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Title

Study of Wave Equation

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قال الله تعالى :" **يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنْكُمْ وَالَّذِينَ أُوتُوا الْعِلْمَ دَرَجَاتٍ** "

اللّهم ليس بجهدي و إجتهادي إنما بتوفيقك و كرمك و فضلك عليّ، الحمد لله بارئ النسمة الخالق من الكلمة الناطق بالبيان و الحكمة لأهل العلم، **الحمد لله شكرا و حبا و إمتنانا عند البدء و الختام.**

إلى من بلغ الرسالة وأدى الأمانة ... ونصح الأمة ... إلى نبي الرحمة و نور العالمين

سيدنا محمد صلى الله عَلَيْهِ وسَلَّم

إلى أصلي الأصيل، ظلي الطويل، كهفي الآمن... **جدي رحمة الله عليه** الذي عزّ عليّ غيابه طيلة هذه السنين، ورافقتني دعواته في كل حين

إلى من إبتدأني الله فيها و منها ، إلى من كانت وطني و لم يكن لي وطن سواها ، إلى جبروت البحر و طهر السماء إلى سندي و ملاكي في الحياة إلى قدوتي منذ الصغر و حتى الممات ، إلى من بسمتها غايتي و ما تحت أقدامها جنتي ،إلى من حملتني في بطنها و سقتني من صدرها و أسكنتني قلبها فغمرتني بحبها ، إلى من كان دعاؤها و رضاها و سيبقى مفتاحا لعسرتي و جبرا لكسرتي ، إلى من علمتني معنى الحنان و العطاء معنى الصبر و القوة و الحب نجمتي المضيئة في سماء العمر، إلى من كانت في كواليس نجاحي دوما و الداعم الأول في كل خطواتي، إلى من سهرت معي و ساندتني فجري المشرق بعتمة الأيام، إلى من بذلت و كابدت لتوصلني لبر الأمان إلى المرأة التي صنعتني إلى من ظلت تنفق عمرها لتخيط لي طريقا مستقيما **جنتي و نور قلبي محبوبة العمر و الروح أمي الغالية** النعمة التي أدعو الله أن يحفظها لي.

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آية طالبي

ABSTRACT

The aim of this memory is to study some physical problems related to wave equations.

These wave phenomena appear in numerous applications such as: sound waves and electromagnetic waves...

In the first chapter, we recalled some definitions and properties of PDEs, as well as some generalities that we used in the following chapters.

The second chapter was devoted to solving the wave equation, where we presented three methods of resolution, namely the D'Alembert method, the Fourier method and the Kirchhoff method.

Finally, and through the last chapter, we have proposed a numerical solution of the equation in question via the finite difference method.

Keywords:

Partial differential equations - The wave equations - CHAUCHY problem – D'ALEMBERT's formula - FOURIER separation - Kirchhoff formula.

RÉSUMÉ

L’objectif de ce mémoire s’inscrit dans l’étude de certains problèmes physiques liés aux équations des ondes.

Ces équations apparaissent le plus souvent dans de nombreuses applications telles que : les ondes sonores et les ondes électromagnétiques, etc...

Dans le premier chapitre, nous rappelons quelques définitions et propriétés sur les EDPs, ainsi que quelques généralités que nous utiliserons par la suite

Le deuxième chapitre est consacré à l’étude de ces équations, où nous avons présenté trois méthodes de résolutions à savoir la méthode D'Alembert, la méthode de Fourier ainsi que celle de Kirchhoff.

Enfin, et à travers le dernier chapitre, nous proposons une résolution numérique de l’équation des ondes via la méthode des différences finies.

Mots clés :

Équations aux dérivées partielles - Les équations des ondes -problème de CHAUCHY - La formule de D’ALEMBERT - Séparation de FOURIER- formule de KIRCHHOFF.

ملخص

تهدف هذه المذكرة إلى دراسة بعض المشاكل الفيزيائية المرتبطة بمعادلات الامواج.

وتظهر هذه الظواهر الموجية في العديد من التطبيقات مثل: الامواج الصوتية، الامواج الكهرومغناطيسية وغيرها.

في الفصل الأول ، تذكرنا بعض التعاريف والخصائص حول المعادلات التفاضلية الجزئية ، وكذلك بعض العموميات التي استخدمناها في الفصول التالية.

وخصصنا الفصل الثاني لحل معادلة الامواج، حيث عرضنا ثلاث طرق للحل، وهي طريقة دالمبرت وطريقة فورييه وطريقة كيرشوف.

أخيرا ، ومن خلال الفصل الأخير ، اقترحنا حلا عدديا للمعادلة المعنية عبر طريقة الفرق المحدود.

الكلمات المفتاحية :

المعادلات التفاضلية الجزئية - المعادلات الموجية - مشكلة كوشي - معادلة دالمبرت - فصل فوريي -صيغة كيرشوف.

Contents

Introduction

Many phenomena and problems in the world today are based on partial differential equations, hereafter abbreviated as PDEs.

When did PDEs first appear? They were probably first formulated at the birth of rational mechanics in the 17th century (Newton, Leibniz...). The "catalogue" of PDEs then grew with the development of the sciences, in particular physics. If we have to choose just a few names, we should mention Euler, followed by Navier and Stokes for the equations of fluid mechanics, Fourier for the heat equation, Maxwell for the equations of electromagnetism, Schrodinger and Heisenberg for the equations of quantum mechanics and, of course, Einstein for the PDEs of relativity.

Partial differential equations translate the laws of phenomena in mechanics, fluid dynamics, elasticity and meteorology, among others, into mathematical language. Furthermore, due to technological advancement, PDEs have been applied to image processing. Hence, physics and mathematics have a close correlation. Poincare's words affirm this relationship: "**All laws are derived from experience, but to state them, a special language is needed; ordinary language is too poor, and moreover too vague, to express such delicate, rich and precise relationships. This, then, is the first reason why the physicist cannot do without mathematics it is the only language he can speak**". (Poincare, 1910).In 1765, Euler wrote a dissertation on pure mathematics dedicated to a specific PDE." Lagrange gave a summary of PDEs encountered in physical problems and provided innovative integration methods. He was less systematic and mathematical in his teaching of EDP. D'Alembert was the only one to simultaneously:

- consider PDEs outside of any physical problem.

- undertake the study of very broad classes of PDEs.

Wave equation is one of partial differential equations that express many of the fundamental laws of nature, and generally arise in the mathematical analysis of problems in science and technology.

We essentially encounter three types of waves: acoustic waves, i.e. waves that propagate in a fluid (water or air for example). Elastic waves, i.e. waves that propagate in a solid; electromagnetic waves, for example light. We will mainly study the acoustic wave equation which is the simplest model (scalar model).

This memoir is divided into three chapters:

In the first chapter, we review some definitions and properties of PDEs, as well as some generalities such as Fourier series.

In the second chapter, we looked at solving the wave equation.We present this resolution using three methods Fourier's method, D'Alembert's method and Kirchouff's method.

In the last chapter, we establish a numerical resolution of the equation under consideration using the finite-difference method.

Chapter 1

Preliminaries [1] [2]

In this part, we introduce some fundamental notions about partial differential equations and different properties necessary for the rest of our work.

1.1 Generalities about partial differential equations

1.1.1 Definitions and properties

Definition 1 *A partial differential equation of the first order with the unknown u and n independent variables $x_1, ..., x_n$ is an equation of the form:*

$$F(x_1, ..., x_n, u, \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}) = 0 \tag{1.1}$$

where $(x_1, ..., x_n) \in \Omega$ is an open of $\mathbb{R}^n$.

Definition 2 *The general form of a second-order PDE is:*

$$F(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y \partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}) = 0 \tag{1.2}$$

for $(x, y) \in \Omega$ is an open of $\mathbb{R}^2$.

1.2 Some mathematical physics equations

PDEs are commonly found in science and engineering, particularly in physics. Here are a few examples of them:

1.2.1 Transport equation :

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

where $u(x, t)$ and c is a positive real number.

The transport equation is used to model air pollution, the dispersion of dyes and even traffic flows.

1.2.2 Wave equation :

$$\frac{\partial^2 u}{\partial t^2} - c^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}) = 0$$

where $u(x, y, z, t)$

•The wave equation is utilised in various phenomena such as string theory, gravitational waves, light waves and sound waves.

•It helps us determine the movement of strings and fluid surfaces such as water waves.

•The wave theory has its application in numerous fields, be it wireless communications, musical instruments or the detection of speeding vehicles.

1.2.3 Heat equation :

$$\frac{\partial u}{\partial t} - k(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}) = 0$$

where $u(x, y, z, t)$

The heat equation is used in real life to predict temperature changes in various objects and systems, such as in engineering design, climate modeling, heat transfer processes and thermal conduction.

1.2.4 Burgers equation :

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

where $u\,(x, t)$ and u is the speed.

The Burgers equation is a widely used mathematical model for physical phenomena such as gas dynamics, traffic flow, shallow water waves, diffusion of chemical reactions.

1.2.5 Laplace equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

where $u\,(x, y, z)$

Laplace's equation is a fundamental second-order partial differential equation that finds extensive application in the field of physics and engineering.These include electrostatics, gravitation, fluid mechanics and steady-state heat conduction.

1.2.6 Euler-Bernoulli equation :

$$\frac{\partial^2 u}{\partial t^2} - c^4\frac{\partial^4 u}{\partial x^4} = 0$$

where $u\,(x, t)$

Bernoulli's equation has many real-life applications. One common application is in the study of fluid dynamics, it is also used in the design of water distribution systems.

