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Title

Investigation of Relaxation Methods and Iterative Schemes Befitting Matrices of Bounded Linear Operators

Defended on: October 16, 2024, before the jury composed of:

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People's Democratic Republic of Algeria
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A thesis submitted in partial fulfilment of the requirements for the degree of

Doctor of Philosophy in Applied Mathematics

Investigation of Relaxation Methods and Iterative Schemes Befitting Matrices of Bounded Linear Operators

Laboratoire des Mathématiques Appliquées et de Modélisation
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"With Him are the keys of the unseen —no one knows them except Him. And He knows what is in the land and sea. Not even a leaf falls without His knowledge, nor a grain in the darkness of the earth or anything–green or dry–but is written in a perfect Record."

Noble Qur'an, Al-An'am, 59.

Declaration of Authorship

I, Mohammed Ghaïth MAHCENE, member of Laboratoire des Mathém -atiques Appliquées et de Modilisation (LMAM), hereby declare that the present work, *Investigation of Relaxation Methods and Iterative Schemes Befitting Matrices of Bounded Linear Operators*, is the result of my original research during the period of my candidature to acquire the Doctorate of Philosophy in Applied Mathematics degree at the University of Guelma (08 May 1945).

I attest that, except for what has been publicly participated in a scientific article or communicated at national or international conferences and/or events, which have added to the qualifying fulfilments of acquiring the PhD degree aforementioned, never has the work at hand, partially or entirely, been a subject of sharing, submitting, or publishing as a previous academic endeavour by myself nor by any other fellow member from the scientific community.

Unforgettably, I greatly cherish the constructive intervention(s) from any one fellow researcher in the crafting of the present manuscript or the development of the scientific reasoning found within; therefore, I testify that an appropriate amount of care has been regarded to the proper attribution and/or indication of all sorts of external and/or non-personal contributions.

Furthermore, an adequate effort has been dedicated to the scrutiny[1] of references, citations, sources, and/or any other bibliographic material in accordance with the extent of interest they were used in.

1. Except for what has been explained in a footnote, this is especially demonstrated in the digital PDF version of the manuscript, where, occasionally, an expression might appear faintly coloured to infer that it should be clickable referring the reader to external sources.

Incorporation of Artificial Intelligence (AI)

Ultimately, I hereby touch on the human-originality of the present work as a scientific contribution of self-effort(s) and signal its independence from all artificially generated contents ; especially, those by generative artificial intelligence tools (Gen. AIs) like : OpenAI's ChatGPT, or the like, except for the AI services and platforms that have been leveraged to provide impetus and assistance for personal development thus helping to better one's skills on either of scientific and linguistic aspects, and that have as well been used as alternatives to conventional and/or regular methods of online information retrieval.

Date	*Signature*
24/10/2024	Mohammed Ghaith MAHCENE

To mother, and grandmother.

TO ALLAH, THE MOST-GRACIOUS, THE OMNIBENEVOLENT, BE ALL THE GRACES AND VIRTUES. TO HIM, THE OMNIPOTENT, I DECLARE THE EVERLASTING SUBMISSION. I ACKNOWLEDGE THE ROLE MY PARENTS PLAYED, AND CONTINUE PLAYING, IN MY LIFE, AND I AM, ONCE AGAIN, MOSTLY GRATEFUL TO ALLAH (SWT[2]) BECAUSE OF IT.

I acknowledge the efforts of many of my past teachers. From my early days of studentship between the walls of schools at different levels, I have been acquainted with kind and good-hearted people who supported me on my journey ahead ; the thing which continued at university. I am especially grateful to my MSc.'s Degree supervisor, respectful Mr Saïd MAZOUZI, professor at University of Annaba, for everything he was.

I am in as much debt to my PhD degree supervisor, Pr Mohamed Zine AISSAOUI, who has been a guiding compass throughout my years of the degree ; he has been the poised, firm gentleman who radiated enthusiasm explaining the overlapping of PDEs, Fluid Mechanics, and Functional Analysis. To him, I attribute one of my mottos throughout the degree[3] :

Almost everything in Maths amounts to treating numbers, eventually. For reproducibility, minimise your hypotheses then attempt bringing your problems to the realm of numbers ; only then, many patterns could be recognised.

—My supervisor

I have to report my feelings of thanks to my co-supervisor, Dr Ammar KHELLAF, senior lecturer at École Nationale Polytéchniques de Constantine, for his patience, his encouragement, and the fact that he almost always indicated that *only* a little more was *almost* needed to be *there*. Moreover, I cannot downplay his little side notes that did wonders to my crafting of documents, at times, nor could I dismiss the opportunity to have had fruitful and rich correspondences with him that helped me deepen my understanding of the thesis subject.

I acknowledge my contentment for having roads crossed with LMAM's head, Pr Hamza GUEBBAÏ, who, additional to being my first encounter in the early days of the degree, he was the gentleman with attentive ears whenever I needed support, and who, in light-hearted complex and distinguished smart humour, supplied me with sources and *hopeful determination* that undoubtedly softened my times all throughout. I cannot but thank him for having accepted to be one of my juries.

2. Subhanahu Wa Taa'la.

3. The saying is a synthesised assembly of other fragment sayings drawn across the multitude of the given lectures to maintain readability and cohesion.

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ملخص

عملا بأسلوب فكري جديد تم اعتماده في مخبر الرياضيات التطبيقية والنمذجة، يختلف جوهريا عن سابقه المعتمد بشكل واسع ومتعارف عليه في الأبحاث والذي يرجع أصل شعلته لأفكار تم استقائها من مقالة حول معادلات غير خطية، نعنى في مشوارنا بأمر مهمة رياضياتية متداخلة المناحي في البحث النظري والرقمي عن امكانيات تقريب الحل لمعادلة فريدهولم الخطية الموضوعة في شكل عام والمعرفة على مجال جد كبير. خطوتنا الأولى تبرز كيف يكون لتطبيق فكرة اقتطاع سلسلة نويمان الهندسية للمؤثرات الخطية المدروسة أثر في تحسين جودة النتائج على الصعيد الرقمي رغم المحافظة على إطار العمل المنتهج في أعمال آنفة ضمن محيط من فضاءات باناخ محكمة البناء. في خطوتنا الثانية، يكون الغوص أعمق في حيثيات الأمر وتحويل الدفة تجاه فضاء هيلبرت L^2 سببا في توضيح كيف تنقلب الأمور متشابكة حين العزم على تعميم طرق التراخي المعهودة في الجبر الخطي إلى إطار العمل الجديد؛ خاصة ما يعنى به بحثنا حول تعميم طريقة جاكوبي المرتخية. نختم عملنا بتوضيح لما واجهناه من عقبات وبتطلعاتنا نحو المستقبل في البحث عن تذليل العقبات المتبقية والبحث أكثر في ما يبدو أننا فتحنا بابا له في أطروحتنا.

كلمات مفتاحية معادلات فريدهولم التكاملية، تحليل دالي، تحليل رقمي، نظرية المؤثرات الخطية المحدودة، نظرية التقريب، الطرق التراجعية، الجبر الخطي، البرمجة بماتلاب.

Abstract

Following a newly established paradigm in precursor works at LMAM, diverging from widely recognised conventions, and inspired by an article on non-linear equations, we embark on the interdisciplinary mathematical mission to carry on the pursuit of numerically and theoretically discussing the approximation of solutions to the general Fredholm integral equation of the second kind defined on a large interval. Firstly, we show how efficient it is to truncate a Neumann's Series resulting in enhancing the outcomes and reducing numerical costs further all whilst manoeuvring the same well-constructed environment of the Banach spaces from previous works in order to approximate the solution of the equation cited hereinabove. Secondly, delving deeper into it, we demonstrate that with a shift in focus towards the Hilbert space L^2, new horizons emerge. The need for more generalisations of classically known algebraic iterative methods, with a particular care landing on generalising those of relaxation; namely, the Jacobi Over-relaxation (JOR) scheme, uncovers various theoretical corners, demonstrating that, with implicit analogies to $\mathbb{R}^n$, we provide coherent and consistent findings as well as highlight the promising possibility of additional investigations despite the handful of limitations encountered. Our work concludes with enticing perspectives and inviting goals for richer and more comprehensive explorations.

Keywords Fredholm Integral Equations, Functional Analysis, Numerical Analysis, Bounded Linear Operators Theory, Spectral Theory, Approximation Theory, Iterative Methods, Linear Algebra, MATLAB Programming.

Résumé

En suivant un paradigme nouvellement établi dans des travaux précurseurs au LMAM, s'écartant des conventions largement reconnues, et inspiré par un article sur les équations non linéaires, nous entamons une mission mathé -matique interdisciplinaire pour poursuivre l'étude numérique et théorique de l'approximation des solutions de l'équation intégrale de Fredholm gén -érale du second ordre définie sur un large intervalle. Tout d'abord, nous démontrons l'efficacité de la troncation d'une série de Neumann, améliorant ainsi les résultats et réduisant les coûts numériques tout en manœuvrant dans le même environnement bien construit des espaces de Banach issus de travaux précédents pour approcher la solution de l'équation citée ci-dessus. Ensuite, en nous immergeant plus avant dans ce sujet, nous démontrons que lorsque notre attention se porte sur l'espace de Hilbert L^2, de nouveaux horizons apparaissent. La nécessité de généraliser davantage les méthodes itératives algébriques classiques, avec un soin particulier accordé à la généralisation des méthodes de relaxation, et plus précisément la méthode de Jacobi sur-relaxation (JOR), révèle divers aspects théoriques. Cela démontre que, en tirant des analogies implicites avec , nos résultats sont cohérents et consistants, et souligne les possibilités prometteuses d'investigations supplémentaires malgré les limitations rencontrées. Notre travail se conclut par des perspectives alléchantes et des objectifs invitants pour des explorations plus riches et plus exhaustives.

Mots Clés Equations intégrales de Fredholm, Analyse Fonctionelle, Analyse numérique, Théorie des operateurs linéaires bornés, Théorie spéctrale, Théorie d'approximation, Méthodes itératives, Algèbre linéaire, Programmation MATLAB.

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Introduction

With Latin roots, one of Cambridge Dictionary's definitions for the word *equation* is

> **Equation** *n.* (statement) A mathematical statement in which you show that two amounts are equal using mathematical symbols.

The lexical definition does not fall short on conveying the specialised technical essence of how the concept manifests and is realistically drawn upon per mathematical structures. Inarguably, for as far back in history as from when historians, historian mathematicians, and etymologists would agree, the concept of equating or looking for how to equate two sides (or quantities) has been very intrinsic an idea to mathematics that, owing to which, many theories, within or beyond the scope of this manuscript, have flourished and known stellar advancements that continue today.

Incidentally, our statement is upheld in the shear diversity of this one concept by which it manifests throughout the field, where an equation could be something algebraic having a polynomial expression, like the fifth-degree three-unknown Diophantine equation of

$$m^3 + n^3 - k^3 = 0, \quad (m, n, k) \in \mathbb{N}^3,$$

for which there are no non-trivial triplet solutions (m, n, k) of natural numbers [4]. Or, it could be something algebraic involving a transcendental concept like that of the exponential, as in the equation

$$e^z + 1 = 0, \quad z \in \mathbb{C},$$

whose one of the solutions takes part in producing one of Mathematics' beautiful equations. However, equations could enjoy a little more in-depth

4. This mathematical assertion is due to the mathematician Andrew J. Wiles' breakthrough paper *Modular elliptic curves and Fermat's Last Theorem*. However, Number Theory certainly exceeds our knowledge.

sophistication, when they could be condensely assembled to formulate (non-)linear systems of them, by which way researchers attempt to capture some of the real world's fascinating intricacies, thus introducing formulae having very physical interpretations as in Navier-Stokes (non-)linear system of partial differential equations that is conventionally given, for $(x,t) \in \Omega = \mathbb{R}^d \times [0;\infty)$, with $d = 2,3$, as

$$\begin{cases} \rho\partial_t V(x,t) = \nabla_x p(x,t) + \nu\Delta_x V(x,t) + \rho f(x,t), \\ \operatorname{div}(V(x,t)) = 0, \\ \text{+Initial and Boundary Conditions,} \end{cases}$$

resulting in one of six unsolved problems from seven millennial problems proposed by Clay Mathematics Institute.

Building upon this, often being on par with differential equations (ordinary or partial), from taking a share in formulating the solutions for some, performing the role of their alternatives, to mixing together with them producing even more potent mathematical tools, Integral Equations exhibit qualities and behaviours that have captivated the attention since early in the 19th century; the reason underpinning why the ball kept rolling for continuous theoretical discoveries in different areas of mathematics as well as in other fields all along the historical accounts until the present days; this could be traced in the current times, and the reader may refer to [35] for newer research works on these equations.

Presumably, the historical narrative had began with Fourier's works on thermal transport along his integral transformation (cf. [13]), but much clearer with Abel's integral equation and his mechanical problem touching upon the tautochrone curve[5]. His mission of identifying a trajectory curve given the distance travelled by or the time it took a moving object on it to reach the level from a given initial height led to studying the solution(s) of what in modern times would be classified as a weakly-singular, first-kind, linear Volterra integro-differential equation. Here, we provide a general formulation of it as,

$$\text{For a given } T,\ \gamma > 0,\ \text{find } \psi: \quad T(h_0) = \int_{h_0}^{0} \frac{\psi'(h)}{\sqrt{\gamma(h_0 - h)}}\, dh, \quad T(0) = 0.$$

For further details on this, see [8, 12, 39], and [32] for an exclusive focus on the different aspects of Volterra integral equations. But, for the more

5. A tautochrone curve is one type of curve that, if being the shape of a movement trajectory with specific physical considerations, objects on it reach its endpoint at the same time regardless of their initial positions on the curve; exactly as its Latin-engineered name indicates: same time. Digging deeper, the curious may have his feet dragged towards Fractional Calculus or even Horology.

curious reader, in [18, 19], one could find how integral equations might be favoured over differential ones in what showcases the effect of memory possessed by integrals and lacked in ordinary differentiation.

Circa these periods, treating infinite-dimensionality problems stemming from attempts in solving differential equations, the stationary, perturbed partial differential equation of Poisson:

$$\lambda\, u(x,y) + \Delta\, u(x,y) = g(x,y), \quad \lambda \neq 0,\ (x,y) \in \Omega \subset \mathbb{R}^2, \tag{1}$$

stimulated works from pioneering mathematicians like Poincaré, Fredholm, then Hilbert, leading to the launch of further investigations into the integral equation they posed in a retrospect analogy to (1), as they all had concentrated a great share of their foci on the following problem,

$$\text{Given } g, \text{ find } \psi\text{:} \quad \psi(x) - \lambda \int_a^b k(x,s)\psi(s)\, ds = g(x), \quad \lambda \neq 0, \tag{2}$$

creating both a historical and a historic trend within the mathematical community, thus paving a way to what finished, after extensive rigorous works, in the establishment of a rather new discipline at those periods; it was the birth of Functional Analysis. For more details, we refer the reader to [8, 10, 13].

In spite of this , integral equations' theory did not halt receiving care. Multifacetedly, those equations can be legible for analytical closed forms of solutions via the use of classical methods like Laplace's Transform or The Adomian Decomposition, whenever possible, on which subject the reader may confer [24, 25, 32, 39]. However, in [5], one reads on Atkinson's historical account of how the conventions that once propelled forwards the theory of those equations [12, 16] became the de facto initial step in studying the harnessing of new machine technologies and computational capabilities[6], hence spawning a racing interest in developing competing methods to help accelerate the approximations of the equations' solutions, provided that they existed. It was primarily developed alongside iterative methods and their theory that germinated in Gauss' works; the reader may confer [41].

As will be addressed later on this thesis manuscript, albeit modified to align with modern times conventions, the integral equation (2), which is classified as an inhomogeneous linear Fredholm integral equation of the second kind, has known extensive examinations; one of which is [4].

6. With Artificial Intelligence booming and Quantum Computing being on the horizon, Numerical Mathematics may know unprecedented and unparalleled leaps in how the methods are developing or yet to be developed.

This equation has been known to be, in many cases, related to boundary-value differential problems. Moreover, being an integral equation, with a smooth working environment; i.e, at one's disposal, the solution sought-after complies in its characteristics with traits and the regular, nearly-regular, weakly-singular, or singular behaviours of the kernel and the source functions where (bi-)continuity, continuity, or square-integrability are predominantly the prevalent conditions of the milieu one sets for ulterior numerical approximations to develop[7].

Furnishing an a priori guaranteed stability by means of the integration operation and a manageable consistency with respect to what approach applied in the discretisation phase, the numerical methods applied on the Fredholm's integral equation of the second kind, having more or less moderate error analyses, have been principally classified under three main categories:

- Degenerate Kernel Methods,
- Projection Methods, and
- Nyström's Method(s) (for richer contents, cf. [4, 5]).

Nevertheless, all these methods, or reportedly their variations, have always had the same structured road to follow:

⋆ Provided its solvability, tend to the integral term of the studied equation and approximate it,
⋆ Then, adequately create a discrete quantity out of it,
⋆ Build an algebraic system using your findings,
⋆ With a proper iterative scheme, use your machine to iteratively approximately solve the linear algebraic system you created.

This structured process finds cradle in a long-lived idea that was adopted in early theoretical discussions of the integral equations area and other overlapping areas, as well as being all-present in the numerical transition that we aforementioned.

In his thesis [43], O. Titaud undertook the modelling and the numerical study of the generally static, stratified, atmospheric, stellar energy transfer boundary-value problem. Complying with his aim, his careful considerations brought him about treating *"une équation de Fredholm de seconde éspèce à noyeau de convolution faiblement singulier"*. Although exceptionally successful, given the milieu he operated within, Titaud's conclusion involved his solemn declaration,

7. More on this in the Preliminaries chapter of this manuscript.

"Ce point de vue nous a d'ailleurs poussé à étudier et mettre en œuvre des méthodes asymptotiques efficaces pour essayer de surmonter la difficulté majeure de ce problème qui réside en l'amplitude trés importante de l'intervalle d'intégration."

—Olivier Titaud, *PhD Thesis*.

Such a statement supported the question of how sophisticated was it to operate

$$\lambda\phi(x) - \int_a^b k(x,t)\phi(t)\,dt = f(x), \quad \lambda \neq 0, \tag{3}$$

where x belonged to an interval $[a,b] \subset \mathbb{R}$ having a finite but substantial length $(b-a)$; i.e,

$$1 \ll b - a < \infty?$$

At LMAM, it was examined at length that, sitting on the fence between regularity and singularity (regardless of its kernel type), equation (3) posed a challenging numerical mission when the conventional discretise-iterate process listed above was used for numerical approximation but failed to deliver good and/or satisfying outcomes.

On this and on other motivations, like being inspired by [20] on nonlinear equations, Lemita's works [26, 27] and thesis [28] were erected. Concerning this matter, the so-called Chasles Property of definite integration, if applied $(N-1)$ times, yields:

$$\int_a^b = \sum_{i=1}^{N} \int_{c_{i-1}}^{c_i}, \quad a \leq c_j \leq b, \quad j = 0, \cdots, N,$$

which proved useful in transforming (3), having transitioned towards mathematical abstraction, into a linear system that was not of algebraic type; the instance which needed a thorough investigation of its invertibility and the discussion of the potential applicability of iterative schemes and techniques known from Linear Algebra on the resulting mathematical object. It was indeed feasible, and generalisations to Jacobi's and Gauss-Seidel's schemes were made within the new frame as well as the meaning of a *matrix of bounded linear operators* to be *strictly diagonally dominant by rows* was established. The rewards were promising.

Despite these, the question whether one could possibly consider generalising more iterative schemes and establish more meaning to fundamental concepts in Linear Algebra like that of symmetry and that of positive definiteness for matrices of bounded linear operators was left unaddressed. In this manuscript, we venture in addressing those questions and set three main chapters where:

In Chapter 1, we provide the utilised mathematical prerequisites and concepts found all throughout.

In Chapter 2, we present the extreme of extending the boundaries and introduce a refinement in the both introduced methods in Lemita's works, [27, 28], using the idea of truncation and relying on numerical quadrature rules' stability results.

In Chapter 3, we move to establishing the generalisation of a relaxation scheme known in Linear Algebra by the acronym of JOR; half of what was similarly discussed in [29]. Resorting to a new environment of setups that should not be astray from real scenarios, the setting out on the quest to address those queries left behind by precursor works of Lemita, with the transition from Riemann integrability to Lebesgue's, channels our ways to craft new adequate definitions and properties resembling their finite-dimensional counterparts; in particular we introduce the generalisation of the concepts of symmetry and positive definiteness for a matrix of bounded linear operators which shall be harnessed a priori to ensure the convergence of the said generalised iterative JOR as well as contribute to determining region for the relaxation parameter on that matter.

Chapter 1

Preliminaries: A General Miscellany

Contents

Per mathematical customs, it is indispensable to set ahead the ground for all that is necessary to allow for the development of whichever mathematical endeavour to come. Hence, with much-regarded attention to citation instances, this preliminary offers a general miscellany that aims at reviving the reader's knowledge and equipping them with the toolkit needed to navigate one's way through the manuscript.

We admit that the present inaugural chapter does not find interest in reiterating foundational concepts and axiomatic builds from undergraduate courses; namely, proofs for theorems at this level are either omitted or referred to in external sources. Precisely, our work does not target university first-years or second-years of the mathematics speciality, nor does it most certainly target less than graduates in other fields of science and engineering. Moreover, to when it should fit, symbolic notations used all throughout the text are not all clarified; that is, massively used conventional mathematical notations are not defined and the reader is considered aware of them by default. But, conventional notations that are specific to our uses and are found in the subsequent chapters may be initiated and established early at this stage. Embedded, the reader may find dedicated passages (including footnotes to that matter) for this whenever the need occurs.

Finally, we signal that, occasionally, sections of the chapter, with or without notice, may overlap at some point or another. To illustrate this, overlaps may happen between: section 1.1 and section 1.3, or section 1.3 and section 1.2, etc.

1.1 Univalued Single-variable Functions: Integration and Approximations

Let I be a non-empty real interval of any kind. Although a little flexible to fit where it is needed, a function $w(\cdot) : I \mapsto \mathbb{R}$ is termed a *weight-function* if it is at least

1. Non-negative but not identically null,
2. Both integrable and essentially bounded regardless of what the type of I was, and
3. Having a prevalent monotony over I with a special character about some of its points, including its endpoints, with the presence of a certain asymptotic behaviour about them—the endpoints.

For this, such a weight-function w renders a mapping like

$$\|f\|_{w,\infty} = \sup\{w(x)|f(x)| : x \in I\}$$

a norm over the vector-space of continuous real-valued functions $f(\cdot) : I \mapsto \mathbb{R}$. Moreover, such a norm-application offers the specific property that whenever I is both closed and bounded, a sequence of functions $\{f_k\}_k$, defined over I, verifying

$$\|f_p - f_{p+q}\|_{w,\infty} \to 0, \quad q \geq 1, \ p \to \infty,$$

would always have a limit f in the vector-space from above. This is what defines the space completion and makes the normed space of continuous functions mapping the compact I into $\mathbb{R}$ a Banach space. We refer the reader to one of [8, 10, 13] for more history on the origins and development of Banach spaces.

