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Présentée par

SEDKA Ilyes

Intitulée

Généralisation des techniques de linéarisation pour approcher les systèmes d'équations intégrales non linéaires

Soutenue le : 08/07/2023 Devant le Jury composé de :

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TO MY FAMILY

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Abstract

The objective of this thesis is to improve numerical linearization techniques for solving systems of nonlinear integral equations, which play a crucial role in modeling various problems across domains such as physics, biology, and machine learning. The study focuses on Banach spaces. Traditionally, most methods involve discretizing the problem first and then proceeding with the linearization process to solve nonlinear functional problems. However, we propose a novel approach that reverses this order, starting with the linearization process using Newton's iterative method, and then discretizing the iterative linear system obtained from the first phase. We apply this approach to solve a variety of problems, starting with a system of nonlinear Fredholm integral equations of the second kind with regular kernels, using the Nyström method for discretization. Additionally, we introduce a new scheme with a double discretization process that utilizes the Kantorovich projection method to approximate solutions for nonlinear functional equations, specifically nonlinear integro-differential equations of the second kind. Finally, we apply our approach to solve a system of nonlinear integro-differential equations with a weakly singular kernel, estimating all integrals with weakly singular kernels using product integration rules. We have provided necessary conditions for each application to ensure the convergence of our methods.

Keywords: System of nonlinear Fredholm integral equations, System of nonlinear integro-differential equations, Kantorovich projection, Newton Iterative Method, Nystöm quadrature method, Product integration process, Linearization-Discretization

Mathematics Subject Classification: 45B05, 47G20, 45F05, 65R10, 65F10, 64R20

Résumé

L'objectif de cette thèse est d'améliorer des techniques de linéarisation numérique pour résoudre les systèmes d'équations intégrales non linéaires, qui jouent un rôle crucial dans la modélisation de divers problèmes dans des domaines tels que la physique, la biologie et l'apprentissage automatique. L'étude se concentre sur les espaces de Banach.
Traditionnellement, la plupart des méthodes impliquent d'abord la discrétisation du problème, puis le processus de linéarisation pour résoudre les problèmes fonctionnels non linéaires. Cependant, nous proposons une nouvelle approche qui inverse cet ordre, en commençant par le processus de linéarisation à l'aide de la méthode itérative de Newton, puis en discrétisant le système linéaire itératif obtenu lors de la première phase.
Nous appliquons cette approche pour résoudre une variété de problèmes, en commençant par un système d'équations intégrales non linéaires de Fredholm du second type avec des noyaux réguliers, en utilisant la méthode de Nyström pour la discrétisation. En outre, nous introduisons un nouveau schéma avec un processus de double discrétisation qui utilise la méthode de projection de Kantorovich pour approximer les solutions des équations fonctionnelles non linéaires, en particulier les équations intégro-différentielles non linéaires du second type. Enfin, nous appliquons notre approche pour résoudre un système d'équations intégro-différentielles non linéaires avec un noyau faiblement singulier, en estimant toutes les intégrales avec des noyaux faiblement singuliers à l'aide de règles d'intégration de produits. Nous avons fourni les conditions nécessaires pour chaque application afin d'assurer la convergence de nos méthodes.

Mots clé : Système d'équations intégrales non linéaires de Fredholm, Système d'équations intégro-différentielles non linéaires, Projection de Kantorovich, Méthode itérative de Newton, Méthode de quadrature de Nystöm, Processus d'intégration des produits, Linéarisation-Discrétisation.

Classification des sujets de mathématiques :45B05, 47G20, 45F05, 65R10, 65F10, 64R20

الملخص

هدف هذه الأطروحة هو تحسين تقنيات التخطيط العددي لحل نظم المعادلات التكاملية غير الخطية، التي تلعب دورًا حاسمًا في نمذجة مشاكل متنوعة في مجالات مثل الفيزياء والبيولوجيا وتعلم الآلة.

يركز البحث على فضاءات باناخ. تقوم معظم الأساليب التقليدية بتقسيم المشكلة أولاً ثم المتابعة بعملية التخطيط الخطي لحل المشاكل الوظيفية غير الخطية. نقترح نهجًا جديدًا يعكس هذا الترتيب، بدءًا من عملية التخطيط الخطي باستخدام طريقة نيوتن التكرارية، ثم تقسيم النظام الخطي التكراري الذي تم الحصول عليه في المرحلة الأولى.

نطبق هذا النهج لحل مجموعة متنوعة من المشاكل، بدءًا من نظام المعادلات التكاملية لفريدولم غير الخطية من النوع الثاني بنوى منتظمة، باستخدام طريقة نيشتروم. بالإضافة إلى ذلك، نقدم مخططًا جديدًا يستخدم عملية تقسيم مزدوجة بإستخدام طريقة الإسقاط لكانتوروفيتش لتقريب حلول المعادلات الوظيفية غير الخطية، على وجه التحديد المعادلات التفاضلية التكاملية غير الخطية من النوع الثاني.

أخيرًا، نقوم بتطبيق نهجنا لحل نظام من المعادلات التفاضلية التكاملية غير الخطية ذو نواة مفردة ضعيفة ، مقدرين جميع التكاملات ذات النوى المفردة الضعيفة باستخدام قواعد انتاج التكامل. لقد قدمنا الشروط اللازمة لكل تطبيق لضمان تقارب طرقنا.

الكلمات المفتاحية: نظام المعادلات التكاملية غير الخطية لفريدولم، نظام المعادلات التفاضلية التكاملية غير الخطية، طريقة إسقاط كانتوروفيتش، طريقة نيوتن التكرارية، طريقة تكامل نيشتروم، عملية انتاج التكامل، عملية التخطيط الخطي-التقسيم.

تصنيف مواضيع الرياضيات: 45B05, 47G20, 45F05, 65R10, 65F10, 64R20

Notations

$\mathbb{R}$: Set of real numbers.

$\mathbb{C}$: Set of complex numbers.

$C^0([a,b],\mathbb{R})$: The Banach space of continuous functions.

$C^1([a,b],\mathbb{R})$: The Banach space of continuously differentiable functions.

$C^m([a,b],\mathbb{R})$: The Banach space of continuously m differentiable functions.

$W^{1,p}([a,b],\mathbb{R}),\ \ p\ \in [1,+\infty[$: The Sobolev space, which vector space of functions that have weak derivative.

$H^1([a,b],\mathbb{R})$: The Sobolev space with $p=2$.

$L^P([a,b],\mathbb{R}),\ \ p\ \in [1,+\infty[$: The vector space of classes of functions whose exponent power p is integrable in the Lebesgue sense.

T^{-1}: The inverse of operator T.

$0_{\mathcal{X}}$: The null operator of the Banach space $\mathcal{X}$.

$re(T)$: The resolvant set of T.

$R(T,\lambda)$: The resolvant of operator T.

$sp(T)$: The spectrum set of T.

$\mathcal{L}(\mathcal{X},\mathcal{Y})$: The space of linear and bounded operator.

M_T: The block operator matrix.

M_{T_n}: The block approximate operator matrix of M_T.

$\langle\cdot,\cdot\rangle$: Dual product.

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Introduction

The field of integral equations (IEs) has a rich and lengthy past in both pure and applied mathematics, dating as far back as the 1800s and attributed to Fourier's [1] Theorem [12]. One notable early application of IEs can be found in the solution of Dirichelet's [2] problem, a partial differential equation (PDE), through its integral formulation. The work of prominent mathematicians such as Fredholm [3], Volterra [4], Hilbert [5] and Schmidt [6] has further advanced and solidified the foundations of this theory.

Integral equations (IEs) have a broad range of applications across various fields such as physics, chemistry, biology, and engineering, as evidenced by works of [1, 12]. Examples of such applications include potential theory, diffraction, and inverse problems like scattering in quantum mechanics [1, 12, 52]. Furthermore, given the highly non-local nature of neural field equations that describe brain activity, they can be effectively described using IEs and integro-differential equations (IDEs) [47].

Integral equations (IEs) share a connection with the theories of ordinary differential equations (ODEs) and partial differential equations (PDEs), but they also possess distinct properties. Unlike ODEs and PDEs, which describe local behavior, IEs capture global (long-distance) spatiotemporal relationships. Additionally, IEs offer better stability properties for solving ODE and PDE models, making them more effective and efficient compared to ODE and PDE solvers [53, 54].

Integral equations are being increasingly used to model a variety of problems in data science, as demonstrated by the recent trend in utilizing them to address learning dynamics problems, which involve understanding how a system changes over time based on available data. To tackle such problems, researchers have proposed using integral equations and introduced two models: the Neural Integral Equation (NIE) and the Attentional

[1] Joseph Fourier (1768-1830) was a French mathematician and physicist who developed the Fourier series, a method for representing periodic functions as a sum of sine and cosine functions.

[2] Peter Gustav Lejeune Dirichlet (1805-1859) was a prominent German mathematician who made significant contributions to number theory, Fourier analysis, and the theory of functions.

[3] Erik Ivar Fredholm (1866-1927) was a Swedish mathematician known for his contributions to the theory of integral equations and linear operators

[4] Vito Volterra (1860-1940) was an Italian mathematician and physicist known for his contributions to mathematical biology, integral equations, and mathematical physics.

[5] David Hilbert (1862-1943) was a German mathematician who made profound contributions to many areas of mathematics, including algebraic number theory, geometry, mathematical physics and logic.

[6] Erhard Schmidt (1876-1959) was a German mathematician who made significant contributions to many areas of mathematics, including analysis, geometry, and number theory.

Neural Integral Equation (ANIE). The aim of these models is to learn an integral operator that can accurately represent the dynamics observed in a given data set.

The passage refers to the "operator learning problem", which arises when we have observations of a system's behavior over time, but lack a mathematical formula to describe how the system evolves. To address this problem, researchers suggest using integral equations to learn the system's behavior solely from the available observations.

Their model has the ability to predict how a system will evolve over time by generating new dynamics, and it can also be utilized to infer the spatiotemporal relationships responsible for producing the observed data. Essentially, the integral operator learned from the model is capable of describing the fundamental patterns and connections that control the behavior of the system.

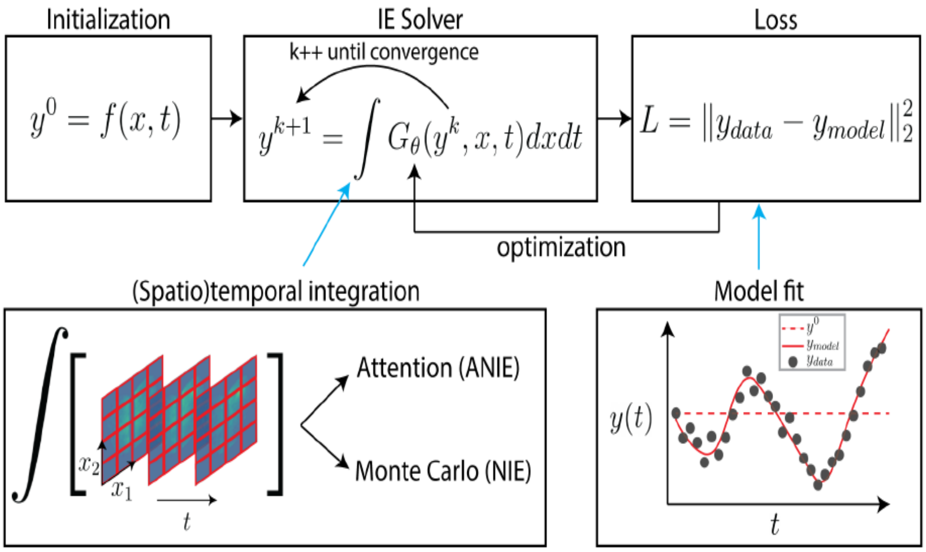

Figure 1: Diagrammatic representation of the model.

The proposed method presents a novel approach to the operator learning problem for dynamics, in which we formulate the problem as an optimization problem for an IE solver. This sets the method apart from other operator learning methods that learn dynamics as a mapping between fixed time points in function spaces, represented by $\mathcal{X}_i$ and $\mathcal{Y}_j$, denoted as $T : \prod_i \mathcal{X}_i \longrightarrow \prod_j \mathcal{Y}_j$. Unlike these methods, NIE and ANIE enable the continuous learning of dynamics with arbitrary time resolution. As IEs are functional equations, our solver adopts an iterative procedure [1] to output solutions, which ultimately converge to a solution of the IE.

Integral equations are a versatile mathematical tool that has the potential to model and analyze diverse phenomena across various fields. They offer a comprehensive understanding of system behavior, rendering them an indispensable tool for researchers and practitioners in different domains. Owing to advancements in computational and numerical techniques, they are becoming more accessible and useful in an expanding range of applications.

The recent trend in research has been focused on proposing and developing numerical methods for solving nonlinear integral equations of the second kind and nonlinear integro-differential equations. Various techniques have been introduced to approximate the solutions of these equations, as demonstrated by recent literature. In [37], a new approach combining the modified Adomian decomposition method and quadrature rules has been proposed for approximating the solutions of the nonlinear Volterra-Fredholm integral equations of the second kind with a phase lag. Similarly, in [29], a class of Hermite interpolation polynomials has been utilized to obtain an approximate solution for nonlinear Fredholm integral equations of the second kind.

In [38], the authors have employed the quasilinearization technique to generate a sequence of linear equations with low smooth solutions that quadratically converge to the unique solution of the nonlinear equation. Another proposed numerical solution for Fredholm integral equations of the second kind with weakly singular kernel is presented in [57], utilizing the spectral collocation method. In [26], the dual series equation method has been used to transform the mixed problem to a Fredholm integral equation of the second kind. Lastly, in [4], the Legendre-Kantorovich method has been introduced for solving Fredholm integral equations of the second kind, with the order of convergence of the proposed method and the super convergence of the iterated versions being established.

Recent research articles have proposed novel solutions and approaches to solve different types of integro-differential equations. In [14], a new general solution has been suggested for linear Fredholm integro-differential equations along with its applications in solving boundary value problems, and solvability criteria for the equation are provided. In [50], the product integration method has been used to approach the solution of nonlinear integro-differential Fredholm equations with a weakly singular kernel. [18] introduces a new approach for solving nonlinear Fredholm integro-differential equations using rationalized Haar wavelet bases in a complex plane. In [40], efficient numerical methods are proposed for solving second-kind Fredholm integral equations over infinite intervals using the Sinc quadrature formula. In [8], various methods, including the Adomian decomposition method, modified Adomian decomposition method, variational iteration method, and homotopy perturbation method, are utilized to solve fuzzy integro-differential equations. Finally, in [43], a 2D-BPF method is employed to convert nonlinear two-dimensional mixed Volterra-Fredholm integro-differential equations into an algebraic system of equations that can be computed.

This thesis consists of four crucial chapters. The first chapter serves as a foundation that introduces important preliminary concepts which are vital for understanding the following chapters. In other words, the first chapter is necessary to comprehend before moving on to the subsequent ones.

The second chapter of the thesis introduces a novel method for solving second-kind nonlinear Fredholm integral equation systems. Our approach (LN.DN) involves two stages, where we first use the Newton[7] method to linearize the integral equations system, followed by utilizing the Nyström[8] method to discretize a specific integrals operator. In addition to presenting our new approach, we also compare it to the classical process (DN.LN) where the discretization phase is performed first followed by the linearization process. We have established that our method converges under specific requirements. To demonstrate the effectiveness of our approach compared to the classical one, we worked on numerical system of nonlinear Fredholm integro-differential equations and a system of Fredholm equations of the second kind.

Chapter three presents a novel numerical scheme for solving systems of nonlinear integral equations of the second kind. After confirming the superiority of linearizing before discretizing over the inverse approach, we introduce the new method (L.D.D). It involves beginning with the linearization stage, applying the Newton iterative process, and then using a double discretization process with Kantorovich[9] projection. This approach provides a distinct theoretical framework that ensures convergence under certain assumptions. To demonstrate its effectiveness, we provide numerous examples, including a comparison with the outcomes of a recently published study [31], and its application to systems of nonlinear integro-differential equations. The numerical results show that the (L.D.D) method is more efficient in solving systems of nonlinear integral equations of the second kind.

In the fourth chapter, we focus on a specific type of nonlinear integro-differential Fredholm system of equations that feature a weakly singular kernel. Our proposed method, referred to as (LN.DK), builds upon the findings of the previous chapter, which established that the linearizing-discretizing (L.D) scheme is superior to traditional methods. In our approach, we use Newton's iterative method for the linearization stage and Kantorovich's projection method for discretization. The weakly singular kernels that arise in the linear discretized problem are estimated using the product integral rule, while regular kernels are approximated using the Nyström method. To guarantee convergence, we establish specific conditions that must be satisfied throughout the process. These conditions are essential for the success of our algorithm. Finally, we present practical examples that demonstrate the effectiveness of our new algorithm in solving systems of weakly singular nonlinear integro-differential equations. Our results confirm the efficiency and accuracy of our approach in tackling such problems.

[7]Sir Isaac Newton (1643-1727) was an English mathematician, physicist, and astronomer. He is best known for his work on the laws of motion and gravity, which laid the foundation for classical mechanics and transformed our understanding of the physical world.

[8]Nils Aall Barricelli Nyström (1904-1975) was a Swedish mathematician who made important contributions to the theory of approximation, numerical analysis, and differential equations.

[9]Leonid Vitaliyevich Kantorovich (1912-1986) was a Soviet mathematician and economist who made significant contributions to the theory of optimization, linear programming, and functional analysis.

Chapter 1

Preliminary Concepts

Contents

This chapter presents an essential overview of the background required for the rest of the thesis. It will cover notation, terminology, and fundamental concepts, including key theorems from operator theory that will be used later. We assume the reader has a basic understanding of linear spaces, normed spaces, and Banach[1] spaces.

The chapter begins with the fundamentals of linear operators, including notations and theorems, all these presented in section 1.1. Next, we provide a concise overview of differential calculus for nonlinear operators in section 1.2 before discussing the linearization of nonlinear equations through section 1.3. It's important to classify the Fredholm integral equations before proceeding, where we do this in section 1.4. Finally, in section 1.5 we will define the principal numerical methods for solving Fredholm integral equations.

1.1 Notions and Fundamental Theorems of Linear Operators

Let $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{Y}}$ denotes the norm on a Banach space $\mathcal{X}$ and $\mathcal{Y}$ respectively. Also the (induced uniform) operator norm in $\mathcal{L}(\mathcal{X},\mathcal{Y})$ the Banach space of linear and bounded operators from $\mathcal{X}$ into $\mathcal{Y}$ is given for all $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ by

$$\|T\| = \sup\left\{\|T\phi\|_{\mathcal{Y}} \,:\, \|\phi\|_{\mathcal{X}} \leq 1\right\} = \sup\left\{\|T\phi\|_{\mathcal{Y}} \,:\, \|\phi\|_{\mathcal{X}} = 1\right\} = \sup\left\{\frac{\|T\phi\|_{\mathcal{Y}}}{\|\phi\|_{\mathcal{X}}} \,:\, \phi \neq 0_{\mathcal{X}}\right\}.$$

Definition 1.1.1. *[3] The approximations sequence $(T_n)_{n\in\mathbb{N}^*} \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ of T converges pointwise (or simply) to T, if and only if, for all $\phi \in \mathcal{X}$*

$$\lim_{n\to+\infty} \|(T_n - T)\,\phi\|_{\mathcal{Y}} = 0.$$

Definition 1.1.2. *[3] $(T_n)_{n\in\mathbb{N}^*}$ is called a uniform approximations sequence of $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ if and only if*

$$\lim_{n\to+\infty} \|T_n - T\| = 0.$$

Theorem 1.1.1 (Banach's Fixed Point Theorem). *[2]*
Let $T : \mathcal{X} \longrightarrow \mathcal{Y}$ be a contraction, i.e., there is a number $0 \leq \lambda < 1$ such that

$$\|T\phi - T\psi\|_{\mathcal{Y}} \leq \lambda\,\|\phi - \psi\|_{\mathcal{X}}\,, \quad \forall \phi, \psi \in \mathcal{X}.$$

Then the following holds:

(i) *The fixed point equation*

$$\varphi = T\varphi, \tag{1.1}$$

has exactly one solution $\varphi_{ext} \in \mathcal{X}$.