1.3 Classification of second-order PDEs

Second-order partial differential equations of the following form :

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = 0$$

where **A**, **B** and **C** are constants of x and y. Partial differential equations of the second order are classified into three types, depending on the value of the determinant: $\mathbf{B^2 - 4AC}$

where we distinguish:

If $\mathbf{B}^2 - \mathbf{4AC} < \mathbf{0}$, the PDE is called Elliptic.

If $\mathbf{B}^2 - \mathbf{4AC} = \mathbf{0}$, the PDE is called Parabolic.

If $\mathbf{B}^2 - \mathbf{4AC} > \mathbf{0}$, the PDE is called Hyperbolic.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$B^2 - 4AC = 0^2 - 4\,(1)\,(1) = -4 < 0$ so the **Laplace equation** is elliptic.

$$\frac{\partial u}{\partial x} - d\frac{\partial^2 u}{\partial x^2} = 0$$

$B^2 - 4AC = 0^2 - 4\,(-d)\,(0) = 0$ so **the diffussion equation** is parabolic.

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

$B^2 - 4AC = 0^2 - 4\,(-c^2)\,(1) = 4c^2 > 0$ so **the wave equation** is hyperbolic.

$$Y\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

tricomi equation.

if $Y < 0 \Longrightarrow$ the PDE is **elliptic**.

if $Y = 0 \Longrightarrow$ the PDE is **parabolic**.

if $Y > 0 \Longrightarrow$ the PDE is **hyperbolic**.

1.4 Reminder of Fourier series:[1] [3]

1.4.1 Periodic Applications :

Definition 3 *A function f:*$\mathbb{R} \longrightarrow \mathbb{C}$ *is called periodic with period* $T > 0$ *if*

$$f(x+T) = f(x)$$

$\forall x \in \mathbb{R}$

Example 4 *1) Constant functions are periodic, for any period* T.

2) $t \longmapsto \exp(it)$ *is periodic with period* 2π.

3) The functions $x \longmapsto \cos x$ *and* $x \longmapsto \sin x$ *are* 2π *periodic.*

1.4.2 Fourier series and Fourier coefficients

Definition 5 *A Fourier series is a series that takes the form of the following formula*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

(Real form)

Definition 6 *Let* f *be a piecewise continuous function on* $[-\pi, \pi]$ *.The exponential Fourier coefficients of f are the complex numbers* c_n *defined for all* $n \in \mathbb{Z}$ *by*

$$c_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx$$

and we call the trigonometric Fourier coefficients of f *the numbers* a_n *and* b_n *are defined by*

$$a_0(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx$$

$$\begin{aligned} a_n(f) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \\ \text{and} \quad b_n(f) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \end{aligned}$$

Example 7 *let* $f(x) = x + x^2 \in [-\pi, \pi]$ *so*

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2\pi^2}{3}$$

if $n \geq 1$ *so*

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x + x^2\right)\cos(nx)dx \\ &= \frac{1}{n\pi}(\left(x + x^2\right)\sin(nx)]_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi}(1 + 2x)\sin(nx)dx \\ &= -\frac{1}{n^2\pi}\left\{(-(1 + 2x)\cos(nx)]_{-\pi}^{\pi} + \frac{1}{n}2\sin(nx)]_{-\pi}^{\pi}\right\} \\ &= \frac{4\cos(n\pi)}{n^2} = \frac{4(-1)^n}{n^2} \\ \text{and } b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x + x^2\right)\sin(nx)dx = \frac{2(-1)^{n+1}}{n} \end{aligned}$$

therefore the series of $f(x)$ *is*

$$\frac{\pi^2}{3} + \sum_{n=1}^{+\infty}\left(\left(\frac{4(-1)^n}{n^2}\right)\cos(nx) + \left(\frac{2(-1)^{n+1}}{n}\right)\sin(nx)\right)$$

Remark 8 *1) If* f *is an even function* $\Rightarrow (b_n = 0)$, *then its Fourier series is as follows:*

$$\frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n\cos(nx) \quad \text{with } a_n = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \text{ for } n \geq 0$$

2) If f *is an odd function* $\Rightarrow (a_n = 0)$, *then its Fourier series is as follows:*

$$\sum_{n=1}^{+\infty} b_n\sin(nx) \text{ with } b_n = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \text{ for } n \geq 0$$

1.4.3 Dirichlet's simple convergence theorem

Definition 9 *A piecewise continuous function on* $[0, 2\pi]$ *is piecewise continuously differentiable (of class* C^1 *) on this interval if there exists a partition of* $[0, 2\pi]$ *consisting of a finite number of intervals* $[a, b]$ *such that*

1: f is continuous on $]a,b[$, and f has a right-hand limit at a and a left-hand limit at b.

2: f is differentiable on $]a,b[$ and f' is piecewise continuous on this interval.

Theorem 10 "Dirichlet's" *Let $f\colon \mathbb{R} \longrightarrow \mathbb{C}$ be a periodic 2π function that is continuously piecewise differentiable on $\mathbb{R}$. Then the Fourier series of f converges simply and its sum is for all $x \in \mathbb{R}$.*

$$a_0(f) + \sum_{n\geq 1} [a_n(f)\cos(nx) + b_n(f)\sin(nx)] = \frac{f(x_0^+) + f(x_0^-)}{2}$$

where $f(x_0^+)$ and $f(x_0^-)$ are the right and left limits of f at x_0. In particular if f is continuous at a point x, then we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} [a_n(f)\cos(nx) + b_n(f)\sin(nx)]$$

Chapter 2

wave equation[6]

A wave is the phenomenon of propagation in a medium that does not involve the transport of matter.

The wave equation is a second-order partial differential equation that describes wave propagation phenomena such as sound waves, light waves ....This equation appears in many fields.

In this chapter, we study the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = f$$

where c is a constant.

2.1 Wave equations in one dimension

The evolutionary P.D.E. of second order in time (t) and space (x) is called the linear wave equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad t \in \mathbb{R}^+, x \in \mathbb{R}$$

where c is a given positive real number, homogeneous to the wave propagation speed.

- In dimension one (of space), the wave equations model the equations of vibrating strings.

-The general solution to this equation is the sum of two functions:

$$u(x,t) = F(x+ct) + G(x-ct)$$

2.2 Types of wave equations

2.2.1 Acoustic waves

If Ω denotes an open space of $\mathbb{R}^3$ (bounded or not) filled with fluid, the propagation of acoustic waves in this fluid depends on $\rho(x)$ the fluid density at the point x, $c(x)$ the local velocity of the acoustic waves and the function $g\,(x,t)$ is the source.

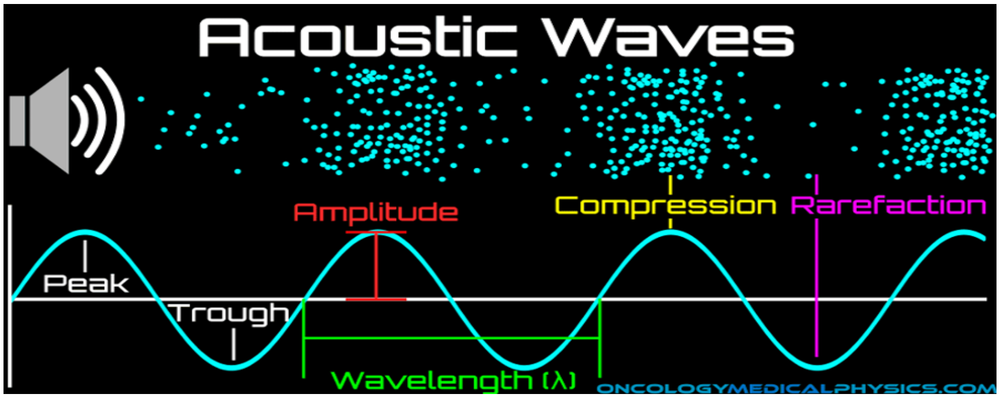
The equation that governs variations in the pressure $P(x,t)$ of the fluid at a point x and at time t is the wave equation

$$\frac{1}{\mu(x)}\frac{\partial^2 P}{\partial t^2} - \operatorname{div}\left(\frac{1}{\rho(x)}\nabla P\right) = g \quad x \in \Omega, t > 0$$

where $\mu(x) = \rho(x)c^2(x)$

Applications:

Acoustic waves are used, for example, in musical acoustics, room acoustics, underwater acoustics or to study wave phenomena.



Acoustic wave

2.2.2 Elastic waves

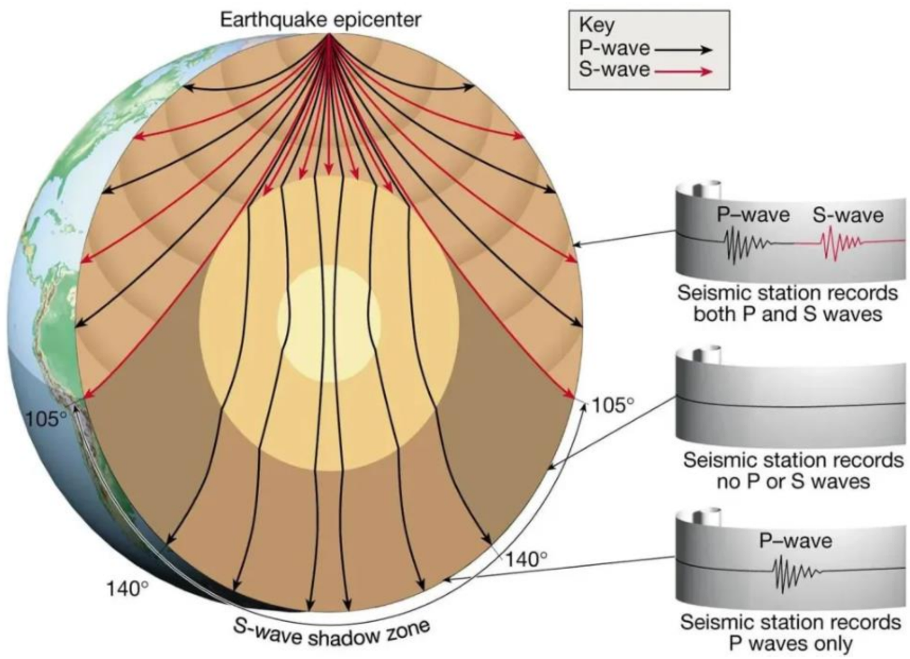
The equations of motion are

$$\rho\frac{\partial^2 \overrightarrow{u}}{\partial t^2} - \mathrm{div}\,(\sigma) = \rho\,\overrightarrow{g}$$

where $\overrightarrow{u}\,(x,t)$ is a vector representing the displacement of a particle from point x to time t, σ is the tensor of opposites , linked to the deformation tensor $\varepsilon(\overrightarrow{u})$ by the material behaviour law.