In our manuscript, we shall not wander non-deterministically. For, fixing I to be a compact of $\mathbb{R}$, and taking the *neutral* weight function

$$w(x) = 1, \quad x \in I,$$

we introduce the *uniform-convergence* norm[1] by $\|\cdot\|_\infty$:

$$\|\cdot\|_{w,\infty} \equiv \|\cdot\|_\infty.$$

Thus, denoted $C(I, \mathbb{R})$, or simply $C(I)$ so long as we are not mapping into the complex numbers, the Banach space in mention does not lose its possession of the subsequent two characteristics:

One is a concept encountered in general metric spaces, but becomes especially sophisticated for subsets of functions. We talk about compactness for subsets of $C(I)$ that is framed differently. Without much verbiage, this particular matter is given in the Arzelà-Ascoli Theorem:

Theorem 1.1.1. *(Arzelà-Ascoli) A subset $S \subset C(I)$ is compact if its elements are both*

1. *Uniformly bounded, such that*

$$\exists M > 0: \quad \|f\|_\infty \leq M, \quad f \in S,$$

and

2. *Equi-continuous, where*

$$\forall \epsilon > 0,\ \exists \delta_\epsilon: \quad f \in S,\ x, y \in I,\ |x-y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

In literature, this theorem may be formulated in terms of sequences of functions and the result may be given in terms of the possibility of extracting a uniformly convergent subsequent from all sequences of functions in S verifying the two conditions of the theorem; a result, if weakened, may require the concept of *Quasi-uniform Convergence* [8]. In addition, as will be indicated in the rest of this section and later, Arzelà-Ascoli Theorem plays a role in proving compactness for various mappings between normed functional vector-spaces on the broader scale; namely, the vector-space $C(X)$ of all uniformly continuous real-valued functions taking from the compact set X with the appropriate topology for completeness.

Two is an approximation property independent of any peculiarity displayed by the chosen element of $C(I)$. Indeed, it is always possible to approximate elements of this Banach space with an adjustable polynomial function. A result rigorously stated in the well-known theorem of

1. Starting at this stage, the symbol $\equiv$ should infer a sense of equivalence and/or identification between two functions, applications, sets, vector-spaces, etc.

Theorem 1.1.2. *(Weierstrass) For every element f of $C(I)$, and for any positive real ϵ, there exists a polynomial function P of a degree dependent on ϵ such that*

$$\|f - P\|_\infty < \epsilon.$$

This cornerstone theorem, in its core, sets the ground for the essentials of Approximation Theory in modern times. Indeed, such a result can be very much observed for sufficiently smooth functions over I; i.e, functions having continuous derivatives up to some positive integer order k about some point $x_0 \in I$, such that these functions are approximated using their Taylor k-polynomial expansions which are an excellent approximating guess provided one is restrained within a neighbourhood of the point x_0.

However, in general, Weireistrass' Theorem can be approached independently of Taylor's expansions. For instance, we note that for a, b the endpoints of I, $a < b$, the p-family of Bernstein's polynomials, which is originally defined over $[0, 1]$, is defined over I with individual terms given by

$$B_{q,p}(x) = \binom{p}{q} \left(\frac{x-a}{b-a}\right)^q \left(\frac{b-x}{b-a}\right)^{p-q}, \quad 0 \le q \le p, \quad a \le x \le b,$$

where $\binom{p}{q}$ is the binomial coefficient. These polynomials are often used to provide a proof for Weireistrass' Theorem, and at this stage, this may help us construct an idea of the abundance of polynomials that share the trait of approximating the function f of $C(I)$!

Of this abundance, let be a mesh $\{x_i\}_{i=0}^n$ of equidistant nodes over I:

$$x_i = a + ih, \quad h = (b-a)/n, \quad 0 \le i \le n, \quad n \ge 1,$$

Such a construction helps define Lagrange's polynomial of degree n with the expression

$$\ell_i(x) = \prod_{\substack{j=0 \\ i \neq j}}^{n} \frac{x - x_j}{x_i - x_j}, \quad x_i \text{ of the mesh and } x \in I,$$

which is yet another approach to approximate a function of $C(I)$ using a linear combination of the linearly independent family of polynomial functions $\{\ell_i(\cdot)\}_{i=0}^n$ of degree n in what builds *Lagrange's polynomial interpolant* *of f at the mesh points* such that[2]

$$f(x) \approxeq P(x) = \sum_{j=0}^{n} f(x_i)\ell_i(x), \quad x \in I,$$

with

$$f(x_i) = P(x_i).$$

Furthermore, not only do Lagrange's polynomials play an interpolating role for f, but also prove quite useful in building the expression for the interpolatory[3] Newton-Cotes quadrature rules formulae where a general statement is given by

$$Q = \sum_{i=0}^{n} w_i f(x_i),$$

with w_i being precisely determined *weight* numbers w.r.t. $\ell_i(\cdot)$, such that

$$w_i = \int_a^b \ell_i(x)\,dx.$$

Q is termed a *quadrature rule* and is used to approximate the definite integral of f over I.

Reliant on polynomial interpolation and the use of the polynomial family $\{\ell_i(\cdot)\}_i$ to compute the corresponding weights w_i, quadrature rules Q with equidistant meshes of nodes x_i's, that may or may not contain both endpoints[4], provide basic and generally practical rules for their goal. For instance, as will be used later in this manuscript, we shall make use of the two-point closed Newton-Cotes rule known as the trapezoidal rule which is generally given by

$$Q = \sum_{i=0}^{1} w_i f(x_i) = \frac{f(a) + f(b)}{2}.$$

2. The symbol $\approxeq$ may be used, starting at this stage, to indicate approximate equality between eventually numerical quantities. The symbol $\approx$ may serve other in-text purposes that we shall clarify ulteriorly.

3. It is true that the reader may deliberately sense the hidden emphasis put on this term. For, rules of approximating the definite integral of a continuous function over a compact real interval may not be interpolatory; namely, Riemann Sums are one such example, and, to a lesser extent, Darboux Sums as well. Moreover, by all means, these numerical integration rules mentioned make apparent the *deterministic* character of theirs contrary to the *probabilistic* one encountered in Monte Carlo's or Bayesian quadratures!

4. The rule is termed *closed* if it involves both endpoints, and *open* if it excludes them.

One such rule is known to be exact for all polynomial functions of order at most one and has a convergence order of three. Moreover, in its composite version over N subintervals of equal length, where the rule is generally concatenated subinterval-by-subinterval, the trapezoidal rule is given by

$$Q = \frac{h}{2} \sum_{m=1}^{N} \left(f(x_{m-1}) + f(x_m) \right).$$

This rule is rather simple and easy to implement; perfect for our needs discussed in Chapter 2. However, with a low order, trapezoidal quadrature is often not the best option for an efficient numerical definite integration of sufficiently smooth functions as other higher order methods like the three-point Simpson's rule of order two, that is exact for all polynomials of an order at most three[5], exist to augment the efficiency of the method like augmenting the convergence order (Simpson's has a convergence order of five), irrespective of their composite versions that refine the outcomes a step further.

On another hand, even though we have

$$\sum_{i=0}^{n} w_i = b - a,$$

we read in [14] that the quantity

$$\kappa = \sum_{i=0}^{n} |w_i|,$$

determines whether the Newton-Cotes variant one manipulates is stable or not and thus determines its convergence or divergence, respectively. Notably, due to the appearance of negative weights, closed Newton-Cotes rules of a degree at least nine are not stable and thus not convergent.

In summary, Newton-Cotes do not fulfil every need one has and may create more issues than they solve. Hence, in response, a first remedy to one of the two problems with these methods is to break free of the uniformity of the quadrature nodes distribution and choose to operate on a non-uniform mesh. In other terms, the special character of some functions where their behaviour around the endpoints is quite oscillatory resulting,

5. In general, depending on the parity of the nodes number n, those methods exhibit different exactness degrees; namely, their exactness is always of an odd degree, such that if n is even, then they are exact with degree $(n-1)$, and if it is odd, then they are exact with degree n.

at times, in what is known as *Runge's Phenomenon* solicits the consideration of other methods; notably, The Clenshaw-Curtis quadrature is one example of such remedies and, albeit still based on the integration of the Lagrangian interpolant of the function f like Newton-Cotes, the nodes in this method that are predefined as the zeros of the Chebyshev polynomials (originally defined on $[-1, 1]$) of which, for

$$y = \frac{2x - b - a}{b - a}, \quad x \in I,$$

those of the first kind are

$$T_{n-1}(y) = \cos\left((n-1)\arccos(y)\right), \quad y \in [-1, 1],$$

and those of the second kind are

$$R_{n-1}(y) = \frac{2}{n(b-a)}\frac{d}{dy}\left[T_{n-1}(y)\right], \quad y \in [-1, 1],$$

The clustering behaviour the zeros of these polynomials show around the endpoints helps mitigate the undesired effects of the oscillations and lead to better results.

Notwithstanding the above, the Clenshaw-Curtis methods with the advantages they provide, still depend on predefining the nodes with a special character. A better approach that would guarantee both stability and treatment of the endpoints at once, all whilst maintaining the convention of using Lagrange's polynomials for weights determination, would be crowned in Gaussian quadrature rules that raise the order of exactness to $(2n + 1)$ and echo Clenshaw-Curtis' rules in *seeking* the nodes as the zeros of special polynomials that are termed *orthogonal*.

Indeed, generated by Rodrigues formula:

$$L_{n-1}(y) = \frac{2^{1-n}}{(n-1)!}\frac{d^{n-1}}{dy^{n-1}}\left[\left(y^2 - 1\right)^{n-1}\right], \quad y \in [-1, 1],$$

Legendre's polynomials $L_{n-1}(\cdot)$ verify the orthogonality criterion

$$\int_{-1}^{1} L_i(y)L_j(y)\,dy = 0, \quad 0 \le i \ne j \le n,$$

and prompt the generation of special zeros that serve the role of the nodes choice over which the Gaussian rules are built. One may refer to [45] for a systematic review of the above two methods; i.e, Gaussian vs. Clenshaw-Curtis quadrature rules.

With this being stated, orthogonality is not a concept exclusive to Legendre's polynomials and numerical quadratures. In fact, outside the above context, letting

$$\Omega = I \setminus \{a, b\},$$

the weighted real inner-product

$$\langle f, g \rangle_w := \int_\Omega w(x) f(x) g(x)\, dx,$$

where the weight function w is subject to the three conditions listed at the beginning of this section[6], helps define the weighted vector-space of square-integrable functions that we denote[7] $L^2_w(\Omega)$ over which an induced norm follows as,

$$\|f\|^2_w := \langle f, f \rangle_w, \quad f \in L^2_w(\Omega),$$

leading to the functional Cauchy-Schwarz inequality:

$$|\langle f, g \rangle_w| \leq \langle f, f \rangle_w \langle g, g \rangle_w, \quad f, g \in L^2_w(\Omega).$$

It is that we have the property of

$$\{f_k\}_{k \geq 0} \subset L^2_w(\Omega), \quad \langle f_p - f_q, f_p - f_q \rangle_w \to 0, \quad p, q \to \infty,$$

that if f is an element of the space, limit to the sequence above, then it is so in the strong sense:

$$\langle f_k - f, f_k - f \rangle_w \to 0, \quad k \to \infty,$$

and such that we accord the *Hilbert Space* label to $L^2_w(\Omega)$.

To [8], not only does the Arzelà-Ascoli Theorem extend to the mentioned Hilbert space, but the concept of orthogonality that had been more or less purely geometric since the times of Euclid until the birth of Functional Analysis also finds room for application to functions of $L^2_w(\Omega)$ such that two vectors (functions) are said to be orthogonal if they verify

$$f, g \in L^2_w(\Omega), \quad f \perp g \iff \langle f, g \rangle_w = 0.$$

Moreover, going for the neutral weight

$$w(x) = 1, \quad x \text{ almost everywhere (a.e.w.) in } \Omega,$$

6. The conditions are rather non-exhaustive. Had more specialised emphasis been required, one may check W. Gautschi, *Orthogonal polynomials: Computations and Approximation*, Numerical Mathematics and Scientific Computations Series, Oxford University Press, 2004.

7. Like previously, since we do not intend to manipulate mappings into the complex plane, then $L^2_w(\Omega)$ shall infer $L^2_w(\Omega, \mathbb{R})$

as indicated previously, the performance of the Gram-Schmidt orthogonalisation procedure on the canonical family of monomials $\{x^m\}_{m\geq 0}$ naturally generates Legendre's polynomials which underlies their fulfilment of the two-term recursive formula:

$$(n+1)L_{n+1}(y) + nL_{n-1}(y) = (2n+1)xL_n(y), \quad y \in [-1,1].$$

For more essential details, one could confer [6].

Subsequently, if normalised with respect to their corresponding inner-product, the resulting denumerable family of normalised Legendre polynomials $\{\tilde{L}_{n-1}(\cdot)\}_{n\geq 1}$:

$$\tilde{L}_{n-1}(y) = \sqrt{\frac{2n-1}{2}} L_{n-1}(y), \quad y \in [-1,1],$$

defines an orthonormal system over $L^2(\Omega)$ whose closure spans the Hilbert space:

$$\overline{\text{Span}\{\tilde{L}_{n-1}(\cdot)\}_{n\geq 1}} = L^2(\Omega),$$

making it separable and allowing for the *Fourier expansion*,

$$f(x) = \sum_{j=0}^{\infty} \langle f, \tilde{L}_j \rangle \tilde{L}_j(x), \quad x \text{ a.e.w. in } \Omega,\, f \text{ in } L^2(\Omega),$$

where, clearly, Bessel's inequality holds:

$$\sum_{j=0}^{m} |\langle f, \tilde{L}_j \rangle|^2 \leq \|f\|^2, \quad f \in L^2(\Omega).$$

On the other hand, considered an indexing parameter, the weight function w, in general, determines what polynomial family to result of the performance of the Gram-Schmidt ortho-normalisation procedure on the canonical family of monomials from above; for instance, Chebyshev polynomials of the first kind are the result obtained in regards to the weight function

$$w(y) = \frac{1}{\sqrt{1-y^2}}, \quad y \text{ a.e.w. in } (-1,1),$$

and we obtain the same with them as with their counterparts of Legendre!

1.2 Finite-dimensional Mappings: Matrix Linear Algebra, Square Linear Systems, and Elements of Spectral Theory

Let be an integer $N \geq 1$. Over the real vector-space $\mathbb{R}^N$, the bilinear form

$$\langle \mathbf{u} | \mathbf{v} \rangle := \sum_{i=1}^{N} u_i v_i, \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}^N,$$

defines a discrete scalar-product by which *the Euclidean norm* defines as

$$\|\mathbf{u}\|_2^2 := \langle \mathbf{u} | \mathbf{u} \rangle, \quad \mathbf{u} \in \mathbb{R}^N.$$

Being complete in the Cauchy sense, both of the above real-valued functionals of the scalar-product and its induced norm grant $\mathbb{R}^N$ the Hilbert structure needed subsequently.

Endomorphic over $\mathbb{R}^N$, the mappings $\mathbb{R}^N \overset{\varphi}{\to} \mathbb{R}^N$ constitute a vector-space $\mathscr{L}(\mathbb{R}^N)$ that is bijectively identified with that of *square matrices* $\mathbb{R}^{N\times N}$. Elements A of $\mathbb{R}^{N\times N}$ have a rich theory of their own and such that the invertibility of the linear system

$$A\mathbf{u} = \mathbf{0}, \quad \mathbf{u} \in \mathbb{R}^N,$$

with $\mathbf{0}$ denoting the null vector of $\mathbb{R}^N$, is more than often the first step to delve into the subject. Namely, the solution $\mathbf{u} = \mathbf{0}$ may be the only solution to the system, hence allowing for terming A *invertible* and writing

$$\det(A) \neq 0.$$

Otherwise, for $\mathrm{Id}_{\mathbb{R}^N}$ being the identity square matrix of $\mathbb{R}^N$, one is brought into treating the case where

$$\exists \mathbf{u} \in \mathbb{R}^N \setminus \{\mathbf{0}\} : \; A\mathbf{u} = \mathbf{0} \implies \exists \lambda \in \mathbb{C} : \; (\lambda \mathrm{Id}_{\mathbb{R}^N} - B)\, \mathbf{u} = \mathbf{0}.$$

It is far from ambiguous that one has the view of

$$A = \lambda \mathrm{Id}_{\mathbb{R}^N} - B, \quad B \in \mathbb{R}^{N\times N},$$

and that, due to $\mathbb{R}$ being non-algebraically closed, λ could be a complex number depending on the type of B. Of course, we shall not explore this and refer the reader to introductory or intermediate courses and textbooks on Matrix Linear Algebra or Linear Algebra in general, like [11, 38, 40].

Attributed the German term *eigen*, the above λ with its $\mathbf{u}_\lambda$ make an eigen-couple $(\lambda, \mathbf{u}_\lambda)$ where, in general, the eigenvalue λ is root to the characteristic polynomial $\chi_B(\cdot)$ of B defined by

$$\chi_B(t) = \det(t\mathrm{Id}_{\mathbb{R}^N} - B), \quad t \in \mathbb{C}.$$

Overlooking the potential hardships that may arise when solving

$$\chi_B(t) = 0,$$

this is usually the standard method to find[8] the eigenvalues λ of B. However, accentuating the use of the scalar-product of $\mathbb{R}^N$, the *Rayleigh Quotient* that is defined by

$$R_B(\mathbf{v}) := \frac{\langle \mathbf{v} | B\mathbf{v} \rangle}{\langle \mathbf{v} | \mathbf{v} \rangle}, \quad \mathbf{v} \neq \mathbf{0},$$

makes a rather delicate approach for this aim.

Indeed, it is easily observed that any eigenvalue of B must pertain to the image set of the quotient:

$$W(B) = R_B\left(\mathbb{R}^N \setminus \{\mathbf{0}\}\right),$$

termed *the numerical range of B*; that is, denoted $\sigma(B)$, the spectrum of B, being the complex subset of the eigenvalues of B, satisfies:

$$\sigma(B) \subset W(B) \subset \mathbb{C}.$$

If B is symmetric, then the quotient clarifies the containment

$$\sigma(B) \subset \mathbb{R} : \quad \mathrm{card}(\sigma(B)) = N.$$

A fact that fuels the hallmark theorem of

Theorem 1.2.1. *(Spectral Decomposition: Finite Dimensions) For any symmetric matrix $B \in \mathbb{R}^{N \times N}$, its spectrum is a subset of the reals, and we have the dirsct-sum decomposition*

$$\mathbb{R}^N = \bigoplus_{j=1}^{N} E_j,$$

where

$$E_j = \mathrm{Span}\{u_{\lambda_j}\}, \quad 1 \leq j \leq N,$$

are the pairwise orthogonal unidimensional eigenspaces of B.

8. Unless otherwise indicated in the text, the explanation involving the matrix B can be made in the account of all other elements of $\mathbb{R}^{N \times N}$.

Moreover, the symmetry of B helps Rayleigh quotient serve another significant role, such that one obtains the equivalence

$$B = B^T \implies R_B(\mathbf{v}) > 0 \iff \langle \mathbf{v} | B\mathbf{v} \rangle > 0, \quad \mathbf{v} \neq \mathbf{0},$$

giving rise to a specific type of square matrices: the *symmetric and positive-definite (SPD)* matrices.

In general, for all $(N \times N)$-matrices B, the quotient may be equivalently defined by the expression:

$$R_B(\mathbf{w}) := \langle \mathbf{w} | B\mathbf{w} \rangle, \quad \mathbf{w} = \frac{\mathbf{v}}{\|\mathbf{v}\|_2}, \quad \mathbf{v} \neq \mathbf{0},$$

such that a quantity of interest, termed *the numerical radius of B*, is defined by

$$\nu(B) := \max_{\mathbf{w}} |\langle \mathbf{w} | B\mathbf{w} \rangle|,$$

and is notably qualified to establish a norm over $\mathbb{R}^{N \times N}$ which is tightly related to the *spectral* quantity termed *the spectral radius of B* that defines as

$$\rho(B) = \max\{|\lambda| : \ \lambda \in \sigma(B)\},$$

and, in turn, establishes another norm over $\mathbb{R}^{N \times N}$, where, with consideration of the suprema manipulated, applying the discrete Cauchy-Schwarz inequality yield the inequalities

$$\rho(B) \leq \nu(B) \leq \|B\|_{2,2}, \quad B \in \mathbb{R}^{N \times N},$$

with

$$\|B\|_{2,2} := \max_{\mathbf{x} \neq \mathbf{0}} \frac{\|B\mathbf{x}\|_2}{\|\mathbf{x}\|_2}.$$

As one so remarks, the three of $\rho(\cdot)$, $\nu(\cdot)$, and $\|\cdot\|_{2,2}$ introduce a topological (metric) structure over the vector-space of $\mathbb{R}^{N \times N}$, and, in general, one introduces such a structure with the use of the (operator) matrix-norm:

$$\|M\|_{\text{mat}} := \max_{\mathbf{x} \neq \mathbf{0}} \frac{\|M\mathbf{x}\|_{\mathbb{R}^N}}{\|\mathbf{x}\|_{\mathbb{R}^N}}, \quad M \in \mathbb{R}^{N \times N}.$$

Hence, we should note that with the p-norms over $\mathbb{R}^N$ such that

$$\|\mathbf{x}\|_p^p := \sum_{i=1}^{N} |x_i|^p, \quad \mathbf{x} \in \mathbb{R}^N, \quad p \in [1; \infty),$$

infinitely many norms could be induced over the matrices following the formula

$$\|M\|_{\text{mat}} \equiv \|M\|_{pq} := \max_{\mathbf{x}\neq\mathbf{0}} \frac{\|M\mathbf{x}\|_p}{\|\mathbf{x}\|_q}, \quad M \in \mathbb{R}^{N\times N}, \quad p, q \in [1;\infty),$$

where the Banach-space structure generated over $\mathbb{R}^N$ with the two vector norms[9] $\|\cdot\|_1$ and $\|\cdot\|_\infty$ provides what Chapter 2 features an analogy of:

$$\|M\|_{\infty,1} = \max_{i=1}^{N} \sum_{j=1}^{N} |m_{ij}|.$$

A further exploration could be referred to in [30]. However, not all matrix norms are vector-induced, and such that in Chapter 3 we shall be interested in the analogous of the so-called Frobenius norm:

$$\|M\|_F^2 := \sum_{i=1}^{N}\sum_{j=1}^{N} |m_{ij}|^2, \quad M \in \mathbb{R}^{N\times N}.$$

Though, with the scalar-product and its induced Hilbert structure over $\mathbb{R}^N$, we always have

$$\|M\|_{2,2} \leq \|M\|_F, \quad M \in \mathbb{R}^{N\times N}.$$

Under all circumstances, intrinsic of a property to its corresponding matrix, the spectral radius of a matrix satisfies the Gelfand's formula of

$$\rho(M) := \lim_{k\to\infty} \sqrt[k]{\|M^k\|_{\text{mat}}}, \quad M \in \mathbb{R}^{N\times N},$$

which easily establishes another bound for the spectral radius:

$$\rho(M) \leq \|M\|_{\text{mat}}.$$

Thus, irrespective of the matrix norm used[10], Gelfand's formula serves on the list of mathematical machinery to prove the Geometric Series Theorem:

Theorem 1.2.2. *(The Geometric Series Theorem: Finite Dimensions) Let be M and $\mathrm{Id}_{\mathbb{R}^N}$ be elements of $\mathbb{R}^{N\times N}$, with the latter being the identity matrix with*

9. $\|\cdot\|_\infty = \lim_{p\to\infty} \|\cdot\|_p$ uniformly over the whole of $\mathbb{R}^N$.