[1]Stefan Banach (1892-1945) was a Polish mathematician who made significant contributions to functional analysis, topology, measure theory. He is best known for his development of the theory of Banach spaces

(ii) *For each starting value $\varphi^{(0)} \in \mathcal{X}$, the sequence $\varphi^{(k)}$ defined by*

$$\varphi^{(k+1)} = T\varphi^{(k)}, \quad k = 0, 1, 2, \cdots$$

converges to the exact solution φ_{ext} of (1.1).

(iii) *The following error estimate holds*

$$\left\|\varphi_{ext} - \varphi^{(k)}\right\|_{\mathcal{X}} \leq \frac{\lambda^k}{1-\lambda} \left\|\varphi^{(1)} - \varphi^{(0)}\right\|_{\mathcal{X}}.$$

Proof. See [2].

□

Theorem 1.1.2 (**Banach Steinhaus**[2]). *[7]*
Let $(T_n)_{n\in\mathbb{N}^*} \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ a sequence of linear and continuous operators. Assuming

$$\sup_{n\in\mathbb{N}^*} \|T_n\phi\|_{\mathcal{Y}} < \infty, \quad \forall \phi \in \mathcal{X}.$$

Then,

$$\sup_{n\in\mathbb{N}^*} \|T_n\| < \infty.$$

In other words, there is a positive constant $\alpha > 0$, such that

$$\|T_n\phi\|_{\mathcal{Y}} \leq \alpha\|\phi\|_{\mathcal{X}}, \quad \forall\phi \in \mathcal{X}, \forall n \in \mathbb{N}^*.$$

This theorem is commonly referred to as the principle of uniform boundedness and its proof and further details can be found in [7, 11].

Definition 1.1.3. *[3] Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, we define*

- *The kernel (or null space) of a linear operator T as*

$$Ker(T) = \left\{\phi \in \mathcal{X} \ : \ T\phi = 0_{\mathcal{Y}}\right\}.$$

- *The range of T as the image of $\mathcal{X}$ under T, i.e.,*

$$Ran(T) = T(\mathcal{X}) = \left\{\psi \in \mathcal{Y} : \psi = T\phi, \forall\phi \in \mathcal{X}\right\}.$$

The operator T is called a finite rank operator if $dim(Ran(T)) < +\infty$.

Proposition 1.1.1. *[3] The operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is*

- *Injective if $Ker(T) = \{0_{\mathcal{X}}\}$.*
- *Surjective if $Ran(T) = \mathcal{Y}$.*

[2]Hugo Steinhaus (1887-1972) was a Polish mathematician known for his work on the foundations of mathematics, game theory, and the theory of partitions of integers. He was also a prolific writer and popularize of mathematics.

- *Bijective if it is injective and surjective.*

Definition 1.1.4. *Let $T \in \mathcal{L}(\mathcal{X})$, the resolvent set of T is defined by*

$$re(T) = \Big\{ \lambda \in \mathbb{C} : \lambda I - T \text{ is bijective } \Big\},$$

and for $\lambda \in re(T), R(T, \lambda) = (\lambda I - T)^{-1}$, is called the resolvent operator of T at λ.

Definition 1.1.5. *Let $T \in \mathcal{L}(\mathcal{X})$, the spectrum of T is the set*

$$sp(T) = \mathbb{C} \backslash re(T) = \Big\{ \lambda \in \mathbb{C} : \lambda I - T \text{ has no inverse } \Big\},$$

and an element of $sp(T)$ will be called a spectral value of T. However, $\lambda \in sp(T)$ is called an eigenvalue, if the equation $(\lambda I - T)\phi = 0_{\mathcal{X}}$ has a non null solution $\phi \in \mathcal{X}$.

Theorem 1.1.3 (Inverse Mapping Theorem). *[34]*
Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ a bijective operator, then $T^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$.

Proof. An invertible transformation T is precisely a bijective one. Since the inverse T^{-1} of an invertible linear operator is linear, and since an invertible transformation is open if and only if it has a continuous inverse, the stated result follows from the Open Mapping Theorem (See [7], pp 71). □

Theorem 1.1.4. *[39]*
Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. If $\|T\| < |\lambda|$, then $\lambda I - T$ is bijective and

$$\left\| (\lambda I - T)^{-1} \right\| \leq \frac{1}{|\lambda| - \|T\|}.$$

Proof. See [39]. □

Theorem 1.1.5 (Neumann[3] Expansion). *[34]*
If $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and if λ is any scalar such that $\|T\| < |\lambda|$. Then $\lambda I - T$ has an inverse in $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ given by the following uniformly convergent series:

$$(\lambda I - T)^{-1} = \frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{T}{\lambda} \right)^k.$$

[3] Carl Neumann (1832-1925) was a German mathematician who made significant contributions to algebraic geometry, differential equations, and mathematical physics.

Proof. Take an operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and a nonzero scalar λ. If $\|T\| < |\lambda|$, then

$$\sup_{n \in \mathbb{N}^*} \sum_{k=0}^{n} \left\| \left(\frac{T}{\lambda} \right) \right\|^k < \infty.$$

Thus, since

$$\sum_{k=0}^{n} \left\| \left(\frac{T}{\lambda} \right)^k \right\| \leq \sum_{k=0}^{n} \left\| \left(\frac{T}{\lambda} \right) \right\|^k, \quad \forall n \geq 0,$$

the increasing sequence of nonnegative numbers $\left\{ \sum_{k=0}^{n} \left\| \left(\frac{T}{\lambda} \right)^k \right\| \right\}$ is bounded, and hence it converges in $\mathbb{R}$. Thus the $\mathcal{L}(\mathcal{X}, \mathcal{Y})$-valued sequence $\left\{ \left(\frac{T}{\lambda} \right)^n \right\}_{n \in \mathbb{N}^*}$ is absolutely summable, and so it is summable (since $\mathcal{L}(\lambda, \mathcal{Y})$ is a Banach space, see e.g., ([33], Proposition 4.4)). That is, the series $\sum_{k=0}^{\infty} \left(\frac{T}{\lambda} \right)^k$ converges in $\mathcal{L}(\mathcal{X}, \mathcal{Y})$. Equivalently, there is an operator in $\mathcal{L}(\mathcal{X}, \mathcal{Y})$, say $\sum_{k=0}^{\infty} \left(\frac{T}{\lambda} \right)^k$, for which

$$\sum_{k=0}^{n} \left(\frac{T}{\lambda} \right)^k \xrightarrow{n \to \infty} \sum_{k=0}^{\infty} \left(\frac{T}{\lambda} \right)^k.$$

Now, as is also readily verified by induction, for every $n \in \mathbb{N}^*$

$$(\lambda I - T) \frac{1}{\lambda} \sum_{k=0}^{n} \left(\frac{T}{\lambda} \right)^k = \frac{1}{\lambda} \sum_{k=0}^{n} \left(\frac{T}{\lambda} \right)^k (\lambda I - T) = I - \left(\frac{T}{\lambda} \right)^{n+1}.$$

However $\left(\frac{T}{\lambda} \right)^n \xrightarrow{n \to \infty} O$ (since $\left\| \left(\frac{T}{\lambda} \right)^n \right\| \leq \left\| \left(\frac{T}{\lambda} \right) \right\|^n \xrightarrow{n \to \infty} 0$ when $\|T\| < |\lambda|$), and so

$$(\lambda I - T) \frac{1}{\lambda} \sum_{k=0}^{n} \left(\frac{T}{\lambda} \right)^k \xrightarrow{n \to \infty} I \quad \text{and} \quad \frac{1}{\lambda} \sum_{k=0}^{n} \left(\frac{T}{\lambda} \right)^k (\lambda I - T) \xrightarrow{n \to \infty} I,$$

where I and O are the identity and the null operators respectively. Hence

$$(\lambda I - T) \frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{T}{\lambda} \right)^k = \frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{T}{\lambda} \right)^k (\lambda I - T) = I,$$

and $\frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{T}{\lambda} \right)^k \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is the inverse of $\lambda I - T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$.

□

Definition 1.1.6 (Compact set). *[2] A subset $\mathcal{W}$ of a normed space X is called compact if every open cover $\Omega = \{\mathcal{O}_j\}_{j \in J}$ of $\mathcal{W}$ contains a finite subcover of $\mathcal{W}$. In other words, for every family $\Omega = \{\mathcal{O}_j\}_{j \in J}$ of an open sets with the property*

$$\mathcal{W} \subset \bigcup_{j \in J} \mathcal{O}_j,$$

there exists a finite subfamily $\Omega = \{\mathcal{O}_{j_k}\}_{j_k \in J}, k = 1, 2, \cdots, n$*, such that*

$$\mathcal{W} \subset \bigcup_{k=1}^{n} \mathcal{O}_{j_k}.$$

Compact sets have the properties of being both closed and bounded, however, not all closed and bounded sets are compact unless the space X *is of finite dimension.*

Definition 1.1.7 (**Relatively compact set**). *[2] A subset* $\mathcal{W}$ *of a normed space* X *is called relatively compact if its closure* $\overline{\mathcal{W}}$ *is compact.*

Definition 1.1.8. *[7] The operator* $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ *is compact, if the set*

$$\left\{ T\phi : \|\phi\|_{\mathcal{X}} \leq 1 \right\}$$

is relatively compact in $\mathcal{Y}$*. This is equivalent to saying that for every bounded sequence* $(\phi_n)_{n \in \mathbb{N}^*} \subseteq \mathcal{X}$*, the sequence* $(T\phi_n)_{n \in \mathbb{N}^*}$ *has a subsequence that is convergent to some point in* $\mathcal{Y}$*. Compact operators are also called completely continuous operators.*

Theorem 1.1.6. *[11] Let* $\mathcal{Z}$ *be a Banach space.*

(i) *Let* $(T_n)_{n \in \mathbb{N}^*}$ *be a compact uniform approximation sequence of* T*, Then the operator* T *is also a compact operator.*

(ii) *The product* $ST \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$ *is compact, if one of the factors* $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ *or* $S \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ *is compact.*

(iii) $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ *is compact, if* T *is a finite rank operator, i.e.,* $\dim(Ran(T)) < +\infty$*.*

Theorem 1.1.7 (**Arezlà[4]-Ascoli[5] theorem**). *Let* $\mathcal{W} \subseteq C(\mathcal{D})$*, with* $\mathcal{D} \subseteq \mathbb{R}^d$ *closed and bounded. Suppose that the functions in* $\mathcal{W}$ *are uniformly bounded and equicontinuous over* $\mathcal{D}$*, meaning that*

$$\sup_{f \in \mathcal{W}} \|f\|_{\infty} < \infty,$$

and for all $f \in \mathcal{W}$

$$|f(\phi) - f(\psi)| \leq c_{\mathcal{W}}(\varepsilon), \quad \text{for } \|\phi - \psi\| \leq \varepsilon,$$

with $c_{\mathcal{W}}(\varepsilon) \xrightarrow{\varepsilon \longrightarrow 0} 0$*. Then* $\mathcal{W}$ *is relatively compact in* $C(\mathcal{D})$*.*

[4]Cesare Arzelà (1847-1912) was an Italian mathematician who made significant contributions to the fields of mathematical analysis and algebraic geometry.

[5]Giulio Ascoli (1843-1896) was an Italian mathematician who made significant contributions to the fields of mathematical analysis and topology.

Definition 1.1.9. *It is said that $(T_n)_{n\in\mathbb{N}^*}$ in $\mathcal{L}(\mathcal{X})$ is a collectively compact sequence of approximations of $T \in \mathcal{L}(\mathcal{X})$, if and only if $(T_n)_{n\in\mathbb{N}^*}$ converges pointwise to T and there exists $n_0 \geq 0$ such that*

$$\bigcup_{n_0\geq 0} \Big\{ (T_n - T)\,\phi \,:\, \phi \in \mathcal{X},\, \|\phi\|_{\mathcal{X}} \leq 1 \Big\},$$

is a relatively compact subset of $\mathcal{X}$.

Theorem 1.1.8 (**Fredholm Alternative**)**.** *[7]*
Let $T : \mathcal{X} \to \mathcal{Y}$ be a compact operator. Then the equation $(\lambda I - T)\phi = f, \lambda \neq 0$, has a unique solution $\phi \in \mathcal{X}$ if and only if the homogeneous equation $(\lambda I - T)\phi = 0_{\mathcal{Y}}$ has only the trivial solution $\phi = 0_{\mathcal{X}}$. In such a case, the operator $\lambda I - T : \mathcal{X} \underset{onto}{\overset{1-1}{\to}} \mathcal{Y}$ has a bounded inverse $(\lambda I - T)^{-1}$.

This theorem is true for any compact operator T. In [7], the proof is only for those compact operators that are the limit of a sequence of bounded finite-rank operators, and for more general proof see ([32], chap. 3).

Theorem 1.1.9. *[39] Supposing that $(T_n)_{n\in\mathbb{N}^*}$ is a collectively compact sequence in $\mathcal{L}(\mathcal{X})$. Then, the following hold.*

1. *Each operator T_n is compact.*
2. *If $T_n \xrightarrow{n\longrightarrow+\infty} T$ pointwise, then T is a compact operator.*

Theorem 1.1.10. *Let Ω be a closed set in $\mathbb{R}^d$ and let the operator K be defined by*

$$K \,:\, C(\Omega) \longrightarrow C(\Omega), K\varphi = \int_\Omega \kappa(t,s)\varphi(s)ds,$$

if the following assumptions holds

(i) $\lim\limits_{h\longrightarrow 0} \sup\limits_{|t-\tau|\leq h} \int_\Omega |\kappa(t,s) - \kappa(\tau,s)|\, ds = 0, \quad t,\tau \in \Omega,$

(ii) $\sup\limits_{t\in\Omega} \int_\Omega |\kappa(t,s)|\, ds < +\infty.$

Then the operator K is a compact operator.

Proof. The proof is an application of the Arezlà-Ascoli theorem. So, if the function φ is bounded and integrable, then $K\varphi$ is continuous with

$$|K\varphi(t) - K\varphi(\tau)| \leq \alpha\, |t-\tau|\, \|\varphi\|\,, \quad t,\tau \in \Omega, \tag{1.2}$$

and using the second condition (ii), then the operator K is bounded such that

$$\|K\| = \max_{t\in\Omega} \int_{\Omega} |\kappa(t,s)|\, ds.$$

To show that K is compact, we consider the set $\mathcal{K}$ such that

$$\mathcal{K} = \Big\{ K\varphi : \varphi \in C(\Omega),\ \|\varphi\| \leq 1 \Big\},$$

so, it is clear that $\mathcal{K}$ is a bounded set, i.e.,

$$\|K\varphi\| \leq \|K\| \, \|\varphi\| \leq \|K\| \,.$$

So, by (1.2), the set $\mathcal{K}$ is equicontinuous, and it's also a relatively compact set, then the operator K is a compact operator.

□

1.2 Differential Calculus for Nonlinear Operators

In this section, we extend the concept of derivatives for real functions to operators. The primary sources for this material are ([6] Section 4.3). We start by remembering the meaning of the derivative for a function that takes real values. Let's consider an interval I on the set of real numbers $\mathbb{R}$, and let x_0 be a point within that interval. A function $f : I \longrightarrow \mathbb{R}$ is differentiable at x_0 if and only if

$$f'(x_0) = \lim_{h\longrightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} \quad \text{exists.} \tag{1.3}$$

Alternatively, we can write also for a specific value $a \in \mathbb{R}$

$$f(x_0+h) = f(x_0) + ah + O(|h|) \quad \text{as } h \longrightarrow 0, \tag{1.4}$$

where we let $f'(x_0) = a$ denote the derivative.

From the perspective of a student new to calculus, definition (1.3) may appear to be simpler than definition (1.4), despite the fact that both definitions are equivalent. However, definition (1.4) clearly shows that the concept of differentiation is based on local linearization. Additionally, definition (1.4) can be easily applied to define the derivative of a general operator, while definition (1.3) is more useful for defining specific types of derivatives, such as directional or partial derivatives. This is demonstrated by examining a function that takes multiple real variables and outputs a vector.

Let $\mathcal{W}$ be a subset of the space $\mathbb{R}^d$, with x_0 as an interior point. Let $f : \mathcal{W} \longrightarrow \mathbb{R}^m$. Following (1.4), we say f is differentiable at x_0 if there exists a matrix (linear operator) $A \in \mathbb{R}^{m\times d}$ such that

$$f(x_0+\mathbf{h}) = f(x_0) + A\mathbf{h} + O(|\mathbf{h}|) \quad \text{as } \mathbf{h} \longrightarrow 0, \quad \mathbf{h} \in \mathbb{R}^d. \tag{1.5}$$

We can show that $A = \nabla f(x_0)$, the gradient or Jacobian of f at x_0 :

$$A_{ij} = \frac{\partial f_i}{\partial x_j}, \quad i = 1,2,\cdots,m, \quad j = 1,2,\cdots,d.$$

The challenge in applying equation (1.3) to functions with vector inputs is understanding how to interpret the expression (1.3) when h is a vector. An alternative approach is to use equation (1.4) to define the concept of a directional derivative, where instead of linearizing the function in all directions of the variable x as it approaches x_0, we linearize it along a specific, fixed direction towards x_0. By doing this, we will only have to work with a function that takes a single real variable and outputs a vector. This approach allows the use of the divided difference in equation (1.3) to be meaningful. More precisely, let $\mathbf{h}$ be a fixed vector in $\mathbb{R}^d$, and we consider the function $f(x_0 + t\mathbf{h})$, for $t \in \mathbb{R}$ in a neighborhood of 0. We then say f is differentiable at x_0 with respect to $\mathbf{h}$, if there is a matrix A such that

$$\lim_{t \longrightarrow 0} \frac{f(x_0 + t\mathbf{h}) - f(x_0)}{t} = A\mathbf{h}. \tag{1.6}$$

We now turn to the case of an operator $f : \mathcal{W} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ between two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$. Let us adopt the convention that whenever we discuss the differentiability at a point ϕ_0, implicitly we assume ϕ_0 is an interior point of $\mathcal{W}$; by this, we mean there is an $r > 0$ such that

$$B(\phi_0, r) = \left\{ \phi \in \mathcal{X} : \|\phi - \phi_0\|_{\mathcal{X}} \le r \right\} \subseteq \mathcal{X}.$$

Definition 1.2.1. *[6] The operator f is Fréchet[6] differentiable at ϕ_0 if and only if there exists $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ such that*

$$f(\phi_0 + h) = f(\phi_0) + Ah + O(\|h\|_{\mathcal{X}}), \quad h \longrightarrow 0_{\mathcal{X}}. \tag{1.7}$$

The map A is called the Fréchet derivative of f at ϕ_0, and we write $A = f'(\phi_0)$. The quantity $df(\phi_0; h) = f'(\phi_0)h$ is called the Fréchet differential of f at ϕ_0. If f is Fréchet differentiable at all points in $\mathcal{W}_0 \subseteq \mathcal{W}$, we call $f' : \mathcal{W}_0 \subseteq \mathcal{X} \longrightarrow \mathcal{L}(\mathcal{X}, \mathcal{Y})$ the Fréchet derivative of f on $\mathcal{W}_0$.

