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On solution for fractional differential equations using upper and lower solutions method

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Abstract

We are interested in this memorry in nonlinear differential equations,this type of equations describes many phenomena.The objective of this memory is to study the approximate solution for solving a class of nonlinear differential equation Caputo-Hadamard fractional derivative of variable order using upper and lower method it is also discused the upper and lower methode for their solution ,that are applied to FDE and systems of FDE.Upper and lower technique is suggested and studied in detail.However,the propriities of Caputo and Hadamard derivatives are also given with complecte details to approximate the solution finite or infinite functions (trigonometric,exponential,logarithmic,and others) are called infinite.The relation between Caputo and Hadamard of fractional derivative took a big role for simplifying that represents the containts of Integrable varriable problems.The approximate solution are defined on interval and are compared wich the exact solution of order one wich is an important condition to support the working method.Finally ,illutrative examples are included to confirm the eficiency and accuracy of the proposeed method

Key Words: Caputo-Hadamard fractional integrals and derivatives , upper and lower solution

Contents

Introduction

Upper and lower solutions techniques it is important to note that in some of the revious results some kind of the discontinuities on the spatial variable are assumed in this case some thechniques developed are used there is large bibliograhy on papers related with upper lower solutions with nonlinear boundary value conditions for first and higher order equations,problemes withe impules difference equations have been studied under this point of view for an important number of researchers as Thomson's or Frigon's give some generalizations of the concept of lower and upper solutions that ensure the existense of solutions of nonlinear boundry problemes under weaker assumptions,a methode for demonstrating the existence of solutions of boundary value problemes for differential equations,the idea of this method aplied to ordenary differential equations was discussed in the work of G Peano(1880),for the case of the Direchlet Problem and for the case of Laplace equation the idea occurs in H Poincare's balayage methode O,Perron was the first to give a full exposition of the methode of upper and lower functions for this last case
let the problem be posed in a region G

$$\begin{cases} {}^{C}D^{w}_{p^+}\phi(t) = \psi(t,\ \phi(t),\ I^{w}_{p^+}\phi(t)),\ t \in \Lambda := [p,\ T], \\ \phi(p) = \phi_p, \end{cases} \tag{0.0.1}$$

The classes $\Phi\,(G, f)$ and $\Psi\,(G, f)$ of all upper and lower functions respectively are non empty,and

$$if\ v \in \Phi\,(G, f)\ \ and\ w \in \Psi\,(G, f)\ thenv \geq w$$

Ageneralized solution of the Dirichlet problem is defined as the smallest enevlope of the classe $\Phi\,(G, f)$ or as the largest enevlope of the classe $\Psi\,(G, f)$:

$$\begin{aligned} u\,(x) &= \inf\left\{v\,(x) : v \in \Phi\,(G, f)\right\}, \text{for } x \in G \\ &= \sup\left\{w\,(x) : w \in \Psi\,(G, f)\right\}, \text{ for } x \in G \end{aligned}$$

There is a solution u betwen an upper solution w and a lower solution v to study the existense and approximation of solution for the problem(0.0.1)

Remark 0.0.1. if there existe $w > v$ upper and lower solution, then there existe u a solution of the problem(0.0.1)

This method is somme what similar to the intermediate value theorem states that if f is a countinous$[a\ b]$ and $f(a) < c < f(b)$,then there is an $x \in [a, b]$ such that $f(x) = c$.The upper and lower solution method states that if we can find the a certaine couple of function ,then we can fined a solution betwene them.if w and v are upper and lower solution respectivly ,then we can fined a solution u such that $\underline{w(t)} < u(t) < \overline{v(t)}$ for all $t \in [a\ b]$ on nonlinear equation such as CAPUTO -HADAMARD fractional IVP we can not fined an analytical solution because if the equation is solvable we can pick out the upper and lower solution itself .

Chapter 1

The Uniquence And The Stability In Sens Of Ulam Of The Solution Of Our Problem With Fractional IVP:

presenting two types of fixed point theorem: the first is of BANACH to demonstrate the existence and the uniquence,the second is of SCHAUDER to demonstrate the existence .

1.1 The Existence Of Solution:

The existence given by the methode of upper and lower soution ,by numerics we mighte suppose there are multiple solutions bounded by $\underline{w(t)}$ and $\overline{v(t)}$.The existense of solution :

Definition 1.1.1. A mesurable function $U \in C_{\gamma,\ln}$ that satisfies the condition$\phi(p) = \phi_p$,and the equation${}^C D^w_{p^+}\phi(t) = \psi(t,\ \phi(t),\ I^w_{p^+}\phi(t)),\ t \in \Lambda = [p,\ T]$... [11] ,${}^C D^w_{p^+}$Caput FD for w order$\phi : [0,1] \times [0,\infty) \to [0,\infty)$ is a countinous function is kown.

we transforme to integral equation and by the methode upper and lower solution and using the Schauder and Banach fixed point theorem.
For transforming equation[11]using the SCHAUDER fixed point,$\phi : A \to A$ is defined by $\phi X = X$.
Hypothese:$(H1)$ let $X_*, X^* \subset A$,such that $0 \le a \le X_*(t) \le X^*(t) \le b$..... $(H1)$ *and A* a set
Suppose that $(H1)$satisfies then [11]has one solution $X \in X$ such that :$X^*(t) \le X(t) \le X^*(t)$,for $t \in J$.

Proof. Let $C = \{X \in A : X^*(t) \le X(t) \le X^*(t)\,,\text{for } t \in J\}$the norm defined as $\|X\| = \max_{t\in J} |X(t)|$,then we have$\|X\| < b$.and if $X \in C$
Banach spaces ,there existe $c > 0$

such that :$\Phi(c)$is equicontinue$\Phi(c) \subseteq C$
Let $X \in C$ then $X_*(t) \leq X(t) \leq X^*(t)\,, for\ t \in J$ it mense $\Phi(c) \subseteq C$,and $X \in C$ the problem has one positive solotion $x \in X$,such
that :$X_*(t) \leq (\Phi X)(t) \leq X^*(t)$,for $t \in J$.
As consequens, withe the AZELA ASCOLI theorem, we conclud that Φ is countinous and compact ,from application of
Schauders theorem, we deduce that Φ has at least a fixed point U wich is a solution of problem **??** □

1.2 Ulma -Hyers Rassias Stability:

Now we conserned with the generalized Ulma Hyers Rassias Stability of our problem[11] .

Asume that the folowing hypotheses hold ,then problem[11] generalized Ulma Hyers Rassias Stability

$(H2)$ The function $t \rightarrow F(t,u)$ is mesurable on I for each $u \in C_{\gamma,\ln}$ and the function $u \longrightarrow F(t,u)$ is countinous on $C_{\gamma,\ln}$ for $t \in J$.
$(H3)$There existe $\varphi \in C(J,[0,\infty])$ such that for each $t \in J$ and all $U, V \in \mathbb{R}$ we have:

$$|F(t,u) - F(t,u)| \leq L\,|u - v|$$

If $L : (\ln T)^{1-\delta}\, \Phi^* \lambda_\Phi < 1$ where :$\varphi^* = \sup \Phi(t)_{t \in Cf}$,then there existe a unique solution u_0 of problem**??** and the problem**??** generalized Ulma Hyers Rassias Stability.Furthermore,we have :

$$|U(t) - u_0(t)| \leq \frac{\Phi(t)}{1 - L}$$

1.3 The uniquenesse of solution:

Definition 1.3.1. the uniquenesse along with the maximal interval means if $\Psi$is any solution defined on interval $I = [a,b] \subset J$ then $\Psi(t) = \Phi(t)$,for all $t \in I$.

The folowing solution establishes the exsistence and uniquness of solution to CAPUTO-HADAMARD fractional IVP.

Definition 1.3.2. let $(X, \|\|)$be a Banach space and $\Phi : X \rightarrow X$.Φ is contruction operator if $\exists\, \lambda \in (0,1)$ such that $x, y \in X$ imply: $\|\Phi(x) - \Phi(y)\| \leq \lambda \|x - y\|$.

Theorem 1.3.1. *(*BANACH *) let C a nonempty closed convex subset of Banach space X and Φ : $C \rightarrow C$,be a contraction operator,then there is a unique $x \in C$ with $\Phi x = x$.*

Theorem 1.3.2. *(*SCHAUDER*) let C a nonempty closed convex subset of Banach space X and $\Phi : C \rightarrow C$,be a countinous compact application ,then Φ have a fixed point in C.*

Definition 1.3.3. By a solution of problem **??** we mean a mesurable function $U \in C\gamma_{,\ln}$ that satisfies the condition and the equation.

Proof. 1the uniquences ,if x, y are two fixed point then $\Phi(x) = x, \Phi(y) = y$ such that

$$d(x, y) \leq \alpha_n d(x, y) \text{ for } n \in \mathbb{N}$$

□

where $\sum_{n\geq 1} \alpha_n$ is convergent then $\alpha_n$converges to 0 ,so there is n_0 with $\alpha_{n_0} < 1$,then we have :

$$d(x, y) \leq \alpha_{n_0} d(x, y) < d(x, y)$$

That is contraduction
2 For $n \in \mathbb{N}, p \geq 0$ we have :

$$\begin{aligned} d\left(x_{n+p}, y_n\right) &\leq \sum_{k=n}^{n+p-1} d\left(x_{k+1}, y_k\right) \\ &\leq \sum_{k=n}^{n+p-1} \alpha_k d\left(x_1, y_0\right) \\ &= d\left(x_1, y_0\right) \sum_{k=n}^{n+p-1} \alpha_k \end{aligned}$$

$\sum_{k=n}^{n+p-1} \alpha_k$is convergent $\rightarrow \sum_{k=n}^{n+p-1} \alpha_k \leq \varepsilon$ thus,we have $d\left(x_{n+p}, y_n\right) \leq d\left(x_1, y_0\right)\varepsilon$,suposed$d\left(x_1, y_0\right)\varepsilon = \varepsilon'$.That proves
$x_n$converges and $\lim_{n\to\infty} x_n = x_0, for\ n \in \mathbb{N}$,we have :$x_{n+1} = \Phi(x_n)$ the sequense(x_{n+1}) tends to x_0,and Φ is countinous it is α_1-lipschitizien,

$$d\left(\Phi(x_n), \Phi(y)\right) \leq \alpha_1 d(x, y) \text{ then:} \Phi(x_n) \underset{\rightarrow}{cv} \Phi(x_0)$$

By uniquences of the limite we have :

$$\Phi(x_0) = x_0$$

So,Φ has a fixed point that is unique and we have $x_0 \in X$ as limite to $\left(\Phi^n(x_0)\right)_n$ in this inequality $d\left(x_{n+p}, y_n\right) \leq d\left(x_1, y_0\right) \sum_{k=n}^{n+p-1} \alpha_k$ if $p \rightarrow +\infty$ then $d\left(x_0, y_n\right) \leq d\left(x_1, y_0\right) \sum_{k=n}^{n+p-1} \alpha_k$

it result: $d\left(x_0, y_n\right) \leq \varepsilon'$.$x_0$is a unique fixed point of a function Φ.
The existence and the uniquence of a fixed point $x_0 \in X$,$\Phi(x_0) = x_0$implies the existence and the uniquence for solution u of problem **??**.

Chapter 2

Mathematical preliminaries

In this section,i review some spaces definition wich i will need later in the analyses it containes resultes from various researchers.

2.1 Notation and definitions

Sets of natural,integer, real and complex numbers are denoted by: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}, \mathbb{R}_+ = [0, \infty]$.
let Ω be a subset of $\mathbb{R}, C_b(\Omega)$:the set of continuous function of Ω such that :

$$\|f\|_{C_b(\Omega)} = \sup |f(x)| < \infty \text{ ,for } x \in \Omega.$$

$C_b(\Omega)$ is a Banach space,if Ω is open, then:k compact subsets of Ω $(k \subset \Omega)$,we denoted the semi norm of continuous functions f:

$$\|f\|_k = \sup |f(x)|, \text{ for } x \in k.$$

For $[a, b]$ $(-\infty < a < b < +\infty)$ an interval of $\mathbb{R}$

Definition 2.1.1. The space of absolutly continuous functions is denoted by: $AC([a, b]) = AC^1([a, b])$, there
$C([a, b]) \subset AC([a, b])$,moreover :

for $f \in AC([a, b]) \iff f(x) = c +_a^x g(t)dt,\ g = f' \in [a, b] \in \mathbb{R}_+$such that: $\left(g \in L^1([a, b])\right)$

Definition 2.1.2. The space of function f wiche have continuous derivatives up to the order $(n-1) \in [a, b] \in \mathbb{R}_+$ is denoted by:

$$AC^n([a, b]) = \Big\{ f : [a, b] \longrightarrow \mathbb{C} \text{ and } D^{(n-1)} f(x) \in AC^n([a, b]) \text{ for every } b > 0, n \in \mathbb{N}, n \geq 2, \Big(D =$$

$$AC^1([a, b]) = \left\{ f : [a, b] \longrightarrow \mathbb{R}, \frac{d}{dt} f \in AC([a, b]) \right\}.$$

Definition 2.1.3. let $\delta = t\frac{d}{dt}, q > 0, n = [q] + 1$where $[q]$is the integer part of q define the space:

$$AC_{\delta}^{n}([a,b]) = \left\{ f : [1,T] \longrightarrow E, \delta^{n-1}[f(t)] \in AC([a,b]) \right\}.$$

Definition 2.1.4. let $\gamma \in [0,1]$ define the space:

$$C_{\gamma \ \ln} = \left\{ f : (\ln(t))^{1-\gamma} f(t) \in \mathbb{C} \right\}.$$

with the norm

$$\|f\|_{C_{\gamma \ \ln}} = \sup \left| (\ln(t))^{1-\gamma} f(t) \right| \text{ ,for } t \in [a,b].$$

and

$$C_{\gamma} = \left\{ f : [0,T] \longrightarrow E, (t)^{1-\gamma} f(t) \in \mathbb{C} \right\}.$$

with the norm

$$\|f\|_{C_{\gamma}} = \sup \left| (t)^{1-\gamma} f(t) \right| \text{ ,} for \ t \in [a,b].$$

and

$$C_{\gamma}^{1}(f) = \left\{ f \in C : \frac{df}{dt} \in C_{\gamma} \right\}.$$

with the norm

$$\|f\|_{C_{\gamma}^{1}} = \|f\|_{\infty} + \|f'\|_{C_{\gamma}}$$

Definition 2.1.5. The space of mesurable functions is denoted by:

$$L^{p}((a,b)) = L^{p}([a,b]) = \left\{ f \ mesurable \iff \left({}_{a}^{b} |f(x)|^{p} dx \right)^{\frac{1}{P}} < \infty, \text{ for } P \geq 1. \right\}$$

In $L^{p}([a,b])$,for $P \geq 1$the norm is defined by:
$\|f\|_{p} = \left({}_{a}^{b} |f(x)|^{p} dx \right)^{\frac{1}{P}}$
In $L^{\infty}([a,b])$,we have:$\|f\|_{\infty} = \sup |f(x)|$,for $x \in [a,b]$.

