People`s Democratic Republic Algeria
Ministry of Higher Education and Scientific Research

Faculty of Mathematics and Computer Science and Material Sciences

University 8 May 1945 -GUELMA-

Mathematics Department

In view of obtaining :
Academic Master in Mathematics
Option :
Partial Differential Equations and Numerical Analysis

Presented by : MEZARA INES

Entitled

Existence and Stability Results of Fractional Stochastic Differential Equations with Lévy Noise

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June 2023

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Abstract

The aim of this work is to study that the existence of solution of stochastic fractional differential equations with Lévy noise is established by the Picard-Lindelöf successive approximation scheme. The stability of nonlinear stochastic fractional dynamical system with Lévy noise is obtained using Mittag Leffler function. Examples are provided to illustrate the theory.

Résumé

L'objectif de ce mémoire est d'étudier que l'existence de solution d'équations différentielles fractionnaires stochastiques avec le bruit de Lévy est établie par le schéma d'approximation successive Picard-Lindelöf. La stabilité du système dynamique fractionnaire stochastique non linéaire avec bruit Lévy est obtenue en utilisant la fonction Mittag Leffler. Des exemples sont fournis pour illustrer la théorie..

ملخص

الهدف من هذا العمل هو دراسة أن وجود حل للمعادلات التفاضلية الكسرية العشوائية معLévy noise تم إيجاده بواسطة مخطط التقريب المتتالي Picard-Lindelöfيتم الحصول على استقرار النظام الديناميكي الجزئي العشوائي غير الخطي مع Lévy noiseباستخدام وظيفة Mittag Leffler .

يتم توفير أمثلة لتوضيح النظرية.

CHAPTER 1

Introduction and describe problem

1.1 History of Fractional Calculus

Fractional Calculus (FC) is a generalization of classical analysis that deals with integral and derivative operations of non-integer (fractional) orders. The concept of fractional operators was introduced almost simultaneously with the development of classical operators.

In a letter dated 30^{th} September 1695, L'Hopital wrote to Leibniz asking him particular notation he has used in his publication for the n-th derivative of a function $\frac{D^n f(x)}{Dx^n}$,«what would the result be if $n = \frac{1}{2}$» . Leibniz's response «an apparent paradox, from which one day useful consequences will be drawn».That date is regarded as the exact birthday of the fractional calculus. . Following L'Hopital's and Liebniz's first inquisition, fractional calculus was primarily a study reserved for the best mathematical minds in Europe. Euler [31],wrote in 1730: "*When* n *is a positive integer and* p *is a function of* x, $p = p(x)$, *the ratio of* $d^n p$ *to* dx^n *can always be expressed algebraically. But what kind of ratio can then be made if* n *be a fraction?*".

In 1730, Euler mentioned interpolating between integral orders of a derivative. In 1812

Laplace defined a fractional derivative by means of an integral, and in 1819 there appeared the first discussion of a derivative of fractional order in a calculus text written by S. F. Lacroix [31].

During the 19 th century, the theory of fractional calculus was developed primarily in this way, trough insight and genius of great mathematicians. Namely, in 1819 Lacroix , gave the correct answer to the problem raised by Leibnitz and L'Hospital for the first time, claiming that $\frac{d^{1/2}x}{dx^{1/2}} = 2\sqrt{\frac{x}{\pi}}$. In his 700 pages long book on Calculus published in 1819, Lacroix developed the formula for n-th derivative of ,$y = x^m$ with m being a positive integer

$$D_x^n y = \frac{d^n}{d^n x}\left(x^m\right) = \frac{m!}{(m-n)!}x^{m-n},\ m \geq n. \tag{1.1.1}$$

Replacing the factorial symbol by Gamma function (1.1.1), he developed the formula for the fractional derivative of a power function

$$D_x^\alpha x^\beta = \frac{\Gamma\left(\beta+1\right)}{\Gamma\left(\beta-\alpha+1\right)}x^{\beta-\alpha}, \tag{1.1.2}$$

where α and β are fractional numbers and where the gamma function $\Gamma\left(z\right)^1$ is defined for $z > 0$ as:

$$\Gamma\left(z\right) = \int_0^{+\infty} e^{-x}x^{z-1}dx, \tag{1.1.3}$$

In particular, Lacroix calculated

$$D_x^{1/2}x = \frac{\Gamma\left(2\right)}{\Gamma\left(3/2\right)}x^{\frac{1}{2}} = 2\sqrt{\frac{x}{\pi}}. \tag{1.1.4}$$

Surprisingly, the previous definition gives a nonzero value for the fractional derivative of a constant function($\beta = 0$), since

$$D_x^\alpha 1 = D_x^\alpha x^0 = \frac{1}{\Gamma\left(1-\alpha\right)}x^{-\alpha} \neq 0. \tag{1.1.5}$$

Using linearity of fractional derivatives, the method of Lacroix is applicable to any analytic function by term-vise differentiation of its power series expansion. Unfortunately, this class of functions is too narrow in order for the method to be considered general.

It is interesting to note that simultaneously with these initial theoretical developments, first practical applications of fractional calculus can also be found. In a sense, the first of these was the discovery by Abel in 1823,[15]-[20]. Abel considered the solution of the integral equation related to the tautochrone problem . He found that the solution could be accomplished via an integral transform, which could be written as a semi-derivative.

More precisely, the integral transform considered by Abel was

$$K = \int_0^x (x-t)^{-1/2} f(t)\, dt, \quad K = const. \tag{1.1.6}$$

Abel wrote the right hand side of (1.1.6) by means of a fractional derivative of order $\frac{1}{2}$,

$$\sqrt{\pi}\left(\frac{d^{-1/2}}{dx^{-1/2}}\left(f(x)\right)\right). \tag{1.1.7}$$

Abel's solution had attracted the attention of Joseph Liouville, who made the first major study of fractional calculus,[33]-[34]. The most critical advances in the subject came around 1832 when he began to study fractional calculus in earnest and then managed to apply his results to problems in potential theory. Liouville began his theoretical development using the well-known result for derivatives of integer order n

$$D_x^n e^{ax} = a^n e^{ax}. \tag{1.1.8}$$

Expression (1.1.8) can rather easily be formally generalized to the case of non-integer values of n, thus obtaining

$$D_x^\alpha e^{ax} = a^\alpha e^{ax}. \tag{1.1.9}$$

By means of Fourier expansion, a wide family of functions can be composed as a superposition of complex exponentials.

$$f(x) = \sum_{n=0}^{\infty} c_n \exp(a_n x), \quad \operatorname{Re} a_n > 0. \tag{1.1.10}$$

Again, by invoking linearity of the fractional derivative, Liouville proposed the following expression for evaluating the derivative of order α

$$D_x^\alpha f(x) = \sum_{n=0}^{\infty} c_n a_n^\alpha e^{a_n x}. \tag{1.1.11}$$

Formula (1.1.11) is known as the Liouville's first formula for a fractional derivative,[33]. However, this formula cannot be seen as a general definition of fractional derivative for the same reason Lacroix formula could not: because of its relatively narrow scope. In order to overcome this, Liouville labored to produce a second definition. He started with a definite integral (closely related to the gamma function):

$$I = \int_0^{\infty} u^{\beta-1} e^{-xu} du, \ \ \beta > 0, x > 0, \tag{1.1.12}$$

and derived what is now referred to as the second Liouville's formula

$$D_x^{\alpha} x^{-\beta} = (-1)^{\alpha} \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} x^{-\alpha-\beta}, \beta > 0. \tag{1.1.13}$$

None of previous definitions were found to be suitable for a general definition of a fractional derivative. In the consequent years, a number of similar formulas emerged. Greer [15], for example, derived formulas for the fractional derivatives of trigonometric functions using in the form:

$$D_x^{\alpha} e^{iax} = a^{\alpha} \left(\cos\frac{\pi\alpha}{2} + i\sin\frac{\pi\alpha}{2}\right)(\cos ax + i\sin ax). \tag{1.1.14}$$

Joseph Fourier [14] obtained the following integral representations for $f(x)$ and its derivatives

$$D_x^{n} f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\xi)d\xi \int_{-\infty}^{+\infty} t^{n} \cos\left[t(x-\xi) + \frac{n}{2}\pi\right] dt, \tag{1.1.5}$$

By formally replacing integer n by an arbitrary real quantity α he obtained

$$D_x^{\alpha} f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\xi)d\xi \int_{-\infty}^{+\infty} t^{\alpha} \cos\left[t(x-\xi) + \frac{\alpha}{2}\pi\right] dt. \tag{1.1.16}$$

Probably the most useful advance in the development of fractional calculus was due to a paper written by G. F.

Bernhard Riemann [11] during his student days. Unfortunately, the paper was published only posthumously in 1892. Seeking to generalize a Taylor series in 1853, Riemann derived different definition that involved a definite integral and was applicable to power series with non-integer exponents

$$D_{c,x}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_c^x (x-t)^{\alpha-1} f(t)dr + \Psi(x). \qquad (1.1.17)$$

In fact, the obtained expression is the most-widely utilized modern definition of fractional integral. Due to the ambiguity in the lower limit of integration c, Riemann added to his definition a "*complementary*" function $\Psi(x)$ where the present-day definition of fractional integration is without the troublesome complementary function. Since neither Riemann nor Liouville solved the problem of the complementary function, it is of historical interest how today's Riemann-Liouville definition was finally deduced.

The earliest work that ultimately led to what is now called the Riemann-Liouville definition appears to be the paper by N. Ya. Sonin in 1869, [44] where he used Cauchy's integral formula as a starting point to reach differentiation with arbitrary index. A. V. Letnikov [30] extended the idea of Sonin a short time later in 1872, [29]. Both tried to define fractional derivatives by utilizing a closed contour. Starting with Cauchy's integral formula for integer order derivatives, given by

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(t)}{(t-z)^{n+1}} dt, \qquad (1.1.18)$$

the generalization to the fractional case can be obtained by replacing the factorial with Euler's Gamma function $\alpha! + \Gamma(1+\alpha)$. However, the direct extension to non-integer values α results in the problem that the integrand in (1.1.18) contains a branching point, where an appropriate contour would then require a branch cut which was not included in the work of Sonin and Letnikov. Finally, Laurent [28], used a contour given as an open circuit (known as Laurent loop) instead of a closed circuit used by Sonin and Letnikov and thus produced today's definition of the Riemann-Liouville fractional integral

$$D_{c,x}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_c^x (x-t)^{\alpha-1} f(t)dt, \quad \mathrm{Re}(\alpha) > 0. \qquad (1.1.19)$$

In expression (1.1.19) one immediately recognizes Riemann's formula (1.1.17), but without the problematic complementary function. In nowadays terminology, expression (1.1.19) with lower terminal $c = -\infty$ is referred as Liouville fractional integral; by taking $c = 0$ the expression reduces to the so called Riemann fractional integral, where as the expression (1.1.19) with arbitrary lower terminal c is called Riemann-Liouville fractional integral.

