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Et analyse numérique

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Intitulé

Mean values of arithmetic functions

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قال رسول الله صلى الله عليه وسلم:"مَنْ صَنَعَ إِلَيكُم مَعْرُوفًا فَكَافِئُوهُ فَإِنْ لَمْ تَجِدُوا مَا تُكَافِئُونَهُ بِهِ فَادْعُوا لَهُ حَتَّى تَرَوْا أَنَّكُم كَافَأْتُمُوهُ" رواه أبو داوود.

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إلهي لا يطيب الليل إلا بشكرك ... ولا يطيب النهار إلا بطاعتك ... ولا تطيب اللحظات إلا بذكرك ... ولا تطيب الآخرة إلا بعفوك

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إلى من بلّغ الرسالة وأدّى الأمانة ... ونصح الأمة ... إلى نبي الرحمة و نور العالمين

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إلى كل الذين أعرفهم ولم تسعني الكتابة لهم.

Mean values of arithmetic functions

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Résumé

Dans ce travail, nous énonçons les fonctions arithmétiques de base, ainsi que les fonctions de Chebyshev. Ensuite, nous présentons les formules de sommation célèbres avec preuves. Par exemple : La formule de sommation d'Euler et la formule de sommation partielle. A la fin, nous appliquons ces formules pour trouver les valeurs moyennes de certaines fonctions multiplicatives et additives. Les valeurs moyennes de certaines fonctions arithmétiques arbitraires sont également discutées.

Mots clés. Fonctions arithmétiques, La valeur moyenne des fonctions arithmétiques, Moyenne ordres des fonctions arithmétiques.

Abstract

In this work, we understand the basic arithmetic functions, as well as chebyshev's functions. Next, we present famous summation formulas with proofs. For example: The Euler summation formula and partial summation formula. At the end, we apply these formulas to find mean values of some multiplicative and additive functions. The mean values of some arbitrary arithmetic functions are also discussed.

Keywords and phrases Arithmetic functions, Averages order of arithmetic functions, The mean values of arithmetic functions.

Table of notations

We list below page numbers where various notations in the body of the text are introduced.

Notation	**Explanation**
$\tau(n)$, $d(n)$	the number of divisors of n
$\sigma_s(n)$	The sum of s-th power of all divisors of n
$\varphi(n)$	Euler phi function
$\omega(n)$	the number of distinct prime divisors of n
$\Omega(n)$	the total number of prime divisors of n
$d \mid n$, $d \nmid n$	divides (does not divide)
$\mu(n)$	Möbius function
$\zeta(s)$	Riemann zeta function
$[x]$	The integer part of x
$\{x\}$	The fractional part of x
$\lceil x \rceil$	The smallest positive integer $\geq x$
(a, b)	The greatest common divisor of two integers a and b
$\varphi_s(n)$	The generalized Euler's function
$\sigma(n)$	the sum of the divisors of n
$f \sim g$	asymptotic equality
$\Lambda(n)$	Von Mangoldt function
$\psi(x)$	Chebyshev ψ-function
$\theta(x)$	Chebyshev θ-function
$\psi(n)$	The sum of $\Lambda(k)$ over integers $k \leq n$
$\lambda(n)$	The Liouville function
$f \asymp g$	$f(x)/g(x)$ is bounded above and below
O, o	Big (little) oh notation
$\pi(n)$, $\pi(x)$	number of primes $\leq n$ (or x)
log	The logarithm-function

Introduction

An arithmetic function is defined to be a function $f(n)$, defined for $n \in \mathbb{N}$, which maps to a complex number such that $f : \mathbb{N} \to \mathbb{C}$. Examples of arithmetic functions include: the number of divisors of n, the sum of divisors of n, Euler's function, the number of primes less than a given number n and the number of ways n can be represented as a sum of two squares, ... etc. For suitable references, see [2],[4],[6], [7]. While the behavior of values of such arithmetic functions are hard to predict, it is easier to analyze the behavior of the averages of arithmetic functions which is defined as:

$$\lim_{n\to\infty} \frac{f(1) + f(2) + ... + f(n)}{n} = L.$$

Here L is called the average value of $f(n)$ (see, eg [7, Section 6, page 201]). So in this work we will understand how they examine averages of several different arithmetic functions. More precisely, we focus of the following summation formulas:

- $\sum_{n\leq x}$: summation over all positive integers $\leq x$.
- $\sum_{p\leq x}$: summation over all primes $\leq x$.
- $\sum_{p^m}$: summation over all prime powers p^m with p prime and m a positive integer.

- $\sum_{d|n}$: summation over all positive divisors of n (including the trivial divisors $d = 1$ and $d = n$).
- $\sum_{d^2|n}$: summation over all positive integers d for which d^2 divides n.
- $\sum_{p|n}$: summation over all (distinct) primes dividing n.

We need to the following definition (see [4],[7],[5]): Let f and g be functions of x. The notation $f \asymp g$ denotes that $f(x)/g(x)$ is bounded above and below by positive numbers for large values of x. The notation $f = O(g)$ denotes that there exists a constant c such that $|f(x)| \leq cg(x)$. The notation $f \sim g$ denotes that $\lim\limits_{x\to\infty} f(x)/g(x) = 1$.

While the behavior of a number theoretic function $f(n)$ for large n is often difficult to determine because the function values can fluctuate considerably as n increases, it is more fruitful to study partial sums and seek asymptotic formulas of the form

$$\sum_{n\leq x} f(n) = F(x) + O(h(x)),$$

where $F(x)$ is a known function of x and $O(h(x))$ represents the error, a function of smaller order than $F(x)$ for all x in some prescribed range. Some of these arithmetic functions are called multiplicative when they satisfy $f(nm) = f(n) f(m)$ whenever n and m are coprime. Here, we will focus on the following arithmetic functions:

- d : The number of non-negative divisors function.
- σ_s : The sum of the s -th powers of all the non-negative divisors function, for $s \in \mathbb{R}$. In particular, $\sigma_0 = d$.

- φ : The Euler's function.

An **average order** of an arithmetic function is some simpler or better-understood function which takes the same values "on average". So if f is an arithmetic function. We say that an average order of f is g if

$$\sum_{n\leq x} f\left(n\right) \sim \sum_{n\leq x} g\left(n\right)$$

as x tends to infinity. It is conventional to choose an approximating function g that is continuous and monotone. But even so an average order is of course not unique. In cases where the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} f\left(n\right) = A$$

exists, it is said that f has a **mean value** (**average value**) A. Let us study the following facts:

- An average order of $d(n)$, the number of divisors of n, is $\log n$.
- An average order of $\sigma(n)$, the sum of divisors of n, is $n\pi^2 / 6$.
- An average order of $\varphi\left(n\right)$, Euler's totient function of n, is $3n / \pi^2$.
- The average order of representations of a natural number as a sum of three squares is $4\pi n / 3$.
- An average order of $\omega(n)$, the number of distinct prime factors of n, is $\log\log n$.
- An average order of $\Omega(n)$, the number of prime factors of n, is $\log\log n$.

- The prime number theorem is equivalent to the statement that the von Mangoldt function $\Lambda(n)$ has average order 1.
- An average value of $\mu(n)$, the Möbius function, is zero; this is again equivalent to the prime number theorem.

Workplan:

In Chapter 1, we introduce some basic facts and notations that will appear in the rest of this work. That is, the basic arithmetic functions and we illustrate an example for each function. In Chapter 2, we present properties of big-O notations and the famous summation formulas and results with proofs. For example: The Euler summation formula, Abel summation formula and the Maclaurin summation formula. In Chapter 3, we calculate mean values in the case when f is a multiplicative or additive arithmetic function with $f(n) = \sum_{d|n} \overline{f}(d)$. Finally, in Chapter **??** we state some open problems.

Chapter 1

Basic arithmetic functions

In this chapter, we state some basic arithmetic functions (see [2],[7]). First, we state the Fundamental Theorem of Arithmetic and then recall definitions of basic arithmetic functions and we illustrate an example for each function.

Theorem 1.1 (Fundamental Theorem of Arithmetic, see [7, p. 25]) *Every positive integer n greater than 1 can be written uniquely as the product of primes:*

$$n = q_1^{a_1} q_2^{a_2} \cdots q_r^{a_r} = \prod_{i=1}^{r} q_i^{a_i}, \tag{1.1}$$

where $q_1, q_2, ..., q_r$ are distinct primes and $a_1, a_2, ..., a_r$ are natural numbers. The equation (1.1) is often called the prime power decomposition of n, or the standard prime factorization of n.

1.1 Definitions and examples

Definition 1.1 *A real or complex valued function defined in the positive integers (or all integers) is called an arithmetic functions or a number-theoretic* *function.*

Remark 1.1 *An arithmetic function is a function whose domain is the set of natural numbers.*

We give some examples of arithmetic functions as follows and we will discuss their properties in the following section.

1) The divisor function d.