Theorem 2.1.1. *If p and q are real numbers such that* $p, q \geq 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$ *for* $p = 1$,*and* $q = \infty$

$$if \quad \left\{ \begin{array}{c} 1.\ f \in L^{p}([a,b]) \\ 2.g \in L^{q}([a,b]) \end{array} \right\}$$

$$then \quad : \quad \left\{ \begin{array}{c} 1.fg \in L^{1}([a,b]) \ \ and \\ 2.{}_{a}^{b} |f(x)g(x)| \, dx \leq \|f\|_{p} \|g\|_{p} \quad [Holder \ inequality] \end{array} \right\}$$

Definition 2.1.6. A real value function f defined of $[a,b] \subset \mathbb{R}$is said absolutly continuous of $[a,b]$:

$$\text{If for } \varepsilon > 0 \text{ ,ther is } \delta > 0 \text{ ,such that : } \sum_{k=1}^{n} \left(x'_k - x_k\right) < \delta \implies \sum_{k=1}^{n} \left|f\left(x'_k\right) - f\left(x_k\right)\right| < \varepsilon$$

Definition 2.1.7. A function f of $[a,b]$ is holder continuous at $x_0 \in [a,b]$:

$$\text{If there existe } A > 0, \lambda > 0 \text{ such that : } |f(x) - f(x_0)| \leq A\,|x - x_0|^{\lambda}$$

Definition 2.1.8. Holder type spaces of an interval $[a,b]$ are defined as subspaces of integrable functions of this interval is denoted by:

$$H^{\lambda} \equiv H^{\lambda}([a,b]) = \left\{ f \ / \ |f(x_1) - f(x_2)| \leq A\,|x_1 - x_2|^{\lambda}, (x_1, x_2) \in [a,b], \lambda \in [0,1] \right\}$$

$$H \equiv H([a,b]) = \cup_{0<\lambda\leq 1} H^{\lambda}([a,b])$$

Chapter 3

Basic definition and propreties of fructional derivatives and integral

The content of this chapiter is addressed to FD and FI for the case :Riemann liouville,Riemann liouville fractional integrals and derivatives on the real half axis,Caputo in the origine,Caputo fractional integrals and derivatives,Risez potentials and Risez derivatives ,other types of fructional derivatives, Canavati fructional derivatives,Marchaud fructional derivatives,Grunwald-Letnikove fructional derivatives,somme additional properties of the FD,Fermat theorem for FD,Taylor theorem for FD, Variational problems with fractional derivatives and Euler Lagrange equation.
In the folowinng, I will enumerate the basic definition and properties of the FD and FI in the case of VO, it presentes the most important special functions , a special attantion is Gama functions other special function such as Beta functions.

3.1 Introduction:

3.1.1 History of fractional calculus:

A relevant part of the history of fractional calculus began with the papers of Abel and Liouvile.

Leibniz first introduced the idea of a symbolic method and used the symbol $\frac{d' y}{dx'} = D' y$ for the nth derivative ,Where n is a non negative integer .
In a letter to L'Hopital in 1695 Leibniz raisaid the following question "can the meaning of derivatives with integer order be generalized to derivatives with non integer orders ? "
L'Hopital was somezhat curious about that question and replied by another question to Leibniz:"what if the order will be $\frac{1}{2}$?"
Leibniz in a letter dated September 30.1695 the exact birthday of the fractional calculus! replied:"i will lead to a paradox.from which one day useful consequences with be drawn"
.

Euler observed that the result of evaluation of $\frac{d^{'}y}{dx^{'}}$the power function x^p has meaning for non integerρ .
Laplace proposed the idea of differentation of non integer order for functions representable by an integral $\Gamma(t)\, t^{-x}dt$.
Fourier suggested the idea of using his integral representation of $f(x)$ to define the derivative for non integer order.
Grunzald and Letnikov developed an appoach to fractional defferentiation based on the limit of a sum.
Liouville formaly extended the formula for the derivative of arbitrary order $D'e^{xy} = \partial^{'}e^{xy}$.
In a paper written when just a student Riemann,that was published only ten years after is death ,he arrived to an expression for fractional integration that becam one of the main formulate together with Liouville construction .

Definition 3.1.1. (Euleur's Gama function) denoted $\Gamma(,)$ is defined as:

$$\Gamma(p) =_0^\infty e^{-x}x^{p-1}dx$$

represented in fig 111
the plot of y=$\Gamma(p)$ function,

Theorem 3.1.1. *function* $\Gamma(p)$ *is convergent for* $p > 0$.

Proof. the integrale can be written as:$\Gamma(p) = \int_0^1 e^{-x}x^{p-1}dx + \int_1^\infty e^{-x}x^{p-1}dx = I_1 + I_2$
where $I_1 = \int_0^1 e^{-x}x^{p-1}dx$ is convergent from the interval $[0,1]$,e^{-x} is decreasing from $x = 0$,we have :$\int_0^1 e^{-x}x^{p-1}dx < \int_0^1 x^{p-1}dx = \frac{1}{p}$
Moreover,$I_2 = \int_1^\infty e^{-x}x^{p-1}dx$ is also convergent,we obtain:$1 \leq x \implies e^{-x}x^{p-1} \leq e^{-x\backslash 2} \iff x^{p-1} \leq e^{x\backslash 2} \iff \frac{x^{p-1}}{e^{x\backslash 2}} \leq 1$
because $\lim \frac{x^{p-1}}{e^{x\backslash 2}} = 0$,we have :$\int_1^\infty e^{-x}x^{p-1}dx \leq \int_1^\infty e^{-x\backslash 2}dx = 2e^{-1\backslash 2}$
the integral $\int_0^\infty e^{-x}x^{p-1}dx$ is convergent for $p > 0$ and devergent for $p \leq 1$,the function $\Gamma(p)$ is continuous for $p > 0$ □

The folowing relation are valid:

$$\Gamma(p+1) = \int_0^\infty e^{-x}x^p dx = -\left[e^{-x}x^p\right]_0^\infty + P\int_0^\infty e^{-x}x^{p-1}dx = P\Gamma(p)$$

$$
\begin{aligned}
\Gamma(p+n) &= (p+n-1)\,.........\,(p+1)\,P\Gamma(p), \\
\Gamma(1) &= 1 \\
\Gamma(n+1) &= n! \\
\Gamma(0) &= +\infty \\
\Gamma(-n) &= \frac{\Gamma(-n+1)}{-n} \\
&= \frac{\Gamma(-n+2)}{n\,(n-1)} \\
&= \frac{\Gamma(-n+3)}{n\,(n-1)\,(n-2)} \\
&= .... \\
&= (-1)^n\,\frac{\Gamma(0)}{n!} \\
&= (-1)^n\,\infty \\
\Gamma(\frac{1}{2}) &= \sqrt[2]{\pi} \\
\Gamma(-\frac{1}{2}) &= -2\sqrt[2]{\pi} \\
\Gamma(\frac{1}{3}) &= \frac{1}{2}\sqrt[2]{\pi} \\
\Gamma(\frac{3}{2}) &= \Gamma(1+\frac{1}{2}) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt[2]{\pi} \\
\Gamma(\frac{4}{3}) &= \Gamma(1+\frac{1}{3}) = \frac{1}{3}\Gamma(\frac{1}{3}) = \frac{1}{6}\sqrt[2]{\pi} \\
\Gamma(\frac{5}{2}) &= \Gamma(2+\frac{1}{2}) = \frac{4!\Gamma(\frac{1}{2})}{2!2^4} = \frac{3}{4}\sqrt[2]{\pi} \\
\Gamma(m+\frac{1}{2}) &= \frac{(2m-1)!}{2^m}\Gamma(\frac{1}{2}) \\
\Gamma(m+\frac{1}{3}) &= \frac{14........\,(3m-2)}{3^m}\Gamma(\frac{1}{3}) \\
\Gamma(m+\frac{2}{3}) &= \frac{15......\,(3m-1)}{3^m}\Gamma(\frac{2}{3}) \\
\Gamma(m+\frac{1}{4}) &= \frac{25......\,(4m-3)}{4^m}\Gamma(\frac{1}{4}) \\
\Gamma(m+\frac{2}{4}) &= \frac{37.....\,(4m-1)}{4^m}\Gamma(\frac{2}{4}) \\
\frac{\Gamma\,(p+1)}{\Gamma\,(q+1)\,\Gamma\,(p-q+1)} &= \binom{p}{q}
\end{aligned}
$$

We exppres $\exp(-x)$ as:

$$e^{-x} = \lim_{x \mapsto \infty} \left(1 - \frac{x}{k}\right)^k$$

,then we obtaine:

$$\Gamma(p) = \int_0^\infty e^{-x} x^{p-1} dx = \lim_{x \mapsto \infty} \int_0^k \left(1 - \frac{x}{k}\right)^k x^{p-1} dx$$

For $x = tk \implies dx = kdt$ resulting:

$$\Gamma(p) = \lim_{k \longrightarrow \infty} k^p \int_0^1 (1-t)^k t^{p-1} dt$$

Integrating by partes we obtaine:

$$\frac{1}{p} \int_0^1 (1-t)^K dt^p = \frac{1}{p} \left[(1-t)^K t^p\right]_0^1 - \frac{1}{p} \int_0^1 t^p d(1-t)^K = \frac{K}{p} \int_0^1 (1-t)^{K-1} t^p dt$$

Repeting this operation it follows:

$$\Gamma(p) = \lim_{k \longrightarrow \infty} \frac{k^p k!}{p(p+1) \ldots (p+k)}$$

but:

$$\lim_{k \longrightarrow \infty} \frac{(k+1)^p}{k^p} = 1$$

from:

$$\Gamma(p) = \frac{1}{p} \lim_{k \longrightarrow \infty} \frac{1}{(1+p)\left(1+\frac{p}{2}\right) \ldots \left(1+\frac{p}{K}\right)} \frac{2^p 3^p \ldots (k+1)^p}{1^p 2^p \ldots k^p} \tag{3.1.1}$$

It follows:

$$\begin{aligned} \Gamma(p) &= \frac{1}{p}{}_{k=1}^{\infty} \frac{1}{\left(1+\frac{p}{K}\right)} \frac{(k+1)^p}{k^p} \\ &= \frac{1}{p}{}_{K=1}^{\infty} \left(1+\frac{1}{k}\right)^p \left(1+\frac{p}{k}\right)^{-1} \end{aligned}$$

Definition 3.1.2. (Euleur's Beta function) can be defined as:

$$B(p,q) =_0^1 x^{p-1}(x-1)^{q-1} dx \text{ wher } Re(p) > 0 \text{ and } Re(q) > 0$$

fig112
the plot of y=$B(p,q)$

Proprieties 3.1.1. *In the folowing we will enumerate the basic properities of Beta function:*

1. For every $p > 0$ and for the natural number $n \in \mathbb{N}$

$$B(p,n) = B(n,p) = \frac{1.2.3.......(n-1)}{p(p+1)........(p+n)}$$

And also:

$$B(p,1) = \frac{1}{p}$$

2. For every natural numbers m, n we obtaine :

$$B(m,n) = \frac{(n-1)!(m-1)!}{(m+n-1)!}$$

3. For every $p > 0$ and $q > 0$ it is the identity:

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

4. For every $p > 0$ and $q > 0$ we have $B(p,q) = B(q,p)$

5. For every $p > 0$ and $q > 0$ we have $B(p,q) = \frac{q-1}{p+q-1} B(p,q-1)$

Definition 3.1.3. (Fractional integral of order α) for every $\alpha > 0$ and a local integrabl function $f(t)$.
The right FI of order α is defined :

$$_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-u)^{\alpha-1} f(u)\,du \text{ for } -\infty \leq a < t < +\infty$$

It can be defined also the left FI as:

$$_bI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (t-u)^{\alpha-1} f(u)\,du for -\infty < t < b \leq +\infty$$

Definition 3.1.4. (Riemann liouville fractional derivative of order α)
For every α and $n = [\alpha]$ the riemann liouville derivative of order α can be defined as:

$$_aD_t^\alpha = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^t (t-u)^{n-\alpha-1} f(u)\,du$$

Corollaire 3.1.1. *If C is a constant,then the Riemann liouville FD of C is:*

$$_0D_t^\alpha C = \frac{Cx^{-\alpha}}{\Gamma(1-\alpha)} for\ \alpha = 1,2,3.....$$

3.2 Riemann Liouville fractional integrales and derivative on the real half axis $\mathbb{R}_+$

Corollaire 3.2.1. $I_{+}^{\alpha} f$ *:The Right Riemann liouville fractional integrals*

$I_{-}^{\alpha} f$:The left Riemann liouville fractional integrals
$D_{-}^{\alpha} f$:The left Riemann liouville fractional derivative
$D_{+}^{\alpha} f$:The Right Riemann liouville fractional derivative

$$\begin{aligned}
I_{+}^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, t>0, (Re\alpha > 0). \\
I_{-}^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_t^\infty (\tau-t)^{\alpha-1} f(\tau)\, d\tau, t>0, (Re\alpha > 0). \\
D_{-}^{\alpha} f &= (-1)^n \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^\infty \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, t>0, (n-1 \le Re\alpha < n). \\
D_{+}^{\alpha} f &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, t>0, (n-1 \le Re\alpha < n).
\end{aligned}$$

Corollaire 3.2.2.

