People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research
University May 8, 1945 Guelma

Faculty of Mathematics, Informatics and material Sciences.
Departement of Mathematics

THESIS

IN VIEW OF OBTAINING THE DIPLOMA OF A DOCTORATE IN SCIENCE

Speciality: Mathematics

Presented par

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Titled

On the solvability and optimal control of fractional differential equations

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Annee Universitaire : 2023/2024.

The PhD dissertation of **LaTeX**.

To my dear mother's soul,
MAMA
To the soul of my dear sister,
Amina
To my Professor's soul
AJROUD Nacer

Dedication

In the name of Allah the merciful

This thesis is dedicated to:
The sake of ALLAH, my Creator and my Master. My great teacher and messenger, Mohammed (May Allah bless and grant him), who taught us the purpose of life. My great parents, who never stop giving of themselves in countless ways, and my beloved husband, My beloved sister, who has always been by my side. who have supported me all the way since the beginning of my studies, My darling Fadi Amine, Sirine, Adem, and Amira. My dear Nada, Rihab, Hadile, Loya, Ahmed, Heithem, and Nor Farah, Haidar, islem, To everyone in my family and circle of friends My friends who motivate and encourage me. To those who have been deprived of their right to study and to all those who believe in the richness of learning.
♡

Naima. H

Mathematics

Mathematics rightly viewed possesses not only truth but supreme beauty.

[Bertrand Russell]

Mathematics is as much an aspect of culture as it is a collection of algorithms.

[Carl Benjamin Boyer]

Mathematics is the art of giving the same name to different things.

[Henri Poincare]

If I feel unhappy, I do mathematics to become happy. If I am happy, I do mathematics to keep happy."

[Alfred Renyi]

Mathematics is the queen of the sciences.

[Carl Friedrich Gauss]

If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.

[John von Neumann]

Acknowledgements

First of all, I thank the Almighty God for giving me the courage and the determination as well as guidance in conducting this research study, despite all the difficulties.

My supervisor, Prof. Amar DEBBOUCHE, deserves special recognition for his unwavering support of my Ph.D. study and associated research, as well as his patience, enthusiasm, and vast expertise. His advice was invaluable during the research and writing of this thesis, and I would like to extend special thanks to my co-supervisor Prof. Delfim F. M. Torres.

Id want to express my gratitude to my committee members: Prof. Nadjib Boussetila, and Prof. Zoubir Dahmani, and Prof. Tahar Haddad, and Mrs Amel Berhail. for their support, intelligent comments, and challenging questions.

I'd like to convey my gratitude to everyone who, Prof. Rajesh Dhayal, and Prof. M. LAVANYA, and Prof. B.S. VADIVOO. I regret not being able to thank everyone individually.

Contents

Abstract

The aim of this work is to develop recent methods for the solvability of some classes of initial values problems involving fractional operators and optimal controls. In particular, during the project of this doctorate thesis we present the theory of fractional calculus and control theory to prove the questions of existence results, controllability, stability and others properties for new kinds of problems which can be applicable with more accurate and better useful

keywords: Fractional integrals and derivatives; Optimal control; semi groupe theory; fixed point techniques; controllability.

Résumé

L'objectif de ce travail est de développer des méthodes récentes de résolution de certaines classes de problèmes aux valeurs initiales impliquant des opérateurs fractionnaires et des contrôles optimaux.
En particulier, au cours du projet de cette thèse de doctorat, nous présentons la théorie du calcul fractionnaire et la théorie du contrôle pour prouver les questions de résultats d'existence, de contrôlabilité, de stabilité et d'autres propriétés pour de nouveaux types de problèmes qui peuvent être appliqués avec plus de précision et de meilleure utilité.

Mots clés: Intégrales et dérivées fractionnaires; Contrôle optimal; théorie des semi-groupes; techniques du point fixe; contrôlabilité.

ملخص

الهدف من هذا العمل هو استخدام طرق حديثة مطورة لامكانية حل بعض فئات مشاكل القيم الاولية التي تتضمن عوامل التشغيل الكسري و الضوابط المثلى. على وجه الخصوص خلال مشروع اطروحة الدكتوراه هذه, نقدم نظرية حساب التفاضل و التكامل الجزئي و نظرية التحكم لاثبات اسئلة نتائج الوجود و قابلية التحكم و الاستقرار و خصائص اخرى لانواع حديدة من المشكلات التي يمكن تطبيقها بمزيد من الدقة والافادة بشكل افضل.

الكلمات المفتاحية: التكاملات الكسرية و المشتقات, التحكم الامثل, نظرية شبه المحموعة, تقنيات النقطة الثابتة.

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The List Of Works

1. N. Hakkar, R. Dhayal, A. Debbouche, D.F.M. Torres, Approximate Controllability of Delayed Fractional Stochastic Differential Systems with Mixed Noise and Impulsive Effects. Fractal Fract. 2023, 7, 104.https://doi.org/10.3390/fractalfract7020104
2. N. HAKKAR, M. Lavanya, A. Debbouche, AND B.S. Vadivoo, Nonlinear Fractional Order Neutral-type Stochastic Integro-Differential System with Rosenblatt Process - A Controllability Exploration.
   Volume48,SpecialIssue,2022,Pages68-83.10.30546/2409-4994.48.2022.6883

Notations

1. **For abbreviations and expressions**

The abbreviation	**The meaning**
s.t.	such that
iff	if and only if
RHS	right hand side
LHS	left hand side
ONB	orthogonormal basis
"i.i.d"	independent identically distributed
U M D	unconditional for martingale difference
PDEs	partial differential equations
SDEs	stochastic differential equations
SPDEs	stochastic partial differential equations
FSPDEs	fractional stochastic partial differential equations
The expression	The meaning
$x := y$ or $y := x$	x is equal to y by definition
$a \wedge b$	$\min(a, b)$
$a \vee b$	$\max(a, b)$
arg z	argument of the complex number z
Eq.$(n.m)$	an equation of number m exists in chapter n
Prb.$(n.m)$	a problem of number m exists in chapter n
IV P.$(n.m)$	an initial value problem of number m exists in chapter n
Est.$(n.m)$	an estimate of number m exists in chapter n
Cond.$(n.m)$	a condition of number m exists in chapter n

2. **For sets and functions**

The symbol	**The meaning**
$\mathbb{R}_+$	the interval $[0, +\infty)$
$\mathbb{R}_+^*$	the interval $(0, +\infty)$
$\mathbb{R}_+^d$	$\{t = (t_1, \cdots, t_d \in \mathbb{R}, s.t., t_i \geq 0, \forall i\}$
$\mathbb{N}_0$	$\mathbb{N}$ - $\{0\}$
A^*	the adjoint of the operator A
1_E	the identity operator defined on some space E
$D(A)$	domain of definition of the operator A
Γ	gamma function
1_B	the indecator function of the set B
$supp(f)$	support of the function f
(a, b), For $a < b$	an open interval
Domain D	a non empty open set
∂D	the boundary of the domain D

3. **For Stochastic analysis**

The symbol	The meaning
$(\Omega, \mathcal{F}, \mathbb{P})$	Probability space
$\mathcal{N}$	The normal law
$\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$	Normal filtration
$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$	Filtered probability space
$(\beta_t)_{t\in[0,T]}$	Brownian motion
$W := (W_t)_{t\in[0,T]}$	Wiener process
$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W)$	Stochastic basis
$\mathbb{E}(X) := \int_\Omega X(\omega)\mathrm{d}\mathbb{P}(\omega)$	Expectation of the random variable X
$L^p(\Omega, E)$,for a Banach space E	Space of all p- th integrable E-valued random variables on Ω
$\mathcal{M}_T^2$	Space of all continuous square integrable E-valued martingales

4. **For functional spaces**

The symbol	The meaning
$B(E)$	Borel σ- algebra generated by all open sets of the topological space E
$(E, \lvert\cdot\rvert_E)$	Banach space with its norm $\lvert\cdot\rvert_E$
E'	The dual space of E
$(\cdot,\cdot)_{E'\times E}$	the pairing of E and E'
$(H, \langle\cdot,\cdot\rangle_H)$	Hilbert space with its inner product $\langle\cdot,\cdot\rangle_H$
L^p	Lebesgue space, for the special case $p = 2$
C and C^m, for $m \in \mathbb{N}_0$	space of continuous functions and space of all functions of class m respectively
C_0^m, for $m \in \mathbb{N}_0$	space of functions of class m with compact support
W_p^m, $m \in \mathbb{N}$, $1 \leq p \leq \infty$	Sobolev space
W_p^s, $s \in \mathbb{R}_+^*$ - $\mathbb{N}_0$, $1 \leq p < \infty$	fractional Sobolev space
H_2^α and H_0^α, for $\alpha > 0$	fractional Sobolev space for $p = 2$ and the closure of C_0^∞ in H_2^α respectively
B_s^{pq}	Besov space
C^δ, for $\delta \in (0,1)$	Holder space
$\mathcal{L}(E_1, E_2)$	Banach space of linear bounded operators from E_1 to E_2 with its norm $\lVert\cdot\rVert_{\mathcal{L}(E_1,E_2)}$. For $E_1 = E_2$ we simply write $\mathcal{L}(E_1)$
$(HS(H_1, H_2), \lVert\cdot\rVert_{HS(H_1,H_2)})$	space of Hilbert-Schmidt operators from H_1 to H_2
HS	space of Hilbert-Schmidt operators from $L^2(0,1)$
$\mathcal{L}_\mathcal{N}(H_1, H_2)$	space of nuclear operators from H_1 to H_2
$\mathcal{S}$	Schwartz space
D_{RL}^α	Riemann-Liouville fractional derivative of order α.
$n!$	Factorial function.
$\beta(z, w)$	Beta function.

Introduction

Early in 1695, Leibniz and L'Hôpital exchanged letters in which they discussed the relevance of the derivative of order 1. Since numerous eminent mathematicians worked on this and similar issues in the years that followed, including Euler (1738), Laplace (1820), Fourier (1822), and Lagrange (1849), establishing the subject that is now known as fractional calculus.
The theory of derivatives and integrals of any order is known as such calculus (fractions, rational, irrational, complex, etc.). Since L'Hôpital specifically requested the order $n = \frac{1}{2}$, it generalizes the ideas of integer-order differentiation and n-fold integration, where the term "fractional" can be deceptive (i.e., a fraction), actually gave rise to the name of this field of mathematics. Classical analysis is well-known for its use of the integral and integer-order derivative. The fractional derivative and fractional integral are not similarly determined, however. There are numerous definitions that, in general, do not agree with one another. This is because various writers have tried to preserve certain characteristics of the conventional integer-order derivative and integral. One difficulty in this area of mathematics is that there are plainly multiple ways to define such concepts in the fractional calculus.
Fractional Brownian motion (fBm for short) is a family of Gaussian random processes that are indexed by the Hurst parameter $\hat{\mathcal{H}} \in (0,1)$. It is a self-similar stochastic process with long-range dependence ans stationary increment properties when $\hat{\mathcal{H}} > 1/2$. For more recent works on fractional Brownian motion, see [20, 37, 5, 146, 67, 42] and the references therein.

One of the most significant ideas in mathematical control theory is controllability. In both deterministic and stochastic control systems, controllability is critical. Control theory is an area of mathematics that studies how far the state of a system can be changed based on the systemes fundamental qualities and how we can act on it. For example, one might question if a solides temperature can be brought to a constant in a finite amount of time by heating and cooling only a portion of the solid. Since 1995, this problem, known as the null-controllability of the heat equation, has been solved. Controllability roughly translates to the ability to direct a dynamical control system from an arbitrary initial state to an arbitrary final state using the admissible controls. Many mathematical concepts and methods from differential geometry, functional analysis, topology, matrix analysis, theory of ordinary and partial differential equations, and theory of difference equations are used to solve controllability problems for various types of dynamical systems. The controllability of diverse classes of systems can be studied using state-space models of dynamical systems, which give a reliable and general method. On the other hand, optimal control is concerned with the problem of determining a control law for a given system that meets a predetermined optimality condition.
The optimal control theory's goals are as follows: Obtaining necessary (or possibly necessary and sufficient) conditions for the control to be an extreme (or minimum), studying the structure and properties of the equations expressing these conditions, and obtaining constructive algorithms amenable to numerical computations of the admissible controls that determine the inf (such a control is referred to as an "optimal control").Optimal control can be used to a variety of sectors, including biology, economics, ecology, engineering, finance, management, and medicine. See also [86]-[87] and the references therein.
Controllability theory for various systems with fractional derivatives and fractional has advanced significantly since the publication of research publications such as [68]-[77] and monograph [87]. This theory has formed the basis of a very active research topic since it provides a natural framework for mathematical modelling of many physical phenomena and validation of existing ones. Fractional differential equations have recently proved to be strong tools in

the modelling of many phenomena in various fields of engineering, physics, and economics. As a con-sequence, there was an intensive development of the theory of fractional differential equations. Due to this fact, the fractional order models are capable of describing more realistic situations than the integer order models. Many articles have been devoted to the existence of solutions for fractional differential equations. Existence, uniqueness, stability, controllabil-ity, and other quantitative and qualitative features of evolutionary equation solutions have re-cently gotten a lot of attention. For more information see [102, 10, 68, 100, 123, 96, 75]. Fractional differential equations are applied in a variety of fields, including fractals, chaos, electrical engineering, and medicine. In recent years, there has been a lot of progress in the field of fractional differential equations. For instance, we refer to the monographs of Abbas et al. [1], Kilbas et al. [75], Miller and Ross [99], Podlubny [123], and other documents.

This thesis is divided into four chapters.

In the first one, we collect some concepts and results for linear distances that are frequently used in this thesis, which are Sobolev fractional distances and Holder distances, and we introduce some of them. Definitions and basic results of linear factors such as the Hilbert-Schmidt factor are discussed, as is a short review of the partial half-group theory. Calculus and fractional integration and must contain some results in a classical case in a certain way and some definitions and known results about random operations. The random integrals are summed in Hilbert spaces, that is, the Wiener processes and random integrals. Mentioned here for completeness. Approximate controllability and optimal control.

In the second one, we will present Fractional Brownian motion (fBm for short) is a family of Gaussian random processes that are indexed by the Hurst parameter. It is a self-similar stochastic process with long-range dependence and stationary increment properties. For more recent works on fractional Brownian motion, see [20, 42] and the references therein. In order to describe various real-world problems in physical and engineering sciences subject to abrupt changes at certain instants during the evolution process, impulsive frac-tional differential equations have become important in recent years as mathematical models of many phenomena in both physical and social sciences. Impulsive effects begin at any arbitrary fixed point and continue with a finite time interval. The concept of controllability plays a major role in finite dimensional control theory. However, its generalization to infinite dimensions is too strong and has lim-ited applicability, while approximate controllability is a weaker concept completely adequate in applications [155].
The results of this chapter are represented in part by Manuscript entitled:

- ⋆ Hakkar N. Dhayal, R. Debbouche, A. Torres, D.F.M., Approximate Controllability of Delayed Fractional Stochastic Differential Systems with Mixed Noise and Impulsive Effects. Fractal Fract. 2023, 7, 104. `https://doi.org/10.3390/fractalfract7020104`

The third chapter, this work demonstrates nonlinear randomness of neutral order Differential system integrated with Rosenblatt process, controllability is dead The most accessible resource for studies. Our major contributions are highlighted as follows: We have developed a solution for the controllability problem of non-linear fractional order neutral type stochastic integro-differential system with Rosenblatt process.

- We take the terms in the system as a bounded linear operators instead of a matrix, which produces the same results as a matrix.

- The illustration the results on stochastic systems bounded linear opera-tors are more competent.
- We take the stochastic term as driven by the Rosenblatt process which is non-Gaussian and has the properties like self-similarity, stationarity of the increments and has long range dependence.
- We intend to bring new lights to the Rosenblatt process, since many real-life phenomena are modeled by fractional Brownian motion a only Gaussian Hermite process, when the property of Gaussianity is failed one can use Rosenblatt process.
- We define the controllability Grammian operator, which is defined by the Mittag-Leffler function to prove the controllability results.
- By employing Banach contraction principle to prove the controllability criteria instead of semigroup theory which does not applicable to obtain the results on controllability.
- We have provided a numerical example to illustrate the theory.
- Generally speaking, both the Riemann-Liouville and the Caputo frac-tional operators do not possess neither semigroup nor commutative prop-erties, which are inherent to the derivatives on integer order.

The results of this chapter are represented in part by Manuscript entitled:

⋆ N. HAKKAR, M. LAVANYA, A. DEBBOUCHE, AND B.S. VADIVOO., Nonlinear Fractional Order Neutral-type Stochastic Integro-Differential System with Rosenblatt Process-A Controllability Exploration.
`Volume48,SpecialIssue,2022,Pages68-83.10.30546/2409-4994.48.2022.6883`

The fourth chapter, in this work, we want to prove the existence of the theory of existence for light solutions using semi-operated set theory and fixed-point theories for multivalued mapping. Then, by constructing the sequence of minimizations twice, the theory of existence is in the optimal position. Also, pairs are obtained from the state control. It is worth emphasizing that we omit the unique feature of light solutions, which is a basic assumption for this. Our work improves some of the existing literature. If the Riemann-Leuvel fractions Evolution contents involve time delays, it is difficult to prove mild solutions as well as optimal control because the partial Riemann-Liouville derivative The singular is at $t = 0$. It is a valuable topic that we will study in the future.

Finally, we conclude the thesis with a conclusion and views section that summarizes the main findings and offers suggestions for future research studies on the subject.

Chapter 1

Preliminary Background

This chapter's goal is to compile ideas and findings related to many facets of functional analysis. We gather some ideas and findings on linear spaces in Section 1.1, focusing on some of the more significant ones, like Banach spaces, Hilbert spaces, and some functional spaces that are frequently utilized in this thesis. Specifically, Hölder spaces and fractional Sobolev spaces. Definitions and the fundamental findings of linear operators like the Hilbert-Schmidt operator are introduced. The generalizations of the semigroup theory are the topic of Section 1.3. We specify this in the references: [121, 132, 154, 152, 153].

1.1 On some functional aspects

The mathematician's concept of a Hilbert space generalizes the concept of Euclidean space. It is a standard space on which the internal product function is defined, in addition to that it must be a complete standard space or what is called a Banach space. This means that any Hilbert-Space ist a Banach-Space, but the reverse is not true. For example, the Q space is a regular subnormal space but not a Banach space.

Hilbert spaces enable us to generalize the methods of linear algebra and calculus used in two-dimensional and three-dimensional Euclidean spaces to spaces that may be infinite in dimension. A Hilbert space is a vector space with an inner product, and thus allows the definition of a distance and orthogonality function. In addition, the Hilbert space is a complete metric space with the distance function defined in it (in this case, the standard function), which means the availability of limits that allow the use of calculus.

Hilbert spaces appear naturally a lot in mathematics and physics, usually as an infinite-dimensional functional space. The oldest Hilbert spaces were studied by **David Hilbert**, **Erhard Schmidt** and **Frigyes Riesz** in the first decade of the twentieth century. It is an important tool in partial differential equations, quantum mechanics, Fourier analysis (which includes applications to signal processing and heat transfer) and ergodic theory (which forms the mathematical basis of thermodynamics).
John von Neumann coined the term Hilbert space for the abstract concept used in many of these diverse applications. The success of Hilbert's space methods led to the flourishing of functional analysis. Apart from classical Euclidean spaces, examples of Hilbert spaces are L^p space, sequence space, Sobolev space of generalized functions, and Hardy space of fully formed functions.

Definition 1.1.1

For a metric space is an ordered pair (E, d) where E is a set and d is a function of distance, i.e. it is a function

$$d : E \times E \to \mathbb{R}^+$$

where the following properties are combined for any three elements x, y, and z of E

1. $d(x, y) \geq 0$ *(The distance function is a non-negative function).*
2. $d(x, y) = 0$ *If and only if $x = y$.*
3. $d(x, y) = d(y, x)$.
4. $d(x, z) \leq d(x, y) + d(y, z)$. *(trigonometric inequality)*

The set E is provided with a space called a space Metric and denoted by binary (E, d).

Definition 1.1.2 (*series by Cauchy.*)

A sequence $x_1, x_2, x_3, \cdot$ in a metric space $(\mathcal{X}, d)$ is said to be Cauchy if the following is present in it: Whatever r is a definitively positive real number (that is, $r > 0$), there is a natural number N where whenever two natural numbers are greater than this number $m, n > \mathbb{N}$ it provides the following

$$d(x_m, x_n) < r$$

Definition 1.1.3 (*full space.*)

A metric space $(\mathcal{X}, d)$ is said to be complete if one of these equivalent conditions is met with the others:

1. *For every Cauchy sequence made up of points belonging to set $\mathcal{X}$, there is a limit which also belongs to the same set $\mathcal{X}$.*
2. *Every Cauchy sequence defined by $\mathcal{X}$ is a set that converges to $\mathcal{X}$ (that is, it converges to some point in $\mathcal{X}$).*

Definition 1.1.4 (*Lebesgue space.*)

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $f : \Omega \to \mathbb{R}^n$.
A measurable function f is called a p-integrable function if $|f|^p$ is integrable.
These factors specify the L^p-Lebesgue space:

$$L^p := \left\{ f\, measurable\;\; s.t. \quad |f|^p_{L^p} := \int_{\mathbb{R}} |f(x)|^p \mathrm{d}x < \infty \right\},$$

for $0 < p < \infty$ and by

$$L^\infty := \left\{ f\, measurable\; s.t. \quad |f|_{L^\infty} := ess \sup_{\mathbb{R}} |f(x)| < \infty \right\},$$

where $''ess \sup''$ means the essential supremum, i.e.

$$ess \sup f := \inf_{c \in \mathbb{R}} \{c,\, \mu(f^{-1}(c, +\infty)) = 0\},$$

with μ the Lebesgue measure.

Definition 1.1.5 (***Sobolev space.***)

*We define the **Sobolev space** for $1 \leq p \leq \infty$ and $m \in \mathbb{N}$,:*

$$\mathcal{W}_p^m(\Omega) := \left\{ f \in L^p, \quad s.t. \quad |f|^p_{\mathcal{W}_p^m} := \sum_{k=0}^{m} |\mathcal{D}^k f|^p_{L^p} < \infty \right\},$$

where $\mathcal{D}^k f$ is the derivative of f of order k in the distributional sense, that is, for all $\varphi \in C_0^k$,

$$\langle \mathcal{D}^k f \cdot \varphi \rangle = (-1)^k \langle f, d^k \varphi \rangle,$$

with $d^k\varphi$ means the classical derivative of order k of φ.

Definition 1.1.6 (***Spaces of continuous functions.***)

*The **space of continuous function** is defined by:*

$$\mathcal{C} := \{ f \text{ bounded and continuous}, s.t. \, |f|_{\mathcal{C}} := \sup_{\mathbb{R}} |f(x)| < \infty \}.$$

Definition 1.1.7 (***spaces for differentiable functions of order m.***)

*Let $m \in \mathbb{N}$. The definition of the **order** m **spaces of differentiable functions** is:*

$$\mathcal{C}^m := \{ f \in \mathcal{C} \, s.t. \, d^\alpha f \in \mathcal{C}, \text{ for all } \alpha \leq m \},$$

endowed with the norm

$$|f|_{\mathcal{C}^m} := \sum_{\alpha \leq m} |d^\alpha f|_{\mathcal{C}},$$

where d^α means the classical derivative of order α, with the convention $\mathcal{C}^0 = \mathcal{C}$. The notation $\mathcal{C}_0^m$ is reserved for the space of all functions in $\mathcal{C}^m$ with compact support.

Definition 1.1.8 (***Hölder spaces.***)

*For $\delta \in (0,1)$, we define the **Hölder space** by*

$$\mathcal{C}^\delta := \left\{ f \in \mathcal{C} \, s.t. \, |f|_{\mathcal{C}^\delta} = |f|_{\mathcal{C}} + \sup_{x,y \in \mathbb{R}, x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\delta} < \infty \right\}.$$

Definition 1.1.9 (***fixed point theory.***)

If $(\mathcal{X}, d)$ is complete metric space, $T : \mathcal{X} \to \mathcal{X}$ with $d(f(x), f(y)) \leq kd(x,y)$ and $k < 1$ then $\exists! x \in \mathcal{X}$ such that $T(x) = x$.

1.1.1 Basic notions and some useful results

Definition 1.1.10 (*Bounded operator.*)

Let $A : D(A) \subseteq \mathcal{X} \to \mathcal{Y}$ be a linear operator. Then, we say that A is bounded if there exists $C > 0$ s.t.

$$|A(x)|_{\mathcal{Y}} \leq C|x|_{\mathcal{X}}, \ \forall x \in D(A).$$

Definition 1.1.11

We denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ for the Banach space of all linear ***bounded*** *operators defined from $\mathcal{X}$ to $\mathcal{Y}$ endowed by the norm*

$$\parallel A \parallel_{\mathcal{L}(\mathcal{X},\mathcal{Y})} := \sup\{|x|_{\mathcal{X}}^{-1}|A(x)|_{\mathcal{Y}}, \ x \in \mathcal{X}, \ x \neq 0\}.$$

If $\mathcal{X} = \mathcal{Y}$, we write $\mathcal{L}(\mathcal{X}, \mathcal{Y}) = \mathcal{L}(\mathcal{X})$. Moreover, if $\mathcal{Y} = \mathbb{R}$ we call A a ***linear bounded functional*** *on $\mathcal{X}$. We denote the collection of all such functionals by $\mathcal{X}'$, which is the dual space of $\mathcal{X}$. The symbols $\parallel \cdot \parallel_{\mathcal{X}'}$ and $(\cdot, \cdot)_{\mathcal{X}',\mathcal{X}}$ denote the norm in $\mathcal{X}'$ and the duality (pairing) of $\mathcal{X}'$ and $\mathcal{X}$ respectively.*

Definition 1.1.12 (*Compact operatorr.*)

An operator $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is said to be compact if for all bounded subset $B \subset \mathcal{X}$, the closure of $A(B)$ is compact.

Definition 1.1.13 (*Closed operator.*)

We say that, a linear operator A: $D(A) \subseteq \mathcal{X} \to \mathcal{Y}$ is closed if its graph is a closed subspace of $\mathcal{X} \times \mathcal{Y}$.

Theorem 1.1.1

Every linear bounded operator A on $\mathcal{X}$ satisfies $D(A) = \mathcal{X}$. Moreover, A is closed.

Definition 1.1.14 (*Symmetric operator.*)

A densely defined linear operator $A : D(A) \subseteq \mathcal{U} \to \mathcal{U}$ is said to be ***symmetric*** *if*

$$\langle Au, v\rangle_{\mathcal{U}} = \langle u, Av\rangle_{\mathcal{U}}, \ \forall u, v \in D(A).$$

where $\langle \cdot, \cdot \rangle_{\mathcal{U}}$ denotes the inner product in $\mathcal{U}$.

Lemma 1.1.1 (*[44]*)

Let $A : D(A) \subseteq \mathcal{U} \to \mathcal{H}$ s.t. $D(A)$ is dense in $\mathcal{U}$. Then, A admits a closed operator A^* called the ***adjoint****, which is defined on $D(A^*)$ into $\mathcal{U}$ where*

$$D(A^*) := \{v \in \mathcal{H}, \ s.t. \, u \in D(A), u \to \langle Au, v\rangle_{\mathcal{H}} \text{is continuous}\}$$

such that for all $u \in D(A)$ and all $v \in D(A^*)$ it holds

$$\langle u, A^*v\rangle_{\mathcal{U}} = \langle Au, v\rangle_{\mathcal{H}}.$$

Definition 1.1.15 (*Self-adjoint operator.*)

Let $A : D(A) \subseteq \mathcal{U} \to \mathcal{U}$ be a densely defined linear operator. Then, we say that A is ***self-adjoint*** *if $D(A) = D(A^*)$ and $A = A^*$.*

Corollaire 1.1 (*[169]*)

Let $(A, D(A))$ be a symmetric operator on the Hilbert space $\mathcal{U}$. If $D(A) = \mathcal{U}$, then A is self-adjoint.

Next, we present a valuable finding pertaining to the spectrum of a linear, self-adjoint, non-negative operator. The explanation is that defining such a spectrum first is necessary.

Definition 1.1.16 (*Resolvent set of an operator.*)

Let the operator $A \in \mathcal{L}(\mathcal{U})$. The resolvent set of A denoted by $\rho(A)$ is defined by

$$\rho(A) := \{\lambda \in \mathbb{C}\, s.t.(A - \lambda I_{\mathcal{U}}) \text{ is inversible}\}$$

Definition 1.1.17 (*Spectrum of an operator.*)

Let the operator $A \in \mathcal{L}(\mathcal{U})$. The spectrum of A, denoted by $\rho(A)$ is the complement of the resolvent set in $\mathbb{C}$. The spectrum of A is subdivided as follows

Definition 1.1.18

Let the operator $A \in \mathcal{L}(\mathcal{U})$.

1. *The* ***discrete spectrum*** *of A consists of all $\lambda \in \rho(A)$ s.t. $(A - \lambda I_{\mathcal{U}})$ is not one-to-one. In this case λ is called an* ***eigenvalue*** *of A.*
2. *The* ***continuous spectrum*** *of A consists of all $\lambda \in \rho(A)$ s.t. $(A - \lambda I_{\mathcal{U}})$ is one-to-one but not onto and range $(A - \lambda I_{\mathcal{U}})$ is dense in $\mathcal{U}$.*
3. *The* ***residual spectrum*** *of A consists of all $\lambda \in \rho(A)$ s.t. $(A - \lambda I_{\mathcal{U}})$ is one-to-one but not onto and range $(A - \lambda I_{\mathcal{U}})$ is not dense in $\mathcal{U}$.*

Lemma 1.1.2 (*[150]*)

Let A be a linear (not necessarily bounded), self-adjoint and nonnegative operator defined on $D(\mathcal{U}) \subseteq \mathcal{U}$, which has eigenvalues $\{\mu_j\}_{j=1}^{N}$, for $1 < N \leq \infty$ corresponding to a basis of orthogonormal eigenfunctions $\{\varphi_j\}_{j=1}^{N}$. Then, for an arbitrary function $\mathcal{G}$ defined on the spectrum $\sigma(A) = \{\mu_j\}_{j=1}^{N}$ of A, it holds

$$\mathcal{G}(A)v = \sum_{j=1}^{N} \mathcal{G}(\mu_j)\langle v, \varphi_j\rangle_{\mathcal{U}\varphi_j}, \ \forall v \in \mathcal{U}.$$

and

$$\| \mathcal{G}(A) \|_{\mathbb{L}(\mathcal{U})} = \sup_{1 \leq j \leq N} |\mathcal{G}(\mu_j)|.$$

Definition 1.1.19 (*Hilbert-Schmidt operator.*)

Let $(\epsilon_n)_{n\in\mathbb{N}_0}$, be an ONB of $\mathcal{U}$. An operator $A \in \mathcal{L}(\mathcal{U},\mathcal{H})$ is said to be ***Hilbert-Schmidt*** *if*

$$\sum_{n\in\mathbb{N}_0} |A\epsilon_n|^2_{\mathcal{H}} < +\infty.$$

We denote by $\mathcal{HS}(\mathcal{U},\mathcal{H})$ the set of all Hilbert-Schmidt operators from $\mathcal{U}$ to $\mathcal{H}$. In the special case $\mathcal{U} = \mathcal{H}$ we shortly write $\mathcal{HS}(\mathcal{U},\mathcal{H}) = \mathcal{HS}(\mathcal{U})$. The definition of a Hilbert-Schmidt operator and the induced Hilbert-Schmidt norm in $\mathcal{HS}(\mathcal{U},\mathcal{H})$

$$\| A \|_{\mathcal{HS}(U,H)} := \left(\sum_{n\in\mathbb{N}_0} |A\epsilon_n|^2_H \right)^{\frac{1}{2}},$$

are independent of the choice of the basis $(\epsilon_n)_{n\in\mathbb{N}_0}$.

Proposition 1.1.1 (*[127]*)

Let $\mathcal{Q} \in \mathcal{L}(\mathcal{U})$ be a symmetric and nonnegative operator. Then, there exists a unique symmetric and nonnegative operator $\mathcal{Q}^{\frac{1}{2}} \in \mathcal{L}(\mathcal{U})$ satisfies $\mathcal{Q}^{\frac{1}{2}} \circ \mathcal{Q}^{\frac{1}{2}} = \mathcal{Q}$. Moreover, if $\mathcal{Q}$ with finite trace, then $\mathcal{Q}^{\frac{1}{2}} \in \mathcal{HS}(\mathcal{U})$, s.t. $\| \mathcal{Q}^{\frac{1}{2}} \|^2_{\mathcal{HS}} = tr\mathcal{Q}$ and for all $\psi \in \mathcal{L}(\mathcal{U},\mathcal{H})$ it holds $\psi \circ \mathcal{Q}^{\frac{1}{2}} \in \mathcal{HS}(\mathcal{U},\mathcal{H})$.

Corollaire 1.2

Let $\mathcal{Q} \in \mathcal{L}(\mathcal{U})$ be a symmetric and nonnegative operator and $\{\epsilon_n,\, n \in \mathbb{N}\}$ be an ONB of $\mathcal{U}$, consisting of eigenvectors of $\mathcal{Q}$ with corresponding eigenvalues $\{\lambda_n,\, n \in \mathbb{N}\}$. Then, the operator $\mathcal{Q}^{\frac{1}{2}}$ admits the family $\{(\epsilon_n, \lambda_n^{\frac{1}{2}}),\, n \in \mathbb{N}\}$ as an eigenpairs.

Proof : Let $\{\epsilon_n,\, n \in \mathbb{N}\}$ be an ONB of $\mathcal{U}$, consisting of eigenvectors of $\mathcal{Q}$ with corresponding eigenvalues $\{\lambda_n,\, n \in \mathbb{N}\}$. Let $A \in \mathcal{L}(\mathcal{U})$ defined by $A\epsilon_n := \lambda_n^{\frac{1}{2}}\epsilon_n$, for any $n \in \mathbb{N}$. The operator A is symmetric. Indeed, for all $u, v \in \mathcal{U}$ we have

$$\begin{aligned}
\langle Au, v\rangle_{\mathcal{U}} &= \langle A \sum_{n\in\mathbb{N}} \langle u, \epsilon_n\rangle_{\mathcal{U}}\epsilon_n, \sum_{m\in\mathbb{N}} \langle v, \epsilon_m\rangle_{\mathcal{U}}\epsilon_m\rangle_{\mathcal{U}} \\
&= \sum_{n,m\in\mathbb{N}} \langle u, \epsilon_n\rangle_{\mathcal{U}}\langle v, \epsilon_m\rangle_{\mathcal{U}}\langle A\epsilon_n, \epsilon_m\rangle_{\mathcal{U}}. \\
&= \sum_{n,m\in\mathbb{N}} \langle u, \epsilon_n\rangle_{\mathcal{U}}\langle v, \epsilon_m\rangle_{\mathcal{U}}\langle \lambda_n^{\frac{1}{2}}\epsilon_n, \epsilon_m\rangle_{\mathcal{U}} \\
&= \sum_{n\in\mathbb{N}} \lambda_n^{\frac{1}{2}}\langle u, \epsilon_n\rangle_{\mathcal{U}}\langle v, \epsilon_m\rangle_{\mathcal{U}}.
\end{aligned}$$

By the same manner it holds that

$$\langle u, Av\rangle_{\mathcal{U}} = \sum_{n\in\mathbb{N}} \lambda_n^{\frac{1}{2}}\langle u, \epsilon_n\rangle_{\mathcal{U}}\langle v, \epsilon_n\rangle_{\mathcal{U}}.$$

In Addition, A is nonnegative since we can get easily for all $u \in \mathcal{U}$,

$$\langle Au, u\rangle_{\mathcal{U}} = \sum_{n\in\mathbb{N}} \lambda_n^{\frac{1}{2}}\langle u, \epsilon_n\rangle^2_{\mathcal{U}} \geq 0.$$

Moreover, for any $n \in \mathbb{N}$ we have

$$(A \circ A)_{\epsilon_n} = A(\lambda_n^{\frac{1}{2}}\epsilon_n) = \lambda_n^{\frac{1}{2}} A(\epsilon_n) = \lambda_n \epsilon_n = \mathcal{Q}\epsilon_n.$$

Consequently, for all $u \in \mathcal{U}$ it holds $(A \circ A)_u = \mathcal{Q}u$. Thus, by virtue of Proposition 1.44 we obtian that $A = \mathcal{Q}^{\frac{1}{2}}$. ■

The following proposition introduces a significant linear subspace of $\mathcal{U}$, namely the image of $\mathcal{Q}^{\frac{1}{2}}$, which, when fitted with a properly selected inneres Produkt, is also a separable Hilbert space.