Definition 1.2 *The divisors function $d(n)$ is defined as the number of positif divisors of n, i.e .,*

$$d(n) = \sum_{d|n} 1.$$

It is well-known that for the natural number $n \geq 2$ with canonical representation $n = q_1^{a_1} q_2^{a_2} ... q_k^{a_k}$ (where $k, a_1, ..., a_k$ are positive integers and $q_1, q_2, ...q_k$ are different primes), we have

$$d(n) = (a_1 + 1)(a_2 + 1) ... (a_k + 1). \tag{1.2}$$

Let $n = 2023 = 7 \cdot 17^2$, we have by (1.2), $d(n) = (1 + 1)(2 + 1) = 6$. If n is square-free having k distinct primes, that is, $n = q_1 q_2 ... q_k$ then $d(n) = 2^k$.

2) The divisor sum function σ.

Definition 1.3 *The divisor sum function $\sigma(n)$ is defined as the sum of all positif divisors of n, i.e .,*

$$\sigma(n) = \sum_{d|n} d.$$

By a well-known result, note that if $n = q_1^{a_1} q_2^{a_2} ... q_k^{a_k}$, where $k, a_1, ..., a_k$ are positive integers and $q_1, q_2, ...q_k$ are different primes, then

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{a_i+1} - 1}{p_i - 1}. \tag{1.3}$$

Let $n = 2023 = 7 \cdot 17^2$, clearly by (1.3)

$$\sigma(n) = \prod_{i=1}^{2} \frac{p_i^{a_i+1} - 1}{p_i - 1} = \left(\frac{7^{1+1} - 1}{7 - 1} \right) \cdot \left(\frac{17^{2+1} - 1}{17 - 1} \right) = 2456.$$

More generally, the divisor sum function power $\sigma_s(n)$ with ($s \in \mathbb{C}$ and $\sigma_1(n) = \sigma(n)$) is defined as the sum of s power of all positif divisors of n, i.e .,

$$\sigma_s(n) = \sum_{d|n} d^s.$$

3) The Euler totient function φ.

Definition 1.4 *The Euler totient function $\varphi(n)$ is defined as*

$$\varphi(n) = \sum_{\substack{1 \leq k \leq n \\ (k,n)=1}} 1.$$

By a well-known result, note that if $n = q_1^{a_1} q_2^{a_2} ... q_k^{a_k}$, where $k, a_1, ..., a_k$ are positive integers and $q_1, q_2, ...q_k$ are different primes, then

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p} \right). \tag{1.4}$$

Let us take $n = 2023 = 7 \cdot 17^2$. Then by (1.4), we have $\varphi(n) = (7-1) \cdot 17 \cdot (17-1) = 1632$. Note that $\varphi(p) = p - 1$ if and only if p is prime. Here is a short table of values of φ:

n	1	2	3	4	5	6	7	8	9	10
$\varphi(n)$	1	1	2	2	4	2	6	4	6	4

Note if n is a prime number, then $\varphi(p) = p - 1$ and if n is a prime power, say $n = p^a$ then $\varphi(n) = p^{a-1}(p-1)$.

4) Möbuis function $\mu(n)$.

Definition 1.5 *The Möbuis function $\mu(n)$ is defined as follows :*

$$\mu(n) = \begin{cases} 1, \text{ if } \ n = 1, \\ (-1)^r, \text{ if } n = p_1 p_2 \ldots p_r \text{ with distinct primes } p_i, \\ 0, \text{ otherwise.} \end{cases} \tag{1.5}$$

Let $n = 2023 = 7 \cdot 17^2$, then by (1.5) $\mu(n) = 0$. If $n = 2027 \cdot 2029$, then $\mu(n) = 1$. Note that $\mu(n) = 0$ if and only if n has a square factor > 1. Here is a short table of values of μ:

n	1	2	3	4	5	6	7	8	9	10
$\mu(n)$	1	-1	-1	0	-1	1	-1	0	0	1

5) Von Mangoldt function $\Lambda(n)$.

Definition 1.6 *The von Mangoldt function $\Lambda(n)$ is defined as follows:*

$$\Lambda(n) = \begin{cases} \log p, \text{ if } \ n = p^k, \ k \geq 1 \text{and } p \text{ prime}, \\ 0, \text{ otherwise.} \end{cases} \tag{1.6}$$

Let $n = 2023 = 7 \cdot 17^2$. By (1.6), we see that $\Lambda(n) = \log(7 \cdot 17^2) = \log 7 + \log(17^2) = \log 7 + 2\log 17 = 7.6123$. We also have the following lemma:

Lemma 1.1 (see [2]) *For every $n \geq 1$, one has*

$$\sum_{d|n} \Lambda(d) = \log n.$$

Proof. Let $n = q_1^{a_1} q_2^{a_2} ... q_k^{a_k}$, where $2 \leq q_1 < q_2 < ... < q_k$ are distinct primes and $a_1, a_2, ..., a_k$ are positive integers. We write

$$\sum_{d|n} \Lambda(d) = \sum_{i=1}^{k} a_i (\log q_i) = \log n.$$

The proof is finished. ■

6) Number of prime factors $\omega(n)$.

Definition 1.7 *The omega function* $\omega(n)$ *is defined as the number of distinct prime factors of* n*, i.e.,*

$$\omega(n) = r, \tag{1.7}$$

where $n = p_1^{a_1} p_2^{a_2} ... p_r^{a_r}$ *is the prime-power decomposition.*

Let $n = 2023 = 7 \cdot 17^2$, we have $\omega(n) = 2$.

7) Total number of prime factors $\Omega(n)$.

Definition 1.8 *The omega function* $\Omega(n)$ *is defined as the total number of prime factors of* n*, i.e.,*

$$\Omega(n) = a_1 + a_2 + ... + a_r, \tag{1.8}$$

where $n = p_1^{a_1} p_2^{a_2} ... p_r^{a_r}$ *is the prime-power decomposition.*

Let $n = 2023 = 7 \cdot 17^2$, we have $\Omega(n) = 3$.

8) Liouville function $\lambda(n)$.

Definition 1.9 *The Liouville function* λ *is defined as follows :*

$$\lambda(n) = (-1)^{\Omega(n)}. \tag{1.9}$$

Let $n = 2023 = 7 \cdot 17^2$, we have $\lambda(n) = -1$.

9) Riemann zeta function $\zeta(s)$.

Definition 1.10 *Let $s = \sigma + it \in \mathbb{C}$. For $\sigma > 1$, the Riemann zeta function $\zeta(s)$ is defined by the series*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1.10}$$

10) Prime-counting function $\pi(x)$.

Definition 1.11 *The prime-counting function is the function counting the number of prime numbers less than or equal to some real number x. It is denoted by $\pi(x)$ (unrelated to the number π).*

As an example, for $\pi(10) = 4$. Of great interest in number theory is the growth rate of the prime-counting function. It was conjectured in the end of the 18th century by Gauss and by Legendre to be approximately $x/\log x$, where log is the natural logarithm, in the sense that $\lim \frac{\pi(x)}{x/\log x} = 1$ as x tends to infinity.

1.2 Multiplicative functions

An important class of arithmetic functions are multiplicative functions defined as follows.

Definition 1.12 *An arithmetic function f which is not identically zero is said to be multiplicative if*

$$f(mn) = f(m) \cdot f(n) \tag{1.11}$$

whenever $(m,n)=1$. *Moreover, if* (1.11) *holds for all* m,n, *then* f *is called completely multiplicative.*

We have the following property of all multiplicative functions.

Proposition 1.1 *If* f *is multiplicative, then* $f(1)=1$.

Proof. Since f is not identically zero, there exists $n\in\mathbb{N}$ such that $f(n)\neq 0$.We have $f(n)=f(n)f(1)$ as f is multiplicative. Hence, $f(1)=1$. ■

Proposition 1.2 (see [2]) *The function* φ *is multiplicative.*

Proof. For any $m,n\in\mathbb{N}$ such that $(m,n)=1$, we need to prove $\varphi(m\cdot n)=\varphi(m)\varphi(n)$. Assume $m=p_1^{a_1}p_2^{a_2}...p_r^{a_r}$ and $n=q_1^{b_1}q_2^{b_2}...q_s^{b_s}$ with p_i, q_j are distinct primes and a_i,$b_j\in\mathbb{N}$. By (1.4), we have

$$\varphi(m\cdot n)=mn\prod_{p|mn}(1-\frac{1}{p})=m\prod_{i=1}^{r}(1-\frac{1}{p_i})\cdot n\prod_{i=1}^{s}(1-\frac{1}{q_j})=\varphi(m)\varphi(n).$$

This completes the proof. ■

Proposition 1.3 *The functions* d,σ, σ_s *and* μ *are multiplicative.*

The Möbius function arises in many different places in number theory. One of its fundamental properties is a remarkably simple formula for the divisor sum $\sum_{d|n}\mu(d)$.

