or, equivalently $\mathbf{X}(s) = (sI - A)^{-1}(X_0 + F(s))$, this give the formal solution, but this is after we find the inverse Laplace transform $X(t) = \mathcal{L}^{-1}\{X(s)\}$.

Example 3.2. Consider the system

$$\begin{cases} x_1' = 3x_1 - 4x_2 + e^t \\ x_2' = x_1 + x_2 + e^t \end{cases}$$

where $x_1(0) = x_2(0) = 1$. Applying Laplace transform, we obtain

$$\begin{bmatrix} s-3 & 4 \\ -1 & s+1 \end{bmatrix} \mathbf{X} = \begin{bmatrix} \frac{1}{s-1} \\ \frac{1}{s-1} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where $\mathbf{X}(s)$ can be calculated easily since

$$\mathbf{X}(s) = \begin{bmatrix} \frac{s-3}{(s-1)^2} + \frac{s-3}{(s-1)^3} \\ \frac{s-2}{(s-1)^2} + \frac{s-2}{(s-1)^3} \end{bmatrix}.$$

By the partial fraction decomposition, we get

$$\frac{s^2-4s+3+s-3}{(s-1)^3}=\frac{1}{s-1}-\frac{1}{(s-1)^2}-\frac{1}{(s-1)^3},$$

Hence $\mathcal{L}^{-1}\{\cdot\}=e^t-te^t-t^2e^t$, and similarly for the second term $\mathcal{L}^{-1}\{\cdot\}=e^t-\frac{1}{2}t^2e^t$. Thus, the final solution is given by

$$\begin{cases} x_1=(1-t-t^2)\,e^t \\ x_2=\left(1-\frac{1}{2}t^2\right)e^t. \end{cases}$$

3.3.2 Variation of constants

Let e^{At} be a fundamental matrix solution to the homogeneous system $X'=AX$. Hence, the general solution can be written as $X_h(t)=e^{At}\cdot c$, where $c=(c_1,\ldots,c_n)^t$ is an arbitrary constant. Now, suppose that this vector a function depending on t, that is, $c=c(t)$ and set $X(t)=e^{At}c(t)$ into (3.9). We find $\left(e^{At}\right)'(t)\cdot c(t)+e^{At}c'(t)=Ae^{At}c(t)+F(t)$, and since $\left(e^{At}\right)'(t)=Ae^{At}$, we obtain $c'(t)=\left(e^{At}\right)^{-1}F(t)$, which gives the solution $c(t)=c_0+\int_{t_0}^{t}e^{-Ar}F(r)\,dr$, where c_0 is a constant vector. Applying this solution, we get

$$X(t)=\underbrace{e^{At}\cdot c_0}_{X_h(t)}+\underbrace{e^{At}\int_{t_0}^{t}e^{-Ar}F(r)\,dr}_{X_p(t)}.$$

Since e^{At} is a special fundamental matrix, we can derive the following formula

$$X(t)=e^{At}c_0+\int_{t_0}^{t}e^{A(t-r)}F(r)\,dr.$$

If we give the initial condition $X(t_0)=X_0$, we also get

$$X(t)=e^{A(t-t_0)}X_0+\int_{t_0}^{t}e^{A(t-r)}F(r)\,dr.$$

CHAPTER 4

SYSTEMS OF HIGHER ORDER DIFFERENTIAL EQUATIONS

At first, we present a transformation of a problem of order n into a system of n equations of order 1, and vice versa. Suppose that y is the solution of a problem of order n of type (3.5). To simplify, we will assume here that $n = 2$:

$$x'' = a_1 x + a_2 x' + a(t), \, a_1, a_2 \in \mathbb{R}. \tag{4.1}$$

Let $X(t)$ be the vector of $\mathbb{R}^2$ such that $X(t) = \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix}$. By (4.1), we then notice that $X'(t) = \begin{pmatrix} x'(t) \\ x''(t) \end{pmatrix}$ satisfies

$$X'(t) = \begin{pmatrix} x'(t) \\ a_1 x(t) + a_2 x'(t) + a(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a_1 & a_2 \end{pmatrix} \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix} + \begin{pmatrix} 0 \\ a(t) \end{pmatrix}.$$

It is therefore a system of two differential equations of order 1, given the following form:

$$\begin{cases} x_1' = a_{11} x_1 + a_{12} x_2 + f_1(t) \\ x_2' = a_{22} x_1 + a_{21} x_2 + f_2(t) \end{cases} \tag{4.2}$$

where $x_1 = x$, $x_2 = x'$, ...

