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Intitulée

Etude de quelques problèmes fractionnaires nonlinéaires

Sous la direction de : M.C.A. Amara HITTA.

Co-encadreur : Prof. Rabah KHALDI.

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I, Abdourazek Souahi declare that this thesis and the work presented in it are my own and has been generated by me as the result of my own original research:

Study of some nonlinear fractional differential problems.

I confirm that:

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- Where I have consulted the published work of others, this is always clearly attributed;
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

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الملخص

تميزت العشريتان الأخيرتان باهتمام الباحثين بدراسة المعادلات التفاضلية الكسرية. واتجهت دراساتهم في وحدانية ووجود حلول لها وكذلك بعض التقديرات حول استقرار الحلول. بعض المساهمات في هذا المجال تمحورت حول المعادلات غير الخطية والمعادلات الكسرية الضبابية. يرجع الفضل في ذلك إلى قدرة هاته المعادلات في نمذجة الظواهر المشاهدة كبُنْيَاتٍ لها خواصا غير موضعية، كما تظهر أهميتها عند نمذجة تغيرات مواد تحتفظ بمعلومات عن حالاتها القديمة.

الهدف من هذه الأطروحة هو المساهمة في هذا المجال، وذلك من خلال دراسة بعض المعادلات التفاضلية الكسرية في فضاءات بناخ. سنستعمل بعض مبرهنات القيم الصامدة، وبعض الشروط المشابهة لشروط ناقومو.

سنتطرق في بداية الأمر إلى وجود الحلول الموجبة لبعض المعادلات ذات القيم الابتدائية وكذا معادلات ذات قيم حدية باستعمال مبرهنة كراسنوسالسلي وغيو. ثم نهتم بوحدانية حلول بعض المعادلات الأخرى باستعمال شروط مشابهة لناقومو. في الأخير سنساهم في معالجة مسائل ضبابية ذات رتب كسرية كيفية ببرهنة وجود ووحدانية حلولها.

سنرفق بعض الأمثلة التوضيحية ومثالا رقميا لتبرير صحة وفعالية التقنيات المستعملة.

كلمات مفتاحية:

معادلات تفاضلية كسرية، مسألة غير خطية، مبرهنة الوحدانية، التقاريب المتتالية، تقريبات بيكارد، الشروط الابتدائية، شروط حدية، وجود، وحدانية، حلول موجبة، مبرهنة النقط الصامدة، مشتقة كسرية، معادلات تفاضلية ضبابية، شروط كراسنوسالسكي وكراين، شروط كُوي، شروط روغرس، مبرنة كراسنوسالسلي وغيو، التفاضل الضبابي حسب ريمان وليوفيل، مشتقة ضبابية.

Abstract

In the last decades, the qualitative and the quantitative aspect of fractional differential equations got a lot of researchers' attention. Some of the contributions in this fields are in the investigation of solutions to nonlinear fractional differential equations and fuzzy fractional differential equations. The interest in this field is mainly due to the large applicability of the theory in modeling problems presenting nonlocal characteristics, such as viscoelasticity and memory preservation.

The object of this thesis is to contribute in the field of fractional calculus where we study some class of fractional differential equations in Banach spaces. Our results are based on fixed point theorems and Nagumo like conditions.

First, we establish the positivity and existence to a class of fractional initial value problems and boundary value problems using the fixed point theorem of Guo-Krasnoselskii. Later on, we establish the uniqueness alongside the existence of the solution of some initial value problems under Krasnoselskii-Krein like conditions. Finally, we are concerned by the study of a class of fuzzy fractional differential equations with uncertainty and we prove some new results on existence and uniqueness of high order problems.

One numerical example and several other examples are provided to outline the applicability and the usefulness of our results in the field.
Keywords:
Fractional differential equation ; Nonlinear problem ; Uniqueness theorem ; Successive approximations ; Picard's iterates ; Initial value problem ; Boundary value problem ; Existence ; Uniqueness ; Positivity of solution ; Fixed point theorem ; fractional derivative ; Fuzzy fractional differential equations ; Krasnoselskii-Krein conditions ; Kooi conditions ; Rogers conditions ; Guo-Krasnoselskii Theorem ; Riemann-Liouville fuzzy differentiability ; Fuzzy derivative ; Fuzzy fractional differential equation.

Résumé

Ces deux dernières décennies ont connu un développement rapide dans la théorie du calcul fractionnaire. De nombreux chercheurs ont apporté des contributions importantes à cette théorie qui permet de mieux expliquer et modéliser les phénomènes non-locaux qui préservent une certaine mémoire. L'intérêt est surtout porté à l'étude qualitative et quantitative des solutions de problèmes fractionnaires et de quelques équations différentielles fractionnaires floues.

Cette thèse est consacrée à l'étude de quelques problèmes engendrés par des équations différentielles d'ordre non entier. On s'intéresse à l'existence, à l'unicité et à la positivité de la solution. En se basant sur les théorèmes du point fixe et les méthodes itératives. On établit aussi quelques résultats pour les équations différentielles floues.

D'abord, on présente des résultats de positivité et d'existence de solutions de quelques classes d'équations différentielles fractionnaires de Riemann-Liouville, notamment à conditions initiales ou aux limites en utilisant le théorème du point fixe de Guo-Krasnoselskii. Ensuite, on établit l'existence et l'unicité avec d'autres techniques reposant sur des conditions semblables à celle de Nagumo. Enfin, quelques nouveaux résultats d'existence et d'unicité de solutions floues sont démontrés.

Pour bien illustrer leur applicabilité et leur intérêt particulier, on présentera quelques exemples et un cas numérique pour conclure.
Mots clés:
Equation différentielle fractionnaire ; Problème non linéaire ; Théorème d'unicité ; Approximations successives ; Itérations de Picard ; Conditions initiales ; Conditions aux limites ; Existence ; Unicité ; Positivité ; Théorème du point fixe ; Dérivée fractionnaire ; Equations différentielles fractionnaires floues ; Dérivée floue ; Conditions de Krasnoselskii-Krein ; Conditions de Kooi ; Conditions de Rogers ; Théorème de Guo-Krasnoselskii ; Différentiabilité floue au sens de Riemann-Liouville.

To Allah.

The Glorified and Exalted,
The only One worshipped,
The greatest Cherisher and Sustainer.

For his satisfaction, I have read and written.

He is the Best Disposer of affairs.
And, Him alone is sufficient for us.
May Him accept us as
his righteous slaves and submitters,
and pious worshippers.

All the praises to Allah.
My Almighty, Allah.
...

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List of Symbols

Symbol	Description
FDE	Fractional Differential Equation
FFDE	Fuzzy Fractional Differential Equation
IVP	Initial value problem
FIVP	Fuzzy initial value problem
BVP	Boundary value problem
$\Gamma(\cdot)$	Gamma function
$\beta(\cdot)$	Beta function
C_n^r, rCn	Binomial coefficients
$E_\alpha(\cdot)$	One-parameter Mittag-Leffler function
$E_{\alpha,\beta}(\cdot)$	Two-parameter Mittag-Leffler function
${}^{GL}D^\alpha$	Grunwald-Letnikov fractional derivative
I^α	Fractional Riemann-Liouville integral
D^α	Fractional Riemann-Liouville derivative
${}^{R}D^\alpha$	Fractional Riesz derivative
${}^{C}D^\alpha$	Fractional Caputo derivative
$\mathrm{Re}(\cdot)$	Real part of the complex variable
$[\cdot]$	Integer part of the real variable
$\{\cdot\}$	Fractional part of the real variable
AC	Space of absolutely continuous functions
L^1	Space of Lebesgue integrable functions
$\triangleq$	Denoted by
$\mathcal{L}$	Laplace transform

$\mathcal{L}^{-1}$	Inverse Laplace transform
$\mathbf{L}$	Fuzzy Laplace transform
$d(\cdot,\cdot)$	A metric or Hausdorff distance (in fuzzy sets)
$D(\cdot,\cdot)$	Another distance defined in the last chapter
$\mathbb{E}$	Space of fuzzy numbers
$\mu(\cdot)$	Fuzzy membership function
$\odot, \oplus$	Multiplication and addition in fuzzy sets
$\ominus$	H-difference
$C^{\mathbb{F}}$	Space of continuous fuzzy valued functions
$L^{\mathbb{F}}$	Space of all Lebesgue integrable fuzzy valued functions

Author's papers

[1] A. Souahi, A. Ben Makhlouf, and M. A. Hammami. Stability analysis of fractional-order nonlinear systems. Submitted.

[2] A. Souahi, A. Guezane-Lakoud, and A. Hitta. On the existence and uniqueness for high order fuzzy fractional differential equations with uncertainty. *Advances in Fuzzy Systems*, pages 9, 2016.

[3] A. Souahi, A. Guezane-Lakoud, and A. Hitta. Positive solutions for higher-order nonlinear fractional differential equations. *Vietnam Journal of Mathematics*, pages 1–10, 2016.

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[5] A. Souahi, A. Guezane-Lakoud, and R. Khaldi. On a fractional higher order initial value problem. Submitted.

[6] A. Souahi, A. Guezane-Lakoud, and R. Khaldi. On some existence and uniqueness results for a class of equations of order $0 < \alpha \leq 1$ on arbitrary time-scales. *International Journal of Differential Equations*, pages 8, 2016.

Introduction

A huge number of physical phenomena is modeled using differential equations. A great portion of the models has been or is being studied using the fixed point theory.

In the past years, researchers from around the world tried to narrow the gap of error between the physical model and the experiments conducted in adjacency. This resulted in the introduction and application of new theories or extension of old theories, such as time-scales theory, fuzzy sets theory, fractional calculus, etc.

The fractional calculus and fuzzy set theory caught our attention. In a matter of fact, the former one is a generalisation of integration and differentiability to noninteger order numbers and was coined by Leibniz in his famous letter to L'Hôpital. With this theory, models have fewer variables and are better describing nonlocal properties and memory characteristics in models. The Latter theory was introduced by Zadeh and Klaua as an extension of the classic notion of set. In contrast with the classical set theory where the elements either belong or don't belong to the set, an element on fuzzy sets have a gradual membership with intermediate grades: rather belong than not, to more likely to be out of the set. In this suitable concept to cope with reality, the incomplete or imprecise information are better understand and taken into account.

These theories advanced considerably the modeling and efficience of a wide range of applications, for instance signal processing, molecular biology, psychology analysis, game theory, etc.

Motivated by the works on these two theories and as a continuation of the works conducted intensively in these fields, our objectives are to demonstrate the existence and uniqueness of solutions for fractional differential equations with initial value problems and boundary value problem using the definition of Riemann-Liouville in crisp sets and fuzzy sets.

We treat in the chapters of this thesis the following problems under several assumptions. The several notation that arises are defined properly in the main chapters.

$$\begin{cases} D^q x(t) = f(t, x(t), D^{q-1}x(t)) \\ x(0) = 0, D^{(q-i)}x(0) = 0, i = 1, \ldots, [q], \end{cases}$$

$$\begin{cases} D^q u(t) + g(t) f(u(t), u'(t), u''(t), \ldots, u^{(m)}(t), D^\alpha u(t)) = 0, \\ u(0) = 0, u^{(i)}(0) = 0, i = 1, \ldots, n-2, \\ D^\beta u(1) = 0, 2 \leq \beta \leq n-2, 1 \leq m \leq \alpha \leq \beta - 1, \end{cases}$$

$$\begin{cases} D^q x(t) = f(t, x(t), D^{q-1}x(t)) \\ x(0) = y_0, D^{(q-i)}x(0) = \tilde{0}, i = 1, \ldots, [q]. \end{cases}$$

This thesis is organized as follows:

In the preliminaries' Chapter of this thesis, we recall necessary definitions, notions, and two models. Chapter 2 is based on the papers submitted [96] and accepted [95]. We treat two FDE; one is with initial data and the second one is with boundary conditions. After succinctly speaking about the previous researchs in the field, we give new results on the existence of positive solutions. Further, in the following Chapter 3, we prove the existence of uniqueness for a class of FDE using different techniques than those of Chapter 2 and provide a numerical example, see the accepted paper [93]. At last, Chapter 4 is based on the accepted paper [94]. We use the fuzzy Laplace transform and Nagumo-like conditions to obtain several uniqueness results for fuzzy fractional differential equations of high order.

Chapter 1

Preliminaries

In this chapter, we give an introduction to the several concepts that are of a particular interest in what follows. First, we define the fractional differentiation and give some practical properties and lemmas. Next, we give some general definitions and some fixed point theorems. Finally, we introduce fuzzy sets and exhibit some extensions, definitions, and other useful theorems.

1.1 Fractional Calculus on crisp sets

1.1.1 Fractional differentiation and integration

Before introducing a definition of fractional derivative we define some special functions which generalise the factorial and exponential functions, more details on the book [2].

Definition 1.1.1 (Gamma function)
The gamma function $\Gamma(\cdot)$ *is defined by the integral*

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt,$$

which converges in the right half of the complex plan: that is, $\mathrm{Re}(z) > 0$.

One of the basic properties of the gamma function is that it satisfies

$$\Gamma(z+1) = z\Gamma(z),$$

and for every integer $n \geq 0$, we have

$$\Gamma(n+1) = n!.$$

The gamma function can be represented also for every $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > 0$ by the limit

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)\cdots(z+n)}.$$

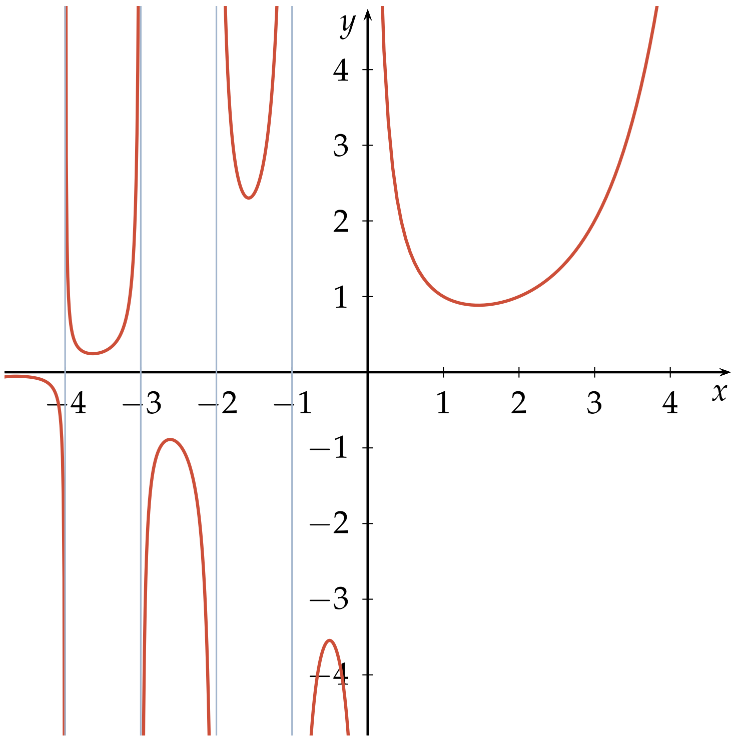

Figure 1.1: Plot of the gamma function.

Properties 1.1.2
The following are some properties of the gamma function, see [42].

(i) $\Gamma'(x) > 0$, *for every* $x > x_{min} > 1$,

(ii) $x_{min} \simeq 1.461$ *and* $\Gamma(x_{min}) \simeq 0.885$,

(iii) $\Gamma(1) = \Gamma(2) = 1$,

(iv) $\Gamma(x)\Gamma(x-1) = \dfrac{\pi}{\sin(\pi x)}$ *for every* $0 < x < 1$.

The binomial coefficients for r, n integers such that $0 \leq r \leq n$ are defined by

$$C_n^r = rCn = \frac{n!}{(n-r)!r!}.$$

Without loss of generality, we denote by $rC\alpha$ the following extended binomial coefficients for $\alpha \in \mathbb{R}_+$

$$rC\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(n-r+1)\Gamma(r+1)}.$$

Definition 1.1.3 (Beta function)
For every $z, w \in \mathbb{C}$ *such that* $\mathrm{Re}(z) > 0, \mathrm{Re}(w) > 0$, *the beta function is defined by*

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt.$$

An interesting formula relating the gamma and beta functions is

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$$

Mittag-Leffler introduced a function that plays the role of the exponential function in the fractional calculus. Several generalisations were introduced by many researchers later. For the convenience of the reader, we define here briefly the one parameter and two parameters Mittag-Leffler functions. In [35, 71, 88, 89], several discussions and properties of this function could be found.

Definition 1.1.4 (One-parameter Mittag-Leffler function)
For every $\alpha > 0$ *and* $z \in \mathbb{R}$, *the one parameter Mittag-Leffler function is defined*

by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},$$

where $\alpha > 0$.

This function coincides with the exponential function when $\alpha = 1$.

Definition 1.1.5 (Two-parameter Mittag-Leffler function)
For every $\alpha, \beta > 0$ *and* $z \in \mathbb{R}$*, the two parameter Mittag-Leffler function is defined by*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

A great number of papers and books on the theory and applications of fractional calculus appeared. They used several approaches to defining integration and differentiation of arbitrary order, see [38, 63, 74, 78, 82, 87]. The following are some of the most used ones:

Definition 1.1.6
The Grunwald-Letnikov fractional derivative of order $\alpha > 0$ *of* f *is given by*

$$^{GL}D_a^\alpha f(t) = \lim_{h \to 0+} \frac{1}{h^\alpha} \sum_{k=0}^{[(t-a)/h]} (-1)^k (\alpha C^k f(t - kh).$$

Definition 1.1.7 (Riemann-Liouville fractional integral)
The fractional integral of the function $h : (0, \infty) \to \mathbb{R}$ *of order* $\alpha \in \mathbb{R}^+$ *is defined by*

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds,$$

provided the right side is pointwise defined on $(0, \infty)$.

Definition 1.1.8 (Riemann-Liouville fractional derivative)
For a function $h \in C((0, \infty), \mathbb{R})$*, the Riemann-Liouville fractional derivative of* h *is defined by*

$$D^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} h(s) ds,$$

where $n \in \mathbb{N}^*$ and $n = [\alpha] + 1$, provided that the right side is pointwise defined on $(0, \infty)$.

Definition 1.1.9 (Riesz fractional derivative)
The Riesz fractional derivative of fractional order α of the function f is given by

$$
\begin{aligned}
{}^{R}D^{\alpha} f(t) = & \frac{-1}{2\cos(\pi\alpha/2)\Gamma(n-\alpha)} \\
& \times \frac{d^n}{dt^n}\left(\int_{-\infty}^{t} (t-s)^{n-\alpha-1} f(s)ds + (-1)^n \int_{t}^{+\infty} (t-s)^{n-\alpha-1} f(s)ds\right).
\end{aligned}
$$

Definition 1.1.10
Let $\alpha > 0$ and $n = [\alpha] + 1$, for a function $f \in C^n([a,b], \mathbb{R})$ the Caputo fractional derivative of f of order α is defined by

$$
{}^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^n(s)s.
$$

Lemma 1.1.11 (Relation between RL and C derivatives)
Let $\alpha \geq 0$ and $n = [\alpha] + 1$. Assume that $h : [a,b] \to \mathbb{R}$ is such that $D^{\alpha}h$ and $D^n h$ exist. Then

$$
{}^{C}D^{\alpha} h(t) = D^{\alpha} h(t) - \sum_{k=0}^{n} \frac{D^k f(a)}{\Gamma(k-\alpha+1)} (x-a)^{k-\alpha}.
$$

In the realm of fractional differential equations, the most used definitions are Caputo and Riemann-Liouville derivatives. The Caputo definition is more welcome due to initial data in integer derivatives, Nevertheless, the Riemann Liouville derivatives are possible to measure and observe as can be found in [57]. Furthermore, fractional order initial conditions of Riemann-Liouville type differential equations can be given by integer order conditions, as usual, see [68, 104]. For the above mentioned reasons, we will focus on it throughout this manuscript. And we refer the reader to the good monographs and book [1, 63, 67, 82, 87], for discussions and more details on the fractional calculus.

Lemma 1.1.12 ([63])
Let $p, q \geq 0$ and $f \in L^1([0,1])$, then

$$
I_{0+}^{p} I_{0+}^{q} h(t) = I_{0+}^{q} I_{0+}^{p} h(t) = I_{0+}^{p+q} h(t)
$$

and

$$D^q I^q_{0+} h(t) = h(t).$$

Lemma 1.1.13 ([63])
Let $p > q > 0$, and $h \in L^1([a,b])$, then for all $t \in [a,b]$ we have

$$D^q I^p_{0+} h(t) = I^{p-q}_{0+} h(t).$$

Theorem 1.1.14 (Leibniz' formula for Riemann-Liouville operators)
Let $n > 0$ and assume that f and g are analytic on $(a-h, a+h)$ with some $h > 0$. Then,

$$D^\alpha[fg](t) = \sum_{k=0}^{[\alpha]} C^n_k (D^k f)(t)(D^{\alpha-k} g)(t) + \sum_{k=[\alpha]+1}^{\infty} C^n_k (D^k f)(t)(I^{k-\alpha} g)(t),$$

for $a < t < a + \frac{h}{2}$.

If $\alpha, \beta \in \mathbb{C}$ such that $\mathrm{Re}\beta > 0$, then

1. $(I^\alpha_{a+}(t-a)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1} \quad (\mathrm{Re}\alpha > 0)$
2. $(D^\alpha_{a+}(t-a)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1} \quad (\mathrm{Re}\alpha \geq 0)$
3. $(I^\alpha_{b-}(b-t)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(b-x)^{\beta+\alpha-1} \quad (\mathrm{Re}\alpha > 0)$
4. $(D^\alpha_{b-}(b-t)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-x)^{\beta-\alpha-1} \quad (\mathrm{Re}\alpha \geq 0)$.

Remark 1.1.15. Unlike the usual differentiation, the fractional derivative in Riemann-Liouville sense of a constant is not zero. However

$$(D^\alpha_{a+} t^{\alpha-m})(x) = \frac{\Gamma(\alpha-m+1)}{\Gamma(n-m+1)}(x^{n-m})^{(n)} = 0,$$

for every $m \in \{1, 2, \ldots, [\mathrm{Re}\alpha]+1\}$. This last equation will be used in proving the equivalence between FDE and the associated integral equation.

1.1.2 Fractional functional spaces

Before speaking of a solution to differential equations, we need the functional space were the solution belongs to obtain some regularity properties. The problems treated in the following chapters use some of the Banach spaces defined in this subsection or based on the definitions below. The list here is not exhaustive and we refer to some books devoted to this branch [36, 48].

Definition 1.1.16
We denote by $AC(0,1)$ the space of absolutely continuous functions defined on $[0,1]$. In fact, $x \in AC(0,1)$ if and only if there exist $\phi \in L^1((0,1),\mathbb{R})$ and $c \in \mathbb{R}$ such that

$$x(t) = c + \int_0^t \phi(s)ds \text{ for } t \in (0,1),$$

where $L^1([0,1],\mathbb{R})$ is the Banach space of Lebesgue integrable functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|h\|_{L^1} = \int_0^1 |h(t)|dt$.
Also, we define $AC^{n-1}(0,1)$ by

$$AC^{n-1}(0,1) = \{x \in C^{n-2}, x^{(n-1)} \in AC(0,1)\}.$$

Theorem 1.1.17 ([83, Theorem 7.21])
If $x : [a,b] \to \mathbb{R}$ is differentiable at every point of $[a,b]$ and $x' \in L^1$ on $[a,b]$, then

$$x(t) - x(a) = \int_a^t x'(\tau)d\tau.$$

Theorem 1.1.18 ([47, Theorem 3.21])
If $x : [a,b] \to \mathbb{R}$, the followings are equivalent:

(i) *x is absolutely continuous on $[a,b]$.*

(ii) *$x(t) - x(a) = \int_a^t y(\tau)d\tau$ for some $y \in L^1([a,b],\mathbb{R})$.*

(iii) *x is differentiable almost everywhere on $[a,b]$, $x' \in L^1([a,b],\mathbb{R})$, and $x(t) - x(a) = \int_a^t x'(\tau)d\lambda(\tau)$.*

1.1.3 Between the FDE and the integral equation

The basic idea of this subsection is to get general results of the following classical standard result from differential and integral calculus.