Applications:

Elastic waves are encountered in particular when studying seismic or geophysical problems and non-destructive testing.



seismic waves

2.2.3 Electromagnetic waves

In a medium that we will assume to be isotropic linear dielectric, an electromagnetic field in a domain Ω is described by the electric field $\overrightarrow{E}(x,t)$ and the magnetic field $\overrightarrow{H}(x,t)$ which satisfy Maxwell's equations

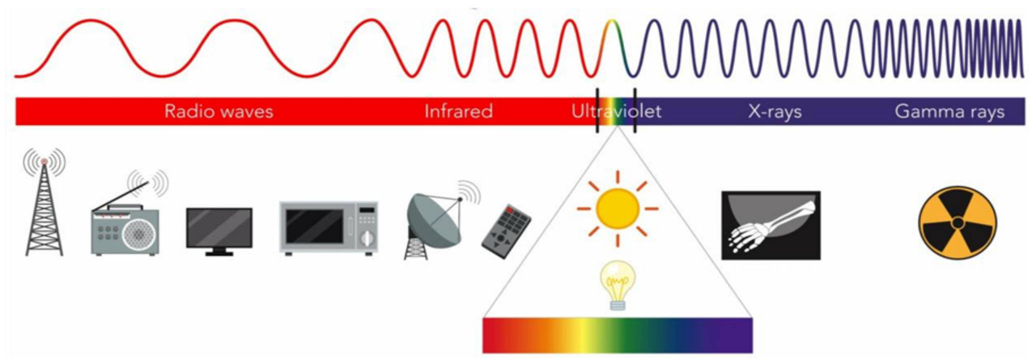
$$\begin{cases} \mu\frac{\partial \overrightarrow{H}}{\partial t} + rot\,\overrightarrow{E} = 0 \\ \varepsilon\frac{\partial \overrightarrow{E}}{\partial t} + \sigma\,\overrightarrow{E} - rot\,\overrightarrow{H} + \overrightarrow{j_s} = 0 \end{cases}$$

$\varepsilon\,(x) > 0$ is the dielectric permittivity of the medium, $\mu(x) > 0$ is the magnetic permeability of the medium, $\sigma(x) \geq 0$ is the conductivity of the material, $\overrightarrow{j_s}$ is a source current. The wave velocity of the medium is given by the relation :

$$c^2 = \frac{1}{\varepsilon\mu}$$

Applications:

Electromagnetic waves are widely used to detect flying objects (radar stealth).



Electromagnetic waves applications

2.3 Canonical form and general solution[5]

The homogeneous wave equation in one dimension space has the form:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad -\infty < x \leq +\infty,\ t > 0 \tag{2.1}$$

where $c \in \mathbb{R}$ is called the wave velocity. To obtain the canonical form of equation (2.1), we use the change of variables

$$\begin{cases} \xi = x + ct \\ \eta = x - ct \end{cases}$$

We pose $\omega(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$. The chain rule gives

$$\begin{cases} \frac{\partial u}{\partial t} = c\left(\frac{\partial \omega}{\partial \xi} - \frac{\partial \omega}{\partial \eta}\right) \\ \frac{\partial u}{\partial x} = \frac{\partial \nu}{\partial \xi} + \frac{\partial \nu}{\partial \eta} \end{cases}$$

and

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 \omega}{\partial \xi^2} - 2\frac{\partial^2 \omega}{\partial \xi \partial \eta} + \frac{\partial^2 \omega}{\partial \eta^2} \right) \\ \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 \omega}{\partial \xi^2} + 2\frac{\partial^2 \omega}{\partial \xi \partial \eta} + \frac{\partial^2 \omega}{\partial \eta^2} \end{cases}$$

Therefore,

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = -4\frac{\partial^2 \omega}{\partial \xi \partial \eta} = 0$$

This is the canonical form of the wave equation. Such as $\frac{\partial^2 \omega}{\partial \xi \partial \eta} = 0$ it follows that $\frac{\partial \omega}{\partial \xi} = f(\xi)$ then $\omega = \int f(\xi)\, d\xi + G(\eta)$.

The general solution of equation $\frac{\partial^2 \omega}{\partial \xi \partial \eta} = 0$ has the form:

$$\omega(\xi, \eta) = F(\xi) + G(\eta).$$

Where $F, G \in \mathbb{C}^2(\mathbb{R})$ are both arbitrary functions, the fact that

$$\omega(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$$

implies

$$u(x, t) = F(x + ct) + G(x - ct)$$

2.4 Cauchy's problem and d'Alembert's formula[5]

The Cauchy problem for the one-dimensional homogeneous wave equation is given by

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad -\infty < x \leq +\infty,\ t > 0 \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad -\infty < x \leq +\infty \end{cases}$$

Theorem 11 *The solution to the Cauchy problem is given by the following formula (called the D'Alembert formula)*

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds \tag{2.2}$$

Proof. we have the general solution of the wave equation is

$$u(x,t) = F(x+ct) + G(x-ct)$$

therefore at $t = 0$, we obtain

$$u(x,0) = F(x) + G(x) = f(x).$$

$$u_t(x,0) = cF'(x) - cG'(x) = g(x).$$

therefore

$$u(x,0) = F(x) + G(x) = f(x). \tag{2.3}$$

$$u_t(x,0) = F'(x) - G'(x) = \frac{1}{c}g(x). \tag{2.4}$$

integration of (2.4) in $[0,x]$ we obtain

$$F(x) - G(x) = \frac{1}{c}\int_0^x g(s)ds + C \tag{2.5}$$

where $C = F(0) - G(0)$

$$(2.3)+(2.5) \quad F(x) = \frac{1}{2}\left(f(x) + \frac{1}{c}\int_0^x g(s)ds\right) + \frac{C}{2}. \tag{2.6}$$

$$(2.3)-(2.5) \quad G(x) = \frac{1}{2}\left(f(x) - \frac{1}{c}\int_0^x g(s)ds\right) - \frac{C}{2}. \tag{2.7}$$

If we substitute these expressions for F and G into the general solution

$$u(x,t) = F(x+ct) + G(x-ct)$$

we obtain the following formula

$$u(x,t) = \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)ds + \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(s)ds$$

where

$$-\frac{1}{2c}\int_0^{x-ct} g(s)ds = +\frac{1}{2c}\int_{x-ct}^{0} g(s)ds$$

therefore

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds. \tag{2.8}$$

■

Example 12 *solve:*

$$\begin{aligned} u_{tt} &= 4u_{xx} \\ u(x,0) &= -x \qquad u_t(x,0) = 0 \end{aligned}$$

here $c^2 = 4 \qquad c = 2 \qquad f(x) = -x \qquad g(x) = 0$

the general solution

$$u(x,t) = \frac{1}{2}\left[f(x+2t) + f(x-2t)\right] + \frac{1}{4}\int_{x-2t}^{x+2t} g(s)ds.$$

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[-(x+2t) - (x-2t)\right] + 0 \\ &= \frac{1}{2}\left[-x - 2t - x + 2t\right] \\ &= \frac{1}{2}\left[-2x\right] \\ u(x,t) &= -x \end{aligned}$$

2.5 The non-homogeneous Cauchy problem[5]

Let's consider the following Cauchy problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = F(x,t), \quad -\infty < x < +\infty, t > 0 \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad -\infty < x < +\infty, t > 0 \end{cases} \tag{2.9}$$

Theorem 13 *The problem* (2.9)*has a unique solution u, which is given by the formula*

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds + \frac{1}{2c}\int_{\tau=0}^{\tau=t}\int_{\xi=x-c(t-\tau)}^{\xi=x+c(t-\tau)} F(\xi,\tau)d\xi d\tau$$