Conversely, if we assume that (x_1, x_2) is the solution of the system (4.2) then by replacing $a_{12} x_2 = x_1' - a_{11} x_1 - f_1(t)$ in the second equation we obtain

$$x_1'' - (a_{11} + a_{22}) x_1' + (a_{11} a_{22} - a_{12} a_{21}) x_1 = f_1' - a_{22} f_1,$$

which is a second order differential equation for x_1. Likewise we show that x_2 verifies the following differential equation

$$x_2'' - (a_{11} + a_{22}) x_2' + (a_{11} a_{22} - a_{12} a_{21}) x_2 = f_2' - a_{11} f_2.$$

4.1 How to solve $X^{(m)} = AX$ with $m \geq 1$?

In this subsection, we deal with a system of higher-order differential equations of the form $X^{(m)}(t) = A \cdot X(t)$ with $m \geq 1$, where $A \in \mathcal{M}_n(\mathbb{R})$. There are two cases to consider:

Case 1. When A is *digonalizable*. Assume further that $A = PDP^{-1}$, where P is invertible and D is diagonal. We put $X(t) = PY(t)$. It follows that

$$\begin{cases} X^{(m)}(t) = PDP^{-1}X(t) = PY^{(m)}(t) \\ Y^{(m)}(t) = DP^{-1}X(t) = DY(t) \end{cases}$$

We first solve $Y^{(m)}(t) = DY(t)$ and then we obtain $X(t)$. As an application, we have the following example:

Example 4.1. We want to solve the system

$$\begin{cases} x_1'' = -x_1 + x_2 + x_3 \\ x_2'' = x_1 - x_2 + x_3 \\ x_3'' = x_1 + x_2 - x_3 \end{cases}$$

After simple computations, the corresponding matrix is diagonalizable. We can easily get

$$\begin{cases} x_1 = c_1e^t + c_2e^{-t} + (c_3 + c_5)\cos 2t + (c_4 + c_6)\sin 2t, \\ x_2 = c_1e^t + c_2e^{-t} - c_3\cos 2t - c_4\sin 2t, \\ x_3 = c_1e^t + c_2e^{-t} - c_5\cos 2t - c_6\sin 2t. \end{cases}$$

Case 2. When A is *non-digonalizable*. Assume that $A = PJP^{-1}$, where P is invertible and J is the Jordan matrix by blocks. Similarly as above, we put $X(t) = PY(t)$. It follows that

$$\begin{cases} X^{(m)}(t) = PJP^{-1}X(t) = PY^{(m)}(t) \\ Y^{(m)}(t) = JP^{-1}X(t) = JY(t) \end{cases}$$

We first solve $Y^{(m)}(t) = JY(t)$ and we find $X(t)$.

4.2 Converting systems of higher order linear equations

Let us deal with the differential equation $y^{(m)} = ay$. Define the system of differential equations

$$\begin{cases} x_1' = ax_2 \\ x_2' = x_3 \\ \vdots \\ x_{m-1}' = x_m \\ x_m' = x_1 \end{cases} \tag{4.3}$$

The corresponding matrix is given by

$$A = \begin{pmatrix} 0 & a & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Clearly if we put $x_1 = y$, we obtain $y^{(m)} = ay$. Then solving the system (4.3) gives the general solution to our higher order differential equation $y^{(m)} - ay = 0$.

Example 4.2. We wish to solve the differential equation $y''' = y$. For this purpose, we consider the system of differential equations:

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_1 \end{cases} \tag{4.4}$$

where x_1, x_2 and x_3 are complex functions depending on the real variable t. After computation using the eigenanalysis of the corresponding matrix we get

$$\begin{cases} x_1 = c_1e^t + c_2e^{jt} + c_3e^{j^2t} \\ x_2 = c_1e^t + c_2je^{jt} + c_3j^2e^{j^2t} \\ x_3 = c_1e^t + c_2j^2e^{jt} + c_3je^{j^2t} \end{cases}$$

where $1 + j + j^2 = 0$ and $c_1, c_2, c_3 \in \mathbb{C}$.