Theorem 1.1.19 (Fundamental Theorem of classical calculus)
Let $f : [a,b] \to \mathbb{R}$ be a continuous function and $F : [a,b] \to \mathbb{R}$ be defined by

$$F(t) \triangleq \int_a^t f(s)ds.$$

Then, F is differentiable and $F' = f$.

Lemma 1.1.20 ([63])
Assume that $u \in C(0,1) \cap L^1(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L^1(0,1)$, then

$$I^{\alpha}_{0+} D^{\alpha}_{0+} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \ldots + C_n t^{\alpha-n}$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$.

Lemma 1.1.21 ([63])
For $\alpha > 0$, the general solution of the fractional differential equation $D^{\alpha}_{0+} u(t) = 0$ is given by

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \ldots + C_n t^{\alpha-n},$$

where $C_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, and $n = [\alpha] + 1$.

Lemma 1.1.22
Let $v \in L^1((0,1), \mathbb{R})$, then

$$u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v(s)ds$$

is the unique continuous solution on $[0,1]$ of the equation

$$D^{q}_{0+} u(t) = v(t), \tag{1.1}$$

satisfying the initial conditions

$$u(0) = 0, D^{(q-i)} u(0) = 0, i = 1, \ldots, [q]. \tag{1.2}$$

Proof. In fact, we may apply Lemma 1.1.20 to reduce equation (1.1) to an equivalent integral equation

$$u(t) = I^q_{0+}v(t) + C_1 t^{q-1} + C_2 t^{q-2} + \ldots + C_n t^{q-n},$$

for some C_i, $i = 1, \ldots n$.
By the initial conditions $u(0) = 0$, $D^{(q-i)}u(0) = 0$, $i = 1, \ldots, [q]$, we obtain $C_1 = C_2 = \cdots = C_n = 0$.
Substituting theses values in the last formula, we obtain the expression of the unique solution of (1.1) subject to the conditions (1.2)

$$u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v(t) ds.$$

□

A function x is called a solution of the nonlinear equation

$$D^q u = f(t, u) \tag{1.3}$$

where f is a continuous function an $[0, 1]$, if $x \in C([0, 1], \mathbb{R})$, $D^q x$ exists and is continuous on $[0, 1]$, $f(t, x)$ is well defined, and x satisfies (1.3).

It's often the case to use some special transforms to obtain the associated integral equation, and one of the main methods is the Laplace transforms as mentioned by A. Jafarian *et al.* in [59].

Definition 1.1.23
We say that a function $f : [0, \infty) \to E$ is of an exponential type, if there exist $t_0, M > 0$, and $\gamma \in \mathbb{R}$ such that

$$\|f(t)\| \leq M e^{\gamma t}, \text{ for all } t \geq t_0.$$

In other words, the function f must not grow faster then a certain exponential function when $t \to \infty$.

Lemma 1.1.24
Let $f : [0,\infty) \to E$ be a locally integrable function of an exponential type. Then there exists $\gamma > 0$ such that

$$\int_0^\infty e^{-\lambda t} f(t)dt$$

is convergent for $\mathrm{Re}(\lambda) > \gamma$.

Denote the subset D by

$$D = \{f \in L^1((0,\infty),E) : f \text{ is of exponential type}\}.$$

Definition 1.1.25 (Laplace transform)
The Laplace transform $\mathcal{L}$ of $f \in D$ is defined by

$$\mathcal{L}\{f(t)\}\{\lambda\} = \int_0^\infty e^{-\lambda t} f(t)dt.$$

Definition 1.1.26 (Inverse Laplace transform)
Let $\tilde{f}$ be an integral function. We define the inverse Laplace transform of $\tilde{f}$ by

$$\mathcal{L}^{-1}\{\tilde{f}(\lambda)\}\{t\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\lambda t}\tilde{f}(\lambda)d\lambda.$$

1.2 Existence and uniqueness theorems

In general cases, when we want to study a fractional differential equation, we have to obtain solution in a suitable functional spaces. Later, we modify the problem suitably and apply one of the fixed point theorems in combination with other useful estimates and known inequalities in order to establish the wanted existence, uniqueness or stability results. In this Section, we recall some known fixed point theorems. We think that the books and notes [6, 28, 50, 56, 61, 67, 90] give sufficiently large number of theorems, other versions of these theorems, their proofs alongside applications of the theory.

1.2.1 Fixed point theorems

Definition 1.2.1
A mapping T of a metric space E into itself is said to satisfy a Lipschitz condition with a Lipschitz constant K if

$$d(Tx,Ty) \leq Kd(x,y),$$

for every $x,y \in E$. If this condition is satisfied with $0 \leq K < 1$ then T is called a contraction mapping.

Theorem 1.2.2 (Banach's contraction mapping principle)
Let (E,d) be a complete metric space and T be a contraction mapping from E into itself. Then T has a unique fixed point x_0 and for each $x \in E$, $\lim_{n\to\infty} T^n(x) = x_0$. Moreover, $d(T^n(x),x_0) \leq \frac{K^n}{1-k} d(x,T(x))$.

Theorem 1.2.3 (Nonlinear alternative of Leray-Schauder [103])
Let E be a Banach space and C be a nonempty convex subset of E. Let U be a nonempty open subset of C with $0 \in U$ and $T : \overline{U} \to C$ continuous and compact operator. Then either

(1) T has fixed points or

(2) There exist $u \in \partial U$ and $\lambda \in (0,1)$ such that $u = \lambda T(u)$.

Theorem 1.2.4 (Brouwer fixed point theorem)
Let B be a non empty compact convex subset of a finite dimensional normed space and let T be a continuous mapping of B into itself. Then T has a fixed point in B.

Theorem 1.2.5 (Schauder)
Let B be a non empty closed convex subset of a normed space. Let T be a continuous mapping of B into a compact subset of B, then T has a fixed point in B.

Theorem 1.2.6 (Altman)
Let E be a normed space, let Q be the closed ball of radius $r > 0$, i.e. : $Q = \{x :$

$\|x\| \leq r\}$ *and let* T *be a continuous mapping of* Q *into a compact subset of* E *such that*

$$\|Tx - x\|^2 \geq \|Tx\|^2 - \|x\|^2,$$

for every x *verifying* $\|x\| = r$. *Then* T *has a fixed point in* Q.

Theorem 1.2.7 (Schaefer)
Let E *be a normed space,* T *a continuous mapping of* E *into* E *which is compact on each bounded subset* B *of* E. *Then either*

(i) the equation $x = \lambda Tx$ *has a solution for* $\lambda = 1$, *or*

(ii) the set of all such solution x, *for* $0 < \lambda < 1$, *is unbounded.*

Theorem 1.2.8 (Krasnoselskii)
Let M *be a closet convex non-empty subset of a Banach space* (E, d). *Suppose that* A *and* B *map* M *into* E *and suppose the following conditions are satisfied*

(i) $Ax + By \in M, \text{ for all } x, y \in M,$

(ii) A *is compact and continuous,*

(iii) B *is a contraction mapping.*

Then there exists y *in* M *such that* $Ay + By = y$.

Lemma 1.2.9 (Arzelà-Ascoli Theorem)
Let Ω *be a bounded subset of* $\mathbb{R}^n$. *A subset* $A \in C(\overline{\Omega})$ *is relatively compact if and only if it is bounded and equicontinuous.*

Lemma 1.2.10 (Arzelà-Ascoli Theorem)
Let X *be a compact metric space and* Y *any metric space. A subset* ϕ *of the space* $C(X, Y)$ *of continuous mappings of* X *into* Y *is totally bounded in the metric of uniform convergence if and only if* Φ *is equicontinuous on* X *and* $\Phi(x) = \{\phi(x) : \phi \in \Phi\}$ *is a totally bounded subset of* Y *for each* $x \in X$.

For the following theorem of Guo-Krasnoselskii we need to define a cone:

Definition 1.2.11
A closed subset P of a Banach space X is called a cone if:

(i) $\forall x, y \in P : x + y \in P$,

(ii) $\forall x \in P, \forall \alpha > 0 : \alpha x \in P$,

(iii) $P \cap (-P) = \{0\}$.

A cone P defines a partial order by $x \leq y \Leftrightarrow y - x \in P$.

Theorem 1.2.12 (Guo-Krasnoselskii)
Let E be a Banach space and $P \subset E$ be a cone. Assume Ω_1 and Ω_2 are open bounded subsets contained in E with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Assume, further, that

$$T : P \cap \left(\overline{\Omega_2} \backslash \Omega_1\right) \to P$$

be a completely continuous operator. If either

(I) $\|Tu\| \leq \|u\|, u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|, u \in P \cap \partial\Omega_2$ *or*

(II) $\|Tu\| \leq \|u\|, u \in P \cap \partial\Omega_2$ *and* $\|Tu\| \geq \|u\|, u \in P \cap \partial\Omega_1$.

Then T has at least one fixed point in $P \cap \left(\overline{\Omega_2} \backslash \Omega_1\right)$.

1.2.2 Nagumo like theorems

Before stating some theorems, we would like to define some subsets.

- $S = \{(t, y) \in [t_0 - a, t_0 + a] \times [y_0 - b, y_0 + b]\}$.
- $S_0 = \{(t, y) \in [t_0, t_0 + a] \times [y_0 - b, y_0 + b]\}$.
- $D_0 = \{(t, y) : 0 \leq t \leq a, |y| < \infty\}$.
- $U = [t_0, t_0 + a] \times \{x \in \mathbb{R}^n : \|x - x_0\| \leq b\}$.

such that $a, b > 0$ and $t_0, y_0 \geq 0$.

Theorem 1.2.13 (Lipschitz, 1876)
Let f be continuous and satisfies a Lipschitz condition with respect to x: that is,

$$|f(t,x) - f(t,\overline{x}| \leq K|x - \overline{x}|$$

in some neighbourhood V of (a,b). Then the differential equation with initial condition

$$\frac{dy}{dt} = f(t,y), y(a) = b \tag{1.4}$$

has a unique solution in some neighbourhood of a.

In the following some uniqueness theorems which assumes different conditions on the function $f(\cdot,\cdot)$ then the Lipschitz condition.

Theorem 1.2.14 (Peano's Uniqueness Theorem)
Let $f(t,y)$ be continuous in S_0 and nonincreasing in y for each fixed $t \in [t_0, t_0 + a]$. Then, IVP (1.4) *has at most one solution in $[t_0, t_0 + a]$.*

We say that a function $f(\cdot,\cdot)$ satisfies Nagumo's condition in a set D if

$$|f(t,y) - f(t,\overline{y})| \leq k|t - t_0|^{-1}|y - \overline{y}|,$$

for all $(t,y), (t,\overline{y}) \in D$, such that $t \neq t_0$ and $k \leq 1$.

Theorem 1.2.15 (Nagumo's uniqueness theorem)
Let $f(t,y)$ be continuous function, which satisfies Nagumo's condition in S. Then, IVP (1.4) *has at most one solution in $|t - t_0| \leq a$.*

The constant in the Nagumo condition k could be greater than 1 provided we add an additional condition, for proofs and other interesting results, see the excellent book of Agarwal *et al.* [4].

Theorem 1.2.16 (Krasnoselskii-Krein' Uniqueness theorem)
Let $f(t,x)$ be a continuous function on S, which satisfies for all $(t,x), (t,\overline{x}) \in S$:

(i) $|f(t,x) - f(t,\overline{x})| \leq k|t-t_0|^{-1}|x-\overline{x}|$, $t \neq t_0$,

(ii) $|f(t,x) - f(t,\overline{x})| \leq c|x-\overline{x}|^{\alpha}$, *where c and k are positive constants, the real number α is such that $0 < \alpha < 1$, and $k(1-\alpha) < 1$.*

Then IVP (1.4) *has at most one solution in* $|t - t_0| \leq a$.

Theorem 1.2.17 (Kooi's Uniqueness theorem)
Let $f(t,x)$ be a continuous function on S, which satisfies for all $(t,x),(t,\overline{x}) \in S$:

(i) $|f(t,x) - f(t,\overline{x})| \leq k|t-t_0|^{-1}|x-\overline{x}|$, $t \neq t_0$,

(ii) $|t-t_0|^{\beta}|f(t,x) - f(t,\overline{x})| \leq c|x-\overline{x}|^{\alpha}$, *where c and k are positive constants, the real number α is such that $0 < \alpha < 1$, $\beta < \alpha$ and $k(1-\alpha) < 1-\beta$.*

Then IVP (1.4) *has at most one solution in* $|t - t_0| \leq a$.

Theorem 1.2.18 (Roger's uniqueness theorem)
Let $f(t,y)$ be continuous in D_0, which satisfies the condition

$$f(t,y) = o(e^{-1/t}t^{-2})$$

uniformly, for $0 \leq y \leq \delta$, $\delta > 0$ arbitrary. Further, let

$$|f(t,y) - f(t,\overline{y})| \leq \frac{1}{t^2}\,|y-\overline{y}|\,, t \neq 0$$

for all $(t,y),(t,\overline{y}) \in D_0$. Then IVP (1.4) *has at most one solution in* $[0,a]$.

Consider the initial value problem

$$x' = h(t,x), \quad x(t_0) = x_0 \tag{1.5}$$

where $h : U \subset \mathbb{R}^n \to \mathbb{R}$ is continuous.

Theorem 1.2.19 (Kamke's uniqueness Theorem)
Assume that for all $(t,x),(t,\overline{x}) \in U$, $t \neq t_0$, if

$$|h(t,x) - h(t,\overline{x})| \leq g(t-t_0, \|x-\overline{x}\|),$$

for some continuous function $g : (0,a] \times [0,2b] \to [0,\infty)$ *such that for every* $c \in (0,a)$*, the trivial solution* $x \equiv 0$ *is the only solution of*

$$x'(t) = g(t,x(t)) \quad \text{for all } t \in (0,c), \text{ and } \lim_{t \to 0^+} \frac{x(t)}{t} = 0.$$

Then IVP (1.5) *has at most one solution in* $[t_0, t_0 + a]$.

Theorem 1.2.20 (Montel-Tonelli uniqueness Theorem)
Assume that for all $(t,x), (t,\overline{x}) \in U$, $t \neq t_0$*, we have*

$$|h(t,x) - h(t,\overline{x})| \leq p(t - t_0)\psi(\|x - \overline{x}\|),$$

where $p : (0,a] \to (0,+\infty)$ *is integrable, and* $\psi : [0,2b] \to [0,+\infty)$ *is continuous and*

$$\int_{0^+} \frac{dr}{\psi(r)} = +\infty.$$

Then IVP (1.5) *has at most one solution in* $[t_0, t_0 + a]$.

Last, we provide two inequalities that will be used in the sequel. The first is a particular case of Jensen's inequality and the second in a generalisation of the Jensen's inequality.

Lemma 1.2.21
For every $a, b \in (0,1)$ *and every* $\alpha \geq 1$*, we have the following estimate*

$$(a+b)^\alpha \leq \frac{2^\alpha}{2}\left(a^\alpha + b^\alpha\right).$$

Lemma 1.2.22 (Jensen's inequality [101])
Let p *be a nonnegative continuous function on* $[a,b]$ *such that* $\int_a^b p(x)dx > 0$*. If* g *and* h *are real-valued continuous functions on* $[a,b]$ *and*

$$m_1 \leq g(x) \leq M_1, \quad m_2 \leq h(x) \leq M_2,$$

for all $x \in [a,b]$*, and* F *is convex on*

$$\Delta = [m_1, M_1] \times [m_2, M_2],$$

then

$$F\left(\frac{\int_a^b g(t)p(t)dt}{\int_a^b p(t)dt}, \frac{\int_a^b h(t)p(t)dt}{\int_a^b p(t)dt}\right) \leq \frac{\int_a^b F(g(t),h(t))p(t)dt}{\int_a^b p(t)dt}. \tag{1.6}$$

1.3 Fuzzy sets

In contrast to odd and even numbers which can be classified as belonging to exactly one class, in many engineering tasks, we are faced with classes such as *tall*, *speed*, *age*, etc. All of these convey a useful semantic meaning that is obvious to a certain community. However, a borderline between the belonging or not of a given object to such classes is not evident. Two-valued logic, used in describing these classes of situations, might be not well-suited, see Figure 1.2.

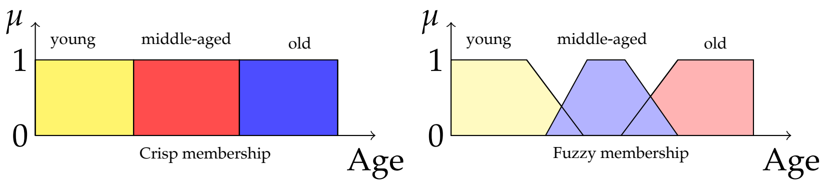

Figure 1.2: Crisp VS fuzzy membership logic.

1.3.1 Basic definitions and properties

First, let us recall some basic definitions about fuzzy numbers, fuzzy sets, and fuzzy logic. Further foundation and applications of this theory could be found in [64, 102, 113].

Let $\mathbb{E} = \{u : \mathbb{R} \to [0,1];\, u$ satisfies (A1)-(A4) $\}$ the space of fuzzy numbers:

(A1) u is normal: that is, there exists $y_0 \in \mathbb{R}$ such that $u(y_0) = 1$,

(A2) u is fuzzy convex i.e. $u(\lambda y + (1-\lambda)z) \geq \min\{u(y), u(z)\}$ whenever $y, z \in \mathbb{R}$ and $\lambda \in [0,1]$,

(A3) u is upper semicontinuous i.e. for any $y_0 \in \mathbb{R}$ and $\epsilon > 0$ there exists $\delta(y_0, \epsilon) > 0$ such that $u(y) < u(y_0) + \epsilon$ whenever $|y - y_0| < \delta$, $y \in \mathbb{R}$,

(A4) The closure of the set $\{y \in \mathbb{R};\, u(y) > 0\}$ is compact.

The set $[u]^{\alpha} = \{x \in \mathbb{R};\ u(y) \geq \alpha\}$ is called an α-level set of u.

It follows from (A1)-(A4) that the α-level sets $[u]^{\alpha}$ are convex compact subsets of $\mathbb{R}$ for all $\alpha \in (0,1]$. The fuzzy zero is defined by

$$\tilde{0} = \begin{cases} 0 & \text{if} y \neq 0, \\ 1 & \text{if} y = 0. \end{cases}$$

Definition 1.3.1 ([113])
A fuzzy number u in the parametric form is a pair $(\underline{u}(r), \overline{u}(r))$ of functions $\underline{u}$, $\overline{u}$, for $0 \leq r \leq 1$, which satisfy the following conditions:

(i) $\underline{u}$ is a bounded nondecreasing left continuous function in $(0,1]$ and right continuous at 0,

(ii) $\overline{u}$ is a bounded nonincreasing left continuous function in $(0,1]$ and right continuous at 0,

(iii) $\underline{u}(r) \leq \overline{u}(r)$, for all $0 \leq r \leq 1$.

The characteristic function of a set D is denoted $\chi : D \to \{0,1\}$ such that

$$\chi(x) = \begin{cases} 1 & \text{if } x \in D \\ 0 & \text{if } x \notin D. \end{cases}$$

Whereas a trapezoid fuzzy membership function $\mu : D \to [0,1]$ verifies

$$\mu(x) = \begin{cases} 0 & \text{if} \quad x \leq a, \\ \dfrac{x-a}{b-a} & \text{if} \quad a \leq x \leq b, \\ \dfrac{d-x}{d-c} & \text{if} \quad c \leq x \leq d, \\ 0 & \text{if} \quad d \leq x, \end{cases}$$

such as $[a,d] \subset D$, below an other membership function is plot on Figure 1.4 and a crisp membership function on Figure 1.3.

Moreover, we also can present the r-cut levels of fuzzy numbers as $[u]^r = [\underline{u}(r), \overline{u}(r)]$ for all $0 \leq r \leq 1$.

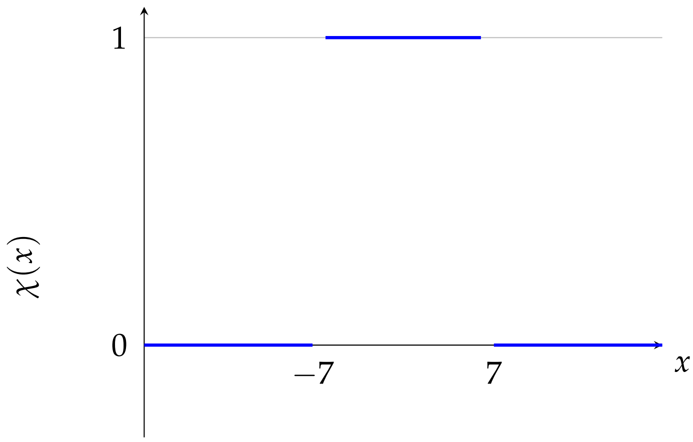

Figure 1.3: Example of a crisp membership function.

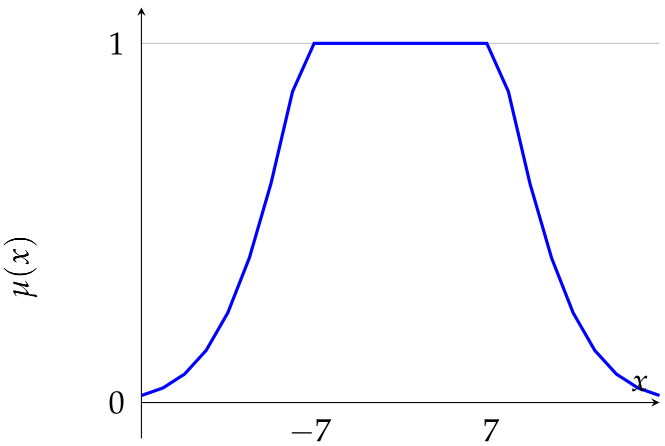

Figure 1.4: Example of a fuzzy membership function.

According to Zadeh's extension principle, we have the following properties of fuzzy addition and multiplication by scalar on $\mathbb{E}$

$$(u \oplus v)(x) = \sup_{y \in \mathbb{R}} \min\{u(y), v(x-y)\}, x \in \mathbb{R},$$

and

$$(k \odot u(x)) = \begin{cases} u\left(\frac{x}{k}\right) & \text{if } k \geq 0, \\ \tilde{0} & \text{if } k = 0. \end{cases}$$

Seeking simplicity, we note $\oplus$, $\odot$ by the usual $+, \cdot$. The Hausdorff distance between the fuzzy numbers is denoted by $d : \mathbb{E} \times \mathbb{E} \to [0, +\infty[$, such that $d(u,v) = \sup_{r \in [0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\}$. And $(d, \mathbb{E})$ is a complete metric space.

Definition 1.3.2
Let $x, y \in \mathbb{E}$. If there exists $z \in \mathbb{E}$ such that $x = y + z$, then z is called the Hukuhara difference of x and y, and it is denoted by $x \ominus y$ and called H-difference.

Remark 1.3.3. Note that the sign $\ominus$ stands for H-difference and $x \ominus y \neq x + (-1)y$.

We denote by $C^{\mathbb{F}}[0,a]$ the space of all fuzzy-valued functions which are continuous on $[0,a]$ and $L^{\mathbb{F}}[0,a]$ the space of all Lebesgue integrable fuzzy-valued functions on $[0,a]$, where $a > 0$. We also denote by $AC^{(n-1)\mathbb{F}}[0,a]$ the space of fuzzy-valued functions f which have continuous H-derivatives up to order $n-1$ on $[0,a]$ such that $f^{(n-1)}$ in $AC^{\mathbb{F}}[0,a]$.