By choosing $c = 0$ in (1.1.19) one obtains the Riemann's formula (1.1.17) without the problematic complementary function $\Psi(x)$ and by choosing $c = -\infty$, formula (1.1.19) is equivalent to Liouville's first definition (1.1.10). These two facts explain why equation (1.1.19) is called Riemann-Liouville fractional integral. While the notation of fractional integration and differentiation only differ in the sign of the parameter α in (1.1.19), the change from fractional integration to differentiation cannot be achieved directly by inserting negative α at the right-hand side of (1.1.19).

The problem originates from the integral at the right side of(1.1.19) which is divergent for negative integration orders. However, by analytic continuation it can be shown that

$$D_{c,x}^{\alpha} f\left(x\right) = D_{c,x}^{n-\beta} f\left(x\right) = D_{c,x}^{n} f\left(x\right) D_{c,x}^{-\beta} f(x) = \frac{d^n}{dx^n}\left(\frac{1}{\Gamma(\beta)}\int_c^x \left(x-t\right)^{\beta-1} f(t)dt\right), \tag{1.1.20}$$

holds, which is known today as the definition of the Riemann-Liouville fractional derivative. In (1.1.20) $n = [\alpha]$ is the smallest integer greater than α with $0 < \beta = n - \alpha < 1$. For either $c = 0$ or $c = \infty$ the integral in (1.1.20) is the Beta-integral for a wide class of functions and thus easily evaluated.

Nearly simultaneously, Grunwald and Letnikov provided the basis for another definition of fractional derivative which is also frequently used today. Disturbed by the restrictions of the Liouville‘s approach Grunwald (1867) adopted the definition of a derivative as the limit of a difference quotient as its starting point.

He arrived at definite-integral formulas for ordinary derivatives, showed that Riemann's definite integral had to be interpreted as having a finite lower limit, and also that the Liouville's definition, in which no distinguishable lower limit appeared, correspond to a lower limit $-\infty$. Formally,

$$ {}^{GL}D_x^\alpha f(x) = \lim_{h\to 0}\frac{(\Delta_h^\alpha f)}{h^\alpha} = \lim_{h\to 0}\frac{\sum_{k=0}(-1)^k \begin{pmatrix}\alpha\\ k\end{pmatrix} f(x-kh)}{h^\alpha}, \alpha > 0, \tag{1.1.21} $$

which is today called *the Grunwald-Letnikov fractional derivative.* In definition (1.1.21), $\begin{pmatrix}\alpha\\ k\end{pmatrix}$ is the generalized binomial coefficient, wherein the factorials are replaced by Euler's Gamma function. Letnikov [18] also showed that definition (1.1.21) coincides, under certain relatively mild conditions, with the definitions given by Riemann and Liouville. Today, the Grunwald-Letnikov definition is mainly used for derivation of various numerical methods, which use formula (1.1.21)) with finite sum to approximate fractional derivatives. Together with the advances in fractional calculus at the end of the nineteenth century the work of O. Heaviside [26] has to be mentioned. The operational calculus of Heaviside, developed to solve certain problems of electromagnetic theory, was an important next step in the application of generalized derivatives. The connection to fractional calculus has been established by the fact that Heaviside used arbitrary powers of p, mostly $\sqrt{p}$, to obtain solutions of various engineering problems.

Weyl and Hardy,[24]-[25\], also examined some rather special, but natural,properties of differintegrals of functions belonging to Lebesgue and Lipschitz classes in 1917. Moreover, Weyl showed that the following fractional integrals could be written for $0 < \alpha < 1$, assuming that the integrals in (1.1.21) are convergent over an infinite interval

$$ I_+^\alpha \varphi(x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-t)^{\alpha-1}\varphi(t)dt, I_-^\alpha \varphi(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(t-x)^{\alpha-1}\varphi(t)dt \tag{1.1.22} $$

Specially, the Riemann-Liouville definition of a fractional integral given in (1.1.19) with lower limit $c = -\infty$, the form equivalent to the definition of fractional integral proposed by Liouville, is also often referred to as Weyl fractional integral. In the modern terminology one recognizes two distinct variants of all fractional operators,left sided and right sided ones.

Weyl operators defined in (1.1.21) are sometimes also referred to as the left and right Liouville fractional integrals, respectively.

Later, in 1927 Marchaud developed an integral version of the Grunwald-Letnikov definition (1.1.21) of fractional derivatives, using

$$\begin{aligned} {}^{M}D_{x}^{\alpha}f(x) &= \frac{\alpha}{\Gamma(1-\alpha)}\int_{0}^{\infty}\frac{\left(\Delta_{t}^{l}f\right)(x)}{t^{1+\alpha}}dt \\ &= \frac{\alpha}{\Gamma(1-\alpha)}\int_{0}^{\infty}\frac{f(x)-f(x-t)}{t^{1+\alpha}}dt, \quad \alpha>0 \end{aligned} \tag{1.1.23}$$

as fractional derivative of a given function f , today known as Marchaud fractional derivative. The term $\left(\Delta_{t}^{l}f\right)(x)$ is a finite difference of order $l>\alpha$ and c is a normalizing constant. Since this definition is related to the Grunwald-Letnikov definition,it also coincides with the Riemann-Liouville definition under certain conditions. M. Riesz published a number of papers starting from 1938 [40]-[41] which are centered around the integral

$${}^{R}I_{+}^{\alpha}\varphi=\frac{1}{2\Gamma(\alpha)\cos(\frac{\alpha\pi}{2})}\int_{-\infty}^{+\infty}\frac{\varphi(t)}{|t-x|^{1-\alpha}}dt, \quad \mathrm{Re}\,\alpha>0, \quad \alpha\neq 1,3,5... \tag{1.1.24}$$

today known as Riesz potential. This integral (and its generalization in the $n-$dimensional Euclidean space) istightly connected to Weyl fractional integrals (1.1.22) and therefore to the Riemann-Liouville fractional integrals by

$${}^{R}I^{\alpha}=\left(I_{+}^{\alpha}+I_{-}^{\alpha}\right)\left(2\cos\left(\frac{\alpha\pi}{2}\right)\right)^{-1}. \tag{1.1.25}$$

In 1949 Riesz [40] also developed a theory of fractional integration for functions of more than one variable.

A modification of the Riemann-Liouville definition of fractional integrals, given by

$$\frac{2x^{-2(\alpha+\eta)}}{\Gamma(\alpha)}\int_{0}^{x}\left(x^{2}-t^{2}\right)^{\alpha-1}t^{2\eta+1}\varphi(t)dt, \quad \frac{2x^{2\eta}}{\Gamma(\alpha)}\int_{0}^{x}\left(x^{2}-t^{2}\right)^{\alpha-1}t^{1-2\alpha-2\eta}\varphi(t)dt, \tag{1.1.26}$$

were introduced by Erdelyi et al. in [13], which became useful in various applications. While these ideas are tightly connected to fractional differentiation of the functions x^2 and $\sqrt{x}$, already done by Liouville 1832, the fact that Erdelyi and Kober used the Mellin's transform for their results is noteworthy.

Among the most significant modern contributions to fractional calculus are those made by the results of M.Caputo in 1967. One of the main drawbacks of Riemann-Liouville definition of fractional derivative is thatfractional differential equations with this kind of differential operator require a rather "strange" set of initial conditions. In particular, values of certain fractional integrals and derivatives need to be specified at the initial time instant in order for the solution of the fractional differential equation to be found. Caputo reformulated the more "*classic*" definition of the Riemann-Liouville fractional derivative in order to use classical initial conditions, the same one needed by integer order differential equations . Given a function f with an $(n-1)$ absolutely continuous integer order derivatives, Caputo defined a fractional derivative by the following expression

$$D_n^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \left(\frac{d}{ds}\right)^n f(s)ds, \tag{1.1.27}$$

Derivative (1.1.27) is strongly connected to the Riemann-Liouville fractional derivative and is today frequently used in applications. It is interesting to note that Rabotnov introduced the same differential operator into the Russian viscoelastic literature a year before Caputo's paper was published. Regardless of this fact, the proposed operator is in the present-day literature commonly named after Caputo.

1.2 History of Stochastic Calculus

Stochastic calculus is a branch of mathematics that deals with the study of processes that involve randomness. It is widely used in various fields, including physics, engineering, finance, and economics.

The foundations of stochastic calculus were laid in the early 20th century by the mathematicians Norbert Wiener and Andrey Kolmogorov. Wiener developed the theory of what is now known as Brownian motion, which is a type of random motion that occurs in many physical systems, such as the movement of particles in a fluid. Wiener also introduced the concept of the stochastic integral, which is used to define the integral of a function with respect to a stochastic process.

Kolmogorov, on the other hand, developed a rigorous mathematical framework for studying stochastic processes. His work laid the foundations for modern probability theory, and he introduced the concept of the stochastic differential equation, which is a type of differential equation that describes the evolution of a stochastic process.

In the 1950s and 1960s, a number of mathematicians, including Paul Lévy, Itô Kiyoshi, and William Feller, further developed the theory of stochastic calculus. Itô introduced the Itô calculus, which is a type of stochastic calculus that is widely used in finance and economics.

Today, stochastic calculus is an active area of research, and it has a wide range of applications in various fields. It is used to model complex systems that involve randomness, and it has led to many important insights in the fields of finance, economics, and physics.