10. Norms are equivalent on Euclidean spaces where Gelfand's formula, in general, concerns all matrix norms, including that of Frobenius.

respect to the canonical basis of $\mathbb{R}^N$. *Provided that M is strictly less than unity in the spectral radius, it is sufficient that we necessarily have the matrix* $(\mathrm{Id}_{\mathbb{R}^N} - M)$ *be invertible with*

$$(\mathrm{Id}_{\mathbb{R}^N} - M)^{-1} = \mathrm{Id}_{\mathbb{R}^N} + M + M^2 + \cdots + M^n + \cdots;$$

and such that

$$\| (\mathrm{Id}_{\mathbb{R}^N} - M)^{-1} \|_{\mathrm{mat}} \leq \frac{1}{1 - \|M\|_{\mathrm{mat}}}.$$

Furthermore, the spectral radius also serves in governing one of applied mathematics most crucial theorem. Based on [6], its finite-dimensional version follows as

Theorem 1.2.3. *(Banach's Fixed-point: Euclidean Spaces) Let be a real* α *such*

$$0 \leq \rho(M) \leq \alpha < 1, \quad M \in \mathbb{R}^{N \times N}.$$

The fixed-point equation of

$$\mathbf{x} = M\mathbf{x} + \mathbf{c}, \quad \mathbf{c}, \mathbf{x} \in \mathbb{R}^N,$$

has a unique solution $\mathbf{x}^*$ *in* $\mathbb{R}^N$ *and such that the generated iterative sequence of*

$$\begin{cases} \mathbf{x}^{k+1} & = M\mathbf{x}^k + \mathbf{c}, \quad k \geq 0, \\ \mathbf{x}^0 & \in \mathbb{R}^N, \end{cases}$$

verifies the error estimation of

$$\|\mathbf{x}^* - \mathbf{x}^k\|_{\mathbb{R}^N} \leq \frac{\alpha^k}{1 - \alpha} \|\mathbf{x}^1 - \mathbf{x}^0\|_{\mathbb{R}^N}.$$

The previous theorem underlies the convergence of iterative methods known to approximate the solutions of algebraic linear systems.

Historically [41], however relatively effective and diverse the direct methods to solve the algebraic square linear system of

$$T\mathbf{x} = \mathbf{b}, \quad \mathbf{b}, \mathbf{x} \in \mathbb{R}^N, \ T \in \mathbb{R}^{N \times N},$$

may be, like Cramer's rule, The LU-Decomposition, The QR-Decomposition, etc., in his *relaxation* process of finding the solution to the above system, Gauss weighed the virtues of iteration and proceeded as such setting the residual vector

$$\mathbf{r}^k = \mathbf{b} - T\mathbf{x}^k, \quad k \geq 0,$$

where he would annihilate a component of the residual vector at each iteration of the scheme. Later, these germinal ideas knew much development that took a final frame in the century that followed prior to the advent of computational machines.

In every instance, the processes that emerged, including the developed version of Gauss, had all rested on the splitting of the *data* matrix T such that one takes

$$T = P - R, \quad \det(P) \neq 0,$$

where the invertible matrix P is often referred to in literature as the *preconditioning* matrix and R the matrix of the rest of data from T. Hence, to this day, building upon this, two famous iterative methods have been established where considering the decomposition

$$T = D - L - U,$$

with D the diagonal part of T, and L, U are the lower and upper parts of it, respectively. Hence, we obtain the following two results:

Jacobi's Iterative Method (J) is the result of choosing

$$P = D, \quad R = L + U,$$

allowing, by a handful of manipulations, the surfacing of fixed-point equation

$$\mathbf{J}_{\mathrm{Eq}}: \quad \mathbf{x} = D^{-1}(L+U)\mathbf{x} + D^{-1}\mathbf{b},$$

which builds an iterative scheme whose component-wise representation is given by:

$$\mathbf{J}: \begin{cases} x_i^{k+1} & = \dfrac{1}{t_{ii}} \left(\displaystyle\sum_{\substack{j=1 \\ i \neq j}}^{N} t_{ij} x_j^k + b_i \right), \quad k \geq 0, \\ x_i^0 & \in \mathbb{R}, \ 1 \leq i \leq N. \end{cases}$$

As demonstrated in theorem 1.2.3, it is sufficient and necessary for the iteration matrix $D^{-1}(L+U)$ to meet

$$\rho(D^{-1}(L+U)) < 1,$$

for the iterative scheme $\mathbf{J}$ to converge to the solution of the fixed-point equation $\mathbf{J}_{\mathrm{Eq}}$ which in turn is the solution of the algebraic linear system under examination.

Gauss-Seidel's Iterative Method (G-S) on the other hand, following the same three-term decomposition of T as indicated previously, is the result of choosing

$$P = D - L, \quad R = U.$$

Thus, the exact same manipulations from above offer the fixed-point equation:

$$\mathbf{G\text{-}S}_{\text{Eq}}: \quad \mathbf{x} = (D-L)^{-1}U\mathbf{x} + (D-L)^{-1}\mathbf{b},$$

which yields the following

$$\mathbf{G\text{-}S}: \begin{cases} x_i^{k+1} &= \sum_{j=1}^{i-1} t_{ij}x_j^{k+1} + \sum_{j=i+1}^{N} t_{ij}x_j^{k} + b_i, \quad k \geq 0, \\ x_i^0 &\in \mathbb{R}, \ 1 \leq i \leq N. \end{cases}$$

Anew, theorem 1.2.3 states that **G-S** converges to the solution of $\mathbf{G\text{-}S}_{\text{Eq}}$, equivalently to the solution of the algebraic linear system in discussion, provided that we sufficiently and necessarily have the iteration matrix $(D-L)^{-1}U$ verify

$$\rho((D-L)^{-1}U) < 1.$$

Even though theorem 1.2.3 may appear sufficient and universal in determining whether schemes **J** and **G-S** converged, it still utilised the spectral radius which is more than often far from reachable without the harnessing of machine power. Consequently, using the dominance of the matrix norm over the radius, a sufficient condition for convergence would be to assume that the iteration matrix, in general, verifies

$$\|P^{-1}R\|_{\text{mat}} < 1.$$

However, an a priori condition would make matters rather approachable. In fact, the type of the data matrix T factors in to be of use, such that the two schemes above converge a priori[11] if T is a *strictly diagonally dominant matrix by rows (SDD-R)*; that is, if it satisfies

$$|t_{ii}| > \sum_{\substack{j=1 \\ i \neq j}}^{N} |t_{ij}|, \quad 1 \leq i \leq N.$$

The **G-S** also always converges if T is an SPD matrix, even though the **J** one may or may not be so.

11. A proof for this may be mediated by means of Gershgorin's Discs.

Since the 1950s (give or take few years), propelled by the especially unprecedented computational triumphs achieved by new technological machines at the times, the ongoing competitive race to create faster schemes caused the *over-relaxation* schemes to see light. In regards to [22], the base idea was to set

$$T = P_\omega - R_\omega : \quad P_\omega = \omega^{-1} P, \quad \omega \in \mathbb{R} \setminus \{0\}.$$

Hence, solving for R_ω, the particular case of the **JOR** scheme following the relaxation of **J** is given the fixed-point equation:

$$\mathbf{JOR}_{\mathrm{Eq}} : \quad \mathbf{x} = \Big((1-\omega)\mathrm{Id}_{\mathbb{R}^N} + \omega D^{-1}(L+U) \Big)\, \mathbf{x} + \omega D^{-1}\mathbf{b},$$

with the component-wise formulation as

$$\mathbf{JOR} : \begin{cases} x_i^{k+1} & = (1-\omega) x_i^k + \dfrac{\omega}{t_{ii}} \left(\sum\limits_{\substack{j=1 \\ i \neq j}}^{N} t_{ij} x_j^k + b_i \right), \quad k \geq 0, \\ x_i^0 & \in \mathbb{R},\ 1 \leq i \leq N. \end{cases}$$

The convergence of the new over-relaxation schemes ensues from merging the predisposed condition on the data matrix and the determination of a specific region for the relaxation parameter ω. Namely, among other scenarios, the successive over-relaxation (SOR), the issue of Gauss-Seidel's scheme, is convergent for SPD data matrices with $\omega \in (0, 2)$, with the sole aim of adjusting the convergence speed of the **G-S**. On the other hand, the **JOR** scheme from above plays two roles of which one is rather significant:

1. Accelerate the convergence of **J** provided it is convergent, with $\omega \in (0, 1]$, and
2. Despite divergence, introduce the convergence of the scheme with an appropriate range for the over-relaxing ω.

Indeed, being SPD but non-SDD-R, matrices like

$$\begin{bmatrix} 15 & 2 & 1 & 2 & 0 \\ 2 & 10 & 2 & 2 & 2 \\ 1 & 2 & 7 & 2 & 2 \\ 2 & 2 & 2 & 5 & 0 \\ 0 & 2 & 2 & 0 & 1 \end{bmatrix}$$

urge for more in-depth investigations on **J**'s convergence. As a result, article [46] offers a list of discussions in which remedies to the above issue are provided discarding, at times, the use of the spectral radius of the iteration matrix in setting bounds for the region of ω over which convergence is guaranteed for the **JOR**. However, of the results surfacing therein, we reword a theorem as follows:

Theorem 1.2.4. *(JOR Convergence for all SPD Matrices) Let $T \in \mathbb{R}^{N\times N}$ be an SPD matrix. The iterative **JOR** scheme converges for all values ω such that*

$$0 < \omega < \frac{2}{\rho(D^{-1}T)}.$$

1.3 Of Linear Operators: Theory and Some Classifications

In this section, we take interest in discussing operators between vector-spaces of finite or infinite dimensions.

Thus, let $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ be a real Banach space. An endomorphism $A : \mathbf{B} \mapsto \mathbf{B}$ is said to be bounded if it verifies the assertion

$$\exists\, C \geq 0, \quad \|Au\|_{\mathbf{B}} \leq C\|u\|_{\mathbf{B}}, \quad u \in \mathbf{B} \setminus \{0_{\mathbf{B}}\}.$$

We always have that the assertion above is satisfied for the *operator-norm* $\|\cdot\|_{op}$:

$$\|A\|_{op} := \sup_{u \neq 0_{\mathbf{B}}} \frac{\|Au\|_{\mathbf{B}}}{\|u\|_{\mathbf{B}}}.$$

The boundedness of A is equivalent to its *continuity* where it is *continuous* if we have

$$\{u_n\}_n \subset \mathbf{B} : \quad \lim_{n\to\infty} u_n = u \implies \lim_{n\to\infty} Au_n = Au.$$

In a loose speech, bounded endomorphisms A, if we notice, preserve the boundedness between the input and the output space.

A more subtle result is that, assumed bijective, a bounded endomorphism A over the Banach $\mathbf{B}$ gains invertibility as well as boundedness of its inverse A^{-1} and such that bounded endomorphisms T that are *in proximity* to A where

$$\|A - T\|_{op} \leq \frac{1}{\|A^{-1}\|_{op}},$$

are invertible. This is especially important when one investigates the solvability of perturbed abstract linear vector-equations like tackled in this manuscript. This also interprets to the fact that bounded and invertible endomorphisms form an open subset of the space of bounded endomorphisms that we denote[12] $\mathscr{L}(\mathbf{B})$. Additionally, of the tools that one likely uses to prove the above result is the following generalisation of theorem 1.2.2 that we state with a relatively weakened hypothesis:

Theorem 1.3.1. *(The Geometric Series Theorem: Inclusive Statement) Let be the identity endomorphism of* $\mathbf{B}$, $\mathrm{Id}_{\mathbf{B}}$, *and an* A *of* $\mathscr{L}(\mathbf{B})$ *with an operator-norm that is less than unity. We have that the endomorphism* $(\mathrm{Id}_{\mathbf{B}} - A)$ *is an invertible element of* $\mathscr{L}(\mathbf{B})$ *such that*

$$(\mathrm{Id}_{\mathbf{B}} - A)^{-1} = \mathrm{Id}_{\mathbf{B}} + A + A^2 + \cdots + A^k + \cdots,$$

and that

$$\| (\mathrm{Id}_{\mathbf{B}} - A)^{-1} \|_{op} \leq \frac{1}{1 - \|A\|_{op}}.$$

A stronger class of endomorphic operators are the *compact* endomorphisms which are not exclusive to Banach spaces. Elements of this class, by definition, not only do they preserve boundedness as in the above, but they also transform it into a stronger property; namely, they produce image sets with *compact closure* in the output space out of bounded sets in the input space. Hence, all compact endomorphisms are bounded. Furthermore, in the case of infinite dimensions, these endomorphisms are often perceived as the counterparts of matrices from Euclidean spaces where both share the property that both their spectra are denumerable and are only constituted of eigenvalues; a fact that leads to a pivotal statement extending the solvability theory of matrices to compact operators in general Banach spaces:

Theorem 1.3.2. *(Fredholm's Alternative) Let be an endomorphism*[13] $(\lambda - A)$ *of* $\mathbf{B}$ *with* A *compact. Then, equation*

$$(\lambda - A)\, u = f, \quad f \in \mathbf{B},$$

is either uniquely solvable for all the terms f, *with* $u = 0_{\mathbf{B}}$ *being the unique solution of*

$$(\lambda - A)u = 0_{\mathbf{B}},$$

12. This is an exclusive notation for its served purpose. In general, we shall denote by $\mathscr{L}(V_1, V_2)$, respectively $\mathscr{L}(V_1)$, the vector-space of all bounded homomorphisms, respectively endomorphisms, between two vector-spaces V_1, V_2, respectively between V_1 and itself.

13. Starting at this point, we shall dismiss indicating clearly the presence of the identity operator throughout the whole of the manuscript, unless it is inevitable as when $\lambda = 1$ or when we have varying indices to operate.

or that there exists a linearly independent family $\{f_n\}_n$ *with*

$$f_n \neq 0_{\mathbf{B}} \quad \textit{and} \quad (\lambda - A)f_n = 0_{\mathbf{B}}, \quad n \geq 0,$$

In terms of the zero of a vector-equation, Fredholm's Alternative is not the only result to extend from normed Euclidean spaces to Banach spaces of arbitrary dimensions, for we also have the extension of Banach's Fixed-point Theorem that deals with the existence, uniqueness, and approximation of the zero of vector-equations involving *contractive mappings*, although not necessarily *compact endomorphisms* as with Fredholm's result:

Theorem 1.3.3. *(Banach's Fixed-point: General Statement in Banach Spaces) Let* $K : \mathbf{B} \mapsto \mathbf{B}$ *be a mapping such that*

$$\|K(u) - K(v)\|_{\mathbf{B}} \leq \alpha \|u - v\|_{\mathbf{B}}, \quad u, v \in \mathbf{B},$$

with $0 \leq \alpha < 1$*, then* K *is a contractive mapping and the equation*

$$u - K(u) = 0_{\mathbf{B}},$$

is uniquely solvable with a guaranteed convergence for the sequence $\{u_k\}_k \subset \mathbf{B}$ *such that*

$$u_{k+1} = K(u_k), \quad k \geq 0, \; u_0 \in \mathbf{B},$$

towards the fixed point u^* *solution to the equation from above and such that one obtains the estimation*

$$\|u^* - u_k\|_{\mathbf{B}} \leq \frac{\alpha^k}{1 - \alpha} \|u_1 - u_0\|_{\mathbf{B}}, \quad k \geq 0.$$

In light of Riesz's theorem[14], a vector-space's dimensionality transgresses the elementary concept of its vector-basis' cardinality. In fact, with topological structures introduced via norm applications, it is very striking of a result to have a close link between dimensionality and the topological compactness such that both concepts become equivalent. This is very well exploited in regards to endomorphisms where, provided that its rank[15] is finite, an endomorphism is always going to be compact. Besides, when blended with the closure of the set of compact endomorphisms, this produces another particular result that the limit of a sequence of finite-rank operators is always a compact operator.

14. In brief, it is the one stating that a unit ball is compact if, an only if, the dimension of the space is finite.

15. The dimension of its image set; also known as codimension.

Nonetheless, this is not a universally reversible result over general Banach spaces; that is, a compact endomorphism is not always the limit of a finite-rank sequence of endomorphisms even that it is indeed a limit of a sequence of compact endomorphisms[16].

Such statements bring us about the mention of *modes of convergences* [1] over $\mathscr{L}(\mathbf{B})$ where if $A \in \mathscr{L}(\mathbf{B})$ then a sequence $\{A_n\}_n \subset \mathscr{L}(\mathbf{B})$ may be

1. *Pointwise-convergent* to A if

$$\lim_{n\to\infty} A_n u = Au, \quad \forall u \in \mathbf{B},$$

2. *Norm-convergent* to A if

$$\lim_{n\to\infty} \|A - A_n\|_{op} = 0,$$

3. *Collectively-compact convergent* to A if
   (a) It is pointwise convergent to it, and
   (b) The sequence elements A_n start to verify operator compactness starting at some sequence rank:

$$\bigcup_{n\geq n_0} \{(A_n - A)u : \quad \|u\|_{\mathbf{B}} \leq 1\}$$

   is relatively compact with

$$\bigcup_{n\geq n_0} \{A_n u : \quad \|u\|_{\mathbf{B}} \leq 1\}$$

   is so in case of compactness of A.

Mode 3. of convergence characterises the specialness and rather nuanced attachments joining integral and numerical quadrature operators. One may refer to [1, 4, 6] for further details, as well as Chapter 2 where we will faintly articulate it in a lemma's proof involving *Nystroöm's Method*.

On the other hand, with real inner-products $(\cdot,\cdot)_{\mathbf{H}}$ we induce the norm applications $\|\cdot\|_{\mathbf{H}}$ which places us onto real Hilbert spaces $\mathbf{H}$ of arbitrary dimensions, where many of the attributes from the extensively covered finite-dimensional case of Euclidean spaces are adequately extended to infinite dimensions, and such that with the (almost) always guaranteed

16. Although stated, the paragraph's substance is particularly advanced and exceeds our current scope and/or capacity in all terms!

existence[17] of an orthonormal family $\{e_j\}_{j\geq 0} \subset \mathbf{H}$ over infinite-dimensional Hilbert spaces, an infinite *Hilbert* system (or loosely, *basis*) proves the approximation result on compact endomorphisms over $\mathbf{H}$, elements of $\mathscr{L}(\mathbf{H})$, which, in this space settings, are (almost) always the limits of finite-rank endomorphisms. To illustrate this, the *orthogonal projection* endomorphisms $\pi_n : \mathbf{H} \mapsto \mathbf{H}$ are usually defined by

$$\pi_n u := \sum_{j=0}^{n} (e_j, u)_{\mathbf{H}}\, e_j, \quad u \in \mathbf{H},$$

making of them, by Bessel's inequality and their finite-ranks, compact elements of $\mathscr{L}(\mathbf{H})$ which verify:

1. Pointwise convergence to the identity endomorphism $\mathrm{Id}_{\mathbf{H}}$, without norm-convergence to it unless $\mathbf{H}$ is finite-dimensional. In fact, the sequence $\{\pi_n\}_n$ does converge to $\pi_\infty : \mathbf{H} \mapsto \mathbf{H}$ which is a compact[18] element of $\mathscr{L}(\mathbf{H})$ defined by the expression

$$\pi_\infty u := \sum_{j=0}^{\infty} (e_j, u)_{\mathbf{H}}\, e_j, \quad u \in \mathbf{H},$$

2. The space's direct decomposition such that

$$\mathbf{H} = \mathrm{Ker}(\pi_n) \bigoplus \mathrm{Ker}(\mathrm{Id}_{\mathbf{H}} - \pi_n), \quad n \geq 0,$$

   or equivalently

$$\mathbf{H} = \mathrm{Range}(\mathrm{Id}_{\mathbf{H}} - \pi_n) \bigoplus \mathrm{Range}(\pi_n), \quad n \geq 0,$$

   where $(\mathrm{Id}_{\mathbf{H}} - \pi_n)$ is a second orthogonal projection operator that is pointwise-convergent to the annihilation endomorphism mapping $\mathbf{H}$ onto the singleton $\{0_{\mathbf{H}}\}$, and

3. The identity

$$(v, \pi_n u)_{\mathbf{H}} = (\pi_n v, u)_{\mathbf{H}}, \quad u, v \in \mathbf{H}.$$

Point 3. weighs in to show the idempotence trait of π_n in addition to how it translates into the *self-adjointness* of the orthogonal projection which, by virtues of Riesz's Representation Theorem, presents us to an especially

17. Exotic enough, *Non-seperable Hilbert Spaces* is yet another advanced topic that the reader may investigate independently!

18. Clearly, π_∞ is not the identity operator $\mathrm{Id}_{\mathbf{H}}$.

important class of endomorphisms over $\mathbf{H}$: the *self-adjoint class* of endomorphisms that necessarily verify 3. unconditionally which establishes the extension of the concept of symmetry from matrices over Euclidean spaces to endomorphisms over arbitrary Hilbert spaces. Consequently, this clarifies the reason underlying the fact that the spectra of self-adjoint operators are embedded into the real line.

Finally, another interesting subclass of compact endomorphisms over Hilbert spaces that usually features within the frame of integral equations (see next section) is the one of Hilbert-Schmidt endomorphisms S for which the Hilbert-Schmidt norm[19] is defined by

$$\|S\|_{HS}^2 := \sum_{j=0}^{\infty}(Sb_j, Sb_j)_{\mathbf{H}}^2 = \sum_{i,j=0}^{\infty}(Sb_j, b_i)_{\mathbf{H}}^2,$$

independently of the choice of the Hilbert system $\{b_j\}_{j\geq 0}$. To conclude, Hilbert-Schmidt norms verify the inequality

$$\|S\|_{op} \leq \|S\|_{HS}, \quad S \text{ a H-S endomorphism},$$

and even though they are not *sub-multiplicative*[20], Hilbert-Schmidt norms also verify a closely related inequality:

$$\|ST\|_{HS} \leq \|T\|_{op}\|S\|_{HS}, \quad S \text{ a H-S endomorphism, and } T \in \mathscr{L}(\mathbf{H}).$$

More aspects and various other results can be referred to in [7].