1.3.2 Fuzzy differentiation and integration

Definition 1.3.4 ([26])
A function $f : (a,b) \to \mathbb{E}$ is called H-differentiable on $x_0 \in (a,b)$ if for $h > 0$ sufficiently small there exist H-differences $f(x_0 + h) \ominus f(x_0)$, $f(x_0) \ominus f(x_0 - h)$ and an element $f'(x_0) \in \mathbb{R}$ such that

$$0 = \lim_{h \searrow 0} d\left(\frac{f(x_0+h) \ominus f(x_0)}{h}, f'(x_0)\right) = \lim_{h \searrow 0} d\left(\frac{f(x_0) \ominus f(x_0+h)}{h}, f'(x_0)\right),$$

where h at denominator means $\frac{1}{h}\odot$.

This usual concept of differentiability has some shortcoming:

Remark 1.3.5. If c is a fuzzy number and $g : [a,b] \to \mathbb{R}$ an usual real valued function differentiable on $x_0 \in (a,b)$ with $g'(x_0) \leq 0$, then $f(x) = c \odot g(x)$ is not differentiable on x_0. In [26], Bede *et al.* defined a generalised concept of differentiability.

Definition 1.3.6

Let $f:(a,b)\rightarrow \mathbb{E}$ and $x_0\in(a,b)$. We say that f is strongly generalised differentiable on x_0 if there exists an element $f'(x_0)\in\mathbb{E}$, such that for all $h>0$ sufficiently small, we have

(i)

$$\begin{aligned} f'(x_0) &= \lim_{h\to 0}\frac{\Phi(x_0+h)\ominus\Phi(x_0)}{h} \\ &= \lim_{h\to 0}\frac{\Phi(x_0)\ominus\Phi(x_0-h)}{h}, \end{aligned}$$

or

(ii)

$$\begin{aligned} f'(x_0) &= \lim_{h\to 0}\frac{\Phi(x_0)\ominus\Phi(x_0+h)}{-h} \\ &= \lim_{h\to 0}\frac{\Phi(x_0-h)\ominus\Phi(x_0)}{-h}. \end{aligned}$$

Definition 1.3.7 ([12])

Let $f\in C^{\mathbb{F}}[0,1]\cap L^{\mathbb{F}}[0,1]$. The fuzzy fractional integral of the fuzzy-valued function f is defined by

$$I^{\beta}f(x;r)=\left[I^{\beta}\underline{f}(x;r),I^{\beta}\overline{f}(x;r)\right],0\le r\le 1,$$

where

$$\begin{aligned} I^{\beta}\underline{f}(x;r) &= \frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}\underline{f}(s;r)ds, \\ I^{\beta}\overline{f}(x;r) &= \frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}\overline{f}(s;r)ds. \end{aligned}$$

Definitions for higher-order derivatives were introduced based on the selection of derivative type in each step of differentiation.

In [26], the authors considered four cases for derivative up to second order. Following [12] and for the sake of convenience, we only consider two cases. The other cases are trivial because they can be reduced to crisp elements. For more details, see [12].

Definition 1.3.8 ([12, Definition 6])
Let $f \in C^{(n)\mathbb{F}}[0,1] \cap L^{\mathbb{F}}[0,1]$, $x_0 \in (0,1)$, *and* $\Phi(x) = (1/\Gamma(n-\beta)) \int_0^x (f(t)dt/(x-t)^{\beta-n+1})$, *where* $n = [\beta]+1$. *We say that* f *is fuzzy Riemann-Liouville fractional differentiable of order* β *at* x_0, *if there exists an element* $\left(D_0^{\beta} f\right)(x_0) \in \mathbb{E}$, *such that for all* $h > 0$ *sufficiently small, we have*

(i)

$$\begin{aligned}\left(D_0^{\beta} f\right)(x_0) &= \lim_{h\to 0} \frac{\Phi^{(n-1)}(x_0+h) \ominus \Phi^{(n-1)}(x_0)}{h} \\ &= \lim_{h\to 0} \frac{\Phi^{(n-1)}(x_0) \ominus \Phi^{(n-1)}(x_0-h)}{h},\end{aligned}$$

or

(ii)

$$\begin{aligned}\left(D_0^{\beta} f\right)(x_0) &= \lim_{h\to 0} \frac{\Phi^{(n-1)}(x_0) \ominus \Phi^{(n-1)}(x_0+h)}{-h} \\ &= \lim_{h\to 0} \frac{\Phi^{(n-1)}(x_0-h) \ominus \Phi^{(n-1)}(x_0)}{-h}.\end{aligned}$$

Denote by $C^{(n-1)\mathbb{F}}([0,a])$ the space of fuzzy-value functions f on the interval $[0,a]$ which have continuous H-derivative up to order $n-2$ such that $f^{(n-1)} \in C^{\mathbb{F}}[0,a]$. $C^{(n-1)\mathbb{F}}([0,a])$ is a complete metric space endowed by the metric D such that for every $g,h \in C^{(n-1)\mathbb{F}}([0,a])$:

$$D(g,h) = \sum_{i=0}^{n-1} \sup_{t\in[0,a]} d\left(g^{(i)}(t), h^{(i)}(t)\right).$$

In the rest of this thesis, we say that a fuzzy-valued function f is ${}^{RL}[(i)-\beta]$-differentiable if it is differentiable as in Definition 1.3.8 case (i) and is ${}^{RL}[(ii)-\beta]$-differentiable if it is differentiable as in Definition 1.3.8 case (ii).

Definition 1.3.9 ([12, Theorem 7])
Let $f \in C^{(n)\mathbb{F}}[0,1] \cap L^{\mathbb{F}}[0,1]$, $x_0 \in (0,1)$, *and* $\Phi(x) = (1/\Gamma(n-\beta)) \int_0^x (f(t)dt/(x-t)^{\beta-n+1})$, *where* $n = [\beta]+1$ *such that* $0 \leq r \leq 1$, *then*

(i) if f is ${}^{RL}[(i)-\beta]$-differentiable fuzzy valued function, then

$$\left(D_0^{\beta} f\right)(x_0;r) = \left[\left(D_0^{\beta} \underline{f}\right)(x_0;r), \left(D_0^{\beta} \overline{f}\right)(x_0;r)\right],$$

or

(ii) if f is ${}^{RL}[(i)-\beta]$-differentiable fuzzy valued function, then

$$\left(D_0^{\beta} f\right)(x_0;r) = \left[\left(D_0^{\beta} \overline{f}\right)(x_0;r), \left(D_0^{\beta} \underline{f}\right)(x_0;r)\right],$$

where

$$\left(D_0^{\beta} \underline{f}\right)(x_0;r) = \left[\frac{1}{\Gamma(n-\beta)}\left(\frac{d}{dx}\right)^n \int_0^x (x-t)^{n-\beta-1} \underline{f}(t;r)dt\right]_{x=x_0}$$
$$\left(D_0^{\beta} \overline{f}\right)(x_0;r) = \left[\frac{1}{\Gamma(n-\beta)}\left(\frac{d}{dx}\right)^n \int_0^x (x-t)^{n-\beta-1} \overline{f}(t;r)dt\right]_{x=x_0}.$$

1.3.3 High order differential equations and limitations

In [14], Allahviranloo and Ahmadi introduced the fuzzy Laplace transform, which they used under the strongly generalised differentiability. Recently, E. ElJaoui *et al.* [43] developed it further. The newly defined fuzzy Laplace transform [14] for high order fuzzy derivatives is one of the most useful methods as mentioned by A. Jafarian *et al.* in [59]: "..., one of the important and interesting transforms in the problems of fuzzy equations is Laplace transforms. The fuzzy Laplace transform method solves fuzzy fractional differential equations and fuzzy boundary and initial value problems [21, 22, 86] ...." See also [9, 85].

Definition 1.3.10
Let f be a continuous fuzzy-valued function. Suppose that $f(x) \odot e^{-px}$ is improper fuzzy Riemann integrable on $[0,\infty)$, then $\int_0^{\infty} f(x) \odot e^{-px}dx$ is called the fuzzy Laplace transform of f and is denoted by

$$\mathbf{L}[f(x)] = \int_0^{\infty} f(x) \odot e^{-px}dx, \quad (p>0 \text{ and integer}).$$

Remark 1.3.11. Using the definition of fuzzy Riemann integration, if follows that

$$\mathbf{L}[f(x)] = \left(\mathcal{L}\left[\underline{f}(x,r)\right], \mathcal{L}\left[\overline{f}(x,r)\right]\right).$$

Theorem 1.3.12
Let f' be an integrable fuzzy valued function and f its primitive on $[0,\infty)$. Then

$$\mathbf{L}[f'(x)] = p \odot \mathbf{L}[f(x)] - f(0)$$

when f is (i)-differentiable
or

$$\mathbf{L}[f'(x)] = (-f(0)) \ominus (-p \odot \mathbf{L}[f(x)]).$$

when f is (ii)-differentiable.

Let us give the following theorem whose prove is essentially based on [12, Theorem 16]

Theorem 1.3.13
Suppose that $f \in C^{(n)\mathbb{F}}[0,\infty) \cap L^{\mathbb{F}}[0,\infty)$, one has the following:

(i) if f is ${}^{RL}[(i)-\beta]$-differentiable fuzzy valued function, then

$$\mathbf{L}\left[\left(D_0^\beta f\right)(x)\right] = p^\beta \mathbf{L}\left[f(t)\right] \ominus \left(\sum_{k=0}^{n-1} p^k D^{\beta-k-1} f\right)(0)$$

or

(ii) if f is ${}^{RL}[(i)-\beta]$-differentiable fuzzy valued function, then

$$\mathbf{L}\left[\left(D_0^\beta f\right)(x)\right] = -\left(\sum_{k=0}^{n-1} p^k D^{\beta-k-1} f\right)(0) \ominus \left(-p^\beta \mathbf{L}\left[f(t)\right]\right).$$

For more details and properties of the fuzzy Laplace transform, we kindly refer the reader to [14, 43, 80] and for an application to fuzzy differential equations see, e.g., [17], Chapter 4, and the references cited therein.

Chapter 2

Existence of positive solutions in a cone

> The value of a mathematical discipline is to be determined by its applicability to the empirical sciences
>
> C. Runge, Doctoral dissertation, Berlin, 23 April 1880.

Initial and boundary value problems of fractional orders have been studied extensively in the last decades and by different methods, for instance, fixed point theorems, upper and lower solution method, coincidence degree theory of Mawhin.

In this Chapter, we establish sufficient conditions for the existence of positive solutions for a class of higher order Riemann-Liouville fractional initial and boundary value problems. By means of Guo-Krasnoselskii fixed point theorem, we investigate the existence of at least one positive solution in suitable cones. The first part is dedicated to the study of IVP where the second one is devoted to BVP.

2.1 Previews works

It goes without saying that ordinary and fractional IVP and BVP are greatly treated in several thousands of papers, we only cite a few: the works of Zhang *et al.* [105–111], Bai [24], Goodrich [49], Sotiris *et al.* [77], Guezane-Lakoud *et al.* [51–55], Chidouh *et al.* [31–33], others [69, 70, 98], and many more details in [3, 8] and the references cited therein. We give an overview of the problems treated:

In [7], Agarwal *et al.* treated the following boundary value problem

$$\begin{aligned}&D^{\alpha}u + f(t, u(t), D^{\mu}u(t)) = 0, \quad 0 \leq t \leq 1, 1 < \alpha < 2,\\&u(0) = u(1) = 0,\end{aligned}$$

with $\mu \leq 1 - \alpha$ and f satisfies Carathéodory conditions.

In [70, 98], the authors used Schauder fixed-point theorem for a coupled system where the nonlinear terms involve a fractional derivative of order $0 < \sigma < 1$.

In [66], Lakshmikantham *et al.* obtained the existence uniqueness of solution to the following IVP, using Krasnoselskii-Krein type conditions

$$D_0^q u(t) = f(t, u(t)).$$

More details are in Chapter 3, when we treat a more general problem.

In [24], Bai considered the existence of positive solutions of the fractional boundary value problem:

$$\begin{aligned}&D_0^{\alpha}u(t) + f(t, u(t)) = 0, 0 < t < 1, 1 < \alpha \leq 2,\\&u(0) = 0, \beta u(\eta) = u(1),\end{aligned}$$

where D_0^{α} denotes the Riemann-Liouville fractional derivative.

In [49], Goodrich studied a similar problem

$$\begin{aligned}&-D_0^{\nu}u(t) = f(t, u(t)), 0 < t < 1,\\&u^{(i)}(0) = 0, [D^{\alpha}u(t)]_{t=1} = 0, 0 \leq i \leq n-2, 1 \leq \alpha \leq n-2, \nu > 3.\end{aligned}$$

The author established the existence of a positive solution using cone theoretic techniques.

In [77], Sotiris *et al.* extended his problem to a nonlinear fractional order differential equation with advanced arguments

$$D_0^{\alpha}u(t)+a(t)f(u(\theta(t)))=0, 0<t<1, n-1<\alpha\leq n,$$
$$u^{(i)}(0)=0, \left[D^{\beta}u(t)\right]_{t=1}=0, 0\leq i\leq n-2, 1\leq\beta\leq n-2, n>3.$$

In [52], Guezane-Lakoud *et al.* studied the following problem

$$^{c}D_0^{\alpha}u(t)=f(t,u(t),D^{\sigma}(t)), 0<t<1, 2<\alpha<3,$$
$$u(0)=u'(0)=u''(0)=0, u'(1)={}^{c}D^{\sigma}u(1),$$

where $^{c}D_0^{\alpha}$ represents the Caputo fractional derivative of order α. And recently in [51], the author established the existence and uniqueness for the following problem

$$D^{q}u(t)=f(t,u(t),u'(t)), 0\leq t\leq 1,$$
$$u(0)=u'(0)=u''(0)=0,$$

where $2<q<3$, using Guo-Krasnoselskii theorem.

Note that in the previously cited works and in many others, the authors trea-ted initial value problems assuming that the nonlinear term f depends on the classical derivatives of u, i.e. $D^{i}u$ for $i\in\mathbb{N}$, or on the fractional derivatives $D^{\sigma}u$, $\sigma\in]0,1[$, also they used some assumptions on the growth of a certain fractional derivative of the solution. In contrast to most of the anterior works, we prove the existence of at least one positive solution without imposing a growth assumption on the fractional derivative $D^{q-1}u$.

A few question arises about quantitative aspect of positive solution in the following cases:

- The nonlinear term depends not only on integer derivatives of the solution but also on its fractional order derivative

- The fractional derivative that the nonlinear term depends on is of high order

Motivated by the active investigations on the IVP and BVP. In the following, we prove some existence results for two problems and answer some of the above questions. For that, first, we define some quantities and recall some lemma.

2.2 Some definitions

We set

$$\begin{aligned} C_u^0 &= \left[I^q (1-t)^{-\beta} g(t) f(u(t), u'(t), u''(t), \dots, u^{(m)}(t), D^\alpha u(t)) \right]_{t=1}, \\ C_u^i &= \frac{\Gamma(q)}{\Gamma(q-i)} C_0, \, C_u^\alpha = \frac{\Gamma(q)}{\Gamma(q-\alpha)} C_0, \\ R_u(t) &= f(u(t), u'(t), u''(t), \dots, u^{(m)}(t), D^\alpha u(t)), \\ \gamma &= \min \left\{ \frac{\left(\frac{1}{2}\right)^{q-\beta-1}}{2^\beta - 1}, \left(\frac{1}{2}\right)^{q-1} \right\}. \end{aligned} \tag{2.1}$$

Definition 2.2.1 ([99])
Let $f : \mathbb{R}^2 \to \mathbb{R}$, define

$$F^* : \mathbb{R}_+ \to \mathbb{R}_+, \, F^*(r) = \max_{0 \le u+v \le r} \{f_1(u,v)\},$$

$$F_0 = \lim_{|u|+|v| \to 0^+} \frac{f_1(u,v)}{u+v}, \quad F_0^* = \lim_{r \to 0} \frac{F^*(r)}{r},$$

$$F_\infty = \lim_{|u|+|v| \to +\infty} \frac{f_1(u,v)}{u+v}, \quad F_0^* = \lim_{r \to \infty} \frac{F^*(r)}{r},$$

The case where $F_0 = \infty$ and $F_\infty = 0$ is called the sublinear case.

Lemma 2.2.2 ([99, Lemma 2.8])
If f_1 is continuous, then $F_0^* = F_0$ and $F_\infty^* = F_\infty$.

Proof. The proof runs along similar lines as in [99].
First, since f is continuous, we obtain that

$$\lim_{|u|+|v|\to 0} \frac{f(u,v)}{u+v} = \lim_{r\to 0} \frac{\max_{|u|+|v|\leq r} f(u,v)}{r} :$$

that is, $F_0^* = F_0$.
Next, to prove that $F_\infty^* = F_\infty$ we distinguish two case.
Case (i): f is bounded.
In this case from $\lim_{|u|+|v|\to\infty}(u+v)^{-1} = 0$, we conclude that $F_\infty = F_\infty^*$.
Case (ii): f is unbounded.
In one hand, Setting for any chosen $\delta > 0$

$$N_\rho = \inf\{|u|+|v| : |u|+|v| \geq \delta \text{ and } f(u,v) \geq F^*(\rho)\} \geq \rho,$$

and noting that

$$F^*(\rho) = \max\{f(u,v) : |u|+|v| \leq N_\rho\} = \max\{f(u,v) : |u|+|v| = N_\rho\},$$

the relation

$$F^*(r) = \max\{f(u,v) : |u|+|v| \leq r\}$$

is reduced to

$$F^*(r) = \max\{f(u,v) : N_\rho \leq |u|+|v| \leq r\},$$

for every $r > N_\rho$. In the other hand, there exists $(u_1,v_1) \in \mathbb{R}^2$ such that $F^*(r) = f(u_1,v_1)$, with $N_\rho \leq |u|+|v| \leq r$. Let us choose $(u_2,v_2) \in \mathbb{R}^2$ such that $|u_2|+|v_2| = r$, then

$$\frac{f(u_2,v_2)}{|u|+|v|} \leq \frac{F^*(r)}{r} = \frac{f(u_1,v_1)}{r} \leq \frac{f(u_1,v_1)}{|u_1|+|v_1|}.$$

Using this last estimate and the fact that f is continuous: that is, for any $\epsilon > 0$ there exists $\rho > 0$ such that for $|u|+|v| > \rho$ the inequality

$$F_\infty - \epsilon \leq \frac{f(u,v)}{u+v} \leq F_\infty + \epsilon$$

yield that
$$F_\infty - \epsilon \leq \frac{F^*(r)}{r} \leq F_\infty + \epsilon$$
for any $r > N_\rho$. Hence $F_\infty = F^*_\infty$. When $F_\infty = \infty$ a similar argument may show that $F^*_\infty = \infty$. $\square$

Definition 2.2.3
Let $f : \mathbb{R}^{m+2} \to \mathbb{R}$ and putting

$$A_0 = \lim_{\sum_{i=1}^{m+2}|v_i| \to 0^+} \frac{f(v_1, \dots, v_{m+2})}{\sum_{i=1}^{m+2}|v_i|}, \quad A_\infty = \lim_{\sum_{i=1}^{m+2}|v_i| \to +\infty} \frac{f(v_1, \dots, v_{m+2})}{\sum_{i=1}^{m+2}|v_i|},$$

The case where $A_0 = 0$ and $A_\infty = \infty$ is called the superlinear case and the case where $A_0 = \infty$ and $A_\infty = 0$ is called the sublinear case.

Lemma 2.2.4 ([98])
Let the space X be as follows

$$X = \{u |\, u \in C[0,1], D^\sigma u \in C[0,1]\}$$

endowed with the norm

$$\|u\|_X = \max_{0 \leq t \leq 1} |u|(t)| + \max_{0 \leq t \leq 1} |D^\sigma u(t)|,$$

where $0 < \sigma < 1$. Then $(X, \| \cdot \|_X)$ is a Banach space.

Definition 2.2.5
We define the space X_1 as follows

$$X_1 = \{u |\, u \in C^{n-2}[0,1]\ , D^{q-1}u \in C[0,1] \text{ and}$$
$$u(0) = 0,\ D^{(q-i)}u(0) = 0,\ i = 1, \dots, n-1\}, n > 2$$

endowed with the norm

$$\|u\|_{X_1} = \sum_{i=0}^{n-2} \left\{ \max_{0 \leq t \leq 1} |u^{(i)}(t)| \right\} + \max_{0 \leq t \leq 1} |D^{q-1}u(t)|,$$

where $[q] = n - 1$.

Lemma 2.2.6

$(X_1, \|\cdot\|_{X_1})$ *is a Banach space.*

Proof. Let $\{u_k\}_{k=1}^{\infty}$ be a Cauchy sequence in the space $(X_1, \|\cdot\|_{X_1})$, obviously $\{u_k^{(i)}\}_{k=1}^{\infty}$ and $\{D^{q-1}u_k\}_{k=1}^{\infty}$ are Cauchy sequences in the space $C([0,1])$. Therefore, $\{u_k^{(i)}\}_{k=1}^{\infty}$ and $\{D^{q-1}u_k\}_{k=1}^{\infty}$ converge to some $v^{(i)}$ and w in $C([0,1])$ uniformly, $v^{(i)}, w \in C([0,1])$ and $v(0) = 0$ and $w(0) = 0$. It suffices to show that $w = D^{q-1}v$. We have

$$\begin{aligned}|I^{q-1}D^{q-1}u_k(t) - I^{q-1}w(t)| \leq & \frac{1}{\Gamma(q-1)}\int_0^t (t-s)^{q-2}|D^{q-1}u_k(s) - w(s)|ds \\ \leq & \frac{1}{\Gamma(q)}\max_{t\in[0,1]}|D^{q-1}u_k(t) - w(t)|.\end{aligned}$$

Since $\{D^{q-1}u_k\}_{k=1}^{\infty}$ converges uniformly, then $\lim_{k\to\infty} I^{q-1}D^{q-1}u_k(t) = I^{q-1}w(t)$ uniformly on $[0,1]$. From Lemma 1.1.20, with $\alpha = q-1$, we get $I^{q-1}D^{q-1}u_k(t) = u_k(t)$. Hence, we have $v(t) = I^{q-1}w(t)$. From Lemma 1.1.13, we obtain $w = D^{q-1}v$. Thus, $\{u_k\}_{k=1}^{\infty}$ converges in X_1. This completes the proof. $\square$

Definition 2.2.7

We define the space X_2 as follows

$$\begin{aligned}X_2 = \{u|\, & u \in C^m[0,1], D^{\alpha}u \in C([0,1]) \textit{ and} \\ & u(0) = 0,\, u^{(i)}(0) = 0,\, i = 1,\ldots,m\Big\},\end{aligned}$$

endowed with the norm

$$\|u\|_{X_2} = \sum_{i=0}^{m}\left\{\max_{0\leq t\leq 1}|u^{(i)}(t)|\right\} + \max_{0\leq t\leq 1}|D^{\alpha}u(t)|,$$

where $u^{(0)} = u$ and $[\alpha] = m$.

Lemma 2.2.8

$(X_2, \|\cdot\|_{X_2})$ *is a Banach space.*

Proof. Let $\{u_k\}_{k=1}^{\infty}$ be a Cauchy sequence in the space $(X_2, \|\cdot\|_{X_2})$, obviously $\{u_k^{(i)}\}_{k=1}^{\infty}$ and $\{D^{\alpha}u_k\}_{k=1}^{\infty}$ are Cauchy sequences in the space $C([0,1])$.