1.3 Describe Problem

the aim of this section,we will study the existence and uniqueness,stability of solution of nonlinear stochastic fractional delay differential equations with Lévy Noise in the form:

$$\begin{aligned} {}^{C}D^{\alpha}x(t) &= \mathsf{b}(t,x(t)) + \sigma(t,x(t))\frac{\mathsf{dW}(t)}{\mathsf{dt}} + \int_{z} g(t,x(t),z)\frac{\mathsf{d}\tilde{\mathsf{N}}(t,z)}{\mathsf{dt}}, t \in J = [0,T] \\ x(0) &= x_0, \end{aligned} \tag{1.3.1}$$

Let $\mathsf{W}(t)$ be an $m-$dimensional motion and $\tilde{\mathsf{N}}(\mathsf{dt},\mathsf{dz}) = \mathsf{N}(\mathsf{dt},\mathsf{dz}) - \mathsf{v}(\mathsf{dz})\mathsf{dt}$ which is the $l-$dimensionall compensated jumb measure of $\eta(.)$ an independent compensated Poisson random measure on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Here $\mathsf{N}(\mathsf{dt},\mathsf{dz})$ is the $l-$dimensional jump measure (or Poisson measure) and $\mathsf{v}(\mathsf{dz})$ is the Lévy measure of $l-$dimensional Lévy process $\eta(.)$. For convenience $x(t,\omega), t \geq 0$ and can be written as $x(t)$ throughout this section.

where $\alpha \in (\frac{1}{2}, 1)$ and $z \in \mathbb{R}_0^n = \mathbb{R}^n / \{0\}$. Here $b : J \times \mathbb{R}^n \to \mathbb{R}^n, \sigma : J \times \mathbb{R}^n \to \mathbb{R}^{nm}$, $g : J \times \mathbb{R}^n \times \mathbb{R}_0^n \to \mathbb{R}^{nl}$ are given functions such that for all $t, b\left(t, x\left(t\right)\right), \sigma\left(t, x\left(t\right)\right), g\left(t, x\left(t\right), z\right)$ are $\mathcal{F}_t$ measurable for all $x \in \mathbb{R}^n$ and $z \in \mathbb{R}_0^n$.

CHAPTER 2

Preliminaries

In this chapter, we present a few well-known concepts and results in the fields of fractional and stochastic differential equations.

2.1 Special functions

The Gamma Function

Definition 2.1.1 *Let $z \in \mathbb{C}$, then we define the Gamma function as*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

This integral converges for $\mathrm{Re}(z) > 0$ *(the right half of the complex plane).*

One of the basic properties of the Gamma function is

$$\Gamma(z+1) = z\Gamma(z).$$

The Beta Function

Definition 2.1.2 *Let $z, w \in \mathbb{C}$, then we define the Beta function as*

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt,$$

For $\mathrm{Re}(z) > 0$ *and* $\mathrm{Re}(w) > 0$*. After we use the Laplace transform for convolutions the Beta function can be expressed in terms of the Gamma function by*

$$B(z,w) = \frac{\Gamma(z)\,\Gamma(w)}{\Gamma(z+w)}.$$

Mittag-Leffler function

Definition 2.1.3 *The one parameter Mittag-Leffler function is defined by*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \left(z \in \mathbb{C},\ \ \mathrm{Re}\,(\alpha) > 0\right). \tag{2.1.1}$$

A two parameter Mittag-Laffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \left(z, \beta \in \mathbb{C},\ \ \mathrm{Re}\,(\alpha) > 0\right). \tag{2.1.2}$$

In particular when $\beta = 1$ then $E_{\alpha,1}(z) = E_\alpha(z)$. The Mittag Leffler function of a matrix A is defined by

$$E_{\alpha,\beta}(At) = \sum_{k=0}^{\infty} \frac{(At)^k}{\Gamma(\alpha k+\beta)}, \left(\alpha, \beta > 0,\ A \in \mathbb{R}^{n\times n}\right).$$

2.2 Fractional derivatives and integrals

Riemann-Liouville fractional derivative

Definition 2.2.1 *The Riemann-Liouville fractional derivative of order $\alpha > 0$ of function $f(t)$ is defined as*

$$D_{o^+}^{\alpha} f(t) = \left(\frac{d}{dt}\right) I_{o^+}^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right) \int_0^t (t-s)^{n-\alpha-1} f(s)ds, \tag{2.2.1}$$

where $n-1 < \alpha < n, n \in \mathbb{N}$, and where the function $f(t)$ has absolutely continuous derivatives upto order $(n-1)$.

Caputo fractional derivative

Definition 2.2.2 *The Caputo fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n, n \in \mathbb{N}$, is defined as*

$$^{C}D^{\alpha}_{o^+}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^{(n)}(s)ds, \qquad 2.2.2$$

where the function $f(t)$ has absolutely continuous derivatives upto order $(n-1)$.

Riemann-Liouville Fractional integral

Definition 2.2.3 *The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f \in L^1(\mathbb{R}^+)$ is defined as*

$$I^{\alpha}_{o^+}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds, \quad t > 0, \qquad (2.2.3)$$

where $\Gamma(.)$ is the Euler gamma function.

2.3 Stochastic Calculus:

Theorem 2.3.1 **(i)** *if $\{X_n\}_{n=1}^{\infty}$ is a submartingale, then*

$$\mathbb{P}\left(\max_{1\le k\le n} X_k \ge \lambda\right) \le \mathbb{E}(X_n^+).$$

for all $n = 1, 2, ...$and $\lambda > 0$.

(ii) *if $\{X_n\}_{n=1}^{\infty}$ is a martingale and $1 < p < \infty$, then*

$$\mathbb{P}\left(\max_{1\le k\le n} |X_k|^p\right) \le \left(\frac{p}{p-1}\right)^p \mathbb{E}(|X_n|^P).$$

for all $n = 1, 2, ...$

Stochastic Process

Definition 2.3.1 *A collection $\{X(t)\,|t \ge 0\}$ of random variables is called a stochastic process.*

Lévy Process

Definition 2.3.2 *Let $\{X(t)\,|t \geq 0\}$ be a stochasticprocess defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Then X is a Lévy Process if*

1. *$X(0) = 0(a.s)$.*

2. *X has an independent and stationary increments.*

3. *X is stochastically continuous. That is, for all $a > 0$ and for all $s > 0$*

$$\lim_{t \to s} P(|X(t) - X(s)| > \alpha) = 0.$$

Poisson process

Itô process

Definition 2.3.3 *An Itô process or stochastic intregral is a stochastic on $(\Omega, \mathcal{F}, \mathbb{P})$ adopted to $\mathcal{F}_t$ which can be written in the form*

$$X_t = X_0 + \int_0^t U_s ds + \int_0^t V_s dB_s, \tag{2.3.1}$$

where $U, V \in \mathcal{L}_2$. As a shorthand notation, we will write (2.3.1) as

$$dX_t = U_t dt + V_t dB_t.$$

2.4 Some Useful Inequalities

Markov inequality

If $U(X) \geq 0$ for non-descreasing function U then for all $r > 0$,

$$\mathbb{P}(X \geq r) \leq \frac{\mathbb{E}(U(X))}{U(r)}.$$

Cauchy-Schwarz inequality

In $[0; +\infty]$, with equality if and only if X and are colinear,

$$\mathbb{E}(XY) \leq \mathbb{E}(|X|^2)^{\frac{1}{2}} \mathbb{E}(|Y|^2)^{\frac{1}{2}}.$$

Hölder inequality

If $p \in [1;\infty]$ and $q = \frac{1}{1-\frac{1}{p}} = \frac{p}{p-1}$ then, in $[0;+\infty]$,

$$\mathbb{E}(|XY|) \leq \mathbb{E}(|X|^p)^{\frac{1}{p}}\mathbb{E}(|Y|^q)^{\frac{1}{q}}.$$

Chebyshev‘s Inequality

If X is a random variable and $1 \leq p < \infty$,then

$$\mathbb{P}(|X| \geq \lambda) \leq \frac{1}{\lambda^p}\mathbb{E}(|X|^P) \text{ for all } \lambda > 0.$$

Borel Cantelli Lemma

Lemma 2.4.1 *If* $\{A_k\} \subset \mathcal{F}$ *and* $\sum_{k=1}^{\infty}\mathbb{P}(A_k) < \infty$, *then*

$$\mathbb{P}\left(\lim_{k\to\infty}\sup A_k\right) = 0.$$

2.5 Delay differential equations

Suppose $\tau \geq 0$ is a given real number, $\mathbb{R} = (-\infty,\infty)$, $\mathbb{R}^n$ is an n-dimensional linear vector space over the reals with norm $|.|$, $C([a,b],\mathbb{R}^n)$ is the Banach space of continuous functions mapping the interval $[a,b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. If $[a,b] = [-\tau,0]$ we let $C = C([-\tau,0],\mathbb{R}^n)$ and designate the norm of an element ϕ in C by $|\phi| = \sup_{-\tau<\theta<0}|\phi(\theta)|$. Even though single bars are used for norms in different spaces, no confusion should arise. If

$$t_0 \in \mathbb{R}, A \geq 0 \text{ and } x \in C\left([t_0-\tau, t_0+A], \mathbb{R}^n\right),$$

then for any $t \in [t_0, t_0+A]$, we let $x_t \in C$ be defined by $x_t(\theta) = x(t+\theta), -\tau \leq \theta \leq 0$.

Definition 2.5.1 *If* Ω *is a subset of* $\mathbb{R}\times C, f : \Omega \to \mathbb{R}^n$ *is a given function and represents the right-hand derivative, we say that the relation*

$$x'(t) = f(t, x_t), \tag{2.5.1}$$

is a retarded functional differential equation on Ω and will denote this equation by $RFDE$.[48] If we wish to emphasize that the equation is defined by f,we write the $RFDE(f)$. A function x is said to be a solution of Equation (2.5.1) on $[t_0 - \tau, t_0 + A)$ if there are $t_0 \in \mathbb{R}$ and $A > 0$ such that $x \in C([t_0 - \tau, t_0 + A], \mathbb{R}^n), (t, x_t) \in \Omega$ and $x(t)$ satisfies Equation (2.5.1) for $t \in [t_0, t_0 + A)$. For given $t_0 \in \mathbb{R}, \phi \in C$, we say $x(t_0, \phi, f)$ is a solution of Equation (2.5.1) with initial value ϕ at t_0 or simply a solution through (t_0, ϕ) if there is an $A > 0$ such that $x(t_0, \phi, f)$ is a solution of Equation (2.5.1) on $[t_0 - \tau, t_0 + A)$ and $x_{t_0}(t_0, \phi, f) = \phi$.