1.4 Integral Equations: Diversity and Some Theory

In this section, we shall not provide extensive explanations on the theory of integral equations since richer details may be referred to in literature like [4, 5, 6, 16, 25, 32, 39].

Nonetheless, we shall indicate that, similar in its aim to all other varying rich types of functional equations in mathematics, an integral equation is an equation of functions who seeks to find the unknown solution function appearing under the integral sign. We report that there exist types of integral equations that are classified with respect to their defining parts:

19. One should note that in finite dimensions, Hilbert-Schmidt norm is identical to Frobenius'.

20. Sub-multiplicity is the fact that one has $\|AB\| \leq \|A\| \cdot \|B\|$, within appropriate settings. Sometimes, textbooks or otherwise may term it *consistency* property.

1. The domain: regular if finite, singular if infinite or contains singularity points, of Volterra if of varying constant length, or of Fredholm if of constant fixed length,
2. The kernel function: regular, nearly-singular, weakly-singular, or singular, in compliance with the domain above,
3. The source-function: homogenous if it is null, non-homogenous otherwise (the regularity plays less important role), and
4. The parameter λ (see below) that makes the first kind of the equation if being null, and introduces the regular/spectral second kind if being non-zero.

To our concern, the present manuscript treats unidimensional regular integral equations of the form

$$\lambda\phi(x) - \int_D k(x,t,\phi(t))\,dt = f(x), \quad x \in D \subset \mathbb{R}.$$

Provided that D is a *fixed*[21], finite-length real domain that may be open or closed, the kernel function $k(\cdot,\cdot,\cdot) : D^3 \mapsto \mathbb{R}$ is linear in its third variable:

$$k(x,y,z) = k(x,y)z, \quad x,y,z \in D,$$

and that the non-zero source-function $f(\cdot) : D \mapsto \mathbb{R}$ exhibits certain regularities with belongingnesses to well-determined spaces, the situation amounts to the treatment of

$$\lambda\phi(x) - \int_D k(x,t)\phi(t)\,dt = f(x), \quad x \in D,$$

where λ is taken a non-zero real number. This is the classic *non-homogenous Fredholm integral equation of the second kind* where the *linear integral operator* A:

$$\mathrm{A}\varphi(x) = \int_D k(x,t)\varphi(t)\,dt, \quad x \in D,$$

that we appropriately define in subsequent chapters, is compact. The equation is often twin to the study of boundary-value problems (BVPs) like the one below[22]

$$\begin{cases} x^2y^{(2)} + xy' + (x^2-4)y = e^{-x}, \quad \pi/2 \le x \le \pi, \\ \begin{cases} y(\pi/2) = y'(\pi), \\ y(\pi) = y'(\pi/2). \end{cases} \end{cases}$$

21. Constant with respect to changes in the variable x.
22. This is BVP with a Bessel's differential equation whose solution involves a linear combination of Bessel's 2-functions of the first and second kind: $J_2(\cdot), Y_2(\cdot)$, respectively.

Ultimately, either produced by BVPs like the one above or are the results of treating and/or modelling real-world phenomena, the application of Fredholm's Alternative, or of Banach's Fixed-point Theorem in its general statement for Banach spaces, entails that Fredholm's integral equations of the second kind are uniquely solvable for all continuous, respectively bi-continuous, functions $f(\cdot)$ and $k(\cdot,\cdot)$ over the compact $D = I = [a,b] \subset \mathbb{R}$ provided that we have λ a regular parameter:

$$|\lambda| > \max_{a \leq x \leq b} \int_a^b |k(x,t)|\, dt.$$

Over the Lebesgue-measured set $D = I \setminus \{a,b\}$, the mentioned equations of Fredholm's second kind are also uniquely solvable when one takes $f(\cdot)$ and $k(\cdot,\cdot)$ square-integrable making of A a Hilbert-Schmidt integral operator to verify

$$\lambda^2 > \iint_{D^2} |k(x,t)|^2\, dA.$$

We close our preliminaries by noting that the unique solution of the above BVP is appropriately determined using the boundary conditions; a situation invoking *Green's function*, $G(\cdot,\cdot)$, that is piecewise defined by

$$G(x,t) = \begin{cases} \frac{J_2(x)Y_2(t)}{W(t)}, & \pi/2 \leq t \leq x, \\ \frac{J_2(t)Y_2(x)}{W(t)}, & x \leq t \leq \pi, \end{cases}$$

with

$$W(t) = \det \begin{bmatrix} J_2(t) & Y_2(t) \\ J_2'(t) & Y_2'(t) \end{bmatrix},$$

being the *Wronskien* of the corresponding second-order linear Bessel's differential equation. Predominantly, as can be observed in $G(x,t)$, Green's functions are, among other states, *symmetric* where

$$G(x,t) = G(t,x), \quad t, x \text{ in the domain of the corresponding BVP},$$

which makes them, being the kernels of integral operators of Fredholm's type, a specimen of a broader class of real-valued[23] kernel functions having the symmetry property that we shall exploit in Chapter 3. For further readings, although not particularly used in our work but may find applications within, one may investigate *Mercer's Theorem* that relates to kernel functions and explore *Reproducing Kernel Hilbert Spaces* (RKHS); a concept drawing curtains on some rather vast area of entangling mathematical subfields.

23. As previously stated, we do not intend to treat complex-valued functions nor do we intend to use complex numbers.

Chapter 2

To Truncate a Series of Operators: A Refinement

Contents

2.1 Prelude

In the manuscript's Introduction chapter, we have already touched upon the motivating key points behind the present work and its predecessor. However, in this prelude section, we shall elaborate a little further on integral equations of the Fredholm's type. Evidently, those equations appear in various physical scenarios; namely, we talk about the intertwinement of partial differential boundary-value problems and their integral-equation representations in Potential Theory [25].

Justifiably, drawing from classical physics, [17] surveys Love-Lieb integral equations where the study of electrostatic potential fields of uniformly charged capacitors [1] of coaxially δ-separated iso-radii [2] end disc plates in-

1. Energy storage devices.
2. Or, different radii as well.

vokes the considerations of mathematical models that lead to investigating a linear Fredholm integral equation of the second kind as:

$$u(x) \pm \frac{1}{\pi} \int_{-1}^{1} \frac{\delta\, u(t)}{(x-t)^2+\delta^2}\, dt = f(x), \quad x \in [-1,1].$$

Such a regular integral equation, despite its recognised simplicity of treatment, formulates an interesting study case when δ decreases; i.e, $\delta \downarrow 0$, pushing towards the creation of a *nearly singular* kernel function[3] about the bisector $x = t$, and posing numerical challenges with the tackling of integrations over the dilated intervals of the form[4] $[-1/\delta, 1/\delta]$ whose substantial lengths are indisputable.

On these steps, the thesis discussing the celestial astrophysical phenomena of the atmospheric stellar energy transfer as well as the matter discussed in [20] that addressed the numerical treatment of non-linear equations stimulated the works in [26, 27] crowned by the thesis [28] where it was demonstrated that Fredholm's integral equations of the second kind defined over *large intervals* were not to be treated the same way the ones defined over *normal-length* interval were. In essence, it was primarily due to the failure of the conventional process of *discretising then iterating* versus the success of the novel one of *iterating then discretising* in what made parallel the situation to that tackled in [20] as mentioned.

Before we proceed, this chapter shall treat the Fredholm integral equation of the second kind defined over the compact real interval $I = [a,b]$ where

$$1 \ll (b-a) < \infty,$$

and such that for the non-zero real parameter λ and the real-valued source function $f(\cdot) \in C(I)$ and regular kernel function $k(\cdot,\cdot) \in C(I \times I)$, we set the mentioned equation of the unknown function $\phi(\cdot) \in C(I)$ as follows

$$\lambda\phi(x) - \int_a^b k(x,t)\phi(t)\, dt = f(x), \quad x \in I. \tag{2.1}$$

As postulated in the Preliminaries' chapter, Section 1.4, such an equation is uniquely solvable provided that

$$|\lambda| > \max_{a \le x \le b} \int_a^b |k(x,t)|\, dt. \tag{2.2}$$

3. $(x-t)^2 + \delta^2 = (t - x - i\delta)(t - x + i\delta)$.
4. Make the change $t/\delta = y$.

2.2 Discretise then Iterate: A Ruptured Convention

Conventionally, for equation (2.1) to be treated numerically, one dedicates much care to the approximation of the integral operator A defined by

$$\mathrm{A}\phi(x) = \int_a^b k(x,t)\phi(t)\,dt, \quad \phi(\cdot) \in \mathcal{C}, \quad x \in I,$$

with $\mathcal{C} = C(I)$ endowed with its norm of uniform convergence being a Banach space. It is noteworthy to mention that, owing to Arzelà-Ascoli's Theorem (see Preliminaries), A is a compact element of $\mathscr{L}(\mathcal{C})$ with

$$\|\mathrm{A}\|_{op} = \max_{a \le x \le b} \int_a^b |k(x,t)|\,dt,$$

which, together with (2.2), one obtains the invertibility of $(\lambda - \mathrm{A})$ and hence the applicability of Fredholm's Alternative Theorem to ensure the existence of one $\phi(\cdot)$ in $\mathcal{C}$ solution of (2.1).

Of the care regarded for A, utilised and discussed throughout this chapter, we have the Nyström's method that consists of replacing (2.1) with a discrete version:

$$m \ge 2, \quad \lambda\phi_m(x) - \mathrm{Q}_m\phi_m(x) = f(x), \quad x \in I, \tag{2.3}$$

where, based on numerical quadrature principles for the approximation of definite integrals of appropriate functions, the endomorphism $\mathrm{Q}_m \in \mathscr{L}(\mathcal{C})$ is the *discrete* integral operator defined by

$$\mathrm{Q}_m\psi(x) = \sum_{j=0}^{m} w_j k(x,t_j)\psi(t_j), \quad x \in I, \ \{t_j\}_{j=0}^{m} \subset I,$$

with t_j a node from the mesh created over I and

$$\|\mathrm{Q}_m\|_{op} = \sum_{j=0}^{m} |w_j k(x,t_j)|.$$

Presumably, the aim is to construct the sequence $\{\phi_m\}_{m\ge 2} \subset \mathcal{C}$ such that

$$\lim_{m\to\infty} \|\phi - \phi_m\|_\infty = 0,$$

which, by the following pointwise convergence

$$\lim_{m\to\infty} \mathrm{Q}_m\psi(x) = \mathrm{A}\psi(x), \quad x \in I, \quad \psi(\cdot) \in \mathcal{C},$$

one obtains a primal convergence result.

However, in terms of operator-norm, this is not a straightforwardly achieved aim as Nyström's method posits a nuanced approach of convergence and error analysis such that the sequence of compact endomorphisms $\{\mathrm{Q}_m\}_{m\geq 2} \subset \mathscr{L}(\mathcal{C})$ does not satisfy

$$\lim_{m\to\infty} \|\mathrm{Q}_m - \mathrm{A}\|_{op} = 0.$$

Since this sequence verifies the collectively-compact convergence towards the endomorphism A which implies that we have

$$\lim_{m\to\infty} \|(\mathrm{Q}_m - \mathrm{A})\mathrm{A}\|_{op} = \lim_{m\to\infty} \|(\mathrm{Q}_m - \mathrm{A})\mathrm{Q}_m\|_{op} = 0.$$

this is a guarantee that, together with the compactness of A, (2.2) entertains the invertibility of the endomorphism $(\lambda - \mathrm{Q}_m)$ for all sufficiently large ranks m. Consequently, one obtains the estimation

$$m \gg 1, \quad \|\phi_m - \phi\|_\infty \leq \|(\lambda - \mathrm{Q}_m)^{-1}\|_{op}\|\mathrm{A}\phi - \mathrm{Q}_m\phi\|_\infty. \tag{2.4}$$

For a profounder explanation, one may refer to [4, 6].

The convention of discretising (2.1) to obtain (2.3) by choosing, without loss of generality, the composite closed two-point trapezoidal rule such that with

$$m \geq 2, \quad d = (b-a)/m, \quad t_j = a + jd, \quad 0 \leq j \leq m,$$

which yields the scheme

$$\lambda\phi_m(x) - \frac{d}{2}\sum_{j=0}^{m} k(x, t_j)\phi_m(t_j) = f(x), \quad x \in I,$$

where a collocation over the nodes t_j of the mesh gives the $(m+1)^2$-linear system of equations:

$$\lambda\phi_m(t_i) - \frac{d}{2}\sum_{j=0}^{m} k(t_i, t_j)\phi_m(t_j) = f(t_i), \quad 0 \leq i \leq m,$$

such that the unknown vector $(\phi_m(t_j))_{j=0}^m$, that could be approximately found using an iterative scheme like Jacobi's, Gauss-Seidel's, etc., is leveraged in the usual Lagrange's interpolating formula:

$$\phi_m(x) \approxeq \sum_{j=0}^{m} \phi_m(t_j)\ell_j(x), \quad x \in I,$$

or in Nyström's interpolating formula[5]:

$$\phi_m(x) \approxeq \frac{1}{\lambda}\left(f(x) + \frac{d}{2}\sum_{j=0}^{m} k(x,t_j)\phi_m(t_j) \right), \quad x \in I,$$

to build an approximation $\phi_m(\cdot)$ of the solution $\phi(\cdot)$.

In [28], it was demonstrated that such a conventional approach failed to meet the expectations when one operated over a substantially large interval.

Algorithm 1 Computation of the approximate solution $\phi_m(\cdot)$ of (2.1) following the discretise-then-iterate method

---

Require: λ, a, b, and $m \geq 2$
Ensure: $\phi_m(t_j)$, for all $0 \leq j \leq m$

$d = \frac{b-a}{m}$
$t_0 = a, t_m = b$
while $1 \leq i \leq m-1$ **do**
  $t_i \leftarrow t_{i-1} + d$
end while
while $0 \leq i \leq m$ **do**
  $F(i) = f(t_i)$
end while
while $0 \leq i \leq m$ **do**
  while $0 \leq j \leq m$ **do**
    $K(i,j) = \frac{d}{2}k(t_i, t_j)$
  end while
end while
$A = \lambda \mathrm{Id}_{\mathbb{R}^{m+1}} - K$
Choose Jacobi's or Gauss-Seidel's scheme to approximate the solution of the linear system $AV = F$, where $V = \left(\phi_m(t_j)\right)_{j=0}^{m}$
Rebuild $\phi_m(\cdot)$ using an interpolation formula

---

5. The weights are w.r.t. the quadrature in use; here, we use the composite closed two-point trapezoidal rule. In general, they are taken as w_j and computed as explained in the Preliminaries' chapter.

2.3 Iterate then Discretise: Generalised Jacobi's and Gauss-Seidel's Schemes

The remedy of the discussed failure results in the break of the conventions that starts in setting, for $N \geq 2$, the mesh[6]

$$H = (b-a)/N, \qquad \begin{cases} x_i = x_{i-1} + H, & 1 \leq i \leq N-1, \\ x_0 = a, \ x_N = b, \end{cases}$$

which, using the so-called Chales rule for definite integration, leads to writing (2.1) in the following equivalent form

$$\lambda\phi(x) - \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} k(x,t)\phi(t)\,dt = f(x).$$

With the premise that $I_i = [x_{i-1}, x_i]$, the notations

$$\phi_i(x) \equiv \phi(x), \quad f_i(x) \equiv f(x), \quad x \in I_i, \quad 1 \leq i \leq N,$$

result in yet another form of (2.1) where one thus creates an $(N \times N)$-linear system of Fredholm's integral equations of the second kind:

$$\lambda\phi_i(x) - \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} k(x,t)\phi_j(t)\,dt = f_i(x), \quad x \in I_i, \quad 1 \leq i \leq N.$$

Hence, denoting by $\mathcal{C}_i$ the Banach space $C(I_i)$ endowed with its uniform convergence i-norm $\|\cdot\|_i$:

$$\|g\|_i = \|g\|_\infty, \quad g \in \mathcal{C}_i,$$

we introduce the family of compact integral operators $\{\mathrm{A}_{ij}\}_{1=i,j}^{N}$ where

$$\begin{aligned} \mathrm{A}_{ij} \colon \mathcal{C}_j &\mapsto \mathcal{C}_i \\ g &\mapsto \mathrm{A}_{ij}g(x) = \int_{x_{j-1}}^{x_j} k(x,t)g(t)\,dt, \quad x \in I_i. \end{aligned}$$

With the agreed-upon definition for the operator-norm, where 0_i denotes the additive neutral element of $\mathcal{C}_i$, we have that:

$$\|\mathrm{A}_{ij}\|_{op} = \sup_{g \neq 0_j} \frac{\|\mathrm{A}_{ij}g\|_i}{\|g\|_j} = \left\| \int_{x_{j-1}}^{x_j} k(\cdot,t)\,dt \right\|_i, \quad 1 \leq i,j \leq N,$$

6. Our choice of the mesh is the basic standard. However, other convexly uniformly formed meshes within I may also work with respectful minor to no effects on the outcome resulting thereof.

implies the belongingness $\mathrm{A}_{ij} \in \mathscr{L}(\mathcal{C}_j, \mathcal{C}_i)$.

Next, defining the product-vector-space $\mathscr{B}$ by

$$\mathscr{B} := \prod_{i=1}^{N} \mathcal{C}_i,$$

one introduces a Banach structure over it with the norm

$$\|G\|_{\mathscr{B}} = \max_{i=1}^{N} \|g_i\|_i, \quad G \in \mathscr{B}.$$

Moreover, denoting by $\mathscr{M}$ the vector-space of matrices of bounded linear operators which is identifiable with the vector-space $\mathscr{L}(\mathscr{B})$, one is allowed to set the norm of operator matrices that is very similar in expression to $\|\cdot\|_{\infty,1}$ (see Preliminaries, Section 2):

$$\|\mathcal{T}\|_{\mathscr{M}} = \max_{i=1}^{N} \sum_{j=1}^{N} \|\mathrm{T}_{ij}\|_{op}, \quad \mathcal{T} \in \mathscr{M}.$$

With Id_{ii} denoting the identity operator of $\mathcal{C}_i$, one defines the identity operator-matrix of $\mathscr{B}$, element of $\mathscr{M}$, by

$$\mathcal{I} = \delta_{ij}\mathrm{Id}_{ii}, \quad 1 \leq i, j \leq N,$$

where δ_{ij} denotes Kronecker's delta. At this stage, one is prepared for the following definition

Definition 2.3.1. *(SDD-R Operator-matrices) Let $\mathcal{M} = (\mathrm{M}_{ij})_{i,j=1}^{N} \in \mathscr{M}$. We say that $\mathcal{M}$ is an SDD-R operator-matrix if and only if we have*

$$\|\mathrm{M}_{ii}\|_{op} > \sum_{\substack{j=1 \\ i \neq j}}^{N} \|\mathrm{M}_{ij}\|_{op}.$$

Equivalently, we say it is SDD-C[7] if $\mathcal{M}^T$ is an SDD-R, where

$$(\mathcal{M}^T)_{ij} = \mathrm{M}_{ji}, \quad 1 \leq i, j \leq N.$$

The structures set hereinabove facilitate the task of treating (2.1) where one produces the following abstract linear matrix-vector system out of it:

$$(\lambda\mathcal{I} - \mathcal{A})\,\Phi = F, \tag{2.5}$$

7. SDD-C: Strictly Diagonally-dominant by Columns.

with $F = (f_i)_{i=1}^N \in \mathscr{B}$, and $\mathcal{A} = (\mathrm{A}_{ij})_{i,j=1}^N \in \mathscr{M}$ constituting the given data, and Φ being the unknown function-vector to be found in $\mathscr{B}$. It is, then, straightforward to notice that (2.2) implies

$$\begin{aligned} |\lambda| &> \|\mathrm{A}\|_{op} \\ &= \max_{a \le x \le b} \int_a^b |k(x,t)|\, dt \\ &= \max_{a \le x \le b} \sum_{j=1}^N \int_{x_{j-1}}^{x_j} |k(x,t)|\, dt \\ &= \max_{i=1}^N \sum_{j=1}^N \|\mathrm{A}_{ij}\|_{op} = \|\mathcal{A}\|_{\mathscr{M}}. \end{aligned} \tag{2.6}$$

Hence, consequence to The Geometric Series Theorem in its inclusive statement and to Fredholm's Alternative Theorem, (2.5) is uniquely invertible with the formal solution

$$\Phi = (\lambda \mathcal{I} - \mathcal{A})^{-1} F.$$

Similar to the algebraic case, the stance of assuming the splitting of $\mathcal{A}$:

$$\mathcal{A} = \mathcal{D} - \mathcal{L} - \mathcal{U},$$

where $\mathcal{D}, -\mathcal{L}, -\mathcal{U}$ are the diagonal, the lower, and the upper parts of $\mathcal{A}$, respectively, allows for:

$$\|\mathcal{A}\|_{\mathscr{M}} \ge \|\mathcal{D}\|_{\mathscr{M}},$$

as well as

$$\|\mathcal{A}\|_{\mathscr{M}} \ge \|\mathcal{D} - \mathcal{L}\|_{\mathscr{M}},$$

which, by (2.6), justify writing $(\lambda \mathcal{I} - \mathcal{D})^{-1}$ and $(\lambda \mathcal{I} - \mathcal{D} + \mathcal{L})^{-1}$; thus generating the generalised versions of the Jacobi (**GJ**) and the Gauss-Seidel (**GG-S**) schemes as follows:

$$\mathbf{GJ} : \begin{cases} \Phi^{k+1} & = (\lambda \mathcal{I} - \mathcal{D})^{-1} \left((\mathcal{L} + \mathcal{U})\, \Phi^k + F \right), \quad k \ge 0, \\ \Phi^0 & \in \mathscr{B}, \end{cases} \tag{2.7}$$

and

$$\mathbf{GG\text{-}S} : \begin{cases} \Phi^{k+1} & = (\lambda \mathcal{I} - \mathcal{D} + \mathcal{L})^{-1} \left(\mathcal{U} \Phi^k + F \right), \quad k \ge 0, \\ \Phi^0 & \in \mathscr{B}. \end{cases} \tag{2.8}$$