Therefore, $\{u_k^{(i)}\}_{k=1}^{\infty}$ and $\{D^\alpha u_k\}_{k=1}^{\infty}$ converge to some $v^{(i)}$ and w in $C([0,1])$ uniformly, $v^{(i)}, w \in C([0,1])$ and $v(0)=0$ and $w(0)=0$. It suffices to show that $w = D^\alpha v$. We have

$$\begin{aligned}|I^\alpha D^\alpha u_k(t) - I^\alpha w(t)| \leq & \frac{1}{\Gamma(q-1)} \int_0^t (t-s)^{q-2} |D^\alpha u_k(s) - w(s)| ds \\ \leq & \frac{1}{\Gamma(q)} \max_{t\in[0,1]} |D^\alpha u_k(t) - w(t)|.\end{aligned}$$

Since $\{D^\alpha u_k\}_{k=1}^{\infty}$ converges uniformly, then $\lim_{k\to\infty} I^\alpha D^\alpha u_k(t) = I^\alpha w(t)$ uniformly on $[0,1]$. From Lemma 1.1.20, we get $I^\alpha D^\alpha u_k(t) = u_k(t)$. Hence, we have $v(t) = I^\alpha w(t)$. From Lemma 1.1.13, we obtain $w = D^\alpha v$. Thus, $\{u_k\}_{k=1}^{\infty}$ converges in X_2. This completes the proof. □

We define a cone $P_1 \subset X_1$ by

$$P_1 = \{u \in X_1 : u(t) \geq 0, \forall t \in [0,1]\}.$$

Also, we define the cone $P_2 \subset X_2$ by

$$P_2 = \left\{ u \in X_2 : u(t) > 0 \text{ and } \min_{t\in\left[\frac{1}{2},1\right]} u(t) \geq \frac{\gamma}{(m+2)\Gamma(q)} \|u\|_{X_2} \right\}.$$

2.3 Positive solutions to IVP

In this section, we investigate the existence of positive solutions of the following initial value problem with higher order Riemann-Liouville type fractional differential equation:

$$\begin{cases} D^q u(t) = f(t, u(t), D^{q-1}u(t)), q > 2, 0 \leq t \leq 1 \\ u(0) = 0, D^{(q-i)}u(0) = 0, i = 1, \ldots, n-1, \end{cases} \tag{2.2}$$

where $n = [q] + 1$.

We assume that f satisfies the following conditions:

(H1) $f(t,x,y) = g(t) f_1(x,y)$, $g \in L^1([0,1], \mathbb{R}_+^*)$ and $f_1 \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$.

(H2) f_1 is a convex, nonnegative, and continuous function, decreasing according to each of its variables.

(H3) $f_1(0,0) \neq 0$.

We seek a solution u of IVP (2.2), such that $u \in C([0,1],\mathbb{R})$ and $D^q u$ exists and is Lebesgue integrable on $[0,1]$ and u satisfies (2.2).

Define $T : X_1 \to X_1$ by

$$Tu(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,u(s),D^{q-1}u(s))ds,$$

where f satisfies (H1)-(H3).

Clearly from Lemma 2.2.6, a function u is a solution of IVP (2.2) if and only if it is a fixed point of T;

$$u(t) = Tu(t), \quad \forall t \in [0,1].$$

We first introduce the following Lemma which is crucial in the proof of the main results.

Lemma 2.3.1

Assume conditions (H1)-(H3) *are satisfied, then the operator $T : P_1 \to P_1$ is completely continuous.*

Proof. Step 1: Firstly, from conditions (H1)-(H3), and noting that I^q maps $L^1([0,1],\mathbb{R})$ into $AC^{n-1}([0,1],\mathbb{R})$, for every $q > 1$, with $n = [q]+1$ it yields that for every $u \in X_1$, $Tu \in C^{n-2}([0,1],\mathbb{R})$ and $D^{q-1}(Tu) \in C([0,1],\mathbb{R})$.

We have for every $i \in 1,\ldots,n-1$

$$D^{(q-i)}(Tu)(t) = \frac{1}{\Gamma(q-i)} \int_0^t (t-s)^{q-i-1} g(s) f_1(u(s),D^{q-1}u(s))ds. \quad (2.3)$$

Since g and f_1 are bounded at 0, we get from the previous equation (2.3) and the definition of T that $Tu(0) = D^{q-i}(Tu)(0) = 0$, for every $i \in \{1,\ldots,n-1\}$.

Next, if $u(t) \geq 0$, then $D^{q-1}u(t) \geq 0$. Moreover, it follows from conditions

(H1)-(H3) that $Tu(t) \geq 0, \forall t \in [0,1]$. Finally, we conclude that $T(P_1) \subset P_1$.
Step 2: we show that T maps bounded sets into bounded sets in P_1.
In fact, set B be a bounded set in P_1, then for $u \in B$ we have

$$|Tu(t)| = \frac{1}{\Gamma(q)} \left| \int_0^t (t-s)^{q-1} g(s) f_1(u(s), D^{q-1}u(s)) ds \right|.$$

And from condition (H1), $g \in L^1([0,1], \mathbb{R}_+^*)$ nonnegative, then there exist constants $g_1, g_2 > 0$, such that $g_1 \leq \int_0^t g(t) \leq g_2$, for all $t \in [0,1]$. Hence, we obtain

$$|Tu(t)| \leq \frac{g_2 f_1(0,0)}{\Gamma(q)}.$$

and for each integer i such that $0 \leq i \leq n-2$, we have

$$\begin{aligned}
|(Tu)^{(i)}(t)| =& \frac{1}{\Gamma(q-i)} \left| \int_0^t (t-s)^{q-1-i} g(s) f_1(u(s), D^{q-1}u(s)) ds \right| \\
\leq& \frac{1}{\Gamma(q-i)} f_1(0,0) \left| \int_0^t g(s) ds \right| \\
\leq& \frac{g_2 f_1(0,0)}{\Gamma(q-i)},
\end{aligned}$$

and

$$\begin{aligned}
|D^{q-1}Tu(t)| =& \left| \int_0^t g(s) f_1(u(s), D^{q-1}u(s)) ds \right| \\
\leq& g_2 f_1(0,0).
\end{aligned}$$

So, $\|Tu\|_{X_1} \leq g_2 f_1(0,0) \left(1 + \sum_{i=0}^{n-2} \frac{1}{\Gamma(q-i)}\right)$. Hence, $T(B)$ is uniformly bounded.
Step 3: Now, we prove that the operator T is equicontinuous.
For each $u \in B$, any $\epsilon > 0$, $t_1, t_2 \in [0,1]$, $t_2 > t_1$, we have

$$\begin{aligned}
|Tu(t_1) - Tu(t_2)| =& \frac{1}{\Gamma(q)} \left| \int_0^{t_1} (t_1-s)^{q-1} g(s) f_1(u(s), D^{q-1}u(s)) ds \right. \\
& \left. - \int_0^{t_2} (t_2-s)^{q-1} g(s) f_1(u(s), D^{q-1}u(s)) ds \right| \\
\leq& \frac{1}{\Gamma(q)} f_1(0,0) \left(\int_0^{t_1} g(s) \left((t_2-s)^{q-1} - (t_1-s)^{q-1} \right) ds \right.
\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_1}^{t_2} g(s)(t_2-s)^{q-1}ds\Big) \\
\leq &\frac{f_1(0,0)}{\Gamma(q)}\left(\int_0^{t_1}(q-1)(t_2-t_1)g(s)ds+\int_{t_1}^{t_2}(t_2-s)^{q-1}g(s)ds\right) \\
\leq &\frac{g_2 f_1(0,0)}{\Gamma(q-1)}(t_2-t_1)+\frac{f_1(0,0)}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}g(s)ds \to 0, \text{ as } t_1 \to t_2,
\end{aligned}
$$

and for each integer i such that $1 \leq i \leq n-2$, we have

$$
\begin{aligned}
|(Tu)^{(i)}(t_1)-(Tu)^{(i)}(t_2)| =&\frac{1}{\Gamma(q-i)}\left|\int_0^{t_1}(t_1-s)^{q-1-i}g(s)f_1(u(s),D^{q-1}u(s))ds\right. \\
&\left.-\int_0^{t_2}(t_2-s)^{q-1-i}g(s)f_1(u(s),D^{q-1}u(s))ds\right| \\
\leq&\frac{1}{\Gamma(q-i)}f_1(0,0)\left(\int_0^{t_1}g(s)\left((t_2-s)^{q-1-i}\right.\right. \\
&\left.\left.-(t_1-s)^{q-1-i}\right)ds+\int_{t_1}^{t_2}g(s)(t_2-s)^{q-1-i}ds\right) \\
\leq&\frac{g_2 f_1(0,0)}{\Gamma(q-1-i)}(t_2-t_1)+\frac{f_1(0,0)}{\Gamma(q-i)}\int_{t_1}^{t_2}(t_2-s)^{q-1}g(s)ds
\end{aligned}
$$

which tends to 0 as t_1 tends to t_2.

$$
\begin{aligned}
&|D^{q-1}Tu(t_1)-D^{q-1}Tu(t_2)| \\
=&\left|\int_0^{t_1}g(s)f_1(u(s),D^{q-1}u(s))ds-\int_0^{t_2}g(s)f_1(u(s),D^{q-1}u(s))ds\right| \\
=&\int_{t_1}^{t_2}g(s)f_1(u(s),D^{q-1}u(s))ds \to 0, \text{as } t_1 \to t_2
\end{aligned}
$$

Therefore, $T(B)$ is equicontinuous.
From Arzelà-Ascoli Theorem, we deduce that the operator T is compact.

□

2.3.1 Existence result

Now, we establish an existence result for positive solutions to problem (2.2). The results of this section are submitted for publication, see [96].

Theorem 2.3.2

Suppose that f satisfies (H1)-(H3)*, then problem* (2.2) *has at least one non trivial positive solution in the cone P_1 in the sublinear case.*

Proof. From Lemma 2.3.1, T is completely continuous. Since $F_0 = +\infty$, choosing $M_1 > \frac{\Gamma(q+1)}{\int_0^1 (1-s)^q g(s)ds}$, there exists $r_1 > 0$, for any $0 < u + v \leq r_1$, then $f_1(u,v) \geq M(u+v)$. Let $\Omega_1 = \{u \in E, \|u\|_{X_1} < r_1\}$. Assume that $\exists u_1 \in P_1 \cap \partial\Omega_1$, such that $\|Tu_1\|_{X_1} < \|u_1\|_{X_1}$, then we can write

$$\begin{aligned}
r_1 =& \|u_1\|_{X_1} > \|Tu_1\|_{X_1} \geq \|Tu_1\|_\infty \geq \int_0^1 Tu_1(t)dt \\
\geq& \frac{1}{\Gamma(q)} \int_0^1 \int_0^1 (t-s)^{q-1} g(s) f_1(u_1(s), D^{q-1}u_1(s))dsdt \\
\geq& \frac{1}{\Gamma(q)} \int_0^1 \left(\int_s^1 (t-s)^{q-1} dt \right) g(s) f_1(u_1(s), D^{q-1}u_1(s))ds \\
\geq& \frac{1}{\Gamma(q+1)} \int_0^1 (1-s)^q g(s) f_1(u_1, D^{q-1}u_1)ds.
\end{aligned}$$

From the convexity of f_1 and using Lemma 1.2.22, we get

$$\begin{aligned}
r_1 \geq& \frac{1}{\Gamma(q+1)} \left(\int_0^1 (1-s)^q g(s)ds \right) \\
&\times f_1 \left(\frac{\int_0^1 (1-s)^q g(s) u_1(s)ds}{\int_0^1 (1-s)^q g(s)ds}, \frac{\int_0^1 (1-s)^q g(s) D^{q-1}u_1(s)ds}{\int_0^1 (1-s)^q g(s)ds} \right). \qquad (2.4)
\end{aligned}$$

Using the estimations

$$\begin{aligned}
\int_0^1 (1-s)^q g(s) u_1(s)ds &\leq \|u_1\|_\infty \int_0^1 (1-s)^q g(s)ds \\
\int_0^1 (1-s)^q g(s) D^{q-1}u_1(s)ds &\leq \|D^{q-1}u_1\|_\infty \int_0^1 (1-s)^q g(s)ds,
\end{aligned}$$

and taking (H2) into account and the fact that f_1 is decreasing according to each of its variables, inequality (2.4) becomes

$$r_1 \geq \frac{\int_0^1 (1-s)^q g(s)ds}{\Gamma(q+1)} f_1(\|u_1\|_\infty, \|D^{q-1}u_1\|_\infty).$$

Using (H3), we obtain

$$\begin{aligned} r_1 \geq & \frac{\int_0^1 (1-s)^q g(s) ds}{\Gamma(q+1)} f_1 \left(\sum_{i \leq n-2} \|u_1^{(i)}\|_\infty, \|D^{q-1} u_1\|_\infty \right) \\ \geq & \frac{\int_0^1 (1-s)^q g(s) ds}{\Gamma(q+1)} M \left(\sum_{i \leq n-2} \|u_1^{(i)}\|_\infty + \|D^{q-1} u_1\|_\infty \right) = \frac{g_1}{\Gamma(q+1)} M r_1. \end{aligned}$$

Using the fact that $M > \frac{\Gamma(q+1)}{\int_0^1 (1-s)^q g(s) ds}$, then it follows that

$$r_1 \geq \frac{\int_0^1 (1-s)^q g(s) ds}{\Gamma(q+1)} M r_1 > r_1,$$

which is a contradiction. Thus, $\|Tu\|_{X_1} \geq \|u\|_{X_1}, \forall u \in P_1 \cap \partial\Omega_1$.
From Lemma 2.2.2, we have $F_\infty = 0 \Leftrightarrow F_\infty^* = 0$, so for every $\epsilon > 0$ verifying

$$\epsilon < \left(g_2 \left(1 + \sum_{i=0}^{n-2} \frac{1}{\Gamma(q-i)} \right) \right)^{-1},$$

there exists $R > 0$ such that for $r > R$, then $F^*(r) \leq \epsilon r$. Let $r_2 > \max(r_1, R)$ and set $\Omega_2 = \{u \in E : \|u\|_{X_1} < r_2\}$, it is easy to see that $\overline{\Omega_1} \subset \Omega_2$.
Assume that $\exists u_2 \in P_1 \cap \partial\Omega_2$, such that $\|Tu_2\|_{X_1} > \|u_2\|_{X_1}$, then we have

$$\begin{aligned} r_2 = & \|u_2\|_{X_1} < \|Tu_2\|_{X_1} \\ = & \max_{t \in [0,1]} \sum_{i=0}^{n-2} \frac{1}{\Gamma(q-i)} \int_0^t (t-s)^{q-i-1} g(s) f_1(u_2(s), D^{q-1} u_2(s)) ds \\ & + \max_{t \in [0,1]} \int_0^t g(s) f_1(u(s), D^{q-1} u(s)) ds \\ \leq & F^*(r_2) g_2 \left(1 + \sum_{i=0}^{n-2} \frac{1}{\Gamma(q-i)} \right) \leq \epsilon r_2 g_2 \left(1 + \sum_{i=0}^{n-2} \frac{1}{\Gamma(q-i)} \right) < r_2 \end{aligned}$$

which is a contradiction. So $\|Tu\|_{X_1} \leq \|u\|_{X_1}, \forall u \in P_1 \cap \partial\Omega_2$.
It's clear that the assumptions of Theorem 1.2.12 are satisfied, it follows that problem (2.2) has at least one positive solution in $P_1 \cap \left(\overline{\Omega_2} \backslash \Omega_1\right)$. □

2.3.2 An example

In this example we illustrate the usefulness of our result. Let us consider the following IVP

$$D^{\alpha}u(t) = \begin{cases} \dfrac{1+t^2}{1+|u(t)D^{\alpha-1}u(t)|^2}, & t \in [0,1]\cap\mathbb{Q}, \qquad 3<\alpha<4, \\ \\ \dfrac{1}{1+u^4(t)(D^{\alpha-1}u(t))^2}, & t \in [0,1]\backslash\mathbb{Q}, \end{cases} \tag{2.5}$$

$$u(0) = 0, D^{\alpha-1}u(0) = 0, D^{\alpha-2}u(0) = 0, D^{\alpha-3}u(0) = 0. \tag{2.6}$$

Set $f(t,x,y) = g(t)f_1(x,y)$, where

$$g(t) = \begin{cases} 1+t^2, & t \in [0,1]\cap\mathbb{Q} \\ 1, & t \in [0,1]\backslash\mathbb{Q} \end{cases}$$

and

$$f_1(x,y) = \frac{1}{1+x^4y^2}.$$

Thus $g \in L^1((0,1),\mathbb{R}_+)$, with $g_1 = 1$ and $g_2 = 2$ and clearly f is a continuous decreasing nonnegative convex function and $f(0,0) = 1$.

Hence, conditions (H1)-(H3) hold. Then, by Theorem 2.3.2, problem (2.5) has at least one continuous positive solution on $[0,1]$.

2.4 Positive solutions to BVP

In this section, we consider the following BVP

$$\begin{cases} D^q u(t) + g(t)f(u(t),u'(t),u''(t),\ldots,u^{(m)}(t),D^{\alpha}u(t)) = 0, 0 \le t \le 1, \\ u(0) = 0,\, u^{(i)}(0) = 0,\, i = 1,\ldots,n-2, \\ D^{\beta}u(1) = 0,\, 2 \le \beta \le n-2,\, 1 \le m \le \alpha \le \beta - 1, \end{cases} \tag{2.7}$$

where $n-1 < q \le n$ and $n > 3$ and the functions g and f satisfy the conditions:

(H1) $g \in L^1([0,1],\mathbb{R}_+^*)$,

(H2) $f \in C(\mathbb{R}_+^{m+2},\mathbb{R}_+)$.

Note that the nonlinear term doesn't depend solely on the unknown solution but on its fractional derivatives of high orders too.

Firstly, we convert the problem into an equivalent integral equation, followed by some estimating on te Green function and finally we apply Guo-Krasnoselskii's fixed-point theorem in cones to establish some new results. The results of this section appeared in [95].

2.4.1 New estimates on Green function

In [49], Goodrich proved the following Lemma and obtained estimates on the minimum of the Green function.

Lemma 2.4.1
Assume that $y \in C([0,1])$, then the following problem

$$\begin{cases} D^{\nu}u(t) + y(t) = 0, n-1 < \nu \leq n, \\ u(0) = 0, u^{(i)}(0) = 0, i = 1,\ldots,n-2, \\ D^{\beta}u(1) = 0, 1 \leq \beta \leq n-2, \end{cases}$$

has the unique solution

$$u(t) = \int_0^1 G(t,s)y(s)ds,$$

where

$$G(t,s) = \begin{cases} \dfrac{t^{q-1}(1-s)^{\nu-\beta-1} - (t-s)^{\nu-1}}{\Gamma(\nu)}, 0 \leq s \leq t \leq 1, \\ \dfrac{t^{q-1}(1-s)^{\nu-\beta-1}}{\Gamma(\nu)}, 0 \leq t \leq s \leq 1. \end{cases} \tag{2.8}$$

He proved, under the assumption that $\beta \leq \nu - 1$ the following estimate:

Lemma 2.4.2 ([49])
There exists a constant $\gamma \in (0,1)$ *such that*

$$\min_{t\in\left[\frac{1}{2},1\right]} G(t,s) \geq \gamma \max_{t\in[0,1]} G(t,s) = \gamma G(1,s),$$

where

$$\gamma = \min\left\{ \frac{\left(\frac{1}{2}\right)^{\nu-\beta-1}}{2^{\beta}-1}, \left(\frac{1}{2}\right)^{\nu-1} \right\}.$$

By substituting β and q by $\beta_\alpha = \beta - \alpha$ and $q_\alpha = q - \alpha$, then $\beta_i = \beta - i$, $q_i = q - i$, where $i \in \{1,\ldots,m\}$, we obtain the following estimates:

Lemma 2.4.3
Let $G(t,s)$ *be as given in* (2.8). *Then for every* $i = 1,\ldots,m$ *there exist constants* $\gamma, \gamma_i, \gamma_\alpha$ *given by*

$$\gamma = \min\left\{ \frac{\left(\frac{1}{2}\right)^{q-\beta-1}}{2^{\beta}-1}, \left(\frac{1}{2}\right)^{q-1} \right\}$$

$$\gamma_i = \min\left\{ \frac{\left(\frac{1}{2}\right)^{q-\beta-1}}{2^{\beta-i}-1}, \left(\frac{1}{2}\right)^{q-1-i} \right\}$$

$$\gamma_\alpha = \min\left\{ \frac{\left(\frac{1}{2}\right)^{q-\beta-1}}{2^{\beta-\alpha}-1}, \left(\frac{1}{2}\right)^{q-1-\alpha} \right\},$$

such that

$$\min_{t\in\left[\frac{1}{2},1\right]} G(t,s) \geq \gamma \max_{t\in[0,1]} G(t,s) = \gamma G(1,s)$$

$$\min_{t\in\left[\frac{1}{2},1\right]} G_i(t,s) \geq \gamma_i \max_{t\in[0,1]} G_i(t,s) = \gamma_i G_i(1,s)$$

$$\min_{t\in\left[\frac{1}{2},1\right]} G_\alpha(t,s) \geq \gamma_\alpha \max_{t\in[0,1]} G_\alpha(t,s) = \gamma_\alpha G_\alpha(1,s),$$

where

$$G_i(t,s) = \frac{d^i}{dt^i}G(t,s) = \begin{cases} \dfrac{t^{q-1-i}(1-s)^{q-\beta-1} - (t-s)^{q-1-i}}{\Gamma(q-i)}, 0 \le s \le t \le 1, \\ \dfrac{t^{q-1-i}(1-s)^{q-\beta-1}}{\Gamma(q-i)}, 0 \le t \le s \le 1, \end{cases}$$

$$G_\alpha(t,s) = D^\alpha G(t,s) = \begin{cases} \dfrac{t^{q-1-\alpha}(1-s)^{q-\beta-1} - (t-s)^{q-1-\alpha}}{\Gamma(q-\alpha)}, 0 \le s \le t \le 1, \\ \dfrac{t^{q-1-\alpha}(1-s)^{q-\beta-1}}{\Gamma(q-\alpha)}, 0 \le t \le s \le 1. \end{cases}$$

Moreover,

$$\min_{t\in\left[\frac{1}{2},1\right]} G(t,s) \ge \gamma G(1,s) \ge \frac{\gamma}{(m+2)\Gamma(q)}\left(G(1,s) + G_\alpha(1,s) + \sum_{i=1}^{m} G_i(1,s)\right). \tag{2.9}$$

Proof. Using the same process as in [49], we get for every $i = 1,\dots,m$ the desired constants γ, γ_i and γ_α. Moreover, using the fact that $1 \le i \le m \le \alpha \le \beta - 1$, it follows for every $0 \le s \le 1$ that

$$G(1,s) \ge \frac{\Gamma(q-i)}{\Gamma(q)}G_i(1,s) \ge \frac{1}{\Gamma(q)}G_i(1,s)$$
$$G(1,s) \ge \frac{\Gamma(q-\alpha)}{\Gamma(q)}G_\alpha(1,s) \ge \frac{1}{\Gamma(q)}G_\alpha(1,s).$$

Noting that $\Gamma(.)$ is increasing in $(2,\infty)$, we obtain the desired inequality (2.9). □

We define $T : X_2 \to X_2$ by

$$Tu(t) = \int_0^1 G(t,s)g(s)f(u(s),u'(s),u''(s),\dots,u^{(m)}(s),D^\alpha u(s))ds,$$

where g and f satisfy (H1)-(H2).
Clearly from Lemma 2.4.1, a function u is a solution of BVP (2.7) if and only if it is a fixed point of T,

$$u(t) = Tu(t), \quad \forall t \in [0,1].$$

2.4.2 Existence result

In this section, we deduce the existence of a positive solution to problem (2.7). To accomplish this we first introduce the following Lemma which is crucial to the proof of our result based on the Guo-Krasnoselskii fixed point theorem.