Equation (2.5.1) is a very general type of equation and includes ordinary differential equations $(\tau = 0)$.

We say Equation (2.5.1) is linear if $f(t, \phi) = L(t, \phi) + h(t)$ where $L(t, \phi)$ is linear in ϕ; is homogeneous if $h \equiv 0$ and nonhomogeneous $h \neq 0$. We claim Equation (2.5.1) is autonomous if $f(t, \phi) = g(\phi)$ where g does not depend on t.

For example, the following equations are delay differential equations

$$x'(t) = 2x(t) + 5x(t-1), \tag{2.5.2}$$

$$x'(t) = a(t)x(t) + b(t)x'(t - \tau(t)) + h(t), \tag{2.5.3}$$

$$x'(t) = \int_{-\tau}^{0} x(t+s)ds. \tag{2.5.4}$$

a, b, τ are continuous functions. Equation (2.5.2) is an linear autonomous delay differential equation with constant $\tau = 1$, Equation (2.5.3) is nonhomogeneous, linear nonautonomous delay functional differential equations and Equation (2.5.4) is a delay linear integro-differential equation.

If $t_0 \in \mathbb{R}, \phi \in C$ are given and $f(t, \phi)$ is continuous, then finding a solution of Equation (2.5.1) through (t_0, ϕ) is equivalent to solving the integral equation

$$\begin{aligned} x_{t_0} &= \phi, \\ x(t) &= \phi(0) + \int_{t_0}^{t} f(s, x_s)ds, \ t \geq t_0. \end{aligned} \tag{2.5.5}$$

we define Tx by

$$\begin{aligned} Tx(t) &= \phi(0) + \int_{t_0}^{t} f(s, x_s)ds, \ t \geq t_0, \\ x_{t_0} &= \phi. \end{aligned}$$

To prove the existence of the solution through a point $(t_0, \phi) \in \mathbb{R} \times C$, we consider an $\eta > 0$ and all functions x on $[t_0 - \tau, t_0 + A]$ which are continuous and coincide with ϕ on $[t_0 - \tau, t_0]$; that is, $x_{t_0} = \phi$. The values of these functions on $[t_0, t_0 + \eta]$ are restricted to the class of x such that $|x(t) - \phi(0)| < \delta$ for $t \in [t_0, t_0 + \eta]$. The usual mapping T obtained from the corresponding integral equation is defined and it is then shown that η and δ can be so chosen that T maps this class into itself and is completely continuous. Thus, Schauder's fixed-point theorem implies existence .

Theorem 2.5.1 *(Existence) In (2.5.1), suppose Ω is an open subset in $\mathbb{R} \times C$ and f is continuos on Ω. If $(t_0, \phi) \in \Omega$, then there is a solution of (2.5.1) passing through (t_0, ϕ).*

Definition 2.5.2 *We say $f(t, \phi)$ is Lipschitz in ϕ in a compact set K of $\mathbb{R} \times C$ if there is a constant $k > 0$ such that, for any $(t, \phi_i) \in K$, $i = 1, 2$,*

$$|f(t, \phi_1) - f(t, \phi_2)| \leq k\, |\phi_1 - \phi_2|\,. \tag{2.5.6}$$

Theorem 2.5.2 *(Uniqueness) Suppose Ω is an open set in $\mathbb{R} \times C$, $f : \Omega \to \mathbb{R}^n$ is continuous, and $f(t, \phi)$ is Lipschitz in ϕ in each compact set in Ω. If $(t_0, \phi) \in \Omega$, then there is a unique solution of Eq. (2.5.1) through (t_0, ϕ).*

Neutral delay differential equations

In order to define a general class of neutral delay differential equations ($NDDEs$) (or neutral functional differential equations ($NFDEs$)), we need the definition of atomic.

Definition 2.5.3 *Suppose $\Omega \subseteq \mathbb{R} \times C$ is open with elements (t, ϕ). A function $\Psi : \Omega \to \mathbb{R}^n$ is said to be atomic at β on Ω if Ψ is continuous together with its first and second Fréchet derivatives with respect to ϕ: and Ψ_ϕ, the derivative with respect to ϕ, is atomic at β on Ω.*

Definition 2.5.4 *Suppose* $\Omega \subseteq \mathbb{R}\times C$ *is open,* $f:\Omega\to\mathbb{R}^n$, $\Psi:\Omega\to\mathbb{R}^n$ *are given continuous functions with* Ψ *atomic at zero. The equation*

$$\frac{d}{dt}\Psi(t,x_t)=f(t,x_t), \tag{2.5.7}$$

is called the neutral delay differential equation $NDDE(\Psi,f)$.

Definition 2.5.5 *A function* x *is said to be a solution of the* $NDDE(\Psi,f)$ *or Equation (2.5.7), if there are* $t_0\in\mathbb{R}, A>0$, *such that* $x\in C([t_0-\tau,t_0+A),\mathbb{R}^n)$, $(t,x_t)\in\Omega, t\in[t_0,t_0+A), \Psi(t,x_t)$ *is continuously differentiable and satisfies Eq. (2.5.7) on* $[t_0,t_0+A)$. *For a given* $t_0\in\mathbb{R}$, $\phi\in C$, *and* $(t_0,\phi)\in\Omega$, *we say* $x(t_0,\phi)$ *is a solution of Eq. (2.5.7) with initial value* ϕ *at* t_0, *or simply a solution through* (t_0,ϕ) *,if there is an* $A>0$ *such that* $x(t_0,\phi)$, *is a solution of (2.5.7) on* $[t_0-\tau,t_0+A)$ *and* $x_{t_0}(t_0,\phi)=\phi$.

Theorem 2.5.3 *(Existence) if* Ω *is an open set in* $\mathbb{R}\times C$ *and* $(t_0,\phi)\in\Omega$, *t hen there exists a solution of the* $NDDE(\Psi,f)$ *through* (t_0,ϕ).

Theorem 2.5.4 *(Uniqueness). If* $\Omega\subseteq\mathbb{R}\times C$ *is open and* $f:\Omega\to\mathbb{R}^n$ *as Lipschitz in* ϕ *on compact sets of* Ω, *then, for any* $(t_0,\phi)\in\Omega$, *there exists a unique solution of the* $NDDE(\Psi,f)$ *through* (t_0,ϕ).

For example

$$\begin{aligned} x'(t) &= -x'(t-1),\\ x'(t) &= x(t-1)+\left[x'(t-3)+1\right]^3,\\ x''(t) &= x(\frac{t}{2})+x'(t-1)-x'(t-3), \end{aligned}$$

are neutral delay differential equations.

CHAPTER 3

New Results of Existence and Stability for stochastic fractional time using Lévy noise

3.1 Introduction

The problem of existence and uniqueness of solutions to differential equations forms the basis for this validation of the model and further investigation of the corresponding dynamic processes. many authors [7]-[23]-[32]-[38] discussed the existence and uniqueness of solving stochastic differential equations. This problem was studied by Pedjeu and Ladde [27] using independent timescales.

The concept of stability is very important because almost all functioning control systems are designed with stability in mind, this means that the system remains in a constant state and returns to its original state unless it is affected by an external force, steady state when the external action is removed. Luo [46], Khasminskii [12], Balachandran et al [25\]-[26] discussed the stability of stochastic differential equations. Exponential stability for stochastic neutrality partial function equations were obtained by Govindan using semigroup theory [46]-[47]. Zhu et al [50] studied the stability of stochastic systems with Poisson jumps. Then fractional stabilityof dynamical systems have been studied by many authors [12]-[27]-[43]. Abouagwa and Li [35]-[36] discussed probability theoryof fractional system with Levy noise under Caratheodory conditions.

The fractional Brownian motion introduced by Mandelbrot and Van Ness [6], considering memory with randomness. These fractional Brownian motions are the fractional integrals or fractional derivatives of Brownian motion. However, these models only consider sound memory effects, It's in the system, not in memory relative to system state dynamics. Lee et al [39] is a comparative Study of the Classical Stochastic Model of European Option Prices, Black-Scholes models using stochastic equations with fractional Brownian motion and stochastic equations with fractional motion time derivative. It is shown that the time derivative stochastic model is replaced by the fractional model.

This derivative performs better than the fractional brown noise model. In this work we prove that Existence and Stability of Solutions of Stochastic Fractional Differential Equations with Lévy Noise equation [10].

3.2 Existence and Uniqueness

In this chapter, we will use the classical Picard-Lindelöf method of successive approximation scheme to prove the existence and uniqueness of solution of nonlinear stochastic fractional differential equations [16] and stochastic fractional delay differntial equations with Lévy noise [17]-[4].

We define $t, b(t, x(t)), \sigma(t, x(t)), g(t, x(t), z)$ are $\mathcal{F}_t$ measurable for all $x \in \mathbb{R}^n$ and

$z \in \mathbb{R}_0^n$. We can rewrite the equation (1.3.1) in its equivalent integral form as

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathsf{b}(s, x(s))\, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sigma(s, x(s))\, d\mathsf{W}(s) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathsf{g}(s, x(s), z)\, d\tilde{\mathsf{N}}(\mathsf{d}s, \mathsf{d}z). \quad (3.2.1)$$

Here the results are obtained by using [42]-[43].

Theorem 3.2.1 *Assume that* $(t, x) \in J \times \mathbb{R}^n, \alpha \in \left(\frac{1}{2}, 1\right), z \in \mathbb{R}_0^n, b \in C(J \times \mathbb{R}^n, \mathbb{R}^n), \sigma \in C(J \times \mathbb{R}^n, \mathbb{R}^{nm}), g \in C\left(J \times \mathbb{R}^n \times \mathbb{R}_0^n, \mathbb{R}^{nl}\right)$ *and* $\mathsf{W} = \{\mathsf{W}(t), t \geq 0\}$ *is an* $m-$*dimensional Brownian motion on a complete probability space* $(\Omega, \mathcal{F}, \mathcal{P})$. *Suppose the following inequalities hold[48]:*

(i) Linear growth condition:

$$|\mathsf{b}(t,x)|^2 + |\sigma(t,x)|^2 + \int_z |g(t,x,z)|^2\, v(\mathrm{d}z) \leq K^2(1+|x|^2) \quad (3.2.2)$$

for some constant $K > 0$.