Now, if we put $y = x_1$ then by (4.4) we see that $y''' = y$, so $y = c_1e^t + c_2e^{jt} + c_3e^{j^2t}$.

Proposition 4.1. *Every n order differential equation of the form*

$$x^{(n)} = F\left(t, x, x', \dots, x^{(n-1)}\right)$$

can be represented as a system of differential equations of the form $X' = A \cdot X$.

Proof. We introduce new variables, namely $y_1, y_2, ..., y_n$ by setting

$$y_1 = x,\ y_2 = x', y_3 = x'', ..., y_n = x^{(n-1)} \tag{4.5}$$

It follows that

$$y_n' = x^{(n)} = F\left(t, x, x', \ldots, x^{(n-1)}\right)$$

We derive each term from each equation of (4.5), we obtain for the following system:

$$\begin{cases} y_1' = x' = y_2 \\ y_2' = x'' = y_3 \\ y_3' = x''' = y_4 \\ \vdots \\ y_{n-1}' = x^{(n-1)} = y_n \\ y_n' = x^{(n)} = F\left(t, y_1, y_2, \ldots, y_n\right) \end{cases}$$

The proof is finished. □

Next, we transform systems of the form $X'' = AX + F$ to $Y' = BY + \widetilde{F}$. Consider the system

$$X''(t) = AX(t) + F(t),$$

where $A \in \mathcal{M}_n(\mathbb{R})$. We can rewrite the above system as

$$\frac{\partial}{\partial t}\begin{pmatrix} x(t) \\ x'(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}\begin{pmatrix} x(t) \\ x'(t) \end{pmatrix} + \begin{pmatrix} 0 \\ F(t) \end{pmatrix}. \tag{4.6}$$

Then we put $U(t) = (x(t), x'(t))$, we get $U'(t) = \widetilde{A}U(t) + \widetilde{F}(t)$. Also, if we consider the system

$$AX''(t) = BX'(t) + CX(t) + F(t),$$

where A is invertible. We can rewrite it as

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix} x(t) \\ x'(t) \end{pmatrix} = \begin{pmatrix} 0 & A \\ C & B \end{pmatrix}\begin{pmatrix} x(t) \\ x'(t) \end{pmatrix} + \begin{pmatrix} 0 \\ F(t) \end{pmatrix}.$$

So this is a first order system define by block matrices. We can also obtain

$$\frac{\partial}{\partial t}\begin{pmatrix} x(t) \\ x'(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ A^{-1}C & A^{-1}B \end{pmatrix}\begin{pmatrix} x(t) \\ x'(t) \end{pmatrix} + \begin{pmatrix} 0 \\ A^{-1}F(t) \end{pmatrix}.$$

Similarly, if we have

$$AX'''(t) = BX''(t) + CX'(t) + DX(t) + F(t),$$

then

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} x(t) \\ x'(t) \\ x''(t) \end{pmatrix} = \begin{pmatrix} 0 & A & 0 \\ 0 & 0 & A \\ D & C & B \end{pmatrix} \begin{pmatrix} x(t) \\ x'(t) \\ x''(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ F(t) \end{pmatrix}.$$

Or, equivalently,

$$\frac{\partial}{\partial t} \begin{pmatrix} x(t) \\ x'(t) \\ x''(t) \end{pmatrix} = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ A^{-1}D & A^{-1}C & A^{-1}B \end{pmatrix} \begin{pmatrix} x(t) \\ x'(t) \\ x''(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ A^{-1}F(t) \end{pmatrix}.$$

More generally, if we have

$$AX^{(m)}(t) = A_{m-1}X^{(m-1)}(t) + A_{m-2}X^{(m-2)}(t) + ... + A_1X'(t) + A_0X(t) + F(t), \quad (4.7)$$

which can be written in the matrix form

$$\begin{pmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} x(t) \\ x'(t) \\ \vdots \\ x^{(m-1)}(t) \end{pmatrix} = \begin{pmatrix} 0 & A & 0 & \cdots & 0 \\ 0 & 0 & A & & 0 \\ & & \ddots & \ddots & \\ & & & 0 & A \\ A_0 & & & A_{m-2} & A_{m-1} \end{pmatrix} \begin{pmatrix} x(t) \\ x'(t) \\ \vdots \\ x^{(m-1)}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ F(t) \end{pmatrix}$$