Definition 1.2.2. *[6] The operator f is Gâteaux[7] differentiable at ϕ_0 if and only if there exists $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ such that*

$$\lim_{t \longrightarrow 0} \frac{f(\phi_0 + th) - f(\phi_0)}{t} = Ah, \quad \forall h \in \mathcal{X}, \quad \|h\|_{\mathcal{X}} = 1. \tag{1.8}$$

The map A is called the Gâteaux derivative of f at ϕ_0, and we write $A = f'(\phi_0)$. The quantity $df(\phi_0; h) = f'(\phi_0)h$ is called the Gâteaux differential of f at ϕ_0. If f is Gâteaux differentiable at all points in $\mathcal{W}_0 \subseteq \mathcal{W}$, we call $f' : \mathcal{W}_0 \subseteq \mathcal{X} \longrightarrow \mathcal{L}(\mathcal{X}, \mathcal{Y})$ the Gâteaux derivative of f on $\mathcal{W}_0$.

[6] Maurice Fréchet (1878-1973) was a French mathematician who made significant contributions to the fields of topology, functional analysis, and measure theory. Fréchet introduced the concept of metric spaces and studied their properties, including completeness and compactness.

[7] René Gateaux (1889-1977) was a French mathematician who made significant contributions to the fields of functional analysis and optimization theory.

Example 1.2.1. *[6] Let $\mathcal{X} = C([a,b],\mathbb{R})$ with the maximum norm $\|\cdot\|_\infty$. Assume $g \in C([a,b],\mathbb{R})$, $\kappa \in C([a,b]^2 \times \mathbb{R},\mathbb{R})$. Then we can define the operator $K : \mathcal{X} \longrightarrow \mathcal{X}$ by the formula*

$$K(\phi)(t) = \int_a^b \kappa\left(t, s, \phi(s)\right) ds + g(t).$$

Let $\phi_0 \in C([a,b],\mathbb{R})$ be such that $\frac{\partial \kappa}{\partial \phi_0}\left(\cdot,\cdot,\phi_0(\cdot)\right) \in C([a,b]^2 \times \mathbb{R},\mathbb{R})$. Then K is Fréchet differentiable at ϕ_0, and its Fréchet differential $T = K'$ is given by

$$\left[T(\phi_0)h\right](t) = \int_a^b \frac{\partial \kappa}{\partial \phi_0}\left(t, s, \phi_0(s)\right) h(s) ds, \quad h \in \mathcal{X}, t \in [a,b]. \tag{1.9}$$

It is necessary to generalize the Mean Value Theorem to encompass differentiable functions of a real variable, in order to investigate the impact of variations in the argument on the behavior of non-linear functions.

Theorem 1.2.1 (Mean Value Theorem). *[6] Assume $\mathcal{X}$ and $\mathcal{Y}$ are real linear spaces. Let $F : \mathcal{O} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ with $\mathcal{O}$ an open set. Assume F is differentiable on $\mathcal{O}$ and that $F'(\phi)$ is a continuous function of ϕ on $\mathcal{O}$ to $\mathcal{L}(\mathcal{X},\mathcal{Y})$. Let $\phi, \psi \in \mathcal{O}$ and assume the line segment joining them is also contained in $\mathcal{O}$. Then*

$$\|F(\phi) - F(\psi)\|_{\mathcal{Y}} \leq \sup_{0\leq x\leq 1} \|F'((1-x)\phi + x\psi)\|_{\mathcal{Y}} \|\phi - \psi\|_{\mathcal{X}}.$$

Proof. See ([6]). □

1.3 Linearization Process of Nonlinear Equations

Let F be a Fréchet differentiable operator mapping a convex open subset $\mathcal{O}$ of a infinite dimensions Banach space $\mathcal{X}$ into a infinite dimensions Banach space $\mathcal{Y}$. We are interested in solving the equation

$$F(\varphi) = 0_{\mathcal{Y}}. \tag{1.10}$$

The solutions of this equation are not often expressible in a simple closed-form expression, so numerical approximations are typically used. As a result, the methods for solving the equation are typically iterative.

The linearization of the equation (1.10) is the principal method for constructing successive approximations $(\varphi^{(k)})_{k\geq 1}$ to the exact solution φ_{ext} (if it exists). Different iterative methods can be used to find a sequence of approximations $(\varphi^{(k)})_{k\geq 1}$. One of the most commonly used and well-known methods is Newton-Kantorovich (NK) method, which has a specific algorithm defined for a given initial approximation $\varphi^{(0)}$ as follows

$$\varphi^{(k+1)} = \varphi^{(k)} - \left[F'(\varphi^{(k)})\right]^{-1} F(\varphi^{(k)}), \quad k = 0, 1, 2, \cdots, \tag{1.11}$$

where this scheme is the generalization of Newton's method when the operator F is a real function f of a real variable and knowing by the following algorithm

$$\varphi^{(k+1)} = \varphi^{(k)} - \frac{f(\varphi^{(k)})}{f'(\varphi^{(k)})}, \quad k = 0, 1, 2, \cdots .$$

Newton's method has a well-documented long history [58]. Kantorovich's greatest achievement is the utilization of functional analysis techniques to tackle nonlinear problems in numerical analysis, including resolving integral equations and ordinary and partial differential equations. This method has broadened the range and potential of numerical analysis.

1.3.1 Recurrence relations of Kantorovich

In 1948 [28], Kantorovich proved a semilocal convergence theorem for Newton's method in a Banach space, given certain conditions for the operator F and the initial point $\varphi^{(0)}$:

(W1) For $\varphi^{(0)} \in \mathcal{O} \subseteq \mathcal{X}$, there exists $\varGamma_0 = [F'(\varphi^{(0)})]^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X}), \beta > 0$, such that $\|\varGamma_0\|_{\mathcal{X}} \leq \beta < +\infty$,

(W2) there exists $\eta > 0$, such that $\left\|\varGamma_0 F(\varphi^{(0)})\right\|_{\mathcal{X}} \leq \eta < +\infty$,

(W3) there exists $R > 0, M > 0$, such that $\|F''(\varphi)\|_{\mathcal{Y}} \leq M < +\infty$, for all $\varphi \in B_R(\varphi^{(0)})$,

(W4) $h = M\beta\eta \leq \frac{1}{2}$,

where $B_R(\varphi^{(0)})$ is the ball of center $\varphi^{(0)}$ and a radius R. It is important to consider the Kantorovich condition (W4), as it requires that the magnitude of $\left\|F(\varphi^{(0)})\right\|_{\mathcal{Y}}$ at $\varphi^{(0)}$ be small, meaning that the initial approximation $\varphi^{(0)}$ must be close to a solution of the equation (1.10).

Theorem 1.3.1. *(The Newton-Kantorovich theorem)[19, 20]*

Let $F : \mathcal{O} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ be a twice continuously Fréchet differentiable operator defined on a non-empty open convex domain $\mathcal{O}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$. Suppose that conditions (W1)-(W2)- (W3)-(W4) are satisfied and $B_{\rho^*}(\varphi^{(0)}) \subset B_R(\varphi^{(0)})$ with

$$\rho^* = \frac{1 - \sqrt{1-2h}}{h}\eta.$$

Then, Newton's sequence defined in (1.11) and starting at $\varphi^{(0)}$ converges to a exact solution φ_{ext} of the equation $F(\varphi) = 0_{\mathcal{Y}}$ and the solution φ_{ext} and the iterates $\varphi^{(k)}$ belong to $\overline{B_{\rho^*}(\varphi^{(0)})}$, for all $k \in \mathbb{N}$. Moreover, if $h < \frac{1}{2}$, the solution φ_{ext} is unique in $B_{\rho^{**}}(\varphi^{(0)}) \cap B_R(\varphi^{(0)})$, where

$$\rho^{**} = \frac{1 + \sqrt{1-2h}}{h}\eta,$$

and, if $h < \frac{1}{2}, \varphi_{ext}$ is unique in $\overline{B_{\rho^*}(\varphi^{(0)})}$. Furthermore, we have the following error estimates:

$$\left\|\varphi^{(k)} - \varphi_{ext}\right\|_{\mathcal{X}} \leq 2^{1-k}(2h)^{2^k-1}\eta, \quad k = 0, 1, 2, \cdots .$$

Kantorovich provided two distinct demonstrations of the Newton-Kantorovich theorem, utilizing either recurrence relations or majorant functions. In 1948, he published his original proof, which utilized recurrence relations [28]. In 1951, he presented an alternative proof that relied on the notion of a majorant function [27]. Both proofs are thoroughly explained in [20].

1.4 Classification of Fredholm Integral Equations

In this section, we will focus on the classification of Fredholm integral equations based on the properties of their kernels. Understanding the different types of Fredholm integral equations is important for developing appropriate solution methods and for solving real-world problems in various fields of science and engineering. The most standard form of Fredholm linear integral equations is given by the following form

$$y(t)\varphi(t) = \lambda \int_a^b \kappa(t, s)\varphi(s)ds + g(t), \quad t \in [a, b], \tag{1.12}$$

with g is a known real-valued function and the limit of integration a and b are constants and the unknown function φ appears under the integral sign, where κ is the kernel of the integral equation and λ is a parameter. The equation (1.12) is called linear because the unknown function φ under the integral sign occurs linearly, i.e., the power of φ is one.

The value of $y(\cdot)$ will give the following kinds of Fredholm integral equations:

- If $y(\cdot) = 0$, then equation (1.12) yields

  $$\lambda \int_a^b \kappa(t, s)\varphi(s)ds + g(t) = 0, \quad t \in [a, b],$$

  which is called Fredholm integral equation of the first kind.

- If $y(\cdot) = 1$, then equation (1.12) becomes simply

  $$\varphi(t) = \lambda \int_a^b \kappa(t, s)\varphi(s)ds + g(t), \quad t \in [a, b],$$

  and this equation is called Fredholm integral equation of second kind. But if $g(\cdot) = 0$, equation (1.12) will be take the form

  $$\varphi(t) - \lambda \int_a^b \kappa(t, s)\varphi(s)ds = 0, \quad t \in [a, b],$$

  where this equation is called homogeneous Fredholm integral equation.

Nonlinear Fredholm integral equations are a type of integral equations in which the unknown function φ appears both inside and outside the integral sign. They can be expressed in the general form

$$\varphi(t) = \lambda \int_a^b \kappa(t, s, \varphi(s))ds + g(t), \quad t \in [a, b], \tag{1.13}$$

where in this case the kernel κ is a given nonlinear real-valued function. Another type of nonlinear equation exists, known as the nonlinear Fredholm-Hammerstein[8] integral equation, which can be expressed in the following form

$$\varphi(t) = \lambda \int_a^b \kappa(t, s)H(s, \varphi(s))ds + g(t), \quad t \in [a, b].$$

A weakly singular Fredholm integral equation of the second kind can take several forms depending on the specific problem being considered. However, the general form of such an equation can be written as:

$$\varphi(t) = \lambda \int_a^b p(t-s)\kappa(t, s, \varphi(s))ds + g(t), \quad t \in [a, b],$$

where depending on the specific problem being considered, the kernel function $p(\cdot)$ may take different forms, such as

- $p(t-s) = log(|t-s|)$: Logarithmic kernels is well-defined for all values of t and s, but its derivative becomes unbounded as t approaches s. Logarithmic kernels can arise in integral equations that involve potentials or electrostatics.
- $p(t-s) = \dfrac{1}{|t-s|^{1-\gamma}}$: The Power-law kernels with $0 < \gamma < 1$, and this form of kernel can arise in a variety of nonlinear integral equations, such as those arising in the study of diffusion or chemical reactions.

Lastly, the nonlinear Fredholm integro-differential equations is given by the following form

$$\varphi(t) = \lambda \int_a^b \kappa(t, s, \varphi(s), \varphi'(s), \cdots, \varphi^{(m-1)}(s))ds + g(t), \quad t \in [a, b]. \tag{1.14}$$

1.4.1 Technique of Creating a System of Nonlinear Fredholm Integro-Differential Equations

In order to build a system of nonlinear Fredholm integro-differential equations, we can follow a certain procedure based on Example (1.2.1). Specifically, we can differentiate

[8] Axel Hammerstein (1878-1945) was a German mathematician who made significant contributions to the fields of applied mathematics, theory of differential equations and the calculus of variations.

Equation (1.14) $m-1$ times and use the resulting expressions to form a system of equations as follows

$$\begin{cases} \varphi(t) = \lambda \displaystyle\int_a^b \kappa\left(t, s, \varphi(s), \varphi'(s), \cdots, \varphi^{(m-1)}(s)\right) ds + g(t), \\ \varphi'(t) = \lambda \displaystyle\int_a^b \frac{\partial \kappa}{\partial t}\left(t, s, \varphi(s), \varphi'(s), \cdots, \varphi^{(m-1)}(s)\right) ds + g'(t), \quad t \in [a,b], \\ \quad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ \varphi^{(m-1)}(t) = \lambda \displaystyle\int_a^b \frac{\partial^{(m-1)} \kappa}{\partial t^{(m-1)}}\left(t, s, \varphi(s), \varphi'(s), \cdots, \varphi^{(m-1)}(s)\right) ds + g^{(m-1)}(t). \end{cases} \tag{1.15}$$

If we set, for all $i = 1, 2, \cdots, m$, $g_i = g^{(i-1)}$, $\varphi_i = \varphi^{(i-1)}$ and $\kappa_i = \dfrac{\partial^{(i-1)} \kappa}{\partial t^{(i-1)}}$, we get the following system

$$\begin{cases} \varphi_1(t) = \varphi(t) = \lambda \displaystyle\int_a^b \kappa_1\left(t, s, \varphi_1(s), \varphi_2(s), \cdots, \varphi_m(s)\right) ds + g_1(t), \\ \varphi_2(t) = \varphi'(t) = \lambda \displaystyle\int_a^b \kappa_2\left(t, s, \varphi_1(s), \varphi_2(s), \cdots, \varphi_m(s)\right) ds + g_2(t), \quad t \in [a,b], \\ \quad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ \varphi_m(t) = \varphi^{(m-1)}(t) = \lambda \displaystyle\int_a^b \kappa_m\left(t, s, \varphi_1(s), \varphi_2(s), \cdots, \varphi_m(s)\right) ds + g_m(t). \end{cases} \tag{1.16}$$

If $\varphi \in C^{m-1}([0,1], \mathbb{R})$, system (1.16) can be rewritten as the ensuing structure

$$\text{Find } \varPsi \in \prod_{i=1}^{m} C^{(m-i)}([0,1], \mathbb{R}), \qquad \varPsi = \lambda K(\varPsi) + G, \tag{1.17}$$

where

$$\varPsi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_m \end{pmatrix}, G = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_m \end{pmatrix}, K = \begin{pmatrix} K_1 \\ K_2 \\ \vdots \\ K_m \end{pmatrix},$$

and for all $i = 1, 2, \cdots, m$

$$K_i(\varPsi)(t) = \int_0^1 \kappa_i(t, s, \varPsi(s)) ds.$$

Any Fredholm integro-differential equation is characterized by the existence of one or more of the derivatives $\varphi', \varphi'', \cdots$ outside the integral sign. Overall, constructing a system of nonlinear Fredholm integro-differential equations can be a complex and challenging task, but it is an important step in modeling and analyzing various physical and engineering systems such as the theory of signal processing and neural networks [60].

1.4.2 Converting BVP to Fredholm Integral Equations

In this part, we demonstrate how a boundary value problem (BVP) can be converted to an equivalent Fredholm integral equation. It should be emphasized that this technique for transforming a BVP into a Fredholm integral equation is intricate and infrequently employed. Let us consider the following boundary value problem

$$\begin{cases} \varphi''(t) & = \kappa(t, \varphi(t)), \quad t \in [0,1], \\ \varphi(0) & = \varphi_0, \quad \varphi(1) = \varphi_1, \end{cases} \tag{1.18}$$

Integrating equation (1.18) with respect to s from 0 to t two times yields

$$\varphi(t) = \varphi(0) + t\varphi'(0) + \int_0^t \int_0^t \kappa(s, \varphi(s)) ds ds \tag{1.19}$$

$$= \varphi_0 + t\varphi'(0) + \int_0^t (t-s)\kappa(s, \varphi(s)) ds. \tag{1.20}$$

To determine the unknown constant $\varphi'(0)$, we use the condition at $t = 1$, i.e., $\varphi(1) = \varphi_1$. Hence equation (1.19) becomes

$$\varphi(1) = \varphi_1 = \varphi_0 + \varphi'(0) + \int_0^1 (1-s)\kappa(s, \varphi(s)) ds,$$

and the value of $\varphi'(0)$ is obtained as

$$\varphi'(0) = (\varphi_1 - \varphi_0) - \int_0^1 (1-s)\kappa(s, \varphi(s)) ds.$$

Thus, equation (1.19) can be written as

$$\varphi(t) = \varphi_0 + t(\varphi_1 - \varphi_0) - \int_0^1 \kappa(s, \varphi(s)) H(t,s) ds, \quad t \in [0,1], \tag{1.21}$$

in which the kernel is given by

$$H(t,s) = \begin{cases} s(1-s) & 0 \leq s \leq t, \\ t(1-s) & t \leq s \leq 1. \end{cases}$$

Once again we can reverse the process and deduce that the function φ which satisfies the integral equation also satisfies the BVP. If we now specialize equation (1.18) to the simple linear BVP $\varphi''(t) = -\lambda\varphi(t), 0 < t < 1$ with the boundary conditions $\varphi(0) = \varphi_0, \varphi(1) = \varphi_1$, then equation (1.21) reduces to the second kind Fredholm integral equation

$$\varphi(t) = \lambda \int_0^1 \kappa(t,s)\varphi(s) ds + g(t), \quad t \in [0,1],$$

where $g(t) = \varphi_0 + t(\varphi_1 - \varphi_0)$. It can be easily verified that $\kappa(t,s) = \kappa(s,t)$ confirming that the kernel is symmetric.

1.4.3 Existence of the Solution for Nonlinear Fredholm Integral Equations

In this section, we will present an existence theorem for the solution of nonlinear Fredholm integral equations which is presented in (1.13) as the following form

$$\varphi(t) = \lambda \int_a^b \kappa(t, s, \varphi(s))ds + g(t), \quad t \in [a, b], \tag{1.22}$$

where λ is a purameter. From the theory of the linear Fredholm equations, we know that the parameter λ plays a significant role [13]. In order to establish criteria under which a solution exists for (1.22), we make essentials assumptions as follows:

- The function g is bounded in the interval $a \leq t \leq b$, that is,

$$|g(t)| < S.$$

- The kernel $\kappa(\cdot, \cdot, \varphi(\cdot))$ is integrable and bounded,

$$|\kappa(t, s, \varphi(s))| < z, \quad t, s \in [a, b],$$

  in the domain $D = \Big\{(t, s, \varphi(s)) \in [a, b]^2 \times \mathbb{R} : \|\varphi\|_{\mathcal{X}} < c\Big\}$.