(ii) The Lipschitz condition:

$$|\mathsf{b}(t,x) - \mathsf{b}(t,y)|^2 + |\sigma(t,x) - \sigma(t,y)|^2 + \int_z |g(t,x,z) - g(t,y,z)|^2\, v(\mathrm{d}z) \leq L^2(|x-y|^2) \quad (3.2.3)$$

for some constant $L > 0$.

Let x_0 *be a random variable defined on* $(\Omega, \mathcal{F}, \mathcal{P})$ *and independent of the* $\sigma-$*algebra* $\mathcal{F}_s^t \subset \mathcal{F}$ *generated by* $\mathsf{W} = \{\mathsf{W}(t), t \geq 0\}$ *and such that* $\mathbb{E}\left(\left|x_0^2\right|\right) < \infty$. *Then the initial value problem (1.3.1) has a unique solution[18]-[37] which is* $t-$*continuous with the property that* $x(t, \omega)$ *is adapted to the filtration* $\mathcal{F}_t^{x_0}$ *generated by* x_0 *and* $\{\mathsf{W}(s)(.), s \leq t\}$ *and*

$$\sup_{0 \leq t \leq T} \mathbb{E}\left[\left|x(t)^2\right|\right] < \infty \quad (3.2.4)$$

Proof 3.2.1 (i) *First we establish the existence of solution of the initial value problem [8]-[9]. Let us define* $x^{(0)}(t) = x_0$ *and* $x^{(k)}(t) = x^{(k)}(t,\omega)$ *inductively as follows:*

$$\begin{aligned} x^{(k+1)}(t) &= x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathsf{b}\left(s, x^{(k)}(s)\right) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sigma\left(s, x^{(k)}(s)\right) d\mathsf{W}(s) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathsf{g}\left(s, x^{(k)}(s), z\right) d\tilde{\mathsf{N}}(\mathsf{d}s, \mathsf{d}z), \end{aligned} \tag{3.2.5}$$

for $k = 0, 1, 2, \ldots$*If,for fixed* $k \geq 0$*, the approximation* $x^{(k)}(t)$ *is* $\mathcal{F}_t-$*measurable and continuous on* J*, then it follows from (3.2.2)-(3.2.3), that the integrals in (3.2.5) are meaningful and the resulting process* $x^{(k+1)}(t)$ *is* $\mathcal{F}_t-$*measurable and continuous on J. As* $x^{(0)}(t)$ *is obviously* $\mathcal{F}_t-$*measurable and continuous on J, it follows by induction that so too is each* $x^{(k)}(t)$ *for* $k = 1, 2, \ldots$

Since x_0 *is* $\mathcal{F}_t-$*measurable with* $\mathbb{E}\left(\left|x_0^2\right|\right) < \infty$*, it is clear that*

$$\sup_{0\leq t\leq T} \mathbb{E}\left[\left|x(t)^2\right|\right] < \infty$$

Applying the algebraic inequality $(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2)$*, the Cauchy-Schwartz inequality, the Itô isometry and the linear growth condition (3.2.2) we obtain from (3.2.5) that*

$$\begin{aligned} \mathbb{E}\left(\left|x^{(k+1)}(t)\right|^2\right) &\leq 4\mathbb{E}\left[|x_0|^2\right] + \frac{4}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\mathbb{E}\left[\int_0^t \left|\mathsf{b}\left(s, x^{(k)}(s)\right)\right|^2 ds\right] \\ &+ \frac{4}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\mathbb{E}\left[\int_0^t \left|\sigma\left(s, x^{(k)}(s)\right)\right|^2 ds\right] \\ &+ \frac{4}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\mathbb{E}\left[\int_0^t \left|\mathsf{g}\left(s, x^{(k)}(s), z\right)\right|^2 ds\right] \end{aligned}$$

Therefore

$$\mathbb{E}\left(\left|x^{(k+1)}(t)\right|^2\right) \leq 4\mathbb{E}\left[|x_0|^2\right] + 3K^2\frac{4}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\mathbb{E}\left(\int_0^t \left(1 + \left|x^{(k)}(s)\right|^2\right) ds\right),$$

for $k = 0, 1, 2, \ldots$ *and* $m > 0$*. By induction,we have*

$$\sup_{0\leq t\leq T} \mathbb{E}\left(\left|x^{(k)}(t)\right|^2\right) \leq C_0 < \infty,$$

for $k = 0, 1, 2, \ldots$ *Let*

$$\mathsf{d}^{(k)}(t) = \mathbb{E}\left(\left|x^{(k+1)}(t) - x^{(k)}(t)\right|\right).$$

We claim that

$$\mathsf{d}^{(k)}(t) \leq \frac{(\mathsf{M}t)}{(k+1)!}, \text{ for all } k = 0, 1, 2, \ldots.., \tag{3.2.6}$$

for some constants M, *depending in* K, L *and* x_0.

From equation (3.2.5) by applying the Schwarz inequality and Itô isometry and the Lipchitz condition (3.2.3) we obtain

$$\begin{aligned}
\mathsf{d}^{(k)}(t) &= \mathbb{E}\left[\left|x^{(k+1)}(t) - x^{(k)}(t)\right|^2\right] \\
&\leq \frac{4}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\left[\left|\mathsf{b}\left(s, x^{(k)}(s) - \mathsf{b}(s, x^{(k-1)}(s)\right)\right|^2\right]\mathsf{d}s \\
&\quad + \frac{4}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\left[\left|\sigma\left(s, x^{(k)}(s) - \sigma(s, x^{(k-1)}(s)\right)\right|^2\right]\mathsf{d}s \\
&\quad + \frac{4}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\int_z \left|\mathsf{g}\left(s, x^{(k)}(s), z\right) - \mathsf{g}(s, x^{(k-1)}(s), z)\right|^2 v(\mathsf{d}z)\mathsf{d}s \\
&\leq 4\frac{L^2}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\left[\left|x^{(k)}(s) - x^{(k-1)}(s)\right|^2 \mathsf{d}s\right] \\
&\quad + 4\frac{L^2}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\left[\left|x^{(k)}(s) - x^{(k-1)}(s)\right|^2 \mathsf{d}s\right] \\
&\quad + 4\frac{L^2}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\left[\left|x^{(k)}(s) - x^{(k-1)}(s)\right|^2 \mathsf{d}s\right].
\end{aligned} \tag{3.2.7}$$

By applying again the Schwarz inequality, the Itô isometry together with the growth conditions

(3.2.2) for $k = 0$,

$$\begin{aligned}
\mathsf{d}^{(0)}(t) &= \mathbb{E}\left[\left|x^{(1)}(t) - x^{(0)}(t)\right|^2\right] \\
&\leq \frac{4}{(\Gamma(\alpha))^2}\mathbb{E}\left(\left|\int_0^t (t-s)^{\alpha-1}\,\mathsf{b}\left(s, x^{(0)}(s)\right)\mathsf{d}s\right|^2\right) \\
&\quad + \frac{4}{(\Gamma(\alpha))^2}\mathbb{E}\left(\left|\int_0^t (t-s)^{\alpha-1}\,\sigma\left(s, x^{(0)}(s)\right)\mathsf{dW}(s)\right|^2\right) \\
&\quad + \frac{4}{(\Gamma(\alpha))^2}\mathbb{E}\left(\left|\int_0^t (t-s)^{\alpha-1}\int_z \mathsf{g}\left(s, x^{(0)}(s), z\right)d\tilde{\mathsf{N}}(\mathsf{d}s, \mathsf{d}z)\right|^2\right) \\
&\leq \frac{4}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\left[\left|\mathsf{b}\left(s, x^{(0)}(s)\right)\right|^2 \mathsf{d}s\right] \\
&\quad + \frac{4}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\left[\left|\sigma\left(s, x^{(0)}(s)\right)\right|^2 \mathsf{d}s\right] \\
&\quad + \frac{4}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\left[\left|\int_z \mathsf{g}\left(s, x^{(0)}(s), z\right)v(\mathsf{d}z)\right|^2 \mathsf{d}s\right] \\
&\leq K^2\frac{4^2}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\left(\int_0^t \left(1 + |x_0|^2\right)\mathsf{d}s\right) \\
&\leq K^2\frac{4^2}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\left(1 + \mathbb{E}\left(|x_0|^2\right)\right).
\end{aligned} \tag{3.2.8}$$

Now, for $k = 1$, replacing $\mathbb{E}\left[\left|x^{(1)}(t) - x^{(0)}(t)\right|^2\right]$ in the inequality (3.2.7) with the value on the right hand side of inequality(3.2.8) and integrating, we obtain

$$\begin{aligned}
\mathbb{E}\left[\left|x^{(2)}(t) - x^{(1)}(t)\right|^2\right] &\leq L^2\frac{4}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\left[\left|x^{(1)}(s) - x^{(0)}(s)\right|^2 \mathsf{d}s\right] \\
&\leq K^2\left(1 + \mathbb{E}\left(|x_0|^2\right)\right)\left(L^2\frac{4^2}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\right)^2\int_0^t s\mathsf{d}s \\
&\leq K^2\left(1 + \mathbb{E}\left(|x_0|^2\right)\right)\left(L^2\frac{4^2}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\right)^2 \times \frac{t^2}{2!}.
\end{aligned} \tag{3.2.9}$$

For $k = 2$, proceeding as before, we have

$$\mathbb{E}\left[\left|x^{(3)}(t) - x^{(2)}(t)\right|^2\right] \leq K^2\left(1 + \mathbb{E}\left(|x_0|^2\right)\right)\left(L^2\frac{4^2}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\right)^2 \times \frac{t^3}{3!}. \tag{3.2.10}$$

Thus, by the principle of mathematical induction, we have

$$\mathsf{d}^{(k)}(t) = \mathbb{E}\left[\left|x^{(k+1)}(t) - x^{(k)}(t)\right|^2\right] \leq \frac{\mathsf{BM}^{k+1}t^{(k+1)}}{(k+1)!}, k = 0, 1, 2, ..., 0 \leq t \leq T, \tag{3.2.11}$$

where $\mathsf{B} = \mathsf{K}^2\left(1+\mathbb{E}\left(|x_0|^2\right)\right)$ *and* $\mathsf{M} = \left(L^2\frac{4^2}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\right)$ *is a constant depending only on* α, T, L^2 *and* $\mathbb{E}\left|x_0\right|^2$.