Component-wise, without invoking the inverse of the operator-matrices as shown above, (2.8) and (2.9) can be expressed as

$$\mathbf{GJ} : \begin{cases} \lambda \phi_i^{k+1} & = \mathrm{A}_{ii}\phi_i^{k+1} + \sum_{\substack{j=1 \\ i \neq j}}^{N} \mathrm{A}_{ij}\phi_j^{k} + f_i, \quad k \geq 0, \\ \phi_i^0 & \in \mathcal{C}_i, \quad 1 \leq i \leq N, \end{cases} \tag{2.9}$$

and

$$\mathbf{GG\text{-}S} : \begin{cases} \lambda \phi_i^{k+1} & = \sum_{j=1}^{i} \mathrm{A}_{ij}\phi_j^{k+1} + \sum_{j=i+1}^{N} \mathrm{A}_{ij}\phi_j^{k} + f_i, \quad k \geq 0, \\ \phi_i^0 & \in \mathcal{C}_i, \ 1 \leq i \leq N. \end{cases} \tag{2.10}$$

(2.6) implies that $(\lambda \mathcal{I} - \mathcal{A})$ is SDD-R since

$$|\lambda| = \|\lambda \mathcal{I}\|_{\mathscr{M}},$$

and that, for all the rows $1 \leq i \leq N$,

$$\begin{aligned} \sum_{\substack{j=1 \\ i \neq j}}^{N} \|\mathrm{A}_{ij}\|_{op} &< |\lambda| - \|\mathrm{A}_{ii}\|_{op} \\ &\leq \Big| \|\lambda \mathrm{Id}_{ii}\|_{op} - \|\mathrm{A}_{ii}\|_{op} \Big| \\ &\leq \|\lambda - \mathrm{A}_{ii}\|_{op}, \end{aligned}$$

which allows for the predetermination of convergence for both of **GJ** and **GG-S**, as well as its allowance of the application of Banach's Fixed-point Theorem in its general statement where the general statement of The Geometric Series Theorem clarifies how the combination of (2.2) and (2.6) results in obtaining the necessary *spectral* condition off the sufficient *norm* condition for the convergence of either of (2.7) and (2.8):

$$\rho(\mathcal{J}) \leq \|\mathcal{J}\|_{\mathscr{M}} < 1,$$

with $\mathcal{J}$ denoting the iteration matrix of **GJ**:

$$\mathcal{J} = (\lambda \mathcal{I} - \mathcal{D})^{-1} (\mathcal{L} + \mathcal{U}),$$

or of **GG-S**:

$$\mathcal{J} = (\lambda \mathcal{I} - \mathcal{D} - \mathcal{L})^{-1} \mathcal{U}.$$

Subsequently, it is inevitable that one be cast into the discrete world of numbers where the iterative process discussed previously is followed by the discretisation that is based on Nyström's method, such that, for $n \geq 2$, the sub-mesh nodes over the compact subinterval I_i:

$$1 \leq i \leq N, \quad h = (x_i - x_{i-1})/n, \quad \begin{cases} x^i_q = x^i_{q-1} + h, & 1 \leq q \leq n-1, \\ x^i_0 = x_{i-1}, \ x^i_n = x_i, \end{cases}$$

help define the discrete integral operator

$$\mathrm{A}_{iin} g(x) = \sum_{q=0}^{n} w_q k(x, x^i_q) g(x^i_q), \quad x \in I_i, \quad g \in \mathcal{C}_i, 1 \leq i \leq N.$$

That A_{iin} is a compact element of $\mathscr{L}(\mathcal{C}_i)$ is clear with

$$\|\mathrm{A}_{iin}\|_{op} = \max_{x \in I_i} \sum_{q=0}^{n} |w_q k(x, x^i_q)|, \quad 1 \leq i \leq N,$$

and such that the family $\{\mathrm{A}_{iin}\}_n \subset \mathscr{L}(\mathcal{C}_i)$ is collectively-compact convergent to A_{ii} which de facto offers the pointwise convergence:

$$\mathrm{A}_{iin} g \xrightarrow{n \to \infty} \mathrm{A}_{ii} g, \quad g \in \mathcal{C}_i, \quad 1 \leq i \leq N,$$

and establishes the discrete versions of **GJ**:

$$\mathbf{GJ}_n : \begin{cases} \lambda \phi^{k+1}_{in} & = \mathrm{A}_{iin} \phi^{k+1}_{in} + \sum\limits_{\substack{j=1 \\ i \neq j}}^{N} \mathrm{A}_{ij} \phi^k_{jn} + f_i, \quad k \geq 0, \\ \phi^0_{in} & \in \mathcal{C}_i, \quad 1 \leq i \leq N, \end{cases} \tag{2.11}$$

and of **GGS**:

$$\mathbf{GG\text{-}S}_n : \begin{cases} \lambda \phi^{k+1}_{in} & = \mathrm{A}_{iin} \phi^{k+1}_{in} + \sum\limits_{j=1}^{i-1} \mathrm{A}_{ij} \phi^{k+1}_{jn} + \sum\limits_{j=i+1}^{N} \mathrm{A}_{ij} \phi^k_{jn} + f_i, \quad k \geq 0, \\ \phi^0_{in} & \in \mathcal{C}_i, \ 1 \leq i \leq N. \end{cases} \tag{2.12}$$

On the other hand, we have the estimation

$$k \geq 0, \quad n \gg 2 \implies \|\phi^k_i - \phi^k_{in}\|_i \leq \|(\lambda - \mathrm{A}_{iin})^{-1}\|_{op} \|\mathrm{A}_{ii}\phi^k_i - \mathrm{A}_{iin}\phi^k_i\|_i.$$

Referring to [28], one observes how the above estimation combines with the convergence of **GJ** or of **GG-S** to produce a function-sequence $\{\Phi^k_n : n \geq 2, \ k \geq 0\}$ of $\mathscr{B}$ that converges to the exact solution Φ of (2.5) which is equivalent to finding the solution of (2.1), where we write

$$\lim_{k,n \to \infty} \|\Phi - \Phi^k_n\|_{\mathscr{B}} = 0.$$

2.4 Truncation: The Refinement

Even though the invertibility of $(\lambda - \mathrm{A}_{ii})$ implies that of $(\lambda - \mathrm{A}_{iin})$ when n is sufficiently great, it is not clear whether one has

$$|\lambda| > \|\mathrm{A}_{iin}\|_{op},$$

hence the indecisive applicability of The Geometric Series Theorem in its general statement. In other terms, it would still remain a question to write or not to write the following

$$(\lambda - \mathrm{A}_{iin})^{-1} = \sum_{r=0}^{\infty} \lambda^{-r-1}\mathrm{A}_{iin}^{r}, \quad \mathrm{A}_{iin}^{0} := \mathrm{Id}_{ii}, \quad 1 \le i \le N, \quad n \ge 2. \tag{2.13}$$

Consequently, the answer is provided in the following cornerstone lemma that feeds off quadrature rules' stability where both of the following two assertions hold:

1. The weights of the quadrature rule are positive, and
2. The sums of the weights of the quadrature rule are uniformly bounded regardless of the increasing number of the weights; i.e,

$$\sup \sum (\text{weights of the quadrature}) < \infty.$$

As a result, we have

Lemma 2.4.1. *Denote the resolvent set of* A_{iin} *by* $\mathrm{re}(\mathrm{A}_{iin})$ *where*

$$\mathrm{re}(\mathrm{A}_{iin}) = \mathbb{C} \setminus \sigma(\mathrm{A}_{iin}), \quad 1 \le i \le N, \quad n \ge 2.$$

Then, using stable quadrature rules, for all rank $n \ge n_0$, *with* n_0 *sufficiently greater than 2, the operator* $(\lambda - \mathrm{A}_{iin})$ *is invertible with*

$$n \ge n_0 \gg 2, \quad \lambda \in \mathrm{re}(\mathrm{A}_{iin}) \quad \textit{and} \quad |\lambda| > \|\mathrm{A}_{iin}\|_{op}.$$

Proof. By hypothesis, the positivity of the utilised stable quadrature rule's weights allows for the following consideration

$$\begin{aligned} \|\mathrm{A}_{iin}\|_{op} &= \max_{x \in I_i} \sum_{q=0}^{n} |w_q k(x, x_q^i)| \\ &= \max_{x \in I_i} \sum_{q=0}^{n} w_q |k(x, x_q^i)|. \end{aligned}$$

It is easily noticed that, setting

$$\psi_x(\cdot) = |k(x,\cdot)|, \quad x \in I_i,$$

we introduce an element of $\mathcal{C}_i$ which implies that, in a different statement, if

$$\kappa_n(x) = \sum_{q=0}^{n} w_q \psi_x(x_q^i), \quad x \in I_i,$$

the convergence of stable quadrature rules to the exact definite integral of continuous functions over compact intervals yields

$$\lim_{n\to\infty} \kappa_n(x) = \kappa_\infty(x) = \int_{x_{i-1}}^{x_i} \psi_x(t)\, dt = \int_{x_{i-1}}^{x_i} |k(x,t)|\, dt.$$

Hence,

$$\begin{aligned}
\lim_{n\to\infty} \|\mathrm{A}_{iin}\|_{op} &= \lim_{n\to\infty} \max_{x\in I_i} \kappa_n(x) \\
&= \max_{x\in I_i} \kappa_\infty(x) \\
&= \max_{x\in I_i} \int_{x_{i-1}}^{x_i} |k(x,t)|\, dt \\
&= \|\mathrm{A}_{ii}\|_{op}.
\end{aligned}$$

Consequently, setting the sequence of positive real numbers

$$\{\alpha_n = \|\mathrm{A}_{iin}\|_{op}\}_n \subset \mathbb{R},$$

we have by the above convergence that

$$\lim_{n\to\infty} \alpha_n = \alpha = \|\mathrm{A}_{ii}\|_{op},$$

which translates into the following ϵ-statement:

$$\forall \epsilon > 0, \quad \exists n_0 \gg 2 : \quad \forall n \geq n_0, \quad |\alpha_n - \alpha| < \epsilon.$$

In particular, choosing

$$\epsilon = |\lambda| - \alpha,$$

guarantees the existence of a rank $n_0 \gg 2$ starting from which we have

$$\alpha_n - \alpha \leq |\alpha_n - \alpha| < \epsilon \implies \alpha_n < |\lambda|, \quad n \geq n_0 \gg 2.$$

□

However, it is worthy to note that such a lemma finds place only when the used quadrature rule's weights are positive. The reader may investigate this area independently and we refer to [14].

As a consequence, it is justified that one endorses (2.13) permitting for the truncation of the geometric series of operators into the following geometric p-sum of operators:

$$(\lambda - \mathrm{A}_{iin})^{-1} \approx \sum_{r=0}^{p} \lambda^{-r-1}\mathrm{A}_{iin}^{r}, \quad \mathrm{A}_{iin}^{0} := \mathrm{Id}_{ii}, \quad 1 \leq i \leq N, \quad n \geq 2.$$

Ensuing, two three-parameter discrete schemes follow as attributes to both of $\mathbf{GJ}_n$ [8]:

$$\mathbf{GJ}_n(p) : \begin{cases} \phi_{in}^{k+1}(p) &= \left[\sum_{r=0}^{p} \lambda^{-r-1}\mathrm{A}_{iin}^{r}\right]\left(\sum_{\substack{j=1 \\ i\neq j}}^{N} \mathrm{A}_{ij}\phi_{jn}^{k}(p) + f_i\right), \quad k \geq 0, \\ \phi_{in}^{0}(p) &\in \mathcal{C}_i, \quad 1 \leq i \leq N, \end{cases} \tag{2.14}$$

and $\mathbf{GG\text{-}S}_n(p)$:

$$\mathbf{GG\text{-}S}_n(p) : \begin{cases} \phi_{in}^{k+1}(p) = \left[\sum_{r=0}^{p} \lambda^{-r-1}\mathrm{A}_{iin}^{r}\right]\Big(\sum_{j=1}^{i-1} \mathrm{A}_{ij}\phi_{jn}^{k+1}(p) \\ \qquad\qquad + \sum_{j=i+1}^{N} \mathrm{A}_{ij}\phi_{jn}^{k}(p) + f_i\Big), \quad k \geq 0, \\ \phi_{in}^{0}(p) \in \mathcal{C}_i, \quad 1 \leq i \leq N, \end{cases} \tag{2.15}$$

Although falsely trivial at first glance, it is inevitable to examine the convergence of both $\mathbf{GJ}_n(p)$ and $\mathbf{GG\text{-}S}_n(p)$ subsequent to the application of the truncation's concept; a result supported by:

Theorem 2.4.1. *Over the Banach space $\mathscr{B}$, both of the schemes $\boldsymbol{GJ}_n(p)$ and $\boldsymbol{GG\text{-}S}_n(p)$ generate a function-vector sequence $\{\Phi_n^k(p) \ : \ p,k \geq 0, n \geq 2\}$ that converges in the $\mathscr{B}$-norm to the function-vector $\Phi \in \mathscr{B}$ solution of* (2.5), *equivalently prompting the required data to build the solution of* (2.1).

8. However missing its own conference proceedings paper, $\mathbf{GJ}_n(p)$ had been the substance of discussion around which we took part (online) in a hybrid international conference held in İstanbul, Türkiye, under the name *"6th International Conference on Mathematics: An İstanbul Meeting for World's Mathematicians"*.

Proof. To avoid redundancy, we opt for omitting the proof on the convergence of $\mathbf{GJ}_n(p)$ since it is identical to what shall follow with adequate adjustments in summation indices. Hence, in regards to $\mathbf{GG\text{-}S}_n(p)$, observe that we have

$$\begin{aligned}\|\phi_i - \phi_{in}^k(p)\|_i \leq & \|\phi_i - \phi_i^k\|_i + \|\phi_i^k - \phi_{in}^k\|_i \\ & + \|\phi_{in}^k - \phi_{in}^k(p)\|_i, \quad 1 \leq i \leq N,\ n \geq 2,\ p \geq 0.\end{aligned}$$

Then, establish the following three sums over $\mathcal{C}_i$:

$$S_i^k = \sum_{j=1}^{i-1} \mathrm{A}_{ij}\phi_j^{k+1} + \sum_{j=i+1}^{N} \mathrm{A}_{ij}\phi_j^k + f_i,$$

$$S_{in}^k = \sum_{j=1}^{i-1} \mathrm{A}_{ij}\phi_{jn}^{k+1} + \sum_{j=i+1}^{N} \mathrm{A}_{ij}\phi_{jn}^k + f_i,$$

$$S_{in}^k(p) = \sum_{j=1}^{i-1} \mathrm{A}_{ij}\phi_{jn}^{k+1}(p) + \sum_{j=i+1}^{N} \mathrm{A}_{ij}\phi_{jn}^k(p) + f_i.$$

We shall follow an inductive proof, such that at iteration $k = 0$, since one has

$$p \geq 0, \quad \phi_{in}^0 = \phi_{in}^0(p),$$

then it follows that

$$\|\phi_{in}^0 - \phi_{in}^0(p)\|_i = 0 = \lim_{p\to\infty} \|\phi_{in}^0 - \phi_{in}^0(p)\|_i, \quad n \geq 2, \quad 1 \leq i \leq N.$$

Next, suppose that the assertion holds up to iteration $k \geq 1$:

$$\lim_{p\to\infty} \|\phi_{jn}^{k-1} - \phi_{jn}^{k-1}(p)\|_i = 0, \quad 1 \leq j \leq i-1,$$

and, fact that we are treating the Gauss-Seidel concept[9], we also have that

$$\lim_{p\to\infty} \|\phi_{jn}^{k} - \phi_{jn}^{k}(p)\|_i = 0, \quad i+1 \leq j \leq N.$$

9. In $\mathbf{GJ}_n(p)$, we would not assume the same, as the only used terms originate in the previous iteration and one cannot predetermine anything on the current iteration.

Consequently, we have the following manipulation

$$\begin{aligned}\phi_{in}^{k} - \phi_{in}^{k}(p) &= (\lambda - \mathrm{A}_{iin})^{-1} S_{in}^{k-1} - \left(\sum_{r=0}^{p} \lambda^{-r-1}\mathrm{A}_{iin}^{r}\right) S_{in}^{k-1}(p) \\ &= \left(\sum_{r=0}^{p} \lambda^{-r-1}\mathrm{A}_{iin}^{r}\right)\left(S_{in}^{k-1} - S_{in}^{k-1}(p)\right) \\ &+ \left((\lambda - \mathrm{A}_{iin})^{-1} - \sum_{r=0}^{p} \lambda^{-r-1}\mathrm{A}_{iin}^{r}\right) S_{in}^{k-1}.\end{aligned}$$

On one hand, by virtue of The Geometric Series Theorem, we have that

$$(\lambda - \mathrm{A}_{iin})^{-1} - \sum_{r=0}^{p} \lambda^{-r-1}\mathrm{A}_{iin}^{r} = \sum_{r=p+1}^{\infty} \lambda^{-r-1}\mathrm{A}_{iin}^{r} \xrightarrow{p\to\infty} 0_{\mathscr{L}(\mathcal{C}_i)}, \quad \mathscr{L}(\mathcal{C}_i).$$

On the other, we have

$$S_{in}^{k-1} - S_{in}^{k-1}(p) = \sum_{j=1}^{i-1}\left(\mathrm{A}_{ij}(\phi_{in}^{k} - \phi_{jn}^{k}(p))\right) + \sum_{j=i+1}^{N}\left(\mathrm{A}_{ij}(\phi_{in}^{k-1} - \phi_{jn}^{k-1}(p))\right),$$

which is a quantity convergent to $0_{\mathcal{C}_i}$ for all iterations $k \geq 1$. Hence, the result also holds at iteration $(k+1)$, and by principle of induction, we have

$$\lim_{p\to\infty} \|\phi_{in}^{k} - \phi_{in}^{k}(p)\|_i = 0, \quad n \geq 2, \quad k \geq 0, \quad 1 \leq i \leq N.$$

Next, similar in reasoning to the above, the convergence w.r.t. parameter n is consequence to following the manipulation

$$\begin{aligned} k \geq 1, \quad \phi_{i}^{k} - \phi_{in}^{k} &= (\lambda - \mathrm{A}_{ii})^{-1} S_{i}^{k-1} - (\lambda - \mathrm{A}_{iin})^{-1} S_{in}^{k-1} \\ &= (\lambda - \mathrm{A}_{iin})^{-1}\left(S_{i}^{k-1} - S_{in}^{k-1}\right) \\ &+ \left((\lambda - \mathrm{A}_{ii})^{-1} - (\lambda - \mathrm{A}_{iin})^{-1}\right) S_{i}^{k-1},\end{aligned}$$

where, by virtues of

$$\mathrm{D}_n = \left((\lambda - \mathrm{A}_{ii})^{-1} - (\lambda - \mathrm{A}_{iin})^{-1}\right) = (\lambda - \mathrm{A}_{ii})\,(\mathrm{A}_{iin} - \mathrm{A}_{ii})\,(\lambda - \mathrm{A}_{iin}),$$

and the pointwise convergence of the family $\{\mathrm{A}_{iin}\}_n$ towards A_{ii} in $\mathcal{C}_i$, we have that

$$\lim_{n\to\infty} \mathrm{D}_n = 0_{\mathscr{L}(\mathcal{C}_i)}, \quad 1 \leq i \leq N.$$

Moreover, considerate of the offerings by Gauss-Seidel scheme, the application of inductive reasoning proves that

$$\lim_{n\to\infty}\left(S_i^{k-1} - S_{in}^{k-1}\right) = 0_{\mathcal{C}_i}, \quad k \geq 0, \quad 1 \leq i \leq N.$$

Thus,

$$\lim_{n\to\infty} \|\phi_i^k - \phi_{in}^k\|_i = 0, \quad k \geq 0, \quad 1 \leq i \leq N.$$

Additionally, owing to the estimation from Banach's Fixed-point Theorem, we obtain

$$\|\phi_i - \phi_i^k\|_i \leq \frac{\|\mathcal{J}\|_{\mathscr{M}}^k}{1 - \|\mathcal{J}\|_{\mathscr{M}}} \|\phi_i^1 - \phi_i^0\|_i, \quad k \geq 0, \quad 1 \leq i \leq N,$$

from which, as discussed previously in the text, we have a priori determined that $\|\mathcal{J}\|_{\mathscr{M}} < 1$; that is, **GG-S** is convergent. Thus, we have

$$\lim_{k\to\infty} \|\phi_i - \phi_i^k\|_i = 0, \quad 1 \leq i \leq N.$$

As a result, taking the maximum over $1 \leq i \leq N$, one concludes that

$$\lim_{p,n,k\to\infty} \|\Phi - \Phi_n^k(p)\|_{\mathscr{B}} = 0.$$

□

2.5 Numerical Implementations: Concluding Remarks

Realised on a Windows 10, ×64 bits system, 6th-Gen i5-PC, we carry out some illustrative numerical examples, and, once again, all definite integrals are approximated using composite closed two-point trapezoidal quadrature rule.

For a first example, letting $\tau \gg 0$, we set over the interval $[0, \tau]$ the following

$$\lambda\phi(x) - \int_0^\tau \sin(\alpha x t)\phi(t)\,dt = f(x), \quad 0 \leq x \leq \tau, \tag{2.16}$$

where for positivity of the bi-continuous sine-kernel over the compact interval taken, we have $\alpha = \frac{\pi}{\tau^2}$, and to operate continuous functions, note that since we have the following maximisation procedure

$$\max_{[0,\tau]} \int_0^\tau |\sin(\alpha x t)|\,dt = \max_{[0,\tau]} \left(\frac{\tau^2}{\pi x} - \frac{\tau^2}{\pi x}\cos\left(\frac{\pi x}{\tau}\right)\right) \approxeq \frac{\tau\pi}{4.3355},$$

which is achieved for $x^* \approxeq \dfrac{\tau\pi}{4.232}$, it is justified that, for the value $\lambda = 2\tau$, the ensuing unique exact solution of (2.7):

$$\phi(x) = x, \quad 0 \leq x \leq \tau,$$

is the result of choosing the source-function as

$$f(x) = \begin{cases} \lambda x + \dfrac{\tau}{\alpha x}\cos(\tau\alpha x) - \dfrac{\sin(\tau\alpha x)}{\alpha^2 x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Consequently, setting $\tau = 200$, fixing $N = 200$, increasingly varying the p, iterating over k, and making the following table

Scheme	Error Notation	Error Formula
$\mathbf{GJ}_n$	e	$\lVert\Phi - {}^{J}\Phi_n^k\rVert_{\mathscr{B}}$
$\mathbf{GJ}_n(p)$	e_p	$\lVert\Phi - {}^{J}\Phi_n^k(p)\rVert_{\mathscr{B}}$
$\mathbf{GG\text{-}S}_n$	E	$\lVert\Phi - {}^{GS}\Phi_n^k\rVert_{\mathscr{B}}$
$\mathbf{GG\text{-}S}_n(p)$	E_p	$\lVert\Phi - {}^{GS}\Phi_n^k(p)\rVert_{\mathscr{B}}$

With a 10^{-8}-tolerance, one obtains tables 2.1, 2.2, 2.3, and 2.4:

n	e	Time (sec)
9	2.478061560395872E-07	66.33
21	3.933695325031295E-08	218.69
41	9.561290426063351E-09	718.57

Table 2.1 – The Error of $\mathbf{GJ}_n$

n	e_p	Time (sec)
9	2.460836299178482E-07	64.51
21	3.775389245674887E-08	214.25
41	8.223963732234552E-09	701.15

Table 2.2 – The Error of $\mathbf{GJ}_n(p)$

We notice the effects of truncation comparing Table 2.1 with 2.2 and Table 2.3 with 2.4. The comparison between tables 2.1 and 2.3 or the eventual 2.2 and 2.4 is not of the greatest importance or interest as such point has been already discussed in [28], and it was made clear, as it is the case for the

n	E	Time (sec)
9	2.478061560395872E-07	18.41
21	3.933695325031294E-08	83.97
41	9.561290426063351E-09	292.01

Table 2.3 – The Error in $\mathbf{GG\text{-}S}_n$

n	E_p	Time (sec)
9	2.460836299178482E-07	17.79
21	3.775389245674887E-08	82.04
41	8.223963732234552E-09	289.52

Table 2.4 – The Error in $\mathbf{GG\text{-}S}_n(p)$

algebraic scenario, that Gauss-Seidel achieves what Jacobi does in lesser time.