- $\kappa(\cdot, \cdot, \varphi(\cdot))$ satisfies the Lipschitz[9] condition in D, namely,

$$|\kappa(t, s, \phi(s)) - \kappa\left(t, s, \psi(s)\right)| < M \left\|\phi - \psi\right\|_{\mathcal{X}}, \quad t, s \in [a, b].$$

By successive approximations we now have

$$\begin{aligned}
\varphi_0(t) &= g(t) - g(a) \\
\varphi_1(t) &= \lambda \int_a^b \kappa\left(t, s, \varphi_0(s)\right) ds + g(t),
\end{aligned}$$

and, in general,

$$\varphi_n(t) = \lambda \int_a^b \kappa\left(t, s, \varphi_{n-1}(s)\right) ds + g(t).$$

From these we obtain

$$\begin{cases}
\varphi_1(t) - \varphi_0(t) = & \lambda \displaystyle\int_a^b \kappa(t, s, \varphi_0(s))ds + g(a), \\
\varphi_2(t) - \varphi_1(t) = & \lambda \displaystyle\int_a^b \left[\kappa(t, s, \varphi_1(s)) - \kappa(t, s, \varphi_0(s))\right] ds, \\
\quad\vdots & \quad\vdots \qquad\qquad \vdots \\
\varphi_n(t) - \varphi_{n-1}(t) = & \lambda \displaystyle\int_a^b \left[\kappa(t, s, \varphi_{n-1}(s)) - \kappa(t, s, \varphi_{n-2}(s))\right] ds.
\end{cases}$$

[9]Rudolf Lipschitz (1832-1903) was a German mathematician who made significant contributions to the fields of mathematical analysis, differential equations, and geometry.

From the conditions given above, we have

$$\|\varphi_1 - \varphi_0\|_{\mathcal{X}} < |\lambda|\, z(b-a) + |g(a)| = |\lambda|\,(b-a)z\left[1 + \frac{g(a)}{|\lambda|\, z(b-a)}\right] = |\lambda|\,\alpha(a-b),$$

where $\alpha = z\left[1 + g(a)/\left(|\lambda| z(b-a)\right)\right]$.

From this inequality and the Lipschitz condition, we get

$$\|\varphi_2 - \varphi_1\|_{\mathcal{X}} < |\lambda| M \int_a^b |\varphi_2(s) - \varphi_1(s)|\, ds < |\lambda|^2 M\alpha(b-a)^2 < |\lambda|^2 l^2 (b-a)^2,$$

where l is the larger of the two numbers M and α.

Similarly we obtain the inequalities:

$$\|\varphi_3 - \varphi_2\|_{\mathcal{X}} < |\lambda|^3 l^3 (b-a)^3,$$
$$\|\varphi_n - \varphi_{n-1}\|_{\mathcal{X}} < |\lambda|^n l^n (b-a)^n.$$

A majorante for the series

$$\varphi(t) = \varphi_0(t) + [\varphi_1(t) - \varphi_0(t)] + \cdots + [\varphi_n(t) - \varphi_{n-1}(t)] + \cdots,$$

is furnished by the sum

$$Y = g(a) + \sum_{n=1}^{\infty} |\lambda|^n l^n (b-a)^n,$$

and thus the series converges uniformly for all values of λ for which we have

$$|\lambda| < \frac{1}{l(b-a)}.$$

1.5 Numerical Methods for Solving Fredholm Integral Equations

The task of solving nonlinear integral equations has posed a significant challenge to mathematicians, engineers, and scientists due to the complexity of these equations. However, with the advancements in computational power, various numerical methods have been developed to tackle this issue.

In this section, we introduce and analyze different numerical methods for solving nonlinear integral equations. The core idea behind these methods is the discretization of the integral equation $\varphi = K\varphi + g$ into a series of finite-dimensional approximation problems $\varphi_n = K\varphi_n + g$. As n approaches infinity, this process provides an approximation to the solution of the equation. Three primary methods are employed for this discretization process: Projection methods, which include the widely used Collocation and Galerkin[10]

[10] Boris Galerkin (1871-1945) was a prominent Russian mathematician and engineer who developed the Galerkin method in the early 20th century.

methods, Nyström method, and Product Integration Rule. These projection methods project the original equation onto a finite-dimensional subspace, and the approximation is sought within this subspace. Nyström method, on the other hand, evaluates the integral by discretizing the kernel function, and it has shown to be particularly useful for certain classes of integral equations. By carefully analyzing these methods, one can determine which one is best suited for a particular problem and achieve accurate solutions to nonlinear integral equations.

1.5.1 Projection Operators

Projection operators are mathematical tools that are used to project vectors or operators onto a subspace. They extract the components of the vector or operator that correspond to the subspace, while eliminating the components that are orthogonal to it [7]. Projection operators have many applications in various fields, such as linear algebra, functional analysis, quantum mechanics, signal processing, and optimization[6].

Definition 1.5.1. *Let $\mathcal{A}$ be a normed space and $\mathcal{B}$ a non trivial subspace, a projection operator from $\mathcal{A}$ into $\mathcal{B}$ is a bounded linear operator $\pi : \mathcal{A} \longrightarrow \mathcal{B}$ such that*

$$\pi(\varphi) = \varphi, \quad \forall \varphi \in \mathcal{A}, \tag{1.23}$$

and from (1.23), we have

$$\pi^2(\varphi) = \pi(\pi(\varphi)) = \pi(\varphi),$$

thus

$$\pi^2 = \pi,$$

furthermore

$$\|\pi\| = \left\|\pi^2\right\| \leq \|\pi\|^2,$$

this mean that

$$\|\pi\| \geq 1.$$

1.5.1.1 Orthogonal projection operators

Definition 1.5.2. *Let $\mathcal{A}$ be a Hilbert space, and let $< \cdot, \cdot >$ denote the inner product for $\mathcal{A}$, a projection operator π is orthogonal if and only if*

$$< \pi\phi, (I - \pi)\psi >= 0, \quad \forall \phi, \psi \in \mathcal{A}.$$

Example 1.5.1. *Let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal basis of the space $\mathcal{A}$ and $\mathcal{A}_n = span\{e_1, e_2, \cdots, e_n\}$, for all $\phi \in \mathcal{A}$, the formula*

$$\pi_n\phi = \sum_{i=1}^{n} < \phi, e_i > e_i,$$

defines an orthogonal projection from $\mathcal{A}$ onto $\mathcal{A}_n$.

1.5.1.2 Interpolatory projection operators

Let $\mathcal{A}$ be a normed vector space, and $\mathcal{A}_n$ is an n-dimensional subspace of $\mathcal{A}$ with a basis $\{e_1, e_2, \cdots, e_n\}$, and let $\{t_1, t_2, \cdots, t_n\}$ be be interpolation nodes in the interpolation region Ω, the interpolation function $\varphi_n \in \mathcal{A}_n$ is given to approximate a given function $\varphi \in \mathcal{A}$ where the interpolation problem is setting as follows:

Knowing data $\{y_1, y_2, \cdots, y_n\}$. Then find $\varphi_n \in \mathcal{A}_n$, such that

$$\varphi_n(t) = \sum_{j=1}^{n} c_j e_j(t),$$

such that the interpolation conditions

$$\varphi_n(t_i) = y_i, \quad i = 1, 2, \cdots, n,$$

are satisfied. Thus, we find the coefficient $\{c_1, c_2, \ldots, c_n\}$ by solving the system

$$\sum_{j=1}^{n} c_j e_j(t_i) = y_i, \quad i = 1, 2, \cdots, n. \tag{1.24}$$

The necessary and sufficient condition for the system (1.24) to have a unique solution is

$$det\,[e_j(t_i)] \neq 0.$$

Thus, the solution of the interpolation problem is

$$\pi_n \varphi(t) = \sum_{i=1}^{n} \varphi(t_i) e_i(t),$$

where π_n is the interpolatory projection operator from $\mathcal{A}$ into $\mathcal{A}_n$.

Example 1.5.2. *(Lagrange[11] interpolation) Let $\mathcal{A}$ is $C([a,b],\mathbb{R})$ and $\mathcal{P}_m$ the space of all polynomials of degree not more than m. For any $\varphi \in C([a,b],\mathbb{R})$ wa can define the Lagrange interpolating polynomials $L_m\varphi \in \mathcal{P}_m$ as*

$$L_m \varphi(t) = \sum_{j=0}^{m} \varphi(t_j) \ell_j(t),$$

where

$$\ell_j(t) = \prod_{i \neq j} \frac{t - t_i}{t_j - t_i},$$

and $\{\ell_1, \ell_2, \cdots, \ell_m\}$ is the Lagrange interpolation basic functions associated with Gauss[12]-Jacobi[13] quadrature points $\{t_1, t_2, \cdots, t_m\}$.

[11] Joseph-Louis Lagrange (1736-1813) was an Italian-French mathematician and astronomer who made significant contributions to a wide range of fields, including algebra, calculus, and celestial mechanics.

[12] Johann Carl Friedrich Gauss (1777-1855) was a German mathematician and physicist who made fundamental contributions to a wide range of fields, including number theory, algebra, statistics.

[13] Carl Gustav Jacob Jacobi (1804-1851) was a German mathematician. He is best known for his work on elliptic functions, the theory of differential equations, and the Jacobi identity in vector calculus.

For all $i, j = 1, 2, \cdots, m$, the functions ℓ_j satisfy the special interpolation conditions

$$\ell_j(t_i) = \delta_{ji} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases}$$

and they form a basis in $\mathcal{P}_m$.

1.5.2 Principle of Projection Methods

Let $\mathcal{X}$ be as all times a Banach space, where the most choices are $C([a, b], \mathbb{R})$ and $L^2([a, b], \mathbb{R})$, consider the Uryshon integral operator

$$K(\varphi)(t) = \int_a^b \kappa(t, s)\varphi(s)ds, \quad t \in [a, b],$$

and we are interested in a solution of the operator integral equation

$$\varphi = K\varphi. \tag{1.25}$$

Let $\mathcal{X}_n, n \in \mathbb{N}^*$, be a sequence of finite dimensional subspaces, with $\mathcal{X}_n$ having dimension n for notation simplicity, and let π_n be a projection of $\mathcal{X}$ onto $\mathcal{X}_n$, it is usually assumed that

$$\pi_n\varphi \longrightarrow \varphi, \text{ as } n \longrightarrow \infty, \quad \forall \varphi \in \mathcal{X}. \tag{1.26}$$

The projection method for solving (1.25) is to seek an approximate solution $\varphi_n \in \mathcal{X}_n$, such that φ_n satisfies the operator equation

$$\varphi_n = \pi_n K\varphi_n,$$

thus, we find an approximate fixed point problem which can be written in the following equivalent form

$$\pi_n(I - K)\varphi_n = 0_{\mathcal{X}_n}, \quad \varphi_n \in \mathcal{X}_n.$$

Lemma 1.5.1. *[6] Let $\mathcal{X}$ be a Banach space and let $(\pi_n)_{n\in\mathbb{N}^*}$ be a family of bounded projections on $\mathcal{X}$ where are satisfy the (1.26) condition, if $K : \mathcal{X} \longrightarrow \mathcal{X}$ is a compact operator, then*

$$\|K - \pi_n K\| \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

Theorem 1.5.1. *[6] Assume that the operator $K : \mathcal{X} \longrightarrow \mathcal{X}$ is bounded, and also that*

$$\lambda I - K : \mathcal{X} \longrightarrow \mathcal{X}.$$

Further assume

$$\|K - \pi_n K\| \longrightarrow 0, \ \ as \ n \longrightarrow \infty.$$

Then for all sufficiently large n, the operator $(\lambda I - \pi_n K)^{-1}$ exists as a bounded operator from $\mathcal{X}$ to $\mathcal{X}$, moreover, it is uniformly bounded

$$\sup_{n \in \mathbb{N}^*} \left\| (\lambda I - \pi_n K)^{-1} \right\| < \infty.$$

1.5.3 Collocation Method

The collocation method is a numerical technique used to approximate the solutions of operator equations by choosing a set of points (collocation points) and requiring that the equation holds at these points. The method is based on the idea that if the equation holds at a set of points, then it should hold approximately everywhere between these points as well. We consider the Urysohn[14] integral equation

$$\varphi(t) = \int_a^b \kappa(t,s)\varphi(s)ds + g(t), \quad t \in [a,b]. \tag{1.27}$$

and let $\mathcal{X}$ be $C([a,b],\mathbb{R})$, and let $\mathcal{X}_n, n \in \mathbb{N}^*$, be a finite dimensional subsequence of $\mathcal{X}$, define π_n to be the interpolatory projection operator of $\mathcal{X}$ onto $\mathcal{X}_n$. Let

$$\varphi_n(t) = \sum_{j=1}^{n} \alpha_j e_j(t),$$

where $\{e_1, e_2, \cdots, e_n\}$ is a basis of $\mathcal{X}_n$. The expression is inserted into (1.27), and the values of the coefficients $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ are ascertained by ensuring that the equation is nearly precise in some manner. Now, we introduce

$$\begin{aligned}
\mathcal{R}_n(t) =& \varphi_n(t) - \int_a^b \kappa(t,s)\varphi_n(s)ds - g(t) \\
=& \sum_{j=1}^{n} \alpha_j e_j(t) - \int_a^b \kappa(t,s) \sum_{j=1}^{n} \alpha_j e_j(s)ds - g(t) \\
=& \sum_{j=1}^{n} \alpha_j \left\{ e_j(t) - \int_a^b \kappa(t,s) e_j(s)ds \right\} - g(t),
\end{aligned}$$

where $\mathcal{R}_n$ is called the residual in the approximation of the equation when using $\varphi \simeq \varphi_n$. So, we define $\mathcal{R}_n$ in the collocation points as follows

$$\mathcal{R}_n(t_i) = \sum_{j=1}^{n} \alpha_j \left\{ e_j(t_i) - \int_a^b \kappa(t_i,s) e_j(s)ds \right\} - g(t_i), \quad i = 1, 2, \cdots, n.$$

[14]Pavel Samuilovich Urysohn (1898-1924) was a Russian mathematician. He is best known for his work on the concept of compactness in topology, which led to the development of the Urysohn lemma and the Urysohn metrization theorem.

By forcing $\mathcal{R}_n$ be approximately 0 at the collocation points $\{t_1, t_2, \cdots, t_n\}$, we can determine the coefficients $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. The hope is for the resulting function φ_n to have a good approximation of the exact solution φ.

1.5.4 Galerkin Method

The Galerkin method is a numerical technique used to solve functional equations by approximating the solution as a linear combination of a finite number of basis functions. Let $\mathcal{X}$ be $L_2([a,b],\mathbb{R})$ or other Hilbert space, and let $\mathcal{X}_n$ be a finite dimensional subspace of $\mathcal{X}$. Define π_n to be the orthogonal projection operator of $\mathcal{X}$ onto $\mathcal{X}_n$, based on using the inner product of $L_2([a,b],\mathbb{R})$, thus

$$< \pi_n\varphi, \psi >=< \varphi, \psi >, \quad \forall\psi \in \mathcal{X}_n.$$

Let $\{e_1, e_2, \cdots, e_n\}$ be a basis of $\mathcal{X}_n$, the Galerkin method for solving the Urysohn integral equation (1.27) is to seek an approximate solution $\varphi_n \in \mathcal{X}_n$ where,

$$\varphi_n = \sum_{j=1}^{n} \alpha_j e_j,$$

such that

$$< \mathcal{R}_n, e_i >= 0, \quad i = 1, 2, \cdots, n.$$

This yields to the non-trivial system

$$\sum_{j=1}^{n} \alpha_j < e_j, e_i >= \sum_{j=1}^{n} \alpha_j < \int_a^b \kappa(\cdot, s)e_j(s)ds, e_i > + < g, e_i >, \quad i = 1, 2, \cdots, n,$$

that can be rewriting clearly as

$$\sum_{j=1}^{n} \alpha_j \left\{< e_j, e_i > - < \int_a^b \kappa(\cdot, s)e_j(s)ds, e_i >\right\} =< g, e_i >, \quad i = 1, 2, \cdots, n.$$

Solving this system leads to determine the coefficients $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, therefore we find the resulting function φ_n which we hope will be very close to the exact solution φ of the equation (1.27).

1.5.5 The Nyström Methods

In the following subsection, the quadrature or Nyström method will be employed to estimate a collection of second-kind nonlinear integral equations. This technique involves approximating the integral terms using an ordinary quadrature rule.

Let $\mathcal{Q} : \mathcal{X} = C([a,b],\mathbb{R}) \longrightarrow \mathbb{R}$ be an integral operator defined by

$$\mathcal{Q}(f) = \int_a^b f(t)dt,$$

and let $\mathcal{Q}_n : \mathcal{X} \longrightarrow \mathbb{R}$ be a discrete operator defined by the quadrature rule

$$\mathcal{Q}_n(f) = \sum_{j=1}^{n} \omega_j f(t_j), \tag{1.28}$$

where the values $\{t_1, t_2, \cdots, t_n\}$ are called the quadrature nodes and $\{\omega_1, \omega_2, \cdots, \omega_n\}$ are called weights.

Definition 1.5.3. *A sequence of quadrature rules $\mathcal{Q}_n(f)$ is called convergent if*

$$\mathcal{Q}_n(f) \longrightarrow \mathcal{Q}(f), \ \ \text{as } n \longrightarrow \infty, \forall f \in \mathcal{X},$$

i.e., if the sequence of linear functionals $\mathcal{Q}_n(f)$ converges pointwise to the integral $\mathcal{Q}(f)$.

Theorem 1.5.2. *[3] The quadrature rules $(\mathcal{Q}_n)$ converge if and only if,*
$\mathcal{Q}_n(f) \longrightarrow \mathcal{Q}(f)$, as $n \longrightarrow \infty$, for all f in some dense subset $\mathcal{W} \subset C([a,b], \mathbb{R})$, and

$$\sup_{n \in \mathbb{N}^*} \sum_{j=1}^{n} |\omega_j| < \infty.$$

1.5.5.1 Principle of Nyström methods

Consider the Urysohn integral equation of the second kind

$$\varphi(t) = \int_a^b \kappa(t,s)\varphi(s)ds + g(t), \quad t \in [a,b], \tag{1.29}$$

where $g \in C([a,b], \mathbb{R})$ and we assuming the appropriate level of smoothness, the right-hand side of equation (1.29), which involves the kernel κ, forms a completely continuous operator on an open domain $\mathcal{W} \subset \mathcal{X}$ into $\mathcal{X}$, explicitly

$$K(\varphi)(t) = \int_a^b \kappa(t,s)\varphi(s)ds, \quad t \in [a,b].$$

Hence, solving the equation (1.29) is equivalent to solving the operator equation

$$\varphi = K\varphi + g. \tag{1.30}$$

The process of approximating the integral in (1.29) using the quadrature formula (1.28) and applying the Nyström method to (1.29) can be expressed as:

Finding φ_n such that

$$\varphi_n(t) = \sum_{j=1}^{n} \omega_j \kappa(t, t_j)\varphi_n(t_j) + g(t), \quad t \in [a,b]. \tag{1.31}$$

By considering φ_n as an approximation to φ, we can represent the numerical integral equation (1.31) in operator notation as follows:

$$\varphi_n = K_n\varphi_n + f, \tag{1.32}$$

where the discrete integral operators $K_n, n \in \mathbb{N}^*$, is defined by

$$K_n(\varphi)(t) = \sum_{j=1}^{n} \omega_j \kappa(t, t_j)\varphi(t_j), \quad t \in [a, b].$$

By determining the values of $\{\varphi_n(t_1), \varphi_n(t_2), \cdots, \varphi_n(t_n)\}$, the functional equation (1.31) can be solved, leading to the conversion of (1.31) into a finite nonlinear system in the following manner

$$\varphi_n(t_i) = Z_i = \sum_{j=1}^{n} \omega_j \kappa(t_i, t_j) Z_j + g(t_i), \quad i = 1, 2, \cdots, n, \tag{1.33}$$

where the interpolatory function

$$Z(t) = \sum_{j=1}^{n} \omega_j \kappa(t, t_j) Z_j + g(t), \quad t \in [a, b],$$

satisfies (1.31).