Theorem 1.2 (see [2]) *If* $n\geq 1$, *then we have*

$$\sum_{d|n}\mu(d)=I(n)=\begin{cases}1, & \text{if } n=1,\\ 0, & \text{otherwise.}\end{cases}$$

Proof. If $n = 1$, then both sides are equal to 1. If $n > 1$, then we can write $n = p_1^{a_1} p_2^{a_2} ... p_r^{a_r}$. By definition and Proposition 1.3, we have

$$\begin{aligned}
\sum_{d|n} \mu(d) &= \sum_{0 \leq c_1 \leq a_1} \sum_{0 \leq c_2 \leq a_2} \cdots \sum_{0 \leq c_r \leq a_r} \mu\left(p_1^{c_1} p_2^{c_2} ... p_r^{c_r}\right) \\
&= \sum_{0 \leq c_1 \leq 1} \sum_{0 \leq c_2 \leq 1} \cdots \sum_{0 \leq c_r \leq 1} \mu\left(p_1^{c_1}\right) \mu(p_2^{c_2}) ... \mu(p_r^{c_r}) \\
&= \prod_{i=1}^{r} \sum_{0 \leq c_i \leq 1} \mu(p_i^{c_i}) = \prod_{i=1}^{r} (1-1) = 0.
\end{aligned}$$

This proves the theorem. ■

Theorem 1.3 (see [2]) *If $n \geq 1$, then we have*

$$\varphi(n) = \sum_{d|n} \mu(d) \frac{n}{d}.$$

Proof. By Theorem 1.2, we have

$$\varphi(n) = \sum_{\substack{1 \leq k \leq n \\ (k,n)=1}} 1 = \sum_{1 \leq k \leq n} 1 \sum_{d|(n,k)} \mu(d)$$

Exchanging the order of the sums above, we get

$$\varphi(n) = \sum_{d|n} \mu(d) \sum_{\substack{1 \leq k \leq n \\ d|k}} 1 = \sum_{d|n} \mu(d) \frac{n}{d},$$

as claimed. ■

Theorem 1.4 (see [2]) *If $n \geq 1$, then we have*

$$n = \sum_{d|n} \varphi(d).$$

Proof. By Theorem 1.3, we have

$$\sum_{d|n}\varphi(d)=\sum_{d|n}\sum_{l|d}\mu(l)\frac{d}{l}=\sum_{d|n}\sum_{l|d}\mu\left(\frac{d}{l}\right)l=\sum_{l|n}l\sum_{l|d|n}\mu\left(\frac{d}{l}\right).$$

Making a change of variable $k=l\mid d$, we get

$$\sum_{d|n}\varphi(d)=\sum_{l|n}l\sum_{k|n/l}\mu\left(k\right)$$

By Theorem 1.2, we also have

$$\sum_{d|n}\varphi(d)=\sum_{l|n}l\cdot I(n/l)=n.$$

This completes the proof. ■

1.3 Additive functions

Definition 1.13 *An arithmetic function f which is not identically zero is said to be additive if*

$$f(m\cdot n)=f(m)+f(n). \tag{1.12}$$

whenever $(m,n)=1$. Moreover, if (1.12) holds for all m,n, then f is called completely additive.

Proposition 1.4 *The function ω is additive and the function Ω is completely additive.*

Proof. Write $m=p_1^{a_1}...p_r^{a_r}$ and $n=q_1^{b_1}...q_s^{b_s}$ with prime p_i , q_j and positive integers a_i, b_j. Clearly, If $(m,n)=1$, then $\omega(m\cdot n)=r+s=\omega(m)+\omega(n)$. Moreover, we have $\Omega(m\cdot n)=\sum_i a_i+\sum_j b_j=\Omega(m)+\Omega(n)$. ■

Remark 1.2 *The function $\log n$ is completely additive, since $\log(m\cdot n)=\log m+\log n$. In [3], Erdös proved that if a function $f(n)$ is additive and increasing then there is some $\alpha\geq 0$ such that $f(n)=\alpha\log n$.*

1.4 Chebyshev's Functions

The first, denoted $\theta(x)$ or $\vartheta(x)$, is defined for a real variable x by

$$\theta(x) = \sum_{p \leq x} \log p \tag{1.13}$$

where log denotes the natural logarithm, with the sum extending over all prime numbers p that are less than or equal to x. As an example, $\theta(10) = \log 2 + \log 3 + \log 5 + \log 7$.

The second Chebyshev function $\psi(x)$ is actually the summation function of $\Lambda(n)$. That is,

$$\psi(x) = \sum_{n \leq x} \Lambda(n). \tag{1.14}$$

As an example, $\psi(10) = 3\log 2 + 2\log 3 + \log 5 + \log 7$. This function is defined similarly, with the sum extending over all prime powers not exceeding x. Further for a given prime $p \leq x$ the number of times $\log p$ is counted in the sum for $\psi(x)$ is $\left[\frac{\log x}{\log p}\right]$. Hence, $\psi(x)$ can also be expressed as

$$\psi(x) = \sum_{p \leq x} \left[\frac{\log x}{\log p}\right] \log p.$$

There are certain immediate relationships between these three functions. We have the following corollary:

Corollary 1.1 *For $x \geq 5$, we have*

$$\theta(x) \leq \psi(x) \leq \pi(x) \log x.$$

Proof. First, if $p^k \leq x$ then $p \leq x$ so clearly $\theta(x) \leq \psi(x)$. Further since $1 \leq \log p$ for $p \geq 3$ we have $\pi(x) \leq \theta(x)$ for $x \geq 5$. Now if $p^k \leq x$ then

$k \leq \left[\frac{\log x}{\log p}\right]$. It follows that

$$\psi(x) = \sum_{\substack{p^k \leq x \\ k \geq 1}} \log p = \sum_{p \leq x} \left(\sum_{\substack{p^k \leq x \\ k \geq 1}} 1 \right) \log p = \sum_{p \leq x} \left[\frac{\log x}{\log p}\right] \log p = \sum_{p \leq x} \log x = \pi(x) \log x.$$

Therefore, $\psi(x) \leq \pi(x) \log x$. ■

Chapter 2

Summation Formulas

The basic idea for handling the sums $\sum_{n \le x} f(n)$ is to approximate the sum by a corresponding integral and investigate the error made in the process. The following important result, known as Euler's summation formula, gives an exact formula for the difference between such a sum and the corresponding integral. In fact, these notions are some tools from real analysis and are found in [1], [7], [8] and [5].

Theorem 2.1 (Euler's summation formula [1]) *If f has a continuous derivative f' on the interval $[y, x]$, where $0 < y < x$, then*

$$\sum_{y<n\le x} f(n) = \int_y^x f(t)\,dt + \int_y^x (t-[t])\,f'(t)\,dt + f(x)([x]-x) - f(y)([y]-y), \tag{2.1}$$

where $[t]$ denotes the integer part of t.

Proof. Let $m = [y]$ and $k = [x]$. For integers n and $n-1$ in $[y, x]$ we have

$$\begin{aligned} \int_{n-1}^{n} [t]\,f'(t)\,dt &= \int_{n-1}^{n} (n-1)\,f'(t)\,dt = (n-1)\{f(n) - f(n-1)\} \\ &= \{nf(n) - (n-1)\,f(n-1)\} - f(n). \end{aligned}$$

Summing from $n = m+1$ to $n = k$, we obtain :

$$\begin{aligned}
\int_m^k [t]\, f'(t)\, dt &= \int_m^{m+1} [t]\, f'(t)\, dt + \int_{m+1}^{m+2} [t]\, f'(t)\, dt + ... + \int_{k-1}^{k} [t]\, f'(t)\, dt \\
&= \sum_{n=m+1}^{k} \{nf(n) - (n-1) f(n-1)\} - \sum_{n=m+1}^{k} f(n) \\
&= \sum_{n=m+1}^{k} \{nf(n) - (n-1) f(n-1)\} - \sum_{y<n\leq x} f(n) \\
&= kf(k) - mf(m) - \sum_{y<n\leq x} f(n).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{y<n\leq x} f(n) &= -\int_m^k [t]\, f'(t)\, dt + kf(k) - mf(m) \qquad (2.2) \\
&= -\int_m^k [t]\, f'(t)\, dt + kf(x) - mf(y).
\end{aligned}$$

Integration by parts gives us

$$\int_y^x f(t)\, dt = xf(x) - yf(y) - \int_m^k tf'(t)\, dt.$$

and when this is combined with (2.2) we obtain (2.1). The proof is finished.

■

In most applications, one needs to estimate a sum of the form $\sum_{n\leq x} f(n)$, taken over all positive integers $n \leq x$. In this case, Euler's summation formula reduces to the following result:

Corollary 2.1 (Euler's summation formula, special case) *Let $x \geq 1$ and suppose that $f(t)$ is defined on $[1, x]$ and has a continuous derivative on* *this interval. Then we have*

$$\sum_{n\leq x} f(n) = \int\limits_{1}^{x} f(t)\,dt + \int_{y}^{x} (t-[t])\, f'(t)\,dt + f(x)\,([x]-x) + f(1)\,.$$

Theorem 2.2 (Euler-Maclaurin formula [1]) *Let $a < b$ and $a,\ b \in \mathbb{Z}$. Let $f : [a,b] \longrightarrow \mathbb{C}$. If f is of class C^1 on $[a,b]$. Then we have*

$$\sum_{a<n\leq b} f(n) = \int_a^b (f(x) + \Psi_1(x) f'(x))dx + \frac{1}{2}(f(b) - f(a)),$$

where $\Psi_1(x) = x - [x] - 1/2$ is the saw function.