Proposition 1.1.2 (*[126] and [127]*)

Let $\mathcal{Q} \in \mathcal{L}(\mathcal{U})$ be a symmetric, nonnegative and finite trace operator and let $\{\epsilon_n, n \in \mathbb{N}\}$ be an ONB of $\mathcal{U}$, consisting of eigenvectors of $\mathcal{Q}$ with corresponding eigenvalues $\{\lambda_n, n \in \mathbb{N}\}$. We define $\mathcal{S} := \{n \in \mathbb{N}, \lambda_n > 0\}$ the index set of non-zero eigenvalues. Then, the space $\mathcal{U}_0 := \mathcal{Q}^{\frac{1}{2}}(\mathcal{U})$ defined by

$$\mathcal{U}_0 = \left\{ u \in ker(\mathcal{Q}^{\frac{1}{2}})^{\perp}, \sum_{n \in S} \lambda_n^{-1} \langle u, \epsilon_n \rangle_{\mathcal{U}}^2 < +\infty \right\},$$

is a Hilbert space endowed with the following inner product

$$\begin{aligned} \langle u, v \rangle_{\mathcal{U}_0} &:= \langle \mathcal{Q}^{-\frac{1}{2}} u, Q^{-\frac{1}{2}} v \rangle_{\mathcal{U}} \\ &= \sum_{n \in \mathcal{S}} \lambda_n^{-1} \langle u, \epsilon_n \rangle_{\mathcal{U}} \langle v, \epsilon_n \rangle_{\mathcal{U}}, \ \forall u, v \in \mathcal{U}_0, \end{aligned}$$

where $\mathcal{Q}^{-\frac{1}{2}}$ is the pseudo-inverse of $\mathcal{Q}^{\frac{1}{2}}$. In addition, the space $\mathcal{U}_0$ admits the family $\{\lambda_n^{\frac{1}{2}}\epsilon_n, n \in \mathcal{S}\}$ as an ONB.

Corollaire 1.3

For any $A \in \mathcal{HS}(\mathcal{U}_0, \mathcal{H})$, it holds

$$\| A \|_{\mathcal{HS}(\mathcal{U}_0,\mathcal{H})} = \| A \circ \mathcal{Q}^{\frac{1}{2}} \|_{\mathcal{HS}(\mathcal{U}_0,\mathcal{H})} .$$

Moreover, let the space $\mathcal{L}_0(\mathcal{U}, \mathcal{H}) := \{A_{|\mathcal{U}_0}, \ A \in \mathcal{L}(\mathcal{U}, \mathcal{H})\}$. Then, we have $\mathcal{L}_0(\mathcal{U}, \mathcal{H}) \subset \mathcal{HS}(\mathcal{U}_0, \mathcal{H})$.

Proof: Let $A \in \mathcal{HS}(\mathcal{U}_0, \mathcal{H})$. Then,

$$\| A \|_{\mathcal{HS}(\mathcal{U}_0,\mathcal{H})} = \sum_{n \in \mathcal{S}} \left| A(\lambda_n^{\frac{1}{2}}\epsilon_n) \right|_{\mathcal{H}}^2 .$$

as $\{\mathcal{Q}^{\frac{1}{2}}\epsilon_n, \ n \in \mathcal{S}\}$ is on ONB of $\mathcal{U}_0$ and $\mathcal{Q}^{\frac{1}{2}}\epsilon_n = 0_{\mathcal{U}}$ for all n not in $\mathcal{S}$ yields,

$$\| A \|_{\mathcal{HS}(\mathcal{U}_0,\mathcal{H})} = \sum_{n \in \mathcal{S}} \left| A \circ \mathcal{Q}^{\frac{1}{2}}(\epsilon_n) \right|_{\mathcal{H}}^2 + \sum_{nnotin\mathcal{S}} \left| A \circ \mathcal{Q}^{\frac{1}{2}}(\epsilon_n) \right|_{\mathcal{H}}^2 = \left\| A \circ \mathcal{Q}^{\frac{1}{2}} \right\| \mathcal{HS}(\mathcal{U}_0, \mathcal{H}).$$

Moreover, it holds

$$\| A \|_{\mathcal{HS}(\mathcal{U}_0,\mathcal{H})} \leq \| A \|_{\mathcal{L}(\mathcal{U},\mathcal{H})} \| \mathcal{Q}^{\frac{1}{2}} \|_{\mathcal{HS}(\mathcal{U})} .$$

The fact that $\| \mathcal{Q}^{\frac{1}{2}} \|_{\mathcal{HS}(\mathcal{U})}^2 := \sum_{n \in \mathcal{S}} \| \mathcal{Q}^{\frac{1}{2}}\epsilon_n \|_{\mathcal{U}}^2 = tr\mathcal{Q} < \infty$ leads to $A \in \mathcal{HS}(\mathcal{U}_0, \mathcal{H})$. ■

1.1.2 Laplace operator

This part, we deal with the **Laplace operator** (or Laplacian). The Laplacian is a differential operator represents the simplest elliptic operators occur in differential equations that describe many physical phenomena, such as the diffusion equation for heat and fluid flow. It is denoted by Δ and is given in the d-dimensional case by

$$\Delta u(x) := \sum_{i=1}^{d} \frac{\partial^2 u(x)}{\partial x_i^2},\ x \in D \subseteq \mathbb{R}^d,$$

Proposition 1.1.3 (*[21]*)

The Laplacian $A : D(A) \to \mathcal{L}^2(D)$, is unbounded, nonnegative and self adjoint operator.

Proposition 1.1.4 (*[149]*)

The Laplacian $A : D(A) \to \mathcal{L}^2(D)$, is an isomorphism, its inverse A^{-1} is self-adjoint and compact on $\mathcal{L}^2(D)$.

1.1.3 Fractional Laplacian

The presence of the long range interactions appear in various applications like nonlocal heat conduction allows the nonlocal diffusion operators to arise to replace the standard Laplace operator. The new operators act by a global integration with respect to a singular kernel instead of acting by pointwise differentiation, in that way the nonlocal character of the process is preserved. The **fractional Laplacian** denoted here by $A_\alpha := (-\Delta)^{\frac{\alpha}{2}}$, for $\alpha > 0$ is one of the famous nonlocal diffusion operators. We can find in the literature many definitions of A_α which reflects its extensive use in applications. Throughout this section we let $\alpha \in (0, 2]$.

Fractional Laplacian on $\mathbb{R}$

The fractional Laplacian can be defined in several equivalent ways in the whole space $\mathbb{R}$, see for example [?]. However, when these definitions are restricted to bounded domains, the associated boundary conditions lead to different operators. Here, we introduce two equivalent definitions of the fractional Laplacian, the first is represented via Fourier trans-form and its inverse, whereas the second is based on the singular integral representation.

Definition 1.1.20 (***Pseudo-differential representation.***)

The fractional Laplacian A_α is defined as a pseudo-differential operator,

$$A_\alpha u(x) := \mathcal{F}^{-1}(|\xi|^\alpha \mathcal{F}(u(x), \xi), x), \tag{1.1}$$

where $u \in \mathcal{L}^p(\mathbb{R})$, for $p > 1$.

Definition 1.1.21 (***Singular integral representation.***)

We define the fractional Laplacian A_α as a singular integral operator

$$A_\alpha u(x) := \mathcal{C}_\alpha \lim_{r \to 0^+} \int_{\mathbb{R} \quad B(x,r)} k_\alpha(x, y)(u(x) - u(y))\mathrm{d}y. \tag{1.2}$$

for any $u \in \mathcal{S}$, where $K_\alpha(x, y) := |x - y|^{-(\alpha+1)}$ for any $x \in \mathbb{R}$ and any $y \in \mathbb{R}\backslash B(x, r)$ with

$B(x,r)$ is the open ball of center x and radius r, and $\mathcal{C}_\alpha := \dfrac{\alpha 2^{\alpha-1}\Gamma(\frac{\alpha+1}{2})}{\pi^{\frac{1}{2}}\Gamma(1-\frac{\alpha}{2})}$ is a constant whith Γ is the gamma function.

Theorem 1.1

The two defintions Identity. 1.1 and Identity. 1.2 of the fractional Laplacian A_α are equivalent.

1.2 Fractional calculus

Fractional calculus is a theory of integrals and derivatives of arbitrary real or even complex orders. It is a generalization of the classical calculus and therefore preserves many of its basic properties. Fractional calculus was first mentioned in a letter from L'Hospital to Leibniz in 1695. In this letter, L'Hospital inquires about Leibniz's essay from 1646, in which he defines the derivative of order n of a function f with $n \in \mathbb{N}$. When L'Hospital asks what happens if $n = \dfrac{1}{2}$, Leibniz says, "This leads to a conundrum from which we shall one day extract valuable conclusions ". Many mathematicians have studied the issue since its discovery, with the goal of generalizing the findings established for integer-order derivatives to the case of arbitrary-order derivatives. Fractional calculus is the name given to the theory of arbitrary order integrals and deriva-tives, which unifies and generalizes integer-order differentiation and n-fond integration. In other words, fractional derivatives and integrals can be considered as an interpolation of the infinite sequence,[123]

$$\cdots, \int_a^t \int_a^{\tau_1} f(\tau_2)\mathrm{d}\tau_2\mathrm{d}\tau_1,\ \int_a^t f(\tau_1)\mathrm{d}\tau_1,\ f(t),\ \frac{\mathrm{d}f(t)}{\mathrm{d}t},\ \frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2}, \cdots$$

of the classical n fold integrals and n fold derivatives. Let's review some fundamental fractional calculus definitions and results. We'll go through the definitions and desired outcomes that will help us introduce integral and fractional derivatives, as well as solve our diffusion and fractional wave equations. For more information, look up the references [75, 99, 123].

1.2.1 Special functions

This section is about the collection of functions we'll use in fractional theory. To begin, the Gamma function will be defined as follows:

Gamma function

Definition 1.2.1

The Gamma function, denoted by $\Gamma(z)$ is a generalization of the factorial function $n!$, i.e.,

$$\Gamma(n) = (n-1)! \quad \forall n \in \mathbb{N}.$$

For complex arguments with positive real part it is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \qquad \mathcal{R}e(z) > 0.$$

This function has the following essential results:

Proposition 1.2.1

For a complex argument z with positive real part $\mathcal{R}e(z) > 0$. So we have the following result:

$$\Gamma(z+1) = z\Gamma(z).$$

Some of the most important values are

$$\begin{aligned}\Gamma(1) = \Gamma(2) &= 1,\\ \Gamma\left(\frac{1}{2}\right) &= \sqrt{\pi},\\ \Gamma\left(n+\frac{1}{2}\right) &= \frac{\sqrt{\pi}(2n-1)!}{2^n}, \ \forall n \in \mathbb{N}.\end{aligned}$$

Beta function

Definition 1.2.2

The Beta function is defined by the integral

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\mathrm{d}t, \quad \mathcal{R}e(z) > 0, \quad \mathcal{R}e(w) > 0.$$

The Beta function is used sometimes for convenience to replace a combination of Gamma function. This relation between the Gamma function and Beta function is given by (see [52])

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$$

It should also be mentioned that the Beta function is symmetric, i.e.

$$B(z,w) = B(w,z).$$

The complementary error function (erfc)

Definition 1.2.3

The complementary error function is an entire function, defined as [123]

$$erfc(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}dt.$$

Special values of the complementary error function are

$$\begin{aligned}\mathrm{erfc}(-\infty) &= 2,\\ \mathrm{erfc}(0) &= 1,\\ \mathrm{erfc}(+\infty) &= 0.\end{aligned}$$

The following relations are interesting to be mentioned

$$\begin{aligned} \operatorname{erfc}(-x) &= 2 - \operatorname{erfc}(x), \\ \int_0^\infty \operatorname{erfc}(x)\mathrm{d}x &= \frac{1}{\sqrt{\pi}}, \\ \int_0^\infty \operatorname{erfc}^2(x)\mathrm{d}x &= \frac{2-\sqrt{2}}{\sqrt{\pi}}. \end{aligned}$$

The Mittag-Leffler function

While the Gamma function is a generalization of the factorial function, the Mittag-Leffler function is a generalization of the exponential function

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)}.$$

First introduced as a one parameter function by the series [123]

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad z, \alpha \in \mathbb{C},\ \mathcal{R}e(\alpha) > 0.$$

Later, the two parameter generalization is introduced by *Agarwal*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad z, \alpha, \beta \in \mathbb{C},\ \mathcal{R}e(\alpha) > 0,\ \mathcal{R}e(\beta) > 0,$$

which is of great importance for the fractional calculus. It is called two parameter function of Mittag-Leffler type. Some of its interesting values are [123]

$$\begin{aligned} E_{1,1}(z) &= e^z, \\ E_{2,1}(z^2) &= \cosh(z), \\ E_{2,2}(z^2) &= \frac{\sinh(z)}{z}, \\ E_{\alpha,2}(z) &= E_\alpha(z), \\ E_{\frac{1}{2},1}(z) &= e^{z^2}\operatorname{erfc}(-z). \end{aligned}$$

This function has the following essential results:

Proposition 1.2.2

For a complex argument z with $\mathcal{R}e(z) > 0$,we have the following result:

$$\begin{aligned} E_{\alpha,\beta}(z) &= zE_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)}, \\ \frac{\mathrm{d}}{\mathrm{d}z}E_{\alpha,\beta}(z) &= \frac{1}{\alpha z}\Big[E_{\alpha,\beta-1}(z) + (\beta-1)E_{\alpha,\beta}(z)\Big]. \end{aligned}$$

We'll need to build estimates in order to illustrate the uniqueness of each solution in the subsequent sections. We'll use the following two outcomes to do so:

Lemma 1.2.1

For positive integers m, λ and α, we have

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}E_{\alpha,1}(-\lambda z^{\alpha}) = -\lambda z^{\alpha-n}E_{\alpha,\alpha-n+1}(-\lambda z^{\alpha}), \quad z>0,$$
$$\frac{\mathrm{d}}{\mathrm{d}z}(zE_{\alpha,2}(-\lambda z^{\alpha})) = E_{\alpha,1}(-\lambda z^{\alpha}), \quad z>0.$$

As well as

Theorem 1.2.1

Let $0<\alpha<2$, β is an arbitrary real , and we assume that μ is such that

$$\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}.$$

Then there exists a constant $C = C(\alpha,\beta,\mu) > 0$ such that

$$|E_{\alpha,\beta}(z)| \leq \frac{C}{1+|z|}, \quad \mu \leq |\arg(z)| \leq \pi.$$

The definition of the generalized Mittag-Leffler function is now given.

Definition 1.2.4

Let $\alpha,\beta,\rho \in \mathbb{C}$ such as $\mathcal{R}e(\alpha)>0$ and $\mathcal{R}e(\beta)>0$. The generalized Mittag-Leffler function is thus defined as follows:

$$\varsigma^{\rho}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{(\rho)_n z^n}{\Gamma(\alpha n+\beta)n!}, \quad \forall z \in \mathbb{C},$$

where

$$(\rho)_n = \rho(\rho+1)...(\rho+n-1).$$

Remark 1.2.1

Note that when $\rho = 1$ we have

$$\varsigma^{\rho}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z).$$

We'll need the following Lemma in the sequel:

Lemma 1.2.2

Let $\alpha,\beta,\rho \in \mathbb{C}$ such as $\mathcal{R}e(\alpha)>0$ and $\mathcal{R}e(\beta)>0$. Then, we have

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}\varsigma^{\rho}_{\alpha,\beta}(z) = (\rho)_n\varsigma^{\rho+n}_{\alpha,\beta+\alpha n}(z), \quad z \in \mathbb{C}, \ n \in \mathbb{N},$$
$$\alpha\rho\varsigma^{\rho+1}_{\alpha,\beta}(z) = (1+\alpha\rho-\beta)\varsigma^{\rho}_{\alpha,\beta}(z) + \varsigma^{\rho}_{\alpha,\beta-1}(z), \quad z \in \mathbb{C}.$$

We utilize the Laplace transform to solve our fractional differential equations, just as we did with integer differential equations. As a result, we provide the following definition:

Definition 1.2.5

Let $f : \mathbb{R}^+ \to \mathbb{R}$. The Laplace transform of function is defined by:

$$(\mathcal{L}f)(s) = \mathcal{L}[f(t)](s) = \hat{f}(s) := \int_0^\infty \exp(-st) f(t)\mathrm{d}t, \quad s > 0.$$

On occasion, we will run across transforms of the form,

$$H(s) = F(s)G(s),$$

that can't be dealt with easily using partial fractions. We would like a way to take the inverse transform of such a transform. We can use a convolution integral to do this.

Definition 1.2.6

If $f(t)$ and $g(t)$ are piecewise continuous function on $[0, +\infty]$ then the convolution integral of $f(t)$ and $g(t)$ is,

$$(f \star g)(t) = \int_0^t f(t-s)g(s)\mathrm{d}s.$$

A nice property of convolution integrals is

$$(f \star g)(t) = (g \star f)(t).$$

Or,

$$\int_0^t f(t-s)g(s)\mathrm{d}s = \int_0^t f(s)g(t-s)\mathrm{d}s.$$

The following fact will allow us to take the inverse transforms of a product of transforms.

$$\mathcal{L}\{f \star g\}(t) = F(s)G(s), \quad \mathcal{L}^{-1}F(s)G(s) = \{f \star g\}(t).$$

Lemma 1.2.3

Let $\alpha, \beta, \rho \in \mathbb{C}$ such as $\mathcal{R}e(\alpha) > 0$, $\mathcal{R}e(\rho) > 0$ and $\mathcal{R}e(\beta) > 0$. Then, we have

$$\mathcal{L}^{-1}\left[\frac{s^{\rho-1}}{s^\alpha + as^\beta + b}; z\right] = t^{\alpha-\rho} \sum_{k=0}^{\infty} (-a)^k z^{k(\alpha-\beta)} \varsigma^{k+1}_{\alpha,\alpha+(\alpha-\beta)k-\rho+1}(-bz^\alpha),$$

where $|\frac{as^\beta}{s^\alpha + b}| < 1$. We also assume that the preceding equality's series is convergent.

1.2.2 Riemann-Liouville fractional integral

Calculations of integrals and derivatives of arbitrary real or complex order are referred to as "fractional calculations." In this thesis, we are only concerned with Riemann-Liouville and Caputo derivatives.ă

Definition 1.2.7 (*See [96]*)

Cauchy's formula for repeated integration is given by

$$I^n f(t) := \int_a^t \int_a^{\tau_1} \cdots \int_a^{\tau_{n-1}} f(\tau)\mathrm{d}\tau \cdots \mathrm{d}\tau_2 \mathrm{d}\tau_1$$

$$= \frac{1}{(n-1)!}\int_a^t f(\tau)(t-\tau)^{n-1}\mathrm{d}\tau, \ \forall n \in \mathbb{N}_0, a, t \in \mathbb{R}, t > 0.$$

If n is substituted by a positive real number α and $(n-1)!$ by its generalization $\Gamma(\alpha)$ a formula for fractional integration is obtained.

Definition 1.2.8

The fractional operator

$$I^\alpha f(t) := \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s)\mathrm{d}s, \quad t > a, \quad \alpha > 0.$$

is referred to as Riemann-Liouville fractional integral of order α.

Proposition 1.2.3

- *By convention*

$$I^0 f(t) := f(t), \ \textit{i.e.}, \ I^0 := I \ \textit{is the identity operator.}$$

- *The linearity*

$$I^\alpha(\lambda f(t) + g(t)) = \lambda I^\alpha f(t) + I^\alpha g(t), \ \alpha \in \mathbb{R}_+, \lambda \in \mathbb{C}.$$

- *If $f(t)$ is continuous for $t \geq 0$ the following equalities hold*

$$\begin{aligned} lim_{\alpha\to 0} I^\alpha f(t) &= f(t), \\ I^\alpha(I^\beta f(t)) = I^\beta(I^\alpha f(t)) &= I^{\alpha+\beta} f(t) \ \ \alpha, \beta \in \mathbb{R}_+, \lambda \in \mathbb{C}. \end{aligned}$$

Definition 1.2.9

The Laplace transform of Riemann-Liouville fractional integral is defined by:

$$\begin{aligned} \mathcal{L}\Big[I^\alpha f(x)\Big] &= \frac{1}{\Gamma(\alpha)}\mathcal{L}(x^{\alpha-1} \star f(x)) \\ &= \frac{1}{s^\alpha}\mathcal{L}\Big[f(x)\Big]. \end{aligned}$$

1.2.3 Riemann-Liouville fractional derivative operator

Definition 1.2.10

Let f be a real function, the Riemann-Liouville fractional derivative or the Riemann-Liouville fractional differential operator of order α is defined by

$$\begin{aligned} D^{\alpha}_{RL} f(t) &= \frac{\mathrm{d}^n}{\mathrm{d}t^n}(I^{n-\alpha} f(t)) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t (t-s)^{n-\alpha-1} f(s)\ \mathrm{d}s, \quad t>0, \quad \alpha \in (n-1,n),\ n \in \mathbb{N}. \end{aligned}$$

In the following lemma, we give some relations between the Riemann-Liouville fractional derivative and the Riemann-Liouville fractional integral.

Lemma 1.2.4

Let $u \in \mathbb{C}^n([0,T]),\ \alpha \in (n-1,n),\ n \in \mathbb{N}$ and $v \in \mathbb{C}^1([0,T])$.

- *The Riemann-Liouville fractional differential operator D^{α}_{RL} is the left inverse operator of the fractional integral I^{α}, i.e.,*

$$D^{\alpha}_{RL} I^{\alpha} = I,$$

By convention it is defined

$$D^{0}_{RL} v(t) := v(t),\ \text{i.e., } D^{0}_{RL} := I \text{ is the identity operator.}$$

-

$$\begin{aligned} D^{\alpha}_{RL} v(t) &= \frac{\mathrm{d}}{\mathrm{d}t} I^{1-\alpha} v(t), \quad n=1, \\ D^{\alpha}_{RL} v(t) &= \frac{\mathrm{d}^2}{\mathrm{d}t^2} I^{2-\alpha} v(t), \quad n=2, \\ I^{\alpha} D^{\alpha}_{RL} u(t) &= u(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)}(I^{\alpha-1}u)(0). \end{aligned}$$

Remark 1.2.2

As we can see from the previous definition, the Riemann-Liouville fractional derivative of a constant is non-zero, unlike the integer order derivative of a constant C. To be more specific, the Riemann-Liouville fractional derivative of order $0<\alpha<1$ of a constant C is given by

$$D^{\alpha}_{RL} I^{\alpha} C = \frac{C t^{-\alpha}}{\Gamma(1-\alpha)}.$$

Definition 1.2.11

The Laplace transform of the Riemann-Liouville fractional derivative is defined by:

$$\begin{aligned}\mathcal{L}\Big[D^{\alpha}_{RL}f(t)\Big] &= \mathcal{L}\left[\frac{\mathrm{d}^n}{\mathrm{d}t^n}(I^{n-\alpha}f(t))\right]\\ &= s^{\alpha}\mathcal{L}(f(t)) - \sum_{k=0}^{n-1} s^k D^{\alpha-k-1}f(0)\\ &= s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^k D^{\alpha-k-1}f(0).\end{aligned}$$

Remark 1.2.3

- *The Laplace transform of $f^{(n)}$ is defined as follows:*

$$\mathcal{L}\Big[f^{(n)}(t)\Big] = s^n\mathcal{L}\Big[f(t)\Big] - \sum_{k=0}^{n-1} s^k f^{n-k-1}(0).$$

- $$D^{\alpha}_{RL}f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}(I^{n-\alpha}f(t)) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}(D^{(\alpha-n)}f(t)).$$

As a result, we've arrived at the following two theorems:

Theorem 1.2.2 (*See [104, 65]*)

Let $0 < \alpha < 1$. The derivative Riemann-Liouville fractional equation of order α is then transformed by the Laplace transform:

$$\mathcal{L}\Big[D^{\alpha}_{RL}f(t)\Big] = s^{\alpha}F(s) - \lim_{t\to 0} I^{1-\alpha}f(t).$$

Theorem 1.2.3 (*See [104, 65]*)

Assume that $1 < \alpha < 2$. The derivative Riemann-Liouville fractional equation of order α is then transformed by the Laplace transform:

$$\mathcal{L}\Big[D^{\alpha}_{RL}f(t)\Big] = s^{\alpha}F(s) - s\lim_{t\to 0} I^{2-\alpha}f(t) - \lim_{t\to 0}\frac{\mathrm{d}}{\mathrm{d}t}I^{2-\alpha}f(t).$$

In the formulation of the Laplace transforms, we can see the terms $\lim_{t\to 0} I^{1-\alpha}f(t)$, $\lim_{t\to 0} I^{2-\alpha}f(t)$ and $\lim_{t\to 0}\frac{\mathrm{d}}{\mathrm{d}t}I^{2-\alpha}f(t)$. Contrary, in integer order derivatives, where we can see the initial values of the functions f and f'.

1.2.4 Fractional Green's formula

Let Ω be a bounded open subset of $\mathbb{R}^n$, with a smooth boundary Γ of class $\mathcal{C}^2$. For all $T > 0$, we denote by $Q = \Omega \times (0,T), \Sigma = \partial\Omega \times (0,T)$. Let $y, \phi \in \mathcal{C}^{\infty}([0,T] \times \bar{\Omega})$, $T > 0$. We have the two following results:

Lemma 1.2.5

We set $n = 1$. Then, for all $0 < \alpha < 1$, and, for any $y, \phi \in \mathcal{C}^\infty([0,T] \times \bar{\Omega})$, we have

$$\begin{aligned}
&\int_0^T \int_\Omega \Big[D^\alpha_{RL} y(x,t) - \Delta y(x,t) \Big] \phi(x,t) \mathrm{d}x\mathrm{d}t \\
&\quad = \int_\Omega \phi(x,T) I^{1-\alpha} y(x,T) \mathrm{d}x - \int_\Omega \phi(x,0) I^{1-\alpha} y(x,0) \mathrm{d}x + \int_0^T \int_{\partial\Omega} y(\sigma,t) \frac{\partial \phi}{\partial \nu}(\sigma,t) \mathrm{d}\sigma\mathrm{d}t \\
&\qquad - \int_0^T \int_{\partial\Omega} \phi(\sigma,t) \frac{\partial y}{\partial \nu}(\sigma,t) \mathrm{d}\sigma\mathrm{d}t + \int_0^T \int_\Omega \Big[-\mathcal{D}^\alpha_C \phi(x,t) - \Delta \phi(x,t) \Big] y(x,t) \mathrm{d}x\mathrm{d}t,
\end{aligned}$$

where $\mathcal{D}^\alpha_C$ is the right fractional Caputo derivative of order $0 < \alpha < 1$.

Lemma 1.2.6

We set $n = 2$. Then, for all $1 < \alpha < 2$, and, for any $y, \phi \in \mathcal{C}^\infty([0,T] \times \bar{\Omega})$, we have

$$\begin{aligned}
&\int_0^T \int_\Omega \Big[D^\alpha_{RL} y(x,t) - \Delta y(x,t) \Big] \phi(x,t) \mathrm{d}x\mathrm{d}t \\
&\quad = \int_\Omega \phi(x,T) \frac{\partial}{\partial t} I^{2-\alpha} y(x,T) \mathrm{d}x - \int_\Omega \phi(x,0) \frac{\partial}{\partial t} I^{2-\alpha} y(x,0^+) \mathrm{d}x - \int_\Omega I^{2-\alpha} y(x,T) \frac{\partial \phi}{\partial t}(x,T) \mathrm{d}x \\
&\qquad + \int_\Omega I^{2-\alpha} y(x,0) \frac{\partial \phi}{\partial t}(x,0) \mathrm{d}x + \int_0^T \int_{\partial\Omega} y(\sigma,t) \frac{\partial \phi}{\partial \nu}(\sigma,t) \mathrm{d}\sigma\mathrm{d}t - \int_0^T \int_{\partial\Omega} \phi(\sigma,t) \frac{\partial y}{\partial \nu}(\sigma,t) \mathrm{d}\sigma\mathrm{d}t \\
&\qquad + \int_0^T \int_\Omega \Big[\mathcal{D}^\alpha_C \phi(x,t) - \Delta \phi(x,t) \Big] y(x,t) \mathrm{d}x\mathrm{d}t,
\end{aligned}$$

where $\mathcal{D}^\alpha_C$ is the right fractional Caputo derivative of order $1 < \alpha < 2$.

1.2.5 The left and right Caputo fractional derivatives

The concepts of left and right Caputo fractional derivatives will be discussed here.

Definition 1.2.12

If $f(t)$ is defined in $\mathcal{C}^n[a,\infty)$, then the left Caputo fractional derivative or left Caputo fractional differential operator of order α is defined as

$$\begin{aligned}
\mathcal{D}^\alpha_C f(t) &= I^{n-\alpha} \left(\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t) \right) \\
&= \frac{1}{\Gamma(n-\alpha)} \int_0^t f^{(n)}(s)(t-s)^{n-\alpha-1} ds, \quad t > 0, \quad \alpha \in (n-1,n), \ n \in \mathbb{N}.
\end{aligned}$$

A constant's Caputo derivative is equal to zero.

Definition 1.2.13

The right Caputo fractional derivative or the right Caputo fractional differential operator of order α is defined by

$$\mathcal{D}^\alpha_C f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^T f^{(n)}(s)(s-t)^{n-\alpha-1} ds, \quad 0 < t < T, \quad \alpha \in (n-1,n), \ n \in \mathbb{N}.$$

The adjoint operator of the right fractional derivative is represented by the left fractional derivative. In the following lemma, we give some relations between the Riemann-Liouville fractional derivative and the Caputo fractional integral:

Lemma 1.2.7

Let $u \in \mathbb{C}^n([0,T])$, $\alpha \in (n-1,n)$, $n \in \mathbb{N}$ and $v \in \mathbb{C}^1([0,T])$.

$$\mathcal{D}_C^\alpha I^\alpha v(t) = v(t);$$
$$I^\alpha \mathcal{D}_C^\alpha u(t) = u(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0);$$
$$I^\alpha \mathcal{D}_C^\alpha u(t) = u(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)}(I^{1-\alpha-}u)(0), \quad n=1;$$
$$I^\alpha \mathcal{D}_C^\alpha u(t) = u(t) - u(0), \quad n=1.$$

Lemma 1.2.8

Let $(n-1) < \alpha < n$, $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and $f(t)$ be such that $\mathcal{D}_C^\alpha f(t)$ exists. Then

$$\mathcal{D}_C^\alpha f(t) = I^{n-\alpha} D^n f(t) = I^{n-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t).$$

This implies that the Caputo fractional differential operator is equivalent to an $(n-\alpha)$-fold integration following an n-th order differentiation.

Proposition 1.2.4

In general, the two operators, Riemann-Liouville and Caputo, do not coincide, i.e.,ă

$$D_{RL}^\alpha f(t) \neq \mathcal{D}_C^\alpha f(t).$$

Lemma 1.2.9

Let $(n-1) < \alpha < n$, $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and $f(t)$ be such that $\mathcal{D}^\alpha f(t)$ exists. Then the following properties for the Caputo operator hold:

$$\lim_{\alpha \to n} \mathcal{D}_C^\alpha f(t) = f^{(n)}(t),$$
$$\lim_{\alpha \to n-1} \mathcal{D}_C^\alpha f(t) = f^{(n-1)}(t) - f^{(n-1)}(0).$$

Proof: We refer the reader to[123]. ■

For the Riemann-Liouville fractional differential operator, the corresponding interpolation property readsă

$$\lim_{\alpha \to n} D_{RL}^\alpha f(t) = f^{(n)}(t),$$
$$\lim_{\alpha \to n-1} D_{RL}^\alpha f(t) = f^{(n-1)}(t).$$

Lemma 1.2.10

- *Let* $(n-1) < \alpha < n,\ n, m \in \mathbb{N},\ \alpha \in \mathbb{R}$ *and the functions* $f(t)$ *and* $g(t)$ *be such that both* $\mathcal{D}_C^\alpha f(t)$ *and* $\mathcal{D}_C^\alpha g(t)$ *exist. Then the Caputo fractional derivative is a linear operator, i.e.,*

$$\mathcal{D}_C^\alpha((\lambda f(t) + g(t)) = \lambda \mathcal{D}_C^\alpha f(t) + \mathcal{D}_C^\alpha g(t),\ \alpha \in \mathbb{R}_+, \lambda \in \mathbb{C}.$$

- *The Riemann-Liouville fractional differential operator satisfies*

$$D_{RL}^\alpha(\lambda f(t) + g(t)) = \lambda D_{RL}^\alpha f(t) + D_{RL}^\alpha g(t),\ \alpha \in \mathbb{R}_+, \lambda \in \mathbb{C}.$$

- *Let* $(n-1) < \alpha < n,\ n, m \in \mathbb{N},\ \alpha \in \mathbb{R}$ *and the functions* $f(t)$ *is such that* $\mathcal{D}_C^\alpha f(t)$ *exists. Then in general*

$$\mathcal{D}_C^\alpha D^m f(t) = \mathcal{D}_C^{\alpha+m} f(t) \neq D^m \mathcal{D}_C^\alpha f(t).$$

- *Suppose that* $(n-1) < \alpha < n,\ 0 < \beta = \alpha - (n-1) < 1,\ n \in \mathbb{N},\ \alpha, \beta \in \mathbb{R}$ *and the functionf*(t) *is such that both* $\mathcal{D}_C^\alpha f(t)$ *exists. Then*

$$\mathcal{D}_C^\alpha f(t) = \mathcal{D}_C^\beta D^{n-1} f(t).$$

Proof: We refer the reader to[96]. ■

Definition 1.2.14

The Laplace transform of Caputo's fractional derivative is defined by:

$$\begin{aligned}
\mathcal{L}\left[\mathcal{D}_C^\alpha f(t)\right] &= \mathcal{L}\Big[I^{n-\alpha}(\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t))\Big] \\
&= s^{\alpha-n} \mathcal{L}\left[\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t)\right] \\
&= s^\alpha \mathcal{L}(f(t)) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0) \\
&= s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0).
\end{aligned}$$

1.3 Generalities on the semigroup theory

1.3.1 Semigroups of linear operators

In this section we present the basic notions of the theory of semi-groups which will be used throughout this work. Let $\mathcal{H}$ be a real or complex Hilbert space endowed with a norm denoted $\|\cdot\|$ and the dot product $\langle \cdot, \cdot \rangle$. $\mathcal{L}(\mathcal{H})$ is the space of bounded linear operators of $\mathcal{H}$ in it even whose standard is

$$\| \mathcal{U} \|_{\mathcal{L}(\mathcal{U})} = \sup_{x \neq 0} \frac{\| \mathcal{U}_x \|}{\| x \|}$$

for all $\mathcal{U} \in \mathcal{L}(\mathcal{H})$, $\mathcal{L}(\mathcal{H})$ is a Banach space.

Definition 1.3.1 (*Semigroup.*)

A collection $(S(t))_{t \in \mathbb{R}_+}$ in $\mathcal{L}(\mathcal{X})$ is called a ***semigroup*** *if*

1. $S(0) = I_{\mathcal{X}}$, *where $I_{\mathcal{X}}$ is the identity operator on x.*
2. $S(t+s) = S(t)S(s)$, *for all $t, s \in \mathbb{R}_+$.*
3. $\lim_{t \to 0} \| S(t)x - x \| = 0$, *for all x in $\mathcal{H}$*

If in replaces (3) by

$$\lim_{t \to 0} \| S(t) - I \| = 0, \ t \geq 0$$

it is a uniformly continuous semigroup.