Note that

$$\begin{aligned}\max_{0\leq t\leq T}\left|x^{(k+1)}(t)-x^{(k)}(t)\right|^2 &\leq 4\max_{0\leq t\leq T}\int_0^t (t-s)^{\alpha-1}\left|\mathsf{b}\left(s,x^{(k)}(s)-\mathsf{b}(s,x^{(k-1)}(s)\right)\right|^2\mathsf{d}s\\ &+4\max_{0\leq t\leq T}\int_0^t (t-s)^{\alpha-1}\left|\sigma\left(s,x^{(k)}(s)-\sigma(s,x^{(k-1)}(s)\right)\right|^2\mathsf{dW}(s)\\ &+4\max_{0\leq t\leq T}\int_0^t (t-s)^{\alpha-1}\int_z\left|\mathsf{g}\left(s,x^{(k)}(s),z\right)-\mathsf{g}(s,x^{(k-1)}(s),z)\right|^2 d\tilde{N}(\mathsf{d}s,\mathsf{d}z).\end{aligned}$$

Taking expectation on both sides we have

$$\begin{aligned}\mathbb{E}\left(\max_{0\leq t\leq T}\left|x^{(k+1)}(t)-x^{(k)}(t)\right|^2\right) &\leq 4L^2\frac{T^{2\alpha-1}}{2\alpha-1}\mathbb{E}\left(\max_{0\leq t\leq T}\int_0^t\left|x^{(k)}(s)-x^{(k-1)}(s)\right|^2\mathsf{d}s\right)\\ &+4\mathbb{E}\left(\max_{0\leq t\leq T}\int_0^t (t-s)^{\alpha-1}\left|\sigma\left(s,x^{(k)}(s)-\sigma(s,x^{(k-1)}(s)\right)\right|^2\mathsf{dW}(s)\right)\\ &+4L^2\frac{T^{2\alpha-1}}{2\alpha-1}\mathbb{E}\left(\max_{0\leq t\leq T}\int_0^t\left|x^{(k)}(s)-x^{(k-1)}\right|^2\mathsf{d}s\right).\end{aligned}$$

Using second part of the Theorem 2.3.1 gives

$$\begin{aligned}\mathbb{E}\left(\max_{0\leq t\leq T}\left|x^{(k+1)}(t)-x^{(k)}(t)\right|^2\right) &\leq 3L^2\frac{T^{2\alpha-1}}{2\alpha-1}\mathbb{E}\left(\max_{0\leq t\leq T}\int_0^T\left|x^{(k)}(s)-x^{(k-1)}(s)\right|^2\mathsf{d}s\right)\\ &+12L^2\frac{T^{2\alpha-1}}{2\alpha-1}\mathbb{E}\left(\int_0^T\left|x^{(k)}(s)-x^{(k-1)}(s)\right|^2\mathsf{d}s\right)\\ &\leq \mathsf{B}\frac{\mathsf{M}^{k+1}}{(k+1)!}T^{(k+1)}, \qquad (3.2.12)\end{aligned}$$

where B *is a constant depending on* L *and* T. *By using Chebyshev's inequality gives*

$$\mathbb{P}\left(\max_{0\leq t\leq T}\left|x^{(k+1)}(t)-x^{(k)}(t)\right|^2>\frac{1}{k^2}\right)\leq\sum_{k=0}^{\infty}\frac{\mathsf{B}\mathsf{M}^{k+1}k^4T^{(k+1)}}{(k+1)!},$$

using the equation (3.2.12) and summing up the resultant inqualities gives

$$\sum_{k=0}^{\infty}\mathbb{P}\left(\max_{0\leq t\leq T}\left|x^{(k+1)}(t)-x^{(k)}(t)\right|^2>\frac{1}{k^2}\right)\leq\frac{1}{(1/k^2)^2}\mathbb{E}\left(\max_{0\leq t\leq T}\left|x^{(k+1)}(t)-x^{(k)}(t)\right|^2\right),$$

where the series on the right side converges by ration test. Hence the series on the left side also converges, so by the Borel-Cantelli lemma, we conclude that $\left(\max_{0\leq t\leq T}\left|x^{(k+1)}(t)-x^{(k)}(t)\right|^2\right)$ *converges to* 0, *almost surely,that is, the successive approximations* $x^{(k)}(t)$ *converge, almost* *surely, uniformly on J to a limit $x(t)$ defined by*

$$\lim_{n\to\to\infty}\left(x^{(0)}(t)+\sum_{k=1}^{n}\left(x^{(k)}(t)-x^{(k-1)}(t)\right)\right)=\lim_{n\to\to\infty}x^{(n)}(t)=x(t). \tag{3.2.13}$$

From (3.2.5), we have

$$\begin{aligned} x(t) &= x_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\,\mathsf{b}s,x(s))+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\sigma(s,x(s))\mathsf{d}\mathsf{W}(s)\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\,\mathsf{g}(s,x(s),z)\mathsf{d}\tilde{\mathsf{N}}(\mathsf{d}s,\mathsf{d}z). \end{aligned} \tag{3.2.14}$$

for all t$\in J$. This completes the proof of the existence of solution of (1.3.1)

Proof 3.2.2 (ii) *The uniqueness follows from Itô isometry[2]-[3], the Lipschitz conditions (3.2.3).*

Let $x(t,\omega)$ and $y(t,\omega)$ be solution processes through[5] the initial data $(0,x_0)$ and $(0,y_0)$ respectively, that is $x(0,\omega)=x_0(\omega)$ and $y(0,\omega)=y(\omega)$, $\omega\in\Omega$. Let

$$\begin{aligned} \gamma_1(s,\omega) &= \mathsf{b}(s,x(s))-\mathsf{b}(s,y(s)),\\ \gamma_2(s,\omega) &= \sigma(s,x(s))-\sigma(s,y(s)),\\ \gamma_3(s,\omega) &= \int_z\mathsf{g}(s,x(s),z)\,v(\mathsf{d}z)-\int_z\mathsf{g}(s,y(s),z)\,v(\mathsf{d}z) \end{aligned}$$

Then by virtue of the Schwarz inequality and Itô isometry, we have

$$\begin{aligned} \mathbb{E}\left[|x(t)-y(t)|^2\right] &\le \frac{4}{(\Gamma(\alpha))^2}\mathbb{E}\left[|x_0-y_0|^2\right]+\frac{4}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\mathbb{E}\left[\int_0^t|\gamma_1(s,\omega)|^2\,\mathsf{d}s\right]\\ &\quad+\frac{4}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\mathbb{E}\left[\int_0^t|\gamma_2(s,\omega)|^2\,\mathsf{d}s\right]\\ &\quad+\frac{4}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\mathbb{E}\left[\int_0^t|\gamma_3(s,\omega)|^2\,\mathsf{d}s\right]\\ &\le \frac{4}{(\Gamma(\alpha))^2}\mathbb{E}\left[|x_0-y_0|^2\right]+2^2L^2\frac{4}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t\mathbb{E}\left[|x(s)-y(s)|^2\right]\mathsf{d}s. \end{aligned}$$

we define $v(t)=\mathbb{E}\left[|x(s)-y(s)|^2\right]$. Then the function v satisfies $v(t)\le F+A\int_0^t v(s)\mathsf{d}s$, where $F=\frac{4}{(\Gamma(\alpha))^2}\mathbb{E}\left[|x_0-y_0|^2\right]$ and $A=2^2L^2\frac{4}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}$. By the application of the Gronwall inequality, we conclude that

$$v(t)\le F\exp(At).$$

Now assume that $x_0 = y_0$. Then $F = 0$ and so $v(t) = 0$ for all $t \geq 0$. That is,

$$\mathbb{E}\left[|x(s) - y(s)|^2\right] = 0.$$

which gives

$$\int_0^t |x(t) - y(t)|^2 \, \mathsf{d}\mathbb{P} = 0.$$

This implies that $x(t) = y(t)$ a.s for all $t \in J$. That is

$$P\{|x(t,\omega) - y(t,\omega)| = 0 \qquad \text{for all } t \in J\} = 1,$$

that is, the solution is unique. This completes the proof of existence and uniqueness of solution of the given stochastic fractional differential equation (3.2.1).

3.3 Delay Differential Equations

Delay and Poisson jumps always coexict in real dynamic systems. Thus is reasonable to consider them together leading us to investigate the existence of solution of stochastic fractional delay differntial equations with Lévy noise [50]. Let $\xi(.) \in\in C[-\delta, 0]$ be the initial path of x, where $\delta > 0$ is a given processes time delay. Moreover, denote by $\mathsf{L}^p_{\mathcal{F}_0}([-\delta, 0]; \mathbb{R}^n)$ the family of $\mathbb{R}^n$ valued adapted stochastic processes such that is $\mathcal{F}-$measurable and $\mathbb{E}\left(\sup_{-\delta\leq t\leq 0} |\xi(t)|^2\right) < \infty$.[24]

Consider the nonlinear stochastic fractional delay differential equations of the form

$$\begin{aligned} {}^{C}\mathsf{D}^{\alpha}x(t) &= \mathsf{b}(t, x(t), x(t-\delta)) + \sigma(t, x(t), x(t-\delta)) \tfrac{\mathsf{dW}(t)}{\mathsf{d}t} \\ &+ \textstyle\int_z \mathsf{g}(t, x(t), x(t-\delta), z) \tfrac{\mathsf{d}\tilde{\mathsf{N}}(t,z)}{\mathsf{d}t}, t \in J = [0, T] \\ x(t) &= \xi(t), \quad t \in [-\delta, 0], \end{aligned} \tag{3.3.1}$$

where $\alpha \in (\frac{1}{2}, 1)$ and $z \in \mathbb{R}^n_0 = \mathbb{R}^n/\{0\}$. Here $b : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \sigma : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{nm}$, $g : J \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n_0 \to \mathbb{R}^{nl}$ are given functions such that for all t, $\mathsf{b}(t, x(t), x(t-\delta))$, $\sigma(t, x(t), x(t-\delta))$ and $g(t, x(t), x(t-\delta), z)$ are $\mathcal{F}_t$ measurable for all $x \in \mathbb{R}^n, y \in \mathbb{R}^n$ and