That is,

$$\frac{\partial}{\partial t} \begin{pmatrix} x(t) \\ x'(t) \\ \vdots \\ x^{(m-1)}(t) \end{pmatrix} = \begin{pmatrix} 0 & I & & 0 \\ & \ddots & \ddots & \\ & & 0 & I \\ A^{-1}A_0 & \dots & A^{-1}A_{m-2} & A^{-1}A_{m-1} \end{pmatrix} \begin{pmatrix} x(t) \\ x'(t) \\ \vdots \\ x^{(m-1)}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ A^{-1}F(t) \end{pmatrix}.$$

This proves that (4.7) can rewritten as in (3).

Proposition 4.2. *Let B be the matrix of* (4.6). *Then* $p_B(x) = (-1)^n p_A(x^2)$.

Proof. We see that

$$\begin{pmatrix} -xI & I \\ A & -xI \end{pmatrix} \begin{pmatrix} I & 0 \\ xI & I \end{pmatrix} = \begin{pmatrix} 0 & I \\ A - x^2I & -xI \end{pmatrix}.$$

By applying the determinant product formula, we get

$$\det\begin{pmatrix} -xI & I \\ A & -xI \end{pmatrix}\det\begin{pmatrix} I & 0 \\ xI & I \end{pmatrix} = \det\begin{pmatrix} 0 & I \\ A-x^2I & -xI \end{pmatrix} = (-1)^n \det\left(A - x^2 I\right).$$

Or, equivalently, $\det(B - xI) = \det(A - x^2 I)$. This proves the result. □

Remark 4.1. Let $A \in \mathcal{M}_n(\mathbb{R})$ and define the $2n \times 2n$ matrix by blocks:

$$B = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}, \tag{4.8}$$

where $Y' = BY$ with $Y = \begin{pmatrix} X \\ X' \end{pmatrix}$. Then

$$(B - \lambda I)\begin{pmatrix} u \\ v \end{pmatrix} = 0 \Leftrightarrow \begin{cases} Au = \lambda^2 u \\ v = \lambda u \end{cases}$$

From the above remark, we deduce the following corollary:

Corollary 4.1. Let $A \in \mathcal{M}_n(\mathbb{R})$ and let B as in (4.8). Suppose that B has eigenpairs (λ_i, u_i) for $i = 1, 2, ..., 2n$ with $u_1, u_2, ..., u_{2n}$ are independent. Let I denote the n by n identity and define $v_i = diag\{I, 0\} u_i$ for $i = 1, 2, ..., 2n$. Then $Y' = BY$ and $X'' = AX$ have general solutions:

1. $Y(t) = c_1 e^{\lambda_1 t} u_1 + c_2 e^{\lambda_2 t} u_2 + ... + c_{2n} e^{\lambda_{2n} t} u_n.$
2. $X(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + ... + c_{2n} e^{\lambda_{2n} t} v_n.$

When A has simple eigenvalues, we have the following result:

Proposition 4.3. *Let $A \in \mathcal{M}_n(\mathbb{R})$ with $Sp(A) = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ which are real or complex. Define the matrices $R_1, R_2, ..., R_n$ by*

$$R_i = \prod_{i \neq j} \frac{A - \lambda_j I}{\lambda_i - \lambda_j},\ i = 1, 2, ..., n.$$

Then $e^{At} = e^{\lambda_1 t} R_1 + e^{\lambda_2 t} R_2 + ... + e^{\lambda_n t} R_n$ with $\sum_{i=1}^{k} R_i = I$.

Proof. If we put $P = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}$ the corresponding invertible matrix formed by the eigenvalues of A, then P^{-1} exists and so

$$P^{-1} = \begin{bmatrix} v_1' \\ v_2' \\ \vdots \\ v_n' \end{bmatrix}.$$

Now, define $R_i = v_i v_i'$ for $1 \leq i \leq n$. Clearly, $A = \lambda_1 R_1 + \lambda_2 R_2 + ...\lambda_n R_n$. Since the eigenvalues of e^A are the values $\left\{e^{\lambda_1}, e^{\lambda_2}, ..., e^{\lambda_n}\right\}$ then there exists matrices R_i' $(1 \leq i \leq n)$ such that

$$e^{At} = e^{\lambda_1 t} R_1' + e^{\lambda_2 t} R_2' + ... + e^{\lambda_n t} R_n'.$$