Theorem 1.3.1

For $(S(t))_{t \geq 0}$ a $\mathcal{C}_0$-semigroup on $\mathcal{H}$, then we have the properties following:

(i) $t \to |S(t)|_{\mathcal{L}(\mathcal{H})}$ *is bounded on any compact interval* $[0; t_1]$

(ii) For all x in $\mathcal{H}$, the function $t \to S(t)x$ is continuous on $\mathbb{R}_+$

(iii) There are constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that

$$|S(t)|_{\mathcal{L}(\mathcal{H})} \leq M e^{\omega t}, \quad \forall t \in \mathbb{R}_+.$$

Definition 1.3.2

The operator A defined by $\mathcal{D}(A) = \{x \in \mathcal{H} : \lim_{t \to 0} \frac{S(t)x - x}{t}$ *exists for everything* $t > 0\}$ *and*

$$Ax = \lim_{t \to 0} \frac{S(t)x - x}{t} = \frac{d}{dt} S(t)x|_{t=0}, \text{ For } x \in \mathcal{D}(A).$$

is said to be the infinitesimal generator of the $\mathcal{C}_0$-semigroup

The space $\mathcal{D}(A)$ is endowed with the norm of the graph $\| x \|_{\mathcal{D}(A)} = \| x \| + \| Ax \|$, $x \in \mathcal{D}(A)$.

Remark 1.3.1

$(S(t))_{t \geq 0}$ is a $\mathcal{C}_0$-semigroup of bounded linear operators of infinitesimal generator A, then it is unique..

Example 1.3.1 (*Example of a C_0-semigroup.*)

In $\mathcal{L}^p(\mathbb{R})(1 \leq p \leq +\infty)$, the family $(S(t))_{t \geq 0}$ is defined by:

$$[S(t)x](s) = x(t+s), \forall t \geq 0, \ s \in \mathbb{R} \text{ and } x \in \mathcal{L}^p(\mathbb{R}).$$

We then define the operator A on $\mathcal{L}^p(\mathbb{R})$ by

$$\mathcal{D}(A) = \{x \in \mathcal{L}^p(\mathbb{R}) : x \text{ is locally absolutely continuous, and } x' \in \mathcal{L}^p(\mathbb{R})\}$$

$Ax = x'$ for everything $x \in \mathcal{D}$.

Proposition 1.3.1

Properties of a $\mathcal{C}_0$-semigroup

(i) If $x \in \mathcal{D}(A)$, $S(t)x \in \mathcal{D}(A)$, $\quad 0 \leq t < \infty$.

(ii) A is a dense domain closed linear operator in $\mathcal{H}$ $\left(\mathcal{D}\bar{(}A) = \mathcal{H}\right)$.

(iii) For all $x \in \mathcal{H} : t > 0$ we have

$$\int_0^t S(s)x\mathrm{ds} \in \mathcal{D}(A) \text{ and } A\Big(\int_0^t S(s)x\mathrm{ds}\Big) = S(t)x - x.$$

(iv) If $x \in \mathcal{D}(A)$, then the function $t \to S(t)x$ is continuously differentiable from $\mathbb{R}_+ \to \mathcal{H}$, and we have

$$\frac{d}{dt}S(t)x = AS(t)x = S(t)Ax.$$

(v) For $\lambda \in \mathbb{C}$ with $Re(\lambda) > \omega$ and $x \in \mathcal{H}$, the resolver operator is defined by

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t}S(t)x\mathrm{dt}$$

where $R(\lambda, A) = (\lambda I - A)^{-1}$(this is the transform of the place of the semi-group).

Theorem 1.3.2

Let A be a linear operator on $\mathcal{X}$. Then, A is an infinitesimal generator of uniformly continuous semigroup $(S(t))_{t\in\mathbb{R}_+}$ iff A is bounded, (see [122]). Moreover, any $A \in \mathcal{L}(\mathcal{X})$ is a generator of unique uniformly continuous semigroup, (see [122]).

1.3.2 Hille-Yosida theorem

We present the Hille-Yosida theorem which constitutes a characterization of a generator of a $\mathcal{C}_0$-semigroup.

Theorem 1.3.3 (*Hille-Yosid.*)

The necessary and sufficient condition for closed operator A dense domain in $\mathcal{H}\left(\mathcal{D}\bar{(}A) = \mathcal{H}\right)$ be infinitesimal generator of a $\mathcal{C}_0$- semigroup $\{S(t)\}_{t\geq 0}$ is that there constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that

(i) $\{\lambda : \lambda \in \mathbb{C}, Re(\lambda) > \omega\} \subset \rho(A)$ $\quad$ (the solver set of A)

(ii) $|R(\lambda, A)^n|_{\mathcal{L}(\mathcal{H})} \leq \dfrac{M}{(Re(\lambda) - \omega)^n}$, $\forall Re(\lambda) > \omega$, $n = 1, 2, \cdots$ where $\rho(A)$ is the resolvent set defined by

$$\rho(A) = \{\lambda \in \mathbb{C}/ \quad (\lambda I - A)^{-1} \text{ exists and bounded in } \mathcal{H}\}.$$

Proof: See [122]. ■

Given a linear operator A satisfying the conditions of Theorem (1.3.3), it is convenient to introduce a sequence of linear operators (called the Yosida approximations of A). They are defined by

$$A_n = nAR(n, A) = n^2 R(n, A) - n$$

Lemma 1.3.1

$$\lim_{n\to\infty} nR(n, A)x = x \quad \text{for everything}\, x \in \mathcal{H},$$

and

$$\lim_{n\to\infty} A_n x = Ax \quad \text{for everything}\, x \in \mathcal{D}(A),$$

Proposition 1.3.2 (*[149]*)

Let $(A, D(A))$ be a nonnegative and self-adjoint operator. Then, $(-A)$ is an infinitesimal generator of semigroup of contraction $(S(t) := e^{-tA})_{t\in\mathbb{R}_+}$.

We mention some basics. Concepts and facts about stochastic processes in Hilbert-spaces. We give Stochastic Itô definitions are included in the Hilbert spaces, which allow us to introduce them The concept of random differential equations. Key references For the materials presented here are [7].

Remark 1.3.2

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(E, |.|_E)$ be a separable Banach space and $B(E)$ be the σ-field of its Borel subsets. We fix $T > 0$.

Definition 1.3.3 (*Normal filtration.*)

*Let $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ be a filtration (i.e., an in-creasing family of σ-fields defined on$(\Omega, \mathcal{F}, \mathbb{P})$. We say that $(\mathcal{F}_t)_{t\in[0,T]}$ is a **normal filtration** or say that, it satisfies the usual conditions if*

- *for all $B \in \mathcal{F}$ s.t. $\mathbb{P}(B) = 0$, then $B \in \mathcal{F}_0$.*
- *for all $t \in [0, T]$, $\mathcal{F}_{t^+} := \mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$.*

The space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is also known as a filtered probability space.

Definition 1.3.4 (*E-valued random variable.*)

*Let the mapping $\mathcal{X} : (\Omega, \mathcal{F}) \to (E, B(E))$. We say that $\mathcal{X}$ is an E-**valued random variable** if it is measurable, i.e. for any $B \in B(E)$ it holds $\mathcal{X}^{-1}(B) \in \mathcal{F}$.*

1.3.3 A C_0-semigroup's dualities

Definition 1.3.5

The adjoint of A denoted A^* generates the semigroup $\{S^*(t)\}_{t\geq 0} \subseteq \mathcal{L}(H)$, where, for everything $t \geq 0$, $S^*(t)$ is the deputy of $S(t)$ and which powerfully continues on $\mathcal{H}$.

Lemma 1.3.2

Let $\mathcal{U} \in \mathcal{L}(\mathcal{H})$ so $\mathcal{U}^* \in \mathcal{L}(\mathcal{H})$ and we have

$$\| \mathcal{U} \|_{\mathcal{L}(\mathcal{U})} = \| \mathcal{U}^* \|_{\mathcal{L}(\mathcal{U})} .$$

Lemma 1.3.3

Let $A : D(A) \subseteq \mathcal{H} \to \mathcal{H}$ a dense-domain closed linear operator in $\mathcal{H}$. whether $\lambda \in \rho(A)$, so $\lambda \in \rho(A^*)$, and we have

$$R(\lambda, A^*) = R(\lambda, A)^*.$$

1.3.4 Semi-compact operator group

Definition 1.3.6

In infinite dimension, a $\mathcal{C}_0$-semigroup, $\mathcal{S}(t)$ is said to be compact for $t > t_0$ if for all $t > t_0$, $\mathcal{S}(t)$ is a compact operator. $\mathcal{S}(t)$ is said to be compact if it is compact for $t > 0$

Remark 1.3.3

If $\mathcal{S}(t)$ is compact for $t \geq 0$, then the identity is compact and $\mathcal{H}$ is of finite dimension. Moreover, if there exists $t_0 > 0$ such that $\mathcal{S}(t_0)$ is compact then $\mathcal{S}(t)$ is also for all $t \geq t_0$ because $\mathcal{S}(t) = \mathcal{S}(t - t_0)$ and $\mathcal{S}(t - t_0)$ is bounded. We recall an interesting result concerning compact semigroups.

Theorem 1.2

Let $\mathcal{S}(t)$ be a $\mathcal{C}_0$-semigroup. If $\mathcal{S}(t)$ is compact for $t > t_0$, then $\mathcal{S}(t)$ is continuous by relation to the uniform topology of operators for $t > t_0$.

Corollaire 1.4

Let $\mathcal{S}(t)$ be a semigroup $\mathcal{C}_0$ and let A be its infinitesimal generator. If $R(\lambda, A)$ is compact for some $\lambda \in \rho(A)$ and $S(t)$ is continuous with respect to the uniform topology of operators for $t > t_0$, then $\mathcal{S}(t)$ is compact for $t > t_0$. We conclude this section by introducing the concept of a "mild" solution.

1.3.5 Solution mild (Solution in the sense of semigroups)

Consider the following deterministic problem:

$$\begin{cases} \dfrac{du(t)}{dt} & = Au(t), \quad 0 < t < T, \\ u(0) & = x, \quad x \in \mathcal{H}. \end{cases} \tag{1.3}$$

where $\mathcal{H}$ is a separable real Hilbert space and A is an unbounded operator which generates a $\mathcal{C}_0$-semi-group $\mathcal{S}(t)$

Definition 1.3.7

The function $u : [0,T] \to \mathcal{H}$ is a (classical) solution of problem (1.3) on $[0,T[$ if u is continuous on $[0,T[$, continuously differentiable on $]0,T[$ and $u(t) \in D(A)$ for $t \in]0,T[$. If A is an infinitesimal generator of a $\mathcal{C}_0$-semigroup $\{\mathcal{S}(t)\}_{t\geq 0}$, so for everything $x \in D(A)$, the function $u^x(t) = \mathcal{S}(t)x$, $t \geq 0$ is a solution of (1.3). On the other hand, for $x \notin D(A)$, is not a classic solution, but it can be considered as a "generalized solution" which will be called a "mild solution". In fact, the concept of mild solution can be introduced to study the problem at an inhomogeneous initial value next

$$\begin{cases} \dfrac{du(t)}{dt} &= Au(t) + f(t), \quad 0 < t < T, \\ u(0) &= x, \quad x \in \mathcal{H}. \end{cases} \tag{1.4}$$

Or $f : [0,T[\to \mathcal{H}$

We now define the concept of a mild solution

Definition 1.3.8

Let A be an infinitesimal generator of a $\mathcal{C}_0$-semigroup $\{\mathcal{S}(t)\}_{t\geq 0}$, on $\mathcal{H}$, $x \in \mathcal{H}$, and $f \in \mathcal{L}^1([0,T],\mathcal{H})$ the space of Bochner functions-integrable on $[0,T]$ with values in $\mathcal{H}$. The function $u \in C([0,T],\mathcal{H})$ given by

$$u(t) = \mathcal{S}(t)x + \int_0^t \mathcal{S}(t-s)f(s)\mathrm{d}s, \quad 0 \geq t \geq T$$

is the mild solution of the initial-valued problem (1.4) on $[0,T]$.

1.4 Wiener processes and stochastic integrals in a Hilbert space

In this section, we define the Wiener processes and develop the integral stochastic in a Hilbert space.

1.4.1 Wiener processes on Hilbert spaces

There are many types of stochastic processes, among them; Wiener process, Markov process and Poisson Process. However, Wiener process without any doubt is one of the most important processes both in the theory and in the applications. Originally it was introduced by the mathematician Norbert Wiener in 1920 as a mathematical model of the **Brownian motion**[1] The current section gives a short review on such process which is a generalization of Brownian motion taking values in a general functional space. To do so, we need first to fix some tools; let $T > 0$, $(\mathcal{U}, \langle \cdot, \cdot \rangle_{\mathcal{U}}, |\cdot|_{\mathcal{U}})$ be a separable real Hilbert space and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. As a starting point of this section, let us introduce the notion of Wiener process in a linear topological space $(E, B(E))$ by following [126]

[1]The Brownian motion is named after the biologist Robert Brown who observed in 1827 the irregular motion of pollen particles floating in water.

Definition 1.4.1 (*E-valued Wiener process.*)

Let $\mathcal{W} := (\mathcal{W}_t)_{t\in[0,T]}$ be an E-valued stochastic process. We say that $\mathcal{W}$ is an E-valued Wiener process if the following holds

1. $\mathbb{P}(\mathcal{W}(0) = 0) = 1$,
2. *The trajectories of $\mathcal{W}$ are continuous,*
3. *For any finite increasing sequence $(t_i)_{i=0}^{k} \subset [0,T]$ the increments $(\mathcal{W}_{t_{i+1}} - \mathcal{W}_{t_i})_{i=0}^{k-1}$ are independent,*
4. *For any $t \in [0,T]$ and any $h \in (0,T]$, it holds $\mathbb{P}^{\mathcal{W}_{t+h}-\mathcal{W}_t} = \mathbb{P}^{\mathcal{W}_h}$,*
5. *For any $t \in [0,T]$ it is true that $\mathbb{P}^{\mathcal{W}_t} = \mathbb{P}^{-\mathcal{W}_t}$.*

1.4.2 Q-Wiener processes

Fix $Q \in \mathcal{L}(U)$ be a symmetric, nonnegative and finite trace operator.

Definition 1.4.2 (*Standard Q-Wiener processes*)

*Let $\mathcal{W} := (\mathcal{W}_t)_{t\in[0,T]}$ be an U-valued stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$. Then, $\mathcal{W}$ is said to be a **standard Q-Wiener process** if the conditions of Definition 1.4.1 are true, such that the law of the increments $(\mathcal{W}_t - \mathcal{W}_s)$, for all $0 \leq s < t \leq T$, is Gaussian with zero mean and covariance operator $(t-s)Q$.*

Proposition 1.4.1

An U-valued Q-Wiener process, $\mathcal{W} = (\mathcal{W}_t)_{t\in[0,T]}$ is a Gaussian process with mean zero and covariance operator $tQ, \forall t \geq 0$.

Proposition 1.4.2

For any symmetric, nonnegative and finite trace operator $Q \in \mathcal{L}(\mathcal{U})$, there exists a Q-Wiener process on $\mathcal{U}$.

Theorem 1.4.1

Let $(\epsilon_n)_{n\in\mathbb{N}}$ be an ONB of $\mathcal{U}$ consisting of eigenvectors of Q corresponding to the nonnegative eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$, and let $S := \{n \in \mathbb{N}, \lambda_n > 0\}$ be the index set of non-zero eigenvalues. Then, an $\mathcal{U}$-valued stochastic process $\mathcal{W}$, is a Q-Wiener process iff it can be written for any $t \in [0,T]$ as

$$\mathcal{W}(t) = \sum_{n\in S} \sqrt{\lambda_n}\beta_n(t)\epsilon_n, \tag{1.5}$$

where $(\beta_n)_{n\in\mathcal{S}}$ is a sequence of independent $\mathbb{R}$-valued Brownian motions on $(\Omega,\mathcal{F},\mathbb{P})$ s.t.

$$\beta_n(t) = \frac{1}{\sqrt{\lambda_n}}\langle \mathcal{W}(t), \epsilon_n\rangle_{\mathcal{U}}, \text{for any}\, n \in \mathcal{S}.$$

The series in (1.5) converges in $L^2(\Omega, C([0,T],\mathcal{U}))$, where $C([0,T],\mathcal{U})$ is equipped with the supremum norm.

Proof: see [127] ∎

In the case of filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$, we need to know how the Q-Wiener process behaves in connection with the filtration $\mathbb{F}$. This is leads to the next definition.

Definition 1.4.3 (*Stochastic basis.*)

Let $(\mathcal{U}, \langle\cdot,\cdot\rangle_{\mathcal{U}}, |\cdot|_{\mathcal{U}})$ be a separable Hilbert space. We call $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},\mathcal{W})$ a stochastic basis if, $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ be a filtered complete probability space with respect to the normal filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ and $(\mathcal{W}_t)_{t\in[0,T]}$ be a $\mathcal{U}$-valued Wiener process on it.

1.4.3 Cylindrical Wiener processes

Cylindrical Wiener process appears in many models in infinite dimensional spaces as a source of random noise or random perturbation. In this subsection, we introduce by following [127] a result which ensures the existence of such type of processes. To do so, let $(\mathcal{U}, \langle\cdot,\cdot\rangle_{\mathcal{U}})$ be a separable Hilbert space, $Q \in \mathcal{L}(\mathcal{U})$ be a symmetric and non-negative operator, possibly with $trQ = +\infty$ and let $(\epsilon_n)_{n\in\mathbb{N}}$ be an ONB of $\mathcal{U}$ that consists of eigenvectors of Q with corresponding eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$. Additionally, let $(\mathcal{U}, \langle\cdot,\cdot\rangle_{\mathcal{U}_1}, |\cdot|_{\mathcal{U}_1})$ be an arbitrary separable Hilbert space with $\mathcal{U} \subset \mathcal{U}_1$ continuously and let $J : (\mathcal{U}_0, \langle\cdot\rangle_{\mathcal{U}_0}) \to (\mathcal{U}_0, \langle\cdot\rangle_{\mathcal{U}_0})$ be a Hilbert-Schmidt embedding, besides we define the operator $Q_1 := JJ^* : \mathcal{U}_1 \to \mathcal{U}_1$.

Proposition 1.4.3 (*See [127]*)

Let $Q_1 := JJ^*$. Then, Q_1 is a linear, bounded, nonnegative, symmetric and finite trace operator on $\mathcal{U}_1$, and the operator $J : \mathcal{U}_0 \to Q_1^{\frac{1}{2}}\mathcal{U}_1$ is an isometry, i.e.

$$\| u_0 \|_{\mathcal{U}_0} = \|Q_1^{-\frac{1}{2}}J(u_0)\|_{\mathcal{U}_1} = \| J(u_0) \|_{Q_1^{\frac{1}{2}}(\mathcal{U}_1)}, \text{ for all}\, u_0 \in \mathcal{U}_0$$

Moreover, let $\tilde{\epsilon_n} := Q^{\frac{1}{2}}\epsilon_n$, where $(\epsilon_n)_{n\in\mathbb{N}}$ be an ONB of $\mathcal{U}$ and let $(\beta_n)_{n\in\mathbb{N}}$ be a sequence of independent, real-valued Brownian motions. Then

$$\mathcal{W}(t) := \sum_{n=1}^{\infty} B_n(t)J(\tilde{\epsilon_n}), \forall t \in [0,T], \tag{1.6}$$

is a Q_1-Wiener process on $\mathcal{U}_1$ with $trQ_1 < +\infty$, where the series in (1.6) is convergent in $\mathcal{M}_2^T(\mathcal{U}_1)$

1.4.4 Some notions in the one dimensional case

In this subsection we recall two usefull notions of real-valued processes.

Definition 1.4.4 (*White noise.*)

*Let $(\varepsilon, \beta(\varepsilon), \mu)$ be a σ-finite measurable space and let the random set function $\mathbb{W}$ defined on $\{B \in \beta(\varepsilon), s.t., \mu(B) < \infty\}$. Then, we call $\mathbb{W}$ a **white noise**[2] if it satisfies*

1. *for any $B \in \beta(\varepsilon)$, $\mathbb{W}(B)$ is a Gaussian (or normal) random variable with mean 0 and variance $\mu(B)$.*
2. *for any two disjoint sets $B_1, B_2 \in \beta(\varepsilon)$, the random variables $\mathbb{W}(B_1)$ and $\mathbb{W}(B_2)$ are independent and $\mathbb{W}(B_1) \cap \mathbb{W}(B_2) = \mathbb{W}(B_1) + \mathbb{W}(B_2)$*

Definition 1.4.5 (*Brownian sheet.*)

*Let $d \in \mathbb{N}_0$, $\varepsilon = \mathbb{R}^d_+ := \{t = (t_1, \cdots, t_d), s.t., t_i \geq 0, \ \forall i \in \{1, \cdots, d\}$ and μ be a Lebesgue measure on $\mathbb{R}^d$. The process $(\beta_t)_{t \in \mathbb{R}^d_+}$ is said to be **Brownian sheet** if it is defined by*

$$\beta_t = \mathbb{W}(\Pi_{i=1}^d]0, t_i])$$

where $\mathbb{W}$ is a white noise. This means that, it is a zero-mean Gaussian process with covariance function defined for $t = (t_1, \cdots, t_d)$ and $s = (s_1, \cdots, s_d)$ by

$$\mathbb{E}(\beta_t \beta_s) = \Pi_{i=1}^d t_i \wedge s_i.$$

Remark 1.4.1

There is another way to define white noise. In the special case; $\varepsilon = \mathbb{R}$ and μ is Lebesgue measure, it is informally described as the weak derivative of Brownian motion, since such motion is nowhere-differentiable in the classical sense. Such description is also possible in higher dimensions, but it involves the Brownian sheet instead of Brownian motion.

Definition 1.4.6 (*Stochastic process.*)

*A family $\mathcal{X} := (\mathcal{X}_t)_{t \in [0,T]}$ of E-valued random variables $\mathcal{X}_t$, $t \in [0, T]$ defined on Ω is called a **stochastic process**.*
The stochastic process $\mathcal{X}$ depends on two variables, the temporal variable $t \in [0, T]$ and the probabilistic variable $\omega \in \Omega$.

Definition 1.4.7 (*Adaptation, continuity and measurability of stochastic process.*)

Let $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ be a normal filtration and $\mathcal{X} = (\mathcal{X}_t)_{t\in[0,T]}$ be a stochastic process.

1. *$\mathcal{X}$ is $\mathbb{F}$-**adapted** (or simply adapted) if each X_t is measurable with respect to $\mathcal{F}_t$ for every $t \in [0,T]$,*
2. *$\mathcal{X}$ is **continuous** with probability one if its trajectories are continuous almost surely, i.e.*
$$\mathbb{P}\{\omega \in \Omega,\, t \in \mathcal{X}_t(\omega) \text{ is continuous on } [0,T]\} = 1,$$
3. *$\mathcal{X}$ is **measurable** if the mapping $\mathcal{X}(\cdot,\cdot) : ([0,T]\times\Omega, \beta([0,T])\otimes\mathcal{F}) \to (E, \beta(E))$ is measurable with respect to $\beta([0,T])\otimes\mathcal{F}$ (which is σ-algebra generated by the product).*

Definition 1.4.8 (*Gaussian process.*)

*Let $(\mathcal{U}, \langle\cdot,\cdot\rangle_{\mathcal{U}})$ be a separable Hilbert space. An $\mathcal{U}$-valued stochastic process $\mathcal{X}$ on Ω is called **Gaussian** if for any $k \in \mathbb{N}_0$ and any $t_1, \cdots, t_k \in [0,T]$ the $\mathcal{U}^k$-valued random variable $(\mathcal{X}_{t_1}, \cdots, \mathcal{X}_{t_k})$ is Gaussian.*

Definition 1.4.9 (*Predictable σ-field.*)

Let $\Omega_T := [0,T]\times\Omega$ endowed with the σ-field $\beta([0,T])\otimes F$. The δ-field P_T generated by the sets of the form

$$((s,t]\times F_s : 0 \le s < t \le T, F_s \in \mathcal{F}_s) \text{ and } (\{0\}\times F_0).\, F_0 \in \mathcal{F}_0$$

is known as a predictable σ-field, and its constituent parts are referred to as predictable sets.

Definition 1.4.10 (*p-integrable process*)

An E-valued stochastic process $\mathcal{X} = (\mathcal{X}_t)_{t\in[0,T]}$ is called p-integrable, for $p \ge 1$ if the random variable $\mathcal{X}(t)$, for all $t \in [0,T]$ is p-thintegrable.

Proposition 1.4.4

The space of all continuous square integrable E-valued martingales denoted by $\mathcal{M}^2_T(E)$ is a Banach space endowed with the norm

$$\| M \|_{\mathcal{M}^2_T(E)} = \sup_{t\in[0,T]} \| M(t) \|_{L^2(\Omega,E)} .$$

Definition 1.4.11 (*Gaussian random variable*)

*Let $(\mathcal{U}, \langle\cdot,\cdot\rangle_{\mathcal{U}})$ be a separable Hilbert space. An $\mathcal{U}$-valued random variable $\mathcal{X}$ on Ω is said to be **Gaussian** if the $\mathbb{R}$-valued random variable $\langle\mathcal{X}, u\rangle_{\mathcal{U}})$, for any $u \in \mathcal{U}$ is Gaussian. Hence, $\exists m \in \mathcal{U}$ called the **mean** and a nonnegative and symmetric operator $Q : \mathcal{U} \to \mathcal{U}$ called the covariance operator s.t. the law $\mathbb{P}^{\mathcal{X}} := \mathbb{P}\circ\mathcal{X}^{-1} : B(\mathcal{U}) \to [0,1]$ of $\mathcal{X}$ is denoted by $\mathcal{N}(m,Q)$.*

1.4.5 Stochastic integrals in Hilbert spaces

In this part, we discuss the Q-Wiener process with respect to stochastic integration in Hilbert spaces. We study these ideas in detail by adopting this methodology

Stochastic integral with respect to Q-Wiener process

Throughout this subsection, we fix $T > 0$ and a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Further, we consider two separable Hilbert spaces $(\mathcal{U}, \langle \cdot, \cdot \rangle_{\mathcal{U}}, |\cdot|_{\mathcal{U}})$ and $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}}, |\cdot|_{\mathcal{H}})$. Let $Q \in \mathcal{L}(\mathcal{U})$ be a symmetric, nonnegative and finite trace operator and let $\mathcal{W} = (\mathcal{W}(t)_{t\in[0,T]})$ be a $\mathcal{U}$-valued Q-Wiener process with respect to the normal filtration $\mathbb{F}$. In order to shed a light on the notion of **stochastic integral** with respect to $\mathcal{W}$, we give first the meaning of an elementary process.

Definition 1.4.12 (*Elementary process.*)

*Let $\varphi = (\varphi(t))_{t\in[0,T]}$ be a $\mathcal{L}(\mathcal{U}, \mathcal{H})$-valued process defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. We say that φ is an **elementary process** if there exists a partition $0 = t_0 < t_1 < \cdots < t_m = T$, $m \in \mathbb{N}$ and a sequence $(\varphi_k)_{k=0}^{m-1}$ of $\mathcal{L}(\mathcal{U}, \mathcal{H})$-valued random variables that are taking only a finite number of values in $\mathcal{L}(\mathcal{U}, \mathcal{H})$ s.t. φ_k is $\mathcal{F}_{t_k}$-measurable for any $k \in \{0, \cdots, m-1\}$ and $\varphi(t) = \varphi_k$, for $t \in]t_k, t_{k+1}]$ with the convention $\varphi(0) = 0$. Mathematically, we write*

$$\varphi(t) = \sum_{k=0}^{m-1} \varphi_k I_{]t_k,t_{k+1}]}(t), \text{ fort} \in [0, T]$$

We denote the class of such processes by E.

Now, we are ready to give the meaning of the stochastic integral on E.

Definition 1.4.13 (*stochastic integral on E.*)

Let $\varphi \in E$. Then, the stochastic integral $\mathcal{I}(\varphi) := ((\mathcal{I}(\varphi)(t))_{t\in[0,T]}$ of φ with respect to $\mathcal{W}$ is an $\mathcal{H}$-valued stochastic process defined for all $t \in [0, T]$ by

$$(\mathcal{I}(\varphi))(t) := \int_0^t \varphi(s)\mathrm{d}\mathcal{W}(s) := \sum_{k=0}^{m-1} \varphi_k(\mathcal{W}(t_{k+1} \wedge t) - \mathcal{W}(t_k \wedge t))$$

such that φ_k is acting on $(\mathcal{W}(t_{k+1} \wedge t) - \mathcal{W}(t_k \wedge t))$ as an operator in $\mathcal{L}(\mathcal{U}, \mathcal{H})$.

Definition 1.4.14 (*stochastic integral on $\mathcal{P}^2_{\mathcal{W}}$.*)

Let $\varphi \in \mathcal{P}^2_{\mathcal{W}}$. Then, the stochastic integral of φ with repect to $\mathcal{W}$ is defined for every $t \in [0, T]$ by

$$(\mathcal{I}(\varphi))(t) = \int_0^t \varphi(s)\mathrm{d}\mathcal{W}(s) := \lim_{n\to\infty} \int_0^t \varphi_n(s)\mathrm{d}\mathcal{W}(s),$$

where the limit is taken with respect to the norm $\|\cdot\|_{\mathcal{M}^2_T(H)}$

Theorem 1.4.2

Let $\varphi \in \mathcal{P}^2_{\mathcal{W}}$. Then, $\mathcal{I}(\varphi) \in \mathcal{M}^2_T(H)$.

Definition 1.4.15 (*Stochastically integrable process.*)

Let $\varphi \in \mathcal{P}_{\mathcal{W}}$. Then, the stochastic integral of φ with respect to $\mathcal{W}$ is defined for every

$t \in [0, T]$ by

$$\int_0^t \varphi(s) \mathrm{d}\mathcal{W}(s) := \int_0^t (I_{[0,\Theta_n]}\varphi)(s) \mathrm{d}\mathcal{W}(s),$$

on the set $\{\Theta_n \geq t\}$, for every $n \in \mathbb{N}$, and we say that φ is ***stochastically integrable****.*

The consistency of definition (1.4.8) is ensured. Indeed, let $m \in \mathbb{N}$ s.t. $m > n$. If $\Theta_m \geq t$, then

$$\int_0^t (I_{[0,\Theta_n]}\varphi)(s) \mathrm{d}\mathcal{W}(s), := \int_0^t (I_{[0,\Theta_n]}(1_{[0,\Theta_m]})(s) \mathrm{d}\mathcal{W}(s),$$

we have

$$\int_0^t (I_{[0,\Theta_n]}\varphi)(s) \mathrm{d}\mathcal{W}(s), := \int_0^{\Theta_n \wedge t} (I_{[0,\Theta_m]}\varphi)(s) \mathrm{d}\mathcal{W}(s),$$

and since $\Theta_n \geq t$ we get

$$\int_0^t (I_{[0,\Theta_n]}\varphi)(s) \mathrm{d}\mathcal{W}(s), := \int_0^t (I_{[0,\Theta_m]}\varphi)(s) \mathrm{d}\mathcal{W}(s).$$

for every $n, m \in \mathbb{N}$ s.t. $m > n$.

1.4.6 Stochastic differential equations in infinite dimension

In this part, we deal with stochastic differential equations in the infinite dimension.

Abstract parabolic stochastic partial differential equations.

We give by following [130], a short review about the theory of solvability for a class of SPDEs with globally Lipschitz nonlinearities. let $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}}, |\cdot|_{\mathcal{H}})$ and $(\mathcal{U}, \langle\cdot,\cdot\rangle_{\mathcal{U}}, |\cdot|_{\mathcal{U}})$ be two real separable Hilbert spaces, and let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \mathcal{W})$ be a stochastic basis, where $\mathcal{W} = (\mathcal{W}_t)_{t\in[0,T]}$ be a $\mathcal{U}$-valued cylindrical Wiener process. We consider the following stochastic partial differential equation, perturbed by a multiplicative noise,

$$\begin{cases} \dfrac{du(t)}{dt} &= Au(t) + F(u(t)) + G(u(t))\dfrac{d\mathcal{W}(t)}{dt}, \quad t \in (0, T], \\ u(0) &= u_0, \end{cases} \tag{1.7}$$

where

- $A : D(A) \subset \mathcal{H} \to \mathcal{H}$, is in general an unbounded Linear operator (not necessary the Laplacian),
- $F : \mathcal{H} \to \mathcal{H}$, is $B(\mathcal{H})\backslash B(\mathcal{H})$-measurable operator,
- $G : H \to \mathcal{L}(\mathcal{U}, \mathcal{H})$,be an operator,
- u_0 be an $\mathcal{H}$-valued, $\mathcal{F}_0$-measurable random variable.

There are different concepts of the solutions, namely strong, weak and mild.

Definition 1.4.16 (*Strong solution.*)

Let $u := (u(t))_{t\in[0,T]}$ be an $D(A)$-valued predictable process. We say that u is a ***strong solution*** *of Prb. (1.7) if*

- *for all* $t \in [0,T]$, $(0,t) \ni s \rightarrowtail (Au(s) + F(u(s))) \in \mathcal{H}$ *is* $\mathbb{P}$*-a.s. Bochner integrable,*
- $G(u) : \Omega_T \to \mathcal{HS}(\mathcal{U},\mathcal{H})$ *is continuous predictable s.t.*

$$\mathbb{P}\left\{\int_0^T \| G(u(s)) \|^2_{\mathcal{HS}(\mathcal{U},\mathcal{H})} \mathrm{d}s < \infty\right\} = 1,$$

- *the following equality holds* $\mathbb{P}$*-a.s. in* $\mathcal{H}$

$$u(t) = u_0 + \int_0^t (Au(s) + F(u(s)))\mathrm{d}s + \int_0^t G(u(s))\mathrm{d}\mathcal{W}(s), \text{ for every } , t \in [0,T]$$

Definition 1.4.17 (***Weak solution.***)

Let $u := (u(t))_{t\in[0,T]}$ *be an* $\mathcal{H}$*-valued predictable process. We say that* u *is a weak solution of Prb.*(1.7) *if*

- *for all* $t \in [0,T]$*, and all* $\xi \in D(A^*)$, $(0,t) \ni s \rightarrowtail (\langle u(s), A^*\xi\rangle_{\mathcal{H}} + \langle F(u(s)), \xi\rangle_{\mathcal{H}}) \in \mathbb{R}$ *is* $\mathbb{P}$*-a.s. Lebesgue integrable,*
- *the following equality holds* $\mathbb{P}$*-a.s.in* $\mathbb{R}$*,*

$$\begin{aligned}\langle u(t), \xi\rangle_{\mathcal{H}} = \langle u_0, \xi\rangle_{\mathcal{H}} \quad &+ \int_0^t (\langle u(s), A^*\xi\rangle_{\mathcal{H}} + \langle F(u(s), \xi)\rangle_{\mathcal{H}})\mathrm{d}s \\ &+ \int_0^t \langle G(u(s))\mathrm{d}\mathcal{W}(s), \xi\rangle_{\mathcal{H}}\end{aligned}$$

for every $t \in [0,T]$*, and every* $\xi \in D(A^*)$

Definition 1.4.18 (***Mild solution.***)

Let $u := (u(t))_{t\in[0,T]}$ *be an* $\mathcal{H}$*-valued predictable process. We say that* u *is a mild solution of Prb.*(1.7) *if*

- *for all* $t \in [0,T]$, $(0,t) \ni s \rightarrowtail \mathcal{S}(t-s)F(u(s)) \in \mathcal{H}$ *is* $\mathbb{P}$*-a.s. Böchner integrable,*
- *for all* $t \in [0,T]$, $I_{[0,t[}(\cdot)\mathcal{S}(t-\cdot)G(u(\cdot)) : \Omega_T \to \mathcal{HS}(\mathcal{U},\mathcal{H})$ *is continuous predictable s.t.*

$$\mathbb{P}\left\{\int_0^T \| I_{[0,t[}(s)\mathcal{S}(t-s)G(u(s)) \|^2_{\mathcal{HS}(\mathcal{U},\mathcal{H})} \mathrm{d}s < \infty\right\} = 1$$

- *the following equality holds in* $\mathcal{H}$*,* $\mathbb{P}$*-a.s.,*

$$u(t) = \mathcal{S}(t)u_0 + \int_0^t \mathcal{S}(t-s)F(u(s))\mathrm{d}s + \int_0^t \mathcal{S}(t-s)G(u(s))\mathrm{d}\mathcal{W}(s),$$

for every $t \in [0,T]$*, where* $(\mathcal{S}(t))_{t\in[0,T]}$ *is the semigroup generated by the operator* A*.*

It is worth noticing here to introduce the notion of the mild solution in a general sense framework as .