$z \in \mathbb{R}_0^n$. we can rewrite the equation (1.3.1) in its equivalent integral form as:

$$\begin{aligned} x(t) &= \xi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathsf{b}(s, x(s), x(s-\delta)) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sigma(s, x(s), x(s-\delta)) d\mathsf{W}(s) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \int_z \mathsf{g}(s, x(s), x(s-\delta), z) \tilde{\mathsf{N}}(\mathsf{d}s, \mathsf{d}z). \end{aligned} \tag{3.3.2}$$

Assume the following conditions

(H1) There exists a constant $K_i > 0, i = 1, 2$ such that

$$\begin{aligned} |\mathsf{b}(t,x,y)|^2 - |\sigma(t,x,y)|^2 &\le K_1 \left(1 + |x|^2 + |y|^2\right), \\ \int_z |g(t,x,y)|^2 v(\mathsf{d}z) &\le K_2 \left(1 + |x|^2 + |y|^2\right). \end{aligned}$$

(H2) There exists a constant $\mathsf{L}_i > 0, i = 1, 2, 3$ such that

$$\begin{aligned} |\mathsf{b}(t,x_1,y_1) - \mathsf{b}(t,x_2,y_2)|^2 &\le \mathsf{L}_1^2 \left(|x_1 - x_2|^2 + |y_1 - y_2|^2\right), \\ |\sigma(t,x_1,y_1) - \sigma(t,x_2,y_2)|^2 &\le \mathsf{L}_2^2 \left(|x_1 - x_2|^2 + |y_1 - y_2|^2\right), \\ \int_z |\mathsf{g}(t,x_1,y_1) - \mathsf{g}(t,x_2,y_2)|^2 v(\mathsf{d}z) &\le \mathsf{L}_3^2 \left(|x_1 - x_2|^2 + |y_1 - y_2|^2\right). \end{aligned}$$

Theorem 3.3.1 *Assume that **(H1)** and **(H2)** holds. Let $\xi(t) \in \mathsf{L}^p_{\mathcal{F}_0}([-\delta, 0]; \mathbb{R}^n)$be a random variable defined on $(\Omega, \mathcal{F}, \mathcal{P})$ and independent of the $\sigma-$algebra $\mathcal{F}_s^t \subset \mathcal{F}$ generated by $\{\mathsf{W}(s), t \ge s \ge 0\}$ and such that $\mathbb{E}(\sup_{-\delta \le t \le 0} |\xi(t)|^2) < \infty$. Then the initial value problem (3.3.1) has a unique solution which is t−continuous with the property that $x(t,\omega)$ is adapted to the filtration $\mathcal{F}_t^{x_0}$ generated by x_0 and $\{\mathsf{W}(s)(.), s \le t\}$ and $\sup_{0 \le t \le T} \mathbb{E}\left[|x(t)|^2\right] < \infty$.*

By using successive approximation technique one can prove the existence and uniqueness of solutions.

3.4 Stability Analysis

In this section we study the exponentially asymptotic stability in the quadratic mean of a trivial solution [1]-[43].

Consider the following stochastic fractional nonlinear system with Lévy noise of the form

$$\left.\begin{array}{c} {}^{C}D^{\alpha}x(t) = Ax(t) + f(t, x(t)) + \sigma\left(t, x(t)\right) \frac{\mathsf{d}\mathsf{W}(t)}{\mathsf{d}\mathsf{t}} + \int_{z} g\left(t, x(t), z\right) \frac{d\tilde{N}(t,z)}{dt}, t \in J \\ x(0) = x_{0,} \end{array}\right\} \quad (3.4.1)$$

where $\alpha \in (\frac{1}{2}, 1)$ and $z \in \mathbb{R}_0^n = \mathbb{R}^n / \{0\}$, and $f \in C\left(J \times \mathbb{R}^n, \mathbb{R}^n\right)$,$\sigma \in C\left(J \times \mathbb{R}^n, \mathbb{R}^{nm}\right)$, $g \in C\left(J \times \mathbb{R}^n \times \mathbb{R}_0^n, \mathbb{R}^{nl}\right)$, $\tilde{\mathsf{N}}\left(\mathsf{d}t, \mathsf{d}z\right) = \mathsf{N}\left(\mathsf{d}t, \mathsf{d}z\right) - v(\mathsf{d}z)\mathsf{d}t$ which is the $l-$dimensional compensated jumb measure of $\eta\left(.\right)$ an independent compensated Poisson random measure and $\mathsf{W} = \{\mathsf{W}(t), t \geq 0\}$ is an $m-$dimensional Brownian motion on a complete probability space $\Omega = \left(\Omega, \mathcal{F}, \mathcal{P}\right), A \in \mathbb{R}^{n\times n}$ is a diagonal stability matrix. Assume from now on that $f(t, 0) = \sigma(t, 0)$ a.e t so that the equation (3.4.1)admits a trivial solution.

Definition 3.4.1 *The trivial solution of equation (3.4.1) is said to be exponentially stable in the quadratic mean if there exist positive constants C, v such that*

$$\mathbb{E}\left(\left|x(t)\right|^2\right) \leq C\mathbb{E}\left(\left|x_0\right|^2\right) \exp\left(-vt\right), t \geq 0.$$

The following lemmas are necessary to obtain the main results. For that we assume the following hypothesis [45]:

(H3) There exists a constant $\mathsf{M} > 0$ such that for $t \geq 0$,

$$\left|\mathsf{E}_{\alpha,\beta}\left(At^{\alpha}\right)\right| \leq \mathsf{M}e^{-\alpha t},$$

where $0 < \alpha < 1$ and $\beta = 1, 2,$ and α.

Lemma 3.4.1 *Assume that the hypothesis **(H3)** holds. Then for any stochastic process $F : [0, \infty) \to \mathbb{R}^n$ which is stongly measurable with $\int_0^t \mathbb{E}\left|F(t)\right|^2 \mathsf{d}s < \infty, t \geq 0,$ the following inequality holds for $0 < t \leq T$,*

$$\mathbb{E}\left|\int_0^t \mathsf{E}_{\alpha,\beta}\left(A\left(t-s\right)^{\alpha}\right) F\left(s\right) \mathsf{d}\left(s\right)\right|^2 \leq \left(\frac{\mathsf{M}^2}{a}\right) \int_0^t \exp\left(-a\left(t-s\right)\right) \mathbb{E}\left|F\left(s\right)\right|^2 \mathsf{d}s,$$

where $\alpha \in \left(\frac{1}{2}, 1\right)$ and $\beta = 1, 2$ and α.

Lemma 3.4.2 *Assume that the hypothesis **(H3)** holds. Then for any* B_t-*adapted predictable process* $\Phi : [0,\infty) \to \mathbb{R}^n$ *with* $\int_0^t \mathbb{E}\left|\Phi(s)\right|^2 \mathsf{d}s < \infty, t \geq 0$*, the following inequality holds for* $0 < t \leq T$,

$$\mathbb{E}\left|\int_0^t \mathsf{E}_{\alpha,\beta}\left(A\left(t-s\right)^{\alpha}\right)\Phi\left(s\right)\mathsf{dW}\left(s\right)\right|^2 \leq \mathsf{M}^2\int_0^t \exp\left(-a\left(t-s\right)\right)\mathbb{E}\left|\Phi\left(s\right)\right|^2 \mathsf{d}s,$$

where $\alpha \in \left(\frac{1}{2},1\right)$ *and* $\beta = 1,2$ *and* α.

Theorem 3.4.1 *Let the assumptions of Theorem 3.3.1 holds. Then the solution of equation (3.4.1) is exponentially stable in the quadratic mean provided*

$$a > \beta = \beta\left(a,K,\mathsf{M}\right) = \frac{4\mathsf{M}^2\left(2K^2/a+K^2\right)T^{2\alpha-1}}{2\alpha-1}.$$

Proof 3.4.1 *The integral form of the equation (3.4.1) can be given by [24]-[26]*

$$\begin{aligned} x(t) &= \mathsf{E}_{\alpha}\left(At^{\alpha}\right)x_0 + \int_0^t (t-s)^{\alpha-1}\mathsf{E}_{\alpha,\alpha}\left(A\left(t-s\right)^{\alpha}\right)\mathsf{b}\left(s,x(s)\right)\mathsf{d}s \\ &+ \int_0^t (t-s)^{\alpha-1}\mathsf{E}_{\alpha,\alpha}\left(A\left(t-s\right)^{\alpha}\right)\sigma\left(s,x(s)\right)\mathsf{dW}\left(s\right) \\ &+ \int_0^t (t-s)^{\alpha-1}\mathsf{E}_{\alpha,\alpha}\left(A\left(t-s\right)^{\alpha}\right)\int_z \mathsf{g}\left(s,x(s),z\right)\tilde{\mathsf{N}}\left(\mathsf{d}s,\mathsf{d}z\right). \end{aligned} \tag{3.4.2}$$

Applying the algebraic inequality $(a+b+c+d)^2 \leq 4\left(a^2+b^2+c^2+d^2\right)$ *we have*

$$\begin{aligned} |x(t)|^2 &\leq 4\left(\left|\mathsf{E}_{\alpha}\left(At^{\alpha}\right)x_0\right|\right)^2 + 4\left(\left|\int_0^t (t-s)^{\alpha-1}\mathsf{E}_{\alpha,\alpha}\left(A\left(t-s\right)^{\alpha}\right)\mathsf{b}\left(s,x(s)\right)\mathsf{d}s\right|\right)^2 \\ &+4\left(\left|\int_0^t (t-s)^{\alpha-1}\mathsf{E}_{\alpha,\alpha}\left(A\left(t-s\right)^{\alpha}\right)\sigma\left(s,x(s)\right)\mathsf{dW}\left(s\right)\right|\right)^2 \\ &+4\left(\left|\int_0^t (t-s)^{\alpha-1}\mathsf{E}_{\alpha,\alpha}\left(A\left(t-s\right)^{\alpha}\right)\int_z \mathsf{g}\left(s,x(s),z\right)\tilde{\mathsf{N}}\left(\mathsf{d}s,\mathsf{d}z\right)\right|\right)^2. \end{aligned}$$