Lemma 2.4.4
Assume that conditions (H1)-(H2) *are satisfied, then the operator* $T : P_2 \to P_2$ *is completely continuous.*

Proof. Firstly, it follows from (H1)-(H2) and (2.1) that $Tu(t) \geq 0$, $\forall t \in [0,1]$, and

$$\begin{aligned}
\min_{t\in\left[\frac{1}{2},1\right]} Tu(t) \geq & \frac{\gamma}{(m+2)\Gamma(q)} \int_0^1 \left(G(1,s) + G_\alpha(1,s) + \sum_{i=1}^{m} G_i(1,s) \right) g(s)R_u(s)ds \\
\geq & \frac{\gamma}{(m+2)\Gamma(q)} \max_{t\in[0,1]} \int_0^1 G(t,s)g(s)R_u(s)ds \\
& + \frac{\gamma}{(m+2)\Gamma(q)} \max_{t\in[0,1]} \int_0^1 G_\alpha(t,s)g(s)R_u(s)ds \\
& + \frac{\gamma}{(m+2)\Gamma(q)} \max_{t\in[0,1]} \sum_{i=1}^{m} \int_0^1 G_i(t,s)g(s)R_u(s)ds \\
\geq & \frac{\gamma}{(m+2)\Gamma(q)} \max_{t\in[0,1]} \left(Tu(t) + \sum_{i=1}^{m} (Tu)^{(i)}(t) + D^\alpha Tu(t) \right).
\end{aligned}$$

Thus

$$\min_{t\in\left[\frac{1}{2},1\right]} Tu(t) \geq \frac{\gamma}{(m+2)\Gamma(q)} \|Tu\|_{X_2},$$

which implies that $T : P_2 \to P_2$.
Secondly, we show that T maps bounded sets into bounded sets in P_2.
In fact, for every bounded set B in P_2 there exists

$$\delta = g_2 M_f \left(1 + \sum_{i=0}^{m} \frac{1}{\Gamma(q-i)} \right) + C_\alpha + \sum_{i=0}^{m} C_i > 0,$$

such that

$$
\begin{aligned}
|Tu(t)| =& \left| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s) R_u(s) ds + t^{q-1} C_0 \right| \\
\leq & \frac{1}{\Gamma(q)} M_f \left| \int_0^t g(s) ds \right| + t^{q-1} C_0 \\
\leq & \frac{g_2 M_f}{\Gamma(q)} + C_0,
\end{aligned}
$$

and for each $1 \leq i \leq m$, we have

$$
\begin{aligned}
|(Tu)^{(i)}(t)| =& \left| \frac{1}{\Gamma(q-i)} \int_0^t (t-s)^{q-1-i} g(s) R_u(s) ds + C_i t^{q-1-i} \right| \\
\leq & \frac{1}{\Gamma(q-i)} M_f \left| \int_0^t g(s) ds \right| + C_i t^{q-1-i} \\
\leq & \frac{g_2 M_f}{\Gamma(q-i)} + C_i,
\end{aligned}
$$

and

$$
\begin{aligned}
|D^\alpha Tu(t)| =& \left| \int_0^t g(s) R_u(s) ds + C_\alpha t^{q-1-\alpha} \right| \\
\leq & g_2 M_f + C_\alpha.
\end{aligned}
$$

So, $\|Tu\|_{X_2} \leq \delta$. Hence, $T(B)$ is bounded.
Now, we prove that the operator T is equicontinuous.
For each $u \in B$, any $\epsilon > 0$, $t_1, t_2 \in [0,1]$, $t_2 > t_1$. Let

$$
\delta = \left(\delta_\alpha + \sum_{i=0}^{m} \delta_i \right),
$$

where

$$
\begin{aligned}
\delta_\alpha = \min \Bigg\{ & \frac{1}{2} \left(\frac{\epsilon_m}{C_\alpha} \right)^{1/(q-1-\alpha)}, \left(\frac{\epsilon_m}{C_\alpha (q-1-\alpha)} \right)^{1/(q-1-\alpha)}, \\
& \frac{\Gamma(q-1-\alpha)\epsilon_m}{g_2 M_f}, \left(\frac{\Gamma(q-\alpha)\epsilon_m}{g_2 M_f} \right)^{1/(q-1-\alpha)} \Bigg\}
\end{aligned}
$$

and for every $i = 0, \ldots, m$, let

$$\delta_i = \min\left\{\frac{1}{2}\left(\frac{\epsilon_m}{C_i}\right)^{1/(q-1-i)}, \left(\frac{\epsilon_m}{C_i(q-1-i)}\right)^{1/(q-1-i)}, \right.$$
$$\left.\frac{\Gamma(q-1-i)\epsilon_m}{g_2 M_f}, \left(\frac{\Gamma(q-i)\epsilon_m}{g_2 M_f}\right)^{1/(q-1-i)}\right\},$$

and

$$\epsilon_m = \frac{1}{2(m+2)}\epsilon,$$

then when $|t_2 - t_1| < \delta$, we get

$$\begin{aligned}
|Tu(t_1) - Tu(t_2)| = & \left|\frac{1}{\Gamma(q)}\left(\int_0^{t_1}(t_1-s)^{q-1}g(s)R_u(s)ds\right.\right. \\
& \left.\left. - \int_0^{t_2}(t_2-s)^{q-1}g(s)R_u(s)ds\right) + C_0(t_1^{q-1} - t_2^{q-1})\right| \\
\leq & \frac{1}{\Gamma(q)}M_f\left(\int_0^{t_1} g(s)\left((t_1-s)^{q-1} - (t_2-s)^{q-1}\right)ds\right. \\
& \left. + \int_{t_1}^{t_2} g(s)(t_2-s)^{q-1}ds\right) + |C_0(t_1^{q-1} - t_2^{q-1})| \\
\leq & \frac{M_f}{\Gamma(q)}\left(\int_0^{t_1}(q-1)(t_2-t_1)g(s)ds + \int_{t_1}^{t_2} g(s)(t_2-s)^{q-1}ds\right) \\
& + C_0(t_2^{q-1} - t_1^{q-1}) \\
\leq & \frac{g_2 M_f}{\Gamma(q-1)}(t_2-t_1) + \frac{M_f}{\Gamma(q)}\int_{t_1}^{t_2} g(s)(t_2-s)^{q-1}ds \\
& + C_0(t_2^{q-1} - t_1^{q-1}),
\end{aligned} \tag{2.10}$$

and for each $i \leq m$ we have

$$\begin{aligned}
|(Tu)^{(i)}(t_1) - (Tu)^{(i)}(t_2)| = & \left|\frac{1}{\Gamma(q-i)}\left(\int_0^{t_1}(t_1-s)^{q-1-i}g(s)R_u(s)ds\right.\right. \\
& \left.\left. - \int_0^{t_2}(t_2-s)^{q-1-i}g(s)R_u(s)ds\right) + C_i(t_1^{q-1-i} - t_2^{q-1-i})\right| \\
\leq & \frac{1}{\Gamma(q-i)}M_f\left(\int_0^{t_1} g(s)\left((t_1-s)^{q-1-i} - (t_2-s)^{q-1-i}\right)ds\right. \\
& \left. + \int_{t_1}^{t_2} g(s)(t_2-s)^{q-1-i}ds\right) + |C_i(t_1^{q-1-i} - t_2^{q-1-i})|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{M_f}{\Gamma(q-i)} \left(\int_0^{t_1} (q-1-i)(t_2-t_1) g(s) ds \right. \\
&\quad \left. + \int_{t_1}^{t_2} g(s)(t_2-s)^{q-i-1} ds \right) + C_i (t_2^{q-i-1} - t_1^{q-i-1}) \\
&\leq \frac{g_2 M_f}{\Gamma(q-i-1)} (t_2 - t_1) + \frac{M_f}{\Gamma(q-i)} \int_{t_1}^{t_2} g(s)(t_2-s)^{q-i-1} ds \\
&\quad + C_i (t_2^{q-i-1} - t_1^{q-i-1}),
\end{aligned} \tag{2.11}
$$

and

$$
\begin{aligned}
|D^\alpha Tu(t_1) - D^\alpha Tu(t_2)| = &\left| \frac{1}{\Gamma(q-\alpha)} \left(\int_0^{t_1} (t_1-s)^{q-1-\alpha} g(s) R_u(s) ds \right. \right. \\
&\left. \left. - \int_0^{t_2} (t_2-s)^{q-1-\alpha} g(s) R_u(s) ds \right) + C_i (t_1^{q-1-\alpha} - t_2^{q-1-\alpha}) \right| \\
\leq &\frac{1}{\Gamma(q-\alpha)} M_f \left(\int_0^{t_1} g(s) \left((t_1-s)^{q-1-\alpha} - (t_2-s)^{q-1-\alpha} \right) ds \right. \\
&\left. + \int_{t_1}^{t_2} g(s)(t_2-s)^{q-1-\alpha} ds \right) + |C_\alpha (t_1^{q-1-\alpha} - t_2^{q-1-\alpha})| \\
\leq &\frac{M_f}{\Gamma(q-\alpha)} \left(\int_0^{t_1} (q-1-\alpha)(t_2-t_1) g(s) ds \right. \\
&\left. + \int_{t_1}^{t_2} g(s)(t_2-s)^{q-\alpha-1} ds \right) + C_\alpha (t_2^{q-\alpha-1} - t_1^{q-\alpha-1}) \\
\leq &\frac{g_2 M_f}{\Gamma(q-\alpha-1)} (t_2 - t_1) + \frac{M_f}{\Gamma(q-\alpha)} \int_{t_1}^{t_2} g(s)(t_2-s)^{q-\alpha-1} ds \\
&+ C_\alpha (t_2^{q-\alpha-1} - t_1^{q-\alpha-1}).
\end{aligned} \tag{2.12}
$$

In particular, for every $0 \leq i \leq m$ we divide the estimates $C_i(t_2^{q-1-i} - t_1^{q-1-i})$ in two cases, as in [25].

Case 1. $0 \leq t_1 < \delta_i$, $t_2 < 2\delta_i$:

$$
C_i(t_2^{q-1-i} - t_1^{q-1-i}) \leq C_i (2\delta_i)^{q-1-i} \leq \epsilon_m.
$$

Case 2. $\delta_i \leq t_1 < t_2 < 1$:

$$
C_i(t_2^{q-1-i} - t_1^{q-1-i}) \leq C_i \frac{(q-1-i)}{\delta_i^{2-(q-1-i)}} (t_2 - t_1) \leq C_i (q-1-i) \delta_i^{q-1-i} \leq \epsilon_m.
$$

Doing the same for $C_\alpha(t_2^{q-1-\alpha} - t_1^{q-1-\alpha})$ and then substituting these estimates in (2.10), (2.11) and (2.12), we obtain that $T(B)$ is equicontinuous.

From Arzelà-Ascoli Theorem, we deduce that the operator T is compact.

$\square$

Now, we establish an existence result of positive solutions to problem (2.7).

Theorem 2.4.5
Suppose that f satisfies (H1)-(H2), *then problem* (2.7) *has at least one non trivial non-negative solution in the cone P_2 in either the sublinear case or the superlinear case.*

Proof. From Lemma 2.4.4, T is completely continuous.
(i) Sublinear case: Since $A_0 = +\infty$, there exists $r_1 > 0$, for any $0 < \sum_{i=1}^{m+2} |v_i| \leq r_1$, then $f(v_1, \ldots, v_{m+2}) \geq M(\sum_{i=1}^{m+2} |v_i|)$, where $M > 0$ satisfies

$$M \frac{\gamma}{(m+2)\Gamma(q)} \int_{\frac{1}{2}}^{1} G\left(\frac{1}{2}, s\right) g(s) ds \geq 1.$$

Let $\Omega_1 = \{u \in E, \|u\|_{X_2} < r_1\}$. Take $u \in P_2 \cap \partial\Omega_1$, then

$$\begin{aligned}
\|Tu\|_{X_2} \geq & \|Tu\|_\infty \geq Tu\left(\frac{1}{2}\right) = \int_0^1 G\left(\frac{1}{2}, 1\right) g(s) R_u(s) ds \\
= & \int_0^{\frac{1}{2}} G\left(\frac{1}{2}, s\right) g(s) R_u(s) ds + \int_{\frac{1}{2}}^1 G\left(\frac{1}{2}, s\right) g(s) R_u(s) ds \\
\geq & \int_{\frac{1}{2}}^1 G\left(\frac{1}{2}, s\right) g(s) f(u(s), u'(s), u''(s), \ldots, u^{(m)}(s), D^\alpha u(s)) ds \\
\geq & \int_{\frac{1}{2}}^1 G\left(\frac{1}{2}, s\right) g(s) M \left(|u(s)| + |u'(s)| + |u''(s)|\right. \\
& \left. + \cdots + \left|u^{(m)}(s)\right| + |D^\alpha u(s)|\right) ds \\
\geq & \int_{\frac{1}{2}}^1 G\left(\frac{1}{2}, s\right) g(s) M |u(s)| ds \\
\geq & \|u\|_{X_2} M \frac{\gamma}{(m+2)\Gamma(q)} \int_{\frac{1}{2}}^1 G\left(\frac{1}{2}, s\right) g(s) ds \geq \|u\|_{X_2}.
\end{aligned}$$

Thus, $\|Tu\|_{X_2} \geq \|u\|_{X_2}, \forall u \in P_2 \cap \partial\Omega_1$.
Next, since $A_\infty = 0$, then there exists R such that $f(v_1, \ldots, v_{m+2}) \leq \epsilon \sum_{i=1}^{m+2} |v_i|$

for $\sum_{i=1}^{m+2} |v_i| > r$, where ϵ satisfies

$$\epsilon(m+2)\Gamma(q)\int_0^1 G(1,s)g(s)ds \leq 1.$$

Let $\Omega_2 = \{u \in E : \|u\|_{X_2} < r_2, r_2 > \max\{r_1, R\}\}$, it is easy to see that $\overline{\Omega_1} \subset \Omega_2$.

Now we may choose $u \in P_2 \cap \partial\Omega_2$, and obtain

$$\begin{aligned}
\|Tu\|_{X_2} = \max_{t\in[0,1]} &\left(\int_0^1 G(t,s)g(s)R_u(s)ds + \sum_{i=1}^{m} \int_0^1 G_i(t,s)g(s)R_u(s)ds \right. \\
&\left. + \int_0^1 G_\alpha(t,s)g(s)R_u(s)ds \right) \\
\leq &(m+2)\Gamma(q)\int_0^1 G(1,s)g(s)f(u(s),u'(s),u''(s),\ldots,u^{(m)}(s),D^\alpha u(s))ds \\
\leq &\epsilon(m+2)\Gamma(q)\int_0^1 G(1,s)g(s)\|u\|_{X_2}ds \leq \|u\|_{X_2}.
\end{aligned}$$

Hence, by the second part of Guo-Krasnoselskii fixed point Theorem 1.2.12, we can conclude that (2.7) has at least one positive solution.

(ii) Superlinear case: Since $A_0 = 0$, then there exists r_1 such that

$$f(v_1, \ldots, v_{m+2}) \leq \delta \sum_{i=1}^{m+2} |v_i|$$

for $\sum_{i=1}^{m+2} |v_i| \leq r_1$, where δ satisfies

$$\delta(m+2)\Gamma(q)\int_0^1 G(1,s)g(s)ds \leq 1.$$

Let $\Omega_1 = \{u \in E, \|u\|_{X_2} < r_1\}$. Take $u \in P_2 \cap \partial\Omega_1$, then

$$\begin{aligned}
\|Tu\|_{X_2} = \max_{t\in[0,1]} &\left(\int_0^1 G(t,s)g(s)R_u(s)ds + \sum_{i=1}^{m} \int_0^1 G_i(t,s)g(s)R_u(s)ds \right. \\
&\left. + \int_0^1 G_\alpha(t,s)g(s)R_u(s)ds \right) \\
\leq &(m+2)\Gamma(q)\int_0^1 G(1,s)g(s)f(u(s),u'(s),u''(s),\ldots,u^{(m)}(s),D^\alpha u(s))ds \\
\leq &\delta(m+2)\Gamma(q)\int_0^1 G(1,s)g(s)\|u\|_{X_2}ds \leq \|u\|_{X_2}.
\end{aligned}$$

Next, since $A_\infty = +\infty$, there exists a constant $r > r_1$, such that

$$f(v_1, \dots, v_{m+2}) \geq \lambda(\sum_{i=1}^{m+2} v_i)$$

for any $\sum_{i=1}^{m+2} |v_i| \geq r$, where $\lambda > 0$ satisfies

$$\lambda \frac{\gamma}{(m+2)\Gamma(q)} \int_{\frac{1}{2}}^{1} G\left(\frac{1}{2}, s\right) g(s) ds \geq 1.$$

Let $\Omega_2 = \{u \in E : \|u\|_{X_2} < r_2, r_2 > \frac{\Gamma(q)(m+2)r}{\gamma}\}$, it is easy to see that $\overline{\Omega_1} \subset \Omega_2$. Now we may choose $u \in P_2 \cap \partial\Omega_2$. Then $\frac{\gamma}{(m+2)\Gamma(q)} \|u\|_{X_2} > r$. Hence,

$$\begin{aligned}
\|Tu\|_{X_2} \geq & \|Tu\|_\infty \geq Tu\left(\frac{1}{2}\right) = \int_0^1 G\left(\frac{1}{2}, 1\right) g(s) R_u(s) ds \\
\geq & \int_{\frac{1}{2}}^{1} G\left(\frac{1}{2}, s\right) g(s) f(u(s), u'(s), u''(s), \dots, u^{(m)}(s), D^\alpha u(s)) ds \\
\geq & \int_{\frac{1}{2}}^{1} G\left(\frac{1}{2}, s\right) g(s) \lambda(|u(s)| + |u'(s)| + |u''(s)| \\
& + \cdots + |u^{(m)}(s)| + |D^\alpha u(s)|) ds \\
\geq & \int_{\frac{1}{2}}^{1} G\left(\frac{1}{2}, s\right) g(s) \lambda |u(s)| ds \\
\geq & \|u\|_{X_2} \lambda \frac{\gamma}{(m+2)\Gamma(q)} \int_{\frac{1}{2}}^{1} G\left(\frac{1}{2}, s\right) g(s) ds \geq \|u\|_{X_2}.
\end{aligned}$$

Thus, $\|Tu\|_{X_2} \geq \|u\|_{X_2}, \forall u \in P_2 \cap \partial\Omega_2$.
Therefore, by the second part of Theorem 1.2.12, we conclude that problem (2.7) has at least one positive solution in $P_2 \cap \left(\overline{\Omega_2} \backslash \Omega_1\right)$. □

2.4.3 Some examples

Example 2.4.6

Let us consider the following BVP

$$\begin{cases} D^{\frac{22}{5}}u(t) + u^2(t) + (u'(t))^2 + \left(D^{\frac{4}{3}}u(t)\right)^2 = 0, \\ \hfill t \in [0,1]\backslash\mathbb{Q} \\ D^{\frac{22}{5}}u(t) + \left(t^2+1\right)\left(u^2(t) + (u'(t))^2 + \left(D^{\frac{4}{3}}u(t)\right)^2\right) = 0, \\ \hfill t \in [0,1]\cap\mathbb{Q} \\ u(0) = 0, u'(0) = 0, u^{(2)}(0) = 0, u^{(3)}(0) = 0, D^{\frac{5}{2}}u(1) = 0, \end{cases} \tag{2.13}$$

where $n = 5$, $q = \dfrac{22}{5}$, $\beta = \dfrac{5}{2}$, $\alpha = \dfrac{4}{3}$,

$$g(t) = \begin{cases} 1 + t^2, & t \in [0,1]\cap\mathbb{Q} \\ 1, & t \in [0,1]\backslash\mathbb{Q}, \end{cases}$$

and

$$f(x,y,z) = x^2 + y^2 + z^2.$$

Thus $g \in L^1((0,1),\mathbb{R}_+)$ *and* $f \in C(\mathbb{R}_+^{m+2},\mathbb{R}_+)$. *Since f is superlinear, by Theorem 2.4.5 we deduce that problem* (2.13) *has at least one positive solution on* $[0,1]$.

Example 2.4.7

Let us consider the following BVP

$$\begin{cases} D^{\frac{16}{3}}u(t) + \left(\dfrac{1}{u^2(t)+1} + \dfrac{1}{(u'(t))^2+1} + \dfrac{1}{(u''(t))^2+1} + \dfrac{1}{\left(D^{\frac{5}{2}}u(t)\right)^2+1} \right) = 0 \\ \qquad \text{if}\, t \in [0,1]\backslash\mathbb{Q} \\ D^{\frac{16}{3}}u(t) + \left(t^2+1\right) \left(\dfrac{1}{u^2(t)+1} + \dfrac{1}{(u'(t))^2+1} + \dfrac{1}{(u''(t))^2+1} + \dfrac{1}{\left(D^{\frac{5}{2}}u(t)\right)^2+1} \right) = 0 \\ \qquad \text{if}\, t \in [0,1]\cap\mathbb{Q} \\ u(0) = 0, u'(0) = 0, u^{(2)}(0) = 0, u^{(3)}(0) = 0, D^{\frac{7}{2}}u(1) = 0, \end{cases} \tag{2.14}$$

where $n = 6$, $q = \dfrac{16}{3}$, $\beta = \dfrac{7}{2}$, $\alpha = \dfrac{5}{2}$,

$$g(t) = \begin{cases} 1+t^2, & t \in [0,1]\cap\mathbb{Q} \\ 1, & t \in [0,1]\backslash\mathbb{Q}, \end{cases}$$

and

$$f(x,y,z,w) = \frac{1}{x^2+1} + \frac{1}{y^2+1} + \frac{1}{z^2+1} + \frac{1}{w^2+1}.$$

Thus $g \in L^1((0,1),\mathbb{R}_+)$ *and* $f \in C(\mathbb{R}_+^{m+2},\mathbb{R}_+)$. *Since f is sublinear, by Theorem 2.4.5 we deduce that problem* (2.14) *has at least one positive solution on* $[0,1]$.

Chapter 3

Uniqueness and existence under Nagumo-like conditions

In the book of Agarwal *et al.* [4] published in 1993, a great number of uniqueness and nonuniqueness criteria were detailed and some new refinements were established and proved. Since then, little to no contributions in this field was done.

Nevertheless, a few years ago a new interest debuted by the published works by Lakshmikhantan and Leela in 2009, see [66, 67]. They gave analogues of Nagumo's condition [67] and a Krasnoselskii-Krein type conditions for FDE [66], referred as Nagumo-like conditions for short. They established the existence and uniqueness of the following Riemann-Liouville problem:

$$D^q u(t) = f(t, u(t)), \quad u(t_0) = u^0,$$

where $0 < q < 1$.

Similarly to [66], the authors in [100] established the uniqueness and existence of the following problem

$$\begin{cases} D^q u(t) = f(t, u(t), D^{q-1} u(t)), \\ u(0) = 0, D^{(q-1)} u(0) = 0, \end{cases} \tag{3.1}$$

where $1 \leq q < 2$. One contribution of that paper is in the use of nonlinearity depending on a fractional derivative.

Recently, some works considered Nagumo conditions and Nagumo-like conditions: that is, Krasnoselskii-Krein conditions, Rogers conditions, and Kooi conditions. And in a paper published a year ago [34], Cid and Pouso presented a new uniqueness result for first order systems of ordinary differential equations which contains a generalisation of Montel–Tonelli's uniqueness Theorem.

3.1 Position of the problem

Motivated by the rising interest in the uniqueness theorems [20, 23, 34, 46, 65, 67, 76], we generalise the Krasnoselskii-Krein type of uniqueness theorem to $q > 1$ arbitrary along with Kooi and Rogers ones. The IVP is of the Riemann-Liouville type fractional differential equation, where the nonlinearity is depending on $D^{q-1}x$. This order may take values greater than one. Further, we establish the convergence of successive approximations of the Picard iterations of the equivalent Volterra integral equation. Finally, we give an example illustrating numerically our results. The results of this chapter are accepted for publication [93].