${}_t I_b^{\alpha} f$:The Right Riemann liouville fractional integrals.
${}_a I_t^{\alpha} f$:The left Riemann liouville fractional integrals.
${}_a D_t^{\alpha} f$:The left Riemann liouville fractional derivative.
${}_t D_b^{\alpha} f$:The Right Riemann liouville fractional derivative.
In the case of α is purely imaginary$\alpha = i\theta$:

$${}_a I_t^{i\theta} f(t) = \frac{d}{dt}\left({}_a I_t^{1+i\theta} f(t)\right) = \frac{1}{\Gamma(1+i\theta)} \frac{d}{dt} \int_a^t (t-\tau)^{i\theta} f(\tau)\, d\tau, \text{with } \theta \neq 0.$$

In the case of variable α for $0 \le \alpha(t) \le 1$:

$${}_0 D_t^{\alpha(t)} f(t) = \frac{1}{\Gamma(n-\alpha(t))} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(\tau-t)^{\alpha(t)}} d\tau, \text{ for } [0T] .$$

In the case $\alpha \in \mathbb{C}$of the order $Re(\alpha) > 0, n-1 \le Re(\alpha) < n$:
If $\alpha = 0 \implies$

$${}_0 D_t^0 f(t) = \frac{1}{\Gamma(1)} \frac{d}{dt} \int_0^t f(\tau)\, d\tau = f(t).$$

if $\alpha = n \in \mathbb{N} \implies$

$$D^n f(t) = \frac{1}{\Gamma(1)} \frac{d^{n+1}}{dt^{n+1}} \left(\int_0^t f(\tau)\, d\tau\right) = \frac{d^n}{dt^n} f(t) = f^{(n)}(t).$$

Proposition 3.2.1. *If* $0 < \alpha < 1$ *and* $n = 1 \implies$

$${}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} f(\tau)\, d\tau, \textit{ for } t > 0.$$

Proposition 3.2.2. *The composition of fructional derivatives and fructional integrals:*

$$\begin{aligned} \textit{for all } Re\alpha &> 0, \textit{if } f \in L^p([a,b]), 1 \le p \le \infty \textit{ then}: \\ \left({}_aD_t^{\alpha}\ {}_aI_t^{\alpha}\right) f(t) &= f(t) \textit{ and } \left({}_tD_b^{\alpha}\ {}_tI_b^{\alpha}\right) f(t) = f(t) \end{aligned}$$

${}_tI_b^{\alpha} f$:the Right Riemann liouville fractional integrals
${}_aI_t^{\alpha} f$:the left Riemann liouville fractional integrals
${}_a^CD_t^{\alpha} f$:the left caputo fractional derivative
${}_t^CD_b^{\alpha} f$:the Right caputo fractional derivative

Definition 3.2.1. (The Caputo fractional derivative):
let $\alpha > 0.n = [\alpha]$,the caputo derivative operator of order α is defined as:

$${}_a^CD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} {}_a^t (t-u)^{n-\alpha-1} \left(\frac{d}{du}\right)^n f(u)\, du$$

Definition 3.2.2. (The Caputo fructional derivative in the origin):
for a function $f(t) = 0, if\ t < 0$,it can be defined:

Definition 3.2.3.

$${}_0^CD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} {}_0^t (t-u)^{n-\alpha-1} f^{(n)} du, \text{where } n = [\alpha]$$

if C is a constant,then:

$${}_0^CD_t^{\alpha} C = 0$$

we present the definition fructional derivative from caputo.

Definition 3.2.4. (The left caputo fractional derivative)of a function of order α denoted by ${}_a^CD_t^{\alpha} f$ is:

$${}_a^CD_t^{\alpha} f = \left\{ \begin{array}{c} \frac{1}{\Gamma(n-\alpha)} {}_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \text{for } n-1 < \alpha < n \\ \frac{d^n}{dt^n} f(t) \text{ for } \alpha = n \end{array} \right\} \text{for } (t \in [a,b])$$

Definition 3.2.5. (The Right caputo fractional derivative) of a function of order α denoted by ${}_t^CD_b^{\alpha} f$ is:

$${}_t^CD_b^{\alpha} f = \left\{ \begin{array}{c} (-1)^n \frac{1}{\Gamma(n-\alpha)} {}_t^b \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \text{for } n-1 < \alpha < n \\ (-1)^n \frac{d^n}{dt^n} f(t) \text{ for } \alpha = n \end{array} \right\} \text{for } (t \in [a,b])$$

Corollaire 3.2.3. *it is easy to see that:*

$$
\begin{aligned}
{}_a^C D_t^\alpha f &= {}_a I_t^{n-\alpha}\left(\frac{d^n}{dt^n} f(t)\right) \text{ and} \\
{}_t^C D_b^\alpha f &= (-1)^n \; {}_t I_b^{n-\alpha}\left(\frac{d^n}{dt^n} f(t)\right)
\end{aligned}
$$

where ${}_a I_t^{n-\alpha}$*and*${}_t I_b^{n-\alpha}$ *are the Riemann liouville fractional integrals*

3.3 theHadamard fractional derivative is defined in termes hadamard fractional integral in folowing way:

Definition 3.3.1. (the hadamard fructional derivative) in order $q > 0$ alpplied to the function $f \in AC_\delta^n([a,b])$ is defined as:

$$
{}^H D_1^q f(x) = \delta^n \left({}^H I_1^{n-q} f\right)(x)
$$

if $q \in (0,1]$ then

$$
{}^H D_1^q f(x) = \delta \left({}^H I_1^{n-q} f\right)(x)
$$

let $0 \leq q \leq 1$then:

$$
{}^H D_1^q \ln t = \frac{1}{\Gamma(2-q)} (\ln t)^{n-q} \text{ for} \in [0,e]
$$

in the space of $L^1$the hadamard fructional derivative is the left inverse operator to the hadamard fructional integral

$$
\left({}^H D_1^q\right)\left({}^H I_1^q f\right)(x) = f(x)
$$

$$
\left({}^H I_1^q\right)\left({}^H D_1^q f\right)(x) = f(x) - \frac{{}^H I_1^{1-q} f(1)}{\Gamma(q)} (\ln x)^{q-1}
$$

The Caputo Hadamard fructional derivative is defined in the folowing way:

Definition 3.3.2. (the Caputo Hadamard fructional derivative) of order $q > 0$ applied to the function$f \in AC_\delta^n([a,b])$ is defined as

$$
\left({}^{Hc} D_1^q f\right)(x) = {}^H I_1^{n-q} \delta^n f(x)
$$

if $(0,1]$ then

$$
\left({}^{Hc} D_1^q f\right)(x) = {}^H I_1^{1-q} \delta f(x)
$$

3.3.1 Hilfer fractional derivative:

Hilfer studies applications of a generalised fractional operator having the Rimman liouville and the caputo derivatives as specific cases:

Definition 3.3.3. (The Hilfer fructional derivative)let $\alpha \in (0,1], \beta \in [0,1], f \in L^1([a,b]), I_1^{(1-\alpha)(1-\beta)} f \in AC^1([a,b])$, the Hilfer fructional derivative of order α and type $\beta$of f is defined as

$$\left(D_1^{\alpha,\beta} f\right)(t) = \left(I_1^{\beta,(1-\alpha)} \frac{d}{dt} I_1^{(1-\alpha)(1-\beta)} f\right)(t), for\ t \in [a,b]$$

Proposition 3.3.1. *Let $\alpha \in (0,1], \beta \in [0,1], \gamma = \alpha + \beta - \alpha\beta, f \in L^1([a,b])$*

for $\beta = 0$,

$$D_1^{\alpha,0} = D_1^{\alpha}$$

and $\beta = 1$,

$$D_1^{\alpha,1} =^C D_1^{\alpha}$$

From the HADAMARD fructional integral the and HILFER-HADAMARD fructional derivative is defined in the folowing way:

Definition 3.3.4. (HILFER HADAMARD fructional derivative)let $\alpha \in (0,1], \beta \in [0,1], \gamma = \alpha + \beta - \alpha\beta, f \in L^1([a,b]), {}^H I_1^{(1-\alpha)(1-\beta)} f \in AC^1([a,b])$ the HILFER HADAMARD fructional derivative of order α and type β of f is defined as

$$\begin{aligned} {}^H\left(D_1^{\alpha,\beta} f\right)(t) &= \left({}^H I_1^{\beta(1-\alpha)}\left({}^H D_1^{\gamma} f\right)\right)(t) \\ &= \left({}^H I_1^{\beta(1-\alpha)} \delta \left({}^H I_1^{(1-\gamma)} f\right)\right)(t), \text{for } t \in [a,b] \end{aligned}$$

Remark 3.3.1.

$$\begin{aligned} {}^H D_1^{\alpha,0} &= {}^H D_1^{\alpha} \\ {}^H D_1^{\alpha,1} &= {}^{HC} D_1^{\alpha} \end{aligned}$$

Definition 3.3.5. GRUNWALD-LETNIKOV FRACTIONAL DERIVATIVE OF THE ORDER α is defined as:

$${}_a^{G-L}D_t^{\alpha} f(t) = \lim_{h \longmapsto 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{\left[\frac{t-1}{h}\right]} (-1)^j \binom{\alpha}{j} f(t - jh), t > a, \alpha > 0$$

similarly ,the right grunwald-letnikov FD of the order $\alpha$is defined as:

$${}_t^{G-L}D_b^{\alpha} f(t) = \lim_{h \longmapsto 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{\left[\frac{t-1}{h}\right]} (-1)^j \binom{\alpha}{j} f(t - jh), t > b, \alpha > 0$$

There is a connection between the MARCHAUD AND THE GRUNWALD-LETNIKOV.

3.4 theHadamard fractional derivative is defined in termes hadamard fractional integral in folowing way:

Definition 3.4.1. (the hadamard fructional derivative) in order $q > 0$ alpplied to the function $f \in AC_{\delta}^{n}([a,b])$ is defined as:

$$ {}^{H}D_{1}^{q}f(x) = \delta^{n}\left({}^{H}I_{1}^{n-q}f\right)(x) $$

if $q \in (0,1]$ then

$$ {}^{H}D_{1}^{q}f(x) = \delta\left({}^{H}I_{1}^{n-q}f\right)(x) $$

let $0 \leq q \leq 1$then:

$$ {}^{H}D_{1}^{q}\ln t = \frac{1}{\Gamma(2-q)}(\ln t)^{n-q} \text{ for} \in [0,e] $$

in the space of $L^{1}$the hadamard fructional derivative is the left inverse operator to the hadamard fructional integral

$$ \left({}^{H}D_{1}^{q}\right)\left({}^{H}I_{1}^{q}f\right)(x) = f(x) $$

$$ \left({}^{H}I_{1}^{q}\right)\left({}^{H}D_{1}^{q}f\right)(x) = f(x) - \frac{{}^{H}I_{1}^{1-q}f(1)}{\Gamma(q)}(\ln x)^{q-1} $$

The Caputo Hadamard fructional derivative is defined in the folowing way:

Definition 3.4.2. (the Caputo Hadamard fructional derivative) of order $q > 0$ applied to the function$f \in AC_{\delta}^{n}([a,b])$ is defined as

$$ \left({}^{Hc}D_{1}^{q}f\right)(x) = {}^{H}I_{1}^{n-q}\delta^{n}f(x) $$

if $(0,1]$ then

$$ \left({}^{Hc}D_{1}^{q}f\right)(x) = {}^{H}I_{1}^{1-q}\delta f(x) $$

3.4.1 Hilfer fractional derivative:

Hilfer studies applications of a generalised fractional operator having the Rimman liouville and the caputo derivatives as specific cases:

Definition 3.4.3. (The Hilfer fructional derivative)let $\alpha \in (0,1]$, $\beta \in [0,1]$, $f \in L^{1}([a,b])$, $I_{1}^{(1-\alpha)(1-\beta)}f \in AC^{1}([a,b])$, the Hilfer fructional derivative of order α and type $\beta$of f is defined as

$$ \left(D_{1}^{\alpha,\beta}f\right)(t) = \left(I_{1}^{\beta,(1-\alpha)}\frac{d}{dt}I_{1}^{(1-\alpha)(1-\beta)}f\right)(t), for\ t \in [a,b] $$

Proposition 3.4.1. *Let* $\alpha \in (0,1], \beta \in [0,1], \gamma = \alpha + \beta - \alpha\beta, f \in L^1([a,b])$
for $\beta = 0$,

$$D_1^{\alpha,0} = D_1^{\alpha}$$

and $\beta = 1$,

$$D_1^{\alpha,1} =^C D_1^{\alpha}$$

From the HADAMARD fructional integral the and HILFER-HADAMARD fructional derivative is defined in the folowing way:

Definition 3.4.4. (HILFER HADAMARD fructional derivative)let $\alpha \in (0,1], \beta \in [0,1], \gamma = \alpha + \beta - \alpha\beta, f \in L^1([a,b]), {}^H I_1^{(1-\alpha)(1-\beta)} f \in AC^1([a,b])$ the HILFER HADAMARD fructional derivative of order α and type β of f is defined as

$$\begin{aligned} {}^H\left(D_1^{\alpha,\beta} f\right)(t) &= \left({}^H I_1^{\beta(1-\alpha)}\left({}^H D_1^{\gamma} f\right)\right)(t) \\ &= \left({}^H I_1^{\beta(1-\alpha)} \delta\left({}^H I_1^{(1-\gamma)} f\right)\right)(t), \text{for } t \in [a,b] \end{aligned}$$

Remark 3.4.1.

$$\begin{aligned} {}^H D_1^{\alpha,0} &= {}^H D_1^{\alpha} \\ {}^H D_1^{\alpha,1} &= {}^{HC} D_1^{\alpha} \end{aligned}$$

Definition 3.4.5. GRUNWALD-LETNIKOV FRACTIONAL DERIVATIVE OF THE ORDER α is defined as:

$${}_a^{G-L}D_t^{\alpha} f(t) = \lim_{h \longmapsto 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{\left[\frac{t-1}{h}\right]} (-1)^j \binom{\alpha}{j} f(t - jh), t > a, \alpha > 0$$

similarly ,the right grunwald-letnikov FD of the order $\alpha$is defined as:

$${}_t^{G-L}D_b^{\alpha} f(t) = \lim_{h \longmapsto 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{\left[\frac{t-1}{h}\right]} (-1)^j \binom{\alpha}{j} f(t - jh), t > b, \alpha > 0$$

There is a connection between the MARCHAUD AND THE GRUNWALD-LETNIKOV.