Definition 1.4.19 (***Stochastic convolution.***)

Let $\varphi := (\varphi(t))_{t\in[0,T]}$ *be an* $\mathcal{L}(\mathcal{U},\mathcal{H})$*-valued predictable process and let* $(\mathcal{S}(t))_{t\in[0,T]}$ *be the* *strongly continuous semigroup generated by the operator A s.t. the following integral*

$$\mathcal{W}_s^{\varphi}(t) := \int_0^t \mathcal{S}(t-s)\varphi(s)\mathrm{d}\mathcal{W}(s), \text{ for every } t \in [0,T],$$

*is well defined. Then, the process $\mathcal{W}_s^{\varphi} := (\mathcal{W}_s^{\varphi}(t))_{t\in[0,T]}$ is **called stochastic convolution**.*

1.5 Finite Dimensional Linear Control Dynamical Systems

This Part are mainly concerned with the quadratic cost optimal control problem for distributed parameter systems and systems with time delay, both over a finite and an infinite time interval. For problems over a finite time interval, the main tool used is Dynamic Programming, which leads to a **Hamilton-Jacobi**-Problem equation for the value function.

1.5.1 Ordering system

The object of automatic control is the study of systems on which one can act by means of a command. It results in an input-and-output relationship.

(1) **Commandability**: Is it possible to find a command u that brings the system initially in state x_0, in any state v at time $t=\tau$

(2) **Observability**: Does knowing $y(t)$ and $u(t)$ for all $t \in [0,\tau]$ allow us to determine the state $x(t)$ for all $t \in [0,\tau]$ (or, equivalently, the initial state $x(0)$).

(3) **Stabilization**: Is it possible to construct a command $u(\cdot)$ that asymptotically stabilizes e system around an equilibrium x_0.

1.5.2 Linear systems

We study the following system

$$\begin{cases} \dfrac{dx}{dt}(t) = Ax(t) + Bu(t) \quad t \in [0,T], \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{Σ}$$

We will limit ourselves to the case where the quantities are of finite dimensions: $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$. It follows than $A \in \mathcal{M}_n(\mathbb{R})$, $B \in \mathcal{M}_{n,m}(\mathbb{R})$, $C \in\in \mathcal{M}_{p,n}(\mathbb{R})$, et $D \in\in \mathcal{M}_{p,m}(\mathbb{R})$. The command will be assumed to be **continuous by parts**.

Proposition 1.5.1 (***Formula for the variation of the constant.***)

*Let $u(\cdot)$ be a command and $x_0 \in \mathbb{R}^n$. The **unique** solution of $\dfrac{dx}{dt}(t) = Ax(t) + Bu(t)$ equal to x_0 at $t=0$*

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}Bu(s)\mathrm{d}s$$

Controllability

We consider the system (Σ). We are only interested here in the input state law, i.e.

$$\frac{dx}{dt}(t) = Ax(t) + Bu(t). \tag{1.8}$$

Definition 1.5.1

Given $x_0 \in \mathbb{R}^n$, we say that a state $v \in \mathbb{R}^n$ is reachable in time τ from x_0 if it exists a control law $u : [0, \tau] \to \mathbb{R}^{>}$ such that $x(\tau) = v(x(\cdot))$ being the solution of (1.8) satisfying $x(0) = x_0$. We note $\mathcal{A}(\tau, x_0)$ the set of states reachable from x_0 in time τ, that is to say :

$$\mathcal{A}(\tau, x_0) = \left\{ \begin{array}{cc} x(\tau)/ & x(\cdot)\text{solution of } (\Sigma), \\ & x(0) = x_0 \end{array} \right\}.$$

From the formula of the variation of the constant, it follows that $\mathcal{A}(\tau, 0)$ is a vector-space, and that

$$\mathcal{A}(\tau, x_0)\text{is the affine space}\, e^{\tau A}x_0 + \mathcal{A}(\tau, 0)$$

Definition 1.5.2

On dit que le système (Σ) est commandable en temps T si $\mathcal{A}(\tau, 0) = \mathcal{A}_\tau = \mathbb{R}^\ltimes$

Theorem 1.5.1

$\mathcal{A}_\tau$ space is equal to the image of the matrix $(n \times nm)$

$$C = [BAB \cdots A^{n-1}B]$$

called the controllability matrix.

Proposition 1.5.2 (*Kalman controllability criterion.*)

The system (Σ) is controllable if and only if the controllability matrix has rank n

Observability

The problem: knowing y and u for all $t \in [0, \tau]$, is it possible to determine the initial condition $x(0)$.

Remark 1.5.1

1. *the knowledge of x_0 is equivalent to that of $x(t)$ for all $t \in [0, \tau]$ by virtue of the formulation variation of the constant.*

2. *since $u(\cdot)$ is known, we can restrict ourselves to studying :*

$$\begin{cases} \frac{dx}{dt}(t) & = Ax(t)\, t \in [0, T], \\ y(t) & = Cx(t). \end{cases} \tag{Σ_0}$$

Definition 1.5.3

Let us call space of unobservability $\mathcal{P}_\tau$ of the system (Σ_0) the set of initial conditions

$x(0) \in \mathbb{R}^{\ltimes}$ *For which the solution* $y(\cdot)$ *is identically zero on* $[0, \tau]$ *:*

$$\mathcal{P}_\tau = \left\{ \begin{array}{ccc} x_0(\tau)/ & x(\cdot) \textit{solution of} & (\Sigma_0), \\ & \textit{with}\, x(0) = x_0 \textit{check there}\, y(t) \equiv 0 & \end{array} \right\}.$$

Definition 1.5.4

The system is said to be observable if the unobservability space of (Σ_0) *is reduced to* $\{0\}$

Proposition 1.5.3

If the system (Σ) *is observable, the knowledge of* $y(\cdot)$ *on* $[0, \tau]$ *unequivocally determines* $x(0)$

Proposition 1.5.4 (*Kalman observability criterion.*)

The space of unobservability of the system (Σ_0) *is the kernel of the matrix* $(np \times n)$.

$$\sigma = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

1.5.3 Optimal control

Consider thelinear control system

$$\begin{cases} \dfrac{dx}{dt}(t) & = Ax(t) + Bu(t), \;\; x(s) = x_0, \\ y(t) & = Cx(t) \end{cases} \tag{1.9}$$

on the time interval $[s, T]$, $0 \leq s < T$, where x is an arbitrary initial state in $\mathbb{R}^n$. The notation and terminology are the same as the ones defined in the section on controllability and observability. We shall be primarily concerned with the following optimal control problem:to choxse a control $(\cdot) \in L^2(s, T, \mathbb{R}^m)$ that minimizes the cost functional

$$\mathcal{J}(u, s; x) = \frac{1}{2}\int_s^T [(u(t), Ru(t)) + (x(t), Qx(t))]\mathrm{d}t + \frac{1}{2}(x(T), Sx(T)),$$

where $R = R^* > 0$, $Q = Q^* = C^*C \geq 0$, and $S = S^* \geq 0$. A control $u\hat{(\cdot)}$ minimizing $\mathcal{J}(u, x)$ will be called an optimal control. We shall not be concerned with the question of existence and uniqueness of solutions but with the characterization of the optimal control $(\cdot)$ and the corresponding optimal trajectory $\hat{x}(\cdot)$.

Even though we are interested in solving the optimal control problem over afixed interval $[0, T]$ and for the fixed initial condition x_0, it will turn out to be conceptually important to solve the problem for all initial points (s, x), $0 \leq s < T$.
We shall also be interested in the infinite time problem: Find a control $(\cdot) \in L^2(s, \infty, \mathbb{R}^m)$ that minimizes :

$$\mathcal{J}(u, s; x) = \frac{1}{2}\int_s^\infty [(u(t), Ru(t)) + (x(t), Qx(t))]\mathrm{d}t. \tag{1.10}$$

1.6 Infinite dimension controllability

Controllability is one of the most important qualitative aspects of a system dynamic. The problem of controllability is to prove the existence of a function control, which drives the solution of the system from its initial state to a state final, where the initial and final states can vary throughout space.

Definition 1.6.1

Let $\mathcal{H}$ and $\mathcal{U}$ be two Hilbert spaces and consider the following dynamical system:

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} & = & Ax(t) + Bu(t),\ t \in (0, T], \\ x(0) & = & x_0, \qquad\qquad x_0 \in \mathcal{H}, \end{cases} \tag{1.11}$$

where $T > 0$ fixed, $u \in L^2(0, T, \mathcal{U})$, A is an infinitesimal generator of a C_0-semigroup $\mathcal{S}(\cdot)$ in $\mathcal{U}$ and B is a bounded operator from $\mathcal{U}$ to $\mathcal{H}$. Here $\mathcal{H}$ represents the state space and $\mathcal{U}$ the control space of the system. We know that problem (1.11) has a unique mild solution $x = x(t, x_0, u) \in C([0, T], H)$ stated by

$$x(t, x_0, u) = S(t)x_0 + \int_0^t \mathcal{S}(t-s)Bu(s)\mathrm{d}s,\ t \in [0, T].$$

Definition 1.6.2

We will say that the control u transfers a state a to a state b at time $T > 0$ if

$$x(T, a, u) = b.$$

We also say that state b is reachable from a at time T.

Definition 1.6.3

We will say that the system (1.11) is controllable at time $T > 0$, if for all $a \in H$ and all $b \in H$, there is a control function $u \in L^2(0, T, U)$ such that:

$$x(T, a, u) = b.$$

We also say that the pair (A, B) is controllable at time $T > 0$.
Consider on $(0, T)$ the following dynamical system

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} & = & Ax(t) + Bu(t), \\ x(0) & = & 0. \end{cases}$$

For all $t \in [0, T]$ the solution can be written.

$$x(t, u) = L_t u,$$

where L_t is the bounded linear operator defined by:

$$\begin{cases} L^2(0, t, U) & \to & \mathcal{H}, \\ u & \to & \displaystyle\int_0^t \mathcal{S}(t-s)Bu(s)\mathrm{d}s\ . \end{cases}$$

Proposition 1.6.1

The system (1.11) is controllable at time $T > 0$ if and only if the L_T operator is surjective.

Proof: Let $a, b \in H$ any two states. The equation in u:

$$x(T, a, u) = b. \tag{1.12}$$

has a solution in $L^2(0,T,\mathcal{U})$ if and only if the equation

$$L_{T^u} = b - \mathcal{S}(T)a, \tag{1.13}$$

has a solution in $L^2(0,T,\mathcal{U})$. The equivalence of equations (1.12) and (1.13) leads to the proposition. ■

1.6.1 The Controllability Gramian

The assistant operator of L_T

The L_T operator is defined from the Hilbert space $L^2([0,T],\mathcal{U})$ in Hilbert space H. It is a bounded operator. We have

$$L_T^* : \begin{cases} H & \rightarrow & L^2(0,T,\mathcal{U}), \\ x & \rightarrow & L_T^* x = \vartheta, \end{cases}$$

where ϑ is defined by:

$$\langle L_T^* x, u\rangle = (x, L_T),\ \forall u \in L^2(0,T,\mathcal{U}),\ \forall x \in \mathcal{H}.$$

where $\langle \cdot,\cdot\rangle$ denotes the scalar product in $L^2(0,T,\mathcal{U})$ and $(\cdot,\cdot)$ denotes the scalar product in $\mathcal{H}$.

$$\begin{aligned} (x, L_{T^u}) &= \left(x, \int_0^T S(t-s)Bu(s)\mathrm{d}s\right) \\ &= \int_0^T (x, \mathcal{S}(t-s)Bu(s))\mathrm{d}s \\ &= \int_0^T (B^* S^*(T-s), u(s))\mathrm{d}s \\ &= \langle B^*\mathcal{S}^*(T-\cdot)x, u\rangle. \end{aligned}$$

where B^* (resp. $\mathcal{S}^*(t-s)$) is the adjoint operator of B (resp. $\mathcal{S}(t-s)$). So

$$L_T^* = B^*\mathcal{S}^*(T-\cdot).$$

Definition 1.6.4

We define, $R_T(a)$, the set of states reachable at time T from To. We have :

$$R_T(a) = \mathcal{S}(T)a + R(L_T).$$

The study of controllability at time T comes down to the study of $\cup_{a\in\mathcal{H}} R_T(a) = R_{\mathcal{H}}(a)$. From fact of infinite dimension, we can have

$$R_T(\mathcal{H}) \neq \bar{R_T}(\mathcal{H})\ \mathrm{et}\ \mathcal{S}(T)\mathcal{H} \neq \mathcal{H}.$$

We now introduce the controllability operator called "Gramian of controllability". ***Controllability Gramian*** *Let's pose*

$$Q_T := L_T L_T^* = \int_0^T \mathcal{S}(T-s)BB^*\mathcal{S}^*(T-s)\mathrm{d}s,\ T > 0.$$

The Q_T operator is in $\mathcal{L}(\mathcal{H})$ and

$$\langle Q_T x, x\rangle = \int_0^T |B^*\mathcal{S}^*(T-s)x|^2 \mathrm{d}s = \| L_T^* x \|^2 \geq 0, \quad \forall x \in \mathcal{H}.$$

Definition 1.6.5

$Q_T := L_T L_T^*$ is called Controllability Gramian.

Proposition 1.6.2

$$R(L_T) = R\Big(Q_T^{\frac{1}{2}}\Big)$$

.

1.6.2 Approximate controllability

Definition 1.6.6

The pair $(A; B)$ is approximately controllable at time $T > 0$ if

$$R(\bar{L}_T) = \mathcal{H}.$$

Chapter 2

Approximate Controllability of Delayed Fractional Stochastic Differential Systems with Mixed Noise and Impulsive Effects

The work presented in this Chapter (2) is report a new class of impulsive fractional stochastic differential systems driven by mixed fractional **Brownian motions** with infinite delay and Hurst parameter $\hat{\mathcal{H}} \in (\frac{1}{2}, 1)$ Using fixed point techniques, a q-resolvent family, and fractional calculus, we discuss the existence of a piecewise continuous mild solution for the proposed system. Moreover, under appropriate conditions, we investigate the approximate controllability of the considered system, This work is attributed to the [53]

2.1 Introduction

For a long time, the subject of fractional calculus and its applications has gained a lot of importance,mainly because fractional calculus has become a powerful tool with more accurate and successful results in modeling several complex phenomena in numerous, seemingly diverse and widespread fields of science and engineering. It was found that various, especially interdisciplinary, applications can be elegantly modeled with the help of fractional derivatives [59, 75, 123, 178].See also the recent works of [85, 37, 161, 71].
Fractional Brownian motion (fBm for short) is a family of Gaussian random processes that are indexed by the Hurst parameter $\hat{\mathcal{H}} \in (0, 1)$. It is a self-similar stochastic process with long-range dependence ans stationary increment properties when $\hat{\mathcal{H}} > 1/2$. For more recent works on fractional Brownian motion, see [20, 37, 5, 146, 67, 42] and the references therein.
In order to describe various real-world problems in physical and engineering sciences subject to abrupt changes at certain instants during the evolution process, impulsive fractional differential equations have become important in recent years as mathematical models of many phenomena in both physical and social sciences. Impulsive effects begin at any arbitrary fixed point and continue with a finite time interval, known as non-instantaneous impulses. for more details, we refer the reader to [58, 159, 166, 167, 91, 10, 38, 19, 160].
The concept of controllability plays a major role in finite dimensional control theory. How-

ever, its generalization to infinite dimensions is too strong and has limited applicability, while approximate controllability is a weaker concept completely adequate in applications[155]. Recently, many authors have established approximate controllability results of (fractional) impulsive systems[143, 11]. For example, Kumar [81] investigated the approximate controllability for impulsive semilinear control systems with delay; Anukiruthika et al.[81] analyzed the approximate controllability of semilinear stochastic systems with impulses. Elthough several works exist in this area, the study of the approximate controllability of impulsive fractional stochastic differential suystems driven by mixed noise with infinite delay and Hurst parameter $\hat{\mathcal{H}} \in (1/2, 1)$ is still an understudied topic in the literature, This fact provides the motivation of our current work.
we consider an impulsive fractional stochastic delay differential equation with mixed fractiona Brownian motion defined by

$$\begin{cases} {}^c\mathcal{D}_t^q\, z(t) = \mathcal{P}z(t) + \mathcal{F}(t, z_t) + \mathcal{G}(t, z_t)\dfrac{d\hat{\mathcal{W}}(t)}{dt} + \sigma(t)\dfrac{d\mathcal{B}^{\hat{\mathcal{H}}}(t)}{dt}, \quad t \in \bigcup_{i=0}^{m}(s_i, t_{i+1}], \\ z(t) = K_i(t, z_t), \quad t \in \bigcup_{i=1}^{m}(t_i, s_i], \\ z(t) = \phi(t), \ \phi(t) \in \mathcal{D}_h, \end{cases} \tag{2.1}$$

where $\mathcal{P} : \mathcal{D}(\mathcal{P}) \subset \mathcal{Z} \to \mathcal{Z}$ is the generator of an q-resolvent family $\{\mathcal{S}_q(t) : t \geq 0\}$ on the separable Hilbert space
$\mathcal{Z}$, ${}^c\mathcal{D}_t^q$ is the Caputo fractional derivative of order $1/2 < q < 1$, and state $z(\cdot)$ takes values in the space $\mathcal{Z}$, and $0 = t_0 = s_0 < t_1 < s_1 < t_2 < \cdots < t_m < s_m < t_{m+1} = T < \infty$. The functions $\mathcal{K}_i(t, z_t)$ represent the non-instantaneous impulses during the intervals $(t_i, s_i]$, $i = 1, 2, \cdots, m$, $\hat{\mathcal{W}} = \{\hat{\mathcal{W}}(t) : t \geq 0\}$ is a Q-Wiener process defined on a separable Hilbert space $\mathcal{Y}_1$, and $B^{\hat{\mathcal{H}}} = \{B^{\hat{\mathcal{H}}}(t) : t \geq 0\}$ is a Q-fBm with the Hurst parameter $\hat{\mathcal{H}} \in (1/2, 1)$, defined on a separable Hilbert space $\mathcal{Y}_2$. The history-valued function $z_t : (-\infty, 0] \to \mathcal{Z}$ is defined as $z_t(\theta) = z(t + \theta), \forall\theta \leq 0$, and belongs to an abstract phase space $\mathcal{D}_h$. The initial data $\phi = \{\phi(t), t \in (\infty, 0]\}$ are $\mathcal{F}_0$-measurable, $\mathcal{D}_h$-valued random variable independent of $\hat{\mathcal{W}}$ and $B^{\hat{\mathcal{H}}}$. The functions $\mathcal{F}, \mathcal{G}, \sigma$, and $\mathcal{K}_i$ satisfy several suitable hypotheses, which will be specified later
The work is arranged as follows. In Section 2.2, relevant preliminaries are given that will be used later. we prove the existence os a piecewise continuous mild solution for the proposed system (2.1). Then, we study the approximate controllability for problem (2.1). an example is given to show the application of the obtained results. We end with Section, in which we present the conclusion of our results and also suggest directions of possible future research.

2.2 Preliminaries

Let $L(\mathcal{Y}_i, \mathcal{Z})$ denote the space of all linear and bounded operators from $\mathcal{Y}_i$ to $\mathcal{Z}, j = 1, 2$. The notation $\|\cdot\|$ represents the norms of $\mathcal{Z}, \mathcal{Y}_i, L(\mathcal{Y}_i, \mathcal{Z})$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ be a filtered complete probability space, where $\mathcal{F}_t$ is the σ-algebra generated by
$\{\mathcal{B}^{\hat{\mathcal{H}}}(e), \hat{\mathcal{W}}(e) : e \in [0, t]\}$ and $\mathcal{P}$-null sets. Let $\mathcal{Q}_j \in L(\mathcal{Y}_j, \mathcal{Y}_j)$ be the operators defined by $\mathcal{Q}_j e_i^j = \lambda_i^j e_i^j$ with finite trace $Tr(\mathcal{Q}_j) = \sum_{i=1}^{\infty} \lambda_i^j < \infty$, where $\{\lambda_i^j\}_{i\geq 1}$ are non-negative real numbers and $\{e_i^j\}_{i\geq 1}$ is a complete orthonormal basis in $\mathcal{Y}_j$. Then, there exists a real independent

sequence $\mathscr{B}_i(t)$ of the standard Wiener process such that

$$\hat{\mathcal{W}} = \sum_{i=1}^{\infty} \sqrt{\lambda_i^1} \mathcal{B}(t) e_i^1.$$

The infinite dimensional $\mathcal{Y}_2$-valued fBm $\mathcal{B}^{\hat{\mathcal{H}}}(t)$ is defined as

$$\mathcal{B}^{\hat{\mathcal{H}}}(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i^2} \mathscr{B}_i^{\hat{\mathcal{H}}}(t) e_i^2,$$

where $\mathcal{B}^{\hat{\mathcal{H}}}(t)$ are real, independent fBms. Let $\beta = \{\beta(t), t \in \mathcal{J}\}, \mathcal{J} = [0, T]$ be a Wiener process and $\mathcal{B}^{\hat{\mathcal{H}}} = \{\mathcal{B}^{\hat{\mathcal{H}}}(t), t \in \mathcal{J}\}$ be the one-dimensional fBm with Hurst index $\hat{\mathcal{H}} \in (1/2, 1)$. The fBm $\mathcal{B}^{\hat{\mathcal{H}}}(t)$ has the following integral representation:

$$\mathcal{B}^{\hat{\mathcal{H}}}(t) = \int_0^t \mathscr{K}_{\hat{\mathcal{H}}}(t, e) d\beta(e),$$

where the kernel $\mathscr{K}_{\hat{\mathcal{H}}}(t, e)$ is defined as

$$\mathscr{K}_{\hat{\mathcal{H}}}(t, e) = \mathfrak{X}_{\hat{\mathcal{H}}} e^{1/2 - \hat{\mathcal{H}}} \int_e^t (\mathcal{T} - e)^{\hat{\mathcal{H}} - 3/2} \mathcal{T}^{\hat{\mathcal{H}} - 1/2} d\mathcal{T} \text{ for } t > e.$$

We apply $\mathscr{K}_{\hat{\mathcal{H}}}(t, e) = 0$ if $t \leq e$, Note that $\dfrac{\partial \mathscr{K}_{\hat{\mathcal{H}}}}{\partial t}(t, e) = \mathfrak{X}_{\hat{\mathcal{H}}}(t/e)^{\hat{\mathcal{H}} - 1/2}(t - e)^{\hat{\mathcal{H}} - 3/2}$. Here, $X_{\hat{\mathcal{H}}} = [\hat{\mathcal{H}}(2\hat{\mathcal{H}} - 1)/\xi(2 - 2\hat{\mathcal{H}}, \hat{\mathcal{H}} - 1/2)]^{1/2}$ and $\xi(.,.)$ is the Beta function. for $\Lambda \in L^2([0, T])$, is follows from [108] that the Wiener-type integral of the function Λ w.r.t. fBm $\mathscr{B}^{\mathcal{H}}$is defined by

$$\int_0^T \Lambda(e) d\mathscr{B}^{\mathcal{H}}(e) = \int_0^T \mathscr{K}_{\mathcal{H}}^* \Lambda(e) d\mathscr{B}(e),$$

where $\mathscr{K}_{\hat{\mathcal{H}}}^* \Lambda(e) = \displaystyle\int_e^T \Lambda(t) \frac{\partial \mathscr{K}_{\hat{\mathcal{H}}}}{\partial t}(t, e) dt.$
Let $\varphi_j \in L(\mathcal{Y}_j, \mathcal{Z})$ and define

$$\| \varphi_j \|_{\mathcal{L}_{2^1}} = \left[\sum_{i=1}^{\infty} \| \sqrt{\lambda_i^j} \varphi_j e_i^j \|^2 \right]^{1/2}$$

if $\| \varphi_{j \mathcal{L}_2^j} < \infty$, then φ_j are called Q_j-Hilbert-Schmidt operators, and the spaces $\mathcal{L}_2^j(\mathcal{Y}_j, \mathcal{Z})$ are real ansd separable Hilbert spaces with inner product $\langle \varphi^1, \varphi^2 \rangle_{\mathcal{L}_2^j} = \langle \varphi^1 e_i^j, \varphi^2 e_i^j \rangle$. The stochastic integral of function $\Psi : \mathcal{J} \to \mathcal{L}_2^2(\mathcal{Y}_2, \mathcal{Z})$ w.r.t fBm $\mathcal{B}^{\hat{\mathcal{H}}}$ is defined by

$$\int_0^t \Psi(e) d\mathcal{B}^{\hat{\mathcal{H}}}(e) = \sum_{i=1}^{\infty} \int_0^t \sqrt{\lambda_i^2} \Psi(e) e_i^2 d\mathcal{B}_i^{\hat{\mathcal{H}}}(e) = \sum_{i=1}^{\infty} \int_0^t \sqrt{\lambda_i^2} \mathscr{K}_{\mathcal{H}}^*(\Psi e_i^2) d\mathscr{B}_i(e). \tag{2.2}$$

Lemma 2.2.1

if $\Psi : \mathcal{J} \to \mathcal{L}_2^2(\mathcal{Y}_2, \mathcal{Z})$ satisfies $\displaystyle\int_0^T \| \Psi(e)_{\mathcal{L}_2^2} \|^2 de < \infty$, then Equation (2.2) is a well-defined $\mathcal{Z}$-valued random variable such that

$$\mathbb{E} \left\| \int_0^T \Psi(e) d\mathcal{B}^{\hat{\mathcal{H}}}(e) \right\|^2 \leq 2\hat{\mathcal{H}} t^{2\hat{\mathcal{H}} - 1} \int_0^T \| \Psi(e) \|_{\mathcal{L}_2^2}^2 de.$$

Lemma 2.2.2 (*See[42]*)

For any $\alpha \geq 1$ and for an arbitrary $\mathcal{L}_2^1$-valued predictable process $Y(.)$,

$$\sup_{e\in[0,t]} \mathbb{E}\left\|\int_0^e Y(\mathcal{T})d\mathcal{W}(\mathcal{T})\right\|^{2\alpha} \leq (\alpha(2\alpha-1))^{\alpha}\left(\int_0^t \left(\mathbb{E} \parallel Y(e) \parallel_{\mathcal{L}_2^1}^{2\alpha}\right)^{1/\alpha} de\right)^{\alpha}, t \in [0,T].$$

For $\alpha = 1$ we obtain

$$\sup_{e\in[0,t]} \mathbb{E}\left\|\int_0^e Y(\mathcal{T})d\hat{\mathcal{W}}(\mathcal{T})\right\|^{2} \leq \int_0^t \mathbb{E} \parallel Y(e) \parallel_{\mathcal{L}_2^1}^{2} de.$$

Assume that $h : (-\infty, 0] \to (0,\infty)$ with $\omega = \int_{-\infty}^0 h(t)dt < \infty$ is a continuous function. We define $\mathcal{D}_h$ by

$$\mathcal{D}_h = \Big\{\phi : (-\infty, 0) \to \mathcal{Z},\ \text{for any } a > 0,\ (\mathbb{E}\,|\phi(\theta)|^2)^{1/2} \text{ is a measurable and bounded function on } [-a, 0] \text{ with } \phi(0) = 0,\ \text{and } \int_{-\infty}^0 h(e) \sup_{e\leq\theta\leq 0}(\mathbb{E}\,|\phi(\theta)|^2)^{1/2}de < \infty\Big\}.$$

if $\mathcal{D}_h$ is endowed with the norm

$$\parallel \phi \parallel_{\mathcal{D}_h}= \int_{-\infty}^0 h(e) \sup_{e\leq\theta\leq 0}(\mathbb{E} \parallel \phi(\theta) \parallel^2)^{1/2}de, \phi \in \mathcal{D}_h,$$

then $\left(\mathcal{D}_h, \|.\|_{\mathcal{D}_h}\right)$ is a Banach space [143].

Define the space $\mathcal{D}_{\mathcal{T}} = \{z : (-\infty, T] \to \mathcal{Z}, z|_{\mathcal{J}_i} \in C(\mathcal{J}_i, \mathcal{Z}), i = 0, 1, \cdots, m,$ and there exist $z(t_i^-)$ and $z(t_i^+)$ with $a(t_i^-) = z(t_i)$, and $z_0 = \phi \in \mathcal{D}_h\}$, with the norm

$$\parallel z \parallel_{\mathcal{D}_{\mathcal{T}}}=\parallel \phi \parallel_{\mathcal{D}_h} + \sup_{t\in[0,T]}(\mathbb{E} \parallel z(t) \parallel^2)^{1/2},$$

where $\mathcal{J}_i = (t_i, t_{i+1}], i = 0, 1, \cdots, m.$

Lemma 2.2.3

if for all $t \in [0,T], z_t \in \mathcal{D}_h, z_0 \in \mathcal{D}_h$. then

$$\parallel z_t \parallel_{\mathcal{D}_h}\leq \omega \sup_{t\in[0,T]}(\mathbb{E} \parallel z(t) \parallel^2)^{1/2}+ \parallel z_0 \parallel_{\mathcal{D}_h}.$$

Definition 2.2.1 (*see [56]*)

Let $\mathcal{M} > 0, \theta \in [\pi/2, \pi]$, and $\omega \in \mathbb{R}$. A closed and linear operator $\mathcal{P}$ is called a sectorial operator if

1. $\varrho(\mathcal{P}) \subset \sum_{(\theta,\omega)} = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |arg(\lambda-\omega)| < \theta\},$
2. $\parallel \mathcal{R}(\lambda, \mathcal{P}) \parallel\leq \mathcal{M}/|\lambda-\omega|,\ \lambda \in \sum_{(\theta,\omega)}$

Lemma 2.2.4 (*see [27]*)

Let $\mathcal{P}$ be a sectorial operator. Then, the unique solution of the linear fractional system

$$\begin{cases} {}^{c}\mathcal{D}_t^q\, z(t) = \mathcal{P}z(t) + \mathcal{F}(t), \quad t > t_0 \geq 0, \quad 0 < q < 1 \\ z(t) = \phi(t), \quad t \leq t_0. \end{cases}$$

is given by

$$z(t) = \mathcal{T}_q(t - t_0)z(t_0) + \int_{t_0}^{t} \mathcal{S}_q(t - e)\mathcal{F}(e)de,$$

where

$$\mathcal{T}_q(t) = \frac{1}{2\pi i}\int_{B_r} e^{\lambda t}\frac{\lambda^{q-1}}{\lambda^q - \mathcal{P}}d\lambda,$$

$$\mathcal{S}_q(t) = \frac{1}{2\pi i}\int_{B_r} \frac{e^{\lambda t}}{\lambda^q - \mathcal{P}}d\lambda,$$

Here, $\mathcal{B}_r$ denotes the Bromwich path.

2.3 Solvability Results

We assume the following hypotheses.

Hypothesis 1 (*H1*)

if $q \in (0,1)$ and $\mathcal{P} \in \mathcal{P}^q(\theta_0, \omega_0)$, then , for any $z \in \mathcal{Z}$ and $t > 0$, we have $\| \mathcal{T}_q(t) \| \leq C_1 e^{\omega t}$ and $\| \mathcal{S}_q(t) \| \leq C_2 e^{\omega t}(1 + t^{q-1}), \omega > \omega_0$. Thus, we have

$$\| \mathcal{T}_q(t) \| \leq \mathcal{M}_1 \text{ and } \| \mathcal{S}(t) \| \leq \mathcal{M}_2 t^{q-1},$$

where $\mathcal{M}_1 = \sup\limits_{0\leq t\leq T} C_1 e^{\omega t}$ and $\mathcal{M}_2 = \sup\limits_{0\leq t\leq T} C_2 e^{\omega t}(1 + t^{q-1})$.

Hypothesis 2 (*H2*)

There exists a constant $N_{\mathcal{F}} > 0$ such that

$$\mathbb{E} \| \mathcal{F}(t, \psi_1) - \mathcal{F}(t, \psi_2) \|^2 \leq N_{\mathcal{F}} \| \psi_1 - \psi_2 \|^2_{\mathcal{D}_h}, \qquad \forall t \in \mathcal{J}, \quad \psi_1, \psi_2 \in \mathcal{D}_h$$

Hypothesis 3 (*H3*)

function $\sigma : \mathcal{J} \to \mathcal{L}_2^2(\mathcal{Y}_2, \mathcal{Z})$ satisfies $\int_0^t \| \sigma(e) \|^2_{\mathcal{L}_2^2}\, de < \infty$, for every $t \in \mathcal{J}$, and there exists a constant $\Lambda_\sigma > 0$ such that $\| \sigma(e) \|^2_{\mathcal{L}_2^2} \leq \Lambda_\sigma$, uniformly in $\mathcal{J}$.

Hypothesis 4 (*H4*)

There exists a constant $N_{\mathcal{G}} > 0$ such that

$$\mathbb{E} \| \mathcal{G}(t, \psi_1) - \mathcal{G}(t - \psi_2) \|^2_{\mathcal{D}_h} \leq N_{\mathcal{G}_i} \| \psi_1 - \psi_2 \|^2_{\mathcal{D}_h}, \quad \forall t \in \mathcal{J}, \quad \psi_1, \psi_2 \in \mathcal{D}_h$$

Hypothesis 5 (*H5*)

There are constants $L_{\mathcal{K}_i} > 0, i = 1, 2, \cdots, m$, such that

$$\mathbb{E} \| \mathcal{K}_i(t, \psi_1) - \mathcal{K}_i(t, \psi_2) \|^2 \leq L_{\mathcal{K}_i} \| \psi_1 - \psi_2 \|^2_{\mathcal{D}_h}, \quad \forall t \in \mathcal{J}, \quad \psi_1, \psi_2 \in \mathcal{D}_h.$$

Definition 2.3.1

An $\mathcal{F}_t$-adapted random process $z : (-\infty, T] \to \mathcal{Z}$ is called the mild solution of (2.1) if, for every $t \in \mathcal{J}$, $z(t)$ satisfies $z_0 = \phi \in \mathcal{D}_h$, $z(t) = \mathcal{K}_i(t, z_t)$ for all $t \in (t_i, s_i]$, $i = 1, 2, \cdots, m$, and

$$z(t) = \int_0^t \mathcal{S}_q(t-e)\mathcal{F}(e, z_e)de + \int_0^t \mathcal{S}_q(t-e)\mathcal{G}(e, z_e)d\hat{\mathcal{W}}(e) + \int_0^t \mathcal{S}_q(t-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e), .$$

for all $t \in [0, t_1]$, and

$$\begin{aligned} z(t) = \mathcal{T}_q(t-s_i)\mathcal{K}_i(s_i, z_{s_i}) + \int_{s_i}^t \mathcal{S}_q(t-e)\mathcal{F}(e, z_e)de \\ + \int_{s_i}^t \mathcal{S}_q(t-e)\mathcal{G}(e, z_e)d\hat{\mathcal{W}}(e) + \int_{s_i}^t \mathcal{S}_q(t-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e), \end{aligned} \tag{2.3}$$

for all $t \in (s_i, t_{i+1}]$, $i = 1, 2, \cdots, m$.