While both equations (1.31) and (1.33) are equally solvable (for more details, see [6]), (1.33) is typically used in practical applications, whereas (1.31) is relied upon for theoretical convergence analysis.

1.5.6 Convergence Analysis of the Nyström Methods

This part will discuss the existence and convergence of an approximate solution for equation (1.32) in the vicinity of an isolated solution of equation (1.30). To determine the convergence order, we will employ the theory of collectively compact operators. To begin, we will assume that equation (1.30) has an isolated solution $\varphi_{ext} \in \mathcal{X}$, e.i., there is some ball of radius $r > 0$,

$$B_r(\varphi_{ext}) = \Big\{\phi \in \mathcal{X} : \|\phi - \varphi_{ext}\|_{\mathcal{X}} \leq r\Big\},$$

that contains no solution of (1.30) other than φ_{ext}, and the compact operator K is possesses a continuous first and a bounded second derivative on $B_r(\varphi_{ext})$.

To facilitate easy reference, the necessary assumptions are cited from [3, 55] as follow

$\mathcal{A}_1$. $\{K_n : n \in \mathbb{N}^*\}$ is collectively compact family on $\mathcal{X}$.

$\mathcal{A}_2$. K_n is pointwise convergent to K on $\mathcal{X}$, i.e.,

$$K_n\phi \longrightarrow K\phi, \text{ as } n \longrightarrow \infty, \forall \phi \in \mathcal{X}.$$

$\mathcal{A}_3$. For $n \in \mathbb{N}^*, K_n$ possesses a continuous first and a bounded second Fréchet derivatives on $B_r(\varphi_{ext})$.

Lemma 1.5.2. *[55] Assume that $(I - K'(\varphi_{ext}))$ is non singular and that the hypotheses $\mathcal{A}_1 - \mathcal{A}_3$ hold. Then the linear operator $(I - K'_n(\varphi_{ext}))$ are non singular for sufficiently large n, say $n \geq n_1$, and*

$$\left\|(I - K'_n(\varphi_{ext}))^{-1}\right\|_{\mathcal{X}} \leq \beta < \infty.$$

Theorem 1.5.3. *Assume that the assumptions of lemma 1.5.2 hold. Then there exists a positive integer n_1 such that, for all $n \geq n_1$, the equation (1.32) has a unique solution $\varphi_n \in B_r(\varphi_{ext})$. Furthermore, there exists a constant C independent of n such that*

$$\|\varphi_n - \varphi_{ext}\|_{\mathcal{X}} \leq C \|K(\varphi_{ext}) - K_n(\varphi_{ext})\|_{\mathcal{X}} . \tag{1.34}$$

Proof. By subtracting (1.30) from (1.32) we obtain

$$\varphi_{ext} - \varphi_n = K(\varphi_{ext}) - K_n(\varphi_n),$$

by adding the term $K'_n(\varphi_{ext})(\varphi_{ext} - \varphi_n))$ on both sides we have

$$(I-K'_n(\varphi_{ext}))(\varphi_{ext}-\varphi_n) = K(\varphi_{ext})-K_n(\varphi_{ext})-[K_n(\varphi_n) - K_n(\varphi_{ext}) - K'_n(\varphi_{ext})(\varphi_{ext} - \varphi_n))] .$$

We can verified clearly that the term $K_n(\varphi_n) - K_n(\varphi_{ext}) - K'_n(\varphi_{ext})(\varphi_{ext} - \varphi_n))$ is bounded, such that

$$\|K_n(\varphi_n) - K_n(\varphi_{ext}) - K'_n(\varphi_{ext})(\varphi_{ext} - \varphi_n)\|_{\mathcal{X}} \leq \frac{1}{2}\lambda \|\varphi_{ext} - \varphi_n\|^2_{\mathcal{X}} ,$$

then from lemma 1.5.2 we have

$$\|\varphi_n - \varphi_{ext}\|_{\mathcal{X}} \leq \beta \left(\|K(\varphi_{ext}) - K_n(\varphi_{ext})\|_{\mathcal{X}} + \frac{1}{2}\lambda \|\varphi_{ext} - \varphi_n\|^2_{\mathcal{X}}\right),$$

hence

$$\|\varphi_n - \varphi_{ext}\|_{\mathcal{X}} \left[1 - \frac{\beta\lambda}{2} \|\varphi_{ext} - \varphi_n\|_{\mathcal{X}}\right] \leq \beta \|K(\varphi_{ext}) - K_n(\varphi_{ext})\|_{\mathcal{X}} ,$$

that mean

$$\|\varphi_n - \varphi_{ext}\|_{\mathcal{X}} \leq \frac{\beta \|K(\varphi_{ext}) - K_n(\varphi_{ext})\|_{\mathcal{X}}}{1 - \frac{\beta\lambda}{2} \|\varphi_{ext} - \varphi_n\|_{\mathcal{X}}},$$

and as $\varphi_n \in B_r(\varphi_{ext}), \|\varphi_n - \varphi_{ext}\|_{\mathcal{X}} \leq r$. So

$$\|\varphi_n - \varphi_{ext}\|_{\mathcal{X}} \leq \frac{\beta}{1 - \frac{\beta\lambda r}{2}} \|K(\varphi_{ext}) - K_n(\varphi_{ext})\|_{\mathcal{X}} .$$

Hypothesis $(\mathcal{A}_2)$ now leads to convergence.

□

Remarque 1.5.1. *This theorem allows us to infer that the convergence rate of φ_n to φ_{ext} is equivalent to the rate of convergence of the numerical integration method applied to $K(\varphi_{ext})$, which is typically easy to obtain.*

1.5.7 Product Integration Methods

Our attention now shifts to the numerical solution of second-kind integral equations. These equations involve a kernel function κ that may not be continuous; it is a weakly singular kernel. However, the corresponding integral operator K remains compact, mapping from $C([a,b],\mathbb{R})$ to $C([a,b],\mathbb{R})$. Although the methods we present can be applied to functions in multiple variables, we will initially describe them in the context of integral equations for single-variable functions, as this approach is more intuitive. Consider the linear Fredholm integral equation of the second kind:

$$\varphi(t) = \int_a^b \kappa(t,s)\varphi(s)ds + g(t), \quad t \in [a,b]. \tag{1.35}$$

Within this context, the majority of discontinuous kernel functions κ exhibit an infinite singularity. The most significant instances are $log(|t-s|)$ or $\frac{1}{|t-s|^{1-\gamma}}$ for some $0 < \gamma < 1$.

We introduce the idea of product integration by considering the special case as

$$\varphi(t) = \int_a^b p(|t-s|)L(t,s)\varphi(s)ds + g(t), \quad t \in [a,b], \tag{1.36}$$

with the kernel

$$\kappa(t,s) = p(|t-s|)L(t,s),$$

where

$$p(|t-s) = \begin{cases} log(|t-s|), \\ |t-s|^{\gamma-1}. \end{cases}$$

To solve the weak singular equation (1.36), we define a method called the product trapezoidal rule. Let $n \in \mathbb{N}^*$, $h = (b-a)/n$, and $t_j = a + jh$, for $j = 0, 1, \cdots, n$. For general $\varphi \in C([a,b],\mathbb{R})$, define

$$[L(t,s)\varphi(s)]_n = \frac{1}{h}\left[(t_j - s)L(t,t_j)\varphi(t_j) + (s - t_{j-1})L(t,t_j)\varphi(t_j)\right], \tag{1.37}$$

for $t_{j-1} \le s \le t_j$, and $t \in [a,b]$. This is piecewise linear in s, and interpolates $L(t,s)\varphi(s)$ at $s \in \{t_0, t_1, \cdots, t_n\}$ for all $t \in [a,b]$. Define a numerical approximation to the integral operator in (1.36) by

$$K_n\varphi_n(t) = \int_a^b p(|t-s|)\left[L(t,s)\varphi(s)\right]_n ds, \quad t \in [a,b].$$

This can also be written as

$$K_n\varphi(t) = \sum_{j=0}^{n} \beta_j(t) L(t,t_j)\varphi(t_j), \quad \varphi \in C([a,b]), \tag{1.38}$$

with weights

$$\begin{cases} \beta_0(t) = & \frac{1}{h}\displaystyle\int_{t_0}^{t_1}(t_1-s)p(|t-s|)ds, \\ \beta_n(t) = & \frac{1}{h}\displaystyle\int_{t_{n-1}}^{t_n}(s-t_{n-1})p(|t-s|)ds, \\ \beta_j(t) = & \frac{1}{h}\displaystyle\int_{t_{j-1}}^{t_j}(s-t_{j-1})p(|t-s|)ds + \frac{1}{h}\int_{t_j}^{t_{j+1}}(t_{j+1}-s)p(|t-s|)ds. \end{cases} \tag{1.39}$$

To approximate the integral equation (1.36), we use

$$\varphi_n(t) - \sum_{j=0}^{n} \beta_j(t) L(t,t_j)\varphi(t_j) = g(t), \quad t \in [a,b]. \tag{1.40}$$

As with the Nyström method (1.32)-(1.33), this is equivalent to first solving the linear system

$$\varphi_n(t_i) - \sum_{j=0}^{n} \beta_j(t_i) L(t_i,t_j)\varphi_n(t_j) = g(t_i), \quad i = 0,1,\cdots,n,$$

and then using the Nyström interpolation formula

$$\varphi_n(t) = \sum_{j=0}^{n} \beta_j(t) L(t,t_j)\varphi_n(t_j) + g(t), \quad t \in [a,b].$$

1.5.8 Convergence Analysis of the Product Integration Methods

We consider the equation (1.36), with $L(\cdot,\cdot)$ assumed to be continuous. Further, we assume the following for $p(|t-s|)$:

$$c_p = \sup_{t\in[a,b]} \int_a^b |p(|t-s|)ds| < \infty. \tag{1.41}$$

$$\lim_{h\to 0^+} \varpi_p(h) = 0, \tag{1.42}$$

where

$$\varpi_p(h) = \sup_{|t-\tau|\le h} \int_a^b |p(|t-s|) - p(|\tau-s|)|\, ds, \quad t,\tau \in [a,b].$$

These can be show to be true for both $log(|t-s|)$, and $\frac{1}{|t-s|^{1-\gamma}}, 0<\gamma<1$.

Theorem 1.5.4. *[6] Assume the function $p(|t-s|)$ satisfies (1.41)-(1.42), and $L(\cdot,\cdot)$ is continuous. For a given $g \in C([a,b])$, assume in integral equation (1.36) is uniquely solvable. Consider the numerical approximation (1.38), with $[L(\cdot,\cdot)\varphi(\cdot)]_n$ define with piecewise linear interpolation, as in (1.37). Then for all sufficiently large n, say $n \geq N$, the equation (1.40) is uniquely solvable, and the inverse operators $(I - K_n)$ are uniformly bounded for such n. Moreover,*

$$\|\varphi - \varphi_n\|_\infty \leq c \, \|K\varphi - K_n\varphi\|_\infty \,, \quad n \geq N,$$

for suitable $c > 0$.

We show that the operators $\{K_n\}_{n\in\mathbb{N}^*}$ of (1.38) are a collectively compact and pointwise convergent family on $C([a,b],\mathbb{R})$ to $C([a,b],\mathbb{R})$. This will prove the abstract assumptions $\mathcal{A}_1 - \mathcal{A}_2$ in the subsection 1.5.6. By using (Lemma 4.1.2, of [6]), we can then apply (Theorem 4.1.2, of [6]). A comprehensive description of the proof can be found in the reference [6].

Chapter 2

New approach for Solving Nonlinear Fredholm Integral Equation Systems: (LN.DN) vs (DN.LN)

Contents

In this chapter, we aim to modify the numerical approach proposed in [23] for resolving a nonlinear integral equation, so that it can be employed for solving a system of second-kind nonlinear Fredholm equations. These equations are defined in an infinite-dimensional framework, as expressed by the following format:

$$\begin{cases} \varphi_1(t) = \displaystyle\int_0^1 \kappa_1\left(t, s, \varphi_1(s), \varphi_2(s), \cdots, \varphi_N(s)\right) ds + g_1(t), \\ \varphi_2(t) = \displaystyle\int_0^1 \kappa_2\left(t, s, \varphi_1(s), \varphi_2(s), \cdots, \varphi_N(s)\right) ds + g_2(t), \\ \quad\vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ \varphi_N(t) = \displaystyle\int_0^1 \kappa_N\left(t, s, \varphi_1(s), \varphi_2(s), \cdots, \varphi_N(s)\right) ds + g_N(t), \end{cases} \tag{2.1}$$

for all $t \in [0, 1]$, and a given functions g_i, for all $i = 1, 2, \cdots, N$.

Moreover, in order to demonstrate the efficacy of the novel approach, we conduct a comparative analysis of our results against those obtained using the classical approach. To this end, we shall refer to our new strategy as (**LN.DN**) and the conventional approach as (**DN.LN**). However, we shall outline the methodology of both the (**LN.DN**) and (**DN.LN**) approaches as follows:

(LN.DN) process: Our proposed method involves initiating the **L**inearization phase of the system (2.1) within an infinite-dimensional space by employing the **N**ewton method, followed by the **D**iscretization phase using the **N**ystöm method to address certain integral operators.

(DN.LN) process: We commence by implementing the **N**yström method to **D**iscretize the system (2.1), which results in a nonlinear finite-dimensional algebraic system. Subsequently, we utilize in as a **L**inearization phase the classical **N**ewton method to address the algebraic system obtained.

The first step of this chapter involved introducing the necessary notations and fundamental concepts through section 2.1. This was followed by the presentation of the novel process (LN.DN) in section 2.2, and the subsequent examination of the (LN.DN) and (DN.LN) processes in sections 2.3 and 2.4, respectively. Finally, the chapter concludes with numerical examples in section 2.5.

2.1 Notions and Preliminary Results

We consider for all $i = 1, 2, \cdots, N$, a real Banach spaces $\mathcal{X}_i = C^1([0,1], \mathbb{R})$ and $\mathcal{X} = \prod_{i=1}^{N} \mathcal{X}_i$, with Ω_i and Ω be a nonempty open subsets of $\mathcal{X}_i$ and $\mathcal{X}$ respectively. Let $\|\cdot\|_{\mathcal{X}_i}$ be the norm of the Banach space $\mathcal{X}_i$, and $\|\cdot\|_{\mathcal{X}}$ be the norm of $\mathcal{X}$ such as

$$\forall Z = (z_1, z_2, \cdots, z_N) \in \mathcal{X}, \|Z\|_{\mathcal{X}} = \sum_{i=1}^{N} \|z_i\|_{\mathcal{X}_i} = \sum_{i=1}^{N} \left(\|z_i\|_\infty + \|z_i'\|_\infty\right),$$

where $\|\cdot\|_\infty$ is the norm of the uniform convergence represented as

$$\|z_i\|_\infty = \sup_{t\in[0,1]} |z_i(t)|, \quad z_i \in \mathcal{X}_i.$$

We define a nonlinear Fréchet-differentiable operator $K_i : \Omega \subset \mathcal{X} \longrightarrow \mathcal{X}_i$,

$$K_i\left(\varphi_1, \varphi_2, \cdots, \varphi_N\right)(t) = \int_0^1 \kappa_i\left(t, s, \varphi_1(s), \varphi_2(s), \cdots, \varphi_N(s)\right) ds, \quad \varphi_i \in \Omega_i, \quad t \in [0,1].$$

For all $i, j = 1, 2, \cdots, N$, let $T_{ij} = \dfrac{\partial K_i}{\partial \varphi_j}$ denote the Fréchet derivative of K_i associated to φ_j, i.e., for all $\phi = (\phi_1, \phi_2, \cdots, \phi_N) \in \mathcal{X}$,

$$[T_{ij}(\phi) y_i](t) = \int_0^1 \frac{\partial \kappa_i}{\partial \varphi_j}\left(t, s, \phi(s)\right) y_i(s) ds, \quad y_i \in \mathcal{X}_i, \quad t \in [0,1].$$

The Nyström approximation of K_i, denoted by $K_{n,i}$, obtained using the Nyström method with order $n \in \mathbb{N}^*$ is given as follows

$$K_{n,i}(\phi)(t) = \sum_{p=1}^{n} \omega_{n,p}\kappa_i\left(t, t_p, \phi(t_p)\right), \quad \phi \in \mathcal{X}, \quad t \in [0,1].$$

The Nyström approximation of T_{ij}, denoted by $T_{n,ij}$, obtained using the Nyström method with order $n \in \mathbb{N}^*$ is given as follows

$$\left[T_{n,ij}(\phi)y_i\right](t) = \sum_{p=1}^{n} \omega_{n,p}\frac{\partial \kappa_i}{\partial \varphi_j}\left(t, t_p, \phi(t_p)\right) y_i(t_p), \quad \phi \in \mathcal{X}, \quad y_i \in \mathcal{X}_i, \quad t \in [0,1].$$

In practice, for $\kappa \in C^1([0,1]^2 \times \mathbb{R}^N, \mathbb{R})$, the trapezoidal numerical integration rule (see[6], pp.109), gives us the following convergence order

$$\left|\int_0^1 \kappa(t,s,\Phi(s))ds - \sum_{p=1}^{n} \omega_{n,p}\kappa(t,t_p,\Phi(t_p))\right| = \frac{1}{12n^2}\left|\left[\frac{\partial \kappa(t,s,\Phi(s))}{\partial s}\right]_{s=0}^{s=1}\right| + O(h^4), \quad \Phi \in \mathcal{X}, t \in [0,1]. \tag{2.2}$$

Now, by using previous notations, the system of nonlinear equations (2.1) can be rewritten as

$$\begin{cases} \varphi_1(t) = K_1\left(\varphi_1(t), \varphi_2(t), \cdots, \varphi_N(t)\right) + g_1(t), \\ \varphi_2(t) = K_2\left(\varphi_1(t), \varphi_2(t), \cdots, \varphi_N(t)\right) + g_2(t), \\ \quad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ \varphi_N(t) = K_N\left(\varphi_1(t), \varphi_2(t), \cdots, \varphi_N(t)\right) + g_N(t), \end{cases} \tag{2.3}$$

for $t \in [0,1]$ and a given functions $g_i \in \Omega_i$. As well as, the system (2.3) can take a clear and simple form as

$$\text{Find} \quad \varphi \in \Omega \subset \mathcal{X}: \quad \varphi = K(\varphi) + G. \tag{2.4}$$

where

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_N \end{pmatrix}, G = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_N \end{pmatrix} \text{ and } K = \begin{pmatrix} K_1 \\ K_2 \\ \vdots \\ K_N \end{pmatrix}.$$

Let I_{NN} be the block identity operator of the space $\mathcal{L}(\mathcal{X})$ given by

$$I_{NN} = \begin{pmatrix} I & 0_{\mathcal{X}} & \dots & 0_{\mathcal{X}} \\ 0_{\mathcal{X}} & I & \dots & 0_{\mathcal{X}} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{\mathcal{X}} & 0_{\mathcal{X}} & \dots & I \end{pmatrix}$$

and $\mathcal{L}(\mathcal{X})$ refers to the collection of linear bounded operators that are defined from $\mathcal{X}$ to $\mathcal{X}$, while I represents the identity operator.