3.5 Somme additional propertives of F.D:

3.5.1 FERMAT THEOREM FOR FD:

$$\begin{aligned} \text{let } 0 &< \alpha < 1 :_0 D_t^{\alpha} y(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{y(t-\tau)}{\tau^{\alpha}} d\tau \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{y(0)}{t^{\alpha}} + \frac{1}{\Gamma(1-\alpha)} \int_0^t y^{(1)}(t-\tau) \left(\alpha \int_{\Gamma}^t \xi^{(-1-\alpha)} d\xi + \frac{1}{t^{\alpha}}\right) d\tau \\ &= \frac{y(t)}{\Gamma(1-\alpha) t^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{y(t) - y(t-\tau)}{\tau^{1+\alpha}} d\tau, t > 0 \end{aligned}$$

1. similarly for the caputo derivative of an integrable function ,we have :

$$\begin{aligned} {}_0^C D_t^{\alpha} y(t) &= {}_0 D_t^{\alpha} y(t) - \frac{y(0)}{\Gamma(1-\alpha)\, t^{\alpha}} \\ &= \frac{y(t) - y(0)}{\Gamma(1-\alpha)\, t^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)} + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{y(t) - y(t-\tau)}{\tau^{(1+\alpha)}} d\tau, t > 0. \end{aligned}$$

3.5.2 TAYLOR THEOREM FOR FD:

1. the TAYLOR farmula for CAPUTO derivative is given in the folowing proposition:

Proposition 3.5.1. *let* $\alpha \in [0,1]$ *and suppose that* $f \in C[a,b]$ *,such that*${}_a D_t^{\alpha} f \in ([a,b])$ *then* :

$$f(t) = (t-a)^{\alpha-1} \left[(t-a)^{1-\alpha} f(t) \right]_{t=a} + \frac{(t-a)^{\alpha}}{\Gamma(1+\alpha)} \left[{}_a D_t^{\alpha} f(t) \right]_{t=\xi}, t \in (a,b] \ \textit{with} \ a \leq \ \xi \leq b$$

The geniralization of taylor formular for RIEMANN LIOUVILLE derivative has several defferent forms.To state formula we need the folowing deffinition .

Definition 3.5.1. A function $f{:}[a,b] \to sl\mathbb{R}$is said to be α-countinous for every $0 \leq \alpha \leq 1$,at t_0 if there existe $\lambda \in [0, 1-\alpha]$ such that $g(t) = |t - t_0|^{\lambda} f(t)$is countinous at t_0.

- 1. Function f is α-countinous in $[a,b]$ if it is α-countinous for every $t \in [a,b]$.
  2. Let $C_{\alpha} = \{f \ / \ [a,b] \to sl\mathbb{R} f \ \alpha\text{-countinous}\}$note that $C_1([a,b]) = C([a,b])$.
  3. Let${}_a I_b^{\alpha}(a,b) = \{f / \ [a,b] \to sl\mathbb{R}, {}_a I_b^{\alpha} f$ existe and it is finite for all$t \in [a,b]\}$
  4. A function f is singular of order α at $t = t^*$ if $\lim\limits_{t \longrightarrow t^*} \dfrac{f(t)}{(t - t^*)^{\alpha-1}} = k < \infty$ *and* $k \neq 0$
  5. Finally,we use ${}_a D_t^{j\alpha}$ to denote the application of ${}_a D_t^{\alpha}$ j times ${}_a D_t^{j\alpha} = \underbrace{{}_a D_t^{\alpha} \ldots {}_a D_t^{\alpha}}$ j times.

Proposition 3.5.2. *Let* $0 \leq \alpha \leq 1, n \in \mathbb{N}$*,let* f *be a countinous function in* $(a,b]$ *satisfying the folowing conditions:*

- 1. ${}_a D_t^{j\alpha} f \in C([a,b])$ *and* ${}_a D_t^{j\alpha} f \in_a I_b^{\alpha}(a,b)$ *for all j=0,1,...,n.*
  2. ${}_a D^{(n+1)\alpha} f$ *is countinous in* $[a,b]$.
  3. *If* $\alpha < \frac{1}{2}$*,then for each* $j \in \mathbb{N}, 1 \leq j \leq n$*,such that* $(j+1)\alpha < 1, {}_a D_t^{(j+1)\alpha} f$ *is* γ*-countinous at* $t = a$ *for somme* $\gamma, 1 - (j+1)\alpha \leq \gamma \leq 1$*or it is singular of order* α *at* $t = a$*.Then ,for* $t \in (a,b]$

$$f(t) =_{j=0}^{n} \frac{C_j}{\Gamma((j+1)\alpha)}(t-a)^{(j+1)\alpha-1} + \frac{\left[{}_aD_t^{(n+1)\alpha} f(t)\right]_{t=\xi}}{\Gamma((n+1)\alpha+1)}(t-a)^{(n+1)\alpha}, a \le \xi \le b$$

Where

$$C_j = \Gamma(\alpha)\left[(t-a)^{1-\alpha}_a D_t^{j\alpha} f(t)\right]_{t=a^+}, j = 0, 1, ..., n$$

The taylor formular for the caputo derivative is given in the folowing proposition.

Proposition 3.5.3. *Suppose that* ${}_a^C D_t^{j\alpha} f \in C([a,b])$ *for* $j = 0, 1, .., n+1$

$$f(t) =_{j=0}^{n} \frac{(t-a)^{j\alpha}}{\Gamma(\alpha j+1)}\left[{}_a^C D_t^{j\alpha} f(t)\right]_{t=a} + \frac{\left[{}_a^C D_t^{(n+1)\alpha} f(t)\right]_{t=\xi}}{\Gamma((n+1)\alpha+1)}(t-a)^{(n+1)\alpha,} t \in (a,b], a \le \xi \le b$$

Remark 3.5.1. In the special case for the caputo derivative ,the coresponding result is state as follows.Suppose that $f \in C([a,b])$ and ${}_a^C D_t^{\in\alpha} f \in C([a,b])$,for $0 \le \alpha \le 1$.Then

$$f(t) = f(a) + \frac{\left[{}_a^C D_t^{(n+1)\alpha} f(t)\right]_{t=\xi}}{\Gamma(\alpha+1)}(t-1)^{\alpha}, t \in (a,b], \text{ where } a \le \xi \le b$$

3.6 Laplace tronsform of Rieman Liouville Fractional integrables and derivatives:

that is $f \in L^{1(0,\infty)}, |f(t)| \le A\exp(s_0 t), t > 0$ wher $A < 0, s_0 > 0$ then :

$$\begin{aligned}
L\left[{}_0I_t^{\alpha} f(t)\right](s) &= \frac{1}{s^{\alpha}}\widehat{f(s)}, Re(s) \ge s_0 \\
L\left[{}_0D_t^{\alpha} f(t)\right](s) &= \frac{1}{s^{\alpha}}\widehat{f(s)} - \sum_{k=0}^{n-1} s^{n-k-1}\left[D^K\, {}_0I_t^{\alpha} f(t)\right]_{t=0} \\
L\left[{}_0D_t^{\alpha} f(t)\right](s) &= \frac{1}{s^{\alpha}}\widehat{f(s)} - \left[{}_0I_t^{\alpha} f(t)\right]_{t=0} for\ 0 \le \alpha \le 1 \\
&= s^{\alpha}\widehat{f(s)}, Res \ge s_0
\end{aligned}$$

3.7 The laplace transform of the caputo derivatives is given as folows :

suppose that $n-1 \le \alpha \le n$ and let f be such that :$f \in C^n(R), |f(t)|, \left|f^{(1)}(t)\right|, ..., \left|f^{(n)}(t)\right| \le B\exp(s_0 t), B, s_0 > 0, t > 0$

suppose that $\lim_{t \longrightarrow \infty} f^{(k)}(t) = 0$,for $k = 0, ..., m-1$ then :

$$L\left[{}_0^C D_t^\alpha\right](s) = s^\alpha \widehat{f}(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0)$$
$$L\left[{}_0^C D_t^\alpha\right](s) = s^\alpha \widehat{f}(s) - s^{\alpha-1} f(0), Re(s) \geq s_0$$

The laplace transforme of fructional integrals and derivatives:

if $\alpha > 0$,the Riemann and Caputo FI are the same for both cases :

$$I = I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-y)^{\alpha-1} f(y)\, dy$$

using the laplace transform of the convolution product formula we have :

$$L[I] = \frac{1}{\Gamma(\alpha)} L\left[t^{\alpha-1}\right] L[f(t)] = \frac{\widehat{f}(s)}{s^\alpha}$$

Finally, we give an exemple to illustrate our results:

3.8 Variational problmes with fractional derivatives

3.8.1 Euler -Lagrange equations:

Equation is Euer lagrange for:

$${}_{-\infty}D_t^\alpha y(t) + {}_t D_\infty^\alpha y(t) = 0, t \in \mathbb{R}$$

3.8.2 The generalisation Legendre equation:

The generalisation Legendre equation can be defined as:

$$\left(1 - t^{2\alpha}\right) D^{(2\alpha)} y(t) - 2t^\alpha D^{(\alpha)} y(t) + \lambda y(t) = 0$$

3.8.3 Lane -Emden equation:

$$D^{(\alpha)} y(t) + \frac{a_1}{t^{\alpha-\beta_1}} D^{\beta_1} y(t) + \frac{a_2}{t^{\alpha-\beta_2}} D^{\beta_2} y(t) + ... + \frac{a_n}{t^{\alpha-\beta_n}} D^{\beta_n} y(t) + y^m(t) = 0,$$

and the initial conditions are :

$$y(0) = 1, y'(0) = 0$$

3.8.4 The Gene Alized Bessel equation :

the Bessel FDE can be introduced as:

$$t^{2\alpha}D^{(2\alpha)}y\left(t\right)+t^{\alpha}D^{(\alpha)}y\left(t\right)+\left(t^{2\alpha}-p^{2}\right)y\left(t\right)=0,p\in\mathbb{R}$$

3.8.5 the Van Der pol FE:

the Van Der pol FE

$$\begin{cases} D^{\alpha}x_1\left(t\right)=x_2,x_1\left(0\right)=0 \\ D^{\alpha}x_2\left(t\right)=-x_2+2\left(1-x_1^2\right)x_2,x_2\left(0\right)=1 \end{cases}$$

For $\alpha=0,998$

$$\begin{cases} D^{0,998}x_1\left(t\right)=x_2,x_1\left(0\right)=0 \\ D^{0,998}x_2\left(t\right)=-x_2+2\left(1-x_1^2\right)x_2,x_2\left(0\right)=1 \end{cases}$$

3.8.6 The Van Der pol F SYSTEM:

the Van Der pol FE

$$\begin{cases} D^{\frac{1}{2}}x\left(t\right)=y,x\left(0\right)=1 \\ D^{\frac{1}{2}}x_2\left(t\right)=-x+0,25\left(1-x^2\right)y,y\left(0\right)=0 \end{cases}$$

3.8.7 The fractional Duffing system:

showsThe fractional Duffing system

$$\begin{cases} D^{0,998}X\left(t\right)=Y \\ D^{0,998}Y\left(t\right)=-X-X^3 \end{cases}$$

with

$$\begin{cases} X\left(0\right)=1 \\ Y\left(0\right)=0 \end{cases}$$

3.8.8 Lotka system:

Lotka system with initial conditions:{

3.8.9 The F Rossler attractor system :

$$\begin{cases} D^{0,98}X(t) = -Y - Z \\ D^{0,98}Y(t) = X + 0,2Y \\ D^{0,98}Z(t) = 0,2 + Z(X - 8) \end{cases}$$

with IC

$$\begin{cases} X(0) = 1 \\ Y(0) = 1 \\ Z(0) = 1 \end{cases}$$

3.8.10 the Lorenz fractional attractor system:

the Lorenz fractional attractor is defined by the systeme of FDE:

$$\begin{cases} D^{\alpha}X(t) = 10(Y(t) - X(t)) \\ D^{\alpha}Y(t) = X(t)(28 - Z(t)) - Y(t) \\ D^{\alpha}Z(t) = X(t)Y(t) - \frac{8}{3}Z(t) \end{cases}$$

we use the initial conditions:$X(t) = Y(t) = Z(t) = 0,1$

Exemple 3.8.1. For $\alpha = 0,95$ we have:

$$\begin{cases} D^{0,95}X(t) = 10(Y(t) - X(t)) \\ D^{0,95}Y(t) = X(t)(28 - Z(t)) - Y(t) \\ D^{0,95}Z(t) = X(t)Y(t) - \frac{8}{3}Z(t) \\ X(t) = Y(t) = Z(t) = 0,1, IC \end{cases}$$