Proof. *The equation is in the form*

$$A^2 - B^2 = (A - B)(A + B)$$

The equation (2.9) *can be written as:*

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = F(x,t)$$

To simplify, let

$$v(x,t) = \frac{\partial}{\partial t} - c\frac{\partial}{\partial x}$$

so we obtain the problem

$$\begin{cases} \frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} = F(x,t) \\ v(x,0) = u_t(x,0) - cu_x(x,0) = g(x) - cf'(x) \end{cases}$$

we obtain the following two transport equations

$$\begin{cases} v_t + cv_x = F(x,t) \\ v(x,0) = g(x) - cf'(x) \end{cases}$$

$$\begin{cases} u_t - cu_x = v(x,t) \\ u(x,0) = f(x) \end{cases}$$

$$v(x,t) = g(x-ct) - cf'(x-ct) + \int_0^t F(x - ct + c\tau, \tau)d\tau$$

$$\begin{aligned} u(x,t) &= f(x+ct) + \int_0^t v(x + ct - c\tau, \tau)d\tau \\ &= f(x+ct) + \int_0^t \left[\begin{array}{c} g(x + ct - c\tau - c\tau) - cf'(x + ct - c\tau - c\tau) \\ + \int_0^t F(x + ct - c\tau + c\tau, \tau)d\tau \end{array} \right] d\tau \\ &+ \int_0^t \int_0^t F(x + ct - 2c\tau + c\tau, \tau)d\tau d\tau \end{aligned} \tag{2.10}$$

we have

$$\int_0^t g(x + ct - 2c\tau)d\tau$$

we assume

$$\begin{aligned} s &= x+ct-2c\tau \Longrightarrow ds=-2cd\tau \\ d\tau &= \frac{-1}{2c}ds \\ \tau &\to 0 \qquad s=x+ct \\ \tau &\to t \qquad s=x-ct \\ &\Longrightarrow \int_0^t g(x+ct-2c\tau)d\tau = \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds \end{aligned}$$

we have also

$$\begin{aligned} \int_0^t f'(x+ct-2c\tau)d\tau &= \frac{1}{2c}\int_{x-ct}^{x+ct} f'(s)ds \\ &= \frac{1}{2c}\left[f(x+ct)-f(x-ct)\right] \end{aligned}$$

we take

$$I=\int_0^t\int_0^t F(x+ct-2c\tau+c\tau,\tau)d\tau d\tau$$

suppose that

$$\begin{aligned} \xi &= s+c\tau = x+ct-2c\tau+c\tau \\ d\xi &= ds \\ I &= \frac{1}{2c}\int_0^t\int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\xi,\tau)d\xi d\tau \end{aligned}$$

substitute in (2.10) *we find*

$$u(x,t)=\frac{1}{2}\left[f(x+ct)+f(x-ct)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds+\frac{1}{2c}\int_{\tau=0}^{\tau=t}\int_{\xi=x-c(t-\tau)}^{\xi=x+c(t-\tau)} F(\xi,\tau)d\xi d\tau$$

■

Example 14 *consider the following Cauchy problem*

$$\begin{cases} \frac{\partial^2 u}{\partial^2 t}-9\frac{\partial^2 u}{\partial^2 x}=e^x-e^{-x} & -\infty<x\leq+\infty, t>0 \\ u(x,0)=x & -\infty<x\leq+\infty \\ u_t(x,0)=\sin x & -\infty<x\leq+\infty \end{cases}$$

from D'Alembert formula we have

$$u(x,t)=\frac{1}{2}\left[f(x+ct)+f(x-ct)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct}g(s)ds+\frac{1}{2c}\int_{\tau=0}^{\tau=t}\int_{\xi=x-c(t-\tau)}^{\xi=x+c(t-\tau)}F(\xi,\tau)d\xi d\tau$$

so

$$\begin{aligned}u(x,t) &= \frac{1}{2}\left[f(x+3t)+f(x-3t)\right]+\frac{1}{6}\int_{x-3t}^{x+3t}\sin(s)ds\\ &+\frac{1}{2c}\int_{\tau=0}^{\tau=t}\int_{\xi=x-3(t-\tau)}^{\xi=x-3(t-\tau)}\left(e^{\xi}-e^{-\xi}\right)d\xi d\tau\end{aligned}$$

$$u(x,t)=x+\frac{1}{3}\sin x\sin 3t-\frac{2}{9}\sinh x+\frac{2}{9}\sinh x\cosh 3t$$

2.6 Wave equations in a bounded domain[4]

Now we consider the 1D wave equations posed on the space domain $[0,1]$:

$$\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}=0,\quad t>0,\quad x\in\left]0,1\right[\tag{2.11}$$

with initial conditions

$$u\left(x,0\right)=f\left(x\right),\qquad \frac{\partial u}{\partial t}\left(x,0\right)=g\left(x\right),\qquad x\in\left]0,1\right[\tag{2.12}$$

for example, the homogeneous DIRICHLET boundary conditions

$$u\left(0,t\right)=u\left(1,t\right)=0,\quad t>0 \tag{2.13}$$

2.6.1 Separation of variables and FOURIER series

The separation of variables consists in finding solutions to (2.11) (2.12) (2.13) that take the form

$$u\left(x,t\right)=T\left(t\right)X\left(x\right),\quad t>0,\quad x\in[0,1]$$

By formally injecting this formula into the wave equations (2.11), we then obtain

$$T''(t)\,X(x) = c^2 T(t)\,X''(x)$$

necessarily, then we have

$$X''(x) = \lambda X(x)\,, \qquad x \in [0,1]$$

$$T''(t) = \lambda c^2 T(t)\,, \quad t > 0$$

where λ is a real so far undetermined.

The solutions of the first equation take the form

$$u(x) = \alpha e^{\sqrt{\lambda x}} + \beta e^{-\sqrt{\lambda x}}$$

The DIRICHLET conditions reported on u then impose the conditions

$$u(0) = u(1) = 0$$

which requires having

$$\alpha = -\beta \;\text{ and }\; e^{\sqrt{\lambda}} = e^{-\sqrt{\lambda}}$$

Therefore, it appears that no solution can be found if $\lambda < 0$ and that the only suitable values for $\lambda \geq 0$ are the values for which $\sqrt{\lambda} \in i\pi\mathbb{Z}$.

either again

$$\lambda = -(n\pi)^2 \quad \text{for} \;\; n \in \mathbb{N}$$

Then the solution is a multiple of

$$X(x) = \sin(n\pi x)$$

o which corresponds a time factor oscillating of the form

$$T(t) = a_n \cos(n\pi ct) + b_n \sin(n\pi ct)$$

Any finite combination of these functions is reciprocally a solution of (2.11) (2.13).

More generally, if the initial conditions f and g admit FOURIER series expansions of the form

$$f(x) = \sum_{n\in\mathbb{N}} a_n \sin(n\pi x),$$

$$g(x) = \sum_{n\in\mathbb{N}} (n\pi c)\, b_n \sin(n\pi x).$$

then the solution to the DIRICHLET problem is given by :

$$u(x,t) = \sum_{n\in\mathbb{N}} (a_n \cos(n\pi ct) + b_n \sin(n\pi ct)) \sin(n\pi x) \tag{2.14}$$

provided that this series converges and that we can derive two variables under the sum sign with respect to each of the variables.

Which is the case if f and g are sufficiently regular.

A sufficient condition for the series (2.14) to be a solution of class $\mathbb{C}^2([0,1]\times\mathbb{R}^+)$ is for example to have $f \in \mathbb{C}^4([0,1])$, $g \in \mathbb{C}^3([0,1])$ and the compatibility conditions

$$f(0) = f(1) = f''(0) = f''(1) = g(0) = g(1) = 0$$

Example 15 [8] *We consider a rope of length L maintained at each end. The rope is pinched and released at $t = 0$ in such a way that its initial speed is zero. The place $x = a$ where we clamp the rope to a large influence on the harmonics present (and consequently on the sound). It is considered that at the initial instant, the position of the string is as follows (the string is pinched in the middle over a height h and has a triangular shape represented by the function $f(x)$) .*

The laws of physics lead us to solve the following partial differential equation where $y(x,t)$ is the unknown function (position of the string at each point and at each instant) and v is the propagation velocity in m/s which depends on the physical characteristics of the string.

$$\begin{aligned} \textit{Equation to be solved}\,(E) &: \quad \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} \\ \textit{Field of Study}\ (\Omega) &: \quad 0 \le x \le L \end{aligned}$$

$$\textit{Border conditions } (F) \begin{cases} y(0;t) = 0 \\ y(L;t) = 0 \end{cases} \quad (\textit{rope fixed at the ends})$$

$$\textit{Initial conditions } (I) : \begin{cases} y(x; t=0) = f(x) \;\; (\textit{initial position of the rope}) \\ \frac{\partial y}{\partial t}(x; t=0) = 0 \;\; (\textit{zero initial speed}) \end{cases}$$

The function $f(x)$ *is written :*

$$f(x) = \begin{cases} \frac{2h}{L} x \quad \textit{for} \quad 0 \leq x \leq L/2 \\ \frac{2h}{L}(L - x) \quad \textit{for} \quad L/2 \leq x \leq L \end{cases}$$

We are looking for a solution of the form (we separate the variables x *and* t*) :*

$$y(x,t) = X(x)\,T(t)$$

If we replace this expression in equation (E)*, we obtain :*

$$\frac{1}{v^2} X T'' = X'' T$$

We divide by the product XT

$$\frac{1}{v^2}\frac{T''}{T} = \frac{X''}{X} = -\lambda \quad (\lambda \textit{ is a constant})$$

Again, the variables are separable because they are on either side of the equality.