For $\phi \in \mathcal{X}$, let $M_T(\phi) \in \mathcal{L}(\mathcal{X})$ be the Fréchet derivative of the operator K that we give it as the following form:

$$\forall h \in \mathcal{X}, \quad M_T(\phi)h = \begin{pmatrix} T_{11}(\phi) & T_{12}(\phi) & \dots & T_{1N}(\phi) \\ T_{21}(\phi) & T_{22}(\phi) & \dots & T_{2N}(\phi) \\ \vdots & \vdots & \ddots & \vdots \\ T_{N1}(\phi) & T_{N2}(\phi) & \dots & T_{NN}(\phi) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_N \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{N} T_{1j}(\phi)h_j \\ \vdots \\ \sum_{j=1}^{N} T_{Nj}(\phi)h_j \end{pmatrix}$$

where

$$\|M_T(\phi)h\|_{\mathcal{X}} = \sum_{i=1}^{N} \left\| \sum_{j=1}^{N} T_{ij}(\phi)h_j \right\|_{\mathcal{X}_i}.$$

We suppose some conditions that will play an important role in the proof of the convergence analysis. For all $i, j = 1, 2, \cdots, N$, we assume that

$$\begin{cases} (i) & \text{Equation (2.4) has a unique solution } \varphi \in \mathcal{X}, \\ (ii) & (I_{NN} - M_T(\varphi)) \text{ is invertible, and } \|(I_{NN} - M_T(\varphi))^{-1}\| \le \eta < +\infty, \\ (iii) & \dfrac{\partial \kappa_i}{\partial \varphi_j} \in C^2([0,1]^2 \times \mathbb{R}^N, \mathbb{R}), \\ (i\nu) & R = \sum_{i=1}^{N} R_i > 0 \text{ is such that } B_R(\varphi) = \prod_{i=1}^{N} B_{R_i}(\varphi_i) \subset \Omega, \end{cases} \tag{2.5}$$

with $B_R(\varphi)$ is the ball of center φ and radius R for the norm $\|\cdot\|_{\mathcal{X}}$, and $B_{R_i}(\varphi_i)$ is the ball of center φ_i and radius $R_i > 0$ for the norm $\|\cdot\|_{\mathcal{X}_i}$.

2.2 Description of The New Process (LN.DN)

We propose the Newton method to linearize equation (2.4) as a premier phase, by the following scheme

$$\left(I_{NN} - M_T(\varphi^{(k)})\right)\left(\varphi^{(k+1)} - \varphi^{(k)}\right) = -\varphi^{(k)} + K\left(\varphi^{(k)}\right) + G, \qquad \varphi^{(0)} \in \mathcal{X}, \quad k = 0, 1, \cdots \quad . \tag{2.6}$$

In practical applications, it is necessary to compute $\left(I_{NN} - M_T(\varphi^{(k)})\right)^{-1}$ at each iteration k, but finding this operator exactly is not feasible. As a result, we employ the Nyström method in the discretization phase to approximate the integral operators involved in the scheme (2.6).

Let $\varphi_n^{(k)} = \left(\varphi_{n,1}^{(k)}, \varphi_{n,2}^{(k)}, \cdots, \varphi_{n,N}^{(k)}\right) \in \mathcal{X}, n \in \mathbb{N}^*$, be the approximation of iterative solution $\varphi^{(k)} = \left(\varphi_1^{(k)}, \varphi_2^{(k)}, \cdots, \varphi_N^{(k)}\right) \in \mathcal{X}$ obtained by Nyström method. So, the discretization of the scheme (2.6) is presented as follows

$$\left(I_{NN} - M_{T_n}(\varphi_n^{(k)})\right)\left(\varphi_n^{(k+1)} - \varphi_n^{(k)}\right) = -\varphi_n^{(k)} + K\left(\varphi_n^{(k)}\right) + G, \qquad \varphi_n^{(0)} \in \Omega, \tag{2.7}$$

or in the matrix form

$$\begin{pmatrix} I-T_{n,11}(\varphi_n^{(k)}) & -T_{n,12}(\varphi_n^{(k)}) & \dots & -T_{n,1N}(\varphi_n^{(k)}) \\ -T_{n,21}(\varphi_n^{(k)}) & I-T_{n,22}(\varphi_n^{(k)}) & \dots & -T_{n,2N}(\varphi_n^{(k)}) \\ \vdots & \vdots & \ddots & \vdots \\ -T_{n,N1}(\varphi_n^{(k)}) & -T_{n,N2}(\varphi_n^{(k)}) & \dots & I-T_{n,NN}(\varphi_n^{(k)}) \end{pmatrix} \begin{pmatrix} \varphi_{n,1}^{(k+1)} - \varphi_{n,1}^{(k)} \\ \varphi_{n,2}^{(k+1)} - \varphi_{n,2}^{(k)} \\ \vdots \\ \varphi_{n,N}^{(k+1)} - \varphi_{n,N}^{(k)} \end{pmatrix} =$$

$$- \begin{pmatrix} \varphi_{n,1}^{(k)} \\ \varphi_{n,2}^{(k)} \\ \vdots \\ \varphi_{n,N}^{(k)} \end{pmatrix} + \begin{pmatrix} K_1(\varphi_n^{(k)}) \\ K_2(\varphi_n^{(k)}) \\ \vdots \\ K_N(\varphi_n^{(k)}) \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_N \end{pmatrix}$$

However, we can rewritten (2.7) as, for all $i = 1, 2, \cdots, N$

$$\varphi_{n,i}^{(k+1)}(t) - \sum_{j=1}^{N}\sum_{p=1}^{n} \omega_{n,p} \frac{\partial \kappa_i}{\partial \varphi_j}\left(t, t_p, \varphi_n^{(k)}(t_p)\right) \varphi_{n,j}^{(k+1)}(t_p) = f_{n,i}^{(k)}(t), \tag{2.8}$$

where,

$$f_{n,i}^{(k)}(t) = -\sum_{j=1}^{N}\sum_{p=1}^{n} \omega_{n,p} \frac{\partial \kappa_i}{\partial \varphi_j}\left(t, t_p, \varphi_n^{(k)}(t_p)\right) \varphi_{n,j}^{(k)}(t_p) + \int_0^1 \kappa_i\left(t, s, \varphi_n^{(k)}(s)\right) + g_i(t). \tag{2.9}$$

We defined the vector $X_N^{(k+1)} = (x_1^{(k+1)}, x_2^{(k+1)}, \cdots, x_N^{(k+1)}) \in \mathbb{R}^{n.N}$ for saving the collocation of our discretized approximation $\varphi_n^{(k+1)}(t) \in \mathcal{X}$ in the nodes $(t_p)_{1\le p\le n}$, and for all $i = 1, 2, \cdots, N$ we denote by

$$x_i^{(k+1)}(p) = \varphi_{n,i}^{(k+1)}(t_p).$$

The solution of system of equations defined in (2.8) − (2.9) is gotten by two steps:

Step1. Solve the linear algebraic system

$$\underbrace{(I_N - A_n^{(k)})}_{N.n\times N.n}\underbrace{X_N^{(k+1)}}_{N.n\times 1} = \underbrace{b_n^{(k)}}_{N.n\times 1},$$

where for all $i, j = 1, 2, \cdots, N$ and $l, p = 1, 2, \cdots, n$

$$[A_n^{(k)}]_{ij}(l,p) = \omega_{n,p}\frac{\partial \kappa_i}{\partial \varphi_j}\left(t_l, t_p, x_1^{(k)}(p), x_2^{(k)}(p), \cdots, x_N^{(k)}(p)\right), \tag{2.10}$$

$$b_{n,i}^{(k)}(l) = -\sum_{j=1}^{N}\sum_{p=1}^{n} \omega_{n,p}\frac{\partial \kappa_i}{\partial \varphi_j}\left(t_l, t_p, x_1^{(k)}(p), x_2^{(k)}(p), \cdots, x_N^{(k)}(p)\right) x_j^{(k)}(p) \tag{2.11}$$
$$+ \int_0^1 \kappa_i\left(t_l, s, \varphi_{n,1}^{(k)}(s), \varphi_{n,2}^{(k)}(s), \cdots, \varphi_{n,N}^{(k)}(s)\right) ds + g_i(t_l),$$

and I_N is the real block identity matrix given by

$$[I_N]_{ij}(l,p) = \begin{cases} 1 & \text{if } i = j \text{ and } l = p, \\ 0 & \text{else }. \end{cases}$$

Step2. For all $i = 1, 2, \cdots, N$, we recover $\varphi_{n,i}^{(k+1)}$ by the natural interpolation formula

$$\begin{aligned} \varphi_{n,i}^{(k+1)}(t) &= \sum_{j=1}^{N}\sum_{p=1}^{n} \omega_{n,p} \frac{\partial \kappa_i}{\partial \varphi_j}\left(t, t_p, x_1^{(k)}(p), x_2^{(k)}(p), \cdots, x_N^{(k)}(p)\right)\left(x_j^{(k+1)}(p) - x_j^{(k)}(p)\right) \\ &+ \int_0^1 \kappa_i\left(t, s, \varphi_{n,1}^{(k)}(s), \varphi_{n,2}^{(k)}(s), \cdots, \varphi_{n,N}^{(k)}(s)\right) ds + g_i(t). \end{aligned} \tag{2.12}$$

Before studying the convergence of the (LN.DN) process, we give some properties of the operator M_T and its approximate M_{T_n}.

2.2.1 Properties of Operators M_T and M_{T_n}

To present the convergence theorem of our method, we will initially establish vital properties of the operators M_T and its estimate operator M_{T_n}. The outcomes of Proposition 2.2.1 will be utilized in the proof of Proposition 2.2.3 and the main Theorem 2.3.1. Likewise, Proposition 2.2.2 will be utilized to prove Proposition 2.2.4, and the latter will be implemented in the proof of the main Theorem 2.3.1.

Proposition 2.2.1. *Under the assumption (2.5)(iii), we have M_T is Lipschitzian over $B_R(\varphi)$ where*

$$\lambda_R = 2\sup\left\{ \sup_{1\le j\le N} \sup_{(t,s,\widetilde{\psi}_j)\in[0,1]^2\times D_R} \sum_{i=1}^{N}\left|\frac{\partial^2 \kappa_i}{\partial \varphi_j^2}(t,s,\widetilde{\psi}_j)\right|, \sup_{1\le j\le N} \sup_{(t,s,\widetilde{v}_j)\in[0,1]^2\times D_R} \sum_{i=1}^{N}\left|\frac{\partial^3 \kappa_i}{\partial \varphi_j^2 \partial t}(t,s,\widetilde{v}_j)\right| \right\},$$

is the Lipschitz constant, and

$$D_R = [-\|\varphi\|_{\mathcal{X}} - R, \|\varphi\|_{\mathcal{X}} + R].$$

Proof. Let $\psi = (\psi_1, \psi_2, \cdots, \psi_N), \phi = (\phi_1, \phi_2, \cdots, \phi_N) \in B_R(\varphi)$ and $h \in \mathcal{X}$, we have

$$\begin{aligned}
\|(M_T(\psi) - M_T(\phi))h\|_{\mathcal{X}} &= \sum_{i=1}^{N} \left\| \sum_{j=1}^{N} \left[T_{ij}(\psi) - T_{ij}(\phi)\right] h_j \right\|_{\mathcal{X}_i} \leq \sum_{i=1}^{N}\sum_{j=1}^{N} \|\left[T_{ij}(\psi) - T_{ij}(\phi)\right] h_j\|_{\mathcal{X}_i} \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} \left(|| \left[T_{ij}(\psi) - T_{ij}(\phi) h_j\right] ||_\infty + || \frac{d}{dt}(\left[T_{ij}(\psi) - T_{ij}(\phi)\right] h_j)||_\infty \right) \\
&\leq \sup_{t\in[0,1]} \sum_{i=1}^{N}\sum_{j=1}^{N} \int_0^1 \left| \left(\frac{\partial \kappa_i}{\partial \varphi_j}(t,s,\psi(s)) - \frac{\partial \kappa_i}{\partial \varphi_j}(t,s,\phi(s)) \right) \right| |h_j(s)| ds \\
&\quad + \sup_{t\in[0,1]} \sum_{i=1}^{N}\sum_{j=1}^{N} \int_0^1 \left| \left(\frac{\partial^2 \kappa_i}{\partial \varphi_j \partial t}(t,s,\psi(s)) - \frac{\partial^2 \kappa_i}{\partial \varphi_j \partial t}(t,s,\phi(s)) \right) \right| |h_j(s)| ds,
\end{aligned}$$

and by the assumption (2.5)(iii), we can apply the Mean Value Theorem, then for all $j = 1, 2, \cdots, N$; $\exists u_j, \upsilon_j \in [\psi_j, \phi_j]$ $\big($The line segment joining two points $\psi_j, \phi_j \in B_{R_j}(\varphi_j)\big)$ such that

$$\left| \frac{\partial \kappa_i}{\partial \varphi_j}(t,s,\psi(s)) - \frac{\partial \kappa_i}{\partial \varphi_j}(t,s,\phi(s)) \right| \leq \sup_{u_j\in[\psi_j,\phi_j]} \left| \frac{\partial^2 \kappa_i}{\partial \varphi_j^2}(t,s,\varphi_1(s),\cdots,u_j(s),\cdots,\varphi_N(s)) \right| ||\psi_j - \phi_j||_\infty,$$

$$\left| \frac{\partial^2 \kappa_i}{\partial \varphi_j \partial t}(t,s,\psi(s)) - \frac{\partial^2 \kappa_i}{\partial \varphi_j \partial t}(t,s,\phi(s)) \right| \leq \sup_{\upsilon_j\in[\psi_j,\phi_j]} \left| \frac{\partial^3 \kappa_i}{\partial \varphi_j^2 \partial t}(t,s,\varphi_1(s),\cdots,\upsilon_j(s),\cdots,\varphi_N(s)) \right| ||\psi_j - \phi_j||_\infty,$$

it's not difficult to demonstrate that, for all $j = 1, 2, \cdots, N$, we have

$$\begin{aligned}
||\widetilde{u}||_{\mathcal{X}} = \widetilde{u}_j \in D_R = [-R - ||\varphi||_{\mathcal{X}}, R + ||\varphi||_{\mathcal{X}}], \text{where} \quad \widetilde{u} = (\varphi_1, \cdots, u_j, \cdots, \varphi_N) \in B_R(\varphi), \\
||\widetilde{\upsilon}||_{\mathcal{X}} = \widetilde{\upsilon}_j \in D_R = [-R - ||\varphi||_{\mathcal{X}}, R + ||\varphi||_{\mathcal{X}}], \text{where} \quad \widetilde{\upsilon} = (\varphi_1, \cdots, \upsilon_j, \cdots, \varphi_N) \in B_R(\varphi),
\end{aligned}$$

and by these notations, we can compose

$$\sup_{t\in[0,1]} \sup_{u_j\in[\psi_j,\phi_j]} \left| \frac{\partial^2 \kappa_i}{\partial \varphi_j^2}(t,s,\varphi_1(s),\cdots,u_j(s),\cdots,\varphi_N(s)) \right| \leq \sup_{(t,s,\widetilde{u}_j)\in[0,1]^2\times D_R} \left| \frac{\partial^2 \kappa_i}{\partial \varphi_j^2}(t,s,\widetilde{u}_j) \right| \|\psi_j - \phi_j\|_\infty,$$

$$\sup_{t\in[0,1]} \sup_{\upsilon_j\in[\psi_j,\phi_j]} \left| \frac{\partial^3 \kappa_i}{\partial \varphi_j^2 \partial t}(t,s,\varphi_1(s),\cdots,\upsilon_j(s),\cdots,\varphi_N(s)) \right| \leq \sup_{(t,s,\widetilde{\upsilon}_j)\in[0,1]^2\times D_R} \left| \frac{\partial^3 \kappa_i}{\partial \varphi_j^2 \partial t}(t,s,\widetilde{\upsilon}_j) \right| \|\psi_j - \phi_j\|_\infty.$$

So, as for all $j = 1, 2, \cdots, N$, $\|h_j\|_\infty \leq \|h\|_{\mathcal{X}}$, we have

$$\begin{aligned}
\|(M_T(\psi) - M_T(\phi))\| &\leq \sum_{i=1}^{N}\sum_{j=1}^{N} \sup_{(t,s,\widetilde{u}_j)\in[0,1]^2\times D_R} \left| \frac{\partial^2 \kappa_i}{\partial \varphi_j^2}(t,s,\widetilde{u}_j) \right| \|\psi_j - \phi_j\|_\infty \\
&\quad + \sum_{i=1}^{N}\sum_{j=1}^{N} \sup_{(t,s,\widetilde{\upsilon}_j)\in[0,1]^2\times D_R} \left| \frac{\partial^3 \kappa_i}{\partial \varphi_j^2 \partial t}(t,s,\widetilde{\upsilon}_j) \right| \|\psi_j - \phi_j\|_\infty.
\end{aligned}$$

Obviously, $\sum_{j=1}^{N} \|\psi_j - \phi_j\|_\infty \le ||\psi - \phi||_{\mathcal{X}}$, what's more, we take

$$\lambda_R = 2\sup\left\{ \sup_{1\le j\le N} \sup_{(t,s,\widetilde{u}_j)\in[0,1]^2\times D_R} \sum_{i=1}^{N} \left| \frac{\partial^2 \kappa_i}{\partial \varphi_j^2}(t,s,\widetilde{u}_j) \right|, \sup_{1\le j\le N} \sup_{(t,s,\widetilde{\upsilon}_j)\in[0,1]^2\times D_R} \sum_{i=1}^{N} \left| \frac{\partial^3 \kappa_i}{\partial \varphi_j^2 \partial t}(t,s,\widetilde{\upsilon}_j) \right| \right\},$$

to discover at last that

$$\|(M_T(\varphi) - M_T(\phi))\| \le \lambda_R \|\varphi - \phi\|_{\mathcal{X}}.$$

□

Proposition 2.2.2. *Assume that (2.5)(iii), there exists a constant C_N such that, for all $\phi \in B_R(\varphi)$*

$$\|M_T(\phi) - M_{T_n}(\phi)\| \le \frac{C_N}{n^2}, \quad n \in \mathbb{N}^*, \tag{2.13}$$

where

$$C_N = \frac{1}{12n^2} \sup_{t\in[0,1]} \sum_{i=1}^{N}\sum_{j=1}^{N} \left\{ \left| \left[\frac{\partial^2 \kappa_i(t,s,\phi(s))}{\partial \varphi_j \partial s} \right]_{s=0}^{s=1} \right| + \left| \left[\frac{\partial^3 \kappa_i(t,s,\phi(s))}{\partial \varphi_j \partial s \partial t} \right]_{s=0}^{s=1} \right| \right\}.$$