Exemple 3.8.2.

$$D^{\alpha}Y(t) + 1 - (1 + \alpha) = 0$$

whith I C:

$$Y(0) = Y(1) = 0$$

$$Y(t) = -\frac{t^{\alpha}}{\Gamma(\alpha + 1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha + 1)}$$

Exemple 3.8.3.

$$\begin{cases} D^{\alpha}Y(t) - Y(t) = \frac{t}{\exp(t) - 1}, 0 \leq t \leq 1 \\ Y(0) = 1 \\ Y'(0) = 0, \end{cases} \quad \text{where } 0 \leq \alpha \leq 1$$

$$Y_{app}(t) = 1 - \frac{1}{2}\frac{\Gamma(\alpha+2)\Gamma(\alpha+1)\Gamma(3\alpha+1)}{\Gamma(2\alpha+2)\Gamma^2(2\alpha+1)}t^{2\alpha} + ...$$

For $\alpha = 1$ we have :

$$Y_{app}(t) = 1 - \frac{1}{4}t^2$$

Exemple 3.8.4. Rimann-Liouville derivative for t^β :

$$Df =_0 D_t^\alpha t^\beta = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t u^\beta (t-u)^{n-\alpha-1} du$$

we take $u = vt$,$du = tdv$ we have:

$$\begin{aligned}
Df &= {}_0D_t^\alpha t^\beta = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t \left[(vt)^\beta (1-v)\right]^{n-\alpha-1} tdv \\
&= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t v^\beta \left[(1-v)\right]^{n-\alpha-1} t^{n-\alpha+\beta} dv \\
&= \frac{1}{\Gamma(n-\alpha)}\int_0^t v^\beta \left[(1-v)\right]^{n-\alpha-1} \frac{d^n}{dt^n} t^{n-\alpha+\beta} dv, \text{Resulting : } \frac{d^n}{dt^n} t^{n-\alpha+\beta} = \frac{\Gamma(\lambda}{\Gamma(\lambda -} \\
&= \frac{1}{\Gamma(n-\alpha)}\frac{\Gamma(n-\alpha+\beta+1)}{\Gamma(-\alpha+\beta+1)} t^{\beta-\alpha}\int_0^1 \left[(1-v)\right]^{n-\alpha-1} v^\beta dv, \\
\int_0^1 \left[(1-v)\right]^{n-\alpha-1} v^\beta dv &= B(n-\alpha, \beta+1) = \frac{\Gamma(n-\alpha)\Gamma(\beta+1)}{\Gamma(n-\alpha+\beta+1)} \\
Df &= {}_0D_t^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(-\alpha+\beta+1)} t^{\beta-\alpha}
\end{aligned}$$

Find the Rimann-Liouville FD and FI for function $(t-a)^\beta$
For FI we apply the Rimann-Liouville definition:

$$\begin{aligned}
I &= {}_aI_t^\alpha \overbrace{(t-a)^\beta} = \frac{1}{\Gamma(\alpha)}\int_a^t (t-u)^{\alpha-1} \overbrace{(u-a)^\beta} du, the\ change\ of\ variable : v = \frac{u-a}{t-a}, du = (t-a)\,dv \\
&= \frac{(t-a)^{\beta+\alpha}}{\Gamma(\alpha)}\int_0^t (1-v)^{\alpha-1} v^\beta dv \\
&= \frac{(t-a)^{\beta+\alpha}}{\Gamma(\alpha)} B(\alpha, \beta+1) \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(t-a)^{\beta+\alpha}
\end{aligned}$$

For FD we apply the Rimann-Liouville definition:

$$
\begin{aligned}
Df &= {}_aD_t^{\alpha}(t-a)^{\beta} = \frac{d^n}{dt^n}{}_aI^{n-\alpha}(t-a)^{\beta} \\
&= \frac{\Gamma(\beta+1)}{\Gamma(n+\beta-\alpha+1)}\frac{d^n}{dt^n}(t-a)^{\beta+n-\alpha} \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}
\end{aligned}
$$

Exemple 3.8.5.

$$
F(t) =_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}t^k =_{k=0}^{\infty} \frac{f^{(k)}(0)}{\Gamma(k+1)}t^k
$$

the FD will be

$$
\begin{aligned}
D^{(\alpha)}f(t) &= {}_{k=0}^{\infty}\frac{f^{(k)}(0)}{\Gamma(k+1)}D^{(\alpha)}t^k \\
&= {}_{k=0}^{\infty}\frac{f^{(k)}(0)}{\Gamma(k+1)}\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}t^{k-\alpha} \\
&= {}_{k=0}^{\infty}\frac{f^{(k)}(0)}{\Gamma(k-\alpha+1)}t^{k-\alpha}
\end{aligned}
$$

Name :	Forme:
	As we known
Minus order derivative	$\frac{d^{-1}}{dx^{-1}}\sin(x) = \int \sin(x)\,dx$
Semi derivative	$\frac{d^{0,5}}{dx^{0,5}}\frac{d^{0,5}}{dx^{0,5}}\sin(x) = \frac{d^{0,5+0,5}}{dx^{0,5+0,5}}\sin(x) = \frac{d}{dx}\sin(x) = \cos(x)$
Thepi-order derivative	$\frac{d^{\pi}}{dx^{\pi}} - y - 2\exp(x) = 0$
Complex order derivative	$\frac{d^{a+ib}}{dx^{a+ib}}\sin(x) = \frac{d^{ib}}{dx^{ib}}\frac{d^{a}}{dx^{a}}\sin(x)$ $\frac{d^{1+i}}{dx^{1+i}}\frac{d^{1-i}}{dx^{1-i}}\sin(x) = \frac{d^{1+i+1-i}}{dx^{1+i+1-i}}\sin(x) = \frac{d^2}{dx^2}\sin(x) = -\sin(x)$ The
Variable order derivtive	$\frac{d^{\cos(x)}}{dx^{\cos(x)}}\sin(x) = d(\sin(x), x, \cos(x))$

Chapter 4

On the Caputo-Hadamard fractional IVP with variable order using the upper-lower solutions technique

4.1 Introduction

By comparing integer differential equations to fractional differential equations of a constant order , fractional calculus has been the subject of extensive studies for more than three centuries. The main and initial difference of fractional calculus is to replace the natural numbers in the order of derivative by arbitrary real ones. Although such a description of this widely used theory seems very superIicia l, it has a high power in describing physical phenomena. While numerous number of studies have been implemented for analyzing the existence theory in relation to fractional constant-order boundary value problems (BVPs) [**?**],[2],[3],[4],[5],[6],[7],[8],[9],[10],[11],[12],[13],[14],[15], this theory is rarely investigated for variable-order BVPs in other research studies [16],[17],[18],[19],[20]. HENCE, at the same time, the technique we propose in this paper is new and valuable for such variable order structures. About the investigation of the existence theory for variable order BVPs, we mention some of them. JIAHUI et al. [21] addressed unique solutions in relation to an IVP of RIEMANN-LIOUVILLE fractional differential equations in the case of variable order. In [22], BOUAZZA et al. discussed a new stmcture of variable-order RIEMANN-LIOUVILLE BVPs, and after that in [23], BENKERROUCHE et al. performed an analysis about ULAM-HYERS stable solutions for a CAPUTO nonlinear implicit fractional boundary value problem (FBVPs) of variable order. Simultaneously in 2021, REIICE et al. [24] and HRISTOVA et al. [25] focused on some research studies in relation to existence theory for BVPs of HADAMARD FDEs with the help of complicated method of the KURATOWSKI measure of noncompactness in the case of variable order. For more information, we mention [26],[**?**],[27],[28] Of course, the stability analysis is one of the important aspects of fractional calculus, and some researchers have extended this area for constant-order systems [29],[30],[31],[32], and it can be a motivational factor for other studies in variable-order systems. Many real phenomena exist that expect the concept of HADAMARD fractional derivative permitting the useful of physically initial conditions, which contain $\phi(p)$, $\phi'(p)$, etc. The CAPUTO-HADAMARD fractional derivative provides these conditions. Under this property, the basic notions of the CAPUTO-HADAMARD fractional derivative are studied by ALMEIDA [33]. After that, some researchers such as BEN MAKHLOUF and MCHIRI [34]discussed some other properties of these operators. Moreover, ABUASBEH et al. [35],[36], KHAN et al. [37], NIAZI et al. [**?**] and SHAFQAT et al. [38], [39] similarly investigated the existence and uniqueness of solution for the FUZZY fractional evolution equations. For other applications, see [**?**],[41]. In particulx, BAI et al. [42] studied the existence of solution for the following initial value problem

$$\begin{cases} {}^{C}D^{w}_{p^+}\phi(t) = \psi(t,\, \phi(t),\, I^{w}_{p^+}\phi(t)),\ t \in \Lambda := [p,\, T], \\ \phi(p) = \phi_p, \end{cases} \tag{4.1.1}$$

where ${}^{C}D^{w}_{p^+}$ and $I^{w}_{p^+}$ denote the CAPUTO derivative and HADAMARD integral, respectively, $\psi; \Lambda \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\phi_p \in \mathbb{R}$, and $0 < p < T < \infty$. In this paper, we study the existence of solutions for the following fractional nonlinear differential equation involving the CAPUTO-HADAMARD fractional derivative of variable order

$$\begin{cases} {}^{C}D^{w(t)}_{p^+}\phi(t) = \psi(t,\, \phi(t)),\ t \in \Lambda\ := [p,\, T], \\ \phi(p) = \phi_p, \end{cases} \tag{4.1.2}$$

where $0 < p < T < \infty$, $\phi_p \in \mathbb{R}$ and $0 < w(t) \leq 1$ is a variable order, $\psi; \Lambda \times \mathbb{R} \to \mathbb{R}$ is a given function and ${}^{C}D^{w(t)}_{p^+}$ denotes the CAPUTO-HADAMARD fractional derivative of order $w(t)$. The organization of the rest of this paper is as follows. Some definitions and auxiliary results are given in Section 2. In Section 3, we try to obtain an equivalent system of constant order IVP by deriving HADAMARD integral equations on some continuous subintervals and partitions. With the help of piecewise constant functions, we implement the technique of *upper-lower* solutions for such an equivalent system and generalize our results to the given CAPUTO-HADAMARD variable order problem. One example is presented in Section 4, to show the efficiency and validity of the proposed results. Finally, some conclusion notes are given in Section 5. Note that there is no published work in which the technique of *upper-lower* solutions is used on a variable order system. This shows the originality of our research.

4.2 Auxiliary notions

In this section, we list some of definitions and propositions that are used in the following sections. The space $E := slC(\Lambda := [p,\, T],\, \mathbb{R})$ denotes the BANACH space of continuous functions $\phi : \Lambda \to \mathbb{R}$, and by the function space $slAC(p,\ q; \mathbb{R})$, we determines absolutely continuous $\mathbb{R}$-valued functions on $[p,\ q]$.

Definition 4.2.1. *[43], Let $0 < p < q < \infty$ and $\phi : [p,\ q] \to \mathbb{R}$. The* HADAMARD FRACTIONAL *integral of order $w > 0$ of the function ϕ is deffined by*

$$I^{w}_{p^+}\phi(t) = \frac{1}{\Gamma(w)}\int_p^t (ln\frac{t}{s})^{w-1}\frac{\phi(s)}{s}ds \text{ for } t \in [p,\ q],$$

where the well-known GAMMA *function is denoted by*

$$\Gamma(w) = \int_0^{\infty} t^{w-1}e^{-t}dt.$$

Definition 4.2.2. *[43], Let $0 < p < q < \infty$ and $\phi : [p,\ q] \to$ R. The* HADAMARD FRACTIONAL *derivative of the order $w \in (0,\ 1]$ of the function ϕ is deffined by*

$$D^{w}_{p^+}\phi(t) = \frac{1}{\Gamma(1-w)}t\frac{d}{dt}\int_p^t (ln\frac{t}{s})^{-w}\frac{\phi(s)}{s}ds \text{ for } t \in [p,\ q].$$

Clearly, we have

$$I^{w}_{p^+}(ln\frac{t}{p})^{v-1} = \frac{\Gamma(v)}{\Gamma(v+w)}(ln\frac{t}{p})^{v+w-1},\ D^{w}_{p^+}(ln\frac{t}{p})^{v-1} = \frac{\Gamma(v)}{\Gamma(v-w)}(ln\frac{t}{p})^{v-w-1}, \text{for each } t \in [p,\ q].$$

We now state some important characteristics for HADAMARD fractional integral and derivative operators. The proofs of them can be found in.