We are then led to solve the following problems :

♦*1 eigenvalue problem (Sturm-Liouville) on the variable* x *(because the homogeneous boundary conditions are found on this variable) :*

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(L) = 0 \end{cases}$$

where it is necessary to search for all λ_n *eigenvalues and eigenfunctions* $X_n(x)$*.*

♦ *1 differential equation of the* 1*st order on the variable* t *:*

$$T''(t) + \lambda_n v^2 T(t) = 0$$

a) Solving the eigenvalue problem:

The eigenvalues and eigenfunctions are as follows :

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \text{ with } n = 1, 2, 3, ...\infty$$

$$X_n(x) = A_n \sin\left(\frac{n\pi}{L}x\right)$$

b) Solving the differential equation on the variable t:

$$T''(t) + \left(\frac{n\pi v}{L}\right)^2 T(t) = 0$$

The general solution of this differential equation is written :

$$T(t) = C_n \cos\left(\frac{n\pi v}{L}t\right) + D_n \sin\left(\frac{n\pi v}{L}t\right)$$

where C_n and D_n are two arbitrary constants.

c) General solution of the PDE:

The general solution of the partial differential equation is the superposition of the set of solutions (we must consider all the eigenvalues) :

$$y(x,t) = \sum_{n=1}^{\infty} X_n(x)\, T_n(t)$$

By associating the coefficients A_nC_n and B_nD_n with each other, we obtain the general solution of the PDE :

$$y(x,t) = \sum_{n=1}^{\infty} \left\{ A_n \cos\left(\frac{n\pi v}{L}t\right) + B_n \sin\left(\frac{n\pi v}{L}t\right) \right\} \sin\left(\frac{n\pi}{L}x\right)$$

d) Special solution :

The general solution must verify the initial conditions :

$$y(x; t=0) = f(x) \text{ and } \frac{\partial y}{\partial t}(x; t=0) = 0$$

with

$$f(x) = \begin{cases} \frac{2h}{L}x & \text{for} \quad 0 \leq x \leq L/2 \\ \frac{2h}{L}(L-x) & \text{for} \quad L/2 \leq x \leq L \end{cases}$$

Substituting the two initial conditions into the general solution, we obtain :

$$f\left(x\right)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi}{L}x\right)$$

$$0=\sum_{n=1}^{\infty}\frac{B_nL}{n\pi v}\sin\left(\frac{n\pi}{L}x\right)\ \textit{which imposes}\ B_n=0$$

It therefore remains to calculate the coefficients A_n by decomposing $f(x)$ on the basis of the eigenfunctions :

$$A_n=\frac{2}{L}\int_0^L f\left(x\right)\sin\left(\frac{n\pi}{L}x\right)dx$$

Replacing $f(x)$ with its expression, we obtain:

$$A_n=\frac{2}{L}\int_0^{L/2}\frac{2h}{L}x\sin\left(\frac{n\pi}{L}x\right)dx+\frac{2}{L}\int_{L/2}^{L}\frac{2h}{L}\left(L-x\right)\sin\left(\frac{n\pi}{L}x\right)dx$$

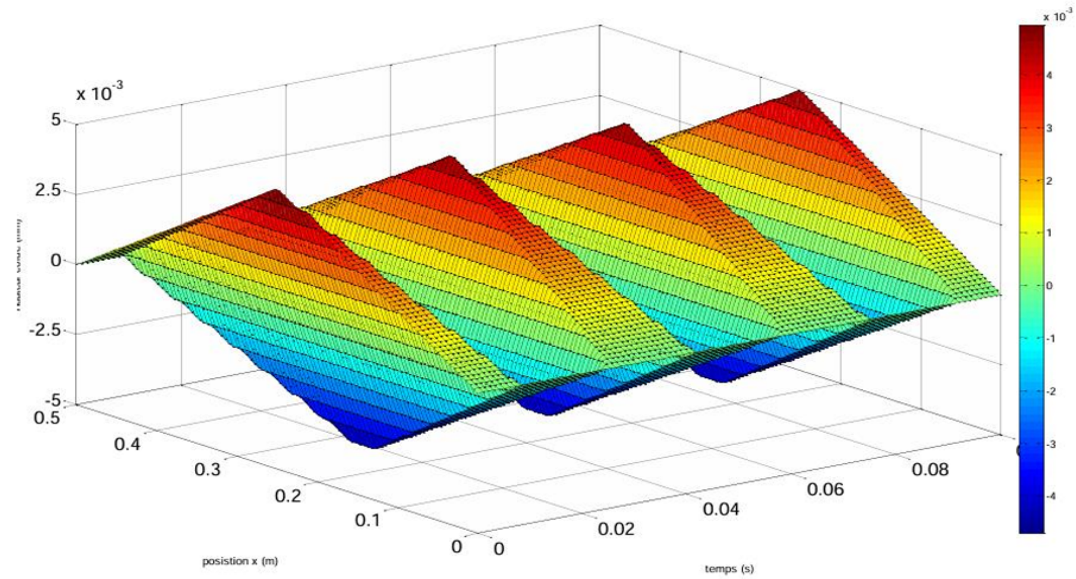
The development of calculations gives us :

$$A_n=\frac{8h}{\left(n\pi\right)^2}\sin\left(\frac{n\pi}{2}\right)$$

The position of the string as a function of x and time is then given by the following function :

$$y\left(x,t\right)=\sum_{n=1}^{\infty}\frac{8h}{\left(n\pi\right)^2}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi v}{L}t\right)\sin\left(\frac{n\pi}{L}x\right)$$

The variations of the position of the rope as a function of x and t are given in the following figure (for $h=5mm, L=0.5m$ and $v=30m/s$).



Evolution of the height of the string as a function of t and x.

2.7 Higher-dimensional wave equation[9]

In this section, let's consider the wave equation in dimension $n \geq 2$,

$$u_{tt} = c^2 \Delta u; \qquad x \in \mathbb{R}^n, t > 0$$

with initial conditions

$$u(x, 0) = f(x), \qquad u_t(x, 0) = g(x); \quad x \in \mathbb{R}^n.$$

Using spherical averaging, we will transform the previous equation given in dimension $n \geq 2$ into an equation in dimension 1.

Let's start with the following definition.

Definition 16 *For a function h continuous on $\mathbb{R}^n$, we define its spherical mean on the sphere of radius r and center x by*

$$M_h(x,r) = \frac{1}{\omega_n}\int_{\partial B(0,1)} h(x+rz)\,dS(z) = \frac{1}{\omega_n}\int_{|z|=1} h(x+rz)\,dS(z)$$

where ω_n is the area of the unit sphere

$$\partial B(0,1) = S^{n-1} = \{z \in \mathbb{R}^n; \mid z \mid = 1\}$$

The function M_h verifies some important properties grouped in the following proposition

Proposition 17 *For a given function $h = h(x) \in C^k(\mathbb{R}^n)\,, k \geq 2$ the function $v : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ defined by $v(x,r) = M_h(x,r)$ is of class C^k on $\mathbb{R}^n \times \mathbb{R}$.*

Moreover, it verifies the following properties:

1.$v(x,0) = h(x)$ for all $x \in \mathbb{R}^n$.

2.$v(x,-r) = v(x,r)$ for all $x \in \mathbb{R}^n, r \in \mathbb{R}$.

3.$\frac{\partial}{\partial r} v(x,0) = 0$ for all $x \in \mathbb{R}^n$.

4.$\frac{\partial^2}{\partial r^2} v(x,r) + \frac{n-1}{r}\frac{\partial^2}{\partial r^2} v(x,r) = \Delta_x v(x,r)$ on $\mathbb{R}^n \times \mathbb{R}^*$.

2.8 Kirchhoff formula:

Either the problem

$$(p_n)\begin{cases} u_{tt} = c^2\Delta u; \qquad x \in \mathbb{R}^n, t > 0 \\ u(x,0) = f(x)\,, \qquad u_t(x,0) = g(x)\,; \quad x \in \mathbb{R}^n. \end{cases}$$

We will solve the problem (p_n) for $n = 3$, in other words we will determine the form of the solution to the problem(p_3).

Using spherical averages, we prove the following theorem.

Theorem 18 *Let $f \in C^3(\mathbb{R}^3)$ and $g \in C^2(\mathbb{R}^3)$. Then the problem (p_3) has a unique solution $u(x,t)$, defined by Kirchhoff's formula*

$$\begin{aligned} u(x,t) &= tG(x,ct) + \frac{\partial}{\partial t}(tF(x,ct)) \\ &= \frac{1}{4\pi c^2 t^2}\int_{|y-ct|=ct}[tg(y) + f(y) + \nabla f(y)(y-x)]\,dS(y) \end{aligned}$$

where $F(x,r)$ and $G(x,r)$ are the spherical means of f and g respectively.

Proof. According to Darboux's formula, since $G(x,r)$ is the spherical mean, then for $n=3$ we have

$$\frac{\partial^2}{\partial r^2}G(x,r) + \frac{2}{r}\frac{\partial}{\partial r}G(x,r) = \Delta_x G(x,r), \quad (x,r) \in \mathbb{R}^n \times \mathbb{R}^*.$$

Multiply both members by r we get

$$rG_{rr}(x,r) + 2G_r(x,r) = \Delta_x(rG(x,r))$$

Write this equation in the form

$$\frac{\partial^2}{\partial r^2}(rG(x,r)) = \Delta_x(rG(x,r))$$

and put

$$\bar{G}(x,t) = tG(x,ct) = \frac{t}{4\pi}\int_{|z|=1} g(x+ctz)\,dS(z).$$

then the function $\bar{G}$ satisfies the wave equation

$$\bar{G}_{tt}(x,t) = c^2\Delta\bar{G}(x,t)\,; x \in \mathbb{R}^3, t > 0.$$

with the initial conditions

$$\bar{G}(x,0) = 0,\ \bar{G}_t(x,0) = g(x) : x \in \mathbb{R}^3.$$

Repeating the same calculations for the spherical mean $F(x,r)$ of $f(x)$ and posing

$$\bar{F}(x,t) = tF(x,ct) = \frac{t}{4\pi}\int_{|z|=1} f(x+ctz)\,dS(z)$$

we can easily check that

$$\bar{F}_{tt}(x,t) = c^2 \Delta \bar{F}(x,t)\,; x \in \mathbb{R}^3, t > 0.$$

As an exercise, we show that since $\bar{F}$ is of class C^3 on $\mathbb{R}^3 \times]0,+\infty[$and satisfies the wave equation, then the function $\bar{F}_t$ also satisfies this equation.