We propose the following IVP

$$\begin{cases} D^q x(t) = f(t, x(t), D^{q-1}x(t)), 0 \leq t \leq 1 \\ x(0) = 0, D^{(q-i)}x(0) = 0, i = 1, \ldots, [q], \end{cases} \tag{3.2}$$

where $f \in C(R_0,)$, such that $R_0 = \{(t, x, y) : 0 \leq t \leq 1, |x| \leq b, |y| \leq d, b, d \in \mathbb{R}^+\}$. Using similar arguments as in the previous Chapter, we prove easily the equivalence of this IVP with the associated Volterra integral equation given by

$$x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s), D^{q-1}x(s)) ds. \tag{3.3}$$

3.2 Uniqueness and existence results

Now, we state the Krasnoselskii-Krein type conditions for IVP (3.2) of the Riemann-Liouville type fractional differential equation which involves derivative term in the function f.

Theorem 3.2.1
Let $f \in C(R_0, \mathbb{R})$ satisfy the following Krein-type conditions:

(A1) $|f(t,x,y) - f(t,\overline{x},\overline{y})| \leq \min\{\Gamma(q),1\} \dfrac{k+\alpha(q-[q])}{2t^{1-\alpha(q-[q])}} [|x-\overline{x}| + |y-\overline{y}|]$, $t \neq 0$ *and* $0 < \alpha < 1$.

(A2) $|f(t,x,y) - f(t,\overline{x},\overline{y})| \leq c\left[|x-\overline{x}|^{\alpha} + t^{\alpha(q-[q])}|y-\overline{y}|^{\alpha}\right]$,

where c and k are positive constants and $k(1-\alpha) < 1+\alpha(q-[q])$. Then the successive approximations given by

$$x_{j+1}(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x_j(s), D^{q-1}x_j(s))ds,$$
$$x_0(t) = 0, n = 0,1,... \tag{3.4}$$

converge uniformly to the unique solution x of (3.2) *on $[0,\eta]$, Let M be the bound for f on R_0 and $\eta = \min\left\{1, \left(\frac{b\Gamma(1+q)}{M}\right)^{1/q}, \frac{d}{M}\right\}$,*

Proof. First, we establish the uniqueness, we suppose x and y are any two solutions of (3.2) on $[0,\eta]$ and let $\phi(t) = |x(t) - y(t)|$ and $\theta(t) = |D^{q-1}x(t) - D^{q-1}y(t)|$. Note that $\phi(0) = \theta(0) = 0$.
We define $R(t) = \int_0^t \left[\phi^{\alpha}(s) + s^{\alpha(q-[q])}\theta^{\alpha}(s)\right] ds$, clearly $R(0) = 0$.
Further, we have for $t \in [0,\eta]$

$$x(t) = I^q f(t, x(t), D^{q-1}x(t)),$$
$$D^{q-1}x(t) = D^{q-1}I^q[f(t,x(t),D^{q-1}x(t))] = \int_0^t f(s,x(s),D^{q-1}x(s))ds.$$

Using this and condition (A2), we get

$$\begin{aligned}\phi(t) \leq & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s,x(s),D^{q-1}x(s)) - f(s,y(s),D^{q-1}y(s))| ds \\ \leq & \frac{c}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[\phi^\alpha(s) + s^{\alpha(q-[q])}\theta^\alpha(s)\right] ds \leq \frac{c}{\Gamma(q)} t^{q-1} R(t).\end{aligned}$$

And

$$\theta(t) \leq \int_0^t |f(s,x(s),D^{q-1}x(s)) - f(s,y(s),D^{q-1}y(s))| ds \leq cR(t).$$

For the sake of simplicity, we use the same symbol C to denote all different constants arising in the rest of the proof.
We have

$$\begin{aligned}R'(t) =& \phi^\alpha(t) + t^{\alpha(q-[q])}\theta^\alpha(t) \\ \leq & C\left[t^{\alpha(q-1)} + t^{\alpha(q-[q])}\right] R^\alpha(t).\end{aligned} \tag{3.5}$$

Since $R(t) > 0$ for $t > 0$, multiplying both sides of (3.5) by $(1-\alpha)R^{-\alpha}(t)$, and then integrating the resulting inequality, we get for $t > 0$

$$R(t) \leq C\left(t^{\left(\frac{\alpha}{1-\alpha}q+1\right)} + t^{\left(\frac{\alpha}{1-\alpha}q+\frac{1-\alpha[q]}{1-\alpha}\right)}\right),$$

where we used Theorem 1.2.2. This leads to the following estimates on $\phi(t)$ and $\theta(t)$, for $t \in [0,\eta]$,

$$\phi(t) \leq C\left(t^{\left(\frac{q}{1-\alpha}\right)} + t^{\left(\frac{q}{1-\alpha}+\frac{\alpha(1-[q])}{1-\alpha}\right)}\right),$$

$$\theta(t) \leq C\left(t^{\left(\frac{\alpha}{1-\alpha}q+1\right)} + t^{\left(\frac{\alpha}{1-\alpha}q+\frac{1-\alpha[q]}{1-\alpha}\right)}\right).$$

Define the function $\psi(t) = t^{-k}\max\{\phi(t),\theta(t)\}$ for $t \in (0,1]$. Either $t^{-k}\phi(t)$ or $t^{-k}\theta(t)$ is the maximum, it follows that

$$0 \leq \psi(t) \leq C\left(t^{\left(\frac{q}{1-\alpha}-k\right)} + t^{\left(\frac{q}{1-\alpha}+\frac{\alpha(1-[q])}{1-\alpha}-k\right)}\right)$$

or

$$0 \leq \psi(t) \leq C\left(t^{\left(\frac{\alpha}{1-\alpha}q+1-k\right)} + t^{\left(\frac{\alpha q}{1-\alpha}+\frac{1-\alpha[q]}{1-\alpha}-k\right)}\right).$$

Since we assumed that $k(1-\alpha) < 1 + \alpha(q - [q])$, we can easily verify that the below inequalities hold

$$\begin{aligned} k(1-\alpha) &< q \\ (k-1)(1-\alpha) &< \alpha q \\ k(1-\alpha) &< q + \alpha - \alpha[q]) \\ k(1-\alpha) &< \alpha q + 1 - \alpha[q]. \end{aligned}$$

So all of the exponents of t in the above inequalities are positive. Hence, $\lim_{t\to 0^+} \psi(t) = 0$. Therefore, if we define $\psi(0) = 0$, the function ψ is continuous in $[0,\eta]$.

We want to prove that $\psi \equiv 0$. Since the function ψ is continuous, if ψ doesn't vanish at some points t that is $\psi(t) > 0$ on $]0,\eta]$, then there exists a maximum $m > 0$ reached when t is equal to some t_1: $0 < t_1 \leq \eta \leq 1$ such that $\psi(s) < m = \psi(t_1)$, for $s \in]0,t_1)$. Therefore, from condition (A1) we get for either cases

$$m = \psi(t_1) = t_1^{-k}\phi(t_1) \leq \min(\Gamma(q),1)mt_1^{q-1+\alpha(q-[q])} < m$$

or

$$m = \psi(t_1) = t_1^{-k}\theta(t_1) \leq \min(\Gamma(q),1)mt_1^{\alpha(q-[q])} < m,$$

which is a contradiction. Thus, the uniqueness of the solution is established.

For the second part, we use Arzelà-Ascoli Theorem. First, we show that the successive approximations $\{x_{j+1}(t)\}, j = 0,1,\dots$ given by (3.4) are well-defined and continuous on $[0,\eta]$. In fact,

$$\begin{aligned} |x_{j+1}(t)| &\leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left| f(s, x_j(s), D^{q-1}x_j(s)) \right| ds \\ |D^{q-1}x_{j+1}(t)| &\leq \int_0^t \left| f(s, x_j(s), D^{q-1}x_j(s)) \right| ds. \end{aligned}$$

For $j = 0$ and $t \in [0,\eta]$, we have

$$|x_1(t)| \leq \frac{Mt^q}{\Gamma(q+1)} \leq b \quad \text{and} \quad |D^{q-1}x_1(t)| \leq Mt \leq d.$$

Moreover, for every $i \in \{0,\dots,n-1\}$ we obtain

$$\begin{aligned}
|x_1^{(i)}(t)| =&|D^i I^q f(t,x_0(t),D^{q-1}x_0(t))|\\
=&|I^{q-i} f(t,x_0(t),D^{q-1}x_0(t))|\\
=&\left|\frac{1}{\Gamma(q-i)}\int_0^t (t-s)^{q-i-1} f(s,x_0(s),D^{q-1}x_0(s))ds\right|\\
\leq&\frac{M}{\Gamma(q-i)}\int_0^t (t-s)^{q-i-1}ds\\
\leq&\frac{Mt^{q-i}}{(q-i)\Gamma(q-i)}\\
\leq&\frac{Mt^{q-i}}{\Gamma(q-i+1)}.
\end{aligned}$$

By induction, the sequences $\{x_{j+1}(t)\}$ and $\{D^{q-1}x_{j+1}(t)\}$ are well-defined and uniformly bounded on $[0,\eta]$. We verify that the family $\{D^{q-1}j_{n+1}(t)\}$ is equicontinuous in $C[0,1]$ and that the family $\{x_{j+1}(t)\}$ is equicontinuous in $C^{n-1}[0,1]$.

We may prove that y and z are continuous functions in $[0,\eta]$, where y and z are defined by

$$\begin{aligned}
y(t) =&\limsup_{j\to\infty} |x_j(t) - x_{j-1}(t)|,\\
z(t) =&\limsup_{j\to\infty} |D^{q-1}x_j(t) - D^{q-1}x_{j-1}(t)|.
\end{aligned}$$

Let us note

$$m(t) = \sup_{i\leq n-1} \limsup_{j\to\infty} |x_j^{(i)}(t) - x_{j-1}^{(i)}(t)|.$$

For $t_1, t_2 \in [0,\eta]$ we have

$$|x_{j+1}(t_1) - x_j(t_1)| \leq |x_j(t_2) - x_{j-1}(t_2)| + \frac{4M}{\Gamma(q+1)}(t_2-t_1)^q.$$

and we get for every $i \in \{0,\dots,n-1\}$

$$|x_{j+1}^{(i)}(t_1) - x_j^{(i)}(t_1)| \leq |x_j^{(i)}(t_2) - x_{j-1}^{(i)}(t_2)| + \frac{4M}{\Gamma(q-i+1)}(t_2-t_1)^{q-i}.$$

In fact, for $0 \leq t_1 \leq t_2$ and for every $i \in \{0, \ldots, n-1\}$, consider the difference

$$
\begin{aligned}
|x_{j+1}^{(i)}(t_1) - x_j^{(i)}(t_1)| - |x_{j+1}^{(i)}(t_2) - x_j^{(i)}(t_2)| \\
\leq & |x_{j+1}^{(i)}(t_1) - x_j^{(i)}(t_1) - x_{j+1}^{(i)}(t_2) + x_j^{(i)}(t_2)| \\
\leq & \frac{1}{\Gamma(q-i)} \left[\left| \int_0^{t_1} (t_1 - s)^{q-1-i} D(s) ds - \int_0^{t_2} (t_2 - s)^{q-1-i} D(s) ds \right| \right] \\
\leq & \frac{2M}{\Gamma(q-i)} \left[\left| \int_0^{t_1} ((t_1 - s)^{q-1-i} - (t_2 - s)^{q-1-i}) ds \right. \right. \\
& \left. \left. - \int_{t_1}^{t_2} (t_2 - s)^{q-1-i} ds \right| \right] \\
\leq & \frac{2M}{(q-i)\Gamma(q-i)} \left[t_1^{q-i} - t_2^{q-i} + 2(t_2 - t_1)^{q-i} \right] \\
\leq & \frac{4M}{\Gamma(q-i+1)} (t_2 - t_1)^{q-i},
\end{aligned}
$$

where $D(s) = |f(s, x_j(s), D^{q-1}x_j(s)) - f(s, x_{j-1}(s), D^{q-1}x_{j-1}(s))| \leq 2M$.
Let us note

$$
\sigma = \max_{i \leq n-1} \left\{ \frac{4M}{\Gamma(q-i+1)} (t_2 - t_1)^{q-i} \right\}.
$$

The right-hand side in the above inequalities is at most $m(t_2) + \epsilon + \sigma(t)$ for large n if $\epsilon > 0$ provided that

$$
|t_2 - t_1| \leq \eta \leq \frac{Mt^q}{\Gamma(q+1)} \leq \sigma.
$$

Since ϵ is arbitrary and t_1, t_2 can be interchangeable, we get

$$
|m(t_1) - m(t_2)| \leq \sigma
$$

The same goes for z: that is,

$$
|z(t_1) - z(t_2)| \leq 2M|t_2 - t_1|.
$$

These imply that y and z are continuous on $[0, \eta]$. Again by using condition (A2) and the definition of successive approximations we obtain

$$
\begin{aligned}
|x_{j+1}(t) - x_j(t)| \leq & c \int_0^t (t-s)^{q-1} \left[|x_j(s) - x_{j-1}(s)|^\alpha \right. \\
& \left. + s^{\alpha(q-[q])} |D^{q-1}x_j(s) - D^{q-1}x_{j-1}(s)|^\alpha \right] ds
\end{aligned}
$$

and

$$
\begin{aligned}
|x_{j+1}^{(i)}(t) - x_j^{(i)}(t)| \leq & C \int_0^t (t-s)^{q-i-1} \left[|x_j(s) - x_{j-1}(s)|^\alpha \right. \\
& \left. + s^{\alpha(q-[q])} |D^{q-1} x_j(s) - D^{q-1} x_{j-1}(s)|^\alpha \right] ds.
\end{aligned}
$$

As a consequence, we obtain the following estimate for a certain $i = i_0$

$$
\begin{aligned}
\|x_{j+1} - x_j\| \leq & C \int_0^1 (1-s)^{q-i_0-1} \left[|x_j(s) - x_{j-1}(s)|^\alpha \right. \\
& \left. + s^{\alpha(q-[q])} |D^{q-1} x_j(s) - D^{q-1} x_{j-1}(s)|^\alpha \right] ds.
\end{aligned}
$$

All of the Arzelà-Ascoli Theorem conditions are fulfilled for the family $\{x_j\}$ in $C^{n-1}[0,1]$, respectively $\{D^{q-1}x_j\}$ in $C[0,1]$. Hence, there exists a subsequence $\{x_{j_k}\}$, respectively $\{D^{q-1}x_{j_k}\}$ converging uniformly on $[0,\eta]$ as $j_k \to \infty$.

Let us define m^* and z^* for every $t \in [0,\eta]$ by

$$
\begin{aligned}
m^*(t) = & \limsup_{k\to\infty} |x_{j_k}(t) - x_{j_{k-1}}(t)|, \\
z^*(t) = & \limsup_{k\to\infty} |D^{q-1}x_{j_k}(t) - D^{q-1}x_{j_{k-1}}(t)|.
\end{aligned}
$$

Further, if $\{|x_j - x_{j-1}|\} \to 0$ and $\{|D^{q-1}x_j - D^{q-1}x_{j-1}|\} \to 0$ as $j \to \infty$, then (3.4) implies that the limit of any such subsequence is the unique solution x of (3.2). It follows that a selection of subsequences is unnecessary and that the entire sequence $\{x_j\}$ converges uniformly to x. For that, it suffices that $y \equiv 0$ and $z \equiv 0$ which leads to m^* and z^* being null.

Setting

$$
R(t) = \int_0^t \left[y(s)^\alpha + s^{\alpha(q-[q])} z(s)^\alpha \right] ds,
$$

and by defining $\psi^*(t) = t^{-k}\max\{y(t), z(t)\}$, the $\lim_{t\to 0^+} \psi^*(t) = 0$.

We now show that $\psi^* \equiv 0$. If $\psi^*(t) > 0$ at any point in $[0,\eta]$, then there exists t_1 such that $0 < \overline{m} = \psi^*(t_1) = \max_{0\leq t\leq\eta} \psi^*(t)$. Hence, from condition (A1), we obtain

$$
\overline{m} = \psi(t_1) = t_1^{-k} y(t_1) \leq \min(\Gamma(q), 1)\overline{m} t_1^{q-1+\alpha(q-[q])} < \overline{m}
$$

or

$$\overline{m} = \psi(t_1) = t_1^{-k} z(t_1) \le \min(\Gamma(q),1)\overline{m} t_1^{\alpha(q-[q])} < \overline{m}.$$

In both cases, we end up with a contradiction. So $\psi^* \equiv 0$. Therefore, the Picard iterates converge uniformly to the unique solution x of (3.2) on $[0,\eta]$.

$\square$

Corollary 3.2.2
Let $f \in C(R_0,\mathbb{R})$ satisfy the following Krein-type conditions:

(B1) $|f(t,x,y) - f(t,\overline{x},\overline{y})| \le \min\{\Gamma(q),1\} \dfrac{k+\alpha p}{2t^{1-\alpha p}} \left[|x-\overline{x}| + |y-\overline{y}|\right], t \neq 0$ *and* $0 < \alpha < 1$.

(B2) $|f(t,x,y) - f(t,\overline{x},\overline{y})| \le c\left[|x-\overline{x}|^{\alpha} + t^{\alpha p}|y-\overline{y}|^{\alpha}\right]$,

where c and k are positive constants, $0 < p < 1$ and $k(1-\alpha) < 1 + \alpha \min(q - [q], p)$. Then the successive approximations given by (3.4) *converge uniformly to the unique solution x of* (3.2) *on $[0,\eta]$,*
where $\eta = \min\left\{1, \left(\frac{b\Gamma(1+q)}{M}\right)^{1/q}, \frac{d}{M}\right\}$, *$M$ is the bound for f on R_0.*

Remark 3.2.3. For the case $1 < q < 2$, Theorem 3.2.1 is reduced to the uniqueness result provided in [100].

Theorem 3.2.4 (Kooi's type uniqueness theorem)
Let f satisfy the following conditions:

(C1) $|f(t,x,y) - f(t,\overline{x},\overline{y})| \le \min\{\Gamma(q),1\} \dfrac{k+\alpha(q-[q])}{2t^{1-\alpha(q-[q])}} \left[|x-\overline{x}| + |y-\overline{y}|\right]$, $t \neq 0$ *and* $0 < \alpha < 1$.

(C2) $t^{\beta}|f(t,x,y) - f(t,\overline{x},\overline{y})| \le c\left[|x-\overline{x}|^{\alpha} + t^{\alpha(q-[q])}|y-\overline{y}|^{\alpha}\right]$,

where c and k are positive constants and $k(1-\alpha) < 1 + \alpha(q-[q]) - \beta$. Then the successive approximations given by (3.4) *converge to the unique solution x on $[0,\eta]$.*

Proof. The proof is similar to that of Theorem 3.2.1, thus we omit it. $\square$

Lemma 3.2.5
Let ϕ and θ be two nonnegative continuous functions in $[0,a]$ for a real number $a>0$. Let $\psi(t)=\int_0^t \frac{\phi(s)+s^{q-[q]}\theta(s)}{2s^{q-[q]+2}}ds$. Assume the following hold:

(i) $\phi(t)\leq t^{q-[q]}\psi(t)$,

(ii) $\theta(t)\leq \psi(t)$,

(iii) $\phi(t)=o(t^{q-[q]}e^{-1/t})$,

(iv) $\theta(t)=o(e^{-1/t})$.

Then $\phi\equiv\theta\equiv 0$.

Proof. Let $\psi(t)=\int_0^t \frac{\phi(s)+s^{q-[q]}\theta(s)}{2s^{q-[q]+2}}ds$. After differentiating ψ and using (ii), we obtain for $t>0$, $\psi'(t)\leq \frac{1}{t^2}\psi(t)$, so that $e^{1/t}\psi(t)$ is decreasing. Now, from (iii) and (iv), if $\epsilon>0$ then for a small t, we have

$$e^{1/t}\psi(t)\leq e^{1/t}\int_0^t \frac{1}{2s^2}2\epsilon e^{-1/s}ds=\epsilon.$$

Hence, $\lim_{t\to 0}e^{1/t}\psi(t)=0$ which implies that $\psi(t)\leq 0$. Finally, ψ is nonnegative due to (i), thus $\psi\equiv 0$. $\square$

Theorem 3.2.6 (Rogers' type uniqueness theorem)
Let f be such that the following conditions hold:

(D1) $f(t,x,y)\leq \min\{\Gamma(q),1\}o\left(\frac{e^{-1/t}}{t^2}\right)$, *uniformly for positive and bounded x and y*

(D2) $|f(t,x,y)-f(t,\overline{x},\overline{y})|\leq \min\{\Gamma(q),1\}\frac{1}{2t^{q-[q]+2}}\left[|x-\overline{x}|+t^{(q-[q])}|y-\overline{y}|\right]$.

Then the problem has at most one solution.

The proof of this theorem is essentially based on the Lemma 3.2.5.

Proof. Suppose x, y are two solutions of 3.2, we get for $t \in [0,a] \subset [0,1]$

$$
\begin{aligned}
|x(t)-y(t)| \leq & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s,x(s),D^{q-1}x(s)) - f(s,y(s),D^{q-1}y(s))| \\
\leq & \int_0^t \frac{(t-s)^{q-1}}{2s^{q-[q]+2}} [|x(s)-y(s)| + s^{q-[q]}|D^{q-1}x(s) - D^{q-1}y(s)|]ds \\
\leq & t^{q-1} \int_0^t \frac{1}{2s^{q-[q]+2}} [|x(s)-y(s)| + s^{q-1}|D^{q-1}x(s) - D^{q-1}y(s)|]ds \\
\leq & t^{q-[q]} \int_0^t \frac{1}{2s^{q-[q]+2}} [|x(s)-y(s)| + s^{q-1}|D^{q-1}x(s) - D^{q-1}y(s)|]ds,
\end{aligned}
$$

and

$$
\begin{aligned}
|D^{q-1}x(s) - D^{q-1}y(s)| \leq & \int_0^t |f(s,x(s),D^{q-1}x(s)) - f(s,y(s),D^{q-1}y(s))| \\
\leq & \int_0^t \frac{\min\{\Gamma(q),1\}}{2s^{q-[q]+2}} [|x(s)-y(s)| \\
& + s^{q-[q]}|D^{q-1}x(s) - D^{q-1}y(s)|]ds \\
\leq & \int_0^t \frac{1}{2s^{q-[q]+2}} [|x(s)-y(s)| \\
& + s^{q-[q]}|D^{q-1}x(s) - D^{q-1}y(s)|]ds.
\end{aligned}
$$

Also, if $\epsilon > 0$, then from condition (D1) for small t, we have

$$
\begin{aligned}
|x(t)-y(t)| \leq & \frac{t^{q-1}}{\Gamma(q)} \int_0^t |f(s,x(s),D^{q-1}x(s)) - f(s,y(s),D^{q-1}y(s))| \\
< & t^{q-1} 2\epsilon \int_0^t \frac{e^{-1/s}}{s^2} ds \leq t^{q-1} e^{-1/t} 2\epsilon \\
< & \epsilon t^{q-[q]} e^{-1/t} 2
\end{aligned}
$$

$$
\begin{aligned}
|D^{q-1}x(t) - D^{q-1}y(t)| \leq & \int_0^t |f(s,x(s),D^{q-1}x(s)) - f(s,y(s),D^{q-1}y(s))| \\
< & 2\epsilon \min\{1,\Gamma(q)\} \int_0^t \frac{e^{-1/s}}{s^2} ds \leq 2\epsilon e^{-1/t}.
\end{aligned}
$$

We may apply Lemma 3.2.5 to find that $|(x-y)(\cdot)| \equiv 0$, and this proves the uniqueness of the solution. $\square$

3.3 Numerical example

In particularly relevant problems, it is very likely that we will be forced to use numerical methods to approximate the solutions since an analytical method is unavailable. In this section, we compare the successive approximations of the Picard iterates with an efficient numerical algorithm to solve some fractional differential equations verifying Krein conditions and show the validity of the convergence of the Picard iterate.