Consequently, the p-truncations, although reliant on finite summations of a geometric series of endomorphisms, offer some gain in time as well in their achieving the same rank of errors as is observed; a profit set to accumulate the more n grows. However, to investigate it further, observe Table 2.5 where over $[0, 10]$ ($\tau = 10$), for ten subintervals ($N = 10$), and for 81 sub-nodes ($n = 81$), one obtains given the same tolerance criterion:

$\tau = 10, \quad N = 10, \quad n = 81$				
E	Time (sec)	p	E_p	Time (sec)
1.216945193505126E-07	13.62	2	6.111517816798084E-05	13.61
		3	3.103139251336984E-06	13.50
		4	2.677355901425926E-07	13.51
		5	1.288582343050848E-07	13.49
		6	1.220462735318506E-07	13.58
		7	1.217118050789168E-07	13.59
		8	1.216953684490818E-07	13.58
		9	1.216945637594335E-07	13.61
		10	1.216945229032262E-07	13.57

Table 2.5 – Varying the p

It is more clear how the truncation does not affect convergence and such that even as small as a three-term geometric sum, an error of rank

10^{-5} makes entrance with a monotonically decreasing behaviour that is parallel to a sequence of monotonically non-increasing elapsed execution time. Ultimately, as it is one of the crucial factors, by preserving the same tolerance of the example and choosing $p = 3$, we also investigate the effect of the interval's size on how well of an approach the truncation would be: As one notices, in general cases, synchronous to the interval's growth

$[0,\tau]$: $H = 1$, $\phi(x) = x$, $x \in [0,\tau]$					
τ	n	E	Time (sec)	E_p	Time (sec)
100	21	8.36226E-08	19.18	7.41688E-08	18.97
150	31	2.10201E-08	89.41	1.74087E-08	88.81
200	41	9.56129E-09	292.01	8.22396E-09	289.52

Table 2.6 – Influence of the interval's size

in length, λ is set to grow proportionately with the interval's size; a fact that explains the small summation rank at which neglect is exerted on the action of the powers of A_{iin}; i.e, the iterated endomorphic compositions A_{iin}^r, against those of in denominator λ^{r+1}. Hence, as seen in our choice, a value as small as $p = 3$ proves rather influential in gaining solid better convergence compared with invoking the inverse of an $(n+1) \times (n+1)$ matrix when, in practice, n is destined to grow. In other terms, over even larger intervals when $\tau > 200$, n is generally taken rather great, which helps visualise the amount of culminated gain in computational time and error efficiency. Again, this is an indisputably noteworthy aspect given the general difficulty of inverting algebraic square matrices with important sizes.

We conclude the section, as well as the chapter, by a second example where we set the following convolution-kernel Fredholm's integral equation of the second kind:

$$\lambda\psi(x) - \int_0^{500} |t - x|\psi(t)\,dt = h(x), \quad x \in [0, 500]. \tag{2.17}$$

It is clear that

$$\max_{[0,500]} \int_0^{500} |t - x|\,dt = \max_{[0,500]} \left(x^2 - 500x + \frac{500^2}{2} \right) = \frac{500^2}{2},$$

is achieved for $x^{**} = \dfrac{500^2}{2}$. Therefore, letting $\mu \geq 1$, the choice $\lambda = 500^2\mu$ testifies for the unique invertibility of (2.17) as well as its guaranteed convergence. Consequently, let $\mu = 2$. A solution for (2.17) is then given by

$$\psi(x) = x^{1/2}, \quad x \in [0, 500],$$

when the source-function term is chosen as

$$h(x) = \lambda\sqrt{x} + \left(\frac{2(10\sqrt{5})^3}{3}\right) x - \left(\frac{8}{15}\right) x^{5/2} - \left(\frac{2(10\sqrt{5})^5}{5}\right), \quad x \in [0, 500].$$

For a principal step $H = 10$, we have $N = 50$ subintervals, where, preserving the same tolerance as in the previous first example and working with the fixed choice $p = 3$ with the errors

Scheme	Error Notation	Error Formula
GG-S$_n$	Δ	$\|\Psi - {}^{GS}\Psi_n^k\|_{\mathscr{B}}$
GG-S$_n(p)$	Δ_p	$\|\Psi - {}^{GS}\Psi_n^k(p)\|_{\mathscr{B}}$

we construct Table 2.7 as

n	Δ	Time (sec)	Δ_p	Time (sec)
9	1.957418373166320E-06	3.82	1.957429979881908E-06	3.72
21	4.963218209752540E-07	17.49	4.963334383489837E-07	16.80
31	2.704285151367003E-07	35.35	2.704401111941479E-07	34.36
41	1.757889371845067E-07	58.07	1.758005474528090E-07	58.01

Table 2.7 – **GG-S**$_n$ versus **GG-S**$_n(p)$

Anew, considering the interval's size influence that we have discussed above, the method proves useful in terms of a merely continuous kernel function. The gains are clear, and identical results are achieved with no insisting resort to invoke the computation of a matrix's inverse.

Chapter 3

A Fredholm Integral Equation on a Hilbert: Symmetry, Projections, and the Functional GJOR

Contents

Despite having not been the subject of any national or international conference(s), nor being the subject of any peer-reviewed article, this chapter has been, in all cases, the substance of a participation in *Les journées guelmoises pour les mathématiques appliquées* (JGMAs), and such that we fully include it in the manuscript because we estimate having made significant findings in this regard.

Furthermore, we testify that, however intentionally not incorporated, even partially, in the preliminaries, the reader may find applications of elements from the theory of (Non-)Commutative Banach Algebras and the theory of C^*-Algebras. To this end, inclusion of glimpses of these concepts may be aptly indicated whenever required/needed. Nonetheless, for a broader coverage of the subject, it is recommended that one confers [23].

3.1 Motivation

In his article [29], Lemita considered the same problem (2.1)-(2.2) as presented in the previous chapter, where, using the same unconventional procedure of iterating-then-discretising, as well as the same abstract topological structures and attributes, he applied the fact[1] that states

> An ω-relaxation procedure that we attribute to an iterative scheme is the action of constructing a convex combination relating, at each iteration of the process, the $(1-\omega)$-multiple of the previous relaxed iteration with the current ω-multiple of the non-relaxed iteration of the scheme of concern.

Differently stated, based on the previous chapter's Section 2.3's notations, the relaxed k-iteration that we construct from **GJ** shall be expressed as

$$\Phi^k_{\mathbf{GJUR}} = (1-\omega)\Phi^{k-1}_{\mathbf{GJUR}} + \omega\Phi^k_{\mathbf{GJ}}, \quad k \geq 1. \tag{3.1}$$

We also have the relaxed k-iteration of **GG-S** expressed as:

$$\Phi^k_{\mathbf{GSUR}} = (1-\omega)\Phi^{k-1}_{\mathbf{GSUR}} + \omega\Phi^k_{\mathbf{GG\text{-}S}}, \quad k \geq 1. \tag{3.2}$$

Implicitly, the convexity of the combination prematurely implies that the investigations focus shall not deviate very much from the interval $[0,1]$, hence the letter **U** (for **Under**) instead of **O** (for **Over**)[2] in **GJUR** as well as in **GSUR**, and, indeed, the endpoint 1 generates the non-relaxed schemes; the other endpoint 0 generates non-conclusion where the treatment would be stationary at the initial guess Φ^0—therefore, it is omitted.

Subsequently, it was demonstrated in [29] that, provided (2.2) held, both of **GJUR** and **GSUR** discussed above converged for all the values $\omega \in (0,1]$. Although completely defensible, the study only relied on continuous functions (data and unknowns) which prompted the use of seemingly general Banach structures and the implicitly implied SDD-R nature of the data operator-matrix $(\lambda\mathcal{I} - \mathcal{A})$. The reader may refer to the previous chapter (Chapter 1) for notations and/or structures.

1. In the Preliminaries, the relaxation process was introduced by means of the splitting of matrices. This is, in fact, not at all contradictory to the statement hereinafter, as it is the result of applying the process discussed in Section 2 of the mentioned chapter.

2. This is not uncommon in the algebraic scenario, where the use of *under* and/or *over* relates in some sense to the eventual effects made on the convergence speed of the scheme under study.

In practice, functions are more than often the least regular: being square-integrable in one of the best cases, and kernels of the integral operators A may very often exhibit a sense of symmetry in case of real evaluations or of Hermitian-nature in case of complex evaluations. Thus, in this chapter, we shall consider, over the substantial-length open interval $\Omega = (a, b)$, the Fredholm's integral equation of the second kind as

$$\lambda u(x) - \int_\Omega k(x,t)u(t)\,dt = f(x), \quad x \text{ a.e.w in } \Omega, \tag{3.3}$$

where λ is taken a non-zero real number, the source and unknown functions $f(\cdot), u(\cdot)$, respectively, are supposed square-integrable over Ω, and the kernel is as well supposed to possess a square-integrability over the open square Ω^2; that is,

$$\iint_{\Omega^2} |k(x,t)|^2\,dA < \infty,$$

which makes of the endomorphism K of the real-Hilbert $L^2(\Omega) = \mathcal{L}$:

$$\mathrm{K}v(x) = \int_\Omega k(x,t)v(t)\,dt, \quad x \text{ a.e.w. in } \Omega, \quad v \in \mathcal{L},$$

a Hilbert-Schmidt operator:

$$\|\mathrm{K}\|_{HS}^2 := \iint_{\Omega^2} |k(x,t)|^2\,dA < \infty.$$

Consequently, as K is a compact endomorphism by default, the condition

$$\|\mathrm{K}\|_{HS} < |\lambda|, \tag{3.4}$$

implies that Fredholm's Alternative is applicable to $(\lambda - \mathrm{K})$ and thus (3.3) is uniquely invertible in $\mathcal{L}$.

3.2 Beforehand Examination: The Conventional Discretise-Iterate Path

Consistent with notations from Chapter 1 (Preliminaries), the Hilbert space $\mathcal{L}$ equipped with its usual topological structure induced by the usual inner-product $\langle \cdot, \cdot \rangle$,

$$\langle v, w \rangle = \int_\Omega v(s)w(s)\,ds, \quad v, w \in \mathcal{L},$$

such that

$$\|v\|_{\mathcal{L}}^2 := \langle v, v \rangle, \quad v \in \mathcal{L},$$

has the orthonormal Legendre's family, $\{\tilde{L}_q\}_{q \geq 0}$, for a Hilbert basis by which the truncating-projection endomorphism π_m of $\mathcal{L}$ is defined as

$$\pi_m v(x) := \sum_{q=0}^{m} \langle v, \tilde{L}_q \rangle \tilde{L}_q(x), \quad x \text{ a.e.w. in } \Omega, \quad v \in \mathcal{L},$$

creating of it an element of $\mathscr{L}(\mathcal{L})$, with

$$\|\pi_m\|_{op} \leq 1, \quad m \geq 0.$$

Hence, defining

$$\mathrm{K}_m v(x) := (\mathrm{K} \circ \pi_m)\, v(x) = \sum_{q=0}^{m} \langle v, \tilde{L}_q \rangle \mathrm{K}\tilde{L}_q(x), \quad x \text{ a.e.w. in } \Omega, \quad v \in \mathcal{L},$$

we introduce a discrete, finite-rank (compact) element of $\mathscr{L}(\mathcal{L})$ whose linearity is due to the bi-linearity of $\langle \cdot, \cdot \rangle$ and whose Hilbert-Schmidt norm verifies

$$\|\mathrm{K}_m\|_{HS} \leq \|\mathrm{K}\|_{HS}, \quad m \geq 0, \tag{3.5}$$

As a result, equation (3.3) transitions from the continuum into the discrete to have the form,

$$\lambda u_m(x) - \mathrm{K}_m u_m(x) = f(x), \quad x \text{ a.e.w. in } \Omega, \tag{3.6}$$

which, by (3.5), is a uniquely invertible[3] abstract discrete equation over $\mathcal{L}$.

Next, if one performs the inner-product of (3.5) with an arbitrary element $\tilde{L}_i$, $0 \leq i \leq m$, of the Legendre's Hilbert basis, a linear algebraic $((m+1) \times (m+1))$-system generates as follows

$$(\lambda - K)\, \mathbf{U}_m = \mathbf{F}_m, \quad m \geq 0, \tag{3.7}$$

where

$$\mathbf{U}_m = \begin{bmatrix} \langle u_m, \tilde{L}_0 \rangle \\ \vdots \\ \langle u_m, \tilde{L}_m \rangle \end{bmatrix}, \quad \mathbf{F}_m = \begin{bmatrix} \langle f, \tilde{L}_0 \rangle \\ \vdots \\ \langle f, \tilde{L}_m \rangle \end{bmatrix}, \quad m \geq 0,$$

3. Notice that, unlike the treatment of invertibility of discrete abstract equations involving discrete integral operators as shown in Chapter 1, the invertibility in our case is verified regardless of the size of the discretisation index m; i.e, m need not be neither sufficiently nor significantly great.

and

$$(\lambda - K) = \begin{bmatrix} \lambda - \langle \mathrm{K}\tilde{L}_0, \tilde{L}_0 \rangle & \cdots & -\langle \mathrm{K}\tilde{L}_m, \tilde{L}_0 \rangle \\ \vdots & \ddots & \vdots \\ -\langle \mathrm{K}\tilde{L}_m, \tilde{L}_0 \rangle & \cdots & \lambda - \langle \mathrm{K}\tilde{L}_m, \tilde{L}_m \rangle \end{bmatrix}, \quad m \geq 0.$$

Dissimilar with what shall ensue in the next sections, it is easily remarked that $(\lambda - K)$ is automatically symmetric by construction leading to

$$\sigma(K) \subset \mathbb{R}.$$

Moreover, given the fact that

$$\|K\|_F^2 = \sum_{i=1}^{m+1} \sum_{j=1}^{m+1} |\langle \mathrm{K}\tilde{L}_{i-1}, \tilde{L}_{j-1} \rangle|^2 = \|\mathrm{K}_m\|_{HS}^2,$$

it is then justified that, by the application of the Spectral Mapping Theorem[4], one has

$$\sigma(\lambda - K) = \{\lambda - \mu : \ \mu \in \sigma(K)\},$$

where, following (3.4) and (3.5), it is implied that

$$\begin{cases} (\lambda - K) \text{ is SPD}, & \lambda > 0, \\ (\lambda - K) \text{ is SND}, & \lambda < 0. \end{cases}$$

Consequently, it follows that the **JOR** (see Preliminaries) is applicable and such that Theorem 1.2.4 applies to (3.7) making it possible to uniquely approximately find the *coefficients* of the approximate solution $u_m(\cdot)$ within the orthonormal Legendre's basis, which leads to naturally constructing

$$u_m(x) = \sum_{j=0}^{m} \langle u_m, \tilde{L}_j \rangle \tilde{L}_j(x), \quad x \text{ a.e.w. in } \Omega,$$

or, similarly as in Chapter 2, to consider

$$u_m(x) = \frac{1}{\lambda} \left(f(x) + \sum_{j=0}^{m} \langle u_m, \tilde{L}_j \rangle \mathrm{K}\tilde{L}_j(x) \right), \quad x \text{ a.e.w. in } \Omega.$$

4. The application is justified since matrix K is symmetric by construction and hence is a normal element of the C^*-Algebra $\left(\mathbb{C}^{(m+1)\times(m+1)}, *\right)$ of complex-entry matrices, with $*$ denoting in this case the involutive matrix-Hermetian-transpose operation.

Nonetheless, due to having

$$\overline{C(\overline{\Omega}, \mathbb{R})} = \mathcal{L}, \quad \text{w.r.t. } \langle \cdot, \cdot \rangle,$$

findings of [29] demonstrate the inefficiency of the conventional approach and hence an insisting demand for the application of the new reversed path.

Algorithm 2 Computation of the approximate solution $u_m(\cdot)$ of (3.3) following the discretise-then-iterate method

---

Require: λ, a, b, and $m \geq 2$
Ensure: $\langle u_j, \tilde{L}_j \rangle_{\mathcal{L}}$, for all $0 \leq j \leq m$

$d = \frac{b-a}{m}$
$T(0) = t_0 = a, T(m) = t_m = b$
while $1 \leq i \leq m-1$ **do**
  $t_i \leftarrow t_{i-1} + d$
end while
while $0 \leq i \leq m$ **do**
  while $0 \leq j \leq m$ **do**
    $L(i,j) = \tilde{L}_i(t_j)$
  end while
end while
while $0 \leq i \leq m$ **do**
  $F(i) = \text{TrapezoidIntegral}(f(T). * L(i,:), a, b)$
end while
while $0 \leq i \leq m$ **do**
  while $0 \leq j \leq m$ **do**
    while $0 \leq p \leq m$ **do**
      $Q(j) = \text{TrapezoidIntegral}(k(t_p, T). * L(j,:), a, b)$
    end while
    $K(i,j) = \text{TrapezoidIntegral}\,(Q. * L(i,:), a, b)$
  end while
end while
$A = \lambda \text{Id}_{\mathbb{R}^{m+1}} - K$
Choose the finite-dimensional JOR scheme, based on Theorem 1.2.4 to approximate the solution of the linear system $AU = F$, where $V = \left(\langle u, \tilde{L}_i \rangle_{\mathcal{L}}\right)_{j=0}^{m}$
Rebuild $u_m(\cdot)$ using an interpolation formula

3.3 Iterate: Hilbert Spaces, SPD Operator-matrices, and Convergence of the GJOR

Let be an integer $N \geq 2$. Then, introduce the equidistant mesh of points $\{x_i\}_{i=0}^{N} \subset \overline{\Omega}$:

$$H = (b-a)/N, \quad \begin{cases} x_i = x_{i-1} + H, & 1 \leq i \leq N-1, \\ x_0 = a, \ x_N = b, \end{cases}$$

such that by denoting $\Omega_i = (x_{i-1}, x_i)$, $1 \leq i \leq N$, the Lebesgue integration properties allows for setting the linear $(N \times N)$-system of Fredholm's integral equations of the second kind identically as was done in Chapter 2:

$$\lambda u_i(x) - \sum_{j=1}^{N} \int_{\Omega_j} k(x,t) u_j(t)\, dt = f_i(x), \quad x \text{ a.e.w. in } \Omega_i, \quad 1 \leq i \leq N,$$

where

$$u_i(x) \equiv u(x), \quad f_i(x) \equiv f(x), \quad x \text{ a.e.w. in } \Omega_i.$$

Hence, denoting by $\mathcal{L}_i = L^2(\Omega_i)$, $1 \leq i \leq N$, the i-Hilbert space of square-integrable functions over Ω_i endowed with its i-inner-product $\langle \cdot, \cdot \rangle_i$:

$$\langle v, w \rangle_i := \int_{\Omega_i} v(s) w(s)\, ds, \quad v, w \in \mathcal{L}_i,$$

from which the i-norm, $\| \cdot \|_i$, follows:

$$\|v\|_i^2 := \langle v, v \rangle_i, \quad v \in \mathcal{L}_i.$$

We build a family of operators $\{\mathrm{K}_{ij}\}_{i,j=1}^{N}$ where

$$\begin{aligned} \mathrm{K}_{ij} \colon \mathcal{L}_j &\mapsto \mathcal{L}_i \\ w &\mapsto \mathrm{K}_{ij} w(x) = \int_{\Omega_j} k(x,t) w(t)\, dt, \quad x \text{ a.e.w. in } \Omega_i, \end{aligned}$$

such that, denoting the additive neutral element of $\mathcal{L}_i$ by 0_i, we have

$$\begin{aligned}
\|\mathrm{K}_{ij}\|_{op}^2 &:= \left(\sup_{w\neq 0_j} \frac{\|\mathrm{K}_{ij}w\|_i}{\|w\|_j}\right)^2 \\
&\leq \sup_{w\neq 0_j} \frac{\|\mathrm{K}_{ij}w\|_i^2}{\|w\|_j^2} \\
&\leq \|\mathrm{K}_{ij}\|_{HS}^2 \\
&= \iint_{\Omega_i\times\Omega_j} |k(x,t)|^2\, dA \\
&\leq \iint_{\Omega^2} |k(x,t)|^2\, dA \\
&= \|\mathrm{K}\|_{HS}^2,
\end{aligned} \tag{3.8}$$

resulting in the endomorphism K_{ij} being of a Hilbert-Schmidt type, hence being compact as well as an element of $\mathscr{L}(\mathcal{L}_j, \mathcal{L}_i)$.

Next, setting the product-vector-space $\mathscr{H}$ as

$$\mathscr{H} := \prod_{i=1}^{N} \mathcal{L}_i,$$

we introduce a Hilbert-space structure over it by means of the inner-product application $\langle\cdot,\cdot\rangle_{\mathscr{H}}$:

$$\langle V, W\rangle_{\mathscr{H}} := \sum_{i=1}^{N} \langle v_i, w_i\rangle_i, \quad V, W \in \mathscr{H},$$

which, in turn, induces the normed structure such that:

$$\|V\|_{\mathscr{H}}^2 := \langle V, V\rangle_{\mathscr{H}}, \quad V \in \mathscr{H}.$$

The space of all the bounded endomorphisms of $\mathscr{H}$ is denoted $\mathscr{L}(\mathscr{H})$ and is identified with the vector-space of $(N\times N)$-operator-matrices, that, similar to Chapter 2, we also denote[5] as $\mathscr{M}$, where a complete topological structure is induced over it by means of a Frobenius-like norm:

$$|\mathcal{M}|_F^2 := \sum_{i=1}^{N}\sum_{j=1}^{N} \|\mathrm{M}_{ij}\|_{op}^2, \quad \mathrm{M}_{ij} \in \mathscr{L}(\mathcal{L}_j, \mathcal{L}_i).$$

5. Both notations $\mathscr{M}$ and $\mathscr{L}(\mathscr{M})$ may be used interchangeably starting at this stage of the text and onwards.