Furthermore, we have

$$\begin{aligned}
\bar{F}_t(x,t) &= \frac{\partial}{\partial t}\bar{F}(x,t) = \frac{\partial}{\partial t}\left(tF(x,ct)\right) = F(x,ct) + t\frac{\partial}{\partial t}F(x,ct) \\
\bar{F}_t(x,t) &= \frac{1}{4\pi}\int_{|z|=1} f(x+ctz)\,dS(z) + \frac{t}{4\pi}\frac{\partial}{\partial t}\left(\int_{|z|=1} f(x+ctz)\,dS(z)\right) \\
&= \frac{1}{4\pi}\int_{|z|=1} f(x+ctz)\,dS(z) + \frac{t}{4\pi}c\left(\int_{|z|=1} \nabla f(x+ctz)\,z dS(z)\right)
\end{aligned}$$

that means

$$\frac{\partial}{\partial t}\bar{F}_t(x,t) = \frac{1}{4\pi}\int_{|z|=1} f(x+ctz)\,dS(z) + \frac{t}{4\pi}\int_{|z|=1}\left[f(x+ctz) + ct\nabla f(x+ctz)\,z\right]dS(z)$$

we have

$$\bar{F}_t(x,0) = \frac{1}{4\pi}\int_{|z|=1} f(x)\,dS(z) = \frac{f(x)}{4\pi} = f(x)\,,$$

and

$$\frac{\partial}{\partial t}\bar{F}_t(x,0) = \frac{\partial^2}{\partial t^2}\bar{F}(x,0) = c^2\Delta\bar{F}(x,0) = 0$$

We have thus obtained two solutions to the equation in the problem(p_3).

The first is $\bar{G}(x,t)$ satisfying

$$\bar{G}(x,0) = 0, \bar{G}_t(x,0) = g(x)\,;\ x \in \mathbb{R}^3$$

and the second solution is $\bar{F}(x,t)$, with

$$\bar{F}_t(x,0) = f(x)\,, \qquad \bar{F}_{tt}(x,0) = \frac{\partial}{\partial t}\bar{F}_t(x,0) = 0;\ x \in \mathbb{R}^3.$$

Now, since the problem (p_3) is linear, its solution can be written as

$$u(x,t) = \bar{F}_t(x,t) + \bar{G}(x,t)\,.$$

and note that for all $x \in \mathbb{R}^3$

$$\begin{cases} u(x,0) = \bar{F}_t(x,0) + \bar{G}(x,0) = f(x) \\ u_t(x,0) = \bar{F}_{tt}(x,0) + \bar{G}_t(x,0) = g(x) \end{cases}$$

Finally, let's express $u(x,t)$ as a function of $f(x)$ and $g(x)$. We have

$$u(x,t) = \bar{F}_t(x,t) + \bar{G}(x,t)$$

By replacing each term by its expression we obtain

$$u(x,t) = \frac{1}{4\pi}\int_{|z|=1} [f(x+ctz) + ct\nabla f(x+ctz)\, z]\, dS(z) + \frac{t}{4\pi}\int_{|z|=1} g(x+ctz)\, dS(z)$$

let $y = x + ctz$, so $dS(y) = c^2t^2 dS(z)$ and $|z| = 1 \implies |y-x| = ct$.

With this change

$$u(x,t) = \frac{1}{4\pi c^2t^2}\int_{|y-x|=ct} [f(y) + \nabla f(y)(y-x)]\, dS(y) + \frac{t}{4\pi c^2t^2}\int_{|y-x|=ct} g(y)(y-x)\, dS(y).$$

and we arrive at Kirchhoff's formula

$$u(x,t) = \frac{1}{4\pi c^2t^2}\int_{|y-x|=ct} [tg(y) + f(y) + \nabla f(y)(y-x)]\, dS(y).$$

■

2.9 Energy and Unicity

Definition 19 *let u be a solution of $\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$ in the interval I of $\mathbb{R}\,(x \in I)$. The energy of u at time t is the quantity*

$$E(t) = \frac{1}{2}\int_I \left[\left(\frac{\partial u}{\partial t}\right)^2 + c^2\left(\frac{\partial u}{\partial x}\right)^2\right] dx.$$

Proposition 20 *Let* $I = [a, b]$ *then* $\frac{dE}{dt} = c^2 \left[\frac{\partial u}{\partial x}(b,t)\frac{\partial u}{\partial t}(b,t) - \frac{\partial u}{\partial x}(a,t)\frac{\partial u}{\partial t}(a,t)\right]$

$$\text{in fact } \frac{dE}{dt} = \int_a^b \left(\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial t \partial x}\right) dx$$

$$\text{where } c^2 \int_a^b \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial t \partial x} dx = \left[c^2 \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right]_{x=a}^{x=b} - c^2 \int_a^b \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2} dx$$

as $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$ *only the term in square brackets remains.*

We can deduce the following theorem

Theorem 21 *let* u_1 *and* u_2 *be two solutions to the following problem:*

$$(E) \quad \begin{cases} \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \forall x \in [a, b] \\ u(x, 0) = f(x), \quad \frac{\partial u}{\partial t}(x, 0) = g(x) \quad \forall x \in [a, b] \\ u(a, t) = \alpha(t), \quad u(b, t) = \beta(t), \quad \forall t \geq 0 \end{cases}$$

$$\text{then } u_1(x, t) = u_2(x, t) \quad \forall x \in [a, b], \ \forall t \geq 0$$

Proof. $u_1 - u_2$ is a solution of (E), $u_1(a, t) - u_2(a, t) = \alpha(t) - \alpha(t) = 0 \ \ \forall t \geq 0$ so

$$\frac{d}{dt}(u_1 - u_2)(a, t) = 0 \ \ \forall t$$

in the same way

$$\frac{d}{dt}(u_1 - u_2)(b, t) = 0 \ \forall t$$

and according to the previous proposition $\frac{dE}{dt} = 0$: the energy associated with the function $u_1 - u_2$, is therefore a constant

$$\text{calculate } E(0) = \frac{1}{2}\int_a^b \left[\frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial t}\right]^2 (x, 0) + c^2 \left[\frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial x}\right]^2 (x, 0)\, dx;$$

$$\text{where } \frac{\partial u_1}{\partial t}(x, 0) = g(x) \text{ and } \frac{\partial u_2}{\partial t}(x, 0) = g(x), \quad \forall x$$

$$\text{in addition } u_1(x, 0) = f(x) = u_2(x, 0), \quad \forall x \text{ so } E(0) = 0.$$

$$\text{then } 0 = 2E\left(t\right) = \frac{1}{2}\int_a^b \left[\frac{\partial u_1}{\partial t}\left(x,t\right) - \frac{\partial u_2}{\partial t}\left(x,t\right)\right]^2 + c^2\left[\frac{\partial u_1}{\partial x}\left(x,t\right) - \frac{\partial u_2}{\partial x}\left(x,t\right)\right]^2 dx,$$

Since the functions $\frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial x}$ are continuous, this constant is zero due to the initial conditions.

This method can be generalised to other situations and other equations. Thus replacing $\frac{\partial^2 u}{\partial x^2} = \Delta u$ by $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$we obtain in dimension 3 that for a solution of $\frac{\partial^2 u}{\partial t^2} - c^2\Delta u = 0$ on a closed domain Ω of boundary $\partial\Omega$ and for a boundary condition $u\left(M\right) = f\left(M\right)$ independent of t that $E\left(t\right)$ is constant, this allows us to conclude uniqueness.

The previous method also establishes the uniqueness of the problem where (E) is replaced by the equation with second member $\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = h\left(x,t\right)$, in fact if and u_1 and u_2 are two solutions $u_1\left(x,t\right) - u_2\left(x,t\right)$ is the solution to the problem without a second member.

the method can also be used, for example, for the telegrapher's equation. This can be transformed into:

$$a\frac{\partial^2 u}{\partial t^2} = b\frac{\partial^2 u}{\partial x^2} - k^2 u + cg\left(x,t\right)$$

The energy of a solution u is defined by:

$$\frac{1}{2}\int_a^b \left[a\left(\frac{\partial u}{\partial t}\right)^2 + b\left(\frac{\partial u}{\partial x}\right)^2 + k^2u^2\right]\left(x,t\right)dx$$

and we can reason as before to establish the uniqueness of solutions to a problem whose initial and boundary conditions are determined. ■

Chapter 3

Numerical solution of the wave equation

In this chapter, we have numerically studied the wave equation in dimension one in a homogeneous medium. In the homogeneous case, we apply a finite difference numerical scheme to a vibrating string problem. We have studied the consistency, stability and convergence properties of this method.

3.1 The finite difference method[10] [11] [12] [13] [14]

3.1.1 Principle of the method

The finite difference method consists of approximating the derivatives of the equations of physics by means of Taylor's developments and can be deduced directly from the definition of the derivative. It is due to the work of several 18th century mathematicians (Euler, Taylor, Leibniz, etc.).

Advantages :

• Simple to write and low cost of calculation.

- It is widely used and easy to access.

3.1.2 Definitions and properties

Definition 22 *The finite difference method is a common technique for finding approximate solutions to partial differential equations. It involves solving a system of relations (numerical diagram) linking the values of the unknown functions at certain points sufficiently close to each other. This method appears to be the simplest to implement because it involves two stages: firstly, discretisation by finite differences of the derivation operators, and secondly, convergence of the numerical scheme thus obtained convergence of the resulting numerical scheme as the distance between the points decreases.*

Mesh

Since we mentioned the word mesh in the previous paragraph and we'll need it all the time, let's define it here. For an application defined on a segment of $\mathbb{R}$, we will generally add the two extremities of the segment; for a higher-dimensional mesh, we may have to choose points on the contour of the definition domain.

The mesh pitch is the distance between two successive adjacent mesh points.

*Let's consider a one-dimensional case where we want to determine a quantity $u(x)$

on the interval $[0, 1]$. The search for a discretized solution of the quantity u leads to the creation of a mesh of the interval of definition.