Lemma 4.2.1. *. Let $w > 0$ and $v > 0$.*

(i) *For $\phi \in L^r(p,\ q; \mathbb{R})$, if $1 \leq r < \infty$, then we have*
(ii) *For $\phi \in L^r(p,\ q; \mathbb{R})$, if $1 \leq r < \infty$ and $w > v$, then we have*

$$D^{v}_{p^+}I^{w}_{p^+}\phi(t) = I^{w-v}_{p^+}\phi(t) \text{ for } t \in [p,\ q].$$

[43],
Let $0 < p < q < \infty$ and $\phi : [p,\ q] \to \mathbb{R}$. The CAPUTO-HADAMARD *fractional derivative of order $w \in (0,\ 1]$ of the function ϕ is deffined by*

$$^{C}D^{w}_{p^+}\phi(t) = D^{w}_{p^+}[\phi(t) - \phi(p)] \text{ for } t \in [p,\ q].$$

Remark 4.2.1. *It should be obvious that the* CAPUTO-HADAMARD *fractional derivative, i. e., 4.2.3, is equivalent to the following expression that if $\phi \in AC(p, q; \mathbb{R})$, then*

$$^{c}D^{W}_{p^+}\phi(t) = \frac{1}{\Gamma(1-w)}\int_p^t (\ln\frac{t}{s})^{-w}\phi'(s)ds, \text{for} t \in [p,\ q].$$

Definition 4.2.3. [48] *The left variable-order* CAPUTO-HADAMARD *fractional derivative of the functional order $w(t)$ is deffined by*

$$^{c}D_{p^+}^{w(t)}\phi(t) = \frac{tw^r(t)}{\Gamma(2-w(t))}\int_p^t (ln\frac{t}{s})^{1-w(t)}\phi'(s)[\frac{1}{1-w(t)} - \ln(\ln\frac{t}{s})]ds + \frac{1}{\Gamma(1-w(t))}\int_p^t (ln\frac{t}{s})^{-w(t)}\phi'(s)ds.$$

4.2.1. *$Ifw(t) \equiv w$, (w is constant), then Deffinition 4.2.3 is transformed into the* CAPUTO-HADAMARD *derivative given in [43] as*

$$^{c}D_{p^+}^{w}\phi(t) = \frac{1}{\Gamma(1-w)}\int_p^t (ln\frac{t}{s})^{-w}\phi'(s)ds.$$

The component characteristics for the CAPUTO-HADAMARD fractional operators are listed below, and this section is concluded by mentioning them.

Lemma 4.2.2. *Let $n = [w] + 1$ be the case for $w > 0$.*

(i) *If $\phi \in slC(p,\, q; \mathbb{R})$, then*

$$^{C}D_{p^+}^{w}(I_{p^+}^{w}\phi(t)) = \phi(t) \text{ for } t \in [p,\, q].$$

(ii) *If $\phi \in slAC(p,\, q; \mathbb{R})$, then*

$$I_{p^+}^{w}(^{C}D_{p^+}^{w}\phi(t)) = \phi(t) - \phi(p) \text{ for } t \in [p,\, q]$$

4.3 Main results

Let's state the underlying assumptions. It will be the basic step in proving the results of this section. (Hl) For $n \in \mathbb{N}$, the finite sequence of points $\{T_k\}_{k=0}^{n}$ such that $p = T_0 < T_k < T_n = T$, $k = 1, \ldots, n-1$ is given. Denot $\Lambda_k := (T_{k-1},\, T_k]$, $k = 1, 2, \ldots, n$. Consequently, $\mathcal{P} = \bigcup_{k=1}^{n} \Lambda_k$ is a partition of Λ. The symbol $E_{slm} = slC(\Lambda_{slm},\, \mathbb{R})$, $m = 1, 2, \ldots, n$ denotes the Banach space of continuous functions $\phi : \Lambda_{slm} \to \mathbb{R}$ endowed with $||\phi||_{E_{slm}} = \sup_{t\in\Lambda_{slm}} |\phi(t)|$. Suppose that $w(t) : \Lambda \to (0, 1]$ is defined by $w(t) = \sum_{slm=1}^{sln} w_{slm} I_{slm}(t)$, where $0 < w_{slm} \leq 1$ are constants and I_{slm} is the indicator of Λ_{slm} be a piecewise constant function with respect to $\mathcal{P}$, where

$$I_{slm}(t) = \begin{cases} 1, & for\ t \in \Lambda_{slm}, \\ 0, & \text{else where.} \end{cases}$$

The left CAPUTO-HADAMARD derivative for the function $\phi \in slC(\Lambda, \mathbb{R})$ with variable order $w(t)$, given by Definition 4.2.3, might then be stated as the sum of the left CAPUTO-HADAMARD derivatives of the constant orders w_k, $k = 1, 2, \ldots, n$, i.e.,

$$D_{p^+}^{w(t)}\phi(t) = \frac{tw'(t)}{\Gamma(2-w(t))}\int_p^t \frac{t}{s}(\ln\frac{t}{s})]ds + \frac{1}{\Gamma(1-w(t))}\int_p^t (ln\frac{t}{s})^{-w(t)}\phi'(s)ds$$

For each $t \in \Lambda_{slm}$, where $m = 1, 2, \ldots, n$, the CAPUTO-HADAMARD derivative for the system of CHFDEVO (4.1.2) can be stated in the following form (**??**) To solve the integral equation (**??**), let the function $\tilde{\phi} \in C(\Lambda_{slm}, \mathbb{R})$ be such that $\tilde{\phi}(t) \equiv 0$ on $t \in [p, T_{slm-1}]$. Then (3.1) is transformed into

$$D_{T_{slm-1}^+}^{w_{slm}}\tilde{\phi}(t) = \psi(t, \tilde{\phi}(t)),\ t \in \Lambda_m.$$

For obtained CAPUTO-HADAMARD constant order fractional differential equations, we consider the following auxiliary CAPUTO-HADAMARD fractional differential equations (CHFDE) of constant order

$$\begin{cases} {}^C D_{T_{slm-1}^+}^{w_{slm}}\phi_{slm}(t) = \psi(t, \phi_{slm}(t)),\ t \in \Lambda_{slm}, \\ \phi_{slm}(T_{slm-1}) = \phi_{T_{slm-1}}, \end{cases}$$

for each $m = 1, 2, \ldots, n$. The main basic theorem can be stated now.

Theorem 4.3.1. *Assume that $\psi : \Lambda_{slm} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The solution to the integral equation (i.e., $\phi_{slm} \in slC(T_{slm-1}, T_{slm}; \mathbb{R})$) given by*

$$\phi_{slm}(t) = \phi_{\tau_{slm-1}} + \frac{1}{\Gamma(w_{slm})}\int_{T_{slm-1}}^t (\ln\frac{t}{s})^{w_{slm}-1}\frac{\psi(s, \phi_{slm}(s))}{s}dsfort \in \Lambda_{slm} \tag{4.3.1}$$

solves the auxiliary CHFDE of constant order (**??**).

Proof. Assume that $\phi_{slm} \in slC(T_{slm-1}, T_{slm}; \mathbb{R})$ is a solution of (4.3.1). Naturally, we take $\phi(T_{slm-1}) = \phi_{T_{slm-1}}$ and $t \to I_{T_{slm-1}^+}^{w_{slm}}\phi_{slm}(t) \in slC(T_{slm-1}, T_{slm}; \mathbb{R})$. The definition of the HADAMARD integral $I_{T_{slm-1}^+}^{w_{slm}}$ and the continuity of ψguarantee that $t \to \psi(t, \phi_{slm}(t))$ is continuous as well and □

$$I_{T_{slm-1}^+}^{w_{slm}}\psi(t, \phi_{slm}(t))|_{t=T_{slm-1}} = 0.$$

Since $t \to I_{T_{slm-1}^+}^{w_{slm}}\psi(t, \phi_{slm}(t))$ is continuous, we can conclude that ϕ_{slm} is differentiable for a.e. $t \in (T_{slm-1}, T_{slm})$, (see (4.3.1)), i.e., $\phi_{slm} \in slAC(T_{slm-1}, T_{slm}; \mathbb{R})$. From Lemma 4.2.2, we have

$$^{c}D^{w_m}_{T^+_{slm-1}} I^{w_{slm}}_{T^+_{slm-1}} \psi(t,\phi_{slm}(t)) = \psi(t,\ \phi_{slm}(t)) \text{ for } t \in \Lambda_{slm}.$$

On the other hand, Remark 2.5 gives

$$\begin{aligned} {}^{C}D^{w_{slm}}_{T^+_{slm-1}}[\phi_{slm}(t) - \phi_{T_{slm-1}}] &= \frac{1}{\Gamma(1-w_{slm})}\int_{T_{slm-1}}^{t} (\ln\frac{t}{s})^{-w_{slm}}[\phi_{slm}(s) - \phi_{T_{slm-1}}]' ds \\ &= \frac{1}{\Gamma(1-w_{slm})}\int_{T_{slm-1}}^{t} (\ln\frac{t}{s})^{-w_{slm}}\phi'_{slm}(s) ds \\ &= {}^{C}D^{w_{slm}}_{T^+_{slm-1}}\phi_{slm}(t), \end{aligned}$$

for each $t \in \Lambda_{slm}$. By all above, we conclude that $\phi_{slm} \in C(T_{slm-1},\ T_{slm};\mathbb{R})$ is a solution of the auxiliary CHFDE of constant order (**??**).

Definition 4.3.1. *Let* $(\underline{\phi_{slm}},\ \overline{\phi_{slm}}) \in slC(T_{slm-1},\ T_m;\mathbb{R}) \times slC(T_{slm-1},\ T_{slm};\mathbb{R})$. *A pair of-functions* $(\phi_{slm}, \overline{\phi_{slm}})$ *is called an* upper-lower *solutions of the auxiliary CHFDE of constant order* (**??**),respectively,*if*

$$\underline{\phi_{slm}}(t) \leq \phi_{T_{slm-1}} + \frac{1}{\Gamma(w_{slm})}\int_{T_{slm-1}}^{\frac{\psi(s,\underline{\phi_{slm}}(s))}{}} t(ln\frac{t}{s})^{w_{slm}-1}_{s} ds \text{ for all } t \in \Lambda_{slm}$$

and

$$\overline{\phi_{slm}}(t) \geq \phi_{T_{slm-1}} + \frac{1}{\Gamma(w_{slm})}\int_{T_{slm-1}}^{t} (ln\frac{t}{s})^{w_{slm}-1}\frac{\psi(s,\overline{\phi_m}(s))}{s} ds \ \text{ for all } t \in \Lambda_{slm}.$$

Assume that the upper-lower solution to the the auxiliary CHFDE of constant order **??** is $(\underline{\phi_m}, \overline{\phi_m})$. In the following, we define an acceptable set of solutions for the auxiliary CHFDE of constant order **??** which is controlled by two upper-lower solutions $(\underline{\phi_{slm}}, \overline{\phi_{slm}})$ as follows

$$s_{(\underline{\phi_{slm}},\overline{\phi_{slm}})} := \{\phi_{slm} \in C(T_{slm-1},\ T_{slm};\mathbb{R}) : \underline{\phi_{slm}}(t) \leq \phi_{slm}(t) \leq \overline{\phi_{slm}}(t),\ t \in \Lambda_{slm} \text{ and } \phi_{slm} \text{is asolution of}\textbf{??}$$

.

Theorem 4.3.2. *Let* $\psi \in slC(\Lambda_{slm} \times \mathbb{R};\ \mathbb{R})$ *and* $(\underline{\phi_{slm}}, \overline{\phi_{slm}}) \in slC(T_{slm-1},\ T_{slm};\mathbb{R}) \times C(T_{slm-1},\ T_{slm};\mathbb{R})$. *The auxiliary CHFDE of constant order* **??** *has the pair of upper-lower solutions with* $\underline{\phi_m}(t) \leq \overline{\phi_{slm}}(t)$ *and* $t \in \Lambda_{slm}$. *If* $\phi_m \to \psi(t,\ \phi_{slm})$ *is nondecreasing, that is* $\psi(t,\ \phi_1) \leq \psi(t,\ \phi_2)$ *for* $\phi_1 \leq \phi_2$, *then, there are minimum and maximum solutions*

$\phi_{slM,slm},\ \phi_{slL,slm} \in s_{(\underline{\phi_{slm}},\overline{\phi_{slm}})}$ *in* $s_{(\underline{\phi_{slm}},\overline{\phi_{slm}})}$; *i. e.,for each* $\phi_{slm} \in S_{(\underline{\phi_{slm}},\overline{\phi_{slm}})}$, $\phi_{slL,slm}(t) \leq \phi_{slm}(t) \leq \phi_{slM,slm}(t)$

Proof. We provide two sequences $\{\theta_{sln,slm}\}$ and $\{\beta_{sln,slm}\}$ as □

$$\begin{cases} \theta_{0,slm} = \underline{\phi_{slm}}, \\ \theta_{sln+1,slm}(t) = \phi_{T_{slm-1}} + \frac{1}{\Gamma(w_{slm})} \int_{T_{slm-1}}^{t} (ln\frac{t}{s})^{w_{slm}-1} \frac{\psi(s,\theta_{sln,slm}(s))}{s} ds, \; t \in \Lambda_{slm} \text{ and } n = 0, 1, \dots, \end{cases} \tag{4.3.2}$$

and

$$1^{\beta_{0,slm} = \overline{\phi_{slm}}},$$

$$\beta_{sln+1,slm}(t) = \phi_{T_{slm-1}} + \frac{1}{\Gamma(w_{slm})} \int_{T_{slm-1}}^{t} (ln\frac{t}{s})^{w_{slm}-1} \frac{\psi(s,\beta_{sln,slm}(s))}{s} ds, \; t \in \Lambda_{slm} \text{ and } n = 0, 1, \dots. \tag{4.3.3}$$

The proof is now divided into three steps.
Stepl. Sequences $\{\theta_{sln,slm}\}$ and $\{\beta_{sln,slm}\}$ satisfy the following relation:

$$\underline{\phi_{slm}}(t) = \theta_{0,slm}(t) \le \theta_{1,slm}(t) \le \theta_{2,slm}(t) \le \ldots \le \theta_{sln,slm}(t) \le \ldots \le \beta_{sln,slm}(t) \le \ldots \le \beta_{1,slm}(t) \le \beta_{0,slm}(\tag{4.3.4}$$

for each $t \in \Lambda_{slm}$.We will first demonstrate that the sequence $\{\theta_{sln,slm}\}$ is nondecreasing and $\theta_{sln,slm}(t) \le \beta_{0,slm}(t), \; t \in \Lambda_m$ for all $n \in slN$.Therefore, by a recurrence relation, we prove

$$\theta_{sln-1,slm}(t) \le \theta_{sln,slm}(t), \; \forall t \in \Lambda_{slm} \tag{4.3.5}$$