Here, we see that

$$R_i' = \prod_{i \neq j} \frac{A - \lambda_j I}{\lambda_i - \lambda_j}, \; i = 1, 2, ..., n.$$

More generally, for every function f defined on the values $(\lambda_i)_{1 \leq i \leq k}$, we deduce that

$$f(A) = \sum_{i=1}^{k} f(\lambda_i) R_i \, ; R_i \in \mathcal{M}_n(\mathbb{R}) \quad (1 \leq i \leq k).$$

In particular, for $f(x) = \cos^2 x + \sin^2 x$, giving

$$I = \cos^2 A + \sin^2 A = \sum_{1=1}^{k} \left(\cos^2 \lambda_i + \sin^2 \lambda_i\right) R_i = \sum_{1=1}^{k} R_i.$$

Thus, $\sum_{1=1}^{k} R_i = I.$ □

4.3 Examples on the second and third-order linear differential equations

Here we discuss an example on the systems of the form $X''(t) = AX'(t) + BX(t) + F(t)$, where A, B are two matrices.

Example 4.3. Consider the system

$$\begin{cases} x_1''(t) = a_1 \cdot x_1'(t) + b_1 \cdot x_2'(t) + c_1 \cdot x_1(t) + d_1 \cdot x_2(t) + f_1(t) \\ x_2''(t) = a_2 \cdot x_1'(t) + b_2 \cdot x_2'(t) + c_2 \cdot x_1(t) + d_2 \cdot x_2(t) + f_2(t) \end{cases} \tag{4.9}$$

with the initial values $x_1(0)$, $x_1'(0)$, $x_2(0)$ and $x_2'(0)$. This system is equivalent to a system including four equations of the first order. Indeed, we introduce two functions $y_1(t)$ and

$y_2(t)$ which represent $x_1'(t)$ and $x_2'(t)$, respectively. That is, $y_1(t) = x_1'(t)$ and $y_2(t) = x_2'(t)$. We put

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ y_1(t) \\ y_2(t) \end{pmatrix}, A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{pmatrix}, F(t) = \begin{pmatrix} 0 \\ 0 \\ f_1(t) \\ f_2(t) \end{pmatrix}, X(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_1'(0) \\ x_2'(0) \end{pmatrix}.$$

Or, equivalently, the system (4.9) can be written as $X' = A \cdot X + F$. If we take $a_i = b_i = d_i = 1$ $(1 \leq i \leq 2)$, then

$$A = \begin{pmatrix} 0 & 1 & 1 & \frac{\sqrt{3}}{(\sqrt{3}-1)(\sqrt{3}-3)}\left(2\sqrt{3}-4\right) \\ 0 & -1 & 1 & \frac{\sqrt{3}}{(\sqrt{3}-1)(\sqrt{3}-3)}\left(2\sqrt{3}-4\right) \\ 1 & 0 & \frac{2}{\sqrt{3}-1} & \frac{1}{\sqrt{3}-1}\left(2\sqrt{3}-4\right) \\ -1 & 0 & \frac{2}{\sqrt{3}-1} & \frac{1}{\sqrt{3}-1}\left(2\sqrt{3}-4\right) \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix}} & & \\ & \boxed{\sqrt{3}+1} & \\ & & \boxed{1-\sqrt{3}} \end{pmatrix} \times$$

$$\begin{pmatrix} 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{4}-\frac{1}{12}\sqrt{3} & \frac{1}{4}-\frac{1}{12}\sqrt{3} & \frac{1}{12}\sqrt{3} & \frac{1}{12}\sqrt{3} \\ \frac{\sqrt{3}}{12\sqrt{3}-24}-\frac{3}{12\sqrt{3}-24} & \frac{\sqrt{3}}{12\sqrt{3}-24}-\frac{3}{12\sqrt{3}-24} & 2\frac{\sqrt{3}}{12\sqrt{3}-24}-\frac{3}{12\sqrt{3}-24} & 2\frac{\sqrt{3}}{12\sqrt{3}-24}-\frac{3}{12\sqrt{3}-24} \end{pmatrix}$$