The identity operator-matrix is denoted $\mathcal{I}$ and is defined mutatis mutandis[6] as in Chapter 2.

That all necessary constructions are made is clear which make it an invitation for the discussion of the following abstract equation that is equivalent to (3.3):

$$(\lambda\mathcal{I} - \mathcal{K})\, U = F, \tag{3.9}$$

where the source function-vector and the unknown function-vector are respectively $F = (f_i)_{i=1}^N, U = (u_i)_{i=1}^N$, and the matrix $\mathcal{K}$ is an element of $\mathscr{M}$ such that, provided that (3.4) holds, then (3.8) implies that

$$|\mathcal{K}|_F \leq \|\mathrm{K}\|_{HS} < |\lambda|, \tag{3.10}$$

hence the applicability of Fredholm's Alternative and the uniqueness of the solution function-vector U of (3.9) in $\mathscr{H}$.

Unlike the conventional method, the adjoint of the data operator-matrix $(\lambda\mathcal{I} - \mathcal{K})$, that we deliberately term *transpose* and denote $(\lambda\mathcal{I} - \mathcal{K})^T$ is not guaranteed by default. Hence, we necessitate the following definition

Definition 3.3.1. *In the space $\mathscr{M}$, an operator-matrix $\mathcal{M}$ is said to be symmetric if, and only if, we have*

$$\mathrm{M}_{ij} = \mathrm{M}_{ji}, \quad 1 \leq i \neq j \leq N.$$

Moreover, a symmetric matrix $\mathcal{M}$ in $\mathscr{M}$ is said to be positive-definite if, and only if, we have

$$\langle \mathcal{M}\psi, \psi \rangle_{\mathscr{H}} > 0, \quad \psi \in \mathscr{H} \setminus \{0_{\mathscr{H}}\}.$$

It is, however, said to be negative-definite if $-\mathcal{M}$ is positive-definite.

Subsequently, with a practical condition on the kernel function $k(\cdot, \cdot)$, a lemma ensues

Lemma 3.3.1. *Suppose that the kernel function in* (3.3) *verifies*

$$k(x,t) = k(t,x), \quad (x,t) \text{ a.e.w. in } \Omega^2,$$

and that (3.4) *is fulfilled. Then we have*

1. *The operator $(\lambda - \mathrm{K})$ is self-adjoint and the data operator-matrix $(\lambda\mathcal{I} - \mathcal{K})$, as well as the operator-matrix $(\lambda\mathcal{I} - \mathcal{D})$, is symmetric.*
2. *The data operator-matrix $(\lambda\mathcal{I} - \mathcal{K})$ as well as that of $(\lambda\mathcal{I} - \mathcal{D})$ are either positive-definite or negative-definite.*

6. =In a similar manner with necessary changes made.

Proof. The first part in 1. is due to showing that the integral operator K is self-adjoint provided that the symmetry condition on the kernel function holds. Indeed, observe that since the adjoint of K, that we denote K^*, is a Hilbert-Schmidt (compact) element of $\mathscr{L}(\mathcal{L})$, such that

$$\mathrm{K}^* h(t) := \int_\Omega k(x,t) h(x)\, dx, \quad t \text{ a.e.w. in } \Omega, \quad h \in \mathcal{L},$$

then we have

$$\langle \mathrm{K}v, h\rangle = \langle v, \mathrm{K}^* h\rangle, \quad v, h \in \mathcal{L},$$

which, coupled with the condition taken on the kernel function, one ensures the self-adjointness of the endomorphism K; i.e, $\mathrm{K} = \mathrm{K}^*$, and hence that of the endomorphism $(\lambda - \mathrm{K})$; that is,

$$(\lambda - \mathrm{K})^* = \lambda - \mathrm{K}^* = (\lambda - \mathrm{K})\,.$$

Next, the finite length of Ω implies the result

$$L^p(\Omega) \subseteq L^1(\Omega), \quad p \in [1;\infty),$$

and that

$$L^p(\Omega^2) \subseteq L^1(\Omega^2), \quad p \in [1;\infty).$$

Hence, since K is a Hilbert-Schmidt operator, then we have

$$k(\cdot,\cdot) \in L^2(\Omega^2) \implies k(\cdot,\cdot) \in L^1(\Omega^2),$$

and such that Fubini's Theorem is applicable which helps establish the following manipulations, for all $1 \leq i \neq j \leq N$:

$$\begin{aligned}
\langle \mathrm{K}_{ij} h_j, h_i\rangle_i &= \int_{\Omega_i} \int_{\Omega_j} k(x,t) h_j(t)\, dt\, h_i(x)\, dx \\
&= \int_{\Omega_i} \int_{\Omega_j} k(x,t) h_j(t) h_i(x)\, dt\, dx \\
&= \iint_{\Omega_i \times \Omega_j} k(x,t) h_j(t) h_i(x)\, dA \\
&= \int_{\Omega_j} \int_{\Omega_i} k(x,t) h_i(x) h_j(t)\, dx\, dt \\
&= \langle h_j, \mathrm{K}_{ij}^* h_i\rangle_j \\
&= \langle h_j, \mathrm{K}_{ji} h_i\rangle_j,
\end{aligned}$$

and, as a consequence, we obtain

$$\mathrm{K}_{ij}^* = \mathrm{K}_{ji},$$

which, by virtue of the kernel function's symmetry, yields

$$\mathrm{K}_{ij} = \mathrm{K}_{ij}^{*} = \mathrm{K}_{ji}.$$

Therefore, the operator-matrix $\mathcal{K}$ is symmetric and thus the data operator-matrix $(\lambda\mathcal{I} - \mathcal{K})$ is also symmetric. The symmetry of the operator-matrix $(\lambda\mathcal{I} - \mathcal{D})$ follows trivially as it is a diagonal operator-matrix.

In regards to 2., we define the numerical radius of $\mathcal{K}$ by

$$\nu(\mathcal{K}) := \sup_{\|V\|_{\mathscr{H}}=1} |\langle \mathcal{K}V, V\rangle_{\mathscr{H}}| = \sup_{V\neq 0_{\mathscr{H}}} \frac{|\langle \mathcal{K}V, V\rangle_{\mathscr{H}}|}{\langle V, V\rangle_{\mathscr{H}}}.$$

Criterion (3.4), together with (3.10), allows for the application of results from [7], where the Hilbert-structure of $\mathscr{H}$ justifies for the writing

$$\nu(\mathcal{K}) \leq |\mathcal{K}|_F < |\lambda|.$$

Next, since $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ maps into the reals, then we necessarily have

$$\frac{\langle \mathcal{K}V, V\rangle_{\mathscr{H}}}{\langle V, V\rangle_{\mathscr{H}}} < |\lambda|, \quad V \in \mathscr{H} \setminus \{0_{\mathscr{H}}\}.$$

Consequently, if λ is positive, then it is straightforward to see that one obtains

$$\langle (\lambda\mathcal{I} - \mathcal{K})\, V, V\rangle_{\mathscr{H}} > 0, \quad V \in \mathscr{H} \setminus \{0_{\mathscr{H}}\},$$

hence the positive-definiteness of the data operator-matrix $(\lambda\mathcal{I} - \mathcal{K})$. Additionally, if λ is negative, then we have

$$\lambda < -\frac{|\langle \mathcal{K}V, V\rangle_{\mathscr{H}}|}{\langle V, V\rangle_{\mathscr{H}}} \leq \frac{\langle \mathcal{K}V, V\rangle_{\mathscr{H}}}{\langle V, V\rangle_{\mathscr{H}}}, \quad V \in \mathscr{H} \setminus \{0_{\mathscr{H}}\},$$

which implies that

$$\langle (\lambda\mathcal{I} - \mathcal{K})\, V, V\rangle_{\mathscr{H}} < 0, \quad V \in \mathscr{H} \setminus \{0_{\mathscr{H}}\},$$

thus the negative-definiteness of the data operator-matrix $(\lambda\mathcal{I} - \mathcal{K})$. Concerning the positive-definiteness of the operator-matrix $(\lambda\mathcal{I} - \mathcal{D})$, notice that, by (3.10), we have

$$\begin{aligned} \nu^2(\mathcal{D}) \leq |\mathcal{D}|_F^2 &= \sum_{i=1}^{N} \|\mathrm{K}_{ii}\|_{op}^2 \\ &\leq \sum_{i=1}^{N}\sum_{j=1}^{N} \|\mathrm{K}_{ij}\|_{op}^2 \\ &= |\mathcal{K}|_F^2 \leq \|\mathrm{K}\|_{HS}^2 < \lambda^2. \end{aligned}$$

Hence, the result on $(\lambda\mathcal{I} - \mathcal{D})$ follows mutatis mutendis as was done for $(\lambda\mathcal{I} - \mathcal{K})$ replacing $\mathcal{K}$ by $\mathcal{D}$ in the steps. The lemma is, thus, proven. □

Furthermore, it is worthy to bring attention to the fact that the previous building-block lemma gives rise to a rather essential corollary for subsequent treatments:

Corollary 3.3.1. *Provided that Lemma 3.3.1 holds, the two operator-matrices* $(\lambda\mathcal{I} - \mathcal{D})^{-1}$ *and* $(\lambda\mathcal{I} - \mathcal{D})^{-1}(\lambda\mathcal{I} - \mathcal{K})$ *are respectively either SPD and PD or SND and ND.*

Proof. We begin with $(\lambda\mathcal{I} - \mathcal{D})^{-1}$ where its symmetry is a result of it being a diagonal operator-matrix. For its positive-definiteness, observe that if $(\gamma, V_\gamma) \in \mathbb{C} \times (\mathscr{H} \setminus \{0_{\mathscr{H}}\})$ denotes an eigen-couple of the matrix, then we have

$$(\lambda\mathcal{I} - \mathcal{D})^{-1} V_\gamma = \gamma V_\gamma \implies V_\gamma = \gamma (\lambda\mathcal{I} - \mathcal{D}) V_\gamma.$$

Hence, performing the inner-product of the equation with the eigen-function-vector V_γ, we obtain

$$\frac{\langle V_\gamma, V_\gamma \rangle_{\mathscr{H}}}{\langle (\lambda\mathcal{I} - \mathcal{D}) V_\gamma, V_\gamma \rangle_{\mathscr{H}}} = \gamma.$$

But, since the LHS is a real number and that, by Lemma 3.3.1, the operator-matrix $(\lambda\mathcal{I} - \mathcal{D})$ is either SPD or SND w.r.t. the sign of λ, it follows that we have the justified assertion

$$\sigma\left((\lambda\mathcal{I} - \mathcal{D})^{-1}\right) \subset \mathbb{R}_+ \setminus \{0\}.$$

Hence, the diagonal operator-matrix $(\lambda\mathcal{I} - \mathcal{D})^{-1}$ is either SPD or SND.

Next, for the operator-matrix $(\lambda\mathcal{I} - \mathcal{D})^{-1}(\lambda\mathcal{I} - \mathcal{K})$, we proceed in a similar manner, such that, let be an eigen-couple $(\delta, W_\delta) \in \mathbb{C} \times (\mathscr{H} \setminus \{0_{\mathscr{H}}\})$:

$$(\lambda\mathcal{I} - \mathcal{D})^{-1}(\lambda\mathcal{I} - \mathcal{K}) W_\delta = \delta W_\delta \implies (\lambda\mathcal{I} - \mathcal{K}) W_\delta = \delta (\lambda\mathcal{I} - \mathcal{D}) W_\delta.$$

Then, multiplying the equation by the eigen-function-vector, we have

$$\frac{\langle (\lambda\mathcal{I} - \mathcal{K}) W_\delta, W_\delta \rangle_{\mathscr{H}}}{\langle (\lambda\mathcal{I} - \mathcal{D}) W_\delta, W_\delta \rangle_{\mathscr{H}}} = \delta.$$

It is clear that, by Lemma 3.3.1, not only is the LHS a real number but we also have the justified following assertion:

$$\sigma\left((\lambda\mathcal{I} - \mathcal{D})^{-1}(\lambda\mathcal{I} - \mathcal{K})\right) \subset \mathbb{R}_+ \setminus \{0\}.$$

Hence, the operator-matrix $(\lambda\mathcal{I} - \mathcal{D})^{-1}(\lambda\mathcal{I} - \mathcal{K})$ is as supposed; thus, the proof is complete. □

Fact that we have established an a priori nature for the data operator-matrix $(\lambda\mathcal{I}-\mathcal{K})$, we move forwards to discuss the iterative approximation of the solution function-vector U by a sequence of function-vectors $\{U^k\}_{k\geq 0}\subset\mathscr{H}$ using the over-relaxed iterative scheme of **GJ**, that we denote **GJOR**.

Introducing the splitting of $\mathcal{K}$:

$$\mathcal{K}=\mathcal{D}-\mathcal{E}-\mathcal{F},$$

with $\mathcal{D}$, $-\mathcal{E}$, $-\mathcal{F}$ are, respectively, the diagonal, lower part, and upper part of $\mathcal{K}$. Then, for a non-zero real ω, observe that by defining

$$\mathcal{D}_\omega := \omega^{-1}\left(\lambda\mathcal{I}-\mathcal{D}\right), \quad \omega\neq 0,$$

we define the matrix

$$\mathcal{N}_\omega=\omega^{-1}\left(\left(1-\omega\right)\left(\lambda\mathcal{I}-\mathcal{D}\right)+\omega\left(\mathcal{E}+\mathcal{F}\right)\right), \quad \omega\neq 0.$$

Ultimately, since (3.10) entertains that

$$|\mathcal{D}|_F<|\lambda|,$$

then $\mathcal{D}_\omega^{-1}$ exists and the **GJOR** ensues as

$$\textbf{GJOR}:\begin{cases} U^{k+1} & =\mathcal{D}_\omega^{-1}\mathcal{N}_\omega U^k+\mathcal{D}_\omega^{-1}F, \quad k\geq 0,\\ U^0 & \in\mathscr{H}.\end{cases} \tag{3.11}$$

which, component-wise, is also given as

$$\textbf{GJOR}:\begin{cases} \lambda u_i^{k+1} & =\mathrm{K}_{ii}u_i^{k+1}+\omega\left(\sum_{\substack{j=1\\ i\neq j}}^{N}\mathrm{K}_{ij}u_j^k+f_i\right)+(1-\omega)u_i^k, \quad k\geq 0,\\ u_i^0 & \in\mathcal{L}_i, \quad 1\leq i\leq N.\end{cases} \tag{3.12}$$

One already remarks that the convex combination discussed in the Motivation section of this chapter surfaces as a result of applying the matrix-splitting technique. Moreover, by Banach's Fixed-point theorem in its general statement, scheme (3.11) is sufficiently and necessarily convergent if, and only if, the iteration matrix $\mathcal{G}_\omega=\mathcal{D}_\omega^{-1}\mathcal{N}_\omega$ meets the spectral condition

$$\rho\left(\mathcal{G}_\omega\right)<1, \quad \omega\neq 0 \text{ and is appropriate.}$$

However, before establishing the convergence theorem for (3.11) (equivalently for (3.12)), we note that Lemma 1 from [46] holds true in the case of our operator-matrices:

$$\rho\left(\left(\lambda\mathcal{I}-\mathcal{D}\right)^{-1}\left(\mathcal{E}+\mathcal{F}\right)\right)<1 \iff \rho\left(\left(\lambda\mathcal{I}-\mathcal{D}\right)^{-1}\left(\lambda\mathcal{I}-\mathcal{K}\right)\right)<2,$$

and that

$$\rho\left(\mathcal{G}_{\omega}\right)<1 \iff \rho\left(\mathcal{D}_{\omega}^{-1}\left(\lambda\mathcal{I}-\mathcal{K}\right)\right)<2.$$

As a consequence, we obtain the following theorem

Theorem 3.3.1. *Provided that Lemma 3.3.1 holds, the scheme* (3.11) *is convergent for all the positive values ω:*

$$0<\omega<\frac{2}{\rho\left(\left(\lambda\mathcal{I}-\mathcal{D}\right)^{-1}\left(\lambda\mathcal{I}-\mathcal{K}\right)\right)}.$$

Proof. If Lemma 3.3.1 holds, then the data operator-matrix $(\lambda\mathcal{I}-\mathcal{K})$ is either SPD or SND w.r.t. to Definition 3.3.1. Next, without loss of generality, we suppose in what ensues that $\lambda>0$, and observe that we have

$$\mathcal{D}_{\omega}^{-1}\mathcal{N}_{\omega}=(1-\omega)\mathcal{I}-\omega\left(\lambda\mathcal{I}-\mathcal{D}\right)^{-1}\left(\mathcal{E}+\mathcal{F}\right), \quad \omega>0.$$

Next, in view of Corollary 3.3.2's proof, let $\mu\in\sigma\left(\left(\lambda\mathcal{I}-\mathcal{D}\right)^{-1}\left(\mathcal{E}+\mathcal{F}\right)\right)\subset\mathbb{R}$. Then we have that

$$\alpha\in\sigma\left(\mathcal{D}_{\omega}^{-1}\mathcal{N}_{\omega}\right) \iff \alpha=1-\omega+\omega\mu,$$

and consequently, the equivalences on the spectral radii mentioned previously, as well as the positive-definiteness of $\left(\lambda\mathcal{I}-\mathcal{D}\right)^{-1}\left(\lambda\mathcal{I}-\mathcal{K}\right)$ by virtues of Corollary 3.3.2, imply that

$$|\alpha|<1 \iff 0<\eta=(1-\alpha)\in\sigma\left(\mathcal{D}_{\omega}^{-1}\left(\lambda\mathcal{I}-\mathcal{K}\right)\right): \quad \eta<2, \quad \omega>0.$$

Observe that

$$0<\eta<2 \iff 0<|\eta|<2 \iff 0<|1-\alpha|<2 \iff 0<|\omega(1-\mu)|<2.$$

Consequently, if the associated **GJ** is convergent, then

$$|\mu|<1 \implies (1-\mu)>0.$$

Then, noting that

$$\rho\left((\lambda\mathcal{I}-\mathcal{D})^{-1}(\lambda\mathcal{I}-\mathcal{K})\right)=\max\{|1-\mu|\},$$

the choice of ω yields

$$0<|\omega(1-\mu)|=\omega(1-\mu)<\frac{2(1-\mu)}{\rho\left((\lambda\mathcal{I}-\mathcal{D})^{-1}(\lambda\mathcal{I}-\mathcal{K})\right)}\leq 2.$$

Subsequently, our choice for ω results in verifying

$$\rho\left(\mathcal{G}_\omega\right)<1 \iff \rho\left(\mathcal{D}_\omega^{-1}(\lambda\mathcal{I}-\mathcal{K})\right)<2.$$

Hence, the convergence of the **GJOR**.

Next, suppose that the associated **GJ** is divergent; that is,

$$\rho\left((\lambda\mathcal{I}-\mathcal{D})^{-1}(\mathcal{E}+\mathcal{F})\right)\geq 1.$$

Then, we do not necessarily have that

$$|1-\mu|=1-\mu.$$

Nevertheless, with the choice made for the range of ω, we have

$$0<|\omega(1-\mu)|=\omega|1-\mu|<\frac{2|1-\mu|}{\rho\left((\lambda\mathcal{I}-\mathcal{D})^{-1}(\lambda\mathcal{I}-\mathcal{K})\right)}\leq 2.$$

Once again, the choice of ω helps establish the equivalence

$$\rho\left(\mathcal{G}_\omega\right)<1 \iff \rho\left(\mathcal{D}_\omega^{-1}(\lambda\mathcal{I}-\mathcal{K})\right)<2.$$

In light of this, the convergence of the **GJOR** follows. Consequently, the theorem is proven. □

Even that we have not made it clear in the proof, notice that the convergence of the **GJ** implies that

$$\frac{1}{\rho\left((\lambda\mathcal{I}-\mathcal{D})^{-1}(\lambda\mathcal{I}-\mathcal{K})\right)}>\frac{1}{2}\implies\frac{2}{\rho\left((\lambda\mathcal{I}-\mathcal{D})^{-1}(\lambda\mathcal{I}-\mathcal{K})\right)}>1,$$

which translates to the fact that if we have

$$\omega\in(0,1],$$

then one de facto ensures the applicability of Theorem 3.3.1 and hence the convergence of the **GJOR**. In other terms, the usual $(0, 1]$ region, inspired by the algebraic **JOR** and adopted in [29], becomes a sufficient condition for the convergence of the **GJOR**.

Nevertheless, given the rigidity of the unclear task of finding the spectrum and the spectral radius of an operator-matrix, the convergence theorem demonstrated above may be perceived as a little impractical for use. Therefore, before proceeding further in the text, we establish the following lemma

Lemma 3.3.2. *The sub-multiplicative and sub-linear functional $N(\cdot) : \mathscr{M} \mapsto \mathbb{R}$ defined by*

$$N(\mathcal{M}) := \sup_{V \neq 0_{\mathscr{H}}} \frac{\|\mathcal{M}V\|_{\mathscr{H}}}{\|V\|_{\mathscr{H}}}, \quad \mathcal{M} \in \mathscr{M},$$

endows a normed structure over the space $\mathscr{L}(\mathscr{M})$.