By the definition of $\theta_{0,slm}(t)$, we have $\theta_{0,slm}(t) \le \theta_{1,slm}(t)$ for each $t \in \Lambda_{slm}$. We suppose that (4.3.5) is tme for n and we prove for $n+1 : \theta_{sln,slm}(t) \le \theta_{sln+1,slm}(t), \; \forall t \in \Lambda_{slm}$.
We have

$$\theta_{sln,slm}(t) = \phi_{T_{slm-1}} + \frac{1}{\Gamma(w_{slm})} \int_{T_{slm-1}}^{t} (ln\frac{t}{s})^{w_{slm}-1} \frac{\psi(s,\theta_{sln-1,slm}(s))}{s} ds.$$

$$\theta_{sln+1,slm}(t) = \phi_{T_{slm-1}} + \frac{1}{\Gamma(w_{slm})} \int_{T_{slm-1}}^{t} (ln\frac{t}{s})^{w_{m}-1} \frac{\psi(s,\theta_{sln,slm}(s))}{s} ds.$$

Using the monotonicity of ψ, we obtain

$$\theta_{sln,slm}(t) \le \theta_{sln+1,slm}(t).$$

As $\theta_{sln,slm}(t)$ is noncreasing, by the definition of $\beta_{0,slm}(t)$, we have

$$\theta_{sln,slm}(t) \le \theta_{sln+1,slm}(t) \le \beta_{0,slm}(t).$$

Further, we will show that

$\theta_{sln,slm}(t) \leq \beta_{sln,slm}(t)$ for $t \in \Lambda_{slm}$ and $n \in slN$.
Since $n = 0$, it is evident that

$$\underline{\phi_{slm}}(t) = \theta_{0,slm}(t) \leq \beta_{0,slm}(t) = \overline{\phi_{slm}}(t) \text{for each } t \in \Lambda_{slm}$$

. Now, we make an inductive assumption

$$\theta_{sln,slm}(t) \leq \beta_{sln,slm}(t),\ t \in \Lambda_{slm}.$$

Accordingly, given that ψ is monotonic with respect to the second variable, it is simple to conclude that

$$\theta_{sln+1,slm}(t) \leq \beta_{sln+1,slm}(t),\ t \in \Lambda_{slm}.$$

Also, we have that the sequence $\{\beta_{sln,slm}\}$ is nonincreasing.
Step2. Both sequences $\{\theta_{\mathrm{\mathbb{N},m}}\}$ and $w_{\mathrm{\mathbb{N},m}}\}$ are relatively compact in $\mathrm{C}(\mathrm{T_{m-1}},\ \mathrm{T_m};\mathbb{R})$. Because ψ is continuous and $(\underline{\phi_{slm}}, \overline{\phi_{slm}}) \in slC(T_{slm-1},\ T_{slm};\mathbb{R})$, from Step 1, we find out that $\{\theta_{sln,slm}\}$ and $\{\beta_{sln,slm}\}$ belong to $C(T_{slm-1},\ T_{slm};\mathbb{R})$ as well. It follows from (4.3.4) that $\{\theta_{sln,slm}\}$ and $\{\beta_{sln,slm}\}$ are uniformlly bounded. On the other hand, for any $t_1,\ t_2 \in \Lambda_{slm}$, without loss of generality, let $t_1 \leq t_2$. We have

$$\begin{aligned}
|\theta_{sln+1,slm}(t_1) - \theta_{sln+1,slm}(t_2)| &= \frac{1}{\Gamma(w_{slm})}|\int_{T_{slm-1}}^{t_2} (ln\frac{t_2}{s})^{w_{slm}-1}\frac{\psi(s,\theta_{sln,slm}(s))}{s}ds \\
&\quad - \int_{T_{slm-1}}^{t_1} (ln\frac{t_1}{s})^{w_{slm}-1}\frac{\psi(s,\theta_{sln,slm}(s))}{s}ds| \\
&= \frac{1}{\Gamma(w_{slm})}|\int_{T_{slm-1}}^{t_1} [(ln\frac{t_2}{s})^{w_{slm}-1} - (ln\frac{t_1}{s})^{w_{slm}-1}]\frac{\psi(s,\theta_{sln,slm}(s))}{s}ds \\
&\quad + \int_{t_1}^{t_2} (ln\frac{t_2}{s})^{w_{slm}-1}\frac{\psi(s,\theta_{sln,slm}(s))}{s}ds| \\
&\leq \frac{M}{\Gamma(w_{slm})}|\int_{T_{slm-1}}^{t_1} \frac{1}{s}[(ln\frac{t_2}{s})^{w_{slm}-1} - (ln\frac{t_1}{s})^{w_{slm}-1}]ds + \int_{t_1}^{t_2}\frac{1}{s}(ln\frac{t_2}{s})^{w_{slm}-1} \\
&= \frac{M}{\Gamma(w_{slm})}|\frac{1}{w_{slm}}((ln\frac{t_2}{T_{slm-1}})^{w_{slm}} - (ln\frac{t_2}{t_1})^{w_{slm}})) + (\frac{1}{w_{slm}}(-(ln\frac{t_1}{T_{slm-1}})^{w_{sl} \\
&\quad +(\frac{1}{w_{slm}}((ln\frac{t_2}{t_1})^{w_{slm}} - (ln\frac{t_2}{t_2})^{w_{slm}})| \\
&= \frac{M}{\Gamma(w_{slm})}|\frac{1}{w_{slm}}((ln\frac{t_2}{T_{slm-1}})^{w_{slm}} - (ln\frac{t_1}{T_{slm-1}})^{w_{slm}})| \\
&= \frac{M}{\Gamma(w_{slm})}|\frac{1}{w_{slm}}((ln\frac{t_2}{T_{slm-1}})^{w_{slm}} - (ln\frac{t_1}{T_{slm-1}})^{w_{slm}})| \\
&= \frac{M}{\Gamma(w_{slm}+1)}|(ln\frac{t_2}{T_{slm-1}})^{w_{slm}} - (ln\frac{t_1}{T_{slm-1}})^{w_{slm}}| as\ t_1 \to t_2
\end{aligned}$$

where $M > 0$ is a constant independent of $n,\ t_1$, and t_2. It gives this fact that $\{\theta_{sln,slm}\}$ is equicontinuous in $C(T_{slm-1},\ T_{slm};\mathbb{R})$.

We conclude that $\{\theta_{sln,slm}\}$ is relatively compact in $C(T_{slm-1}, T_{slm};\mathbb{R})$ based on the ARZELA-ASCOLI Theorem. Similar to this, we find that $\{\beta_{sln,slm}\}$ is also relatively compact in $C(\Lambda_m;\mathbb{R})$.
Step3. In $S_{(\underline{\phi_m},\overline{\phi_m})}$, there are minimum and maximum solutions.
The sequences $\{\theta_{sln,slm}\}$ and $\{\beta_{sln,slm}\}$ are monotone and relatively compact in $C(T_{slm-1}, T_{slm};\mathbb{R})$, as shown in Steps 1 and 2. Evidently, continuous functions θ_{slm} and β_{slm} exist with $\theta_{sln,slm}(t) \leq \theta_{slm}(t) \leq \beta_{slm}(t) \leq \beta_{sln,slm}(t)$ for all $t \in \Lambda_m$ and $n \in \mathbb{N}$, such that $\{\theta_{sln,slm}\}$ and $\{\beta_{sln,slm}\}$ converge uniformly to θ_{slm} and β_{slm}, respectively, in $C(T_{slm-1}, T_{slm};\mathbb{R})$. Therefore, the solutions to the auxiliary CHFDE of constant order (**??**) are θ_{slm} and β_{slm}; i.e.,

$$\begin{aligned}
\theta_{slm}(t) &= \phi_{T_{slm-1}} + \frac{1}{\Gamma(w_{slm})}\int_{T_{slm-1}}^{t}(ln\frac{t}{s})^{w_{slm}-1}\frac{\psi(s,\theta_{slm}(s))}{s}ds,\\
\beta_{slm}(t) &= \phi_{T_{slm-1}} + \frac{1}{\Gamma(w_{slm})}\int_{T_{slm-1}}^{t}(ln\frac{t}{s})^{w_{slm}-1}\frac{\psi(s,\beta_{slm}(s))}{s}ds,
\end{aligned}$$

for each $t \in \Lambda_{slm}$. Therefore,

$$\underline{\phi_{slm}}(t) \leq \theta_{slm}(t) \leq \beta_{slm}(t) \leq \overline{\phi_{slm}}(t) \; for \; t \in \Lambda_{slm}.$$

Finally, we will prove that θ_{slm} and β_{slm} are the minimum and maximum solutions in $s_{(\underline{\phi_{slm}},\overline{\phi_{slm}})}$. If $\phi_{slm} \in s_{(\underline{\phi_{slm}},\overline{\phi_{slm}})}$, then

$$\underline{\phi_{slm}}(t) \leq \phi_{slm}(t) \leq \overline{\phi_{slm}}(t),\; t \in \Lambda_{slm}.$$

Remembering that the second and third arguments do not cause ψ to decrease, we introduce

$$\underline{\phi_{slm}}(t) \leq \theta_{sln,slm}(t) \leq \phi_{slm}(t) \leq \beta_{sln,slm}(t) \leq \overline{\phi_{slm}}(t) \; for \; t \in \Lambda_{slm} \; and \; n \in slN.$$

As $n \to \infty$ in the above inequality, it implies that

$$\underline{\phi_{slm}}(t) \leq \theta_{slm}(t) \leq \phi_{slm}(t) \leq \beta_{slm}(t) \leq \overline{\phi_{slm}}(t) \text{ for } t \in \Lambda_{slm}.$$

This concludes the proof of theorem by considering $\phi_{L,slm} = \theta_{slm}$ and $\phi_{slM,slm} = \beta_{slm}$, respectively, which are the minimum and maximum solutions in $s_{(\underline{\phi_{slm}},\overline{\phi_{slm}})}$.

Theorem 4.3.3. *Assume that the hypotheses of (4.3.3) to be satisffied. The auxiliary CHFDE of constant order (**??**) has at least one solution in $C(\Lambda_{slm};\mathbb{R})$.*

Proof. According to4.3.3, we get $s_{(\underline{\phi_{slm}},\overline{\phi_{slm}})} \neq \emptyset$, implying that the solution set associated with the auxiliary CHFDE of constant order $(\overline{3.2})$ is not empty in $C(T_{slm-1}, T_{slm};\mathbb{R})$. By proving that the auxiliary CHFDE of constant order (**??**) has at least one solution in $C(T_{slm-1}, T_{slm};\mathbb{R})$, this completes the proof of theorem. We shall now investigate the existence result for the CAPUTO-HADAMARD fractional nonlinear differential equation of variable order (CHFDEVO) ((**??**).

Theorem 4.3.4. . *Let all $m \in \{1, 2, \ldots, n\}$ satisfy the condition (Hl). Then, there is at least one solution for the given nonlinear IVP of CHFDEVO (4.1.2) in E.*

Proof. Based on the above proofs, we know that the nonlinear IVP of constant order CAPUTO-HADAMARD □

fractional differential equation (**??**) has at least one solution $\overline{\phi}_{slm} \in E_{slm}$, $m \in \{1, 2, \ldots, n\}$. This is in accordance with *(4.3.3)* and *(4.3.4)*. We define the solution function for each $m \in \{1, 2, \ldots, n\}$ as

$$\phi_{slm} = \begin{cases} 0, \ t \in [p, \ T_{slm-1}], \\ \overline{\phi}_m, \ t \in \Lambda_{slm}. \end{cases} \tag{4.3.6}$$

Thus, $\phi_{slm} \in C(T_{slm-1}, \ T_{slm}; \mathbb{R})$ solves the HADAMARD integral equation (**??**) for each $t \in \Lambda_{slm}$, which means that

$$\phi_{slm}(p) = 0, \ \phi_{slm}(T_{slm}) = \overline{\phi}_{slm}(T_{slm}) = 0.$$

Then, the function

$$\phi(t) = \begin{cases} \phi_1(t), \ t \in \Lambda_1, \\ \phi_2(t) = [Case] \\ \phi_n(t) = [Case] \end{cases}$$

is a solution of the given nonlinear IVP of CHFDEVO (4.1.2) in E.