Example 4.4. Define the third-order linear differential equation:

$$x''' + f(t)x'' + g(t)x' + h(t)x = k(t), \tag{4.10}$$

where f, g, h, k are continuous functions defined some interval $[a, b]$. From (4.10), it follows that

$$x''' = -f(t)x'' - g(t)x' - h(t)x + k(t).$$

After introducing the following dependent variables: $x_1 = x$, $x_2 = x' = x_1'$ and $x_3 = x'' = x_2'$. Thus,

$$x''' = x_3' = -f(t)x_3 - g(t)x_2 - h(t)x_1 + k(t).$$

Or, equivalently

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = k(t) - h(t)x_1 - g(t)x_2 - f(t)x_3 \end{cases}$$

In the matrix form, we get

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -h(t) & -g(t) & -f(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ k(t) \end{pmatrix}.$$

As required.

Conclusion and open problems

Solving different types of problems, particularly in science and engineering, comes down to characterize all solutions of a system of the first order or higher order differential equations. So these systems have many applications which we have not discussed in this manuscript since we have only concentrated on the eigenanalysis methods to solve them, as well as Brine Tank Cascade, Cascades and Compartment Analysis, Recycled Brine Tank Cascade, Pond Pollution, Home Heating, Chemostats and Microorganism Culturing, Irregular Heartbeats and Lidocaine, Nutrient Flow in an Aquarium, Biomass Transfer, Pesticides in Soil and Trees, Forecasting Prices, Coupled Spring-Mass Systems, Boxcars, Electrical Networks, Logging Timber by Helicopter and Earthquake Effects on Buildings.

In the literature, there are new contributions involving system of ordinary differential equations by special methods as well as Adomian decomposition method [2], some generalization as nonlinear stochastic systems of differential equations and others on solution method for a non-homogeneous fuzzy linear system of differential equations.

For more details on some problems involving differential algebra, integrating differential polynomials and differential eliminations, one can see [7]. A very famous problem is the quadratization problem: If we consider a system of ordinary differential equations with polynomial at the right-hand side and we wish to transform it into a system with at most quadratic equation at the right-hand side. For example, consider $y' = y^{10}$, so if we introduce a new variable $x = y^9$, then we obtain $y' = xy$ and $x' = 9y^8y' = 9y^9x = 9x^2$. Hence,

$$\begin{cases} y' = xy, \\ x' = 9x^2. \end{cases}$$

More generally, it is important to ask on the transforming of the differential equation

$$y^{(m)} = y^k + y^{k-1} + ... + y + 1.$$

That is, it is natural to ask if one could use suitable variables for quadratization if the new variables are arbitrary polynomials. For details, see [3].

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ملخص. الهدف الرئيسي من هذا العمل هو فهم بعض المعادلات التفاضلية ذات الرتبة الأعلى والتي لها حلول مشتركة، حيث نتعامل مع العلاقات الأساسية بين المعادلات الخطية من الرتبة الأولى في البعد n. حيث قمنا باستخدام طرق التحليل الذاتي، كما تعاملنا مع أنظمة المعادلات التفاضلية الخطية المتزامنة ذات المعاملات الثابتة. بالإضافة إلى ذلك استخدمنا أنظمة تحويل المعادلات الخطية ذات الرتبة الأعلى.

كلمات مفتاحيه: المصفوفة الأسية، أساليب التحليل الذاتي، نموذج جوردان.

Abstract. We study some higher order differential equations sharing a common set of solutions, where we deal with the fundamental link between these equations and the linear differential systems of the first order in dimension *n*. That is, by using eigenanalysis methods, we deal with systems of linear simultaneous differential equations with constant coefficients. Moreover, we study converting systems of higher order linear equations.

Key words: Exponential Matrix, Nonhomogeneous ODE systems, Eigenanalysis methods, Jordan Form.

Résumé. Nous étudions quelques équations différentielles d'ordre supérieur partageant un ensemble commun de solutions, où nous traitons du lien fondamental entre ces équations et les systèmes différentiels linéaires du premier ordre en dimension n. Autrement dit, en utilisant des méthodes concernant valeurs et vecteurs propre, nous traitons de systèmes d'équations différentielles simultanées linéaires à coefficients constants. De plus, nous étudions la méthode de conversion de systèmes d'équations linéaires d'ordre supérieur.

Mots clés : Matrice exponentielle, systèmes d’EDO non homogènes, méthodes d'analyse propre, forme de Jordan.