We consider a mesh (or calculation grid) made up of $N + 1$ points x_i for $i = 0, ..., N$ regularly spaced with a step Δx. The points $x_i = i\Delta x$ are called the nodes of the mesh. The initial continuous problem of determining a quantity on an infinite-dimensional set thus reduces to the search for N discrete values of this quantity at the various nodes of the mesh.

Truncation error

The quantity obtained by replacing the derivatives by the divided differences is known as the truncation error. The calculation of truncation errors is usually based on Taylor expansions.

The truncation error is an error that indicates how the equation is approximated by the diagram. It is not an error between the exact solution and the approximate solution (convergence error), but it is an error that quantifies to what order the exact solution verifies the scheme.

Schematic order

A discretization scheme with N discretization points is said to be of order p if there exists $C \in \mathbb{R}$, depending only on the exact solution, such that the consistency error satisfies :

$$\max_{i=1,\ldots,N} (R_i) \leq Ch^p$$

Where R_i is the truncation error and h is the mesh pitch (i.e. the maximum of $x_{i+1} - x_i$).

Digital schematic consistency

A finite difference scheme is consistent if :

$$\max_{i=1,\ldots,N} (R_i) \to 0$$

When $h \to 0$, where N is the number of discretization points.

Numerical schema stability

By definition, a numerical scheme is stable if the errors (rounding, truncation, etc.) cannot increase during the numerical procedure from one time step to the next:

A scheme can be :

- Unconditionally unstable: Whatever Δt and Δx the errors increase with each iteration.This causes completely wrong results.

-Unconditionally stable: Whatever Δt, Δx the errors caused by the numerical scheme do not explode over iterations.

- Conditionally stable: We must set a condition on Δt and Δx so that the solution does not explode, for example the Courant-Friedrichs-Lewy condition, which we will define in the next paragraph.

Interpretation of the Courant-Friedrichs-Lewy condition

1928 (CFL)

This is a necessary and sufficient condition for the stability of the scheme, showing that the stability of a scheme cannot necessarily be ensured no matter how Δt and Δx tend to zero. It is a very constraining condition from a practical point of view, since if we wish to have good accuracy in space, it is necessary to choose Δx small which imposes a choice of Δt significantly smaller.

Convergence

In general:(Consistency + Stability) $\Longrightarrow$ Convergence.

Theorem 23 [7] ***(Lax)*** *In a well-posed problem, with a consistent numerical scheme, stability is a necessary and sufficient condition for convergence.*

3.1.3 Finite differences in one dimension

All numerical methods presuppose discretisations of the geometrical domain in order to move from a continuous problem with an infinite number of unknowns to a discrete problem with a finite number of unknowns.

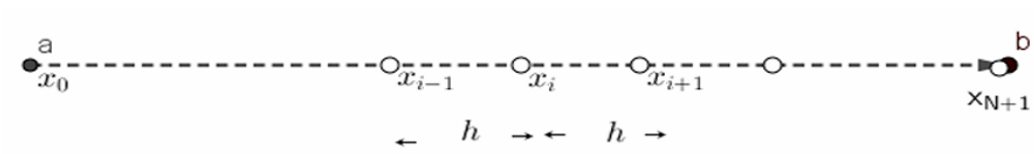
Domain discretisation

In this case, we discretise the continuous interval $[a, b]$ and approach the derivatives at the discretisation points using difference operators. More precisely, we give ourselves an integer N from which we define the discretisation step

$$h = \Delta x = x_i - x_{i-1} = x_{i+1} - x_i$$

We introduce the $N+1$ points $x_i = ih$ for $i = 0, ..., N$ which then form a regular subdivision of the interval$[a, b]$ (uniformly).

The continuous problem is thus replaced by the problem of finding approximate values ui of the solutions $u(x_i)$ at the points x_i of the discretisation.



Discretization by F.D in 1D

3.1.4 Diagram construction

the first derivative

let u be a function of one variable of class $\mathbb{C}^2$

If h tends towards 0, using the Taylor limit expansion of the function u up to order 2 at the point $(x_i + h)$,$(x_i - h)$ we obtain :

$$u_{i+1} = u\left(x_i + \Delta x\right) = u_i + \Delta x \left(\frac{\partial u}{\partial x}\right) i + O\left(\Delta x^2\right) \tag{3.1}$$

$$u_{i-1} = u\left(x_i - \Delta x\right) = u_i - \Delta x \left(\frac{\partial u}{\partial x}\right) i + O\left(\Delta x^2\right) \tag{3.2}$$

indicial case D1 :

The diagram with the 1st order F.D. 'forward' or 'off-centre forward' or 'off-centre right':

$$\left(\frac{\partial u}{\partial x}\right) i = \frac{u_{i+1} - u_i}{\Delta x} + O\left(\Delta x\right) \tag{3.3}$$

The diagram with the 1st order RFs 'backwards' or 'off-centre left'

$$\left(\frac{\partial u}{\partial x}\right) i = \frac{u_i - u_{i-1}}{\Delta x} + O\left(\Delta x\right) \tag{3.4}$$

notation $u_i = u\left(x_i\right)$ and $\left(\frac{\partial u}{\partial x}\right)_{x=x_i} = \left(\frac{\partial u}{\partial x}\right) i = u'_i$

Centred scheme :

First carry out a Taylor expansion in the vicinity of x_i to order 3.

$$u_{i+1} = u\left(x_i + \Delta x\right) = u_i + \Delta x \left(\frac{\partial u}{\partial x}\right) i + \frac{\Delta x^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right) i + O\left(\Delta x^3\right) \tag{3.5}$$

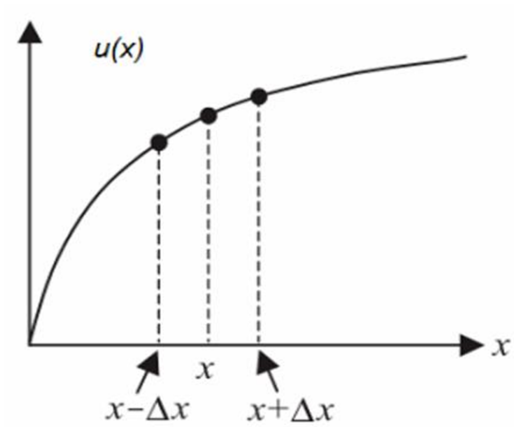
$$u_{i-1} = u\left(x_i - \Delta x\right) = u_i - \Delta x \left(\frac{\partial u}{\partial x}\right) i + \frac{\Delta x^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right) i + O\left(\Delta x^3\right) \tag{3.6}$$

Subtracting these two relationships gives :

$$u_{i+1} - u_{i-1} = 2\Delta x \left(\frac{\partial u}{\partial x}\right) i + O\left(\Delta x^3\right) \tag{3.7}$$

This gives us the second-order 'centred' scheme for approximating the first derivative of u

$$\left(\frac{\partial u}{\partial x}\right) i = \frac{u_{i+1} - u_{i-1}}{2\Delta x} + O\left(\Delta x^2\right) \tag{3.8}$$



Approximation of F.D

Remark 24 *To obtain higher orders, it is necessary to use several neighbouring nodes of* x_i*.The number of points required to write the diagram is called a stencil.*

The second derivative

First performing a Taylor expansion in the vicinity of xi to order 4.Adding the two equalities gives :

$$u_{i+1} = u\left(x_i + \Delta x\right) = u_i + \Delta x \left(\frac{\partial u}{\partial x}\right) i + \frac{\Delta x^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right) i + \frac{\Delta x^3}{6}\left(\frac{\partial^3 u}{\partial x^3}\right) i + O\left(\Delta x^4\right) \quad (3.9)$$

$$u_{i+1} = u\left(x_i - \Delta x\right) = u_i - \Delta x \left(\frac{\partial u}{\partial x}\right) i + \frac{\Delta x^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right) i - \frac{\Delta x^3}{6}\left(\frac{\partial^3 u}{\partial x^3}\right) i + O\left(\Delta x^4\right) \quad (3.10)$$

Adding the two equalities gives :

$$u_{i+1} - u_{i-1} - 2u_i = \Delta x^2 \left(\frac{\partial^2 u}{\partial x^2}\right) i + O\left(\Delta x^4\right) \quad (3.11)$$

This leads to the so-called ‘centred’ second-order scheme for approximating the second derivative of u :

$$\left(\frac{\partial^2 u}{\partial x^2}\right) i = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + O\left(\Delta x^2\right) \quad (3.12)$$

3.1.5 Finite differences in dimension two

Domain discretization

Uniform discretization in time and space is used to reduce to a finite-dimensional system of equations.to a finite-dimensional system of equations. It is desirable for the validity of the calculation that the approximate solution obtained by solving this system. Discretization therefore involves giving a set of points $t^n, n = 0, ..., M$ in the interval $[0, T]$, and a set of points $x_i, i = 0, ..., N$ in the interval $[a, b]$.

For simplicity, we consider a constant step in time and space. Let $h = \frac{b-a}{N} = \Delta x$,be the discretization step in space, and $k = \frac{T}{M} = \Delta t$ the time step. We pose $t^n = nk$ for $n = 0, ..., M$ and $x_i = ih$ for $i = 0, ..., N$.

In the case of $D2$ $u(x,t)$ is decomposed into $N \times P$ pairs of points (x_i, t^n) form a space-time grid are called grid nodes.

u_i^n is the discrete value of the quantity $u(x,t)$ at the nodes (x_i, t^n).

3.2 Solving the wave equation by F.D.[14]

3.2.1 Problem continues

In an unbounded domain, this problem consists of searching for :

$u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$,checked:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(x,t) - c^2 \frac{\partial^2 u}{\partial x^2}(x,t) = f & x \in \mathbb{R}, t > 0 \\ u(x,0) = u_0(x) & \text{initial coditions} \end{cases}$$

The parameter c (wave speed) is assumed to be positive.