Theorem 2.3.1

Assume that conditions $(H1)$-$(H5)$ are satisfies. then, problem (2.1) has a unique mild solution on $(-\infty, T]$, provid that

$$L_{\mathcal{R}} = \max_{1 \le i \le m} \left\{\eta_0, \omega^2 L_{\mathcal{K}_i}, \eta_i\right\} < 1,$$

where

$$\eta_0 = 2\mathcal{M}_2^2\omega^2 \left(\frac{N_{\mathcal{F}} t_1^{2q}}{q^2} + \frac{N_{\mathcal{G}} t_1^{2q-1}}{2q-1}\right),$$

$$\eta_i = \left(3\mathcal{M}_1^2 L_{\mathcal{K}_i}\omega^2 + 3\mathcal{M}_2^2\omega^2 \left\{\frac{N_{\mathcal{F}} t_{i+1}^{2q}}{q^2} + \frac{N_{\mathcal{G}} t_{i+1}^{2q-1}}{2q-1}\right\}\right)$$

.

Proof: We definde the operator Ξ from $\mathcal{D}_{\mathcal{T}}$ to $\mathcal{D}_{\mathcal{T}}$ as follows:

$$(\Xi z)(t) = \begin{cases} \phi(t), & t \in (-\infty, 0] \\ \int_0^t \mathcal{S}_q(t-e)\mathcal{F}(e, z_e)de & \\ + \int_0^t \mathcal{S}_q(t-e)\mathcal{G}(e, z_e)d\hat{\mathcal{W}}(e) + \int_0^t \mathcal{S}_q(t-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e), & t \in [0, t_1] \\ \mathcal{K}_i(t, z_t), & t \in (t_i, s_i] \\ \mathcal{T}_q(t-s_i)\mathcal{K}_i(s_i, z_{s_i}) + \int_{s_i}^t \mathcal{S}_q(t-e)\mathcal{F}(e, z_e)de & \\ + \int_{s_i}^t \mathcal{S}_q(t-e)\mathcal{G} - e, z_e)d\hat{\mathcal{W}}(e) + \int_{s_i}^t \mathcal{S}_q(t-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e), & t \in (s_i, t_{i+1}]. \end{cases}$$

For $\phi \in \mathcal{D}_h$, define

$$g(t) = \begin{cases} \phi(t) & t \in (-\infty, 0] \\ 0, & t \in \mathcal{J} \end{cases}.$$

Then, $g_0 = \phi$. Next we define

$$\overline{y}(t) = \begin{cases} 0, & t \in (-\infty, 0] \\ y(t), & t \in \mathcal{J} \end{cases}$$

for each $y \in C(\mathcal{J}, R)$ with $z(0) = 0$. if $z(\cdot)$ satisfies (2.3), then $z(t) = g(t) + \overline{y}(t)$ for $t \in \mathcal{J}$, which implies that $z_t = g_t + \overline{y}_t$ for $t \in \mathcal{J}$, and the unction $y(\cdot)$ satisfies

$$y(t) = \begin{cases} \int_0^t \mathcal{S}_q(t-e)\mathcal{F}(e, g_e + \overline{y}_e)de + \int_0^t \mathcal{S}_q(1-e)\mathcal{G}(e, g_e + \overline{y}_e)d\hat{\mathcal{W}}(e) & \\ + \int_0^t \mathcal{S}_q(t-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e), & t \in [0, t_1] \\ \mathcal{K}_i(t, g_t + \overline{y}_t), & tt \in (s_i, t_{i+1}] \\ \mathcal{T}_q(t-s_i)\mathcal{K}_i(s_i, g_{s_i} + \overline{y}_e)d\hat{\mathcal{W}}(e) + \int_{s_i}^t \mathcal{S}_q(t-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e) & tt \in (s_i, t_{i+1}]. \end{cases}$$

Set $\mathcal{D}_T^0 = \{y \in \mathcal{D}_T \text{ such that } y_0 = 0\}$. For any $y \in \mathcal{D}_T^0$, we obtain

$$\|y\|_{\mathcal{D}_T^0} = \|y_0\|_{\mathcal{D}_h} + \sup_{t \in \mathcal{J}} \left(\mathbb{E}\|y(t)\|^2\right)^{1/2}.$$

Thus, $\left(\mathcal{D}_T^0, \|\cdot\|_{\mathcal{D}_T^0}\right)$ is a Banach space. define the operator ψ from $\mathcal{D}_T^0$ to $\mathcal{D}_T^0$ as follows:

$$(\psi_y)(t) = \begin{cases} \int_0^t \mathcal{S}_q(t-e)\mathcal{F}(e, g_e + \overline{y}_e)de + \int_0^t \mathcal{S}_q(t-e)\mathcal{G}(e, g_e + \overline{y}_e)d\hat{\mathcal{W}}(e) & \\ + \int_0^t \mathcal{S}_q(t-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e), & t \in [0, t_1] \\ \mathcal{K}_i(t, g_t + \overline{y}_t), & t \in (t_i, s_i] \\ \mathcal{T}_q(t-s_i)\mathcal{K}_i(s_i, g_{s_i} + \overline{y}_{s_i}) + \int_{s_i}^t \mathcal{S}_q(t-e)\mathcal{F}(e, g_e + \overline{y}_e)de & \\ + \int_{s_i}^t \mathcal{S}_q(t-e)\mathcal{G}(e, g_e + \overline{y}_e)d\hat{\mathcal{W}}(e) + \int_{s_i}^t \mathcal{S}_q(t-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e) & t \in (s_i, t_{i+1}] \end{cases}$$

Set $\mathcal{D}_T^0 = \{y \in \mathcal{D}_T \text{ such that } y_0 = 0\}$. For any $y \in D_T^0$, we obtain

$$\|y\|_{D_T^0} = \|y_0\|_{\mathcal{D}_h} + \sup_{t \in [0,T]} \left(\mathbb{E}\|y(t)\|^2\right)^{1/2} = \sup_{t \in [0,T]} \left(\mathbb{E}\|y(t)\|^2\right)^{1/2}.$$

Thus, $\left(D_T^0, \|\cdot\|\right)$ is a Banach space.

Define the operator ψ from D_T^0 to D_T^0 as followsM

$$(\psi_y)(t) = \begin{cases} \int_0^t \mathcal{S}_q(t-e)\mathcal{F}(e, g_e + \overline{y}_e)de + \int_0^t \mathcal{S}_q(t-e)\mathcal{G}(e, g_e + \overline{y}_e)d\hat{\mathcal{W}}(e) & \\ + \int_0^t \mathcal{S}_q(t-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e) & t \in [0, t_1] \\ \mathcal{K}_i(t, g_t + \overline{y}_t), & t \in (t_i, s_i] \\ \mathcal{T}_q(t-s_i)\mathcal{K}_i(s_i, g_{s_i} + \overline{y}_{s_i}) + \int_{s_i}^t \mathcal{S}_q(t-e)\mathcal{F}(e, g_e + \overline{y}_e)de & \\ + \int_{s_i}^t \mathcal{S}_q(t-e)\mathcal{G}(e, g_e + \overline{y}_e)d\hat{\mathcal{W}}(e) + \int_{s_i}^t \mathcal{S}_q(t-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e) & t \in (s_i, t_{i+1}]. \end{cases}$$

in order to prove the existence result, we need to show that ψ has a unique fixed point. Let $y, y^* \in D_T^0$. then, for all $t \in [0, t_1]$, we have

$$\begin{aligned} \mathbb{E}\|\psi_y)(t) - (\psi_1 y^*)(t)\|^2 \leq{} & 2\mathbb{E}\left\|\int_0^t \mathcal{S}_q(t-e)(\mathcal{F}(e, g_e + \overline{y}_e) - \mathcal{F}(e, g_e + \overline{y}_e^*))de\right\|^2 \\ & + 2\mathbb{E}\left\|\int_0^t \mathcal{S}_q(t-e)(\mathcal{G}(e, g_e + \overline{y}_e) - \mathcal{G}(e, g_e + \overline{y}_e^*))d\hat{\mathcal{W}}(e)\right\|^2 \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{2\mathcal{M}_2^2 t_1^q}{q}\int_0^t (t-e)^{q-1} N_{\mathcal{F}} \left\|\overline{y}_e-\overline{y}_e^*\right\|_{\mathcal{D}_h}^2 de \\
&\quad + 2\mathcal{M}_2^2 \int_0^t (t-e)^{2q-2} N_{\mathcal{G}} \left\|\overline{y}_e-\overline{y}_e^*\right\|_{\mathcal{D}_h}^2 de \\
&\leq \frac{2\mathcal{M}_2^2 t_1^q}{q}\int_0^t (t-e)^{q-1} N_{\mathcal{F}}\omega^2 \sup_{e\in\mathcal{J}} \mathbb{E}\left\|y(e)-y^*(e)\right\|^2 de \\
&\quad + 2\mathcal{M}_2^2 \int_0^t (t-e)^{2q-2} N_{\mathcal{G}}\omega^2 \sup_{e\in\mathcal{J}} \mathbb{E}\left\|y(e)-y^*(e)\right\|^2 de \\
&\leq 2\mathcal{M}_2^2\omega^2\left(\frac{N_{\mathcal{F}} t_1^{2q}}{q^2}+\frac{N_{\mathcal{G}} t_1^{2q-1}}{2q-1}\right)\left\|y-y^*\right\|_{D_T^0}^2
\end{aligned}
$$

Hence,

$$
\mathbb{E}\left\|(\Psi_y)(t)-(\psi_{y^*})(t)\right\|^2 \leq 2\mathcal{M}_2^2\omega^2\left(\frac{N_{\mathcal{F}} t_1^{2q}}{q^2}+\frac{N_{\mathcal{G}} t_1^{2q-1}}{2q-1}\right)\left\|y-y^*\right\|_{D_T^0}^2 \tag{2.4}
$$

For $t\in(s_i,t_{i+1}]$,$i=1,2,\cdots,m$, we have

$$
\begin{aligned}
\mathbb{E}\left\|(\psi_y)(t)-(\psi_{y^*})(t)\right\|^2 &\leq \mathbb{E}\left\|\mathcal{K}_i(t,g_t+\overline{y}_t)-\mathcal{K}_i(e,g_e+\overline{y}_e^*)\right\|^2 \\
&\leq L_{\mathcal{K}_i}\left\|\overline{y}_t-\overline{y}_t^*\right\|_{\mathcal{D}_h}^2 \\
&\leq L_{\mathcal{K}_i}\omega^2 \sup_{t\in\mathcal{J}} \mathbb{E}\left\|y(t)-y^*(t)\right\|^2 \\
&\leq L\mathcal{K}_i\omega^2\left\|y-y^*\right\|_{D_T^0}^2 .
\end{aligned}
$$

Hence,

$$
\mathbb{E}\left\|(\psi_y)(t)-(\psi_{y^*})(t)\right\|^2 \leq L_{\mathcal{K}_i}\omega^2\left\|y-y^*\right\|_{D_T^0}^2 . \tag{2.5}
$$

Similarly, for $t\in(s_i,t_{i+1}]$,$i=1,2,\cdots,m$, we have

$$
\begin{aligned}
\mathbb{E}\left\|(\psi_y)(t)-(\psi_{y^*})(t)\right\|^2 &\leq 3\mathbb{E}\left\|\mathcal{T}_q(t-s_i)(\mathcal{K}_i(s_i,g_{s_i}+\overline{y}_{s_i})-\mathcal{K}_i(s_i,g_{s_i}+\overline{y}_{s_i}^*))\right\|^2 \\
&\quad + 3\mathbb{E}\left\|\int_{s_i}^t \mathcal{S}_q(t-e)(\mathcal{F}(e,g_e+\overline{y}_e)-\mathcal{F}(e,g_e+\overline{y}_e^*))de\right\|^2 \\
&\quad + 3\mathbb{E}\left\|\int_{s_i}^t \mathcal{S}_q(t-e)(\mathcal{G}(e,g_e+\overline{y}_e)-\mathcal{G}(e,g_e+\overline{y}_e^*))d\hat{\mathcal{W}}(e)\right\|^2 \\
&\leq 3\mathcal{M}_1^2 L_{\mathcal{K}_i}\omega^2\left\|y-y^*\right\|_{D_T^0}^2 \\
&\quad + \frac{3\mathcal{M}_2^2 t_{i+1}^q}{q}\int_{s_i}^t (t-e)^{q-1} N_{\mathcal{F}}\left\|\overline{y}_e-\overline{y}_e^*\right\|_{\mathcal{D}_h}^2 de \\
&\quad + 3\mathcal{M}_2^2\int_{s_i}^t (t-e)^{2q-2} N_{\mathcal{F}}\left\|\overline{y}_e-\overline{y}_e^*\right\|_{\mathcal{D}_h}^2 de \\
&\leq 3\mathcal{M}_1^2 L_{\mathcal{K}_i}\omega^2\left\|y-y^*\right\|_{D_T^0}^2 \\
&\quad + \frac{3\mathcal{M}_2^2 t_{i+1}^q}{q}\int_{s_i}^t (t-e)^{q-1} N_{\mathcal{F}}\omega^2 \sup_{e\in\mathcal{J}} \mathbb{E}\left\|y(e)-y^*(e)\right\|^2 de \\
&\quad + 3\mathcal{M}_2^2\int_{s_i}^t (t-e)^{2(q-1)} N_{\mathcal{G}}\omega^2 \sup_{e\in\mathcal{J}} \mathbb{E}\left\|y(e)-y^*(e)\right\|^2 de \\
&\leq \left(3\mathcal{M}_1^2 L\mathcal{K}_i\omega^2+3\mathcal{M}_2^2\omega^2\left\{\frac{N_{\mathcal{F}} t_{i+1}^{2q}}{q^2}+\frac{N_{\mathcal{G}} t_{i+1}^{2q-1}}{2q-1}\right\}\right)\left\|y-y^*\right\|_{\mathcal{D}_h}^2 .
\end{aligned}
$$

Hence

$$\mathbb{E}\left\|(\Psi_y)(t)-(\Psi_{y^*})(t)\right\|^2 \leq \left(3\mathcal{M}_1^2 L\mathcal{K}_i\omega^2\left\{\frac{N_{\mathcal{F}}t_{i+1}^{2q}}{q^2}+\frac{N_{\mathcal{G}}t_{i+1}^{2q-1}}{2q-1}\right\}\right)\|y-y^*\|_{\mathcal{D}_h}^2. \tag{2.6}$$

From Equation (2.4)-(2.6), we obtain that

$$\mathbb{E}\left\|\Psi_y-\Psi_{y^*}\right\|_{D_T^0}^2 \leq L_{\mathcal{R}}\left\|y-y^*\right\|_{D_T^0}^2,$$

which implies that Ψ is a contraction. Hence, Ψ has a unique fixed point $y \in D_T^0$, which is a mild solution of problem (2.1) on $(-\infty, T]$. ■

Next, using krasnoselskii's fixed point theorem, we establish the second existence result. At this stage we make the folloing assumptions.

Hypothesis 6 (*H6*)

The map $\mathcal{F}: \mathcal{J}\times\mathcal{D}_h \to \mathcal{Z}$ is a continuous function, and there exists a continuous function $\xi_1: \mathcal{J}\to(0,\infty)$ such that

$$\mathbb{E}\left\|\mathcal{F}(t,\psi)\right\|^2 \leq \xi_1(t)\left\|\psi\right\|_{\mathcal{D}_h}^2,$$

for all $t\in\mathcal{J}$, and $\xi_1^* = \sup_{t\in\mathcal{J}}\xi_1(t)$.

Hypothesis 7 (*H7*)

The map $\mathcal{G}: \mathcal{J}\times\mathcal{D}_h \to \mathcal{L}_2^1(\mathcal{Y}_1,\mathcal{Z})$ is a continuous function, and there exists a continuous function $\xi_2: \mathcal{J}\to(0,\infty)$ such that

$$\mathbb{E}\left\|\mathcal{G}(t,\psi)\right\|_{\mathcal{L}_2^1}^2 \leq \xi_2(t)\left\|\psi\right\|_{\mathcal{D}_h}^2,$$

for all $t\in\mathcal{J}$ and $\xi_2^* = \sup_{t\in\mathcal{J}}\xi_2(t)$.

Hypothesis 8 (*H8*)

The inequality

$$L_{\mathcal{HR}} = 2\mathcal{M}_2^2\omega^2\left(\frac{N_{\mathcal{F}}\mathcal{T}^{2q}}{q^2}+\frac{N_{\mathcal{G}}\mathcal{T}^{2q-1}}{2q-1}\right) < 1$$

holds and

$$\max_{1\leq i\leq m}\{\kappa_0, \upsilon_i\lambda_3, \kappa_i\} < \pi,$$

where

$$\kappa_0 = 3\mathcal{M}_2^2 t_1^{2q}\left(\frac{\lambda_1}{q^2}+\frac{\lambda_2}{t_1(2q-1)}+\frac{2\hat{\mathcal{H}}\lambda_\sigma t_1^{2\hat{\mathcal{H}}-2}}{2q-1}\right),$$

$$\kappa_i = 4\mathcal{M}_1^2\upsilon_i\lambda_3 + 4\mathcal{M}_2^2 t_{i+1}^{2q}\left(\frac{\lambda_1}{q^2}+\frac{\lambda_2}{t_{i+1}(2q-1)}+\frac{2\hat{\mathcal{H}}\Lambda_\sigma t_{i+1}^{2\hat{\mathcal{H}}-2}}{2q-1}\right).$$

Hypothesis 9 (*H9*)

The maps $\mathcal{K}_i: (t_i, s_i]\times\mathcal{D}_h \to \mathcal{Z}, i = 1,2,\cdots,m$, are continuous functions and

i. there exist costants $\upsilon > 0, i = 1,2,\cdots,m$, such that $\mathbb{E}\left\|\mathcal{K}_i(t,\psi)\right\|^2 \leq \upsilon_i\left\|\psi\right\|_{\mathcal{D}_h}^2$ for all $t\in\mathcal{J}$;

ii. *the set* $\{b_i : b_i \in V(\pi, \mathcal{K}_i)\}$ *is an equicontinuous subset of* $C\langle(t_i, s_i], \mathcal{Z}\rangle, i = 1, 2, \cdots, m.$ *where* $V(\pi, \mathcal{K}_i) = \{t \to \mathcal{K}_i(t, y_t) : y \in \mathcal{D}_\pi\}.$

The set $\mathcal{D}_r = \left\{y \in D_T^0 : \|y\|^2_{D_T^0} \le r, r > 0\right\}$ is clearly a convex closed bounded set in D_T^0 for each $uy \in \mathcal{D}_r$. by lemma (2.2.3), we obtain

$$\begin{aligned}
\|x_t + \overline{y}_t\|^2_{\mathcal{D}_h} &\le 2\left(\|x_t\|^2_{\mathcal{D}_h} + \|\overline{y}_t\|^2_{\mathcal{D}_h}\right)\\
&\le 4\left(\omega^2 \sup_{\upsilon\in[0,t]} \mathbb{E}\,\|x(\upsilon)\|^2 + \|x_0\|^2_{\mathcal{D}_h}\right) + 4\left(\omega^2 \sup_{\upsilon\in[0,t]} \mathbb{E}\,\|\overline{y}(\upsilon)\|^2 + \|\overline{y}_0\|^2_{\mathcal{D}_h}\right)\\
&\le 8\left(\|\phi\|^2_{\mathcal{D}_h} + \omega^2 r\right).
\end{aligned}$$

Let

$$\lambda_1 = 8\xi_1^*\left(\|\phi\|^2_{\mathcal{D}_h} + \omega^2 r\right),\ \lambda_2 = 8\xi_2^*\left(\|\phi\|^2_{\mathcal{D}_h} + \omega^2 r\right),\ \lambda_3 = 8\left(\|\phi\|^2_{\mathcal{D}_h} + \omega^2 r\right).)$$

Theorem 2.3.2

Assume conditions (H1)-(H9) are satisfied. Then, problem (2.1) has at least one mild solution on $(-\infty, T]$.

Proof: Let $\mathcal{E}_1 : \mathcal{D}_r \to \mathcal{D}_r$ and $\mathcal{E}_2 : \mathcal{D}_r \to \mathcal{D}_r$ be defined as

$$\mathcal{E}_1(y)(t) = \begin{cases} 0 & t \in [0, t_1]\\ \mathcal{K}_i(t, g_t + \overline{y}_t), & t \in (t_i, s_i]\\ \mathcal{T}_q(t - s_i)\mathcal{K}_i(s_i, g_{s_i} + \overline{y}_{s_i}) & t \in (s_i, t_{i+1}] \end{cases}$$

and

$$\mathcal{E}_2(y)(t) = \begin{cases} \displaystyle\int_0^t \mathcal{S}_q(t-e)\mathcal{F}(e, g_e + \overline{y}_e)de & \\ \displaystyle+ \int_0^t \mathcal{S}_q(t-e)\mathcal{G}(e, g_e + \overline{y}_e)d\hat{\mathcal{W}}(e) + \int_0^t \mathcal{S}_q(t-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e), & t \in [0, t_1]\\ 0, & t \in (t_i, s_i]\\ \displaystyle\int_{s_i}^t \mathcal{S}_q(t-e)\mathcal{F}(e, g_e + \overline{y}_e)de & \\ \displaystyle+ \int_{s_i}^t \mathcal{S}_q(t-e)\mathcal{G}(e, g_e + \overline{y}_e)d\hat{\mathcal{W}}(e) + \int_{s_i}^t \mathcal{S}_q(t-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e) & t \in (s_i, t_{i+1}] \end{cases}$$

For convenience, we divide the proof into various steps.

Step 1. We show that $\mathcal{E}_1 y + \mathcal{E}_2 y^* \in \mathcal{D}_r$. For $y, y^* \in \mathcal{D}_r$ and for $t \in [0, t_1]$, we obtain

$$\begin{aligned}
\mathbb{E}\,\|(\mathcal{E}_1 y)(t) + \mathcal{E}_2 y^*)(t)\|^2 &\le 3\mathbb{E}\left\|\int_0^t \mathcal{S}_q(t-e)\mathcal{F}(e, g_e + \overline{y}_e^*)de\right\|^2\\
&\quad + 3\mathbb{E}\left\|\int_0^t \mathcal{S}_q(t-e)\mathcal{G}(e, g_e + \overline{y}_e^*)d\hat{\mathcal{W}}(e)\right\|^2\\
&\quad + 3\mathbb{E}\left\|\int_0^t \mathcal{S}_q(t-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e)\right\|^2\\
&\le 3\mathcal{M}_2^2\left(\int_0^t (t-e)^{q-1}de\right)\left(\int_0^t (t-e)^{q-1}\xi_1(e)\,\|g_e + \overline{y}_e^*\|^2_{\mathcal{D}_h}\,de\right)\\
&\quad + 3\mathcal{M}_2^2\int_0^t (t-e)^{2q-2}\xi_2(e)\,\|g_e + \overline{y}_e^*\|^2_{\mathcal{D}_h}\,de
\end{aligned}$$

$$+ 6\hat{\mathcal{H}}\Lambda_\sigma \mathcal{M}_2^2 t_1^{2\hat{\mathcal{H}}-1} \int_0^t (t-e)^{2q-2} de$$
$$\leq 3\mathcal{M}_2^2 t_1^{2q} \left(\frac{\lambda_1}{q^2} + \frac{\lambda_2}{t_1(2q-1)} + \frac{2\hat{\mathcal{H}}\Lambda_\sigma t_1^{2\hat{\mathcal{H}}-2}}{2q-1} \right).$$

Hence,

$$\mathbb{E} \|(\mathcal{E}_1 y)(t) + (\mathcal{E}_2 y^*)(t)\|^2 \leq 3\mathcal{M}_2^2 t_1^{2q} \left(\frac{\lambda_1}{q^2} + \frac{\lambda_2}{t_1(2q-1)} + \frac{2\hat{\mathcal{H}}\Lambda_\sigma t_1^{2\hat{\mathcal{H}}-2}}{2q-1} \right). \tag{2.7}$$

For $t \in (s_i, t_{i+1}]$, $i = 1, 2, \cdots, m$, we have

$$\begin{aligned} \mathbb{E} \|(\mathcal{E}_1 y)(t) + (\mathcal{E}_2 y^*)(t)\|^2 &\leq \mathbb{E} \|\mathcal{K}_i(t, g_t + \overline{y}_t)\|^2 \\ &\leq \upsilon_i \|g_t + \overline{y}_t\|^2_{\mathcal{D}_h} \\ &\leq \upsilon i \lambda_3. \end{aligned}$$

Hence,

$$\mathbb{E} \|(\mathcal{E}_1 y)(t) + (\mathcal{E}_2 y^*)(t)\|^2_{\leq} \upsilon_i \lambda_3. \tag{2.8}$$

Similarly, for $t \in (s_i, t_{i+1}]$, $i = 1, 2, \cdots, m$, we have

$$\begin{aligned} \mathbb{E} \|(\mathcal{E}_1 y)(t) + (\mathcal{E}_2 y^*)(t)\|^2 &\leq 4\mathbb{E} \left\| \mathcal{T}_q(t - si)\mathcal{K}_i(s_i, g_{s_i} + \overline{y}_{s_i}) \right\|^2 \\ &+ 4\mathbb{E} \left\| \int_{s_i}^t \mathcal{S}_q(t-e)\mathcal{F}(e, g_e + \overline{y}_e^*) de \right\|^2 \\ &+ 4\mathbb{E} \left\| \int_{s_i}^t \mathcal{S}_q(t-e)\mathcal{G}(e, g_e + \overline{y}_e) d\hat{\mathcal{W}}(e) \right\|^2 \\ &+ 4\mathbb{E} \left\| \int_{s_i}^t \mathcal{S}_q(t-e)\sigma(e) d\mathcal{B}^{\hat{\mathcal{H}}}(e) \right\|^2 \\ &\leq 4\mathcal{M}_2^2 \left(\int_{s_i}^t (t-e)^{q-1} de \right) \left(\int_{s_i}^t (t-e)^{q-1} \xi_1(e) \|g_e + \overline{y}_e^*\|^2_{\mathcal{D}_h} de \right) \\ &+ 4\mathcal{M}_2^2 \int_{s_i}^t (t-e)^{2q-2} \xi_2(e) \|g_e + \overline{y}_e^*\|^2_{\mathcal{D}_h} de \\ &+ 8\hat{\mathcal{H}}\Lambda_\sigma \mathcal{M}_2^2 t_{i+1}^{2\hat{\mathcal{H}}-1} \int_{s_i}^t (t-e)^{2q-2} de \\ &\leq 4\mathcal{M}_1^2 \upsilon_i \lambda_3 + 4\mathcal{M}_2^2 t_{i+1}^{2a} \left(\frac{\lambda_1}{q^2} + \frac{\lambda_2}{t_{i+1}(2q-1)} + \frac{2\hat{\mathcal{H}}\Lambda_\sigma t_{i+1}^{2\hat{\mathcal{H}}-2}}{2q-1} \right). \end{aligned}$$

Therefore,

$$\mathbb{E} \|(\mathcal{E}_1 y)(t) + (\mathcal{E}_2 y^*)(t)\|^2 \leq 4\mathcal{M}_1^2 \upsilon_i \lambda_3 + 4\mathcal{M}_2^2 t_{i+1}^{2q} \left(\frac{\lambda_1}{t_{i+1}(2q-1)} + \frac{2\hat{\mathcal{H}}\Lambda_\sigma t_{i+1}^{2\hat{\mathcal{H}}-2}}{2q-1} \right). \tag{2.9}$$

Equations (2.7)-(2.9) imply that

$$\|\mathcal{E}_1 y + \mathcal{E}_2 y^*\|^2_{D_T^0} \leq r.$$

Thus, $\mathcal{E}_1 y + \mathcal{E}_2 y^* \in \mathcal{D}_r$

Step 2. We show that the operator $\mathcal{E}_1$ is continuous on $\mathcal{D}_r$. Let $\{y^n\}_{n=1}^\infty$ be a sequence such that $y^n \to y$ in $\mathcal{D}_r$. For all $t \in (s_i, t_{i+1}], i = 1, 2, \cdots, m$, we have

$$\mathbb{E} \|(\mathcal{E}_1 y^n)(t) - (\mathcal{E}_1 y)(t)\|^2 \leq \mathbb{E} \|\mathcal{K}_i(t, g_t + \overline{y}_t^n) - \mathcal{K}_i(t, g_t + \overline{y}_t)\|^2.$$

since the maps $\mathcal{K}_i, i = 1, 2, \cdots, m$, are continuous functions, one has

$$\lim_{n\to\infty} \|\mathcal{E}_1 y^n - \mathcal{E}_1 y\|^2_{D^0_T} = 0 \tag{2.10}$$

For all $t \in (s_i, t_{i+1}]$,$i = 1, 2, \cdots, m$, we have

$$\mathbb{E}\,\|(\mathcal{E}_1 y^n)(t) - (\mathcal{E}_1 yu)(t)\|^2 \leq \mathbb{E}\left\|\mathcal{T}_q(t-s_i)(\mathcal{K}_i(s_i, g_{s_i} + \overline{y^n}_{s_i}) - \mathcal{K}_i(s_i, g_{s_i} + \overline{y}_{s_i}))\right\|^2.$$

Therefore,

$$\lim_{n\to\infty} \|\mathcal{E}_1 y^n - \mathcal{E}_1 y\|^2_{D^0_T} = 0. \tag{2.11}$$

Equations (2.10) and (2.11) imply that the operator $\mathcal{E}_1$ is continuous on $\mathcal{D}_r$.

Step 3. The operator $\mathcal{E}_1$ maps bounded sets into bounded sets in $\mathcal{D}_r$. Let us show that for $r > 0$ there exists a $r > 0$ such that, for each $y \in \mathcal{D}_r$, we obtain $\mathbb{E}\,\|\mathcal{E}_1(y)(t)\|^2 \leq r$. for all $t \in (s_i, t_{i+1}]$, $i = 1, 2, \cdots, m$. we have

$$\mathbb{E}\,\|(\mathcal{E}_1 y)(t)\|^2 \leq \mathbb{E}\left\|\mathcal{T}_q(t-s_i)\mathcal{K}_i(s_i, g_{s_i} + \overline{y}_{s_i})\right\|^2 \leq \mathcal{M}_1^2 \upsilon_i \lambda_3.$$

For all $t \in (s_i, t_{i+1}]$, $i = 1, 2, \cdots, m$, we have

$$\mathbb{E}\,\|(\mathcal{E}_1)(t)\|^2 \leq \mathbb{E}\,\|\mathcal{K}_i(t, g_t + \overline{y}_t)\|^2 \leq \upsilon_i \lambda_3.$$

From the above equations, we obtain

$$\|\mathcal{E}_1 y\|^2_{D^0_T} \leq r,$$

where $r = \max\left\{\mathcal{M}_1^2 \upsilon_i \lambda_3, \upsilon_i \lambda_3\right\}$. Hence, the operator $\mathcal{E}_1$ maps bounded sets into bounded sets in $\mathcal{D}_r$..

Step 4. The operator $\mathcal{E}_1$ is equicontinuous. For all $\Delta_1, \Delta_2 \in (t_i, s_i], \Delta_1 < \Delta_2$, and $y \in \mathcal{D}_r$, we obtain

$$\mathbb{E}\,\|(\mathcal{E}_1 y)(\Delta_1) - (\mathcal{E}_1 y)(\Delta_1)\|^2 \leq \mathbb{E}\,\|\mathcal{K}_i(\Delta_2, g_{\Delta_2} + \overline{y}_{\Delta_2}) - \mathcal{K}_i(\Delta_1, g_{\Delta_1} + \overline{y}_{\Delta_1})\|^2. \tag{2.12}$$

For all $\Delta_1, \Delta_2 \in (s_i, t_{i+1}], \Delta_1 < \Delta_2$. and $y \in \mathcal{D}_r$, we obtain

$$\mathbb{E}\,\|(\mathcal{E}_1 y)(\Delta_2) - (\mathcal{E}_1 y)(\Delta_1)\|^2 \leq \mathbb{E}\left\|(\mathcal{T}_q(\Delta_2 - s_i) - \mathcal{T}_q(\Delta_1 - s_i))\mathcal{K}_i(s_i, g_{s_i} + \overline{y}_{s_i})\right\|^2$$

Since $\mathcal{T}_q$ is strongly continuous, it allows us to conclude that

$$\lim_{n\to\infty} \|\mathcal{T}_q(\Delta_2 - s_i) - \mathcal{T}_q(\Delta_1 - s_i)\|^2 = 0 \tag{2.13}$$

Equations (2.12) and (2.13) with (9) (ii) imply that the operator $\mathcal{E}_1$ is equicontinuous on $\mathcal{D}_r$. Finally, combining steps 1-4 together with Ascoli's theorem, we coclude that the operator $\mathcal{E}_1$ is completely continuous.

step 5. The operator $\mathcal{E}_2$ is a contraction map. For $y, y^* \in \mathcal{D}_r$ and for $t \in (s_i, t_{i+1}]$, $i = 1, 2, \cdots, m$, we have

$$\mathbb{E}\,\|(\mathcal{E}_2 y)(t) - (\mathcal{E}_2 y^*)(t)\|^2 = 0. \tag{2.14}$$

Similarly, for $y, y^* \in \mathcal{D}_r$ and for $t \in (s_i, t_{i+1}]$, $i = 1, 2, \cdots, m$, we have

$$\begin{aligned}\mathbb{E}\,\|(\mathcal{E}_2 y)(t) - (\mathcal{E}_2 y^*)(t)\|^2 \leq{}& 2\mathbb{E}\left\|\int_{s_i}^t \mathcal{S}_q(t-e)(\mathcal{F}(e, g_e + \overline{y}_e) - \mathcal{F}(e, g_e + \overline{y}^*_e))de\right\|^2 \\ &+ 2\mathbb{E}\left\|\int_{s_i}^t \mathcal{S}_q(t-e)(\mathcal{G}(e, g_e + \overline{y}_e) - \mathcal{G}(e, g_e + \overline{y}^*_e))d\hat{\mathcal{W}}(e)\right\|^2\end{aligned}$$

$$\leq 2\mathcal{M}_2^2\omega^2\left(\frac{N_{\mathcal{F}}T^{2q}}{q^2}+\frac{N_{\mathcal{G}}T^{2q-1}}{2q-1}\right)\|y-y^*\|^2_{D_T^0}.$$

Hence,

$$\mathbb{E}\|(\mathcal{E}_2y)(t)-(\mathcal{E}_2y^*)(t)\|^2\leq 2\mathcal{M}_2^2\omega^2\left(\frac{N_{\mathcal{F}}T^{2q}}{q^2}+\frac{N_{\mathcal{G}}T^{2q-1}}{2q-1}\right)\|y-y^*\|^2_{D_T^0}. \tag{2.15}$$

From above, we obtain

$$\|\mathcal{E}_2y-\mathcal{E}_2y^*\|^2_{D_T^0}.$$

Thus, $\mathcal{E}_2$ is a contraction map. By Krasnoselskii's fixed point theorem, we obtain that problem (2.1) has at least one solution on $(-\infty,T]$. ∎

2.4 Approximate Controllability

We consider the folloing control system:

$$\begin{cases} {}^cD_t^qz(t)=\mathcal{P}z(t)+\mathcal{A}\hat{u}(t)+\mathcal{F}(t,z_t)+\mathcal{G}(t,z_t)\dfrac{d\hat{\mathcal{W}}(t)}{dt}+\sigma(t)\dfrac{d\mathcal{B}^{\hat{\mathcal{H}}}(t)}{dt}, & t\in\bigcup\limits_{i=0}^{m}(s_i,t_{i+1}],\\ z(t)=\mathcal{K}_i(t,z_t), \quad t\in\bigcup\limits_{i=1}^{m}(t_i,s_i], \\ z(t)=\phi(t), \qquad \phi(t)\in\mathcal{D}_h. \end{cases} \tag{2.16}$$

The control $\hat{u}(\cdot)\in L^2(\mathcal{J},\mathcal{U})$, where $L^2(\mathcal{J},\mathcal{U})$ is the Hilbert space of all admissible control functions. The operator $\mathcal{A}$ is linear and bounded from the separable Hilbert space $\mathcal{U}$ into $\mathcal{Z}$. We consider the linear system

$$\begin{cases} {}^cD_t^qz(t)=\mathcal{P}z(t)+\mathcal{A}\hat{u}(t), \quad t\in[0,T],\\ z(t)=\phi(t), \quad \phi(t)\in\mathcal{D}_h. \end{cases} \tag{2.17}$$

Define the operator $t_{s_i}^{t_{i+1}}$ associated with system of (2.17) as

$$t_{s_i}^{t_{i+1}}=\int_{s_i}^{t}\mathcal{S}_q(t_{i+1}-e)\mathcal{A}\mathcal{A}^*\mathcal{S}_q^*(t_{i+1}-e)de.$$

here, $\mathcal{A}^*$ and $\mathcal{S}_q^*(t)$ are the adjoint of $\mathcal{A}$and $\mathcal{S}_q(t)$, respectively. The operator $t_{s_i}^{t_{i+1}}$ is a bounded and linear operator.