By using Hölder inequality and Lemmas (3.4.1) and (3.4.2) we get

$$\begin{aligned} \mathbb{E}\left|x\left(t\right)\right|^2 &\leq 4\mathsf{M}^2\exp\left(-at\right)\mathbb{E}\left|x_0\right|^2 + 4\left(\mathsf{M}^2/a\right)\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \exp(-a(t-s))\mathbb{E}\left|\mathsf{b}\left(s,x(s)\right)\right|^2\mathsf{d}s \\ &+4\mathsf{M}^2\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \exp(-a(t-s))\mathbb{E}\left|\sigma\left(s,x(s)\right)\right|^2\mathsf{d}s \\ &+4\left(\mathsf{M}^2/a\right)\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \exp(-a(t-s))\mathbb{E}\left|\int_z \mathsf{g}\left(s,x(s),z\right)v\left(\mathsf{d}z\right)\right|^2\mathsf{d}s. \end{aligned}$$

Linear growth assumtion (3.2.2) when $\mathsf{b}(t,0)=\sigma(t,0)\equiv 0$ *a.e* t *yields*

$$\begin{aligned}\exp(at)\mathbb{E}\left|x(t)\right|^{2} &\leq 4\mathsf{M}^{2}\mathbb{E}\left|x_{0}\right|^{2}+4\left(\mathsf{M}^{2}/a\right)K^{2}\frac{T^{2\alpha-1}}{2\alpha-1}\int_{0}^{t}\exp(as)\mathbb{E}\left|x(s)\right|^{2}\mathsf{d}s\\ &+4\mathsf{M}^{2}K^{2}\frac{T^{2\alpha-1}}{2\alpha-1}\int_{0}^{t}\exp(as)\mathbb{E}\left|x(s)\right|^{2}\mathsf{d}s\\ &+4\left(\mathsf{M}^{2}/a\right)K^{2}\frac{T^{2\alpha-1}}{2\alpha-1}\int_{0}^{t}\exp(as)\mathbb{E}\left|x(s)\right|^{2}\mathsf{d}s\\ &\leq 4\mathsf{M}^{2}\mathbb{E}\left|x_{0}\right|^{2}+4\mathsf{M}^{2}\left(\frac{2K^{2}}{a+K^{2}}\right)\frac{T^{2\alpha-1}}{2\alpha-1}\int_{0}^{t}\exp(as)\mathbb{E}\left|x(s)\right|^{2}\mathsf{d}s.\end{aligned}$$

Applying Gronwall's inequality, we obtain

$$\exp(at)\mathbb{E}\left|x(t)\right|^{2}\leq 4\mathsf{M}^{2}\mathbb{E}\left|x_{0}\right|^{2}\exp\left(4\mathsf{M}^{2}\left(\frac{2K^{2}}{a+K^{2}}\right)\frac{T^{2\alpha-1}}{2\alpha-1}t\right).$$

Consequently,

$$\mathbb{E}\left|x(t)\right|^{2}\leq C\mathbb{E}\left|x_{0}\right|^{2}\exp\left(-vt\right),\ \ t\geq 0, \tag{3.4.3}$$

where $v=a-\beta$ *and* $C=4\mathsf{M}^{2}$.

Next consider the nonlinear stochastic fractional delay differential equation of the form

$$\begin{aligned}{}^{C}\mathsf{D}^{\alpha}x\left(t\right)&=Ax\left(t\right)+f\left(t,x\left(t\right),x\left(t-\delta\right)\right)+\sigma\left(t,x\left(t\right),x\left(t-\delta\right)\right)\tfrac{\mathsf{dW}(t)}{\mathsf{dt}}\\ &+\textstyle\int_{z}g\left(t,x\left(t\right),x\left(t-\delta\right),z\right)\tfrac{\mathsf{d\tilde{N}}(t,z)}{\mathsf{dt}},t\in J=[0,T]\\ x\left(t\right)&=\xi\left(t\right),\quad t\in\left[-\delta,0\right]\end{aligned} \tag{3.4.4}$$

where $\alpha\in\left(\frac{1}{2},1\right)$ *and* $z\in\mathbb{R}_{0}^{n}=\mathbb{R}^{n}/\left\{0\right\}$. *Here* $f\in C\left(J\times\mathbb{R}^{n}\times\mathbb{R}^{n},\mathbb{R}^{n}\right),\sigma\in C\left(J\times\mathbb{R}^{n}\times\mathbb{R}^{n},\mathbb{R}^{nm}\right),g\in C\left(J\times\mathbb{R}^{n}\times\mathbb{R}^{n}\times\mathbb{R}_{0}^{n},\mathbb{R}^{nl}\right),\tilde{\mathsf{N}}\left(\mathsf{d}t,\mathsf{d}z\right)=\mathsf{N}\left(\mathsf{d}t,\mathsf{d}z\right)-v\left(\mathsf{d}z\right)\mathsf{d}t$ *which is the* $l-$*dimensional compensated jump measure of* $\eta\left(.\right)$ *an independent compensated Poisson random measure and* $\mathsf{W}=\left\{\mathsf{W}\left(t\right),t\geq 0\right\}$ *is an* $m-$*dimensional Brownian motion on a complete probability space* $\Omega\equiv\left(\Omega,\mathcal{F},\mathcal{P}\right)$, $A\in\mathbb{R}^{n\times n}$ *is a diagonal stability matrix. Assume from now on that* $f\left(t,0\right)=\sigma\left(t,0\right)\equiv 0$ *a.e* t *so that equation (3.4.1) admits a trivial solution. The integral from of the equation (3.4.4) in terms of the Mittag Leffer function is given by*

$$\begin{aligned}x\left(t\right) &= \mathsf{E}_{\alpha}\left(At^{\alpha}\right)\xi\left(0\right)+\int_{0}^{t}\left(t-s\right)^{\alpha-1}\mathsf{E}_{\alpha,\alpha}\left(A\left(t-s\right)^{\alpha}\right)\mathsf{b}\left(s,x\left(s\right),x\left(s-\delta\right)\right)\mathsf{d}s\\ &+\int_{0}^{t}\left(t-s\right)^{\alpha-1}\mathsf{E}_{\alpha,\alpha}\left(A\left(t-s\right)^{\alpha}\right)\sigma\left(s,x\left(s\right),x\left(s-\delta\right)\right)\mathsf{dW}\left(s\right)\\ &+\int_{0}^{t}\left(t-s\right)^{\alpha-1}\mathsf{E}_{\alpha,\alpha}\left(A\left(t-s\right)^{\alpha}\right)\int_{\mathsf{z}}\mathsf{g}\left(s,x\left(s\right),x\left(s-\delta\right)\right)\tilde{\mathsf{N}}\left(\mathsf{d}s,\mathsf{d}z\right).\end{aligned} \tag{3.4.5}$$

Theorem 3.4.2 *Let assumptions of the Theorem 3.3.1 holds. Then the solution of the delay differntial equation (3.4.4) is exponentially stable in the quadratic mean provided*

$$\alpha > \beta = \beta(a, K_1, K_2, \mathsf{M}) = \frac{4\mathsf{M}^2\left(K_1^2\left(\frac{1}{a+1}\right) + K_2^2\right)T^{2\alpha-1}}{2\alpha - 1}$$

Proof 3.4.2 *Using the hypothesis **(H3)**, Lemma [3.4.1] and [3.4.1] one can prove the theorem. The proof is similar to the previous theorem and hence omitted.*

3.5 Examples

Example 1 Consider the followings stochastic fractional differntial equation with Lévy noise of the form

$$\left.\begin{aligned} {}^{C}\mathsf{D}^{\alpha}x(t) + 0.6x(t) = \tfrac{t^{2-\alpha}}{\Gamma(1-\alpha)} + \tfrac{1}{1+t}\tfrac{\mathsf{dW}(t)}{\mathsf{dt}} + \textstyle\int_{\mathbb{R}/\{0\}} tz\tfrac{\mathsf{d\tilde{N}}(t,z)}{\mathsf{dt}}, t \in J \\ x(0) = 1. \end{aligned}\right\} \tag{3.5.1}$$

Here $\mathsf{b}(t, x(t)) = -0.6x(t) + \frac{t^{2-\alpha}}{\Gamma(1-\alpha)}, \sigma(t, x(t)) = \frac{1}{1+t}$ and $g(t, x(t), z) = tz$. It can be easily seen that $\mathsf{b}(t, x(t)), \sigma(t, x(t))$ and $g(t, x(t), z)$ satisfies the condition of (3.2.2) and (3.2.3) of Theorem (3.2.1). Hence by the Theorem (3.2.1) the stochastic fractional differential equation (3.5.1) has a unique solution. Also the equation (3.5.1) satisfy the condition of Theorem (3.4.1) So from Theorem (3.4.1) the stochastic fractional differntial equation with $A = 0.6$ is exponentially stable.

Example 2 Consider the following stochastic fractional differntial equation with Lévy noise of the form

$$\left.\begin{aligned} {}^{C}\mathsf{D}^{\alpha}x(t) + 0.4x(t) = \tfrac{t^3 y}{\Gamma(2-\alpha)} + t^2\tfrac{\mathsf{dW}(t)}{\mathsf{dt}} + \textstyle\int_{\mathbb{R}/\{0\}} zy\tfrac{\mathsf{d\tilde{N}}(t,z)}{\mathsf{dt}}, t \in J \\ x(t) = 0.\ t \in [-t, 0] \end{aligned}\right\} \tag{3.5.2}$$

Here $\mathsf{b}(t, x(t), y(t)) = -0.4x(t) - \frac{t^3 y}{\Gamma(22-\alpha)}, \sigma(t, x(t), y(t)) = t^2$ and $g(t, x(t), y(t), z) = zy$. It can be easily seen that $\mathsf{b}(t, x(t), y(t)), \sigma(t, x(t), y(t))$ and $g(t, x(t), y(t), z)$ satisfies the assumptions **(H1)** and **(H2)** of Theorem []. Hence by the Theorem3.2 the stochastic fractional differential equation (3.5.2)has a unique solution. Also the equation (3.5.2) satisfy the condition of Theorem (3.4.2). So from Theorem (3.4.2) the stochastic fractional differntial equation with $A = 0.4$ is exponentially stable.

Conclusion

In our study, we invertiged Existence and Uniquness solutions of fractional stochastic equations using caputo hadamard derivatives utilize Levy noise definitions, and we finished our results by an example to solve our problem.

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