If the domain is bounded in time and space :

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(x,t) - c^2 \frac{\partial^2 u}{\partial x^2}(x,t) = f \quad x \in [0,L] \ ,t \in [0,T] \\ u(x,0) = u_0(x) \text{ and } \frac{\partial u}{\partial t}(x,0) = v^0 \text{ initial coditions,} \\ u(0,t) = u(L,t) = 0 \text{ boundary conditions.} \end{cases}$$

3.2.2 First approach: discrete problem

Discretization

A first method for solving this evolution problem numerically is to discretize the second-order equation by finite differences. For simplification, let's consider the one-dimensional case of a string of length L. We choose a regular discretization of $[0, L]$ into intervals of length Δx such that $L = M\Delta x$ and a discretization of the time interval $[0, T]$ into time steps of length Δt such that $T = N\Delta t$. Let x_i be the point $i\Delta x$ and t^n the time $n\Delta t$. Let u_i^n be the value of the approximate solution at point x_i and time t^n.

The diagram is explicit in terms of time and centred in space.

Taking $f = 0$ then :

$$\begin{cases} \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} = c^2 \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \\ u_i^0 = u^0(x_i) \text{ and } u_i^1 + \Delta t v_i^0 \text{ such that } i = 1, ..., M-1 \text{ initial conditions} \\ u_i^n = u_M^n = 0 \ \ \forall n = 1, ..., N \text{ homogeneous Dirichlet boundary conditions} \end{cases} \tag{3.13}$$

This scheme is explicit because it gives an explicit formula for calculating the solution at time t^{n+1} as a function of the values of the solution at the previous time. There is no equation to solve to obtain the value at the new time t^{n+1}.

It's an explicit diagram, so :

$$u_i^{n+1} = 2\left(1 - \left(\frac{c\Delta t}{\Delta x}\right)^2\right) u_i^n + \left(\frac{c\Delta t}{\Delta x}\right)^2 \left(u_{i+1}^n + u_{i-1}^n\right) - u_i^{n+1}$$

-Truncation error:

we have $E_i^n = (E^n + E_i)$ such that E_i is the space error and E^n is the time error

$$\mid E_i^n \mid \leq c\left(\Delta t^2 + \Delta x^2\right)$$

Proof. By Taylor's limited expansion for the exact solution u, assumed to be sufficiently regular, we find with respect to the time variable (x_i being fixed) we obtain

$$u\left(x_i, t^{n+1}\right) = u\left(x_i, t^n\right) + \Delta t \frac{\partial u}{\partial t}\left(x_i, t^n\right) + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2}\left(x_i, t^n\right)$$

$$+\frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3}\left(x_i, t^n\right) + \frac{\Delta t^4}{24}\frac{\partial^4 u}{\partial t^4}\left(x_i, \xi^+\right), \xi^+ \in \left[t^n, t^{n+1}\right]$$

$$u\left(x_i, t^{n-1}\right) = u\left(x_i, t^n\right) - \Delta t \frac{\partial u}{\partial t}\left(x_i, t^n\right) + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2}\left(x_i, t^n\right)$$

$$-\frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3}\left(x_i, t^n\right) + \frac{\Delta t^4}{24}\frac{\partial^4 u}{\partial t^4}\left(x_i, \xi^-\right), \xi^- \in \left[t^{n-1}, t^n\right]$$

$$\Longrightarrow \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} = \frac{\partial^2 u}{\partial t^2}\left(x_i, t^n\right) + \frac{\Delta t^2}{12}\frac{\partial^4 u}{\partial t^4}\left(u\left(x_i, \xi^+\right), u\left(x_i, \xi^-\right)\right) \tag{3.14}$$

■

In the same way for space, we get :

$$\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} = \frac{\partial^2 u}{\partial x^2}\left(x_i, t^n\right) + \frac{\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4} u\left(\xi_+, t^n\right), u\left(\left(\xi_-, t^n\right)\right)$$

Since u is a solution, the first terms disappear, leaving a truncation error

$$\mid E_i^n \mid = \mid E^n + E_i \mid = \mid \frac{\Delta t^2}{12} M + \frac{\Delta x^2}{12} M' \mid \leq \frac{\Delta t^2}{12} \mid M \mid + \frac{\Delta x^2}{12} \mid M' \mid$$

$$\mid E_i^n \mid \leq c\left(\Delta t^2 + \Delta x^2\right)$$

and $c = \frac{1}{12}\max\left(M, M'\right)$ such that

$$M = \sup \mid \frac{\partial^4 u}{\partial t^4}\left(u\left(x_i, \xi^+\right), u\left(x_i, \xi^-\right)\right) \mid$$

$$M' = \sup \mid \frac{\partial^4 u}{\partial x^4} u\left(\xi_+, t^n\right), u\left(\left(\xi_-, t^n\right)\right) \mid$$

- Consistency:

The scheme is consistent to order 2 in time and space, and consistent for the truncation error tends to 0 when Δt and Δx tend to 0 .

- Order:

This explicit scheme is of order 2 in time and space such that :

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(x_i, t^n) &= \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} + O\left(\Delta t^2\right) \\ \frac{\partial^2 u}{\partial x^2}(x_i, t^n) &= \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} + O\left(\Delta x^2\right) \end{aligned}$$

- Stability :

In the case of numerical schemes applied to hyperbolic problems, we will choose, as a stability condition, to impose on the vector of approximate solutions to be preserved or to decrease in norm over time.

$$u_i^{n+1} = 2\left(1 - \left(\frac{c\Delta t}{\Delta x}\right)^2\right) u_i^n + \left(\frac{c\Delta t}{\Delta x}\right)^2 \left(u_{i+1}^n + u_{i-1}^n\right) - u_i^{n-1}$$

To get the convex combination :

$$2\left(1 - \left(\frac{c\Delta t}{\Delta x}\right)^2\right) + \left(\frac{c\Delta t}{\Delta x}\right)^2 + \left(\frac{c\Delta t}{\Delta x}\right)^2 - 1 = 1$$

and

$$0 < \left(\frac{c\Delta t}{\Delta x}\right)^2 < 1 \implies 0 < \frac{c\Delta t}{\Delta x} < 1$$

and

$$0 < 2\left(1 - \left(\frac{c\Delta t}{\Delta x}\right)^2\right) < 1$$

for inequality :

$$2\left(1 - \left(\frac{c\Delta t}{\Delta x}\right)^2\right) > 0$$

is verified for $\left(\frac{c\Delta t}{\Delta x}\right)^2 < 1$.

We are left with the second inequality, i.e. :

$$2\left(1-\left(\frac{c\Delta t}{\Delta x}\right)^2\right)<1 \implies \frac{c\Delta t}{\Delta x}<1$$

Therefore: the CFL condition for stability is :

$$\frac{c\Delta t}{\Delta x}<1$$

- Convergence :

According to Lax's theorem, we obtain convergence (consistency and stability imply convergence).

Conclusion 25 *The phenomena of wave propagation are encountered in many applications.There are essentially three types of waves : acoustic waves, that is to say waves that propagate in a fluid, elastic waves, that is to say waves that propagate in a solid, electromagnetic waves. In this memory, and from the properties of the wave equations, we have determined the wave equations in dimension greater than two in particular in dimension one. This equation helped us in the numerical methods used to solve mathematical problems.*

Bibliography

[1] **J. Quint** : Cours Élémentaires de Mathématique Supérieures (4-Equations Différentielles 1977).

[2] **Aude Rondepierre & Adeline Rouchon** : Introduction aux Équations aux Dérivées Partielle 2012-2013.

[3] **B.Helffer** : Cours "Introduction aux EDP Analyse de Fourier" pour Les Étudiants de l'ENS de Bucarest.

[4] **V. Volpert** : Ecole Centrale de Lyon : Dynamique des Systèmes Non Linéaires 2011-2012.

[5] **Abdelkarim KELLECHE**: Cours "Équations de la physique mathématique". Faculté des sciences et de la technologie .Département de mathématiques et informatique.Université Djilali Bounâama.

[6] **Laurent Schwartz**. Méthodes Mathématiques pour les Sciences Physiques. Enseignement des Sciences. Hermann, Paris, 1961.

[7] **H. OUNIS**. Résolution numérique de quelques problèmes paraboliques avec des conditions aux limites non locales non linéaires.Pour l'obtention du diplôme de Master en Mathématiques. Université d'Oum El Bouaghi, 2019.

[8] **Thierry Lubin.** MASTER I2E2I – M1/M2.Spécialité Energie Electrique cours Equations aux dérivées partielles (EDP), Méthode de résolution des EDP par séparation de variables; Applications 2015-2016**.**

[9] **A.Bensedik.** Cours**:**Équations Hyperboliques - Equation des Ondes.Université de Tlemcen.

[10] **Eliane Béchard**, Schéma numérique pour la résolution de l'équation des ondes, Master Simulaion et Modélisation, Janvier 2009.

[11] **Francois Cuvelier**, Méthodes numériques, Notes de cours, Univ Paris XIII.

[12] **Reisser Laurant**, Différences finies pour la résolution numérique des équations de la mécanique des fluides , 4 février 2006.

[13] **Dr Harnane Yamina,** Méthodes numériques I-Méthode des différences finies, Cours,Univ- Larbi Ben M'hidi- Oum Elbouaghi, 2015-2016.

[14] **S.Besseghir & A.Hachani**, thèse de master Schéma aux diérences nies pour l'équation des ondes UNIVERSITÉ Echahid HAMMA LAKHDAR EL OUED.