Definition 2.4.1

System (2.16) is approximately controllable on $[0,T]$ if $\overline{\mathcal{R}(t,\phi,\hat{u})}=L^2(\mathcal{F}_T,\mathcal{Z})$, where $\mathcal{R}(T,\phi,\hat{u})=\Big\{z(\phi,\hat{u})(T):\ z$ is the solution of problem (2.16) and $\hat{u}\in L^2(\mathcal{J},\mathcal{U})\Big\}$.

The following assumption is needed.

AC : System (2.17) is approximate controllability on $\mathcal{J}$.
Note that system (2.17) is approximately controllable on $\mathcal{J}$only if

$$\Delta(\Lambda,t_{s_i}^{t_{i+1}})=(\Lambda I+t_{s_i}^{t_{i+1}})^{-1}\to 0 \text{ as } \Lambda\to 0. \tag{2.18}$$

Definition 2.4.2

An $\mathcal{F}_t$-adapted random process $z : (-\infty, T] \to \mathcal{Z}$ is called the mild solution of (2.16) if for every $t \in \mathcal{J}, z(t)$ satisfies $z_0 = \phi \in \mathcal{D}_h, z(t) = \mathcal{K}_i(t, z_t)$ for all $t \in (s_i, t_{i+1}]$, $i = 1, 2, \cdots, m$, and

$$z(t) = \int_0^t \mathcal{S}_q(t-e)[\mathcal{F}(e, z_e) + \mathcal{A}\hat{u}(e)]de + \int_0^t \mathcal{S}_q(t-e)\mathcal{G}(e, z_e)d\hat{\mathcal{W}}(e) + \int_0^t \mathcal{S}_q(t-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e),$$

for all $t \in [0, t_1]$, and

$$z(t) = \int_0^t \mathcal{S}_q(t-e)\mathcal{K}_i(s_i, z_{s_i}) + \int_{s_i}^t \mathcal{S}_q(t-e)[\mathcal{F}(e, z_e) + \mathcal{A}\hat{u}(e)]de + \int_{s_i}^t \mathcal{S}_q(t-e)\mathcal{G}(e, z_e)d\hat{\mathcal{W}}(e) + \int_{s_i}^t \mathcal{S}_q(t-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e),, \tag{2.19}$$

for all $t \in (s_i, t_{i+1}]$, $i = 1, 2, \cdots, m$.

Lemma 2.4.1

For any $z_{t_{i+1}} \in L^2(\mathcal{F}_T, \mathcal{Z})$, there exist $\phi_1 \in L^2([s_i, t_{i+1}], \mathcal{L}_2^1(\mathcal{Y}_1, \mathcal{Z}))$ and $\phi_2 \in L^2([s_i, t_{i+1}], \mathcal{L}_2^2(\mathcal{Y}_2, \mathcal{Z}))$ such that

$$z_{t_{i+1}} = \mathbb{E}z_{t_{i+1}} + \int_{s_i}^{t_{i+1}} \phi_1(e)d\hat{\mathcal{W}}(e) + \int_{s_i}^{ti+1} \phi_2(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e).$$

Next, we choose the control $\hat{u}^\Lambda(t)$ as follows:

$$\hat{u}^\Lambda(t) = \mathcal{A}^*\mathcal{S}_q^*(t_{i+1} - t)\Delta(\Lambda, t_{s_i}^{t_{i+1}})p(z(.)), \tag{2.20}$$

where

$$\begin{aligned} p(z(.)) = z_{t_{i+1}} - \mathcal{T}_q(t_{i+1} - s_i) - \int_{s_i}^{t_{i+1}} \mathcal{S}_q(t_{i+1} - e)\mathcal{F}(e, z_e)de \\ - \int_{s_i}^{t_{i+1}} \mathcal{S}_q(t_{i+1} - e)\mathcal{G}(e, z_e)d\hat{\mathcal{W}}(e) - \int_{s_i}^{t_{i+1}} \mathcal{S}_q(t_{i+1} - e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e), \\ \forall t \in (s_i, t_{i+1}], \quad i = 1, 2, \cdots, m, \end{aligned}$$

and $\mathcal{K}_0(0, .) = 0, z(t_{m+1}) = z_{y_{m+1}} = z_T$.

Theorem 2.1

Assume the hypotheses $(H1)$-$(H9)$ are satisfied. Then, the problem (2.16) has at least one solution on $(-\infty, T]$.

Proof: The proof is a consequence of Theorem 2.3.2 ∎

Theorem 2.2

Assume the hypotheses $(H1)$-$(H9)$ and [AC] are satisfied. Then functions $\mathcal{F}$ and $\mathcal{G}$are uniformly bounded on their respective domains. Moreover, the system (2.16) is approximately controllable on $[0, T]$.

Proof: Let z^{Λ} be a fixed point of $\mathcal{E}_1 + \mathcal{E}_2$. Using Fubini's theorem, we get

$$z^{\Lambda}(t_{i+1}) = z_{t_{i+1}} - \Lambda\Delta(\Lambda, t_{s_i}^{t_{i+1}})p(z^{\Lambda}(.)), \tag{2.21}$$

where

$$\begin{aligned} p(z^{\Lambda}(.)) = z_{t_{i+1}} - \mathcal{T}_q(t_{i+1} - s_i)\mathcal{K}_i(s_i, z_{s_i}^{\Lambda}) - \int_{s_i}^{t_{i+1}} \mathcal{S}_q(t_{i+1} - e)\mathcal{F}(e, z_e^{\Lambda})de \\ - \int_{s_i}^{t_{i+1}} \mathcal{S}_q(t_{i+1} - e)\mathcal{G}(e, z_e^{\Lambda})d\hat{\mathcal{W}}(e) - \int_{s_i}^{t_{i+1}} \mathcal{S}_q(t_{i+1} - e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e), \\ \forall t \in (s_i, t_{i+1}], i = 1, 2, \cdots, m. \end{aligned}$$

The functions $\mathcal{F}$and $\mathcal{G}$are uniformly bounded. Hence, there exists a subsequence, still represented by $\mathcal{F}(e, z_e^{\Lambda})$ and $\mathcal{G}(e, z_e^{\Lambda})$, that weakly converge to, say, $\mathcal{F}(e)$ and $\mathcal{G}(e)$ in $\mathcal{Z}$and $\mathcal{L}_2^1(\mathcal{Y}_1, \mathcal{Z})$, respectively. Let us define

$$\begin{aligned} \eta = z_{t_{i+1}} - \mathcal{T}_q(t_{i+1} - s_i)\mathcal{K}_i(s_i, z_{s_i}) - \int_{s_i}^{t_{i+1}} \mathcal{S}_q(t_{i+1} - e)\mathcal{F}(e)de \\ - \int_{s_i}^{t_{i+1}} \mathcal{S}_q(t_{i+1} - e)\mathcal{G}(e)d\hat{\mathcal{W}}(e) - \int_{s_i}^{t_{i+1}} \mathcal{S}_q(t_{i+1} - e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e), \\ \forall t \in (s_i, t_{i+1}], i = 1, 2, \cdots, m. \end{aligned}$$

For $t \in (s_i, t_{i+1}]$, $i = 1, 2, \cdots, m$, we have

$$\begin{aligned} \mathbb{E}\left\|p(z^{\Lambda}) - \eta\right\|^2 \leq & 3\mathbb{E}\left\|\mathcal{T}_q(t_{i+1} - s_i)(\mathcal{K}_i(s_i, z_{s_i}^{\Lambda}) - \mathcal{K}_i(s_i, z_{s_i}))\right\|^2 \\ & + 3\mathbb{E}\left\|\int_{s_i}^{t_{i+1}} \mathcal{S}_q(t_{i+1} - e)(\mathcal{F}(e, z_e^{\Lambda}) - \mathcal{F}(e))de\right\|^2 \\ & + 3\mathbb{E}\left\|\int_{s_i}^{t_{i+1}} \mathcal{S}_q(t_{i+1} - e)(\mathcal{G}(e, z_e^{\Lambda}) - \mathcal{G}(e))d\hat{\mathcal{W}}(e)\right\|^2 \end{aligned}$$

By the infinite dimensional version of the Arzela-Ascoli theorem, we obtain that

$$\overline{k}(.) \to \int \mathcal{S}_q(. - e)\overline{k}(e)de$$

is a compact operator. For all $t \in [0, T]$,

$$\mathbb{E}\left\|p(z^{\Lambda}) - \eta\right\|^2 \to 0 \text{ as } \Lambda \to 0^+ \tag{2.22}$$

By Equation (2.21), we get

$$\mathbb{E}\left\|z^{\Lambda}(t_{i+1}) - z_{t_{i+1}}\right\|^2 \leq \mathbb{E}\left\|\Lambda\Delta(\Lambda, t_{s_i}^{t_{i+1}})(\eta)\right\|^2 + \mathbb{E}\left\|\Lambda\Delta(\Lambda, t_{s_i}^{t_{i+1}})\right\|^2 \mathbb{E}\left\|p(z^{\Lambda}) - \eta\right\|^2.$$

By (2.18) and (2.22), we get

$$\mathbb{E}\left\|z^{\Lambda}(t_{i+1}) - z_{t_{i+1}}\right\|^2 \to 0 \text{ as } \Lambda \to 0^+.$$

Thus, the system (2.16) is approximate controllable on the interval $[0, T]$. ■

Example 2.4.1

The functions $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{K}_i$ are continuous functions.
We consider the following fractional stochastic control system:

$$\begin{cases} {}^cD_t^q y(t,z) = \dfrac{\partial^2}{\partial z^2} y(t,z) + \Theta(t,z) + \displaystyle\int_{-\infty}^t e^{4(r-t)} y(r,z)dr \\ \qquad\qquad + \displaystyle\int_{-\infty}^t e^{6(r-t)} y(r,z)dr \dfrac{d\hat{\mathcal{W}}(t)}{dt} + P(t)\dfrac{d\mathcal{B}^{\hat{\mathcal{H}}}(t)}{dt}, \\ \qquad\qquad y \in (0,\pi), t \in [2i, 2i+1], i = 1, 2, \cdots, m, \\ y(t,z) = \displaystyle\int_{-\infty}^t G_i(r-t) y(r,z)dr, t \in [2i-1, 2i], i = 1, 2, \cdots, m, \\ y(t,0) = 0 = y(t,\pi), \\ y(t,z) = \phi(t,z), t \in (-\infty, 0], \end{cases} \tag{2.23}$$

where ${}^cD_t^q$ is the Caputo derivative of order $1/2 < q < 1, 0 = s_0 = t_0 < t_1 < s_1 < t_2 < \cdots < t_m < s_m < t_{m+1} = T < \infty$ with $s_i = 2i, t_i = 2i - 1$.
Let $\mathcal{Z} = L^2([0,\pi])$ and the operator $\mathcal{P}$ be defined by

$$\mathcal{P}w = w'', \mathcal{D}(\mathcal{P}) = H^2(0,\pi) \cap H_0^1(0,\pi).$$

Clearly, $\mathcal{P}$is the generator of an analytic semigroup $\{\mathcal{S}(t) : t \geqslant 0\}$. The soectral representation of $\mathcal{S}(t)$ is given by

$$\mathcal{S}(t)w = \sum_{n\in\mathbb{N}} e^{-2t} \langle w, w_n \rangle w_n,$$

where

$$w_n(y) = \sqrt{2/\pi}\sin(ny), n \in \mathbb{N},$$

is the prthogonal set of eigenvectors corresponding to the eigenvalue $\lambda_n = -n^2$ of $\mathcal{P}$. The semigroup $\{\mathcal{S}(t) : t \geq 0\}$ is compact and uniformly bounded, so that $\mathcal{R}(\lambda, \mathcal{P}) = (\lambda I - \mathcal{P})^{-1}$ is a compact operator for all $\lambda \in \rho(\mathcal{P})$, i.e., $\mathcal{P} \in \mathcal{P}^q(\theta_0, \omega_0)$. Let $h(e) = e^{2e}$, $e < 0$. Then $\omega = \int_{-\infty}^0 h(e)de = 1/2$ and we define

$$\|\phi\|_{\mathcal{D}_h} = \int_{-\infty}^0 h(e) \sup_{e \leq \theta \leq 0} (\mathbb{E}|\phi(\theta)|^2)^{1/2} de, \quad \phi \in \mathcal{D}_h.$$

Hence, $(t,\phi) \in [0,T] \times \mathcal{D}_h$. The bounded linear operator $\mathcal{A}$ is defined by $\mathcal{A}\hat{u}(t)(z) = \Theta(t,z)$. Define the functions $\mathcal{F} : \mathcal{J} \times \mathcal{D}_h \to \mathcal{Z}, \mathcal{G} : \mathcal{J} \times \mathcal{D}_h \to L_2(\mathcal{Y}_1, \mathcal{Z})$, and $\mathcal{K}_i : (t_i, s_i] \times \mathcal{D}_h \to \mathcal{Z}$ as

$$\mathcal{F}(t,\phi)(z) = \int_{-\infty}^0 e^{4\theta}(\phi(\theta)(z))d\theta,$$
$$\mathcal{G}(t,\phi)(z) = \int_{-\infty}^0 e^{6\theta}(\phi(\theta)(z))d\theta,$$
$$\mathcal{K}_i(t,\phi)(z) = \int_{-\infty}^0 G(\theta)(\phi(\theta)(z))d\theta.$$

Assume that

$$\int_0^T \|\sigma(e)\|^2_{\mathcal{L}_2^2} de < \infty.$$

The system (2.23) can be written as an abstract formulation of (2.1), and thus previous theorems can be applied to guarantee both existence and approximate controllability results.

2.5 Conclusions

This Chapter (2) We have investigated impulsive fractional stochastic control systems defined on separable Hilbert spaces. The proposed problem is driven by mixed noise, i.e., it involves both a Q-Wiener process and a Q-fractional Brownian motion with the Hurst parameter $\hat{\mathcal{H}} \in (1/2, 1)$. For our results, we have mainly applied fixed point techniques, a q-resolvent family, and fractional calculus. The obtained results are supported by an illustrative example. As further directions of investigation and continuation to this work, it would be interesting to investigate the sensitivity on the noise range and develop numerical and computational methods to approximate the solution. We also intend to extend our results via discrete fractional calculus.

Chapter 3

Nonlinear fractional order neutral-type stochastic integro-differential system with rosenblatt process a controllability exploration

The work presented in this Chapter (3) is controllability analysis of nonlinear fractional order neutral-type stochastic integro-differential system with non-Gaussian process. We stress out the stochastic term of our system driven by the uncomplicated non-Gaussian Hermite process known as the Rosenblatt process, which is named after by Murray Rosenblatt who first devised this introduced concept. This process is self-similar with consistent accretion and beside emerged as restriction in the non-central limit theorem, and it exists in the second wiener chaos. The necessary and sufficient conditions for the controllability are verified by employ-ing fixed point techniques. This work is attributed to the [54].

3.1 Introduction

We investigate the controllability analysis of nonlinear fractional Order neutral-type stochastic integro-differential system with non-Gaussian process. We investigate the controllability analysis of nonlinear fractional order neutral-type stochastic integro-differential system with non-Gaussian process. We stress out the stochastic term of our system driven by the uncomplicated non-Gaussian Hermite process known as the Rosenblatt process, which is named after by Murray Rosenblatt who first devised this introduced concept. This process is self-similar with consistent accretion and beside emerged as restriction in the non-central limit theorem, and it exists in the second wiener chaos. The necessary and sufficient conditions for the controllability are verified by employing fixed point techniques. At end, we present illustrative examples to clarify the abstract results.

For several centuries ago, the investigators and technologists are always enthusiastic to work on a real-world problem to understand natural phenomenon. Many works are derived by modelling real life problems and try to get solutions for them and then apply the obtained results to real-life that used to live in a better way. Mathematical modelling is one of the best tools to solve this type of situations to investigate the solutions with some accuracies; it is used in different branches of sciences and engineering.

To model a more complex natural phenomenon with more accurate solutions, it is needed to employ modified approaches like a complex system instead of using integer-order derivative, we can replace it by non-integer one. By using a fractional order derivative, we can study the queries of any complex natural phenomenon, fractional differential equations far-reaching applications towards physical phenomena such as fluid dynamics, etc. Recently, the subject of fractional calculus theory and its applications have been arising a considerable interest due to its ability to model many practical systems [49], [50], [74], [75], [151].By using fractional derivatives, we can reveal the modifications in an interval. The fractional by-product is in non-indigenous nature, it makes fractional derivatives appropriate to mimic extra bodily phenomena along with earthquake vibrations, polymers,.∴., etc., see [123], [175]. The applied science has showed that many phenomena have been modelled with the aid of using fractional differential equations coincide with a few uncertainties. There are many fluctuations in the environments and also there are intrinsic and extrinsic noises available in the field. The necessity to clear up veritable issues for greater unique answers, it is recommended the view of stochastic fractional differential equations [45], [78].

Stochastic differential equations of fractional order play an emerging role rather than the integer-order systems and subsequently. The stochastic process is a probability distribution over a space of paths, the theory of stochastic processes was reconciled decades ago. Applications of stochastic processes as virtual identity can be found in numerous disciplines such as control theory, traffic engineering and renewal theory. The mathematical theory of stochastic analysis was developed by Ito. It is regulated via sense of means of boundary and preliminary conditions, however, currently, no longer foreordained via way of means of them. Each time the equation is solved beneath equal preliminary and boundary conditions, the answer takes exclusive numerical values although a particular sample emerges as the answer technique is repeated many times. It has huge applications in various research areas, including environment, finance, and medicine, etc. For important works of fractional stochastic systems and their applications, we may refer to [92], [143], [148], [166],[161]. Here the stochastic process is taken as the uncomplicated non-Gaussian Hermite process known as the Rosenblatt process. The most popular self-similar process is the fractional Brownian motion, which is also the only Hermite Gaussian process. Rosenblatt's process was introduced in 1961 by M. Rosenblatt in the work Independence and dependence [28]. Although outlined throughout the past 60s associated with the later 70s because of their look within the non-central limit theorem, the systematic analysis of Rosenblatt processes has solely been developed during the last 10 years, intended by their specific properties (self-similarity, stationarity of the increments, long-range dependence) since they are non-Gaussian and self-similar with stationary increments. A self-similar object is strictly or about such a region of itself. Self-similar processes are invariant in distribution beneath appropriate scaling. Among the applications of the Rosenblatt processes in statistics or economics, corresponding to the Rosenblatt distribution conjointly seems to be the straight line distribution of an estimation of the long range dependence parameter. More details about this process can be found in [28, 156].

The Hermite process $(Z_H^K(t))$ is in a multiple Wiener-Ito stochastic integral with respect to Brownian motion $B(y)_{y\in\mathbb{R}}$ is given as

$$(Z_H^K(t)) = C(H,k)\int_{\mathbb{R}^K}\int_0^t \left(\prod_{j=1}^{K}(s-y_i\right)_+^{-(\frac{1}{2}+\frac{1-H}{K})}\,)\ \mathrm{d}x\ \mathrm{d}B(y_1)\cdot ...\ \mathrm{d}B(y_K),$$

where $x_+ = \max(H,0)$, the constant $C(H,k)$ is positive, it is H-self similar for any $C>0$, and it has stationary increments. In the above integral, when $K=1$ the process is the fractional Brownian motion with Hurst parameter $H \in \left(\frac{1}{2},1\right)$. For $K \geq 2$ the process is not Gaussian.

In particulaire, when $K = 2$ the process is known as the Rosenblatt process. Because of Gaussianity, numerous practical applications of fractional brownian motion process have been widely studied. But in concrete situations when the Gaussianity is not plausible, one can use a Hermite process living in a higher chaos. More precisely, the Rosenblatt process is not Gaussian, but it has self-similar, stationary of increments and large range dependence property like fractional Brownian motion.

Controllability is an important aspect of mathematical control theory which was introduced by Kalman [69]. The concept of controllability denotes the ability to move the state of the dynamical control system from an initial state to the desired final state by using a suitable control function. In the last years, different aspects of controllability for ordinary as well as fractional dynamic systems, for both deterministic and stochastic structures, have been studied by many researchers [6, 13, 35, 51, 66, 81, 97, 157, 166].

This work is involved with nonlinear fractional-order neutral-type stochastic Integro-differential system with Rosenblatt process, the controllability is deceased in the accessible source of studies. Our main contributions are highlighted as follows:

- We have developed a solution for the controllability problem of non-linear fractional order neutral type stochastic integro-differential system with Rosenblatt process.
- We take the terms in the system as a bounded linear operators instead of a matrix, which produces the same results as a matrix.
- The illustration the results on stochastic systems bounded linear operators are more competent.
- We take the stochastic term as driven by the Rosenblatt process which is non-Gaussian and has the properties like self-similarity, stationarity of the increments and has long range dependence.
- We intend to bring new lights to the Rosenblatt process, since many real-life phenomena are modelled by fractional Brownian motion a only Gaussian Hermite process, when the property of Gaussianity is failed one can use Rosenblatt process.
- We define the controllability Grammian operator, which is defined by the Mittag-Leffler function to prove the controllability results.
- By employing Banach contraction principle to prove the controllability criteria instead of semigroup theory which does not applicable to obtain the results on controllability.
- We have provided a numerical example to illustrate the theory.
- Generally speaking, both the Riemann-Liouville and the Caputo fractional operators do not possess neither semigroup nor commutative properties, which are inherent to the derivatives on integer order.

The paper is organized as follows. In Section 3.2, we review some essential facts from stochastic analysis and fractional calculus that are used to obtain our main results. In Section 3.3, we formulate a suitable solution representation and controllability criteria of Linear system. Then, we will extend the investigation to nonlinear system to be controllable in Section 3.4. Finally, in Section 3.5, we give appropriate examples to illustrate the given theory. We end with Section 3.6 of conclusions to our results of this research.

3.2 Preface

In this section we give some basic definitions and properties which are useful to establish our theoretical results.

Rosenblatt process.

Let $(\Omega, \mathcal{F}, \left\{\mathcal{F}_t\right\}_{t\geq 0}, \mathcal{P})$ be a filtered probability space consists of a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and a filtration $\left\{\mathcal{F}_t\right\}_{t\geq 0}$ contained in $\mathcal{F}$ the filtered probability space is said to satisfy the usual conditions namely

- The probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is complete.
- $\mathcal{F}_0$ contains all $A \in \mathcal{F}$ such that $\mathcal{P}(A) = 0$.
- $\mathcal{F}_t = \mathcal{F}_{t^+}$, $\forall t \in J$, where $\mathcal{F}_{t^+}$ is the intersection of all $\mathcal{F}_s$ where $s > t$, i.e. the filtration is right continuous.

Suppose that $\left\{Z_H(t), t \in [0, b]\right\}$ is the one-dimensional Rosenblatt process with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$. That is $Z_H(t)$ is a Non-Gaussian process with covariance function

$$E(Z_H(t), Z_H(s)) = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right).$$

Moreover, the Rosenblatt process with Hurst parameter $H > \frac{1}{2}$ has the representation as (see [156]):

$$Z_H(t) = d(H) \int_0^t \int_0^t \left\{\int_{Y_1 \vee Y_2}^t \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) \,\mathrm{d}u\right\} \,\mathrm{d}B(Y_1) \,\mathrm{d}B(Y_2).$$

Where $t \leq s$, $\left\{B(t), t \in [0, b]\right\}$is a Brownian motion, and $K^H(t, s)$ is the kernel given by

$$K^H(t, s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} \,\mathrm{d}u.$$

for $t > s$, where

$$c_H = \sqrt{\frac{H(2H-1)}{\Gamma(2-2H, H-\frac{1}{2})}},$$

and $\Gamma(.,.)$ denotes the Gamma function. We put $K_H(t, s) = 0$ if $t \leq s$. Let $\left\{Z_n(t)\right\}_{n\in\mathbb{N}}$ denote a sequence of two-sided one dimensional Rosenblatt process mutually independent on $(\Omega, \mathcal{F}, \mathcal{P})$. Consider a K-valued stochastic process $Z_Q(t)$ given by the series $Z_Q(t) = \sum_{n=1}^{\infty} Z_n(t) Q^{\frac{1}{2}} e_n, t \geq 0$ Moreover, if Q is a non-negative self-adjoint trace class operator, then, this series converges in the space K, that is, it holds that $Z_Q(t) \in L^2(\Omega, K)$ Then, the above $Z_Q(t)$ is a K-valued Q-Rosenblatt process with covariance operator Q.

Definition 3.2.1 (*See [148]*)

Let $\mathcal{L}(K, Y)$ represents the space of all bounded linear operators from K to Y, and $Q \in$

$\mathcal{L}(K,Y)$ represents a non-negative self-adjoint operator in separable Hilbert spaces K and Y. Let $\mathcal{L}_2^0 = \mathcal{L}_2(Q^{\frac{1}{2}K,Y})$ be the space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}}K$ into Y, where $\mathcal{L}_2^0$ is a separable Hilbert space, equipped with the norm $\|\omega\|^2_{\mathcal{L}_2^0} = \|\omega Q^{\frac{1}{2}}\|^2 = Tr(\omega Q \omega^*)$.

Definition 3.2.2 (*See [148]*)

Let $\omega : [\sigma, b] \to \mathcal{L}_2(Q^{\frac{1}{2}}K, V)$ such that

$$\sum_{n=1}^{\infty} \left\| K_H^*(\phi Q^{\frac{1}{2}} e_n) \right\| \mathcal{L}^2([0,b], H) < \infty.$$

Then, for $t \geq 1$, its stochastic integral with respect to the Rosenblatt process $Z_Q(t)$ is defined as

$$\begin{aligned}
\int_0^t \omega(s)\, \mathrm{d}Z_Q(s) &= \sum_{n=1}^{\infty} \int_0^t \omega(s) Q^{\frac{1}{2}e_n\, \mathrm{d}Z_n(s)} \\
&= \sum_{n=1}^{\infty} \int_0^t \int_0^t (K_H^*(\phi Q^{\frac{1}{2}} e_n))(y_1, y_2)\, \mathrm{d}B(y)_1\, \mathrm{d}B(y)_2.
\end{aligned}$$

Let Y and U_2 be Separable Hilbert Spaces. We define,

- *$X := \mathcal{L}_2(\gamma, F_b, Y)$, Which is the Hilbert space of all $\mathcal{L}_{b^-}$ measurable square integrable random variables with values in Y.*
- *H is a closed subspace of $H : J \to L_2(F, Y)$ consisting of all $\mathcal{F}_{t^-}$ measurable processes with values in Y and endowed with the norm*

  $$\|\phi\|_H^2 = \sup E\|\phi\|^2,$$

  where E denotes expectation with respect to $\mathcal{P}$.
- *$H := \mathcal{L}_2(J, U_2)$, which is a Hilbert space of all square integrable and $\mathcal{F}_{t_-}$ measurable processes with values in U_2.*

Fractional calculus.

Definition 3.2.3 (*See [123]*)

For $n \in \mathbb{N}$, the Euler gamma function $\Gamma : \mathbb{C} - \{0, -1, -2, ...\} \to \mathbb{C}$, for complex arguments with positive real part it is defined as

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1}\, \mathrm{d}t, \quad Rez > 0.$$

Definition 3.2.4 (*See [123]*)

Let $[a, b]$, be a finite interval on the real axis $\mathbb{R}$. The Riemann - Liouville fractional integral of order $\alpha > 0$, $n - 1 < \alpha \leq n$ and $n \in \mathbb{N}$ is defined as

$$I_{0^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, \mathrm{d}s,$$

where $\Gamma(.)$ is the Euler gamma function and $g(t)$ a suitable function.

Definition 3.2.5 (***See*** *[123]*)

The Caputo fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$ is defined as

$$({}^C D^{\alpha}_{0+} g)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} g^{(n)}(s)\ \mathrm{d}s,$$

where the function $g(t)$ has absolutely continuous derivatives up to order $(n-1)$.
If $0 < \alpha < 1$, then

$$({}^C D^{\alpha}_{0+} g)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{g'(s)}{(t-s)^{\alpha}}\ \mathrm{d}s.$$

Note: The Caputo fractional derivative is equivalent to the composition of the same operators ($(n-\alpha)$-fold integration and n^{th} orfer differentiation) ${}^C\mathcal{D}^{\alpha} g = I^{n-\alpha} D^n g$.
In particular, for $0 < \alpha < 1$, $(I^{\alpha}_{0+}\ {}^C D^{\alpha}_{0+})g(t) = g(t) - g(0)$.

Definition 3.2.6 (***See*** *[75]*)

Let A be a bounded linear operator, the Mittag-Leffler function is given by,

$$E_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(k\alpha+\beta)}.$$

In particular, for $\beta = 1$,

$$E_{\alpha,1}(A) = E_{\alpha}(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(k\alpha+1)}.$$

Lemma 3.2.1 (***See*** *[92]*)

Suppose that A is a bounded linear operator defined on a Banach space, and assume that $\|A\| < 1$. Then $(I-A)^{-1}$ is linear and bounded.
Also

$$(I-A)^{-1} = \sum_{k=0}^{\infty} A^k.$$

Definition 3.2.7 (***See*** *[13]*)

If X is a Banach space and $T : X \to X$ is a contraction mapping then T has a unique fixed point.

3.3 Main results

Consider the neutral linear stochastic fractional integro-differential equation as

$$\begin{aligned} {}^C D^{\alpha}[y(t) - g(t, y(t))] &= Ay(t) + h(t, y(t)), \int_0^t l(t, s, y(s))\ \mathrm{d}s + Bu(t) \\ &\quad + f(t)\ \mathrm{d}Z_H(t), \quad t \in J := [0, b], \\ y(0) &= \psi_0, \end{aligned} \tag{3.1}$$

- Where, ${}^{C}D^{\alpha}$ represents the caputo derivatives of order $0 < \alpha < 1$,
- $y(.)$ takes the value in a real separable Hilbert space Y with inner product $< ., . >$ and norm $|.|_Y$,
- $A : Y \to Y$ is a bounded linear operator,
- $u(.)$ the control function belongs to the space $L^2(J, U)$,
- $B : U \to H$ is a linear bounded operator,
- $g : J \times Y \to Y$ is continuous,
- $h : J \times Y \times Y \to Y$ and $l : J \times Y \times Y \to Y$ arethe continuous functions,
- $z_H(t)$ is a Rosenblatt process with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$ and $t \in J = [0, b]$ on a real separable space $(K, \|.\|_K, < ., . >_K)$,
- $f(t)$ is a Hilbert-Schmidt operator for all $t \in J$, and ψ_0 is the initial function.

Assumption A_1 : [92]: For solution representation of the system 3.1, we consider this assumption. The operator $A \in \mathbb{L}(Y)$ commutes with the fractional integral operator I^{α} on Y and

$$\|A\|^2 \leq \frac{(2\alpha - 1)(\Gamma(\alpha))^2}{T^{2\alpha}}.$$

Lemma 3.3.1

For $0 < \alpha < 1$, and $f : J \to L_2^0$ is continuous and bounded, then prove that the solution of the system (3.1) can be represented as

$$\begin{aligned} y(t) = {} & E_{\alpha}(At^{\alpha})[\phi_0 + g(0, \phi_0)] - g(t, y(t)) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^{\alpha}) \\ & \times \left[Ag(s, y(s)) + h(s, y(s)), \int_0^s l(s, \tau, y(\tau)) \, \mathrm{d}\tau) + Bu(s) \right] \mathrm{d}s \\ & + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^{\alpha}) \times f(s) \, \mathrm{d}Z_H(s). \end{aligned}$$

Proof: Taking I^{α} on both sides of (3.1) using assumption A_1 and lemma (3.3.1), we can get the solution of (3.1) as,

$$
\begin{aligned}
y(t) &= (\psi_0 + g(0,\psi_0)) - g(t,y(t)) + I^\alpha Ay(t) + I^\alpha Bu(t)\\
&\quad + I^\alpha h(t,y(t)), \int_0^t l(t,s,y(s)\ \mathrm{d}s) + I^\alpha f(t)\ \mathrm{d}Z_H(t).\\
y(t)[I - I^\alpha A] &= (\psi_0 + g(0,\psi_0)) - g(t,y(t)) + I^\alpha Bu(t)\\
&\quad + I^\alpha h(t,y(t)), \int_0^t l(t,s,y(s)\ \mathrm{d}s) + I^\alpha f(t)\ \mathrm{d}Z_H(t).\\
y(t) &= [I - I^\alpha A]\{(\psi_0 + g(0,\psi_0)) - g(t,y(t)) + I^\alpha Bu(t)\\
&\quad + I^\alpha h(t,y(t)), \int_0^t l(t,s,y(s)\ \mathrm{d}s) + I^\alpha f(t)\ \mathrm{d}Z_H(t)\}.\\
y(t) &= \sum_{k=0}^{\infty}[(I^\alpha A^k)]\{(\psi_0 + g(0,\psi_0) - g(t,y(t)) + I^\alpha Bu(t))\\
&\quad + I^\alpha h(t,y(t)), \int_0^t l(t,s,y(s)\ \mathrm{d}s) + I^\alpha f(t)\ \mathrm{d}Z_H(t)\}\\
y(t) &= \sum_{k=0}^{\infty}[(I^\alpha A^k)]\{(\psi_0 + g(0,\psi_0)\} - g(t,y(t)) \sum_{k=0}^{\infty}[(I^\alpha A^k)]g(t,y(t))\\
&\quad + \sum_{k=0}^{\infty}[(I^\alpha A^k)]I^\alpha Bu(t)) + \sum_{k=0}^{\infty}[(I^\alpha A^k)]I^\alpha h(t,y(t)), \int_0^t l(t,s,y(s)\ \mathrm{d}s\\
&\quad + \sum_{k=0}^{\infty}[(I^\alpha A^k)]I^\alpha f(t)\ \mathrm{d}Z_H(t).
\end{aligned}
$$

$$
\begin{aligned}
y(t) = E_\alpha(At^\alpha)[\phi_0 + g(0,\phi_0)] - g(t,y(t))] + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)\\
\left[Ag(s,y(s)) + h(s,y(s)), \int_s^s l(s,\tau,y(\tau)\ \mathrm{d}\tau) + Bu(s)\right]\ \mathrm{d}s\\
+ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)f(s)\ \mathrm{d}Z_H(s).
\end{aligned} \tag{3.2}
$$

. ■

Definition 3.3.1

Define the operator $K_b : U \to Y$ as

$$k_b u = \int_0^b (b-s)^{\alpha-1}E_{\alpha,\alpha}(A(b-s)^\alpha)Bu(s)\ \mathrm{d}s$$

Clearly, the adjoint operator k_*^b of k_b, $k_*^b : Y \to U$ as

$$(k_*^b y)(t) = (b-t)^{\alpha-1}B^*E_{\alpha,\alpha}(A^*(b-t)^\alpha)E\Big\{x/\mathcal{F}_t\Big\}.$$

Definition 3.3.2

The controllability grammian operator $M_b : Y \to Y$

$$M_b(y) = \int_0^b (b-s)^{2\alpha-2}E_{\alpha,\alpha}(A(b-s)^\alpha)BB^*E_{\alpha,\alpha}(A^*(b-s)^\alpha)E\Big\{x/\mathcal{F}_s\Big\}.$$

Here $*$ represents adjoint operator.