Proof. Let $n \in \mathbb{Z}$ such that $a \leq n < b$. By integration by parts, we have

$$\begin{aligned}
\int_n^{n+1} \Psi_1(x) f'(x)\,dx &= \int_n^{n+1} (x-n-1/2)df(x) \\
&= [(x-n-1/2)f(x)]_n^{n+1} - \int_n^{n+1} f(x)dx \\
&= \frac{1}{2}(f(n+1) + f(n)) - \int_n^{n+1} f(x)dx.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_a^b \Psi_1(x) f'(x)\,dx &= \sum_{n=a}^{b-1} \int_n^{n+1} \Psi_1(x) f'(x)\,dx \\
&= \frac{1}{2}(f(b) + f(a)) + \sum_{n=a+1}^{b-1} f(n) - \int_a^b f(x)dx
\end{aligned}$$

So we obtain

$$\sum_{a<n\leq b} f(n) = \int_a^b (f(x) + \Psi_1(x) f'(x))dx + \frac{1}{2}(f(b) - f(a)),$$

as claimed. ■

Theorem 2.3 (see [7, p. 206]) *Let a and b be integers with $a < b$, and let $f(t)$ be a function that is monotonic on the interval $[a, b]$. Then*

$$\min(f(a), f(b)) \leq \sum_{n=a}^{b} f(n) - \int_a^b f(t)\,dt \leq \max(f(a), f(b)). \tag{2.3}$$

Let x and y be real numbers with $y < [x]$, and let $f(t)$ be a nonnegative monotonic function on $[y, x]$. Then

$$\left| \sum_{y<n\leq x} f(n) - \int_y^x f(t)dt \right| \leq \max(f(y), f(x)). \tag{2.4}$$

If $f(t)$ is a nonnegative unimodal (increasing or decreasing) function on $[1, \infty)$, then

$$F(x) = \sum_{n\leq x} f(n) = \int_1^x f(t)dt + O(1). \tag{2.5}$$

Proof. If $f(t)$ is increasing on $[n, n+1]$, then

$$f(n) \leq \int_n^{n+1} f(t)\,dt \leq f(n+1) \tag{2.6}$$

Moreover, if $f(t)$ is increasing on the interval $[a, b]$, then

$$f(a) + \int_a^{a+1} f(t)\,dt + ... + \int_{b-1}^b f(t)\,dt \leq \sum_{n=a}^{b} f(n) \leq f(b) + \int_a^{a+1} f(t)\,dt + ... + \int_{b-1}^b f(t)\,dt$$

and so

$$f(a) + \int_a^b f(t)\,dt \leq \sum_{n=a}^{b} f(n) \leq f(b) + \int_a^b f(t)\,dt \tag{2.7}$$

If $f(t)$ is decreasing on $[n, n+1]$, then

$$f(n+1) \leq \int_n^{n+1} f(t)\,dt \leq f(n) \tag{2.8}$$

Moreover, if $f(t)$ is increasing on the interval $[a, b]$, then

$$f(b)+\int_{a}^{a+1} f(t)\,dt+\ldots+\int_{b-1}^{b} f(t)\,dt \leq \sum_{n=a}^{b} f(n) \leq f(a)+\int_{a}^{a+1} f(t)\,dt+\ldots+\int_{b-1}^{b} f(t)\,dt$$

and so

$$f(b)+\int_{a}^{b} f(t)\,dt \leq \sum_{n=a}^{b} f(n) \leq f(a)+\int_{a}^{b} f(t)\,dt \tag{2.9}$$

Thus, (2.3) follows immediately from (2.7) and (2.9). ■

Summation by parts (also called partial summation or Abel summation) is the analogue for sums of integration by parts. Given a sum of the form $\sum_{n\leq x} a(n)\, f(n)$, where $a(n)$ is an arithmetic function with summatory function $A(x) = \sum_{n\leq x} a(n)$ and $f(n)$ is a "smooth" weight, the summation by parts formula allows one to "remove" the weight $f(n)$ from the above sum and reduce the evaluation or estimation of the sum to that of an integral over $A(t)$. The general formula is as follows:

Theorem 2.4 (Partial Summation, [1],[7]) *Let $f(n)$ and $g(n)$be arithmetic functions. Consider the sum function*

$$F(x) = \sum_{n\leq x} f(n)$$

Let a and b be nonnegative integers with $a < b$. Then

$$\sum_{n=a+1}^{b} f(n)g(n) = F(b)g(b) - F(a)g(a+1) - \sum_{n=a+1}^{b-1} F(n)(g(n+1) - g(n)). \tag{2.10}$$

Let x and y be nonnegative real numbers with $[y] < [x]$, and let $g(t)$ be a function with a continuous derivative on the interval $[y, x]$. Then

$$\sum_{y<n\leq x} f(n)g(n) = F(x)g(x) - F(y)g(y) - \int_{y}^{x} F(t)g'(t)dt. \tag{2.11}$$

In particular, if $x \geq 2$ and $g(t)$ is continuously differentiable on $[1, x]$, then

$$\sum_{n\leq x} f(n)\, g(n) = F(x)g(x) - \int_1^x F(t)g'(t)dt. \tag{2.12}$$

Proof. Identity (2.10) is a straightforward calculation:

$$\begin{aligned}
\sum_{n=a+1}^{b} f(n)g(n) &= \sum_{n=a+1}^{b} (F(n) - F(n-1))g(n) \\
&= \sum_{n=a+1}^{b} F(n)g(n) - \sum_{n=a}^{b-1} F(n)g(n+1) \\
&= F(b)g(b) - F(a)g(a+1) - \sum_{n=a+1}^{b-1} F(n)(g(n+1) - g(n)).
\end{aligned}$$

If the function $g(t)$ is continuously differentiable on $[y, x]$, then

$$g(n+1) - g(n) = \int_n^{n+1} g'(t)dt.$$

Since $F(t) = F(n)$ for $n \leq t < n+1$, it follows that

$$F(n)(g(n+1) - g(n)) = \int_n^{n+1} F(t)g'(t)dt.$$

Let $a = [y]$ and $b = [x]$. Since $a \leq y < a+1 \leq b \leq x < b+1$, we have

$$\begin{aligned}
\sum_{y<n\leq x} f(n)g(n) &= \sum_{n=a+1}^{b} f(n)g(n) \\
&= F(b)g(b) - F(a)g(a+1) - \sum_{n=a+1}^{b-1} F(n)(g(n+1) - g(n)) \\
&= F(x)g(b) - F(y)g(a+1) - \sum_{n=a+1}^{b-1} \int_n^{n+1} F(t)g'(t)dt \\
&= F(x)g(x) - F(y)g(y) - F(x)(g(x) - g(b)) - F(y)(g(a+1) - g(y)) \\
&\quad - \int_{a+1}^{b} F(t)g'(t)dt \\
&= F(x)g(x) - F(y)g(y) - \int_y^x F(t)g'(t)dt.
\end{aligned}$$

This proves (2.11).

Finally, if $x \geq 2$ and $g(t)$ is continuously differentiable on $[1, x]$, then

$$\begin{aligned}\sum_{n\leq x} f(n)g(n) &= f(1)g(1) + \sum_{1<n\leq x} f(n)g(n) \\ &= f(1)g(1) + F(x)g(x) - F(1)g(1) - \int_1^x F(t)g'(t)dt \\ &= F(x)g(x) - \int_1^x F(t)g'(t)dt.\end{aligned}$$

This proves (2.12). ■

2.1 Definitions and notations

First, we focus on some notations and their explanation (for details one can see [1],[4],[6],[7]):

Definition 2.1 *Suppose that $f(x)$ and $g(x)$ are two real-valued functions. Then*

1. *$f(x) = O(g(x))$ (read $f(x)$ is big O of $g(x)$) or $f(x) \ll g(x)$ if there exists a constant A independent of x and an x_0 such that*

$$f(x) \leq A \cdot g(x) \text{ for all } x \geq x_0.$$

or

$$|f(x)| \leq A \cdot g(x) \text{ for all } x \geq x_0.$$

2. *$f(x) = o(g(x))$ or $f(x) \asymp g(x)$ (read $f(x)$ is little o of $g(x)$) if*

$$\frac{f(x)}{g(x)} \to 0 \text{ as } x \to \infty.$$

In other words $g(x)$ is of a higher order of magnitude than $f(x)$.

3. *If $f(x) = O(g(x))$ and $g(x) = O(f(x))$, that is, there exist constants A_1, A_2 independent of x and an x_0 such that*

$$A_1 \cdot g(x) \leq f(x) \leq A_2 \cdot g(x) \text{ for all } x \geq x_0,$$

then we say that $f(x)$ and $g(x)$ are the same order of magnitude and write

$$f(x) = \Theta g(x) \text{ or } f(x) \approxeq g(x).$$

In addition, we say that g is a normal order of f if for every $\varepsilon > 0$, the inequalities

$$(1-\varepsilon) g(x) \leq f(x) \leq (1+\varepsilon) g(x)$$

hold for almost all n. That is, if the proportion of $n \leq x$ for which this does not hold tends to 0 as x tends to infinity.