Proof. That $N(\cdot)$ is a norm-application mapping into the non-negative reals is a straightforward proposition which qualifies it as a sub-linear[7] functional. For sub-multiplicity, observe that since $\mathscr{H}$ is a Hilbert space, with its inner-product $\langle \cdot, \cdot \rangle_{\mathscr{H}}$, then leveraging the Cauchy-Schwarz inequality yields the following reasoning

$$\begin{aligned} \mathcal{S}, \mathcal{T} \in \mathscr{M}, \quad N^2(\mathcal{S}\mathcal{T}) &:= \left(\sup_{V \neq 0_{\mathscr{H}}} \frac{\|\mathcal{S}\mathcal{T}V\|_{\mathscr{H}}}{\|V\|_{\mathscr{H}}} \right)^2 \\ &\leq \sup_{V \neq 0_{\mathscr{H}}} \frac{\|\mathcal{S}\mathcal{T}V\|^2_{\mathscr{H}}}{\|V\|^2_{\mathscr{H}}} \\ &= \sup_{V \neq 0_{\mathscr{H}}} \frac{\langle \mathcal{S}\mathcal{T}V, \mathcal{S}\mathcal{T}V \rangle_{\mathscr{H}}}{\langle V, V \rangle_{\mathscr{H}}} \\ &\leq \frac{N^2(\mathcal{S})N^2(\mathcal{T})\langle V, V \rangle_{\mathscr{H}}}{\langle V, V \rangle_{\mathscr{H}}}, \quad V \neq 0_{\mathscr{H}}. \end{aligned}$$

Hence, taking the square-root, we have

$$N^2(\mathcal{S}\mathcal{T}) \leq N^2(\mathcal{S})N^2(\mathcal{T}) \implies N(\mathcal{S}\mathcal{T}) \leq N(\mathcal{S})N(\mathcal{T}), \quad \mathcal{S}, \mathcal{T} \in \mathscr{M}.$$

This completes the proof. □

7. Sub-linearity is another term for the triangle-inequality for norms.

Again, it is essential to re-note that,

$$\mathscr{M} \equiv \mathscr{L}(\mathscr{H}), \quad \text{up to an isomorphism.}$$

Additionally, since $(\mathscr{H}, \langle\cdot,\cdot\rangle_{\mathscr{H}})$ is a Hilbert space and hence is a Banach space, then the normed $(\mathscr{L}(\mathscr{H}), N(\cdot))$ is complete and thus may be regarded as a non-commutative unital Banach Algebra, where The Generalised Gelfand's Identity applies:

$$\rho(\mathcal{M}) = \lim_{p\to\infty} \sqrt[p]{N(\mathcal{M}^p)}, \quad \mathcal{M} \in \mathscr{M},$$

where

$$\mathcal{M}^p = \mathcal{M} \underbrace{\circ \cdots \circ}_{p \text{ times}} \mathcal{M}, \quad p \geq 1, \quad \mathcal{M}^0 = \mathcal{I},$$

with the non-commutative algebraic-operator $\circ$ is defined by means of the non-commutative operator-composition over the Hilbert spaces $\mathcal{L}_i$, $\mathcal{L}_j$ and the identity operator-matrix $\mathcal{I}$ is the unital element of the Banach algebra in mention. It is also admitted that

$$\rho(\mathcal{M}) = \inf_{p\in\mathbb{N}\setminus\{0\}} \sqrt[p]{\|\mathcal{M}^p\|},$$

where $\|\cdot\|$ is such that one generates the Banach algebra with over $\mathscr{M}$. In other terms, it is all complete norm application that possesses all that is for $N(\cdot)$ defined hereinabove. One may confer [23] for richer details.

Subsequently, perceived as an alternative to Theorem 3.3.1, we opt for the following sufficient corollary where we use more concrete quantities from the given data:

Corollary 3.3.2. *Provided that Lemma 3.3.1 holds, the scheme* (3.11) *converges for all the positive values ω:*

$$0 < \omega \leq 2\left(\frac{|\lambda| - \|\mathrm{K}|_{HS}}{|\lambda| + \|\mathrm{K}|_{HS}}\right).$$

And we have, for $1 < \varepsilon < 2$,

$$\omega \in \begin{cases} (0,1), & |\lambda| < 3\|\mathrm{K}\|_{HS}, \\ (0,1], & |\lambda| = 3\|\mathrm{K}\|_{HS}, \\ (0,\varepsilon), & |\lambda| > 3\|\mathrm{K}\|_{HS}. \end{cases}$$

Proof. Without loss of generality, suppose that $\lambda > 0$, and let be

$$\beta = \rho\left((\lambda\mathcal{I} - \mathcal{D})^{-1}(\lambda\mathcal{I} - \mathcal{K})\right).$$

Then, by the operator-matrix norm-application $N(\cdot)$ defined in Lemma 3.3.2, we consequently have, over $\mathscr{M}$, that the successive and respective applications of the sub-multiplication property of $N(\cdot)$, its triangle-inequality, and the result of The Geometric Series Theorem in its general statement by virtue of results from [7]:

$$N(\mathcal{D}) \leq |\mathcal{D}|_F \leq |\mathcal{K}|_F \leq \|\mathrm{K}\|_{HS} < \lambda,$$

yield that

$$\begin{aligned}\beta \leq N\left((\lambda\mathcal{I} - \mathcal{D})^{-1}(\lambda\mathcal{I} - \mathcal{K})\right) &\leq \frac{N\left((\lambda\mathcal{I} - \mathcal{K})\right)}{\lambda - N\left(\mathcal{D}\right)}\\ &\leq \frac{\lambda + N\left(\mathcal{K}\right)}{\lambda - N\left(\mathcal{D}\right)}\\ &\leq \frac{\lambda + |\mathcal{K}|_F}{\lambda - |\mathcal{D}|_F}.\end{aligned}$$

Therefore, observe that we have

$$\frac{2}{\beta} \geq 2\left(\frac{\lambda - |\mathcal{D}|_F}{\lambda + |\mathcal{K}|_F}\right) > 2\left(\frac{\lambda - |\mathcal{K}|_F}{\lambda + |\mathcal{K}|_F}\right) \geq 2\left(\frac{\lambda - \|\mathrm{K}\|_{HS}}{\lambda + \|\mathrm{K}\|_{HS}}\right),$$

which means that taking ω:

$$0 < \omega < 2\left(\frac{\lambda - \|\mathrm{K}\|_{HS}}{\lambda + \|\mathrm{K}\|_{HS}}\right)$$

suffices to meeting the region-criterion from Theorem 3.3.1 and thus obtaining the convergence of **GJOR**.

Next, observe that for some $1/2 < \varepsilon' < 1$, we have

$$\left(\frac{\lambda - \|\mathrm{K}\|_{HS}}{\lambda + \|\mathrm{K}\|_{HS}}\right) : \begin{cases} < 1/2, & \lambda < 3\|\mathrm{K}\|_{HS},\\ = 1/2, & \lambda = 3\|\mathrm{K}\|_{HS},\\ (> 1/2) \text{ and } (< \varepsilon'), & \lambda > \|\mathrm{K}\|_{HS}.\end{cases}$$

As a result, for some $1 < \varepsilon < 2$ such that $\varepsilon = 2\varepsilon'$, we write the following sufficient region-condition:

$$\omega \in \begin{cases}(0,1), & |\lambda| < 3\|\mathrm{K}\|_{HS},\\ (0,1], & |\lambda| = 3\|\mathrm{K}\|_{HS},\\ (0,\varepsilon), & |\lambda| > 3\|\mathrm{K}\|_{HS}.\end{cases}$$

Which terminates the proof. □

Ultimately, given our premises and the investigations' findings implied hereinabove, we have managed to employ the **GJOR** to build a function-sequence $\{U^k\}_{k\geq 0} \subset \mathscr{H}$:

$$\lim_{k\to\infty} \|U - U^k\|_{\mathscr{H}} = 0.$$

3.4 Discretise: Projections

Unlike with continuous functions, the imposed square-integrability attribute of our data implies that, similar to Section 3.2, we need to invoke the projection concept to discretise our scheme (3.12).

Therefore, for $1 \leq i \leq N$, let the family $\{\tilde{L}^i_q\}_{q\geq 0} \subset \mathcal{L}_i$ denote the Ω_i-family of orthonormal Legendre's polynomials:

$$\overline{\text{Span}\{\tilde{L}^i_q\}_{q\geq 0}} = \mathcal{L}_i, \quad 1 \leq i \leq N.$$

Thus, for $n \geq 2$, define the projection-endomorphism π^i_n by

$$\pi^i_n v_i(x) = \sum_{q=0}^{n} \langle v_i, \tilde{L}^i_q \rangle_i \tilde{L}^i_q(x), \quad x \text{ a.e.w. in } \Omega_i, \quad v_i \in \mathcal{L}_i, \quad 1 \leq i \leq N.$$

By Bessel's inequality, we have that

$$\|\pi^i_n\|_{op} \leq 1 \implies \pi^i_n \in \mathscr{L}(\mathcal{L}_i), \quad 1 \leq i \leq N.$$

Moreover, we have the pointwise-convergence of it:

$$\lim_{n\to\infty} \pi^i_n v_i = v_i, \quad \text{a.e.w. in } \Omega_i, \quad v_i \in \mathcal{L}_i, \quad 1 \leq i \leq N,$$

as well as

$$\lim_{n\to\infty} \left(\text{Id}_{ii} - \pi^i_n\right) v_i = 0_i, \quad \text{a.e.w. in } \Omega_i, \quad v_i \in \mathcal{L}_i, \quad 1 \leq i \leq N.$$

Then, it is justified that we define the finite-rank Hilbert-Schmidt (compact) operator K_{iin}:

$$\begin{aligned}\text{K}_{iin} v_i &:= \left(\text{K}_{ii} \circ \pi^i_n\right) v_i \\ &= \sum_{q=0}^{n} \langle v_i, \tilde{L}^i_q \rangle_i \text{K}_{ii} \tilde{L}^i_q(x), \quad x \text{ a.e.w. in } \Omega_i, \quad v_i \in \mathcal{L}_i, \quad 1 \leq i \leq N.\end{aligned}$$

With the aforementioned, the interrelation between an operator-norm and the Hilbert-Schmidt norm over Hilbert spaces results in

$$\mathrm{K}_{iin} \in \mathcal{L}_i, \quad n \geq 2, \quad 1 \leq i \leq N,$$

as well as[8]

$$\left\| \frac{\mathrm{K}_{iin}}{\lambda} \right\|_{op} < 1,$$

implying that $(\lambda - \mathrm{K}_{iin})$ is invertible by The Geometric Series Theorem in its general statement. However, we do not intend on merging between the two chapters: Chapter 2 (the previous) and Chapter 3 (the current).

It follows that the discretised version of (3.12), denoted **GJOR**$_n$, receives the following form for a scheme

$$\mathbf{GJOR}_n : \begin{cases} \lambda u_{in}^{k+1} & = \mathrm{K}_{iin} u_{in}^{k+1} + \omega \left(\sum_{\substack{j=1 \\ i \neq j}}^{N} \mathrm{K}_{ij} u_{jn}^{k} + f_i \right) + (1-\omega) u_{in}^{k}, \quad k \geq 0, \\ u_{in}^{0} & \in \mathcal{L}_i, \quad 1 \leq i \leq N. \end{cases} \tag{3.13}$$

And, to the systematic construction of the investigation, we provide the examination of its convergence in the following theorem

Theorem 3.4.1. *Provided that Theorem 3.3.1 holds, the sequence of function-vectors* $\{U_n^k : \ n \geq 2, k \geq 0\}$ *converges in the* $\mathscr{H}$*-norm to the solution function-vector of* (3.11)*; equivalently, of* (3.9)*, and we write*

$$\lim_{k,n \to \infty} \|U - U_n^k\|_{\mathscr{H}} = 0.$$

Proof. Observe that for all iteration-rank $k \geq 0$, and all discretisation parameter $n \geq 2$, we have

$$\|U - U_n^k\|_{\mathscr{H}} \leq \|U - U^k\|_{\mathscr{H}} + \|U^k - U_n^k\|_{\mathscr{H}}.$$

On one hand, since the hypothesis is that Theorem 3.3.1 holds, then the scheme **GJOR** converges; that is,

$$\rho\left(\mathcal{G}_\omega\right) < 1, \quad \omega > 0.$$

Moreover, since $N(\cdot)$ of Lemma 3.3.2 is sub-multiplicative, then

$$\rho\left(\mathcal{G}_\omega\right) = \inf_{p \in \mathbb{N} \setminus \{0\}} \sqrt[p]{N\left(\mathcal{G}_\omega^p\right)} \leq \sqrt[p]{N\left(\mathcal{G}_\omega^p\right)}, \quad \omega > 0.$$

8. The writing $\frac{\mathrm{K}_{iin}}{\lambda}$ is an abuse of notation. In fact, we would normally write $\lambda^{-1}\mathrm{K}_{iin}$.

Hence, if we also have that

$$N(\mathcal{G}_\omega) < 1, \quad \omega > 0,$$

then, as an application of the estimate from the Banach's Fixed-point Theorem in its general statement, we obtain

$$\|U - U^k\|_{\mathscr{H}} \le \frac{N^k(\mathcal{G}_\omega)}{1 - N(\mathcal{G}_\omega)} \|U^1 - U^0\|_{\mathscr{H}}, \quad k \ge 0, \quad \omega > 0.$$

and the convergence follows straightforwardly. Otherwise, by the submultiplicity of the norm-application $N(\cdot)$ we have, in all cases, that

$$\|U - U^k\|_{\mathscr{H}} \le N(\mathcal{G}_\omega)\|U - U^{k-1}\|_{\mathscr{H}} \le \cdots \le N(\mathcal{G}_\omega^k)\|U - U^0\|_{\mathscr{H}}, \quad \omega > 0.$$

It follows then that since it is the infimum over $p \in \mathbb{N} \setminus \{0\}$, then since also:

$$N(\mathcal{G}_\omega^k) \le N^k(\mathcal{G}_\omega), \quad k \ge 1, \quad \omega > 0,$$

the spectral radius verifies by means of the aforementioned results,

$$\|U - U^k\|_{\mathscr{H}} \le \rho^k(\mathcal{G}_\omega) \underbrace{\|U - U^0\|_{\mathscr{H}}}_{\text{a fixed known quantity}} \quad k \ge 0, \quad \omega > 0,$$

which concludes the convergence to 0 as we iterate in k up to infinity; that is, we arrive at the result

$$\lim_{k \to \infty} \|U - U^k\|_{\mathscr{H}} = 0.$$

What remains is the term $\|U^k - U_n^k\|_{\mathscr{H}}$, for which we shall proceed inductively over k.

By construction and the employment of the projection concept, we have

$$\lim_{n \to \infty} \|U^0 - U_n^0\|_{\mathscr{H}} = 0.$$

Next, suppose that the result holds up to iteration $k \ge 0$, that is,

$$\lim_{n \to \infty} \|U^m - U_n^m\|_{\mathscr{H}} = 0, \quad 0 \le m \le k,$$

and observe that, at iteration-rank $(k+1)$, we have the two equations

$$\lambda u_i^{k+1} = \mathrm{K}_{ii} u_i^{k+1} + \omega \left(\sum_{\substack{j=1 \\ i \ne j}}^{N} \mathrm{K}_{ij} u_j^k + f_i \right) + (1 - \omega) u_i^k, \quad k \ge 0,$$

and

$$\lambda u_{in}^{k+1} = \mathrm{K}_{iin} u_{in}^{k+1} + \omega \left(\sum_{\substack{j=1 \\ i \neq j}}^{N} \mathrm{K}_{ij} u_{jn}^{k} + f_i \right) + (1-\omega) u_{in}^{k}, \quad k \geq 0.$$

As a result of taking their difference, we obtain

$$\lambda \left(u_i^{k+1} - u_{in}^{k+1} \right) - \mathrm{K}_{ii} u_i^{k+1} + \mathrm{K}_{iin} u_{in}^{k+1} = \mathrm{R}_\omega \left(u_i^k - u_{in}^k \right), \quad k \geq 0,$$

where $\mathrm{R}_\omega(\cdot)$, $\omega > 0$, denotes an endomorphism of $\mathcal{L}_i$, which, by virtues of the induction hypothesis made hereinabove, we have it verify, for all $0 \leq m \leq k$, and all $1 \leq i \leq N$,

$$\begin{aligned} \lim_{n\to\infty} \mathrm{R}_\omega(u_i^m - u_{in}^m) &= \lim_{n\to\infty} \left[\omega \left(\sum_{\substack{j=1 \\ i \neq j}}^{N} \mathrm{K}_{ij} \left(u_j^m - u_{jn}^m \right) \right) + (1-\omega)(u_i^m - u_{in}^m) \right] \\ &= 0_i. \end{aligned}$$

On the other hand, since π_n^i is a projection over the Hilbert $\mathcal{L}_i$, then, by a result from Preliminaries, Section 3, we have the following direct-sum decomposition:

$$\mathcal{L}_i = \mathrm{Range}(\mathrm{Id}_{ii} - \pi_n^i) \bigoplus \mathrm{Range}(\pi_n^i), \quad n \geq 2, \quad 1 \leq i \leq N,$$

which implies that since u_i^{k+1} is an element of $\mathcal{L}_i$ as a linear combination of its elements, then given the orthogonality of the projection operators in manipulation, we have

$$u_i^{k+1} = (\mathrm{Id}_{ii} - \pi_n^i) u_i^{k+1} + \pi_n^i u_i^{k+1}, \quad n \geq 2, \quad k \geq 0, \quad 1 \leq i \leq N.$$

Moreover, setting

$$d_{in}^{k+1} = u_i^{k+1} - u_{in}^{k+1}, \quad n \geq 2, \quad k \geq 0, \quad 1 \leq i \leq N,$$

we obtain

$$\begin{aligned} \lambda d_{in}^{k+1} - \mathrm{K}_{ii} u_i^{k+1} + \mathrm{K}_{iin} u_{in}^{k+1} &= \lambda d_{in}^{k+1} - \mathrm{K}_{ii} \left((\mathrm{Id}_{ii} - \pi_n^i) u_i^{k+1} + \pi_n^i u_i^{k+1} \right) + \mathrm{K}_{iin} u_{in}^{k+1} \\ &= \lambda d_{in}^{k+1} - \mathrm{K}_{ii} \left(\mathrm{Id}_{ii} - \pi_n^i \right) u_i^{k+1} - \mathrm{K}_{iin} d_{in}^{k+1}. \end{aligned}$$

Consequently, we have

$$(\lambda - \mathrm{K}_{iin})\, d_{in}^{k+1} - \mathrm{K}_{ii}\left(\mathrm{Id}_{ii} - \pi_n^i\right) u_i^{k+1} = \mathrm{R}_\omega d_{in}^k, \quad k \geq 0, \quad n \geq 2, \quad 1 \leq i \leq N.$$

Since $(\lambda - \mathrm{K}_{iin})$ is invertible, and that $\left(\mathrm{Id}_{ii} - \pi_n^i\right)$ is pointwise-convergent to the annihilation endomorphism of $\mathcal{L}_i$, the we have, for all $1 \leq i \leq N$

$$\lim_{n\to\infty} d_{in}^{k+1} = \lim_{n\to\infty} (\lambda - \mathrm{K}_{iin})^{-1} \left[\mathrm{K}_{ii}\left(\mathrm{Id}_{ii} - \pi_n^i\right) u_i^{k+1} + \mathrm{R}_\omega(d_{in}^k)\right] = 0_i.$$

As a result, by the principal of induction, if the convergence holds at iteration-rank $k \geq 0$, then it would as well at iteration-rank $(k+1)$. That is, by induction over k, we have

$$\lim_{n\to\infty} \|U^k - U_n^k\|_{\mathscr{H}} = 0.$$

Hence,

$$\lim_{k,n\to\infty} \|U - U_n^k\|\mathscr{H} = 0.$$

The proof is then complete. □

Although potentially redundant, we should note that the discretisation process offers an effect of perturbation in the the **GJOR** in that, despite the linear convergence, one has it on iterating to have built the following estimation

$$\|U - U_n^k\|_{\mathscr{H}} \leq \left(\rho\left(\mathcal{G}_\omega\right) + E_n\right)^k \|U - U^0\|_{\mathscr{H}}, \quad k \geq 0,\, n \geq 2,\, \omega > 0.$$

However, unlike with the non-linearity's effects on whether n had to be fixed or not which is a matter discussed in [20], it is not clear if it followed that convergence of the discretised **GJOR** is not interrupted by the discretisation effect, and such that fixing an n with small values, the convergence is unclear of an option in regards to the convergence of the scheme, and such that we still do not know to what extent is the dominance of the k-convergence over that of the n.

3.5 Numerical Implementations: Concluding Remarks

We sincerely apologise to the reader for the perplexing and intimidating inconvenience of not showcasing any progress in regards of the challenging and ongoing numerical implementations of the both paths: Conventional (Discretise-Iterate) and/or Unconventional (Iterate-Discretise).

Conclusion and Perspectives

In the name of Allah (SWT),

Ultimately, in our thesis, we have undertaken the numerical and theoretical study of solution approximation of the Fredholm's integral equations of the second kind defined over large real intervals, where we have stated that, unlike the conventional literature treatment of these equations, the one adopted in the manuscript has proven more efficient in that, in case of treatment of continuous functions and thus Banach spaces, given a required sufficient criterion for the stability of the numerical integration quadrature rules used, we have applied the truncation of the Neumann's series of the Banach endomorphisms discussed hereinbefore to prove the usefulness of the possibility to avoid inverting an algebraic matrix and computing a sum of a handful of its powers instead.

Secondly, although being a preprint, we have established the necessary theoretical builds to prove that, if the settings are on the real Hilbert space $L^2(\Omega)$, then, by means of the inner-product(s), we have given precise and coherent definition of **symmetry** and of **positive-definiteness** for **matrices of bounded linear operators over Hilbert spaces** on par with the algebraic case and have shown that the relaxed iterative **GJOR** scheme is a priori necessarily and sufficiently convergent for all SPD operator-matrices of our context provided the relaxation parameter ω subjects to the same frame of constraints established in finite dimensions. Moreover, by a projection method, we have shown that our functional Hilbert-iterative-scheme may be discretised where, using some relatively advanced tools from Functional Calculus and in particular those elements on Banach Algebras, its convergence to the *exact solution* would still hold true on the Hilbert space constructed hereinabove.

As perspectives, insha'Allah, with other projection methods for discretisation, we hopefully envision walking the same path to study the possibility to generalise the relaxed version of **G-S**; that is, the **GSOR**, into the new treatment(s). With weaker constraints on the same large-interval real-Hilbert space $L^2(\Omega)$, we conjecture that, in light of the *trace* concept, then provided that one satisfies

$$|\lambda|^2 > \int_\Omega |k(s,s)|^2 \, ds, \quad \lambda \in \mathbb{C} \setminus \{0_\mathbb{R}\},$$

we might be able to significantly reduce the size of λ without risking the existence and uniqueness of the solution, as well as continue leveraging results from Normed Algebras (Banach, C*) to have our spatial constructions weighing in to either positively confirm or negatively deny that

Proposition 1. *If $(\lambda\mathcal{I} - \mathcal{K})$ is symmetric and positive-definite, then, if $\omega \in (0,2)$, the **GSOR** is convergent.*

In the long-term, we may continue exploring the various possible applications of the eminent Continuous Functional Calculus on the study of Integral Equations. On a deeper level, we may attempt tackling the seemingly alerting patterns and close relationships between $\mathbb{R}^N$, or $\mathbb{C}^N$, and the Cartesian-product vector-space

$$\mathbf{H}^N := \mathbf{H} \underbrace{\times \cdots \times}_{N \text{ times}} \mathbf{H}, \quad \mathbf{H} \text{ a Hilbert space.}$$

In particular, if we were to assume that $\mathbf{H} = \ell^2(\mathbb{Z}, \mathbb{C})$, then how sophisticated would it be to discuss the theory underpinning the study of (linear) equations of the form

$$\mathcal{M}X = B, \quad X, B \in \mathbf{H}^N, \quad \mathcal{M} \in \mathbf{H}^{N \times N} \equiv \mathscr{L}(\mathbf{H}^N)?$$

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