Example 3.3.1
We generalise the IVP proposed in [100] *for arbitrary* $q > 1$.

$$\begin{aligned} D^q x =& f(t,x) = \begin{cases} At^{q\alpha/(1-\alpha)}, & 0 \le t \le 1, -\infty < x < 0, \\ At^{q\alpha/(1-\alpha)} - Axt^{-q}, & 0 \le t \le 1, 0 \le x \le t^{q/(1-\alpha)}, \\ 0, & 0 \le t \le 1, t^{q/(1-\alpha)} < x < +\infty, \end{cases} \\ x(0) =& 0, \\ D^{q-1}x(0) =& 0, \end{aligned}$$

where $0 < \alpha < 1$, $A = \min(1, \Gamma(q))(q(k-1)+1)$, $k(1-\alpha) < 1 + \alpha(q - [q])$ *and* $c = 2^{(1-\alpha)}(q(k-1)+1)$.
The function f *is continuous and we shall verify the following estimates:*

$$\begin{aligned} |f(t,x) - f(t,\overline{x})| \le& \frac{A}{t^q}|x - \overline{x}|, \\ |f(t,x) - f(t,\overline{x})| \le& A2^{1-\alpha}|x - \overline{x}|^\alpha, \end{aligned}$$

by considering the following cases:
Suppose $0 < x, \overline{x} < t^{q/(1-\alpha)}$, *then*

$$\begin{aligned} |f(t,x) - f(t,\overline{x})| \le& \left| -A\frac{x}{t^q} + A\frac{\overline{x}}{t^q} \right| \\ \le& \frac{A}{t^q}|x - \overline{x}| \end{aligned}$$

and

$$\begin{aligned}|f(t,x)-f(t,\overline{x})| \leq & \frac{A}{t^q}|x-\overline{x}|^{1-\alpha}|x-\overline{x}|^{\alpha}\\ \leq & \frac{A}{t^q}(|x|+|\overline{x}|)^{1-\alpha}|x-\overline{x}|^{\alpha}\\ \leq & \frac{A}{t^q}2^{1-\alpha}t^q|x-\overline{x}|^{\alpha}\\ \leq & A2^{1-\alpha}|x-\overline{x}|^{\alpha}.\end{aligned}$$

Suppose $t^{q/(1-\alpha)} < x < +\infty, -\infty < \overline{x} < 0$, *then*

$$\begin{aligned}|f(t,x)-f(t,\overline{x})| \leq & \left|-At^{q\alpha/(1-\alpha)}\right| \leq \frac{A}{t^q}x\\ \leq & \frac{A}{t^q}|x-\overline{x}|\end{aligned}$$

and

$$\begin{aligned}|f(t,x)-f(t,\overline{x})| \leq & At^{q\alpha/(1-\alpha)}\\ \leq & A(|x|+|\overline{x}|)^{\alpha}\\ \leq & A2^{1-\alpha}|x-\overline{x}|^{\alpha}.\end{aligned}$$

Suppose $t^{q/(1-\alpha)} < x < +\infty, 0 < \overline{x} < t^{q/(1-\alpha)}$, *then*

$$\begin{aligned}|f(t,x)-f(t,\overline{x})| \leq & \left|-At^{q\alpha/(1-\alpha)} + A\frac{\overline{x}}{t^q}\right| \leq \frac{A}{t^q}\left|t^{q/(1-\alpha)}-\overline{x}\right|\\ \leq & \frac{A}{t^q}|x-\overline{x}|\end{aligned}$$

and

$$\begin{aligned}|f(t,x)-f(t,\overline{x})| \leq & A\left[\frac{t^{q/(1-\alpha)}-\overline{x}}{t^{(1/(1-\alpha))(1-\alpha)}}\right]\\ \leq & A\left[t^{q/(1-\alpha)}-\overline{x}\right]^{\alpha} \leq A(x-\overline{x})^{\alpha}\\ \leq & A2^{1-\alpha}|x-\overline{x}|^{\alpha}.\end{aligned}$$

Suppose $0 < x < t^{q/(1-\alpha)}, -\infty < \overline{x} < 0$, *then*

$$\begin{aligned}|f(t,x)-f(t,\overline{x})| \leq & \left|At^{q/(1-\alpha)} - A\frac{x}{t^q} - At^{q/(1-\alpha)}\right| \leq A\frac{x}{t^q}\\ \leq & \frac{A}{t^q}|x-\overline{x}|\end{aligned}$$

and

$$|f(t,x) - f(t,\overline{x})| \leq A\frac{x}{t^q} < Ax^\alpha)$$
$$\leq A2^{1-\alpha}|x - \overline{x}|^\alpha.$$

Since all the condition of Theorem 3.2.1 are satisfied, the IVP has a unique solution on $[0,1]$ *limit of the successive approximations*

$$x_{n+1}(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, x_n(s))ds.$$

The works presented in [37, 39–41, 44, 112] give several methods to approximate the solution of FDE. They have been of lower order, but the FracPECE [40] attracted our interest because of the relative ease of application and its reliability as far as convergence and stability are concerned.

Figure 3.1 shows the approximate solution x_f given by the FracPECE method (Fde12 function on Matlab) with a stepsize $h = 0.01$ and an order $q = 1.5$, $k = 1.7$ and $\alpha = 0.5$.

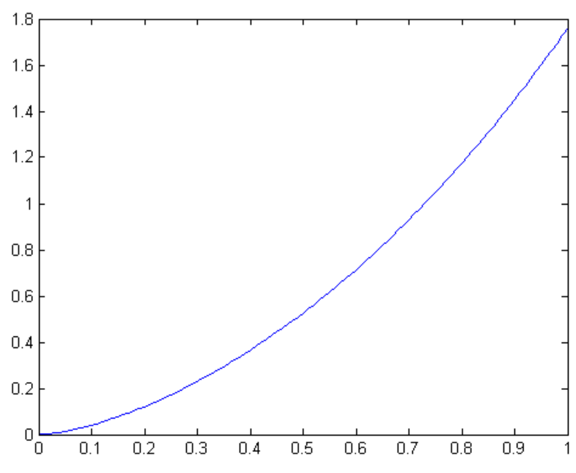

Figure 3.1: Approximate solution x_f given by the FracPECE.

Next, we try to solve the problem using the successive approximations of Picard (3.4). First with different initial vectors x_0 and later by changing the values of q. We calculate the integral with the composite Simpson's rule method with a stepsize $h = 0,01$ and we stop at the nth iteration whenever $err_1 = \|x_n - x_{n-1}\| \leq 0,01$. We then calculate $err_2 = \|x_n - x_f\|$ to see if x_n is converging to x_f or to an other function. The implemented code in the Matlab software is provided in the Appendix.

We stipulated in this chapter that $x_n \to x$ independently of the chosen x_0. Moreover, the following two tables show that after discrediting our example, the Picard's iterations continue to converge to the same solution.

In the first table, we give the error between x_k and x_{k-1} and the error between x_k and the solution x_f for different values of x_0 and the number of iterations k needed to get $err_1 \leq 0,01$ continuing further will not give any divergence in the following iterations.

	err_1	err_2	err_1	err_2	err_1	err_2	err_1	err_2
x_0	0		$rand(0,1)$		x_f		$-x_f$	
$k=1$	4.02e-1	1.35e-0	1.22e-0	4.61e-1	6.97e-3	6.97e-3	2.16e-0	1.35e-0
$k=2$	1.32e-0	3.02e-2	1.90e-1	2.27e-1	0	6.97e-3	1.32e-0	3.02e-2
$k=3$	1.32e-0	3.02e-2	2.73e-1	3.02e-2	0	6.97e-3	0	3.02e-2
$k=4$	0	3.02e-2	0	3.02e-2	0	6.97e-3	0	3.02e-2

Table 3.1: err_1 and err_2 for some values of x_0.

The number of iterations n needed to get an error bounded by 0,01, is shown in the following Table for different values of the fractional order q. Although, one may note that for $x_0 = x_f$ the number of iterations to get to $\hat{x}$ is big but keep in mind that we didn't do any stability analysis and that we only used the Picard's iterations with no adjustments.

x_0 \ q	1.01	1.44	1.5	2	4	4.001	4.98	5
0	3	3	3	3	3	3	5	5
rand(0,1)	4	4	4	4	4	4	4	4
x_f	1	1	1	1	2	2	2	2
$-x_f$	3	3	3	3	3	3	3	3

Table 3.2: Number of iterations to get $err_1 \leq 0.01$.

3.4 Conclusion

To summarize, the fundamental goal of this chapter was to generalise the previous uniqueness results of F. Yoruk *et al.* to arbitrary order using the Krasnoselskii-Krein, Rogers, and Kooi conditions. Obviously, the numerical example showed the convergence of the Picard iterations even if it was somewhat slow for some values of q. The error, the stability analysis, and the use of algorithms to solve this type of equations might be the subject of a future research. We finally hope that this work is a step in the study of the analytical and numerical aspect of fractional differential equations used in applied mathematics' fields.

Chapter 4

Uniqueness and existence of fuzzy solutions to FIVP

> The value of a mathematical discipline cannot be measured by its applicability to the empirical sciences.
>
> F. Rudio, Doctoral dissertation, Berlin, 23 April 1880.

4.1 Introduction

In the previous chapters, we established the existence and uniqueness of the solution to classes of IVP and BVP of fractional order in crisp sets, using different techniques. And in the preliminaries' Chapter we discussed the usefulness of the fuzzy sets and that one has to take uncertainty to obtain more realistic modeling of phenomena.

In this chapter, we give analogues of the results of Chapter 3 for fuzzy sets. First, we present succinctly other authors' results. Next, we present the FFDE that will be studied. Finally, we prove the uniqueness of the solution to the proposed equations followed by a study of the existence of that solution. The results of this chapter are accepted for publication [94].

4.2 Previous works on IVP

The study of fuzzy differential equations was focused on merely the first order ODE [27, 29, 45, 60, 79, 81, 91] and fractional order equation with an order less than one [5, 21] and at most for a second order equation [16, 62, 79]. To solve higher order and obtain the solution the technique used was the fuzzy transforms [18, 85]. And at the best of our knowledge, a very few of articles studied the existence of solution to high order fuzzy differential equations and the highest order treated is three [75].

In [14], Allahviranloo and Ahmadi introduced the fuzzy Laplace transform, which they used under the strongly generalised differentiability. Recently, E. ElJaoui *et al.* [43] developed it further. The newly defined fuzzy Laplace transform [14, 19] for high order fuzzy derivatives is one of the most useful methods as mentioned by A. Jafarian *et al.* in [59]: ″…, one of the important and interesting transforms in the problems of fuzzy equations is Laplace transforms. The fuzzy Laplace transform method solves fuzzy fractional differential equations and fuzzy boundary and initial value problems [21, 22, 86] ….″

This fuzzy transform motivated researchers to contribute to solving and studying the existence of solutions to higher order equations. Many of them worked in the theoretical and numerical aspect of fractional and fuzzy differential equations, the reader is kindly referred to [10, 11, 15, 27, 30, 45, 58, 72, 73, 84] and the references therein.

In the other hand, we already said that there is a rising interest in Nagumo-like conditions these years, see [4, 20, 23, 34, 46, 65–67, 76, 100]. In the fuzzy theory, analogues of Nagumo and Krasnoselskii-Krein condtions were also obtained by Allahviranloo *et al.*, in [13] for order less than 1.

4.3 Position of problem

In this chapter, we are interested by the existence and uniqueness of a solution to the following FFDE) for arbitrary order $q > 1$ with initial conditions:

$$\begin{cases} D^q x(t) = f(t, x(t), D^{q-1}x(t)) \\ x(0) = y_0, D^{(q-i)}x(0) = \tilde{0}, i = 1, \ldots, [q], \end{cases} \tag{4.1}$$

where $y_0 \in \mathbb{E}$ and $f : \mathbb{E}_0 \to \mathbb{E}$ is a continuous fuzzy valued function, with

$$\mathbb{E}_0 = \{(t, x, y) \in \mathbb{R} \times \mathbb{E}^2 : 0 \leq t \leq 1, d(x, y_0) \leq b, d(y, \tilde{0}) \leq d\},$$

where b and d are positive reals and d stands for the Hausdorff distance. Our aim is to both generalise and extend the previous papers [13, 100] and extend the results proved in Chapter 2. For that, we provide some insight on the works done in the study of fuzzy differential equations.

4.4 The associated fuzzy fractional integral equation

Before stating and proving the results of this Chapter, we first study the relation between problem (4.1) and the fuzzy integral form using the well known fuzzy Laplace transform defined in 1.3.13.
By taking the fuzzy Laplace transform on both sides of the following FIVP

$$D^q x(t) = f(t, x(t), D^{q-1}x(t)) \triangleq r(t, x)$$

we get

$$\mathbf{L}\left[D^q x(t)\right] = \mathbf{L}\left[f(t, x(t), D^{q-1}x(t))\right].$$

Based on the type of Riemann-Liouville H-differentiability, we obtain two cases.
Case i:

If $D^q x$ is ${}^{RL}[(i)-q]$-differentiable fuzzy valued function, then

$$\mathbf{L}r(t,x) = -\left(\sum_{k=0}^{n-1} p^k D^{\beta-k-1} x\right)(0) \ominus p^q \mathbf{L}\left[x(t)\right],$$

and based on the lower and upper functions of $D^q x$ the above equation becomes

$$\begin{aligned}
\mathbf{L}\left[\underline{r}(t,x;r)\right] &= p^q \mathbf{L}\left[\underline{x}(t;r)\right] - \sum_{k=0}^{n-1} p^k D^{\beta-k-1} \underline{x}(0;r), \\
\mathbf{L}\left[\overline{r}(t,x;r)\right] &= p^q \mathbf{L}\left[\overline{x}(t;r)\right] - \sum_{k=0}^{n-1} p^k D^{\beta-k-1} \overline{x}(0;r),
\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}
\mathbf{L}\left[\underline{r}(t,x;r)\right] &= \min\left\{r(t,u) \,|\, u \in [\underline{x}(t;r), \overline{x}(t;r)]\right\}, 0 \le r \le 1, \\
\mathbf{L}\left[\overline{r}(t,x;r)\right] &= \max\left\{r(t,u) \,|\, u \in [\underline{x}(t;r), \overline{x}(t;r)]\right\}, 0 \le r \le 1.
\end{aligned}$$

In order to solve system (4.2) and for the sake of simplicity, we assume that

$$\begin{aligned}
\mathbf{L}\left[\underline{x}(t;r)\right] &= H_1(p;r), \\
\mathbf{L}\left[\overline{x}(t;r)\right] &= K_1(p;r),
\end{aligned} \tag{4.3}$$

where $H_1(p;r)$ and $K_1(p;r)$ are solutions of the previous system (4.2), it yields

$$\begin{aligned}
\underline{x}(t;r) &= \mathbf{L}^{-1}\left[H_1(p;r)\right], \\
\overline{x}(t;r) &= \mathbf{L}^{-1}\left[K_1(p;r)\right].
\end{aligned}$$

Case ii:
If $D^q x$ is ${}^{RL}[(ii)-q]$-differentiable fuzzy valued function, then

$$\mathbf{L}r(t,x) = p^q \mathbf{L}\left[x(t)\right] \ominus \left(\sum_{k=0}^{n-1} p^k D^{\beta-k-1} x\right)(0),$$

and based on the lower and upper functions of $D^q x$ the above equation becomes

$$\begin{aligned}
\mathbf{L}\left[\underline{r}(t,x;r)\right] &= p^q \mathbf{L}\left[\underline{x}(t;r)\right] - \sum_{k=0}^{n-1} p^k D^{\beta-k-1} \underline{x}(0;r), \\
\mathbf{L}\left[\overline{r}(t,x;r)\right] &= p^q \mathbf{L}\left[\overline{x}(t;r)\right] - \sum_{k=0}^{n-1} p^k D^{\beta-k-1} \overline{x}(0;r),
\end{aligned} \tag{4.4}$$

where

$$\mathbf{L}\left[\underline{r}(t,x;r)\right] = \min\left\{r(t,u) \,|\, u \in \left[\underline{x}(t;r), \overline{x}(t;r)\right]\right\}, 0 \le r \le 1,$$
$$\mathbf{L}\left[\overline{r}(t,x;r)\right] = \max\left\{r(t,u) \,|\, u \in \left[\underline{x}(t;r), \overline{x}(t;r)\right]\right\}, 0 \le r \le 1.$$

In order to solve system (4.4) and for the sake of simplicity, we assume that

$$\mathbf{L}\left[\underline{x}(t;r)\right] = H_2(p;r),$$
$$\mathbf{L}\left[\overline{x}(t;r)\right] = K_2(p;r),$$

where $H_2(p;r)$ and $K_2(p;r)$ are solutions of the previous system (4.4). Then we obtain

$$\underline{x}(t;r) = \mathbf{L}^{-1}\left[H_2(p;r)\right],$$
$$\overline{x}(t;r) = \mathbf{L}^{-1}\left[K_2(p;r)\right]. \tag{4.5}$$

Taking into account the initial conditions of problem (4.1) and using the linearity of the inverse Laplace transform on systems (4.3) and (4.5), we obtain the following for both cases:
x is solution is a solution for problem (4.1) if and only if x is a solution for the following integral equation

$$x(t) = y_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} r(s,x) ds \tag{4.6}$$

in the sense of ${}^{RL}[(i)-q]$-differentiability, and

$$\hat{x}(t) = y_0 \ominus \frac{-1}{\Gamma(q)} \int_0^t (t-s)^{q-1} r(s,x) ds \tag{4.7}$$

in the sense of ${}^{RL}[(ii)-q]$-differentiability.

4.5 Uniqueness results

Now, we state the Krasnoselskii-Krein type conditions for FFDE (4.1).

Theorem 4.5.1
Let $f \in C(\mathbb{E}_0, \mathbb{E})$ satisfy the following Krein-type conditions:

(H1) $d(f(t,x,y), f(t,\overline{x},\overline{y})) \leq \min\{\Gamma(q), 1\} \dfrac{k + \alpha(q - [q])}{2t^{1-\alpha(q-[q])}} [d(x,\overline{x}) + d(y,\overline{y})]$, $t \neq 0$ *and* $0 < \alpha < 1$,

(H2) $d(f(t,x,y), f(t,\overline{x},\overline{y})) \leq \delta d(x,\overline{x})^{\alpha} + t^{\alpha(q-[q])} d(y,\overline{y})^{\alpha}$,

where δ and k are positive constants and $k(1-\alpha) < 1 + \alpha(q - [q])$, then in the sense of ${}^{RL}[(i) - q]$-differentiability, the solution x is unique and in the sense of ${}^{RL}[(i) - q]$-differentiability, the solution $\hat{x}$ is unique on $[0, \eta]$, where

$$\eta = \min\left\{1, \left(\frac{b\Gamma(1+q)}{M}\right)^{1/q}, \frac{d}{M}\right\}$$

and M is the bound for f on E_0: that is $d(f, \tilde{0}) \leq M$.

Proof. First, we establish the uniqueness. Suppose x and y are any two solutions of (4.1) in ${}^{RL}[(i) - q]$-differentiability and let $\phi(t) = d(x(t), y(t))$ and $\theta(t) = d(D^{q-1}x(t), D^{q-1}y(t))$. Note that $\phi(0) = \theta(0) = 0$.
We define $R(t) = \int_0^t \left[\phi^{\alpha}(s) + s^{\alpha(q-[q])}\theta^{\alpha}(s)\right] ds$, clearly $R(0) = 0$.
Using (4.6) and condition (H2), we get

$$\phi(t) \leq \frac{\delta}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[\phi^{\alpha}(s) + s^{\alpha(q-[q])}\theta^{\alpha}(s)\right] ds \leq \frac{\delta}{\Gamma(q)} t^{q-1} R(t),$$

and

$$\theta(t) \leq \int_0^t \delta\phi(s)^{\alpha} + t^{\alpha(q-[q])}\theta(s)^{\alpha} ds \leq \delta R(t).$$

For the sake of simplicity, we use the same symbol C to denote all different constants arising in the rest of the proof.
We have

$$\begin{aligned} R'(t) =& \phi^{\alpha}(t) + t^{\alpha(q-[q])}\theta^{\alpha}(t) \\ \leq& C\left[t^{\alpha(q-1)} + t^{\alpha(q-[q])}\right] R^{\alpha}(t). \end{aligned} \tag{4.8}$$

Since $R(t) > 0$ for $t > 0$, multiplying both sides of (4.8) by $(1-\alpha)R^{-\alpha}(t)$ and then integrating the resulting inequality, we get

$$R(t)^{1-\alpha} \leq C\left(t^{(\alpha q+1)} + t^{(\alpha q+1-\alpha[q])}\right)^{1-\alpha}. \tag{4.9}$$

Using the fact that

$$(a+b)^{(1-\alpha)} \leq \frac{1}{2^{(1-\alpha)-1}}\left(a^{(1-\alpha)} + b^{(1-\alpha)}\right)$$

for every $a, b \in (0,1)$, equation (4.9) becomes

$$R(t) \leq C\left(t^{\left(\frac{\alpha}{1-\alpha}q+1\right)} + t^{\left(\frac{\alpha}{1-\alpha}q+\frac{1-\alpha[q]}{1-\alpha}\right)}\right).$$

This leads to the following estimates on ϕ and θ, for $t \in [0,\eta]$:

$$\phi(t) \leq C\left(t^{\left(\frac{q}{1-\alpha}\right)} + t^{\left(\frac{q}{1-\alpha}+\frac{\alpha(1-[q])}{1-\alpha}\right)}\right),$$

$$\theta(t) \leq C\left(t^{\left(\frac{\alpha}{1-\alpha}q+1\right)} + t^{\left(\frac{\alpha}{1-\alpha}q+\frac{1-\alpha[q]}{1-\alpha}\right)}\right).$$

Define the function $\psi(t) = t^{-k}\max\{\phi(t), \theta(t)\}$ for $t \in (0,1]$. Either $t^{-k}\phi(t)$ or $t^{-k}\theta(t)$ is the maximum, we get:

$$0 \leq \psi(t) \leq C\left(t^{\left(\frac{q}{1-\alpha}-k\right)} + t^{\left(\frac{q}{1-\alpha}+\frac{\alpha(1-[q])}{1-\alpha}-k\right)}\right),$$

or

$$0 \leq \psi(t) \leq C\left(t^{\left(\frac{\alpha}{1-\alpha}q+1-k\right)} + t^{\left(\frac{\alpha q}{1-\alpha}+\frac{1-\alpha[q]}{1-\alpha}-k\right)}\right).$$

Since $k(1-\alpha) < 1+\alpha(q-[q])$ (by assumption), we have

$$k(1-\alpha) < 1+\alpha(q-[q]) \Longrightarrow \begin{cases} k(1-\alpha) < q \\ (k-1)(1-\alpha) < \alpha q \\ k(1-\alpha) < q+\alpha-\alpha[q] \\ k(1-\alpha) < \alpha q+1-\alpha[q]. \end{cases} \tag{4.10}$$

So all of the exponents of t in the above inequalities are positive. Hence, $\lim_{t\to 0^+}\psi(t) = 0$. Therefore, if we define $\psi(0) = 0$, the function ψ is continuous in $[0,\eta]$.

We want to prove that $\psi \equiv 0$. In fact, since the function ψ is continuous, if ψ doesn't vanish at some points t that is $\psi(t) > 0$ on $]0, \eta]$, then there exists a maximum $m > 0$ reached when t is equal to some t_1: $0 < t_1 \leq \eta \leq 1$ such that $\psi(s) < m = \psi(t_1)$, for $s \in]0, t_1)$. But, from condition (H1) we get for either cases

$$m = \psi(t_1) = t_1^{-k}\phi(t_1) \leq \min(\Gamma(q), 1) m t_1^{q-1+\alpha(q-[q])} < m$$

or

$$m = \psi(t_1) = t_1^{-k}\theta(t_1) \leq \min(\Gamma(q), 1) m t_1^{\alpha(q-[q])} < m$$

which is a contradiction. Thus, the uniqueness of the solution is established in the sense of ${}^{RL}[(i) - q]$-differentiability. The second part of the proof is almost completely similar to the ${}^{RL}[(i) - q]$-differentiability, thus we omit it. $\square$

Remark 4.5.2. For the case $1 < q < 2$, of the deterministic case, Theorem 4.5.1 is reduced to [100, Theorem 3.1] and for the crisp case, Theorem 4.5.1 is reduced to the results proven in Chapter 3

Theorem 4.5.3 (Kooi's type uniqueness theorem)
Let f satisfy the following conditions:

(J1) $d(f(t,x,y), f(t,\overline{x},\overline{y})) \leq \min\{\Gamma(q), 1\} \dfrac{k + \alpha(q-[q])}{2t^{1-\alpha(q-[q])}} [d(x,\overline{x}) + d(y,\overline{y})]$, $t \neq 0$ *and* $0 < \alpha < 1$.