4. *If*

$$\frac{f(x)}{g(x)} \to 1 \text{ as } x \to \infty$$

then we say that $f(x)$ and $g(x)$ are ***asymptotically equal*** *and we write*

$$f(x) \sim g(x).$$

Definition 2.2 *$f(n) = O(g(n))$ if there exist positive constants c and N such that $f(n) \leq cg(n)$ for all $n \geq N$.*

Example 2.1 *For $x \in \mathbb{R}$, we have $[x] \sim x$, $\sin x \ll x$, $\sin x = O(1)$, $2 + \sin x \asymp 1$, $\sqrt{x} = o(x)$, $x^k = o(e^x)$ for every constant k and $\log^k x = o(x^\alpha)$ for every pair of constants k and $\alpha > 0$.*

2.2 Some properties of big-O notations

Properties of Big-O[1] Notation (see [1],[4],[6],[7],[2])

We can easily prove the following facts: $x \in O(x)$, $3x \in O(x)$, $x \in O(x^2)$, $10x + 5 \in O(x^2)$ and $O(x) \subset O(x^2)$.

Transitivity. If $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $f(n) = O(h(n))$. If $f(n) = O(h(n))$ and $g(n) = O(h(n))$, then

$f(n) + g(n) = O(h(n))$.

- $a \cdot n^k = O\left(n^k\right)$.
- The function $n^k = O(n^{k+j})$ for any positive j.
- $2n^2 + O(n) = O(n^2)$.
- Every polynomial is big-O of n raised to the largest power: $2n^3 + 7n^2 + 1 = O(n^3)$.
- If $f(n) = cg(n)$, then $f(n) = O(g(n))$.
- $\log_a n = O(\log_b n)$ for ever positive numbers $a, b \neq 1$.
- $\log_a n = O(\log_2 n)$ for any positive $a \neq 1$.
- 10 is $O(1)$ and 2023 is $O(1)$, and so on.
- $5000000n \in O(n)$ and $0.000005n \in O(n)$.
- If $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$, then $f_1(n) + f_2(n) = \max(O(g_1(n)), O(g_2(n)))$ and $f_1(n)f_2(n) = O(g_1(n))\ O(g_2(n))$.

[1]Big-O expresses an upper bound on the growth rate of a function, for sufficiently large values of n.

- $f(n) = 2^n$ and $g(n) = 3^n$. Then $f(n) = O(g(n))$.

- $f(n) = \log\log n$ and $g(n) = \log n$. Then $f(n) = O(g(n))$.

- If f is $O(g)$, the $f + g$ is $O(g)$. If $f_1, f_2, ..., f_k$ are each $O(g)$, then $f_1 + f_2 + ... + f_k$ is $O(g)$.

- $2n^2+3n+1 = 2n^2+O(n)$ means that there exists a function $f(n) \in O(n)$ such that $2n^2 + 3n + 1 = 2n^2 + f(n)$.

- If $h \in O(g)$ and $g \in O(n^2)$, then $h \in O(n^2)$.

- If $f_1 \in O(g_1)$ and $f_2 \in O(g_2)$, then $f_1 + f_2 \in O(\max\{g_1, g_2\})$.

- $[x] = x + O(1)$.

- $f(x) = O(1)$. This simply means that $f(x)$ is bounded for sufficiently large x (or for all x in a given range). Similarly $f(x) = o(1)$ means that $f(x)$ tends to 0 as $x \to \infty$.

- If $f(x) = g(x) + O(1)$, then $e^{f(x)} \asymp e^{g(x)}$, and vice versa.

- If $f(x) = g(x) + o(1)$, then $e^{f(x)} \sim e^{g(x)}$, and vice versa.

- $\dfrac{1}{n+1}$ is a normal order of $\dfrac{1}{n}$, n is a normal order of $n + 1$ and $\log(n)$ is a normal order of $\log(n + 1)$.

Definition 2.3 *$f(n) = \Omega(g(n))$ if there exist positive constants c and N such that $f(n) \geq cg(n)$ for all $n \geq N$.*

- Note the equivalence $f(n) = \Omega(g(n))$ if and only if $g(n) = O(f(n))$.

Definition 2.4 *$f(n) = \Theta(g(n))$ if there exist positive constants c_1, c_2 and N such that $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq N$.*

Note that $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$. In general, big-O includes the following terms:

$O(1)$	constant
$O(\log n)$	logarithmic
$O((\log n)^c)$	polylogarithmic
$O(n)$	linear
$O(n^2)$	quadratic
$O(n^c)$	polynomial
$O(c^n)$	exponential

We need to use the following lemma:

Lemma 2.1 *We have*

1. $\sum_{d \leq x} 1 = x + O(1)$.
2. $\sum_{p \leq x} O(1) = O(x)$. *In particular,* $O(1) + ... + O(1)_{n\text{-}times} = O(n)$.

Proof. 1. By definition, we have $x = [x] + \{x\}$. It follows that

$$\sum_{d:\ d \leq x} 1 = [x] = x - \{x\} = x + O(1),$$

since $0 \leq \{x\} < 1$.

2. Also if we put $f = O(1)$ (this means that f is bounded), then

$$\sum_{p \leq x} O(1) = [x] f = O(x),$$

since $\left| \frac{[x]f}{x} \right| \leq |f|$. ■

2.3 Applying Euler's summation and Partial summation formulas

Euler's summation formula has numerous applications[2] in number theory and analysis. We will give here three such applications; the first is to the partial sums of the harmonic series. See the references [4], [7], [5].

Theorem 2.5 (Partial sums of the harmonic series, [1]) *For every $x \geq 1$, we have*

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right),$$

where γ is the Euler's constant[3].

[2]An important application of Euler's summation formula is a proof of the socalled Stirling formula, which gives an asymptotic estimate for $n!$. This formula will be an easy consequence of the following estimate for the logarithm of $n!$, $\log n! = \sum\limits_{m \leq n} \log m$, which is a sum to which Euler's summation formula can be applied.

[3]The number $\gamma = 0.577...$ is called Euler's constant. A famous unsolved problem in number theory is to determine whether γ is rational or irrational.

Proof. We take $f(t) = \dfrac{1}{t}$ in Euler's summation formula, which has a continuous derivative on the interval $[y, x]$ with $0 < y < x$, to get

$$\begin{aligned}
\sum_{n\leq x} f(n) &= \int_y^x f(t)\,dt + \int_y^x (t-[t])\,f'(t)\,dt + f(x)([x]-x) - f(y)([y]-y) \\
&= \int_y^x \frac{dt}{t} - \int_y^x \frac{t-[t]}{t^2}dt + \frac{[x]-x}{x} - \left(\frac{[y]-y}{y}\right) \\
&= \int_y^1 \frac{dt}{t} + \int_1^x \frac{dt}{t} - \int_1^y \frac{t-[t]}{t^2}dt - \int_1^x \frac{t-[t]}{t^2}dt + 1 - \frac{x-[x]}{x} \\
&= \int_1^x \frac{dt}{t} - \int_1^x \frac{t-[t]}{t^2}dt + 1 - \frac{x-[x]}{x} \\
&= \log x - \int_1^x \frac{t-[t]}{t^2}dt + 1 + O\left(\frac{1}{x}\right) \\
&= \log x + 1 - \int_1^\infty \frac{t-[t]}{t^2}dt + \int_x^\infty \frac{t-[t]}{t^2}dt + O\left(\frac{1}{x}\right)
\end{aligned}$$

The improper integral $\int_1^\infty \frac{t-[t]}{t^2}dt$ exists since it is dominated by $\int_1^\infty \frac{dt}{t^2}dt$. On the other hand, we see that

$$0 \leq \int_x^\infty \frac{t-[t]}{t^2}dt \leq \int_x^\infty \frac{dt}{t} = \frac{1}{x}.$$

It follows that

$$\sum_{n\leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right),$$

where

$$\gamma = 1 - \int_1^\infty \frac{t-[t]}{t^2}dt.$$

Setting x tends to infinity, we get

$$\gamma = \lim_{x\to\infty}\left(\sum_{n\leq x}\frac{1}{n} - \log x\right),$$

so γ is also equal to Euler's constant. ■

In view of (1.10), which gives the definition of Riemann zeta function. As an application of Euler's summation formula, we now derive an integral representation for this function. This representation will be crucial in deriving deeper analytic properties of the zeta function.

Theorem 2.6 ([1]) *If $x \geq 1$, then*

a) $\sum_{n\leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-s})$, *for $s > 0$ with $s \neq 1$, where*

$$\zeta(s) = \begin{cases} \sum_{n=1}^{\infty} \frac{1}{n^s}, \text{ if } s > 1 \\ \lim\limits_{x\to\infty} \left(\sum_{n\leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} \right), \text{ if } 0 < s < 1. \end{cases}$$

b) $\sum_{n>x} \frac{1}{n^s} = O(x^{1-s})$, *for $s > 1$.*

c) $\sum_{n\leq x} n^{\alpha} = \frac{x^{\alpha+1}}{\alpha+1} + O(x^{\alpha})$, *for $\alpha \geq 0$.*

Proof. We prove this theorem as follows:

a) We apply the Euler's summation formula with $f(x) = x^{-s}$:

$$\begin{aligned} \sum_{n\leq x} \frac{1}{n^s} &= \int_1^x \frac{1}{t^s} dt - s \int_1^x \frac{t-[t]}{t^{s+1}} dt + 1 - \frac{x-[x]}{x^s} \\ &= \frac{x^{1-s}}{1-s} - \frac{1}{1-s} + 1 - s \int_1^{\infty} \frac{t-[t]}{t^{s+1}} dt + O(x^{-s}). \end{aligned}$$

Therefore,

$$\sum_{n\leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \gamma(s) + O(x^{-s}), \tag{2.13}$$

where

$$\gamma(s) = 1 - \frac{1}{1-s} - s \int_1^{\infty} \frac{t-[t]}{t^{s+1}} dt.$$

Now we have to divide into two cases:

If $s > 1$, $\sum_{n\leq x}\frac{1}{n^s}$ approaches $\zeta(s)$ as $x\to\infty$ and the term x^{1-s} and x^{-s} both approach 0. From the definition of $\zeta(s)$ and the fact that $\gamma(s)$ does not depend on x, by making x tend to infinity in (2.13), we obtain that $\gamma(s)=\zeta(s)$ if $s>1$.