4.4 Numerical example

Let $\Lambda := [1, \ e^2]$, $T_0 = 1$, $T_1 = e$, $T_2 = e^2$. Consider the following nonlinear variable order IVP of CHFDE

$$\begin{cases} {}^{C}D_{1^+}^{w(t)}\phi(t) = \dfrac{1}{\pi}(\sqrt{\ln t} + (\ln t)^4) + \phi(t), \ t \in \Lambda, \\ \phi(1) = 0, \end{cases} \tag{4.4.1}$$

where

$$w(t) = \begin{cases} \dfrac{1}{2}, \quad t \in \Lambda_1 := [1, \ e], \\ \dfrac{2}{3}, \quad t \in \Lambda_2; =]e, \ e^2]. \end{cases} \tag{4.4.2}$$

Denote

$$\psi(t,\,\phi)=\frac{1}{\pi}(\sqrt{\ln t}+(\ln t)^4)+\phi(t),\ (t,\,\phi)\in[1,\,e^2]\times\mathbb{R}.$$

Using (4.4.2) and (**??**), we consider two auxiliary constant order IVPs of CHFDEs as

$$\begin{cases} {}^{C}D_{+}^{\frac{1}{12}}\phi(t)=\frac{1}{\pi}(\sqrt{\ln t}+(\ln t)^4)+\phi(t),\ t\in\Lambda_1, \\ \phi(1)=0, \end{cases} \tag{4.4.3}$$

and

$$\begin{cases} {}^{C}D_{+}^{\frac{2}{e3}}\phi(t)=\frac{1}{\pi}(\sqrt{\ln t}+(\ln t)^4)+\phi(t),\ t\in\Lambda_2, \\ \phi(e)=1. \end{cases} \tag{4.4.4}$$

For m = 1: By 4.3.1, the auxiliary IVP of constant order CHFDE (4.4.3) has at least one solution $\overline{\phi}_1\in E_1$ as

$$\phi_1(t)=I_{+}^{\frac{1}{12}}(\frac{1}{\pi}(\sqrt{\ln t}+(\ln t)^4)+\phi_1(t))\text{ for }t\in\Lambda_1. \tag{4.4.5}$$

In fact, as one can see, $(\phi_1(t),\overline{\phi_1(t)})=(\mathrm{O},\ \ln t+(\ln t)^5)$ denotes the *upper-lower* bounds of the solution to (4.4.5). We can calculate the sequences $\{\theta_{sln,1}\}$ and $\{\beta_{sln,1}\}$ by

$$\begin{cases} \theta_{0,1}=\underline{\phi_1} \\ \theta_{sln+1,1}(t)=I_{+}^{\frac{1}{12}}\psi(t,\,\theta_{sln,1}(t)),\ n=0,1,\,\ldots, \end{cases}$$

and

$$\begin{cases} \beta_{0,1}=\overline{\phi_1} \\ \beta_{sln+1,1}(t)=I_{+}^{\frac{1}{12}}\psi(t,\beta_{sln,1}(t)),n=0,1,\,\ldots, \end{cases}$$

for each $t\in\Lambda_1$. We can now use 4.3.3 to determine that $\theta_{sln,1}\to\theta_1\in E_1$ and $\beta_{sln,1}\to\beta_1\in E_1$ as $\pi(\ln t)^2,n\to\infty$. In the meanwhile, we may obtain $t\in\Lambda_1$ for $\beta_1(t)=\theta_1(t)=\overline{3}$. We use Maple to calculate the sequences $\{\theta_{sln,2}\}$ and $\{\beta_{sln,2}\}$ for each n which are defined as integrals with different initial values. Then, we take the values of these sequences at each instant t and plot them with Matlab. In Table 1, we present the error (which is the sup of the absolute value of the defference) between the sequences $\{\theta_{sln,1}\}$, $\{\beta_{sln,1}\}$ and the exact solution for $n=5,10,15,20$. In Figure 1, we plot the sequences $\{\theta_{sln,1}\}$, $\{\beta_{sln,1}\}$ and the exact solution for $n=0,1,2,10,30$.

4.5 code matlab

```
function r=leftFcaputo(x,alpha,a,n)
Dx=diff(x,n);
G=@(t,tau)dx(tau)./(gamma(n-alpha(t,tau)).*(t-tau).^(1+alpha(t,tau)-n));
r=@(t)sum(chebfun(@(tau)g(t,tau),[a,t],'splitting','on',[a t];
end
```

.function r=rightFcaputo(x,alpha ,a,n)

```
G=@(t,tau)dx(tau).\(gama(alpha(tau,t)).^(1-alpha(tau,t)));
R= r =@(t)sum(chebfun(@(tau)g(t,tau),[t b] ,'splitting','on',[t b]) ;
endl
Format long
a=0 ;b=1 ;n=1 ;
Alpha =@(t,tau)t.^2\2;
X=chebfun(@(t)t.^4,[a,b])
Exact =@(t)00000
Approximation =leftFcaputo(x,alpha,a,n) ;
For i=1:9
T=0.1*i;
E=exact (t);
A=approximation (t);
Error=E-A.;
[t E A Error]
```

Table 1.Error analysis for $m = 1$.

	$n = 5$	$n = 10$	$n = 15$	$n = 20$
$\sup_{\{t\in[1,e]\}} \lvert\theta_{\{n,1\}}(t) - \theta_1(t)\rvert$	4.7692×10^{-2}	6.3900×10^{-4}	4×10^{-6}	10^{-15}
$\sup_{\{t\in je,e^2j\}} \lvert\theta_{\{n,2\}}(t) - \theta_2(t)\rvert$	4.6818×10^{-2}	$7.8299 \times 10^{\{e-4\}}$	5×10^{-6}	9×10^{-8}

$itbpF4.8317in3.6841in0inFigure$

$itbpF5.3817in3.6832in0inFigure$

$itbpF5.0643in3.6841in0inFigure$

$itbpF4.9753in3.6832in0inFigure$

$itbpF5.2226in3.685in0inFigure$

$itbpF5.1647in3.685in0inFigure$

$itbpF4.9381in3.6832in0inFigure$

Figure 1. A plot of $\theta_{sln,1}, \beta_{sln,1}$ and exact solution for $n = 0, 1, 2, 10, 20, 30$.
In Figure 1, We notice that when n is larger, the sequences $\{\theta_{sln,1}\}$ and $\{\beta_{sln,1}\}$ are approximated to

$$\pi(\ln t)^2$$

the exact solution $\overline{3}$. Moreover, in Table 1, we confirm our previous remark, because the error approaches to 0 when n converges to $+\infty$.For m = 2: By Theorem 3.1, the auxiliary IVP of constant order CHFDE (4.4.4) has at least one solution $\overline{\phi}_2 \in E_2$ as

$$\phi_2(t) = I^1_{e+}(\frac{1}{\pi}(\sqrt{\ln t} + (\ln t)^4) + \phi_2(t)) fort \in \Lambda_2. \tag{4.5.1}$$

In fact, we are able to observe that $(\phi_2(t), \overline{\phi_2(t)}) = (1,\ \ln t + (\ln t)^5)$ is upper-lower solution to (4.6). We can calculate the sequences $\{\theta_{sln,2}\}$ and $\{\beta_{sln,2}\}$ by

$$\begin{cases} \theta_{0,2} = \phi_2 \\ \theta_{sln+1,2}(t) = I^1_{e+} f(t,\ \theta_{sln,2}(t)),\ n = 0, 1,\ \dots, \end{cases}$$

and

$$\begin{cases} \beta_{0,2} = \overline{\phi_2} \\ \beta_{sln+1,2}(t) = I^{\frac{2}{3}}_{e+} f(t, \beta_{sln,2}(t)), n = 0, 1,\ \dots, \end{cases}$$

for each $t \in \Lambda_2$. We can now use Theorem 3.3 to prove $\theta_{sln,2} \to \theta_2 \in E_2$ and $\beta_{sln,2} \to \beta_2 \in E_2$ as $n \to \infty$. In the meanwhile, we may obtain $t \in \Lambda_2$ for

$$\beta_2(t) = \theta_2(t) = \pi\frac{(\ln t)^5}{30}.$$

In Table 2, we present the error (which is the sup of the absolute value of the defference) between the sequences $\{\theta_{sln,2}\}$, $\{\beta_{sln,2}\}$ and the exact solution for $n = 5, 10, 15, 20$. In Figure 2, we plot the sequences $\{\theta_{sln,2}\}$, $\{\beta_{sln,2}\}$ and the exact solution for $n = 0, 1, 2, 10, 30$. In this figure, we notice that when n is larger,

$$\pi(\ln t)^5$$

the sequences $\{\theta_{sln,2}\}$ and $\{\beta_{sln,2}\}$ are approximated to the exact solution –. In Table 2, we confirm 30 our previous remark, because the error approaches to 0 when n converges to $+\infty$. Consequently, in accordance with Theorem 4.3.4, the given nonlinear IVP of CHFDEVO (4.4.1) has a solution

$$\phi(t) = \begin{cases} \overline{\phi}_1(t), & t \in \Lambda_1, \\ \phi_2(t), & t \in \Lambda_2, \end{cases}$$

where

$$\phi_2(t) = \begin{cases} 0, \ t \in \Lambda_1, \\ \overline{\phi}_2(t), \ t \in \Lambda_2. \end{cases}$$

Table 2. Error analysis for $m = 2$.

In Figure 1, We notice that when n is larger, the sequences $\{\theta_{sln,1}\}$ and $\{\beta_{sln,1}\}$ are approximated to

$$\pi(\ln t)^2$$

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$$\phi_2(t) = I^1_{e+}(\frac{1}{\pi}(\sqrt{\ln t} + (\ln t)^4) + \phi_2(t)) fort \in \Lambda_2. \quad (4.5.2)$$

In fact, we are able to observe that $(\phi_2(t), \overline{\phi_2(t)}) = (1, \ \ln t + (\ln t)^5)$ is upper-lower solution to (4.6). We can calculate the sequences $\{\theta_{sln,2}\}$ and $\{\beta_{sln,2}\}$ by

$$\begin{cases} \theta_{0,2} = \phi_2 \\ \theta_{sln+1,2}(t) = I^1_{e+} f(t, \ \theta_{sln,2}(t)), \ n = 0, 1, \ \dots, \end{cases}$$

and

$$\begin{cases} \beta_{0,2} = \overline{\phi_2} \\ \beta_{sln+1,2}(t) = I^{\frac{2}{3}}_{e+} f(t, \beta_{sln,2}(t)), n = 0, 1, \ \dots, \end{cases}$$

for each $t \in \Lambda_2$. We can now use Theorem 3.3 to prove $\theta_{sln,2} \to \theta_2 \in E_2$ and $\beta_{sln,2} \to \beta_2 \in E_2$ as $n \to \infty$. In the meanwhile, we may obtain $t \in \Lambda_2$ for

$$\beta_2(t) = \theta_2(t) = \pi\frac{(\ln t)^5}{30}.$$

In Table 2, we present the error (which is the sup of the absolute value of the defference) between the sequences $\{\theta_{sln,2}\}$, $\{\beta_{sln,2}\}$ and the exact solution for $n = 5, 10, 15, 20$. In Figure 2, we plot the sequences $\{\theta_{sln,2}\}$, $\{\beta_{sln,2}\}$ and the exact solution for $n = 0, 1, 2, 10, 30$. In this figure, we notice that when n is larger,

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$$\phi(t) = \begin{cases} \overline{\phi}_1(t), & t \in \Lambda_1, \\ \phi_2(t), & t \in \Lambda_2, \end{cases}$$

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$$\phi_2(t) = \begin{cases} 0, \ t \in \Lambda_1, \\ \overline{\phi}_2(t), \ t \in \Lambda_2. \end{cases}$$

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$$\phi_2(t) = I^1_{e+}(\frac{1}{\pi}(\sqrt{\ln t} + (\ln t)^4) + \phi_2(t)) fort \in \Lambda_2. \tag{4.5.3}$$

In fact, we are able to observe that $(\phi_2(t), \overline{\phi_2(t)}) = (1, \ \ln t + (\ln t)^5)$ is upper-lower solution to (4.6). We can calculate the sequences $\{\theta_{sln,2}\}$ and $\{\beta_{sln,2}\}$ by

$$\begin{cases} \theta_{0,2} = \phi_2 \\ \theta_{sln+1,2}(t) = I^1_{e+} f(t, \ \theta_{sln,2}(t)), \ n = 0, 1, \ \ldots, \end{cases}$$

and

$$\begin{cases} \beta_{0,2} = \overline{\phi_2} \\ \beta_{sln+1,2}(t) = I^{\frac{2}{3}}_{e+} f(t, \beta_{sln,2}(t)), n = 0, 1, \ \ldots, \end{cases}$$

for each $t \in \Lambda_2$. We can now use Theorem 3.3 to prove $\theta_{sln,2} \to \theta_2 \in E_2$ and $\beta_{sln,2} \to \beta_2 \in E_2$ as $n \to \infty$. In the meanwhile, we may obtain $t \in \Lambda_2$ for

$$\beta_2(t) = \theta_2(t) = \pi \frac{(\ln t)^5}{30}.$$

In Table 2, we present the error (which is the sup of the absolute value of the defference) between the sequences $\{\theta_{sln,2}\}$, $\{\beta_{sln,2}\}$ and the exact solution for $n = 5, 10, 15, 20$. In Figure 2, we plot the sequences $\{\theta_{sln,2}\}$, $\{\beta_{sln,2}\}$ and the exact solution for $n = 0, 1, 2, 10, 30$. In this figure, we notice that when n is larger,

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where

$$\phi_2(t) = \begin{cases} 0, \ t \in \Lambda_1, \\ \overline{\phi}_2(t), \ t \in \Lambda_2. \end{cases}$$

Table 2. Error analysis for $m = 2$.

	$n = 5$	$n = 10$	$n = 15$	$n = 20$
$\sup_{t \in je,e^2j} \lvert \theta_{n,2}(t) - \theta_2(t) \rvert$	6.8509×10^{-3}	10^{-6}	6×10^{-10}	10^{-10}
$\sup_{t \in Je,e^2j} \lvert \beta_{n,2}(t) - \beta_2(t) \rvert$	3.1123×10^{-2}	10^{-6}	10^{-7}	10^{-12}

$itbpF2.879in3.685in0inFigure$

$itbpF5.1863in3.685in0inFigure$

$itbpF5.1119in3.6832in0inFigure$

$itbpF4.6908in3.685in0inFigure$

Figure 2. A plot of $\theta_{sln,2}$, $\beta_{sln,2}$ and exact solution, for $n = 0, 1, 2, 10, 20, 30$.

Conclusion

In this paper, a CAPUTO-HADAMARD fractional nonlinear differential equation of variable order was considered and discussed. With the help of piece-wise constant order functions on some continuous subintervals of a partition, we converted the main variable order IVP to a constant order IVP of the CAPUTO-HADAMARD differential equation. By calculating and obtaining equivalent solutions in the form of a HADAMARD integral equation, we used the upper-lower solution technique to prove the relevant existence theorems. By plotting some graphs and providing some numerical tables, we presented an example of the variable order IVP to apply and demonstrate the results of our method. In the future, we will extend our studies on different IVPs and BVPs (implicit, resonance, thermostat model, etc.) with changing conditions (terminal, integral conditions, etc.) in the future. Also, if we can define variable order tempered fractional derivative, then it will be a new idea for this purpose .

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