Lemma 3.3.2

The operator $M_b = K_b K_b^*$ is well defined and bounded for any $\alpha \in \left(\frac{1}{2}, 1\right]$,

Proof: The proof of this Lemma is obvious, It is clear that the grammian operator M_b is linear and bounded for all $\alpha \in (\frac{1}{2}, 1]$,. ■

Definition 3.3.3 (*See [97]*)

The stochastic fractional system (3.1) is said to be completely controllable on the interval J if for every $y_1 \in Y$, there exists a control $u \in U$ such that the solution $y(t)$ given in (3.2) satisfies $y(b) = y_1$.

Theorem 3.3.1

Let us assume that A_1 is satisfied, then the linear system (3.1) is completely controllable.

Proof: Using assumption A_1 we obtain the solution of (**??**) as in (**??**). Let y_1 be an arbitrary point in Y. Since the linear operator M_b is invertible, we define the control as.

$$\begin{aligned} u(t) &= (b-t)^{(\alpha-1)} B^* E_{\alpha,\alpha}(A^*(b-t)^\alpha) E\{M_b^{-1}(y_1 - E_\alpha(At^\alpha)[\psi_0 + g(0,\psi_0)] + g(t,y(t)) \\ &\quad - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) \left[Ag(s,y(s)) + h(s,y(s), \int_0^s l(s,\tau,y(\tau))\, \mathrm{d}\tau)\right] \mathrm{d}s \\ &\quad - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) f(s)\, \mathrm{d}Z_H(s)/\mathcal{F}_s\} \end{aligned} \tag{3.3}$$

Substituting (3.3) in (3.2) we get

$$\begin{aligned} y(t) &= E_\alpha(At^\alpha)[\psi_0 + g(0,\psi_0)] - g(t,y(t)) + \int_0^t (t-s)^{2\alpha-2} E_{\alpha,\alpha}(A(t-s)^\alpha)] BB^* \\ &\quad \times E_{\alpha,\alpha}(A^*(t-s)^\alpha) E\{M_b^{-1}(y_1 - E_\alpha(At^\alpha)[\psi_0 + g(0,\psi_0)] + g(t,y(t)) \\ &\quad - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) \left[Ag(s,y(s)) + h(s,y(s)), \int_0^s l(s,\tau,y(\tau)\, \mathrm{d}\tau)\right] \mathrm{d}s \\ &\quad - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) f(s)\, \mathrm{d}Z_H(s)/\mathcal{F}_s\} \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) \left[Ag(s,y(s)) + h(s,y(s)), \int_0^s l(s,\tau,y(\tau)\, \mathrm{d}\tau)\right] \mathrm{d}s \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) f(s)\, \mathrm{d}Z_H(s) \end{aligned}$$

Evaluating $y(t)$ given in the above equation at $t = b$ we obtain

$$\begin{aligned}
y(b) &= E_\alpha(Ab^\alpha)[\psi_0 + g(0,\psi_0)] - g(b,y(b)) \\
&+ M_b M_b^{-1} E\{(y_1 - E_\alpha(Ab^\alpha)[\psi_0 + g(0,\psi_0)] \\
&+ g(b,y(b)) - \int_0^b (b-s)^{\alpha-1} E_{\alpha,\alpha}(A(b-s)^\alpha) \\
&\times \left[Ag(s,y(s)) + h(s,y(s), \int_0^s l(s,\tau,y(\tau))\,\mathrm{d}\tau) \right] \mathrm{d}s \\
&- \int_0^b (b-s)^{\alpha-1} E_{\alpha,\alpha}(A(b-s)^{\alpha)} f(s)\,\mathrm{d}Z_H(s)/\mathcal{F}_s\} \\
&+ \int_0^b (b-s)^{\alpha-1} E_{\alpha,\alpha}(A(b-s)^\alpha) \\
&\times \left[Ag(s,y(s)) + h(s,y(s), \int_0^s l(s,\tau,y(\tau))\,\mathrm{d}\tau) \right] \mathrm{d}s \\
&- \int_0^b (b-s)^{\alpha-1} E_{\alpha,\alpha}(A(b-s)^\alpha) f(s)\,\mathrm{d}Z_H(s) \\
&= y_1
\end{aligned}$$

Since y_1 is an arbitrary point in Y, we infer from the above that $u(t)$ defined in (3.3) steers the system to all points in Y. Thus the proof is completed. ■

3.4 Controllability Criteria of Nonlinear System

Consider the corresponding non-linear system for (3.1), as

$$\begin{aligned}
{}^C D^\alpha[y(t) + g(t,y(t))] &= Ay(t) + h(t,x(t), \int_0^t l(t,s,x(s))\,\mathrm{d}s) + Bu(t) \\
&\quad + f(t,y_t)\,\mathrm{d}Z_H(t)), \quad t \in J := [a,b], \\
y(0) &= \phi_0,
\end{aligned} \tag{3.4}$$

- Where, ${}^C D^\alpha$ represents the caputo derivatives of order $0 < \alpha < 1$,
- $y(.)$ takes the value in a real separable Hilbert space Y with inner product $< .,. >$ and norm $|.|_Y$,
- $A : Y \to Y$ is a bounded linear operator,
- $u(.)$ the control function belongs to the space $L^2(J,U)$,
- $B : U \to H$ is a linear bounded operator,
- $g : J \times Y \to Y$ is continuous,
- $h : J \times Y \times Y \to Y$ and $l : J \times Y \times Y \to Y$ are the continuous functions,
- $z_H(t)$ is a Rosenblatt process with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$ and $t \in J = [0,b]$ on a real separable space $(K, \|.\|_K, < .,. >_K)$,
- $y_t \in \beta$ (where β is the abstract phase space, for details see [148]),

- $f : J \times \beta \to L_0^2$, where $L_0^2 = L_2(Q^{\frac{1}{2}}K, YQ)$ be the space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}}K$, into Y
- ψ_0 is the initial function.

The solution of (3.4) is given by,

$$\begin{aligned} y(t) &= E^{\alpha}(At^{\alpha})[\phi_0 + g(t, \phi_0)] - g(t, y(t)) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)\alpha) \\ &\quad \times \left[Ag(s, y(s)) + h(s, y(s)), \int_0^s l(s, \tau, y(\tau) \, \mathrm{d}\tau) + Bu(s) \right] \, \mathrm{d}s \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)\alpha) f(s, y_s) \, \mathrm{d}Z_H(s) \end{aligned}$$

To prove the controllability results for the nonlinear system (3.4), we consider the following assumptions.
Assumption A_2 : Assume that there exists constants $V_i > 0$ for $i = 1, 2, \therefore, 8$ and W_1, W_2 suche that

$$\begin{aligned} \|h(t, y_1, x_1) - h(t, y_2, x_2)\|^2 &\leq V_1(\|y_1 - y_2\|^2 + \|x_1 - x_2\|^2) \\ \|f(t, y_{t_1}) - f(t, y_{t_2})\|^2 &\leq V_2(\|y_{t_1} - y_{t_2}\|^2) \\ \|l(t, s, y_1) - l(t, s, y_2)\|^2 &\leq V_3 \|y_1 - y_2\|^2 \\ \|g(t, y_1) - g(t, y_2)\|^2 &\leq V_4 \|y_1 - y_2\|^2 \\ V_5 &= \sup_{t \in J} \|f(t, 0)\| \\ V_6 &= \sup_{t \in J} \|h(t, 0, 0)\|, \\ V_7 &= \sup_{t \in J} \| \int_0^t l(t, s, 0) \, \mathrm{d}s \| \\ V_8 &= \sup_{t \in J} \|g(t, 0)\| \\ W_1 &= \sup_{0 \leq t \leq b} \|E_{\alpha}(At^{\alpha})\|^2 \\ W_2 &= \sup_{0 \leq t \leq b} \|E_{\alpha,\alpha}(At^{\alpha})\|^2 \end{aligned}$$

Assumption A_3 : Let $\delta = \dfrac{8b^{2\alpha} W_2}{(2\alpha - 1)} (V_2 b^{-1} + V_2 + V_1 + V_1 V_3 b)$ be such that $0 \leq \delta < 1$.

Theorem 3.4.1

If the assumption A_1 - A_3 are satisfied and if the linear fractional dynamical system (3.3) is controllable, then the non-linear fractional dynamical system (3.4) is controllable.

Proof: Let y_1 be an arbitrary point in Y. Define the operator ϕ on Y by

$$\begin{aligned} \phi y(t) &= E_{\alpha}(At^{\alpha})[\phi_0 + g(0, \phi_0)] - g(t, y(t)) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^{\alpha}) \\ &\quad \times [Ag(s, y(s)) + h(s, y(s)), \int_0^s l(s, \tau, y(\tau)) \, \mathrm{d}\tau) + Bu(s)] \, \mathrm{d}s \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^{\alpha}) f(s, y_s) \, \mathrm{d}Z_H(s). \end{aligned}$$

∎

Since the linear system (3.3) corresponding to the nonlinear system (3.4) is controllable. We have, M_b is invertible and so we can define the control variable u as

$$\begin{aligned} u(t) = (b-t)^{\alpha-1} B^* E_{\alpha,\alpha}(A^*(b-t)^{\alpha}) E\Big\{ M_b^{-1}(y_1 - E_{\alpha}(Ab^{\alpha})[\psi_0 + g(0,\psi_0)] \\ + g(t,y(t)) - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^{\alpha}) \\ \times \Big[Ag(s,y(s)) + h(s,y(s), \int_0^s l(s,\tau,y(\tau))\,\mathrm{d}\tau) \Big]\,\mathrm{d}s \\ - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^{\alpha}) f(s,y_s))\,\mathrm{d}Z_H(s)/\mathcal{F}_s.\} \end{aligned}$$

Clearly, $\phi(y(b)) = y_1$ To show ϕ has a fixed point, our claim is ϕ maps Y into itself. Provided we can obtain a fixed point of the nonlinear operator ϕ.

$$\begin{aligned} \sup_{t\in J} E\|u(t)\|^2 \le 4\|K_b^*\|^2 \|M_b^{-1}\|^2 \Big[E\|Y_1\|^2 + W_1 E\|\psi_0 + g(0,\psi_0)\|^2 \\ + V_7 + W_2 \frac{Zb^2(\alpha-1)}{2\alpha-1} + W_2 \frac{Z^1 b^2(\alpha-1)}{2(\alpha-1)-1} \Big] \\ = T_1 < \infty \end{aligned}$$

Where

$$\begin{aligned} Z = (V_1V_3 + V_1/\sup_{t\in J} E\|y(t)\|^2 + V_5 + V_1V_6) < \infty \\ Z^1 = (V_2 + V_7/\sup_{t\in J} E\|y(t)\|^2 + V_4 + V_8) < \infty. \end{aligned}$$

Further from the assumptions, we have

$$\begin{aligned} \sup_{t\in J} \|\phi y(t)\|^2 \le 4W_1 E\|\phi_0 + g(0,\phi_0)\|^2 + V_7 + 4W_2 T_1 \|B\|^2 \frac{b^{2\alpha}}{2\alpha-1} \\ + 4W_2 Z \frac{b^{2\alpha}}{2\alpha-1} + 4W_2 Z^1 \frac{b^{2\alpha-1}}{2\alpha-1} < \infty. \end{aligned}$$

Now, for $y_1, y_2 \in Y$, we have

$$
\begin{aligned}
\sup_{t\in J} E\|\phi_{y_1}(t) - \phi_{y_2}(t)\|^2 = \sup_{t\in J} E\| & \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(b-\theta)^\alpha) B K_b^* M_b^{-1} \\
& \Big\{ \int_0^b (b-\theta)^{\alpha-1} E_{\alpha,\alpha}(A(b-\theta)^\alpha)[h(\theta, y_1(\theta), \int_0^\theta l(\theta,\tau,y_1(\tau))\, \mathrm{d}\tau) \\
& - h(\theta, y_2(\theta), \int_0^\theta l(\theta,\tau,y_2(\tau))\, \mathrm{d}\theta) + \int_0^b (b-\theta)^{\alpha-1} E_{\alpha,\alpha}(A(b-\theta)^\alpha) \\
& \times [Ag(\theta, y_1(\theta)) - Ag(\theta, y_2(\theta))]\, \mathrm{d}\theta] + \int_0^b (b-\theta)^{\alpha-1} E_{\alpha,\alpha}(A(b-\theta)^\alpha) \\
& [f(\theta, y_{t_1}(\theta)) - f(\theta, y_{t_2}(\theta)\, \mathrm{d}z_H(\theta))\Big\}\, \mathrm{d}s + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(b-s)^\alpha) \\
& \Big[h(s, y_1(s), \int_0^s l(s,\tau,y_1(\tau))\, \mathrm{d}\tau) - h(s, y_2(s), \int_0^s l(s,\tau,y_2(\tau))\, \mathrm{d}\tau)\Big]\, \mathrm{d}s \\
& + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(b-s)^\alpha)[Ag(s, y_1(s)) - A(g(s, y_2(s)))]\, \mathrm{d}s \\
& + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(b-s)^\alpha)[f(s, y_{t_1}(s)) - f(s, y_{t_2}(s))]\|^2\, \mathrm{d}s \\
& \le \frac{8b^{2(\alpha)} W_2}{2\alpha-1}(V_2 b^{-1} + V_2 + V_1 + V_1 V_3 b) \sup_{t\in J} E\|y_1(t) - y_2(t)\|^2 \\
& \le \delta \|y_1 - y_2\|^2.
\end{aligned}
$$

3.5 Examples

Next, we provide an illustration to the above theoretical results.

Example

Consider the fractional order linear stochastic system with Rosenblatt Process,

$$
\begin{aligned}
{}^C D^{\frac{2}{3}} \left(y(t) + \begin{pmatrix} e^{-t} sinty_1(t) \\ (e^t+1) sinty_2(t) \end{pmatrix} \right) \qquad & = Ay(t) + Bu(t) + \left(1 + \int_0^1 2y(s)\, \mathrm{d}s + f(t, y_t)\, \mathrm{d}Z_H(t)\right), \\
& \psi_0 = 0,
\end{aligned}
\tag{3.5}
$$

where $\alpha = \dfrac{2}{3}$, $\quad y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$, for $t \in [0, 1]$,

$A = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}$; $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $f(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $g(t, y(t)) = \begin{pmatrix} e^{-t} sinty_1(t) \\ (e^t+1) sinty_2(t) \end{pmatrix}$.

Here $y(t)$ is the state variable, and $u(t)$ is the control variable. We apply Theorem 3.3.1 to prove that the system 3.5 is controllable on $[0, 1]$. In this example, the solution is given by

$$
\begin{aligned}
y(t) = & E_{\frac{2}{3}}\left(At^{\frac{2}{3}}\right)[\psi_0 + g(0, \psi_0)] - g(t, y(t)) \\
& + \int_0^t (t-s)^{\frac{2}{3}-1} E_{\frac{2}{3},\frac{2}{3}}\left(A(t-s)^{\frac{2}{3}}\right) \\
& \times \left[Ag(s, y(s)) + h(s, y(s), \int_0^s l(s,\tau,y(\tau))\, \mathrm{d}\tau) + Bu(s)\right]\, \mathrm{d}s \\
& + \int_0^t (t-s)^{\frac{2}{3}-1} E_{\frac{2}{3},\frac{2}{3}}\left(A(t-s)^{\frac{2}{3}}\right) f(s)\, \mathrm{d}Z_H(s).
\end{aligned}
$$

We have the control of the system 3.5 as

$$\begin{aligned} u(t) \quad &= (1-t)^{\frac{2}{3}-1}B^*E_{\frac{2}{3},\frac{2}{3}}(A^*(1-t)^{\frac{2}{3}})E\Big\{M_b^{-1}(y_1 - E_{\frac{2}{3}}(Ab^{\frac{2}{3}})[\psi_0 + g(0,\psi_0)] \\ &+ g(t,y(t)) - \int_0^t (t-s)^{\frac{2}{3}-1}E_{\frac{2}{3},\frac{2}{3}}(A(t-s)^{\frac{2}{3}}) \\ &\times \Big[Ag(s,y(s)) + h(s,y(s), \int_0^s l(s,\tau,y(\tau))\,\mathrm{d}\tau)\Big]\,\mathrm{d}s \\ &- \int_0^t (t-s)^{\frac{2}{3}-1}E_{\frac{2}{3},\frac{2}{3}}(A(t-s)^{\frac{2}{3}}f(s)\,\mathrm{d}Z_H(s)/\mathcal{F}_s\Big\}. \end{aligned}$$

By computation, we have the controllability grammian operator as

$$M_1(y) \quad = \int_0^1 (1-s)^{2(\frac{2}{3})-2}E_{\frac{2}{3},\frac{2}{3}}(A(1-s)^{\frac{2}{3}}BB^*E_{\frac{2}{3},\frac{2}{3}}(A^*(1-s)^{\frac{2}{3}}E\Big\{x/\mathcal{F}_s\Big\}\,\mathrm{d}s.$$

$$\begin{aligned} M_1(y) &= \begin{pmatrix} 349.4871 & -99.9949 \\ -99.9949 & 29.8226 \end{pmatrix} \\ |M_1(y)| &= 10422.6139 - 9998.9800 \\ |M_1(y)| &= 423.6339 > 0. \end{aligned}$$

which is positive definite. Hence by Theorem (3.3.1), the system (3.5) given in this example is completely controllable on $[0,1]$.

Example

Consider the fractional order non-linear stochastic system with Rosenblatt Process,

$$\begin{aligned} {}^CD^{\frac{2}{3}}\left(y(t) + \begin{pmatrix} e^{-t}sinty_1(t) \\ (e^t+1)sinty_2(t) \end{pmatrix}\right) &= Ay(t) + Bu(t) + (1 + \int_0^1 2y(s)\,\mathrm{d}s + f(t,y_t)\,\mathrm{d}Z_H(t)), \\ \psi_0 &= 0, \end{aligned} \tag{3.6}$$

where $\alpha = \dfrac{4}{5}$, $y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$, for $t \in [0,1]$,

$A = \begin{pmatrix} 0 & -\dfrac{1}{2} \\ \dfrac{1}{2} & 0 \end{pmatrix}$; $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; $f(t,y_t) = \begin{pmatrix} \dfrac{1}{1+t} \\ \dfrac{e^{-siny_2}}{1+t} \end{pmatrix}$ and $g(t,y(t)) = \begin{pmatrix} e^{-t}sinty_1(t) \\ (e^t+1)sinty_2(t) \end{pmatrix}$.

Here $y(t)$ is the state variable, and $u(t)$ is the control variable. We apply Theorem (3.4.1) to prove that the system (3.6) is controllable on $[0,1]$. In this example, the solution is given by

$$\begin{aligned} y(t) = E_{\frac{4}{5}}(At^{\frac{4}{5}})&[\psi_0 + g(0,\psi_0)] - g(t,y(t)) \\ &+ \int_0^t (t-s)^{\frac{4}{5}-1}E_{\frac{4}{5},\frac{4}{5}}(A(t-s)^{\frac{4}{5}}) \\ &\times \Big[Ag(s,y(s)) + h(s,y(s), \int_0^s l(s,\tau,y(\tau))\,\mathrm{d}\tau\Big]\,\mathrm{d}s \\ &+ \int_0^t (t-s)^{\frac{4}{5}-1}E_{\frac{4}{5},\frac{4}{5}}(A(t-s)^{\frac{4}{5}})f(s,y_s)\,\mathrm{d}Z_H(s). \end{aligned}$$

We have the control of the system (3.6) as

$$\begin{aligned}u(t) = {} & (1-t)^{\frac{4}{5}-1}B^*E_{\frac{4}{5},\frac{4}{5}}(A^*(1-t)^{\frac{4}{5}})E\Big\{M_b^{-1}(y_1 - E_{\frac{4}{5}}(Ab^{\frac{4}{5}})[\psi_0 + g(0,\psi_0)] \\ & + g(t,y(t)) - \int_0^t (t-s)^{\frac{4}{5}-1}E_{\frac{4}{5},\frac{4}{5}}(A(t-s)^{\frac{4}{5}}) \\ & \times \Big[Ag(s,y(s)) + h(s,y(s),\int_0^s l(s,\tau,y(\tau))\, \mathrm{d}\tau)\Big]\, \mathrm{d}s \\ & - \int_0^t (t-s)^{\frac{4}{5}-1}E_{\frac{4}{5},\frac{4}{5}}(A(t-s)^{\frac{4}{5}}f(s,y_s)\, \mathrm{d}Z_H(s)/\mathcal{F}_s\Big\}.\end{aligned}$$

By computation, we have the controllability grammian operator as

$$M_1(y) = \int_0^1 (1-s)^{2(\frac{4}{5})-1}E_{\frac{4}{5},\frac{4}{5}}(A(1-s)^{\frac{4}{5}}BB^*E_{\frac{4}{5},\frac{4}{5}}(A^*(1-s)^{\frac{4}{5}}E\Big\{x/\mathcal{F}_s\Big\}\, \mathrm{d}s.$$

$$\begin{aligned}M_1(y) &= \begin{pmatrix} 0.4754 & -0.5709 \\ -0.5709 & 1.3249 \end{pmatrix} \\ &= 0.62985 - 0.32592 \\ &= 0.30393 > 0.\end{aligned}$$

which is positive definite. We also obtain the value of σ in Assumption $A3$ to be $\sigma = 0.6295 < 1$. All the assumption of Theorem (3.4.1) are verified and hence the system (3.6) is completely controllable on $[0,1]$.

3.6 Conclusions

This Chapter 3 We examined the controllability analysis for both linear and nonlinear frac-tional order neutral-type stochastic integro-differential system with non-Gaussian process, named as Rosenblatt process. We formulate a set of necessary and sufficient conditions for our introduced systems to be controllable by employing standard techniques. The fractional Brownian motion is the foremost studied method within the class of Hermite processes due to its vital importance in mod-eling. Our main interest during this work, from the stochastic calculus purpose of view, was to consider the non-Gaussian Rosenblatt process. Although it received a smaller attention than the half Brownian motion, however this method remains of much interests in sensible applications as results of its self-similarity, stationar-ity of increments and long vary dependence. Truly the terribly giant utilization of the fractional Brownian motion in application (hydrology, telecommunications)

Chapter 4

Optimal control for a class of Sobolev-type fractional nonlocal evolution system

In this Chapter, we show existence and uniqueness of mild solutions to Sobolev type fractional nonlocal evolution equations in Banach spaces. For the main results, we use standard tools such fractional calculus, semigroup theory, fractional power of operators, a singular version of Gronwalls inequality and Leray-Schauder fixed point theorem. We also establish a Lagrange optimal control problem for the considered system by using optimality properties.

4.1 Introduction

Fractional calculus is a very import subject and has confirmed its successful applicability in many fields. More details about theory, methods, and applications can be found in the books [15, 60, 76, 117, 84, 98, 101, 135, 133] and the papers [4, 139, 73, 113, 94, 103, 140, 173]. Nonlocal fractional differential equations and inclusions have considered in [17, 30, 31, 34, 109, 164]. Fractional control systems and fractional optimal control problems were considered in several different works, see for instance [32, 33, 106, 107, 174]. Those control systems are most often based on the principle of feedback, whereby the signal to be controlled is compared to a desired reference signal and the discrepancy used to compute corrective control actions [47]. The fractional optimal control of a distributed system is an optimal control problem for which the system dynamics is defined by means of fractional differential equations [110]. In [34], the authors introduced multi-delay controls and we investigated a nonlocal condition for fractional semilinear control systems, see also[164, 179]. Here we are concerned with the study of fractional evolution equations of Sobolev type with nonlocal conditions. Sobolev type fractional order differential equations have been studied by many researchers, e.g., by Fečkan et al. [48] and Li et al. [90]. Based on above statements, we study here a class of semilinear fractional evolution equations of Sobolev type with nonlocal conditions.

$$ {}^{C}D_t^{\alpha}[Lu(t)] = Mu(t) + \psi(t, u(t)) \tag{4.1} $$

subject to nonlocal conditions

$$ u(0) = u_0 + \varphi(u(t)), \tag{4.2} $$

where ${}^C D_t^\alpha$ is the Caputo fractional derivative with $0 < \alpha \leq 1$ and $t \in J = [0, a]$. Let X and Y be two Banach spaces such that Y is densely and continuously embedded in X, the unknown function $u(\cdot)$ takes its values in X and $u_0 \in X$. We consider the operators $L : D(L) \subset X \to Y$ and $M : D(M) \subset X \to Y$. It is also assumed that $\psi : J \times X \to Y$ and $\varphi : C(J : X) \to X$ are given abstract functions satisfying some conditions to be specified later. In Section 4.2 we present some essential notions and facts that will be used in the proof of our results, such as, fractional operators, fractional powers of the generator of an analytic compact semigroup, and the form of mild solutions of (4.1)–(4.2). In Section 4.3, we prove existence (Theorem 4.1) and uniqueness (Theorem 4.2) of mild solutions to system (4.1)–(4.2). Then, in Section 4.4, we prove existence of optimal pairs for the (*LP*) Lagrange optimal control problem (Theorem 4.3). We end with Section **??**, where an example illustrating the application of the abstract results (Theorems 4.1, 4.2 and 4.3) is given.

4.2 Preliminaries

In this section we introduce some basic definitions, notations and lemmas, which will be used throughout the work. In particular, we give main properties of fractional calculus [76, 117] and well known facts in semigroup theory [63, 116, 170].

Definition 4.2.1

The fractional integral of order $\alpha > 0$ of a function $f \in L^1([a, b], \mathbb{R}^+)$ is given by

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds,$$

where Γ is the classical gamma function. If $a = 0$, we can write $I^\alpha f(t) = (g_\alpha * f)(t)$, where

$$g_\alpha(t) := \begin{cases} \dfrac{1}{\Gamma(\alpha)} t^{\alpha-1}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

and, as usual, $*$ denotes convolution of functions. Moreover, $\lim\limits_{\alpha \to 0} g_\alpha(t) = \delta(t)$, with δ the delta Dirac function.

Definition 4.2.2

The Riemann–Liouville fractional derivative of order $\alpha > 0$, $n - 1 < \alpha < n$, $n \in \mathbb{N}$, is given by

$${}^L D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > 0,$$

where function f has absolutely continuous derivatives up to order $n - 1$.

Definition 4.2.3

The Caputo fractional derivative of order $\alpha > 0$, $n - 1 < \alpha < n$, $n \in \mathbb{N}$, is given by

$${}^C D^\alpha f(t) = {}^L D^\alpha \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0,$$

where function f has absolutely continuous derivatives up to order $n - 1$.

Remark 4.2.1

Let $n-1<\alpha<n$, $n\in\mathbb{N}$. The following properties hold:

(i) If $f\in C^n([0,\infty))$, then

$$^{C}D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds=I^{n-\alpha}f^{(n)}(t),\quad t>0.$$

(ii) The Caputo derivative of a constant function is equal to zero.

(iii) The Riemann–Liouville derivative of a constant function C is given by

$$^{L}D^{\alpha}_{a^+}C=\frac{C}{\Gamma(1-\alpha)}(x-a)^{-\alpha}.$$

If f is an abstract function with values in X, then the integrals which appear in Definitions 4.2.1–4.2.3 are taken in Bochner's sense.

Let $(X,\|\cdot\|)$ be a Banach space, $C(J,X)$ denotes the Banach space of continuous functions from J into X with the norm $\|u\|_J=\sup\{\|u(t)\|:t\in J\}$, and let $\mathcal{L}(X)$ be the Banach space of bounded linear operators from X to X with the norm $\|G\|_{\mathcal{L}(X)}=\sup\{\|G(u)\|:\|u\|=1\}$. We make the following assumptions:

(H_1) $M:D(M)\subset X\to Y$ is a linear closed operator and $L:D(L)\subset X\to Y$ is a linear operator.

(H_2) L is bijective.

(H_3) $L^{-1}:Y\to D(L)\subset X$ is linear, bounded, and compact operator.

Note that (H_3) implies L to be closed. Indeed, if L^{-1} is closed and injective, then its inverse is also closed. From (H_1)–(H_3) and the closed graph theorem, we obtain the boundedness of the linear operator $ML^{-1}:Y\to Y$. Consequently, ML^{-1} generates a semigroup $\{Q(t),t\geq 0\}$, $Q(t):=e^{ML^{-1}t}$. We suppose that $M_0:=\sup\limits_{t\geq 0}\|Q(t)\|<\infty$ and, for short, we denote $C=\|L^{-1}\|$.
According to previous definitions, it is suitable to rewrite problem (4.1)–(4.2) as the equivalent integral equation

$$Lu(t)=Lu(0)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[Mu(s)+\psi(s,u(s))]ds,\tag{4.3}$$

provided the integral in (4.3) exists for a.e. $t\in J$.

Remark 4.2.2

Note that:

(i) For the nonlocal condition, the function $u(0)$ is dependent on t.

(ii) The explicit and implicit integrals given in (4.3) exist (taken in Bochner's sense).

Throughout the paper, $A=ML^{-1}:D(A)\subset Y\to Y$ will be the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $Q(\cdot)$. Then, there exists a constant $M_0 \geq 1$ such that $\|Q(t)\| \leq M_0$ for $t \geq 0$. Without loss of generality, we assume that $0 \in \rho(A)$, the resolvent set of A. Then it is possible to define the fractional power A^q, $0 < q \leq 1$, as a closed linear operator on its domain $D(A^q)$ with inverse A^{-q}. Furthermore, the subspace $D(A^q)$ is dense in X and the expression $\|u\|_q = \|A^q u\|, u \in D(A^q)$ defines a norm on $D(A^q)$. Hereafter, we denote by X_q the Banach space $D(A^q)$ normed with $\|u\|_q$.

Lemma 4.2.1 (*See [116]*)

Let A be the infinitesimal generator of an analytic semigroup $Q(t)$. If $0 \in \rho(A)$, then

(a) $Q(t) : X \to D(A^q)$ for every $t > 0$ and $q \geq 0$.

(b) For every $u \in D(A^q)$, we have $Q(t)A^q u = A^q Q(t)u$.

(c) For every $t > 0$, the operator $A^q Q(t)$ is bounded and $\|A^q Q(t)\| \leq M_q t^{-q} e^{-\omega t}$.

(d) If $0 < q \leq 1$ and $u \in D(A^q)$, then $\|Q(t)u - u\| \leq C_q t^q \|A^q u\|$.

Remark 4.2.3

Note that:

(i) $D(A^q)$ is a Banach space with the norm $\|u\|_q = \|A^q u\|$ for $u \in D(A^q)$.

(ii) If $0 < p \leq q \leq 1$, then $D(A^q) \hookrightarrow D(A^p)$.

(iii) A^{-q} is a bounded linear operator in X with $D(A^q) = Im(A^{-q})$.

Remark 4.2.4

Observe, as in [88], that by Lemma 4.2.1 (a) and (b), the restriction $Q_q(t)$ of $Q(t)$ to X_q is exactly the part of $Q(t)$ in X_q. Let $u \in X_q$. Since $\|Q(t)u\|_q \leq \|A^q Q(t)u\| = \|Q(t)A^q u\| \leq \|Q(t)\|\|A^q u\| = \|Q(t)\|\|u\|_q$, and as t decreases to 0^+, $\|Q(t)u - u\|_q = \|A^q Q(t)u - A^q u\| = \|Q(t)A^q u - A^q u\| \to 0$ for all $u \in X_q$, it follows that $\{Q(t), t \geq 0\}$ is a family of strongly continuous semigroups on X_q and $\|Q_q(t)\| \leq \|Q(t)\| \leq M_0$ for all $t \geq 0$.

In the sequel, we will also use $\|\phi\|_{L^p(J,\mathbb{R}^+)}$ to denote the $L^p(J,\mathbb{R}^+)$ norm of ϕ whenever $\phi \in L^p(J,\mathbb{R}^+)$ for some p with $1 < p < \infty$. We will set $q \in (0,1)$ and denote by Ω_q the Banach space $C(J, X_q)$ endowed with supnorm given by $\|u\|_\infty = \sup_{t\in J} \|u\|_q$ for $u \in \Omega_q$.

Motivated by [34, 48, 176], we give the definition of mild solution to (4.1)–(4.2).

Definition 4.2.4

A function $u \in \Omega_q$ is called a mild solution of system (4.1)–(4.2) if it satisfies the following integral equation:

$$u(t) = S_\alpha(t)L[u_0 + \varphi(u(t))] + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)\psi(s, u(s))ds,$$

where

$$S_\alpha(t) = \int_0^\infty L^{-1}\zeta_\alpha(\theta)Q(t^\alpha\theta)d\theta, \quad T_\alpha(t) = \alpha\int_0^\infty L^{-1}\theta\zeta_\alpha(\theta)Q(t^\alpha\theta)d\theta,$$

$$\zeta_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\varpi_\alpha(\theta^{-\frac{1}{\alpha}}) \geq 0, \quad \varpi_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \theta \in (0,\infty),$$

with ζ_α the probability density function defined on $(0,\infty)$, that is, $\zeta_\alpha(\theta) \geq 0, \theta \in (0,\infty)$ and $\int_0^\infty \zeta_\alpha(\theta)d\theta = 1$.

Remark 4.2.5

For $v \in [0,1]$, ones has

$$\int_0^\infty \theta^v \zeta_\alpha(\theta)d\theta = \int_0^\infty \theta^{-\alpha v}\varpi_\alpha(\theta)d\theta = \frac{\Gamma(1+v)}{\Gamma(1+\alpha v)}$$

(see [177]).

Lemma 4.2.2 (*See [48, 176, 177]*)

The operators $S_\alpha(t)$ and $T_\alpha(t)$ have the following properties:

(a) For any fixed $t \geq 0$, the operators $S_\alpha(t)$ and $T_\alpha(t)$ are linear and bounded, i.e., for any $u \in X$, $\|S_\alpha(t)u\| \leq CM_0\|u\|$ and $\|T_\alpha(t)u\| \leq \dfrac{CM_0}{\Gamma(\alpha)}\|u\|$.

(b) $\{S_\alpha(t), t \geq 0\}$ and $\{T_\alpha(t), t \geq 0\}$ are strongly continuous, i.e., for $u \in X$ and $0 \leq t_1 < t_2 \leq a$, we have $\|S_\alpha(t_2)u - S_\alpha(t_1)u\| \to 0$ and $\|T_\alpha(t_2)u - T_\alpha(t_1)u\| \to 0$ as $t_1 \to t_2$.

(c) For every $t > 0$, $S_\alpha(t)$ and $T_\alpha(t)$ are compact operators.

(d) For any $u \in X$, $p \in (0,1)$ and $q \in (0,1)$, we have $AT_\alpha(t)u = A^{1-p}T_\alpha(t)A^p u$, $t \in J$, and $\|A^q T_\alpha(t)\| \leq \dfrac{\alpha C M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))} t^{-q\alpha}$, $0 < t \leq a$.

(e) For fixed $t \geq 0$ and any $u \in X_q$, we have $\|S_\alpha(t)u\|_q \leq CM_0\|u\|_q$ and $\|T_\alpha(t)u\|_q \leq \dfrac{CM_0}{\Gamma(\alpha)}\|u\|_q$.

(f) $S_\alpha(t)$ and $T_\alpha(t)$, $t > 0$, are uniformly continuous, that is, for each fixed $t > 0$ and $\epsilon > 0$ there exists $g > 0$ such that $\|S_\alpha(t+\epsilon) - S_\alpha(t)\|_q < \epsilon$ for $t+\epsilon \geq 0$ and $|\epsilon| < g$, $\|T_\alpha(t+\epsilon) - T_\alpha(t)\|_q < \epsilon$ for $t+\epsilon \geq 0$ and $|\epsilon| < g$.

Lemma 4.2.3 (*See [172]*)

For each $\phi \in L^p(J,X)$ with $1 \leq p < \infty$,

$$\lim_{g\to 0}\int_0^a \|\phi(t+g) - \phi(t)\|^p dt = 0,$$

where $\phi(s) = 0$ for $s \notin J$.

Lemma 4.2.4 (*See [177]*)

A measurable function $G : J \to X$ is a Bochner integral if $\|G\|$ is Lebesgue integrable.

4.3 Main results

Our first result provides existence of mild solutions to system (4.1)–(4.2). To prove that, we make use of the following assumptions:

(F_1) The linear closed operator A is defined on dense set from X_q into Y.

(F_2) The function $\psi : J \times X_q \to Y$ satisfies: for each $u \in X_q$, the function $t \to \psi(t, u(t))$ is measurable.