Proof. Let $h \in \mathcal{X}$ be the direction of the operator M_T. For $\phi = (\phi_1, \phi_2, \cdots, \phi_N) \in B_R(\varphi)$, we have

$$\begin{aligned}
\|(M_T(\phi) - M_{T_n}(\phi))\, h\|_{\mathcal{X}} &= \sum_{i=1}^{N} \left\| \sum_{j=1}^{N} [T_{ij}(\phi) - T_{n,ij}(\phi)]\, h_j \right\|_{\mathcal{X}_i} \\
&\le \sum_{i=1}^{N}\sum_{j=1}^{N} \|[T_{ij}(\phi) - T_{n,ij}(\phi)]\, h_j\|_{\mathcal{X}_i} \\
&= \sup_{t\in[0,1]} \sum_{i=1}^{N}\sum_{j=1}^{N} |([T_{ij}(\phi) - T_{n,ij}(\phi)]\, h_j)(t)| \\
&\quad + \sup_{t\in[0,1]} \sum_{i=1}^{N}\sum_{j=1}^{N} \left| \frac{d}{dt} ([T_{ij}(\phi) - T_{n,ij}(\phi)]\, h_j)(t) \right| \\
&\le \sup_{t\in[0,1]} \left(\sum_{i=1}^{N}\sum_{j=1}^{N} |[T_{ij}(\phi) - T_{n,ij}(\phi)](t)| \right) ||h_j||_\infty \\
&\quad + \sup_{t\in[0,1]} \left(\sum_{i=1}^{N}\sum_{j=1}^{N} \left| \frac{d}{dt} [T_{ij}(\phi) - T_{n,ij}(\phi)](t) \right| \right) ||h_j||_\infty,
\end{aligned}$$

and by the trapezoidal rule (2.2), we have for all $i, j = 1, 2, \cdots, N$, and $t \in [0,1]$

$$|(T_{ij}(\phi) - T_{n,ij}(\phi))(t)| \le \frac{1}{12n^2} \left| \left[\frac{\partial^2 \kappa_i(t,s,\phi(s))}{\partial \varphi_j \partial s} \right]_{s=0}^{s=1} \right|,$$

$$\left|\frac{d}{dt}(T_{ij}(\phi)-T_{n,ij}(\phi))(t)\right|\leq\frac{1}{12n^2}\left|\left[\frac{\partial^3\kappa_i(t,s,\phi(s))}{\partial\varphi_j\partial s\partial t}\right]_{s=0}^{s=1}\right|,$$

we finish up at last that

$$\|M_T(\phi)-M_{T_n}(\phi)\|\leq\frac{1}{12n^2}\sup_{t\in[0,1]}\sum_{i=1}^{N}\sum_{j=1}^{N}\left\{\left|\left[\frac{\partial^2\kappa_i(t,s,\phi(s))}{\partial\varphi_j\partial s}\right]_{s=0}^{s=1}\right|+\left|\left[\frac{\partial^3\kappa_i(t,s,\phi(s))}{\partial\varphi_j\partial s\partial t}\right]_{s=0}^{s=1}\right|\right\}.$$

□

Proposition 2.2.3. *Assume that (2.5) holds. Let* $r=\min\left(R,\dfrac{1}{2\lambda_R\eta}\right)$*, where* λ_R *is defined in Proposition 2.2.1. Then for all* $\phi=(\phi_1,\phi_2,\cdots,\phi_N)\in B_r(\varphi), I_{NN}-M_T(\phi)$ *is invertible and*

$$\left\|(I_{NN}-M_T(\phi))^{-1}\right\|\leq 2\eta.$$

Proof. For all $\phi=(\phi_1,\phi_2,\cdots,\phi_N)\in\mathcal{X}$, we have

$$\begin{aligned}I_{NN}-M_T(\phi)=&I_{NN}-M_T(\varphi)-M_T(\phi)+M_T(\varphi)\\ =&\left(I_{NN}-M_T(\varphi)\right)\left[I_{NN}-\left(I_{NN}-M_T(\varphi)\right)^{-1}\left(M_T(\phi)-M_T(\varphi)\right)\right],\end{aligned}$$

we have for all $\phi=(\phi_1,\phi_2,\cdots,\phi_N)\in B_r(\varphi)$, (Proposition 2.2.1)

$$\|(M_T(\phi)-M_T(\varphi))\|\leq\lambda_R r.$$

Then

$$\left\|\left(I_{NN}-M_T(\varphi)\right)^{-1}\left(M_T(\phi)-M_T(\varphi)\right)\right\|\leq\eta\lambda_R r\leq\tfrac{1}{2},$$

use the Neummann Expansion Theorem 1.1.5 , we conclude that $I_{NN}-M_T(\phi)$ is invertible such that, for all $\phi=(\phi_1,\phi_2,\cdots,\phi_N)\in B_r(\varphi)$

$$\left(I_{NN}-M_T(\phi)\right)^{-1}=\left(I_{NN}-\left(I_{NN}-M_T(\varphi)\right)^{-1}\left[M_T(\phi)-M_T(\varphi)\right]\right)^{-1}\left(I_{NN}-M_T(\varphi)\right)^{-1},$$

and its inverse is uniformly bounded on $B_r(\varphi)$, where

$$\left\|\left(I_{NN}-M_T(\phi)\right)^{-1}\right\|\leq\eta\sum_{q=0}^{\infty}\left\|\left(I_{NN}-M_T(\varphi)\right)^{-1}\left(M_T(\phi)-M_T(\varphi)\right)\right\|^q\leq 2\eta.$$

□

Proposition 2.2.4. *Assume that* (2.5) *holds. Then for* n *big enough, and for all* $\phi = (\phi_1, \phi_2, \cdots, \phi_N) \in B_r(\varphi)$, $I_{NN} - M_{T_n}(\phi)$ *is invertible, and there exists* $\delta_n \in]0,1[$, *such that*

$$\sup_{\phi \in B_r(\varphi)} \left\| I_{NN} - \left(I_{NN} - M_{T_n}(\phi)\right)^{-1} \left(I_{NN} - M_T(\phi)\right) \right\| \leq \delta_n,$$

$$\sup_{\phi \in B_r(\varphi)} \left\| \left(I_{NN} - M_{T_n}(\phi)\right)^{-1} \right\| \leq 2\eta(1 + \delta_n).$$

Proof. For all $\phi = (\phi_1, \phi_2, \cdots, \phi_N) \in B_r(\varphi)$, we have

$$\begin{aligned} I_{NN} - M_{T_n}(\phi) =& I_{NN} - M_T(\phi) + M_T(\phi) - M_{T_n}(\phi) \\ =& \left(I_{NN} - M_T(\phi)\right) \left(I_{NN} - \left(I_{NN} - M_T(\phi)\right)^{-1} \left[M_{T_n}(\phi) - M_T(\phi)\right]\right), \end{aligned}$$

and as we have in Proposition 2.2.2

$$\| M_T(\phi) - M_{T_n}(\phi) \| \leq \frac{C_N}{n^2} \longrightarrow 0, \text{ as } n \longrightarrow +\infty.$$

So, with n adequately large, plainly $\dfrac{C_N}{n^2} < \dfrac{1}{2\eta}$. Then

$$\left\| \left(I_{NN} - M_{T_n}(\phi)\right)^{-1} \left(M_{T_n}(\phi) - M_T(\phi)\right) \right\| \leq \frac{2\eta C_N}{n^2} < 1,$$

and by the Neumann Expansion Theorem 1.1.5, we have for all $\phi \in B_r(\varphi)$, $(I_{NN} - M_{T_n}(\phi))$ is invertible and

$$\left\| \left(I_{NN} - M_{T_n}(\phi)\right)^{-1} \right\| \leq \frac{2\eta}{1 - 2\eta\xi_n},$$

where $\xi_n = \dfrac{C_N}{n^2}$. As

$$I_{NN} - \left(I_{NN} - M_{T_n}(\phi)\right)^{-1} \left(I_{NN} - M_T(\phi)\right) = \left(I_{NN} - M_{T_n}(\phi)\right)^{-1} \left(M_T(\phi) - M_{T_n}(\phi)\right),$$

we define $\delta_n = \dfrac{2\eta\xi_n}{1 - 2\eta\xi_n}$ and for n large enough, $\delta_n < 1$ we find

$$\sup_{\phi \in B_r(\varphi)} \left\| I_{NN} - \left(I_{NN} - M_{T_n}(\phi)\right)^{-1} \left(I_{NN} - M_T(\phi)\right) \right\| \leq \delta_n,$$

we have

$$\begin{aligned} \left(I_{NN} - M_{T_n}(\phi)\right)^{-1} =& \left(I_{NN} - M_T(\phi)\right)^{-1} \\ & - \left[I_{NN} - \left(I_{NN} - M_{T_n}(\phi)\right)^{-1} \left(I_{NN} - M_{T_n}(\phi)\right)\right] \left(I_{NN} - M_T(\phi)\right)^{-1}. \end{aligned}$$

In this way, we close at last that

$$\sup_{\phi \in B_r(\varphi)} \left\| \left(I_{NN} - M_{T_n}(\phi)\right)^{-1} \right\| \leq 2\eta(1 + \delta_n).$$

□

2.3 Analysis of The (LN.DN) Process

In this section, we study the convergence of (LN-DN) process, where we will prove that our approximate solution $\varphi_n^k = (\varphi_{n,1}^k, \varphi_{n,2}^k, \cdots, \varphi_{n,N}^k) \in \mathcal{X}$ defined in (2.12), converges to the exact solution $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_N) \in \mathcal{X}$ defined in (2.1).

Theorem 2.3.1. *Assume that the assumptions (2.5) are satisfied, set* $r = \min\left(R, \dfrac{1}{2\lambda_R \eta}\right)$. *Then there exist* $\delta_n \in]0,1[$, *and* $\varrho_n > 0$ *such that, if the starting approximation* $\varphi_n^{(0)}$ *is chosen in the closed ball* $B_{\varrho_n}(\varphi)$, *then for all* $k \in \mathbb{N}^*, \varphi_n^{(k)} \in B_{\varrho_n}(\varphi)$, *and*

$$\left\|\varphi_n^{(k)} - \varphi\right\|_{\mathcal{X}} \leq \varrho_n \left(\frac{1+\delta_n}{2}\right)^k \longrightarrow 0 \quad as \quad k \longrightarrow \infty.$$

Proof. We have found in Proposition 2.2.4 that, if $\varphi_n^{(k)} \in B_r(\varphi)$, $\left(I_{NN} - M_{T_n}(\varphi_n^{(k)})\right)$ is invertible. Then $\varphi_n^{(k+1)}$ defined in (2.7) can given by

$$\varphi_n^{(k+1)} - \varphi = \varphi_n^{(k)} - \varphi - \left(I_{NN} - M_{T_n}(\varphi_n^{(k)})\right)^{-1} \left(\varphi_n^{(k)} - \varphi - K(\varphi_n^{(k)}) + K(\varphi)\right).$$

Since

$$K(\varphi) - K(\varphi_n^{(k)}) = -\int_0^1 M_T\left((1-x)\varphi_n^{(k)} + x\varphi\right).\left(\varphi_n^{(k)} - \varphi\right) dx,$$

then, we can write

$$\varphi_n^{(k+1)} - \varphi = \int_0^1 \left[I_{NN} - \left(I_{NN} - M_{T_n}(\varphi_n^{(k)})\right)^{-1}\left[I_{NN} - M_T\left((1-x)\varphi_n^{(k)} + x\varphi\right)\right]\right].\left(\varphi_n^{(k)} - \varphi\right) dx.$$

By added $I_{NN} - M_T(\varphi_n^{(k)})$ to and subtracted from $I_{NN} - M_T\left((1-x)\varphi_n^{(k)} + x\varphi\right)$, we get

$$\begin{aligned}\varphi_n^{(k+1)} - \varphi = &\int_0^1 \left[I_{NN} - \left(I_{NN} - M_{T_n}(\varphi_n^{(k)})\right)^{-1}\left(I_{NN} - M_T(\varphi_n^{(k)})\right)\right].\left(\varphi_n^{(k)} - \varphi\right) dx \\ &+ \int_0^1 \left(I_{NN} - M_{T_n}(\varphi_n^{(k)})\right)^{-1}\left[M_T\left((1-x)\varphi_n^{(k)} + x\varphi\right) - M_T(\varphi_n^{(k)})\right].\left(\varphi_n^{(k)} - \varphi\right) dx,\end{aligned}$$

and

$$\begin{aligned}||\varphi_n^{(k+1)} - \varphi||_{\mathcal{X}} \leq\ &||I_{NN} - \left(I_{NN} - M_{T_n}(\varphi_n^{(k)})\right)^{-1}\left(I_{NN} - M_T(\varphi_n^{(k)})\right)||\ ||\varphi_n^{(k)} - \varphi||_{\mathcal{X}} \\ &+ \left\|\left(I_{NN} - M_{T_n}(\varphi_n^{(k)})\right)^{-1}\right\| \left\|\varphi_n^{(k)} - \varphi\right\|_{\mathcal{X}} \int_0^1 \left\|M_T\left((1-x)\varphi_n^{(k)} + x\varphi\right) - M_T(\varphi_n^{(k)})\right\| dx.\end{aligned}$$

Let $\varphi_n^{(k)} \in B_r(\varphi)$ and according to Proposition 2.2.4

$$\left\| I_{NN} - \left(I_{NN} - M_{T_n}(\varphi_n^{(k)})\right)^{-1} \left(I_{NN} - M_T(\varphi_n^{(k)})\right)\right\| \leq \delta_n,$$

and since $B_r(\varphi)$ is convex, for all $x \in [0,1], (1-x)\varphi_n^{(k)} + x\varphi \in B_r(\varphi)$, and according to Proposition 2.2.1

$$\left\| M_T\left((1-x)\varphi_n^{(k)} + x\varphi\right) - M_T(\varphi_n^{(k)})\right\| \leq \lambda_R x \left\|\varphi_n^{(k)} - \varphi\right\|_{\mathcal{X}}.$$

Hence

$$\int_0^1 \left\| M_T\left((1-x)\varphi_n^{(k)} + x\varphi\right) - M_T(\varphi_n^{(k)})\right\| dx \leq \frac{1}{2}\lambda_R \left\|\varphi_n^{(k)} - \varphi\right\|_{\mathcal{X}}.$$

We use the second inequality of Proposition 2.2.4, we get

$$\left\|\varphi_n^{(k+1)} - \varphi\right\|_{\mathcal{X}} \leq \delta_n \left\|\varphi_n^{(k)} - \varphi\right\|_{\mathcal{X}} + \left(2\eta(1+\delta_n)\left\|\varphi_n^{(k)} - \varphi\right\|_{\mathcal{X}}\right)\frac{1}{2}\lambda_R \left\|\varphi_n^{(k)} - \varphi\right\|_{\mathcal{X}}.$$

We define

$$\varrho_n := \min\left\{r, \left(\frac{1-\delta_n}{2\lambda_R \eta(1+\delta_n)}\right)\right\}.$$

Then if $\varphi_n^{(k)} \in B_{\varrho_n}(\varphi)$, $\frac{1}{2}\lambda_R \left\|\varphi_n^{(k)} - \varphi\right\|_{\mathcal{X}} \leq \frac{1-\delta_n}{4\eta(1+\delta_n)}$. Hence

$$\left\|\varphi_n^{(k+1)} - \varphi\right\|_{\mathcal{X}} \leq \left(\frac{1+\delta_n}{2}\right)\left\|\varphi_n^{(k)} - \varphi\right\|_{\mathcal{X}},$$

since $1+\delta_n < 2$ the previous inequality implies that $\varphi_n^{(k+1)} \in B_{\varrho_n}(\varphi)$ and that

$$\left\|\varphi_n^{(k)} - \varphi\right\|_{\mathcal{X}} \leq \varrho_n \left(\frac{1+\delta_n}{2}\right)^k \longrightarrow 0, \text{ as } k \longrightarrow \infty.$$

□

2.4 Analysis of The (DN.LN) Process

This process begins by the discretization of the system (2.1), where the Nyström method leads to finding $\vartheta_n(t) = (\vartheta_{n,1}(t), \vartheta_{n,2}(t), \cdots, \vartheta_{n,N}(t)) \in \mathcal{X}, n \in \mathbb{N}^*$, such that

$$\begin{cases} \vartheta_{n,1}(t) = \sum_{p=1}^{n} \omega_{n,p}\kappa_1\left(t, t_p, \vartheta_{n,1}(t_p), \vartheta_{n,2}(t_p), \cdots, \vartheta_{n,N}(t_p)\right) + g_1(t), \\ \vartheta_{n,2}(t) = \sum_{p=1}^{n} \omega_{n,p}\kappa_2\left(t, t_p, \vartheta_{n,1}(t_p), \vartheta_{n,2}(t_p), \cdots, \vartheta_{n,N}(t_p)\right) + g_2(t), \\ \quad\vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ \vartheta_{n,N}(t) = \sum_{p=1}^{n} \omega_{n,p}\kappa_N\left(t, t_p, \vartheta_{n,1}(t), \vartheta_{n,2}(t_p), \cdots, \vartheta_{n,N}(t_p)\right) + g_N(t), \end{cases} \tag{2.14}$$

for $t \in [0,1]$ and a given functions $g_i \in \Omega_i$, for all $i = 1, 2, \cdots, N$.

Proposition 2.4.1. *Assume that* (2.5) *holds. Then*

$$\|\vartheta_n - \varphi\|_{\mathcal{X}} \leq \frac{C_N}{n^2}, \quad n \in \mathbb{N}^*,$$

where

$$C_N = \frac{C}{12} \sup_{t\in[0,1]} \sum_{i=1}^{N} \left\{ \left| \left[\frac{\partial \kappa_i(t,s,\varphi(s))}{\partial s} \right]_{s=0}^{s=1} \right| + \left| \left[\frac{\partial^2 \kappa_i(t,s,\varphi(s))}{\partial s \partial t} \right]_{s=0}^{s=1} \right| \right\}.$$

To prove this proposition, we use the numerical integration rule (2.2) and the result (1.34) of the Theorem 1.5.3 .