If instead $0 < s < 1$ and as above taking x tend to infinity in (2.13), we have that $x^{-s}\to 0$. By the fact that $\gamma(s)$ doesn't depend on x we can see that

$$\lim_{x\to\infty}\left(\sum_{n\leq x}\frac{1}{n^s}-\frac{x^{1-s}}{1-s}\right)=\gamma(s).$$

Therefore, by definition, $\gamma(s)$ is also equal to $\zeta(s)$ if $0<s<1$.

b) To prove (b) with $s\geq 1$ we use (a). In fact, we see that

$$\sum_{n>x}\frac{1}{n^s}=\zeta(s)-\sum_{n\leq x}\frac{1}{n^s}=\frac{x^{1-s}}{1-s}+O(x^{-s})=O(x^{1-s}),$$

since $x^{-s}\leq x^{1-s}$.

c) We use Euler's summation formula once more with $f(t)=t^{\alpha}$, we obtain

$$\begin{aligned}\sum_{n\leq x}n^{\alpha} &= \int_1^x t^{\alpha}dt+\alpha\int_1^x t^{\alpha-1}\left(t-[t]\right)dt+1-\left(x-[x]\right)x^{\alpha}\\ &= \frac{x^{\alpha+1}}{\alpha+1}-\frac{1}{\alpha+1}+O\left(\alpha\int_1^x t^{\alpha-1}dt\right)+O\left(x^{\alpha}\right)\\ &= \frac{x^{\alpha+1}}{\alpha+1}+O\left(x^{\alpha}\right).\end{aligned}$$

This completes the proof. ■

Next, we apply Theorem 2.3. In fact, we can prove that:

Proposition 2.1 ([7]) *For $x\geq 2$, we have*

$$\sum_{n\leq x}\log n = x\log x - x + O(\log x).$$

Proof. The function $f(t) = \log t$ is increasing on $[1, x]$. By Theorem 2.3, we get

$$\int_1^x \log t dt \leq \sum_{n\leq x} \log n \leq \int_1^x \log t dt + \log x,$$

and so

$$\sum_{n\leq x} \log n = x \log x - x + O(\log x).$$

This completes the proof. ■

As an application of Abel's summation formula, we have

Theorem 2.7 ([1]) *We have*

$$\sum_{n\leq x} \frac{\Lambda(n)}{n} = \log x + O(1).$$

Proof. Apply Abel's summation formula with $a_n = 1$ and $f(n) = \log n$. Then

$$\begin{aligned}
\sum_{n\leq x} \log n &= \lfloor x \rfloor \log x - \int_1^x \frac{\lfloor u \rfloor}{u} du \\
&= (x - (x - \lfloor x \rfloor)) \log x - \int_1^\infty \frac{u - (u - \lfloor u \rfloor)}{u} du \\
&= x \log x + O(\log x) - (x - 1) + \int_1^x \frac{u - \lfloor u \rfloor}{u} du) \\
&= x \log x - x + O(\log x). \qquad (2.14)
\end{aligned}$$

Also we have

$$\begin{aligned}\sum_{n\leq x}\sum_{n\leq x}\log n &= \log(\lfloor x\rfloor !) = \sum_{p\leq x}\left(\sum_{k=1}^{\infty}\left\lfloor\frac{x}{p^k}\right\rfloor\right)\log p\\ &= \sum_{p^m\leq x}\left\lfloor\frac{x}{p^m}\right\rfloor\log p = \sum_{n\leq x}\left\lfloor\frac{x}{n}\right\rfloor\Lambda(n)\\ &= \sum_{n\leq x}\frac{x}{n}\Lambda(n) - \sum_{n\leq x}\left(\frac{x}{n}-\left\lfloor\frac{x}{n}\right\rfloor\right)\Lambda(n)\\ &= x\sum_{n\leq x}\frac{\Lambda(n)}{n} - O\left(\sum_{n\leq x}\Lambda(n)\right)\end{aligned}$$

But $\sum_{n\leq x}\Lambda(n) = \Psi(x) = O(x)$ and so

$$\sum_{n\leq x}\log n = x\sum_{n\leq x}\Lambda(n) - O(x).$$

By (2.14),

$$x\log x - x + O(\log x) = x\sum_{n\leq x}\Lambda(n) - O(x),$$

hence

$$x\sum_{n\leq x}\frac{\Lambda(n)}{n} = x\log x + O(x).$$

Thus,

$$\sum_{n\leq x}\frac{\Lambda(n)}{n} = \log x + O(1).$$

■

Chapter 3

Average values of arithmetic functions

In this chapter we focus on Mean values of Multiplicative and Additive Arithmetic Functions. Let us start with the following definition:

Definition 3.1 ([1]) *Let f be an arithmetic function. Then the mean value (or the average value) of f over the interval $[1, x]$ is defined to be*

$$g(x) = \frac{1}{x}\sum_{n\leq x} f(n).$$

If $\lim_{x\to\infty} g(x)$ exists, then the limit is called the asymptotic mean of f. In addition, if g is a monotone function such that

$$g(x) \sim \frac{1}{x}\sum_{n\leq x} f(n).$$

Here, we say that $g(n)$ is an average order of $f(n)$.

In other words, let f be an arithmetic function and let $g(x)$ be a monotonic increasing function of x. We say that $g(n)$ is the average order of $f(n)$ if

$$\sum_{n\leq x} f(n) = xg\left(x\right) + o\left(xg\left(x\right)\right).$$

Note that in Chapter 2 (see Theorem 2.5) we showed, by application of partial summation that the average order of $\frac{1}{n}$ is $\frac{\log n}{n}$. In this section we will give the true order of magnitude of τ, φ and σ. We need to present the following lemma:

Lemma 3.1 (see [6]) *Let $f(n)$ be an arithmetic function and*

$$F(x) = \sum_{n \leq x} f(n).$$

Then

$$\sum_{m \leq x} F\left(\frac{x}{m}\right) = \sum_{d \leq x} f(d) \left[\frac{x}{d}\right] = \sum_{n \leq x} \sum_{d|n} f(d).$$

Proof. We see that

$$\begin{aligned} \sum_{m \leq x} F\left(\frac{x}{m}\right) &= \sum_{m \leq x} \sum_{d \leq \frac{x}{m}} f(d) = \sum_{dm \leq x} f(d) \\ &= \sum_{d \leq x} f(d) \sum_{m \leq \frac{x}{d}} 1 = \sum_{d \leq x} f(d) \left[\frac{x}{d}\right] \\ &= \sum_{n \leq x} \sum_{d|n} f(d). \end{aligned}$$

Thus, we have

$$\sum_{m \leq x} F\left(\frac{x}{m}\right) = \sum_{dm \leq x} f(d) = \sum_{n \leq x} \sum_{d|n} f(d).$$

■

3.1 Average order of $d(n)$

Theorem 3.1 ([7]) *Let $d(n)$ be the divisor function. We have:*

(a) The relation $d(n) \ll \log^c n$ is false for every constant c.

(b) The relation $d(n) \ll n^{\delta}$ is true for every fixed $\delta > 0$.

Proof. We prove (a) Let n be any of the numbers $(2 \cdot 3 ... p_r)^m$, $m = 1, 2, ...$; here r is arbitrary but fixed. Then

$$d(n) = \prod_{p|n}(m+1) = (m+1)^r > m^r.$$

But $m = \log n / \log(2 \cdot 3 ... p_r)$, so that

$$d(n) > \frac{\log^r n}{(\log(2 \cdot 3 ... p_r))^r} \gg \log^r n,$$

where the implied constant depends only on r, and not on n.