(F_3) For arbitrary $u, u^* \in X_q$ satisfying $\|u\|_q, \|u^*\|_q \leq \rho$, there exists a constant $L_\psi(\rho) > 0$ and a function $m \in L^1(J, \mathbb{R}^+)$ such that

$$\|\psi(t, u) - \psi(t, u^*)\| \leq L_\psi(\rho) m(t) \|u - u^*\|_q$$

for almost all $t \in J$.

(F_4) There exists a constant $a_\psi > 0$ such that

$$\|\psi(t, u)\| \leq a_\psi(1 + r\|u\|_q) \text{ for all } u \in X_q \text{ and } t \in J.$$

(F_5) The function $\varphi : C(J : X_q) \to X_q$ is Lipschitz continuous and bounded in X_q, i.e., for all $u, v \in C(J, X_q)$ there exist constants $k_1, k_2 > 0$ such that

$$\|\varphi(u) - \varphi(v)\|_q \leq k_1 \|u - v\|_q \text{ and } \|\varphi(u)\|_q \leq k_2.$$

Theorem 4.1

Assume hypotheses (F_1)–(F_5) are satisfied. If $u_0 \in X_q$ and $\alpha q < 1$ for some $\frac{1}{2} < \alpha < 1$, then system (4.1)–(4.2) has a mild solution on J.

The following lemmas are used in the proof of Theorem 4.1.

Lemma 4.3.1

Let operator $P : \Omega_q \to \Omega_q$ be given by

$$(Pu)(t) = S_\alpha(t) L[u_0 + \varphi(u(t))] + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \psi(s, u(s)) ds. \tag{4.4}$$

Then, the operator P satisfies $Pu \in \Omega_q$.

Proof: Let $0 \leq t_1 < t_2 \leq a$ and $\alpha q < \frac{1}{2}$. We have

$$\begin{aligned}
&\|(Pu)(t_1) - (Pu)(t_2)\|_q \\
&= \|[S_\alpha(t_1) - S_\alpha(t_2)] L\,[u_0 + \varphi(u)]\|_q \\
&\quad + \int_0^{t_1} (t_1 - s)^{\alpha-1} \|T_\alpha(t_1 - s)\psi(s, u(s)) - T_\alpha(t_2 - s)\psi(s, u(s))\|_q ds \\
&\quad + \int_0^{t_1} |(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}| \|T_\alpha(t_2 - s)\psi(s, u(s))\|_q ds \\
&\quad + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \|T_\alpha(t_2 - s)\psi(s, u(s))\|_q ds.
\end{aligned}$$

We use Lemma 4.2.2, and fractional power of operators, to get

$$\begin{aligned}
\|(Pu)(t_1)-(Pu)(t_2)\|_q \le & \|L\|\left[k_2+\|u_0\|_q\right]\|S_\alpha(t_1)-S_\alpha(t_2)\|_q \\
& + \int_0^{t_1}(t_1-s)^{\alpha-1}\|A^q[T_\alpha(t_1-s)-T_\alpha(t_2-s)]\|\|\psi(s,u(s))\|ds \\
& + \int_0^{t_1}|(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}|\|A^qT_\alpha(t_2-s)\|\|\psi(s,u(s))\|ds \\
& + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\|A^qT_\alpha(t_2-s)\|\|\psi(s,u(s))\|ds \\
\le & \|L\|\left[k_2+\|u_0\|_q\right]\|S_\alpha(t_1)-S_\alpha(t_2)\|_q \\
& + \frac{\alpha CM_q\Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\|\psi\|_{C(J,X)}\int_0^{t_1}(t_1-s)^{\alpha-1}|(t_1-s)^{-q\alpha}-(t_2-s)^{-q\alpha}|ds \\
& + \frac{\alpha CM_q\Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\int_0^{t_1}|(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}|(t_2-s)^{-q\alpha}\|\psi(s,u(s))\|ds \\
& + \frac{\alpha CM_q\Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\int_{t_1}^{t_2}(t_2-s)^{-q\alpha+\alpha-1}\|\psi(s,u(s))\|ds.
\end{aligned}$$

From Lemma 4.2.2 and Hölder's inequality, one can deduce the following inequality:

$$\begin{aligned}
& \|(Pu)(t_1)-(Pu)(t_2)\|_q \\
& \le \|L\|\left[k_2+\|u_0\|_q\right]\|S_\alpha(t_1)-S_\alpha(t_2)\|_q \\
& \quad + \frac{\alpha CM_q\Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\|\psi\|_{C(J,X)}\left[\left(\int_0^{t_1}|(t_1-s)^{-q\alpha}-(t_2-s)^{-q\alpha}|^2ds\right)^{\frac{1}{2}}\right. \\
& \quad \times\left(\int_0^{t_1}(t_1-s)^{2(\alpha-1)}ds\right)^{\frac{1}{2}}+\left(\int_0^{t_1}|(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}|^2ds\right)^{\frac{1}{2}} \\
& \quad \left.\times\left(\int_0^{t_1}(t_2-s)^{-2q\alpha}ds\right)^{\frac{1}{2}}+\frac{1}{\alpha(1-q)}(t_2-t_1)^{\alpha(1-q)}\right] \\
& \le \|L\|\left[k_2+\|u_0\|_q\right]\|S_\alpha(t_1)-S_\alpha(t_2)\|_q \\
& \quad + \frac{\alpha CM_q\Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\|\psi\|_{C(J,X)}\left[\sqrt{\frac{1}{2\alpha-1}}t_1^{\alpha-\frac{1}{2}}\left(\int_0^a|(t_1-s)^{-q\alpha}-(t_2-s)^{-q\alpha}|^2ds\right)^{\frac{1}{2}}\right. \\
& \quad +\left(\int_0^a|(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}|^2ds\right)^{\frac{1}{2}}\sqrt{\frac{1}{1-2q\alpha}}\left(t_2^{1-2q\alpha}-(t_2-t_1)^{1-2q\alpha}\right)^{\frac{1}{2}} \\
& \quad \left.+\frac{1}{\alpha(1-q)}(t_2-t_1)^{\alpha(1-q)}\right],
\end{aligned}$$

which means that $Pu\in\Omega_q$. ■

Lemma 4.3.2

The operator P given by (4.4) is continuous on Ω_q.

Proof: Let $u, u^* \in \Omega_q$ and $\|u - u^*\|_\infty \leq 1$. Then, $\|u\|_\infty \leq 1 + \|u^*\|_\infty = \rho$ and

$$\begin{aligned}
\|(Pu)(t) - (Pu^*)(t)\|_q &= \|S_\alpha(t)L\left[\varphi(u) - \varphi(u^*)\right]\|_q \\
&\quad + \int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s)[\psi(s,u(s)) - \psi(s,u^*(s))]\|_q ds \\
&\leq \|S_\alpha(t)L\| \, \|A^q[\varphi(u) - \varphi(u^*)]\| \\
&\quad + \int_0^t (t-s)^{\alpha-1}\|A^q T_\alpha(t-s)\|\|\psi(s,u(s)) - \psi(s,u^*(s))\|ds \\
&\leq Ck_1 M_0\|L\|\|u - u^*\|_q \\
&\quad + L_\psi(\rho)m(t)\frac{\alpha C M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\int_0^t (t-s)^{-q\alpha+\alpha-1}\|u - u^*\|_q ds \\
&\leq Ck_1 M_0\|L\|\|u - u^*\|_\infty \\
&\quad + L_\psi(\rho)m(t)\frac{\alpha C M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\frac{1}{\alpha(1-q)}t^{\alpha(1-q)}\|u - u^*\|_\infty.
\end{aligned}$$

Therefore,

$$\|(Pu)(t) - (Pu^*)(t)\|_\infty \leq Ck_1 M_0\|L\|\|u - u^*\|_\infty + L_\psi(\rho)m(t)\frac{\alpha C M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\frac{1}{\alpha(1-q)}t^{\alpha(1-q)}\|u - u^*\|_\infty$$

and we conclude that P is continuous. ■

Lemma 4.3.3

The operator P given by (4.4) is compact.

Proof: Let Σ be a bounded subset of Ω_q. Then there exists a constant η such that $\|u\|_\infty \leq \eta$ for all $u \in \Sigma$. By (F_4), there exists a constant τ such that $\|\psi(t, u(t))\| \leq a_\psi(1 + r\eta) = \tau$. Then $P\Sigma$ is a bounded subset of Ω_q. In fact, let $u \in \Sigma$. Using Lemma 4.2.2 (a) and (d), we get

$$\begin{aligned}
\|(Pu)(t)\|_q &\leq \|S_\alpha(t)L\left[u_0 + \varphi(u)\right]\|_q \\
&\quad + \int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s)\psi(s,u(s))\|_q ds \\
&\leq CM_0\|L\|\left[k_2 + \|u_0\|_q\right] \\
&\quad + \int_0^t (t-s)^{\alpha-1}\|A^q T_\alpha(t-s)\|\|\psi(s,u(s))\|ds \\
&\leq CM_0\|L\|\left[k_2 + \|u_0\|_q\right] \\
&\quad + \frac{\alpha C M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\tau\int_0^t (t-s)^{-q\alpha+\alpha-1}ds \\
&\leq CM_0\|L\|\left[k_2 + \|u_0\|_q\right] \\
&\quad + \frac{\alpha C M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\tau\frac{1}{\alpha(1-q)}t^{\alpha(1-q)}.
\end{aligned}$$

Then, we obtain

$$\|(Pu)(t)\|_\infty \leq CM_0\|L\|\left[k_2 + \eta\right] + \frac{\alpha C M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\frac{\tau a^{\alpha(1-q)}}{\alpha(1-q)}.$$

We conclude that $P\Sigma$ is bounded. Define $\Pi = P\Sigma$ and $\Pi(t) = \{(Pu)(t) | u \in \Sigma\}$ for $t \in J$. Obviously, $\Pi(0) = \{(Pu)(0) | u \in \Sigma\}$ is compact. For each $g \in (0,t)$, $t \in (0,a]$, and arbitrary $\delta > 0$, let us define

$\Pi_{g,\delta}(t) = \{(P_{g,\delta}u)(t) | u \in \Sigma\}$, where

$$\begin{aligned}(P_{g,\delta}u)(t) &= Q(g^\alpha\delta)\int_\delta^\infty L^{-1}\zeta_\alpha(\theta)Q(t^\alpha\theta - g^\alpha\delta)L\left[u_0 + \varphi(u)\right]d\theta \\ &\quad + Q(g^\alpha\delta)\int_0^{t-g}(t-s)^{\alpha-1}\left(\alpha\int_\delta^\infty L^{-1}\theta\zeta_\alpha(\theta)Q((t-s)^\alpha\theta - g^\alpha\delta)d\theta\right)\psi(s,u(s))ds \\ &= \int_\delta^\infty L^{-1}\zeta_\alpha(\theta)Q(t^\alpha\theta)L\left[u_0+\varphi(u)\right]d\theta \\ &\quad + \alpha\int_0^{t-g}\int_\delta^\infty \theta(t-s)^{\alpha-1}L^{-1}\zeta_\alpha(\theta)Q((t-s)^\alpha\theta)\psi(s,u(s))d\theta ds.\end{aligned}$$

Then, since the operator $Q(g^\alpha\delta)$, $g^\alpha\delta > 0$, is compact in X_q, the sets $\{(P_{g,\delta}u)(t)|u \in \Sigma\}$ are relatively compact in X_q. This comes from the following inequalities:

$$\begin{aligned}&\|(Pu)(t) - (P_{g,\delta}u)(t)\|_q \\ &\leq \left\|\int_0^\delta L^{-1}\zeta_\alpha(\theta)Q(t^\alpha\theta)L\left[u_0+\varphi(u)\right]d\theta\right\|_q \\ &\quad + \left\|\int_\delta^\infty L^{-1}\zeta_\alpha(\theta)Q(t^\alpha\theta)L\left[u_0+\varphi(u)\right]d\theta\right\|_q \\ &\quad + \left\|\int_\delta^\infty L^{-1}\zeta_\alpha(\theta)Q(t^\alpha\theta)L\left[u_0+\varphi(u)\right]d\theta\right. \\ &\quad \left. - \int_\delta^\infty L^{-1}\zeta_\alpha(\theta)Q(t^\alpha\theta)L\left[u_0+\varphi(u)\right]d\theta\right\|_q \\ &\quad + \alpha\left\|\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}L^{-1}\zeta_\alpha(\theta)Q((t-s)^\alpha\theta)\psi(s,u(s))d\theta ds\right\|_q \\ &\quad + \alpha\left\|\int_0^t\int_\delta^\infty \theta(t-s)^{\alpha-1}L^{-1}\zeta_\alpha(\theta)Q((t-s)^\alpha\theta)\psi(s,u(s))d\theta ds\right. \\ &\quad \left. - \int_0^{t-g}\int_\delta^\infty \theta(t-s)^{\alpha-1}L^{-1}\zeta_\alpha(\theta)Q((t-s)^\alpha\theta)\psi(s,u(s))d\theta ds\right\|_q \\ &\leq \int_0^\delta \|L^{-1}\zeta_\alpha(\theta)Q(t^\alpha\theta)L\|\left\|A^q\left[u_0+\varphi(u)\right]\right\|d\theta \\ &\quad + \int_\delta^\infty \|L^{-1}\zeta_\alpha(\theta)Q(t^\alpha\theta)L\|\left\|A^q\left[u_0+\varphi(u)\right]\right\|d\theta \\ &\quad + \alpha\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\|L^{-1}\|\zeta_\alpha(\theta)\|A^qQ((t-s)^\alpha\theta)\|\|\psi(s,u(s))\|d\theta ds \\ &\quad + \alpha\int_{t-g}^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\|L^{-1}\|\zeta_\alpha(\theta)\|A^qQ((t-s)^\alpha\theta)\|\|\psi(s,u(s))\|d\theta ds\end{aligned}$$

$$
\begin{aligned}
&\le CM_0\|L\|\left[k_2+\|u_0\|_q\right]\int_0^\delta \zeta_\alpha(\theta)d\theta\\
&\quad + CM_0\|L\|\left[k_2+\|u_0\|_q\right]\int_\delta^\infty \zeta_\alpha(\theta)d\theta\\
&\quad + CM_q\alpha\tau\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta)(t-s)^{-\alpha q}\theta^{-q}d\theta ds\\
&\quad + CM_q\alpha\tau\int_{t-g}^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta)(t-s)^{-\alpha q}\theta^{-q}d\theta ds\\
&\le CM_0\|L\|\left[k_2+\|u_0\|_q\right]\int_0^\delta \zeta_\alpha(\theta)d\theta\\
&\quad + CM_0\|L\|\left[k_2+\|u_0\|_q\right]\\
&\quad + CM_q\alpha\tau\int_0^t\int_0^\delta \theta^{1-q}(t-s)^{-\alpha q+\alpha-1}\zeta_\alpha(\theta)d\theta ds\\
&\quad + CM_q\alpha\tau\int_{t-g}^t\int_\delta^\infty \theta^{1-q}(t-s)^{-\alpha q+\alpha-1}\zeta_\alpha(\theta)d\theta ds\\
&\le CM_0\|L\|\left[k_2+\|u_0\|_q\right]\int_0^\delta \zeta_\alpha(\theta)d\theta\\
&\quad + CM_0\|L\|\left[k_2+\|u_0\|_q\right]\\
&\quad + CM_q\alpha\tau\left(\int_0^t (t-s)^{-\alpha q+\alpha-1}ds\right)\int_0^\delta \theta^{1-q}\zeta_\alpha(\theta)d\theta\\
&\quad + CM_q\alpha\tau\frac{\Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\left(\int_{t-g}^t (t-s)^{-\alpha q+\alpha-1}ds\right)
\end{aligned}
$$

and

$$
\int_0^t (t-s)^{-\alpha q+\alpha-1}ds \le \frac{1}{\alpha(1-q)}t^{\alpha(1-q)},\ \int_{t-g}^t (t-s)^{-\alpha q+\alpha-1}ds \le \frac{1}{\alpha(1-q)}g^{\alpha(1-q)},
$$

so that

$$
\begin{aligned}
\|(Pu)(t)-(P_{g,\delta}u)(t)\|_q \le & CM_0\|L\|\left[k_2+\|u_0\|_q\right]\int_0^\delta \zeta_\alpha(\theta)d\theta\\
& + CM_0\|L\|\left[k_2+\|u_0\|_q\right]\\
& + \frac{CM_q\alpha\tau}{\alpha(1-q)}a^{\alpha(1-q)}\int_0^\delta \theta^{1-q}\zeta_\alpha(\theta)d\theta\\
& + \frac{CM_q\alpha\tau\Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\frac{1}{\alpha(1-q)}g^{\alpha(1-q)}.
\end{aligned}
$$

Therefore, $\Pi(t)=\{(Pu)(t)|u\in\Sigma\}$ is relatively compact in X_q for all $t\in(0,a]$ and, since it is compact at $t=0$, we have relatively compactness in X_q for all $t\in J$.

Next, let us prove that $\Pi=P\Sigma$ is equicontinuous. For $g\in[0,a)$,

$$
\begin{aligned}
\|(Pu)(g)-(Pu)(0)\|_q \le & \|S_\alpha(g)L-I\|_q\\
& + CM_0\|L\|\left[k_2+\|u_0\|_q\right]\\
& + \frac{\alpha CM_q\Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\frac{\tau}{\alpha(1-q)}g^{\alpha(1-q)},
\end{aligned}
$$

and for $0<s<t_1<t_2\le a$, $\|(Pu)(t_1)-(Pu)(t_2)\|_q\le I_1+I_2+I_3+I_4$, where

$$
\begin{aligned}
I_1 &= \|L\|\left[k_2+\|u_0\|_q\right]\|S_\alpha(t_1)-S_\alpha(t_2)\|_q,\\
I_2 &= \frac{\alpha CM_q\Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\|\psi\|_{C(J,X)}\sqrt{\frac{1}{2\alpha-1}}t_1^{\alpha-\frac{1}{2}}\left(\int_0^a |(t_1-s)^{-q\alpha}-(t_2-s)^{-q\alpha}|^2ds\right)^{\frac{1}{2}},
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= \frac{\alpha C M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\|\psi\|_{C(J,X)} \left(\int_0^a |(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}|^2 ds\right)^{\frac{1}{2}} \\
&\quad \times \sqrt{\frac{1}{1-2q\alpha}} \left(t_2^{1-2q\alpha} - (t_2-t_1)^{1-2q\alpha}\right)^{\frac{1}{2}}, \\
I_4 &= \frac{\alpha C M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\|\psi\|_{C(J,X)} \frac{1}{\alpha(1-q)}(t_2-t_1)^{\alpha(1-q)}.
\end{aligned}
$$

Now, we have to verify that I_j, $j = 1, \ldots, 4$, tend to 0 independently of $u \in \Sigma$ when $t_2 \to t_1$. Let $u \in \Sigma$. By Lemma 4.2.2 (c) and (f), we deduce that $\lim\limits_{t_2 \to t_1} I_1 = 0$ and $\lim\limits_{t_2 \to t_1} I_2 = 0$. Moreover, using the fact that $|(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}| \to 0$ as $t_2 \to t_1$, we obtain from Lemma 4.2.3 that

$$
\int_0^a |(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}|^2 ds \to 0 \text{ as } t_2 \to t_1.
$$

Thus, $\lim\limits_{t_2 \to t_1} I_3 = 0$ since $q\alpha < \dfrac{1}{2}$. Also, it is clear that $\lim\limits_{t_2 \to t_1} I_4 = 0$. In summary, we have proven that $P\Sigma$ is relatively compact for $t \in J$ and $\Pi(t) = \{Pu | u \in \Sigma\}$ is a family of equicontinuous functions. Hence, by the Arzela–Ascoli theorem, P is compact. ■

Proof of Theorem 4.1 : We shall prove that the operator P has a fixed point in Ω_q. According to Leray–Schauder fixed point theory (and from Lemmas 4.3.1–4.3.3), it suffices to show that the set $\Delta = \{u \in \Omega_q | u = \beta P u, \beta \in [0,1]\}$ is a bounded subset of Ω_q. Let $u \in \Delta$. Then,

$$
\begin{aligned}
\|u(t)\|_q &= \|\beta(Pu)(t)\|_q \\
&\leq \|S_\alpha(t) L\left[u_0 + h(u)\right]\|_q \\
&\quad + \int_0^t (t-s)^{\alpha-1} \|T_\alpha(t-s)\psi(s,u(s))\|_q ds \\
&\leq C M_0 \|L\| \left[k_2 + \|u_0\|_q\right] \\
&\quad + \int_0^t (t-s)^{\alpha-1} \|A^q T_\alpha(t-s)\| \|\psi(s,u(s))\| ds \\
&\leq C M_0 \|L\| \left[k_2 + \|u_0\|_q\right] \\
&\quad + \frac{a_f \alpha C M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))} \int_0^t (t-s)^{-q\alpha+\alpha-1}(1 + r\|u\|_q) ds \\
&\leq C M_0 \|L\| \left[k_2 + \|u_0\|_q\right] \\
&\quad + \frac{a_f \alpha C M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))} \frac{a^{\alpha(1-q)}}{\alpha(1-q)} + \frac{a_f \alpha r C M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))} \int_0^t (t-s)^{-q\alpha+\alpha-1} \|u\|_q ds.
\end{aligned}
$$

Based on the well known singular version of Gronwall inequality, we can deduce that there exists a constant $R > 0$ such that $\|u\|_\infty \leq R$. Thus, Δ is a bounded subset of Ω_q. By Leray–Schauder fixed point theory, P has a fixed point in Ω_q. Consequently, system (4.1)–(4.2) has at least one mild solution u on J. ■

Theorem 4.2

Mild solution $u(\cdot)$ of system (4.1)–(4.2) is unique.

Proof : Let $u^*(\cdot)$ be another mild solution of system (4.1)–(4.2) with nonlocal initial condition $[u_0 + \varphi(u^*)]$. It is not difficult to verify that there exists a constant $\rho > 0$ such that $\|u\|_q, \|u^*\|_q \leq \rho$. From

$$
\|u(t) - u^*(t)\|_q \leq \|S_\alpha(t) L[\varphi(u) - \varphi(u^*)]\|_q
$$

$$+\int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s)[\psi(s,u(s))-\psi(s,u^*(s))]\|_q ds,$$

we get

$$\|u(t)-u^*\|_q \le Ck_1M_0\|L\|\|u(s)-u^*(s)\|_q$$
$$+L_\psi(\rho)m(t)\frac{\alpha CM_q\Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\int_0^t (t-s)^{-q\alpha+\alpha-1}\|u(s)-u^*(s)\|_q ds.$$

Again, by the singular version of Gronwall's inequality, we arrive at the uniqueness of u. Thus, system (4.1)–(4.2) has a unique mild solution on J. ■

4.4 Optimal controls

Let Z be another separable reflexive Banach space from which the control $\mathfrak{u}$ take its values. We denote by $V_f(Z)$ a class of nonempty closed and convex subsets of Z. The multifunction $\omega : J \to V_f(Z)$ is measurable, $\omega(\cdot) \subset \Lambda$, where Λ is a bounded set of Z. The admissible control set is $U_{ad} = S^p_\omega = \{\mathfrak{u} \in L^p(\Lambda) | \mathfrak{u}(t) \in \omega(t)\ a.e.\}$, $j = \overline{1,k}$, $1 < p < \infty$. Then, $U_{ad} \neq \emptyset$ [61]. Consider the following Sobolev type fractional nonlocal multi-integral-controlled system:

$$^C D_t^\alpha[Lu(t)] = Mu(t) + \psi(t,u(t)) + \int_0^t \mathcal{B}\mathfrak{u}(s)ds, \tag{4.5}$$

$$u(0) = u_0 + \varphi(u(t)). \tag{4.6}$$

Besides the sufficient conditions (F_1)–(F_5) of the last section, we assume:

(F_6) $\mathcal{B} \in L^\infty(J, L(Z, X_q))$, which implies that $\mathcal{B}\mathfrak{u} \in L^p(J, X_q)$ for all $\mathfrak{u} \in U_{ad}$.

Corollaire 4.4.1

In addition to assumptions of Theorem 4.1, suppose (F_6) holds. For every $\mathfrak{u} \in U_{ad}$ and $p\alpha(1-q) > 1$, system (4.5)–(4.6) has a mild solution corresponding to $\mathfrak{u}$ given by

$$u^{\mathfrak{u}}(t) = S_\alpha(t)L\left[u_0 + \varphi(u(t))\right] + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\left[\psi(s,u(s)) + \int_0^s \mathcal{B}\mathfrak{u}(\eta)d\eta\right]ds.$$

Proof: Based on our existence result (Theorem 4.1), it is required to check the term containing multi-integral controls. Let us consider

$$\varphi(t) = \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\left[\int_0^s \mathcal{B}\mathfrak{u}(\eta)d\eta\right]ds.$$

Using Lemma 4.2.2 (d) and Hölder inequality, we have

$$\begin{aligned}
\|\varphi(t)\|_q &\le \left\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\int_0^s \mathcal{B}\mathfrak{u}(\eta)d\eta ds\right\|_q \\
&\le \int_0^t (t-s)^{\alpha-1}\|A^qT_\alpha(t-s)\|\|\mathcal{B}\mathfrak{u}(s)\|a ds \\
&\le \frac{\alpha a CM_q\Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\left[\|\mathcal{B}\|_\infty \int_0^t (t-s)^{-q\alpha+\alpha-1}\|\mathfrak{u}(s)\|_Z ds\right] \\
&\le \frac{\alpha a CM_q\Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\left[\|\mathcal{B}\|_\infty \left(\int_0^t (t-s)^{\frac{p}{p-1}(-q\alpha+\alpha-1)}ds\right)^{\frac{p-1}{p}} \left(\int_0^t \|\mathfrak{u}(s)\|_Z^p ds\right)^{\frac{1}{p}}\right]
\end{aligned}$$

$$\leq \frac{\alpha a C M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\left[\|\mathcal{B}\|_\infty \left(\frac{p-1}{p\alpha(1-q)-1}\right)^{\frac{p-1}{p}} a^{\frac{p\alpha(1-q)-1}{p-1}}\|\mathfrak{u}\|_{L^p(J,Z)}\right],$$

where $\|\mathcal{B}\|_\infty$ is the norm of operators $\mathcal{B}$ in the Banach space $L_\infty(J, L(Z, X_q))$. Thus,

$$\left\|(t-s)^{\alpha-1}T_\alpha(t-s)\int_0^s \mathcal{B}\mathfrak{u}(\eta)d\eta\right\|_q$$

is Lebesgue integrable with respect to $s \in [0,t]$ for all $t \in J$. It follows from Lemma 4.2.4 that

$$(t-s)^{\alpha-1}T_\alpha(t-s)\int_0^s \mathcal{B}\mathfrak{u}(\eta)d\eta$$

is a Bochner integral with respect to $s \in [0,t]$ for all $t \in J$. Hence, $\varphi(\cdot) \in \Omega_q$. The required result follows from Theorem 4.1. ■

Furthermore, let us now assume

(F_7) The functional $\mathcal{L} : J \times X_q \times Z \to \mathbb{R} \cup \{\infty\}$ is Borel measurable.

(F_8) $\mathcal{L}(t,\cdot,\cdot)$ is sequentially lower semicontinuous on $X_q \times Z$ for almost all $t \in J$.

(F_9) $\mathcal{L}(t,u,\cdot)$ is convex on Z for each $u \in X_q$ and almost all $t \in J$.

(F_{10}) There exist constants $d \geq 0$, $C > 0$, such that ψ is nonnegative and $\psi \in L^1(J,\mathbb{R})$ satisfies

$$\mathcal{L}(t,u,\mathfrak{u}) \geq \psi(t) + d\|u\|_q + C\|\mathfrak{u}\|_Z^p.$$

We consider the following Lagrange optimal control problem:

$$\begin{cases} \text{Find } (u^0,\mathfrak{u}^0) \in C(J,X_q) \times U_{ad}^k \\ \text{such that } \mathcal{J}(u^0,\mathfrak{u}^0) \leq \mathcal{J}(u^{\mathfrak{u}},\mathfrak{u}) \text{ for all } \mathfrak{u} \in U_{ad}, \end{cases} \tag{LP}$$

where

$$\mathcal{J}(u^{\mathfrak{u}},\mathfrak{u}) = \int_0^a \mathcal{L}(t,u^{\mathfrak{u}},\mathfrak{u})dt$$

with $u^{\mathfrak{u}}$ denoting the mild solution of system (4.5)–(4.6) corresponding to the multi-integral controls $\mathfrak{u} \in U_{ad}$. The following lemma is used to obtain existence of a fractional optimal multi-integral control (Theorem 4.3).

Lemma 4.4.1

Operator $\Upsilon : L^p(J,Z) \to \Omega_q$ given by

$$(\Upsilon\mathfrak{u})(\cdot) = \int_0^{\cdot}\int_0^s T_\alpha(\cdot - s)\mathcal{B}\mathfrak{u}(\eta)d\eta ds,$$

where $p\alpha(1-q) > 1$ and $j = \overline{1,k}$, are strongly continuous.

Proof: Suppose that $\{\mathfrak{u}^n\} \subseteq L^p(J,Z)$ are bounded. Define $\Theta_n(t) = (\Upsilon\mathfrak{u}^n)(t)$, $t \in J$. Similarly to the proof of Corollary 4.4.1, we can conclude that for any fixed $t \in J$ and $p\alpha(1-q) > 1$, $\|\Theta_n(t)\|_q$ is bounded. By Lemma 4.2.2, it is easy to verify that $\Theta_n(t)$, is compact in X_q and are also equicontinuous. According to the Ascoli–Arzela theorem, $\{\Theta_n(t)\}$ are relatively compact in Ω_q. Clearly, Υ, is linear and continuous. Hence, Υ is strongly continuous operators (see [61, p. 597]). ■

Now we are in position to give the following result on existence of optimal multi-integral controls for the Lagrange problem (LP).

Theorem 4.3

If the assumptions (F_1)–(F_{10}) hold, then the Lagrange problem (LP) admits at least one optimal integral pair.

Proof: Assume that $\inf\{\mathcal{J}(u^{\mathfrak{u}}, \mathfrak{u}) | u^{\mathfrak{u}} \in U_{ad}\} = \epsilon < +\infty$. Using assumptions (F_7)–(F_{10}), we have $\epsilon > -\infty$. By definition of infimum, there exists a minimizing feasible pair $\{(u^m, \mathfrak{u}^m)\} \subset \mathcal{U}_{ad}$ sequence, where $\mathcal{U}_{ad} = \{(u, \mathfrak{u}) | u$ is a mild solution of system (4.5)–(4.6) corresponding to $\mathfrak{u} \in U_{ad}\}$, such that $\mathcal{J}(u^m, \mathfrak{u}^m) \to \epsilon$ as $m \to +\infty$. Since $\{(\mathfrak{u}^m)\} \subseteq U_{ad}$, $m = 1, 2, \ldots, \{(\mathfrak{u}^m)\}$ is bounded in $L^p(J, Z)$ and there exists a subsequence, still denoted by $\{(\mathfrak{u}^m)\}$, $\mathfrak{u}^0 \in L^p(J, Z)$, such that

$$(\mathfrak{u}^m) \overset{\text{weakly}}{\longrightarrow} \left(\mathfrak{u}^0\right)$$

in $L^p(J, Z)$. Since U_{ad} is closed and convex, by Marzur lemma $\mathfrak{u}^0 \in U_{ad}$. Suppose $u^m(u^0)$ is the mild solution of system (4.5)–(4.6) corresponding to $\mathfrak{u}^m(\mathfrak{u}^0)$. Functions u^m and u^0 satisfy, respectively, the following integral equations:

$$u^m(t) = S_\alpha(t) L \left[u_0 + \varphi(u^m(s))\right] \\ + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[\psi(s, u^m(s)) + \int_0^s [\mathcal{B}\mathfrak{u}^m(\eta)] d\eta\right] ds,$$

$$u^0(t) = S_\alpha(t) L \left[u_0 + \varphi(u^0(s))\right] \\ + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[\psi(s, u^0(s)) + \int_0^s [\mathcal{B}\mathfrak{u}^0(\eta)] d\eta\right] ds.$$

It follows from the boundedness of $\{\mathfrak{u}^m\}, \{\mathfrak{u}^0\}$ and Theorem 4.1 that there exists a positive number ρ such that $\|u^m\|_\infty, \|u^0\|_\infty \le \rho$. For $t \in J$, we have

$$\|u^m(t) - u^0(t)\|_q \le \|\xi_m^{(1)}(t)\|_q + \|\xi_m^{(2)}(t)\|_q + \|\xi_m^{(3)}(t)\|_q,$$

where

$$\begin{aligned} \xi_m^{(1)}(t) &= S_\alpha(t) L[\varphi(u^m(s)) - \varphi(u^0(s))], \\ \xi_m^{(2)}(t) &= \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[\psi(s, u^m(s)) - \psi(s, u^0(s))] ds, \\ \xi_m^{(3)}(t) &= \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \int_0^s \mathcal{B}[\mathfrak{u}^m(\eta) - \mathfrak{u}^0(\eta)] d\eta ds. \end{aligned}$$

The assumption (F_5) gives

$$\|\xi_m^{(1)}(t)\|_q \le C M_0 k_1 \|L\| \|u^m - u^0\|_q.$$

Using Lemma 4.2.2 (d) and (F_3),

$$\|\xi_m^{(2)}(t)\|_q \le L_\psi(\rho) m(t) \frac{\alpha C M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))} \int_0^t (t-s)^{-q\alpha+\alpha-1} \|u^m(s) - u^0(s)\|_q ds.$$

From Lemma 4.4.1, we get

$$\xi_m^{(3)}(t) \overset{\text{strongly}}{\longrightarrow} 0.$$

Thus,

$$\|u^m(t) - u^0(t)\|_q \le \|\xi_m^{(3)}(t)\|_q + C M_0 k_1 \|L\| \|u^m - u^0\|_q \\ + L_\psi(\rho) m(t) \frac{\alpha C M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))} \int_0^t (t-s)^{-q\alpha+\alpha-1} \|u^m(s) - u^0(s)\|_q ds.$$

By virtue of the singular version of Gronwall's inequality, there exists $M_* > 0$ such that

$$\|u^m(t) - u^0(t)\|_q \leq M_* \|\xi_m^{(3)}(t)\|_q,$$

which yields that

$$u^m \to u^0 \text{ in } C(J, X_q) \text{ as } m \to \infty.$$

Because $C(J, X_q) \hookrightarrow L^1(J, X_q)$, using the assumptions (F_7)–(F_{10}) and Balder's theorem, we obtain that

$$\begin{aligned}
\epsilon &= \lim_{m\to\infty} \int_0^a \mathcal{L}(t, u^m(t), \mathfrak{u}^m(t))dt \\
&\geq \int_0^a \mathcal{L}(t, u^0(t), \mathfrak{u}^0(t))dt \\
&= \mathcal{J}(u^0, \mathfrak{u}^0) \\
&\geq \epsilon.
\end{aligned}$$

This shows that $\mathcal{J}$ attains its minimum at $\mathfrak{u}^0 \in U_{ad}$. ■

Conclusion

The main purpose of this thesis was to Control is an important aspect of mathematical control theory. It was introduced by [69]. The concept of controllability denotes the ability to transfer the state of the dynamic control system from its initial state to the desired final state using an appropriate control function. In recent years, various aspects of the controllability of ordinary dynamic systems as well as partial dynamic systems, for both deterministic and stochastic structures, have been studied by many researchers. We have developed a solution for the controllability problem of a non-linear fractional order neutral type stochastic integro-differential system with the Rosenblatt process. We take the terms in the system as bounded linear operators instead of a matrix, which produces the same results as a matrix, and the results on stochastic systems using bounded linear operators are more competent. We first proved that the development of modern methods has been used to explore the possibility of solving some classes of initial value problems involving fractional operators and optimal controls. In particular, during this PhD thesis project, we introduced partial calculus theory and control theory to substantiate questions of existence outcomes, controllability, stability, and other properties of new types of problems that can be applied with more precision and better usefulness.

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