(J2) $t^{\beta} d(f(t,x,y), f(t,\overline{x},\overline{y})) \leq c \left[d(x,\overline{x})^{\alpha} + t^{\alpha(q-[q])} d(y,\overline{y})^{\alpha} \right]$,

where c and k are positive constants and $k(1-\alpha) < 1 + \alpha(q-[q]) - \beta$, for $(t,x,y), (t,\overline{x},\overline{y}) \in R_0$, then in the sense of ${}^{RL}[(i) - q]$-differentiability, the solution x is unique and in the sense of ${}^{RL}[(i) - q]$-differentiability, the solution $\hat{x}$ is unique.

Proof. It's similar to that of Theorem 4.5.1, thus we omit it. $\square$

Lemma 4.5.4
Let ϕ and θ be two nonnegative continuous functions in the interval $[0,\eta]$ for a real number $a > 0$. Let $\psi(t) = \int_0^t \frac{\phi(s)+s^{q-[q]}\theta(s)}{2s^{q-[q]+2}}ds$. Assume the following:

(i) $\phi(t) \le t^{q-[q]}\psi(t)$,

(ii) $\theta(t) \le \psi(t)$,

(iii) $\phi(t) = o(t^{q-[q]}e^{-1/t})$,

(iv) $\theta(t) = o(e^{-1/t})$.

Then $\phi \equiv \theta \equiv 0$.

Proof. Let $\psi(t) = \int_0^t \frac{\phi(s)+s^{q-[q]}\theta(s)}{2s^{q-[q]+2}}ds$. After differentiating ψ and using (ii), we obtain, for $t > 0$, $\psi'(t) \le \frac{1}{t^2}\psi(t)$, so that $e^{1/t}\psi(t)$ is decreasing. Now, from (iii) and (iv), if $\epsilon > 0$ then for a small t, we have

$$e^{1/t}\psi(t) \le e^{1/t}\int_0^t \frac{1}{2s^2}2\epsilon e^{-1/s}ds = \epsilon.$$

Hence, $\lim_{t\to 0} e^{1/t}\psi(t) = 0$ which implies that $\psi(t) \le 0$. Finally, ψ is non-negative due to (i), and thus $\psi \equiv 0$. □

Theorem 4.5.5 (Rogers' type uniqueness theorem)
Let the function f verify the following conditions:

(K1) $d\left(f(t,x,y),\tilde{0}\right) \le \min\{\Gamma(q),1\}\, o\left(\frac{e^{-1/t}}{t^2}\right)$, *uniformly for positive and bounded x and y on $\mathbb{E}$*

(K2) $d(f(t,x,y),f(t,\overline{x},\overline{y})) \le \frac{\min\{\Gamma(q),1\}}{2t^{q-[q]+2}}\left[d(x,\overline{x}) + t^{(q-[q])}d(y,\overline{y})\right]$.

Then the problem has at most one solution.

The proof of this theorem is essentially based on Lemma 4.5.4.

Proof. Suppose x and y are any two solutions of (4.1) in ${}^{RL}[(i)-q]$-differentiability, and let $\phi(t)=d(x(t),y(t))$ and $\theta(t)=d(D^{q-1}x(t),D^{q-1}y(t))$, we get for $t\in[0,\eta]\subset[0,1]$

$$\begin{aligned}\phi(t)\leq&\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}d(r(s,x),r(s,y))\\ \leq&\int_0^t\frac{(t-s)^{q-1}}{2s^{q-[q]+2}}[\phi(s)+s^{q-[q]}\theta(s)]ds\\ \leq&t^{q-1}\int_0^t\frac{1}{2s^{q-[q]+2}}[\phi(s)+s^{q-1}\theta(s)]ds\\ \leq&t^{q-[q]}\int_0^t\frac{1}{2s^{q-[q]+2}}[\phi(s)+s^{q-1}\theta(s)]ds\\ \leq&t^{q-[q]}\psi(t),\end{aligned}$$

and

$$\begin{aligned}\theta(s)\leq&\int_0^t d(r(s,x),r(s,y))\\ \leq&\int_0^t\frac{\min\{\Gamma(q),1\}}{2s^{q-[q]+2}}[\phi(s)+s^{q-[q]}\theta(s)]ds\\ \leq&\int_0^t\frac{1}{2s^{q-[q]+2}}[\phi(s)+s^{q-[q]}\theta(s)]ds\\ \leq&\psi(t),\end{aligned}$$

where ψ is defined as in Lemma 4.5.4.

Also, if $\epsilon>0$, then from condition (K1) for small t, we have

$$\begin{aligned}\phi(t)\leq&\frac{t^{q-1}}{\Gamma(q)}\int_0^t d(r(s,x),r(s,y))\\ <&t^{q-1}2\epsilon\int_0^t\frac{e^{-1/s}}{s^2}ds\leq t^{q-1}e^{-1/t}2\epsilon\\ <&\epsilon t^{q-[q]}e^{-1/t}2,\end{aligned}$$

and

$$\begin{aligned}\theta(t)\leq&\int_0^t d(r(s,x),r(s,y))\\ <&2\epsilon\min\{1,\Gamma(q)\}\int_0^t\frac{e^{-1/s}}{s^2}ds\leq 2\epsilon e^{-1/t}.\end{aligned}$$

By applying Lemma 4.5.4, we obtain $d(x(t),y(t)) = 0$ for every $t \in [0,1]$, and this proves the uniqueness of the solution of FFDE (4.1) in ${}^{RL}[(i) - q]$-differentiability. The second part of the proof is almost completely similar, thus we omit it. □

4.6 Existence result

Theorem 4.6.1
Let $f \in C(\mathbb{E}_0,\mathbb{E})$ satisfy the conditions of Theorem 4.5.1. Then the successive approximations

$$x_n(t) = y_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} r(s,x_{n-1})ds \tag{4.11}$$

in the sense of ${}^{RL}[(i) - q]$-differentiability, or

$$\hat{x}_n(t) = y_0 \ominus \frac{-1}{\Gamma(q)} \int_0^t (t-s)^{q-1} r(s,x_{n-1})ds \tag{4.12}$$

in the sense of ${}^{RL}[(ii) - q]$-differentiability converge to the unique solution of FFDE (4.1).

Proof. Without loss of generality, we prove Theorem 4.6.1 for the sequence $\{x_n\}$ in the sens of ${}^{RL}[(i) - q]$-differentiability using Arzelà-Ascoli Theorem. The convergence of the sequence $\{\hat{x}_n\}$ in the sense of ${}^{RL}[(ii) - q]$-differentiability is completely similar so we omit it.
Step 1: The sequence $\{x_j\}_{j\geq 0}$ and $\{D^{q-1}x_j\}_{j\geq 0}$ are well defined and continuous and uniformly bounded on $[0,\eta]$; in fact

$$d(x_{j+1}(t),y_0) \leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} d(r(s,x_j),\tilde{0})ds,$$

and

$$d(D^{q-1}x_{j+1}(t),y_0) \leq \int_0^t d(r(s,x_j),\tilde{0})ds.$$

For $j=0$ and $t\in[0,\eta]$, we have

$$d(x_1(t),y_0)\le\frac{Mt^q}{\Gamma(q+1)}\le b\quad\text{ and }\quad d(D^{q-1}x_1(t),y_0)\le Mt\le d.$$

Moreover, for every $i\in\{0,\dots,n-1\}$ we have

$$\begin{aligned}
d(x_1^{(i)}(t),\tilde{0}) =& d(D^iI^qf(t,x_0(t),D^{q-1}x_0(t)),\tilde{0})\\
=& d(I^{q-i}f(t,x_0(t),D^{q-1}x_0(t)),\tilde{0})\\
=& \frac{1}{\Gamma(q-i)}\int_0^t(t-s)^{q-i-1}d(f(s,x_0(s),D^{q-1}x_0(s)),\tilde{0})ds\\
\le& \frac{M}{\Gamma(q-i)}\int_0^t(t-s)^{q-i-1}ds\\
\le& \frac{Mt^{q-i}}{(q-i)\Gamma(q-i)}\\
\le& \frac{Mt^{q-i}}{\Gamma(q-i+1)}.
\end{aligned}$$

By induction, the sequences $\{x_{j+1}(t)\}$ and $\{D^{q-1}x_{j+1}(t)\}$ are well-defined and uniformly bounded on $[0,\eta]$.

Step 2: We prove that the functions y and z are continuous in $[0,\eta]$, where y and z are defined by

$$\begin{aligned}
y(t) =& \limsup_{j\to\infty}\zeta_j^0(t),\\
z(t) =& \limsup_{j\to\infty}\xi_j(t),
\end{aligned}$$

such that

$$\begin{aligned}
\zeta_j^0(t) =& d(x_j(t),x_{j-1}(t)),\\
\xi_j(t) =& d(D^{q-1}x_j(t),D^{q-1}x_{j-1}(t)).
\end{aligned}$$

Let us note

$$m(t)=\sum_{i\le n-1}\limsup_{j\to\infty}\zeta_j^i(t),$$

where

$$\zeta_j^i(t)=d(x_j^{(i)}(t),x_{j-1}^{(i)}(t)).$$

For $0 \leq t_1 \leq t_2$ and for every $i \in \{0, \ldots, n-1\}$, we obtain

$$\begin{aligned}
\left|\zeta_j^i(t_1) - \zeta_j^i(t_2)\right| =& \left|d\left(x_{j+1}^{(i)}(t_1), x_j^{(i)}(t_1)\right) - d\left(x_{j+1}^{(i)}(t_2), x_j^{(i)}(t_2)\right)\right| \\
\leq& \frac{1}{\Gamma(q-i)}\left[\left|\int_0^{t_1}(t_1-s)^{q-1-i} d(r(s,x_j), r(s,x_{j-1}))ds\right.\right. \\
&\left.\left. - \int_0^{t_2}(t_2-s)^{q-1-i} d(r(s,x_j), r(s,x_{j-1}))ds\right|\right] \\
\leq& \frac{2M}{\Gamma(q-i)}\left[\left|\int_0^{t_1}((t_1-s)^{q-1-i} - (t_2-s)^{q-1-i})ds\right.\right. \\
&\left.\left. - \int_{t_1}^{t_2}(t_2-s)^{q-1-i}ds\right|\right] \\
\leq& \frac{2M}{(q-i)\Gamma(q-i)}\left[t_1^{q-i} - t_2^{q-i} + 2(t_2-t_1)^{q-i}\right] \\
\leq& \frac{4M}{\Gamma(q-i+1)}(t_2-t_1)^{q-i},
\end{aligned}$$

The right-hand side in the above inequalities is at most $\frac{4M}{\Gamma(q-i+1)}(t_2-t_1)^{q-i} + \epsilon$ for large n if $\epsilon > 0$ provided that

$$|t_2 - t_1| \leq \eta \leq \frac{4M}{\Gamma(q-i+1)}(t_2-t_1)^{q-i},$$

for every $i \leq n-1$. And since ϵ is arbitrary and t_1, t_2 can be interchangeable, we get

$$|m(t_1) - m(t_2)| \leq \sum_{i \leq n-1}\left\{\frac{4M}{\Gamma(q-i+1)}(t_2-t_1)^{q-i}\right\} \leq \frac{4M(n-1)}{\Gamma(q+1)}(t_2-t_1)^q$$

The same goes for $z(t)$, and we obtain

$$|z(t_1) - z(t_2)| \leq 2M|t_2 - t_1|.$$

These imply that $y(t)$ and $z(t)$ are continuous on $[0, \eta]$.
Step 3: We verify that the family $\{D^{q-1}j_{n+1}(t)\}$ is equicontinuous in $C^{\mathbb{F}}([0,\eta], \mathbb{E})$ and that the family $\{x_{j+1}(t)\}$ is equicontinuous in $C^{(n-1)\mathbb{F}}([0,\eta], \mathbb{E})$.

We may prove that by using condition (H2) and the definition of successive approximations (4.11) we obtain

$$\zeta_{j+1}^0(t) \leq C\int_0^t (t-s)^{q-1}\left[\zeta_j^0(s)^\alpha + s^{\alpha(q-[q])}\xi_j(s)^\alpha\right]ds,$$

and

$$\zeta_{j+1}^{i}(t) \leq C\int_0^t (t-s)^{q-i-1}\left[\zeta_j^0(s)^\alpha + s^{\alpha(q-[q])}\xi_j(s)^\alpha\right] ds.$$

As a consequence, we obtain the following estimates

$$\begin{aligned} D\left(x_{j+1}, x_j\right) \leq \sum_{i\leq n-1} C\int_0^1 (1-s)^{q-i-1}\left[|x_j(s)-x_{j-1}(s)|^\alpha\right. \\ \left. + s^{\alpha(q-[q])}|D^{q-1}x_j(s) - D^{q-1}x_{j-1}(s)|^\alpha\right] ds. \end{aligned}$$

By Arzelà-Ascoli Theorem, there exists a subsequence of integers $\{j_k\}$, such that

$$\begin{aligned} &d(x_{j_k}(t), x_{j_{k-1}}(t)) \to y(t) \text{ as } j_k \to \infty, \\ &d(D^{q-1}x_{j_k}(t), D^{q-1}x_{j_{k-1}}(t)) \to y(t) \text{ as } j_k \to \infty \end{aligned}$$

Let us note

$$\begin{aligned} m^*(t) =& \limsup_{k\to\infty} d(x_{j_k}(t) - x_{j_{k-1}}(t)), \\ z^*(t) =& \limsup_{k\to\infty} d(D^{q-1}x_{j_k}(t) - D^{q-1}x_{j_{k-1}}(t)). \end{aligned}$$

Further, if $\{d(x_j, x_{j-1})\} \to 0$ and $\{d(D^{q-1}x_j, D^{q-1}x_{j-1})\} \to 0$ as $j \to \infty$, then the limit of any successive approximation of x_n is the solution x of (4.1), which was proved to be unique in Theorem 4.5.1. It follows that a selection of subsequences is unnecessary and that the entire sequence $\{x_j\}$ converges uniformly to $x(t)$. For that, it's sufficient to show that $y \equiv 0$ and $z \equiv 0$ which will lead to m^* and z^* being null.

Setting

$$R(t) = \int_0^t \left[y(s)^\alpha + s^{\alpha(q-[q])}z(s)^\alpha\right] ds,$$

and by defining $\psi^*(t) = t^{-k}\max\{y(t), z(t)\}$, we show that $\lim_{t\to 0^+}\psi^*(t) = 0$.

Now we shall prove that $\psi^* \equiv 0$. Suppose that $\psi^*(t) > 0$ at any point in $[0,\eta]$; then there exists t_1 such that $0 < \overline{m} = \psi^*(t_1) = \max_{0\leq t\leq\eta}\psi^*(t)$. Hence, from condition (H1), we obtain

$$\overline{m} = \psi(t_1) = t_1^{-k}y(t_1) \leq \min(\Gamma(q), 1)\overline{m}t_1^{q-1+\alpha(q-[q])} < \overline{m}$$

or

$$\overline{m} = \psi(t_1) = t_1^{-k} z(t_1) \leq \min(\Gamma(q), 1)\overline{m} t_1^{\alpha(q-[q])} < \overline{m}.$$

In both cases, we end up with a contradiction. So $\psi^* \equiv 0$. Therefore, iteration (4.11) converges uniformly to the unique solution x of (4.1) on $[0, \eta]$.

$\square$

Remark 4.6.2. The presented results in this Chapter generalises and extends the work of F. Yoruk *et al.* to arbitrary order to fuzzy sets.

Conclusion and outlook

All things considered, we think that the results provided in this thesis could be considered as a contribution to the field of fractional differential equations and fuzzy fractional differential equations and even open doors to further studies of uniqueness theorems either in the crisp or the fuzzy set theory as proved by the publication of the works [94, 95]. We summarize the contributions of our results in the following points:

- Existence of positive solution to IVP and BVP.
- Study of nonlinearities depending on the fractional derivative of the solution,
- No assumption was made on the fractional derivative which it depends on.
- Extension of previous results on the uniqueness criterion along existence of solutions.
- Extension of previous fuzzy and crisp results.
- Our results englobes also results for the integer order differential equations.

The obtained results were either submitted [96] or published [94, 95], and the others are to appear in the near future. We believe that the main results of this doctoral thesis contributed to the fractional calculus in several directions and will surely contribute to its further development. As a continuation to this research, we will study the stability and controllability of such problems [92].

Apart from the apparent usage of our results in the existence and uniqueness of solutions, the results can also be employed in several other aspects: The numerical aspect, the investigation of quantitative aspect for fractional coupled systems, dynamical systems, and the study of fractional Lotka-Volterra equations are good starting problems. See for instance our paper [97], where we treat some problems on time-scales.

Matlab code

Main script

```
%clear all
%clf
%clc
t0=0;
t1=1;
h_vett=[1/10 1/20 1/50 1/100 1/200 1/400 1/500];
q=1.01;
alpha=1/2;
k=(1+alpha*(q-ceil(q)))/(1-alpha);
k=k+0.001
%k=1.7

%% test with this function the algorithm
%          f=5
%          y=5/gamma(5/2)*t.^(3/2)
%          fdefun=@(t,x) 5;
%          plot(t,feval(inline('5/gamma(5/2)*t.^(3/2)','t'),t))

%% test with this the step
%for i_h=1:length(h_vett);
%     h=h_vett(i_h);
%     [t,y]=fde12(q,fdefun,t0,t1,x0,h,q);
%     hold on
```

```
%      plot(t,y);
%     %% err(i_h,i_q)=abs(x(end)-fgl_deriv(q,x,h);
%

fdefun=@(t,x,q) (t≠0).*(x<t/(1-alpha)).*((x>0)/((t≠0).^q)+t.^
          (q.*alpha/(1-alpha))).*min(1,gamma(q)).*(q*(k-1)+1);
x0=zeros(1,ceil(q));

n=101;
[t,y]=fde12(q,fdefun,t0,t1,x0,1/100,q);
fdeff=@(s,x,t) (t-s)^(q-1)*fdefun(s,x,q)/gamma(q);

%% test with x0=0
errr=0.01; %erreur de sortie
y0=zeros(1,101); %x0 de iteration de picard
niter=1 %n iteration max
[yn1,nit1,eror1]=suitess(fdeff,y0,errr,niter);
[nit1,eror1,err(y,yn1,101)]
[yn2,nit2,eror2]=suitess(fdeff,yn1,errr,niter);
[nit2,eror2,err(y,yn2,101)]
[yn3,nit3,eror3]=suitess(fdeff,yn2,errr,niter);
[nit3,eror3,err(y,yn3,101)]
niter=20;%n iteration max
errr=-1;
[ynn,nit1,eror1]=suitess(fdeff,y0,errr,niter);
[nit1,eror1,err(y,ynn,101)]

%% test with x0=rand
y0=rand(1,101); %x0 de iteration de picard
niter=1 %n iteration max
[yn1,nit1,eror1]=suitess(fdeff,y0,errr,niter);
[nit1,eror1,err(y,yn1,101)]
[yn2,nit2,eror2]=suitess(fdeff,yn1,errr,niter);
[nit2,eror2,err(y,yn2,101)]
[yn3,nit3,eror3]=suitess(fdeff,yn2,errr,niter);
[nit3,eror3,err(y,yn3,101)]
niter=20;%n iteration max
errr=-1;
```

```
[ynn1,nit1,eror1]=suitess(fdeff,y0,errr,niter);
[nit1,eror1,err(y,ynn1,101)]

%% test with x0=xf
y0=y; %x0 de iteration de picard
niter=1 %n iteration max
[yn1,nit1,eror1]=suitess(fdeff,y0,errr,niter);
[nit1,eror1,err(y,yn1,101)]
[yn2,nit2,eror2]=suitess(fdeff,yn1,errr,niter);
[nit2,eror2,err(y,yn2,101)]
[yn3,nit3,eror3]=suitess(fdeff,yn2,errr,niter);
[nit3,eror3,err(y,yn3,101)]
niter=20;%n iteration max
errr=-1;
[ynn2,nit1,eror1]=suitess(fdeff,y0,errr,niter);
[nit1,eror1,err(y,ynn2,101)]

%% test with x0=-xf
y0=-y;
niter=1 %n iteration max
[yn1,nit1,eror1]=suitess(fdeff,y0,errr,niter);
[nit1,eror1,err(y,yn1,101)]
[yn2,nit2,eror2]=suitess(fdeff,yn1,errr,niter);
[nit2,eror2,err(y,yn2,101)]
[yn3,nit3,eror3]=suitess(fdeff,yn2,errr,niter);
[nit3,eror3,err(y,yn3,101)]
niter=20;%n iteration max
errr=-1;
[ynn3,nit1,eror1]=suitess(fdeff,y0,errr,niter);
[nit1,eror1,err(y,ynn3,101)]

%% n iteration to get errr
errr=0.01; %erreur de sortie
y0=zeros(1,101); %x0 de iteration de picard
niter=100; %n iteration max
[yn1,nit1,eror1]=suitess(fdeff,y0,errr,niter);
[nit1,eror1]
%% n iteration to get errr
```

```
errr=0.01; %erreur de sortie
y0=rand(1,101); %x0 de iteration de picard
niter=100; %n iteration max
[yn1,nit1,eror1]=suitess(fdeff,y0,errr,niter);
[nit1,eror1]

%% n iteration to get errr
errr=0.01; %erreur de sortie
y0=y; %x0 de iteration de picard
niter=100; %n iteration max
[yn1,nit1,eror1]=suitess(fdeff,y0,errr,niter);
[nit1,eror1]

%% n iteration to get errr
errr=0.01; %erreur de sortie
y0=-y; %x0 de iteration de picard
niter=100; %n iteration max
[yn1,nit1,eror1]=suitess(fdeff,y0,errr,niter);
[nit1,eror1]
```

Error estimation

```
function e=err(xn,xm,n)
    errora=0;
    e=0;
    for j=1:n
        errora=abs(xn(j)-xm(j));
        if errora>e
            e=errora;
        end
    end
end
```

Modified composite Simpson rule

```
function inte=simps(f,y,n)
%use an n even number

inte=zeros(1,n);
for j =3:2:n
h = 1/(n-1);
t=0:h:1;
inte(j)=0;
j1=floor((j-1)/2);
for i = 1:j1
    i1=2*i-1;
    i2=2*i;
    i3=2*i+1;
    fsum=[f(t(i1),y(i1),t(j)) f(t(i2),y(i2),t(j)) ...
        f(t(i3),y(i3),t(j))];
    fsum(isnan(fsum))=0;
    inte(j) = inte(j)+fsum(1)+4*fsum(2)+fsum(3);
end
end
for j =4:2:n-1
    inte(j) = (inte(j-1)+inte(j+1))/2;
end
inte = h*inte/3;
end
```

```
function in=inte(ab,r,xn)
    in=0;
    inmid=0;
    if ab≠1 then
        for i=2:ab+1
            inmid=inmid+kern(ab,i-1,r,xn);
        end
        in=(dt/2)*(0+2.*inmid-kern(ab,i-1,r,xn));
```

```
        end
end
```

Loop till one of the criteria attained

```
function [yn,nit,eror]=suitess(f,y0,e_rr,n_i)
yn1=y0+1;
yn=yn1-1;
ii=0;
while ii<n_i & err(yn1,yn,length(yn1))>e_rr
yn1=yn;
yn=simpss(f,yn1,length(y0));
ii=ii+1;
end
nit=ii;
eror=err(yn1,yn,length(yn1));
end
```

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