For the proof of (b), let

$$f(n) = \frac{d(n)}{n^{\delta}}.$$

We see that f is multiplicative. But $f(p^m) = (m+1)/p^{m\delta}$, so that $f(p^m) \to 0$ as $p^m \to \infty$, that is, as either p or m, or both, increases. This clearly implies that $f(n) \to 0$ as $n \to \infty$, which proves the assertion. ■

Theorem 3.2 ([7]) *We have*

$$\sum_{n \leq x} d(n) = x \log x + O(x).$$

Proof. By definition, we get

$$\begin{aligned}
\sum_{n \leq x} d(n) &= \sum_{n \leq x} \sum_{d|n} 1 = \sum_{d \leq x} \sum_{e:\, de \leq x} 1 = \sum_{d \leq x} \sum_{e:\, e \leq \frac{x}{d}} 1 = \sum_{d \leq x} \left[\frac{x}{d}\right] \\
&= \sum_{d \leq x} \left(\frac{x}{d} + O(1)\right) = x \sum_{d \leq x} \frac{1}{d} + O(x) \\
&= x(\log x + O(1)) + O(x) \\
&= x \log x + O(x).
\end{aligned}$$

This means that

$$\frac{1}{x}\sum_{n\leq x} d\left(n\right) = \log x + O\left(1\right) \sim \log x,$$

as x tends to infinity. Thus, the average order of $d\left(n\right)$ is $\log n$. ■

Remark 3.1 *The average order of the number of divisors of natural numbers grows like* $\log n$*. That is,*

$$\frac{d(1)+d(2)+...+d(n)}{n} \sim \log n.$$

In fact, let k *be a fixed integer. If we list the multiples of* k *less than or equal to* n*:*

$$k, 2k, 3k, ..., \left[\frac{n}{k}\right] k,$$

we find that there are $\left[\frac{n}{k}\right]$ *multiples, where* $[t]$ *denotes the floor function. Each of those multiples contributes* 1 *to the sum* $d(1)+...+d(n)$*. If we examine multiples of all integers* $k \leq n$*, it follows that summing over* k *gives*

$$\sum_{k=1}^{n}\left[\frac{n}{k}\right] = d(1)+...+d(n)$$

Now, we want to prove that

$$\lim_{n\longrightarrow\infty}\frac{\sum_{k=1}^{n}\left[\frac{n}{k}\right]}{n\log n} = 1$$

First, we establish the relationship:

$$\frac{n}{k} - 1 < \left[\frac{n}{k}\right] \leq \frac{n}{k}.$$

Summing over k *gives:*

$$\sum_{k=1}^{n}\left(\frac{n}{k}-1\right) < \sum_{k=1}^{n}\left[\frac{n}{k}\right] \leq \sum_{k=1}^{n}\frac{n}{k}.$$

We then factor out n *to get:*

$$n\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{n}\right)<\sum_{k=1}^{n}\left[\frac{n}{k}\right]\leq n\sum_{k=1}^{n}\frac{1}{k}$$

The first and the last term in the above inequality can be rewritten as the integrals

$$n\int_{1}^{n}\left(\frac{1}{t}-\frac{1}{n}\right)dt \quad \text{and } n\int_{1}^{n}\frac{1}{t}dt.$$

Integrating gives

$$n\log n-n+1<\sum_{k=1}^{n}\left[\frac{n}{k}\right]\leq n\log n.$$

So taking $n\rightarrow\infty$, *we have*

$$\lim_{n\rightarrow\infty}\frac{\sum_{k=1}^{n}\left[\frac{n}{k}\right]}{n\log n}=1.$$

and so,

$$\frac{d(1)+d(2)+...+d(n)}{n}\sim\log n.$$

3.2 Average order of $\sigma(n)$

We have:

Theorem 3.3 ([1]) *For every* $x\geq 1$, *we have*

$$\begin{aligned}\sum_{n\leq x}\sigma_1(n) &= \frac{1}{2}\zeta(2)x^2+O\left(x\log(x)\right)\\ &= \frac{\pi^2}{12}x^2+O\left(x\log(x)\right)\end{aligned}$$

Proof.

$$\sum_{n\leq x}\sigma_1(n)=\sum_{n\leq x}\sum_{q|n}q=\sum_{\substack{q,d\\ qd\leq x}}q=\sum_{d\leq x}\sum_{q\leq\frac{x}{d}}q,$$

By (c) of Theorem 2.6, we have

$$\begin{aligned}\sum_{n\leq x}\sigma_1(n) &= \sum_{d\leq x}\left(\frac{1}{2}\left(\frac{x}{d}\right)^2+O\left(\frac{x}{d}\right)\right)\\ &= \frac{x^2}{2}\sum_{d\leq x}\frac{1}{d^2}+O\left(x\sum_{d\leq x}\frac{1}{d}\right),\end{aligned}$$

By Theorem 2.5 and (a) of Theorem 2.6, we get

$$\begin{aligned}&= \frac{x^2}{2}\left(-\frac{1}{x}+\zeta(2)+O\left(\frac{1}{x^2}\right)\right)+O\left(x\log\left(x\right)\right)\\ &= \frac{1}{2}\zeta(2)x^2+O\left(x\log\left(x\right)\right),\end{aligned}$$

Note that $\sum_{d\geq 1}\frac{1}{d^2}=\zeta\left(2\right)=\frac{\pi^2}{6}$. This completes the proof. ∎

3.3 Average order of $\varphi\left(n\right)$

Applying Theorems 1.3, 1.4 and 2.6 we calculate the average order of $\varphi\left(n\right)$.

Theorem 3.4 ([1],[7]) *For $x>1$ we have*

$$\sum_{n\leq x}\varphi\left(n\right)=\frac{3}{\pi^2}x^2+O\left(x\log x\right).$$

That is, the average order of $\varphi\left(n\right)$ is $\frac{3n}{\pi^2}$.

Proof. The method is similar to that used for the divisor function. At first, we have

$$\begin{aligned}
\sum_{n\leq x}\varphi(n) &= \sum_{n\leq x}\sum_{d|n}\mu(d)\frac{n}{d}=\sum_{\substack{e,d\\ ed\leq x}}\mu(d)\,e=\sum_{d\leq x}\sum_{e:\ ed\leq x}\mu(d)\,e=\sum_{d\leq x}\mu(d)\sum_{e:\ e\leq\frac{x}{d}}e\\
&= \sum_{d\leq x}\mu(d)\left(\frac{\left[\frac{x}{d}\right]\left(\left[\frac{x}{d}\right]+1\right)}{2}\right)\\
&= \sum_{d\leq x}\mu(d)\left(\frac{1}{2}\left(\frac{x}{d}\right)^2+O\left(\frac{x}{d}\right)\right)\\
&= \frac{x^2}{2}\sum_{d\leq x}\frac{\mu(d)}{d^2}+O\left(x\sum_{d\leq x}\frac{1}{d}\right)\\
&= \frac{x^2}{2}\left\{\frac{6}{\pi^2}+O\left(\frac{1}{x}\right)\right\}+O(x\log x)\\
&= \frac{3}{\pi^2}x^2+O(x\log x).
\end{aligned}$$

The proof is finished. ■

3.4 Average orders of $\omega(n)$

Based on the following theorem, we present the average order of $\omega(n)$. We will use the result:

Theorem 3.5 ([1]) *We have*

$$\sum_{p\leq x}\frac{1}{p}=\log(\log x)+O(1).$$

In addition, by Theorem 2.2, we have

$$\sum_{n\leq x}\log n=\int_1^x(\log u+\Psi_1(u)\frac{1}{u})du+O(\log x).$$

Since $|\Psi_1(u)| \leq 1$, we have

$$\int_1^x \Psi_1(u)\frac{1}{u}du \ll \int_1^x \frac{1}{u}du = O(\log x).$$

Note that

$$\int_1^x \log u \ du = [(u\log u - u)]_1^x = x\log x - x + 1.$$

Now, we have:

Theorem 3.6 ([7]) *We have*

$$\sum_{n\leq x} \omega(n) = x\log(\log x) + O(x).$$

Proof. We can write

$$\begin{aligned}\sum_{n\leq x}\omega(n) &= \sum_{n\leq x}\sum_{p:\ p|n} 1 = \sum_{p\leq x}\sum_{\substack{n\leq x \\ p|n}} 1 = \sum_{p\leq x}\sum_{e:\ pe\leq x} 1 = \sum_{p\leq x}\sum_{e:\ e\leq \frac{x}{p}} 1 \\ &= \sum_{p\leq x}\left(\frac{x}{p} + O(1)\right) = x\sum_{p\leq x}\frac{1}{p} + O(x),\end{aligned}$$

since $\sum\limits_{p\leq x} O(1) = O(x)$. By Theorem 3.5, we obtain

$$\sum_{n\leq x}\omega(n) = x\log(\log x) + \gamma x + O\left(\frac{x}{\log x}\right),$$

where γ is the euler's constant. Thus, the average order of $\omega(n)$ is $\log(\log n)$.

■

In view of [7, p. 283], applying Chebyshev's theorem and Mertens's theorem, we state the following two results:

Theorem 3.7 ([7]) *For every $x \geq 2$,*

$$\sum_{n \leq x} \omega(n) = x \log(\log x) + Cx + O\left(\frac{x}{\log x}\right),$$

where C is a positive real number.

Similarly, we have

Theorem 3.8 ([7]) *For every $x \geq 2$,*

$$\sum_{n \leq x} \omega^2(n) = x(\log(\log x))^2 + O(x \log(\log x)).$$

3.5 Average orders of some other arithmetic functions

By some summation techniques we can verify the following results (see [1],[4],[6],[7]):

Note that by Proposition 2.1, we have:

$$\sum_{n \leq x} \log n = x \log x - x + O(1). \tag{3.1}$$

Theorem 3.9 (see [7]) *We have*

$$\sum_{p \leq x} \log p \left[\frac{x}{p}\right] = x \log x + O(x).$$

Proof. As before, by Lemma 1.1, we see that

$$\sum_{n \leq x} \log n = \sum_{n \leq x} \sum_{d|n} \Lambda(d) = \sum_{d \leq x} \Lambda(d) \sum_{e:\ pe \leq x} 1 = \sum_{d \leq x} \Lambda(d) \left[\frac{x}{d}\right]$$

Thus,

$$\sum_{d \leq x} \Lambda(d) \left[\frac{x}{d}\right] = x \log x - x + O(1),$$

where

$$\begin{aligned}
\sum_{d\leq x}\Lambda(d)\left[\frac{x}{d}\right] &= \sum_{p\leq x}\log p\left[\frac{x}{p}\right]+\sum_{k\geq 2}\sum_{p:\ p^k\leq x}\log p\left[\frac{x}{p^k}\right] \\
&\leq \sum_{p\leq x}\log p\left[\frac{x}{p}\right]+\sum_{k\geq 2}\sum_{p:\ p^k\leq x}\frac{x\log p}{p^k} \\
&\leq \sum_{p\leq x}\log p\left[\frac{x}{p}\right]+x\sum_{p}\log p\left(\sum_{k=2}^{\infty}\frac{1}{p^k}\right) \\
&= \sum_{p\leq x}\log p\left[\frac{x}{p}\right]+x\sum_{p}\frac{\log p}{p(p-1)} \\
&\leq \sum_{p\leq x}\log p\left[\frac{x}{p}\right]+x\sum_{n=1}^{\infty}\frac{\log n}{n(n-1)} \ll \sum_{p\leq x}\log p\left[\frac{x}{p}\right]+x\sum_{n=1}^{\infty}\frac{\log n}{n^2} \\
&\ll \sum_{p\leq x}\log p\left[\frac{x}{p}\right]+x\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}} \\
&\ll \sum_{p\leq x}\log p\left[\frac{x}{p}\right]+x.
\end{aligned}$$

Thus,

$$x\log x-x+O(1)=\sum_{p\leq x}\log p\left[\frac{x}{p}\right]+O(x)$$

and hence

$$\sum_{p\leq x}\log p\left[\frac{x}{p}\right]=x\log x+O(x).$$

■

As an application of Lemma 3.1, we have

Theorem 3.10 ([7]) *For $x\geq 2$, we have*

$$\sum_{m\leq x}\psi\left(\frac{x}{m}\right)=\sum_{d\leq x}\Lambda(d)\left[\frac{x}{d}\right]=x\log x-x+O(\log x).$$

Proof. Let $f(n) = \Lambda(n)$ in Lemma 3.1, we have

$$F(x) = \sum_{n \le x} \Lambda(n) = \psi(x),$$

and so

$$\begin{aligned}
\sum_{m \le x} \psi\left(\frac{x}{m}\right) &= \sum_{d \le x} \Lambda(d) \left[\frac{x}{d}\right] \\
&= \sum_{n \le x} \sum_{d|x} \Lambda(d) \\
&= \sum_{n \le x} \log n \\
&= x \log x - x + O(\log x).
\end{aligned}$$

The last identity comes from (3.1). ■

3.6 Series of reciprocals of the primes

Let us use the following lemma [4].

Lemma 3.2 (Chebyshev's estimate) $\frac{c_1 x}{\log x} \le \pi(x) \le \frac{c_2 x}{\log x}$, *for all* $x \ge 2$ *(*c_1, c_2 *are constants).*

Theorem 3.11 (see [4]) *There exists positive constants* $B_{1,}$ B_2 *such that*

$$B_1 n \log n \le p_n \le B_2 n \log n.$$

Equivalently, $p_n \asymp n \log n$.

Proof. Let p_n be the n-th prime. Then clearly $\pi(p_n) = n$. From Chebyshev's estimate

$$n = \pi(p_n) \le A_2 \frac{p_n}{\log p_n}, \text{ for all } n \ge 2.$$

This implies

$$\frac{1}{A_2} n \log p_n \leq p_n, \text{ for all } n \geq 2.$$

However, $p_n > n$ and so

$$\frac{1}{A_2} n \log n < \frac{1}{A_2} n \log p_n \leq p_n, \text{ for all } n \geq 2.$$

Therefore, in general we write

$$B_1 n \log p_n \leq p_n$$

for all $n \geq 2$ with $B_1 = 1/A_2$. In the other direction, we have

$$n = \pi(p_n) \geq A_1 \frac{p_n}{\log p_n}.$$

Since $p_n \geq n$ it follows that $\frac{\log p_n}{\sqrt{p_n}} \to 0$ as $n \to \infty$. Therefore, there exists a constant k such that

$$\frac{\log p_n}{\sqrt{p_n}} < A_1 \text{ if } n > k.$$

Hence

$$n \frac{\log p_n}{p_n} \geq A_1 > \frac{\log p_n}{\sqrt{p_n}} \text{ if } n > k.$$

It follows that $n > \sqrt{p_n}$ and so $\log p_n < 2 \log n$ if $n > k$. Let

$$B_2 = \max\{\frac{2}{A_1}, \frac{p_2}{2\log 2}, \frac{p_3}{3\log 3}, ..., \frac{p_{k-1}}{(k-1)\log(k-1)}\}.$$

Then

$$p_n \leq B_2 n \log n \text{ for all } n \geq 2.$$

The proof of Theorem 3.11 is finished. ■

The above result also provides a very simple proof of Euler's Theorem which state that the series $\sum_p \frac{1}{p}$ diverges.

Corollary 3.1 ([2]) *The sum*

$$\sum_p \frac{1}{p}$$

diverges.

Proof. For $n \geq 2$ we have $\frac{1}{p_n} \leq \frac{1}{B_1 n \log n}$ from the last theorem. However the series $\sum_{n=1}^{\infty} \frac{1}{n \log n}$diverges by the integral test. ■

Althought there are infinitely many primes and $\sum_p \frac{1}{p}$ diverges it still diverges very slowly. Using the methods applied in the proof of Chebychev's estimate we can actually bound the growth of the series of reciprocals of the primes.

Theorem 3.12 (see [4]) *There exists a constant k such that*

$$\sum_{2<p\leq x} \frac{1}{p} < k \log\log x \ \textit{if} \ \ x > 3.$$

Proof. From Theorem 3.11, we have $p_n \geq B_1 n \log n$. Therefore,

$$\sum_{2<p\leq x} \frac{1}{p} = \sum_{n=2}^{\pi(x)} \frac{1}{p_n} < \sum_{n=2}^{\pi(x)} \frac{1}{B_1 n \log n} < \frac{1}{B_1} \sum_{n=2}^{[x]} \frac{1}{n \log n}.$$

However,

$$\frac{1}{n \log n} = \int_{n-1}^{n} \frac{dt}{n \log n} \leq \int_{n-1}^{n} \frac{dt}{t \log t}$$

since $\frac{1}{n \log n} \leq \frac{1}{t \ln t}$ on $[n-1, n]$ if $n \geq 3$. Then

$$\sum_{2<p\leq x} \frac{1}{p} < \frac{1}{B_1} \sum_{n=2}^{[x]} \frac{1}{n \log n} \leq \frac{1}{2B_1 \log 2} + \frac{1}{B_1} \sum_{n=3}^{[x]} \int_{n-1}^{n} \frac{dt}{t \log t}$$

and so

$$\begin{aligned}\sum_{2<p\leq x} \frac{1}{p} &\leq \frac{1}{2B_1 \log 2} + \frac{1}{B_1} \int_2^x \frac{dt}{t \log t} = \frac{1}{B_1} \log\log x + \frac{1}{2B_1 \log 2} - \frac{1}{B_1} \log\log 2 \\ &= \frac{1}{B_1} \log\log x + C < k \log\log x.\end{aligned}$$

taking k large enough, where $C = \frac{1}{2B_1 \log 2} - \frac{1}{B_1} \log\log 2$. ■

We finish this work by the following important Mertens Theorems [4],[6],[7]:

.

- For $x \geq 1$, we have

$$\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1).$$

- There exists a constant C such that

$$\sum_{p \leq x} \frac{1}{p} = \log\log x + C + O\left(\frac{1}{\log x}\right),$$

  for $x \geq 2$.

- **Mertens's formula**. There exists a constant c such that for $x \geq 2$,

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} = e^c \log x + O(1).$$

Conclusion and Open Problems

We state three famous open questions on the subject. One can see the reference [6].

- Are there infinitely many prime pairs? Find an asymptotic formula for the number of prime pairs $\leq x$. That is, we ask if there exists an increasing function g such that

$$\sum_{\substack{|p-q|=2 \\ p,q\leq x}} 1 \sim g(x)$$

  for all sufficiently large x, where p, q are prime numbers.

- We ask if the number of *perfect numbers*[1] $\leq n$ is $< c \log n$.

- Pillai Conjecture: $\left| \sum_{n\leq x} (-1)^n p_n \right| \sim \frac{p_{[x]}}{2}$. That is, $-2+3-5+7-11+13-...+(-1)^n p_{[x]} \sim \frac{p_{[x]}}{2}$.

[1]We say $n \in \mathbb{N}$ is a **perfect number** if $\sigma(n) = 2n$, which means the number is equal to the sum of its proper divisors. For examples, 6 and 28. It was of great interest of the Greeks to determine all the perfect numbers. It was known as early as Euclid's time that every number of the form $n = 2^{p-1}(2^p - 1)$, in which both p and $2p - 1$ are prime, is perfect.

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