By utilizing the collocation technique on each equation of system (2.14), we obtain a nonlinear algebraic system of equations in $\mathbb{R}^{n.N}$. Let

$$x_j^{(\infty)}(l) = \vartheta_{n,j}(t_l), \quad j = 1, 2, \cdots, N, \quad l = 1, 2, \cdots, n.$$

For all $i = 1, 2, \cdots, N$. Let $\mathcal{O}_{n,i}$ be open subset of $\mathbb{R}^n$ and let $F_n = (F_{n,1}, F_{n,2}, \cdots, F_{n,N})$ be the nonlinear operator from some open subset $\mathcal{O}_N = \prod_{i=1}^{N} \mathcal{O}_{n,i}$ of $\mathbb{R}^{n.N}$ into $\mathbb{R}^{n.N}$ defined for all $l = 1, 2, \cdots, n$ by

$$F_{n,i}(X)(l) = x_i(l) - \sum_{p=1}^{n} \omega_{n,p} \kappa_i \left(t_l, t_p, X(p)\right) - g_i(t_l), \tag{2.15}$$

where $X = (x_1, x_2, \cdots, x_N) \in \mathcal{O}_N$. The problem is set up as:

$$\begin{cases} \text{Find} \quad X_{n,N}^{\infty} = (x_1^{\infty}, x_2^{\infty}, \cdots, x_N^{\infty}) \in \mathbb{R}^{n.N}, \\ F_n(X_{n,N}^{\infty}) = \left(F_{n,1}(X_{n,N}^{\infty}), F_{n,2}(X_{n,N}^{\infty}), \cdots, F_{n,N}(X_{n,N}^{\infty})\right) = 0_{\mathbb{R}^{n.N}}. \end{cases} \tag{2.16}$$

We have F_n is differentiable and its differentiable is presented by a block of matrices as

$$D_{F_n} = \begin{pmatrix} [D_{F_n}]_{11} & \dots & [D_{F_n}]_{1N} \\ [D_{F_n}]_{21} & \dots & [D_{F_n}]_{2N} \\ \vdots & \ddots & \vdots \\ [D_{F_n}]_{N1} & \dots & [D_{F_n}]_{NN} \end{pmatrix}$$

where for all $i, j = 1, 2, \cdots, N$; $[D_{F_n}]_{ij}$ is a matrix of size $n \times n$ defined for all $l, p = 1, 2, \cdots, n$ as

$$[D_{F_n}]_{ij}(l,p) = \begin{cases} 1 - \omega_{n,p} \dfrac{\partial \kappa_i}{\partial \varphi_j}\left(t_l, t_p, x_1^{(\infty)}(p), x_2^{(\infty)}(p), \cdots, x_N^{(\infty)}(p)\right), & \text{if } i=j \text{ and } l=p, \\ -\omega_{n,p} \dfrac{\partial \kappa_i}{\partial \varphi_j}\left(t_l, t_p, x_1^{(\infty)}(p), x_2^{(\infty)}(p), \cdots, x_N^{(\infty)}(p)\right), & \text{else.} \end{cases}$$

We solve the problem (2.16) by using the Newton-Raphson[1] method. The iterate $X_{n,N}^{(k+1)}$ solves

$$\underbrace{D_{F_n}}_{N.n\times N.n} \underbrace{X_{n,N}^{(k+1)}}_{N.n\times 1} = \underbrace{d_n^{(k)}}_{N.n\times 1}, \quad k = 1, 2, \cdots,$$

where for all $i = 1, 2, \cdots, N$, and all $l = 1, 2, \cdots, n$, we have

$$d_{n,i}^{(k)}(l) = -\sum_{j=1}^{N}\sum_{p=1}^{n} \omega_{n,p} \frac{\partial \kappa_i}{\partial \varphi_j}\left(t_l, t_p, X_{n,N}^{(k)}(p)\right) x_j^{(k)}(p) + \sum_{p=1}^{n} \omega_{n,p}\kappa_i\left(t_l, t_p, X_{n,N}^{(k)}(p)\right) + g_i(t_l). \tag{2.17}$$

We recover the approximation $\vartheta_n^{(k+1)} = \left(\vartheta_{n,1}^{(k+1)}, \vartheta_{n,2}^{(k+1)}, \cdots, \vartheta_{n,N}^{(k+1)}\right) \in \mathcal{X}$ with the natural interpolation formula:

$$\vartheta_{n,i}^{(k+1)}(t) = \sum_{p=1}^{n} \omega_{n,p}\kappa_i\left(t, t_p, X_{n,N}^{(k+1)}(p)\right) + g_i(t), \quad i = 1, 2, \cdots, N, \quad t \in [0,1]. \tag{2.18}$$

After completing these steps, we need to prove the convergence of the iterates $\vartheta_n^{(k+1)}$ towards the solution ϑ_n of the system (2.14).

Let $\|\cdot\|_n$ be the vector norm in $\mathbb{R}^n$, and $\|\cdot\|_{n,N}$ be the vector norm in $\mathbb{R}^{n.N}$ such as

$$\forall V = (v_1, v_2, \cdots, v_N) \in \mathbb{R}^{n.N}, \|V\|_{n,N} = \sum_{j=1}^{N} \|v_j\|_n = \sum_{j=1}^{N}\sum_{p=1}^{n} |v_j(p)|.$$

Let $|||\cdot|||$ be the matrix norm in $\mathcal{M}_{n.N}(\mathbb{R})$ such as

$$\forall M \in \mathcal{M}_{n\times N}(\mathbb{R}), |||M||| = \max_{1\le i\le N} \sum_{j=1}^{N} \max_{1\le l\le n} \sum_{p=1}^{n} |M_{ij}(l.p)|.$$

Now, we define the vector $W = (w_1, w_2, \cdots, w_N) \in \mathbb{R}^{n.N}$ from the exact solution φ by

$$w_i(l) = \varphi_i(t_l), \quad i = 1, 2, \cdots, N, \quad l = 1, 2, \cdots, n,$$

and for $V_{n,N} = (v_1, v_2, \cdots, v_N) \in \mathbb{R}^{n.N}$, we define $\widetilde{V}_{n,N} = (\widetilde{v}_{n,1}, \widetilde{v}_{n,2}, \cdots, \widetilde{v}_{n,N}) \in \mathcal{X}$ by

$$\widetilde{v}_{n,i}(t) = \sum_{p=1}^{n} \omega_{n,p}\kappa_i(t, t_p, V_{n,N}(p)) + g_i(t), \quad i = 1, 2, \cdots, N, \quad t \in [0,1]. \tag{2.19}$$

Let $S_\rho(W)$ the ball of center W and radius ρ in $\mathbb{R}^{n.N}$ for the norm $\|\cdot\|_{n,N}$.

[1] Joseph Raphson (1648-1715) was an English mathematician and surveyor who made important contributions to the development of calculus, as well as to the fields of optics and surveying.

Lemma 2.4.1. *For all* $V_{n,N} \in S_\rho(W) = \prod_{i=1}^{N} S_{\rho_i}(w_i) \subset \mathbb{R}^{n.N}$,

$$\left\|\widetilde{V}_{n,N} - \varphi\right\|_{\mathcal{X}} \leq C_\rho \left\|V_{n,N} - W\right\|_{n,N} + O\left(\frac{N}{n^2}\right). \tag{2.20}$$

Proof. We using (1.34) to write

$$\begin{aligned}\left\|\widetilde{V}_{n,N} - \varphi\right\|_{\mathcal{X}} = \sum_{i=1}^{N} \left\|\widetilde{v}_{n,i} - \varphi_i\right\|_{\mathcal{X}_i} \leq & C\sum_{i=1}^{N} \left\|K_{n,i}(\widetilde{V}_{n,N}) - K_i(\varphi)\right\|_{\mathcal{X}_i} \\ \leq & C\sum_{i=1}^{N} \left(\left\|K_{n,i}(\widetilde{V}_{n,N}) - K_{n,i}(\varphi)\right\|_{\mathcal{X}_i} + \left\|K_{n,i}(\varphi) - K_i(\varphi)\right\|_{\mathcal{X}_i}\right).\end{aligned}$$

For all $i = 1, 2, \cdots, N$, we use (2.2) and the regularity of κ_i, for writing:

$$\left\|K_{n,i}(\varphi) - K_i(\varphi)\right\|_{\mathcal{X}_i} = O\left(\frac{1}{n^2}\right),$$

and

$$\left\|K_{n,i}(\widetilde{V}_{n,N}) - K_{n,i}(\varphi)\right\|_{\mathcal{X}_i} \leq (C_{\rho_i} + C'_{\rho_i}) \left\|v_i - w_i\right\|_n,$$

and by taking $C_\rho = C \max_{1\leq i\leq N}(C_{\rho_i} + C'_{\rho_i})$ and $\rho = \sum_{i=1}^{N} \rho_i$, we have finished the proof. □

In the following step, we fixed $n \gg N$ such that the Propositions 2.2.1 − 2.4.1 are satisfied, and we choose the positive number ρ such that

$$\rho C_\rho + O\left(\frac{N}{n^2}\right) \leq r, \tag{2.21}$$

with r is the parameter defined in Proposition 2.2.3 .Then

$$\forall V_{n,N} \in S_\rho(W) \subset \mathbb{R}^{n.N} \Longrightarrow \widetilde{V}_{n,N} \in B_r(\varphi) \subset \mathcal{X}.$$

As for all $l = 1, 2, \cdots, n$, we have for all $h \in \mathbb{R}^{n.N}$ and $\widetilde{h} \in \mathcal{X}$ given as in (2.19),

$$\left(D_{F_n}(V_{n,N}).h\right)(l, \cdot) = (I_{NN} - M_{T_n}(\widetilde{V}_{n,N}))(t_l, \cdot)\widetilde{h}(t_l), \tag{2.22}$$

and by using the Proposition 2.2.4, we can find that $D_{F_n}(V_{n,N})$ is invertible and

$$\exists \eta_n > 0, \quad |||(D_{F_n}(V_{n,N}))^{-1}||| \leq \eta_n, \quad \forall V_{n,N} \in S_\rho(W). \tag{2.23}$$

Similarly to Proposition 2.2.1, we can demonstrate that

$$|||D_{F_n}(X) - D_{F_n}(Y)||| \leq \lambda_\rho \left\|X - Y\right\|_{n,N}, \quad \forall X, Y \in S_\rho(W), \tag{2.24}$$

$$\lambda_\varrho = 2 \max_{1\leq i\leq N} \sup \left\{ \sup_{1\leq j\leq N} \sup_{(t,s,Z_j)\in[0,1]^2\times D_R} \left|\frac{\partial^2 \kappa_i}{\partial \varphi_j^2}(t, s, Z_j)\right|, \sup_{1\leq j\leq N} \sup_{(t,s,Z'_j)\in[0,1]^2\times D_R} \left|\frac{\partial^3 \kappa_i}{\partial \varphi_j^2 \partial t}(t, s, Z'_j)\right| \right\},$$

and

$$I_\rho = [-\rho - ||W||_{n,N}, \rho + ||W||_{n,N}].$$

Theorem 2.4.1. *Let $\vartheta_n^{(k+1)}$ be the iterate solution defined in (2.18). Assume that the assumption (2.5) are satisfied. Let r be the parameter defined in Proposition 2.2.3 and ρ satisfy (2.21). For $\vartheta_n^{(0)} \in S_\rho(W)$, let the positives constants $r_n, \beta_n, \eta_n, \lambda_\rho$ and τ_n be given with the accompanying properties:*

$$S_{r_n}(\vartheta_n^{(0)}) \subset S_\rho(W), \quad \tau_n = \frac{\beta_n \eta_n \lambda_\rho}{2} < 1, \quad r_n = \frac{\beta_n}{1-\tau_n},$$

the inequalities (2.23) and (2.24) are satisfied, and

$$|||(D_{F_n}(\vartheta_n^{(0)}))^{-1} D_{F_n}(\vartheta_n^{(0)})||| \leq \beta_n.$$

Then $\vartheta_n^{(k)} \in S_{r_n}(\vartheta_n^{(0)})$ and

$$\left\| \vartheta_n^{(k)} - \varphi \right\|_{\mathcal{X}} \leq c\beta_n \frac{\tau_n^{2^k-1}}{1-\tau_n^{2^k}} + \frac{c}{n^2}.$$

This is the Newton theorem for several variables, and the proof is well detailed in [48] (see Theorem 5.3.2, pp.270).

Remarque 2.4.1. ***(Comparison between (LN.DN) and (DN-LN) processes)*** *The difference between processes (LN-DN) and (DN-LN) is due to the fact that integrals on the right-hand side of the system of equations (2.8) – (2.9) in (LN.DN) process are approximated by the Nyström method, i.e., for all $i = 1, 2, \cdots, N$*

$$\int_0^1 \kappa_i\left(t, s, \varphi_n^{(k)}(s)\right) ds \approx \sum_{q=1}^{m} \omega_{m,q} \kappa_i\left(t, t_q, \varphi_n^{(k)}(t_q)\right), \quad \varphi_n^{(k)} \in \mathcal{X}, \quad k = 1, 2, \cdots \quad ,$$

where we choose a finer grid according the number of nodes m in the subdivision too big to n ($m \gg n$). This choice that give (LN-DN) process the preference over the other (DN-LN) process.

2.5 Numerical Examples

The objective of this section is to assess the effectiveness of our new approach (LN.DN) in comparison to the traditional approach (DN.LN) through two examples. In the first example, we use (LN.DN) to solve a nonlinear Fredholm integro-differential equation, which was previously solved using the classical approach (DN.LN) in [10]. We then compare our results to theirs. In the second example, we solve a system of nonlinear integral equations using both (LN.DN) and (DN.LN) approaches, and compare the outcomes.

Let $(\varphi_{n,1}^{(k)}, \varphi_{n,2}^{(k)}, \cdots, \varphi_{n,N}^{(k)}) \in \mathcal{X}$ and $(\vartheta_{n,1}^{(k)}, \vartheta_{n,2}^{(k)}, \cdots, \vartheta_{n,N}^{(k)}) \in \mathcal{X}, k \in \mathbb{N}^*$, the k order approximative solution of our system of equations (2.1) according to the scheme (2.12) of

(LN.DN) process, and to the scheme (2.18) of (DN.LN) process, respectively.

First, let $n \in \mathbb{N}^*$, and considering the equidistant subdivision Δ_n of $[0,1]$ defined by:

$$\Delta_n = \left\{ t_p = (p-1)h,\ h = \frac{1}{n-1}, p = 1, 2, \cdots, n \right\}.$$

We define the stopping condition on the parameter k as

For the (LN.DN) process: $\quad E^k_{LD} = \sum_{i=1}^{N} \max_{1\le p\le n} \left| \varphi^{(k+1)}_{n,i}(t_p) - \varphi^{(k)}_{n,i}(t_p) \right| \le 10^{-09}.$

For the (DN.LN) process: $\quad E^k_{DL} = \sum_{i=1}^{N} \max_{1\le p\le n} \left| \vartheta^{(k+1)}_{n,i}(t_p) - \vartheta^{(k)}_{n,i}(t_p) \right| \le 10^{-09}.$

We denote the obtained error using the both process by:

For the (LN.DN) process: $\quad E_{LD} = \sum_{i=1}^{N} \max_{1\le p\le n} \left| \varphi_i(t_p) - \varphi^{(k)}_{n,i}(t_p) \right|.$

For the (DN.LN) process: $\quad E_{DL} = \sum_{i=1}^{N} \max_{1\le p\le n} |\varphi_i(t_p) - \vartheta^{(k)}_{n,i}(t_p)|,$

where, $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_N) \in \mathcal{X}$ is the exact solution of the initial system of equations (2.1). We pass now to the numerical examples.

Example 2.5.1. *Consider the nonlinear Fredholm integro-differential equation presented in [10]*

$$\varphi(t) = \frac{1}{5}\int_0^1 sin[2(s + t + \varphi(s)) + (1-s)e^s - \varphi'(s)]ds + g(t), \quad \forall t \in [0,1], \tag{2.25}$$

with $\varphi \in C^1([0,1], \mathbb{R})$ and

$$g(t) = te^t - \frac{1}{5}[sin^2(1+t) - sin^2(t)].$$

By following the procedure outlined in Section 1.4.1, it becomes apparent that solving equation (2.25) necessitates solving a corresponding system of equations. So, we notice by

$$\begin{cases} \varphi(t) = \varphi_1(t), \\ \varphi'(t) = \varphi_2(t), \\ g_1(t) = g(t), \\ g_2(t) = g'(t) = (1+t)e^t - \dfrac{2}{5}[\cos(1+t)\sin(1+t) - \cos(t)\sin(t)], \end{cases}$$

then we obtain the following system

$$\begin{cases} \varphi_1(t) = \dfrac{1}{5}\displaystyle\int_0^1 sin[2(s + t + \varphi_1(s)) + (1-s)e^s - \varphi_2(s)]ds + g_1(t), \\ \varphi_2(t) = \dfrac{2}{5}\displaystyle\int_0^1 cos[2(s + t + \varphi_1(s)) + (1-s)e^s - \varphi_2(s)]ds + g_2(t), \end{cases} \tag{2.26}$$

where $\varphi_{ext} = (te^t, (1+t)e^t)$ is its exact solution. However, we solve this system (2.26) by using (LD.ND) process and compare our results with the results obtained in [10].

Example 2.5.2. *Consider the following system of equations, for all $t \in [0,1]$,*

$$\begin{cases} \varphi_1(t) = \displaystyle\int_0^1 \frac{\varphi_1(s)^2}{2+t+\varphi_2(s)\varphi_3(s)} ds + t + \frac{1}{3}\log\left(\frac{t+1}{t+2}\right), \\ \varphi_2(t) = \displaystyle\int_0^1 \frac{t\varphi_3(s)}{2+t+\varphi_1(s)+\varphi_2(s)} ds - t\left(1+\frac{1}{3(t+2)}\right), \\ \varphi_3(t) = \displaystyle\int_0^1 \frac{2t\varphi_2(s)-t}{5+\varphi_1(s)+\varphi_3(s)} ds + t^2 + t\log\left(\frac{7}{5}\right), \end{cases} \tag{2.27}$$

where $\varphi_{ext} = (t, -t, t^2)$ is its exact solution. In the same way of Example 2.5.1, we solve this system (2.27) by using the both processes (LN.DN) and (DN.LN), then we compare between the obtained results.

The Errors E_{LD} and E_{DL} with $m = 9 \times n$			
n	(LN.DN) process	CPU time	(DN.LN) process([10])
5	9.6417 e-05	4.4778 e-02 s	8.5244 e-02
10	2.3565 e-05	2.3515 e-02 s	4.2957 e-02
50	9.2595 e-07	2.2092 e-01 s	8.6250 e-03
100	2.3097 e-07	6.0163 e-01 s	4.3136 e-03
500	9.2225 e-09	1.2988 e 01 s	8.6289 e-04

Table 2.1: Numerical results of Example 2.5.1.

The Errors E_{LD} and E_{DL} with $m = 9 \times n$				
n	(LN.DN) process	CPU time	(DN.LN) process	CPU time
5	3.8751 e-04	1.4710 e-03 s	4,8409 e-02	6.3312 e-04 s
10	9.4735 e-05	4.3120 e-03 s	9,2095 e-03	1.0071 e-03 s
50	3.7221 e-06	6.9896 e-02 s	3.1248 e-04	1.6346 e-02 s
100	9.2821 e-07	2.5423 e-01 s	7.6564 e-05	5.0301 e-02 s
500	3.6736 e-08	5.7045 e 00 s	3.0139 e-06	2.5125 e 00 s

Table 2.2: Numerical results of Example 2.5.2.

Tables 2.1 and 2.2 display the errors of both (LN.DN) and (DN.LN) methods applied to Example 2.5.1 and 2.5.2. These tables provide evidence that the (LN.DN) approach is more accurate than the classical (DN.LN) method. However, the convergence of the (LN.DN) approximate solutions to the exact solutions can be observed in Figures 2.2 and 2.4. On the other hand, Figures 2.3 and 2.5 depict $log_{10}(E^k_{LD})$ and $log_{10}(E^k_{DL})$, the log_{10} of the distance between two successive iterations using (LN.DN) and (DN.LN) methods for Example 2.5.1 and 2.5.2, respectively. These figures demonstrate that the (LN.DN) process has a linear convergence, which is worse than that of the (DN.LN) method. Consequently, our findings align with those of [23], which affirm the validity of our approach.

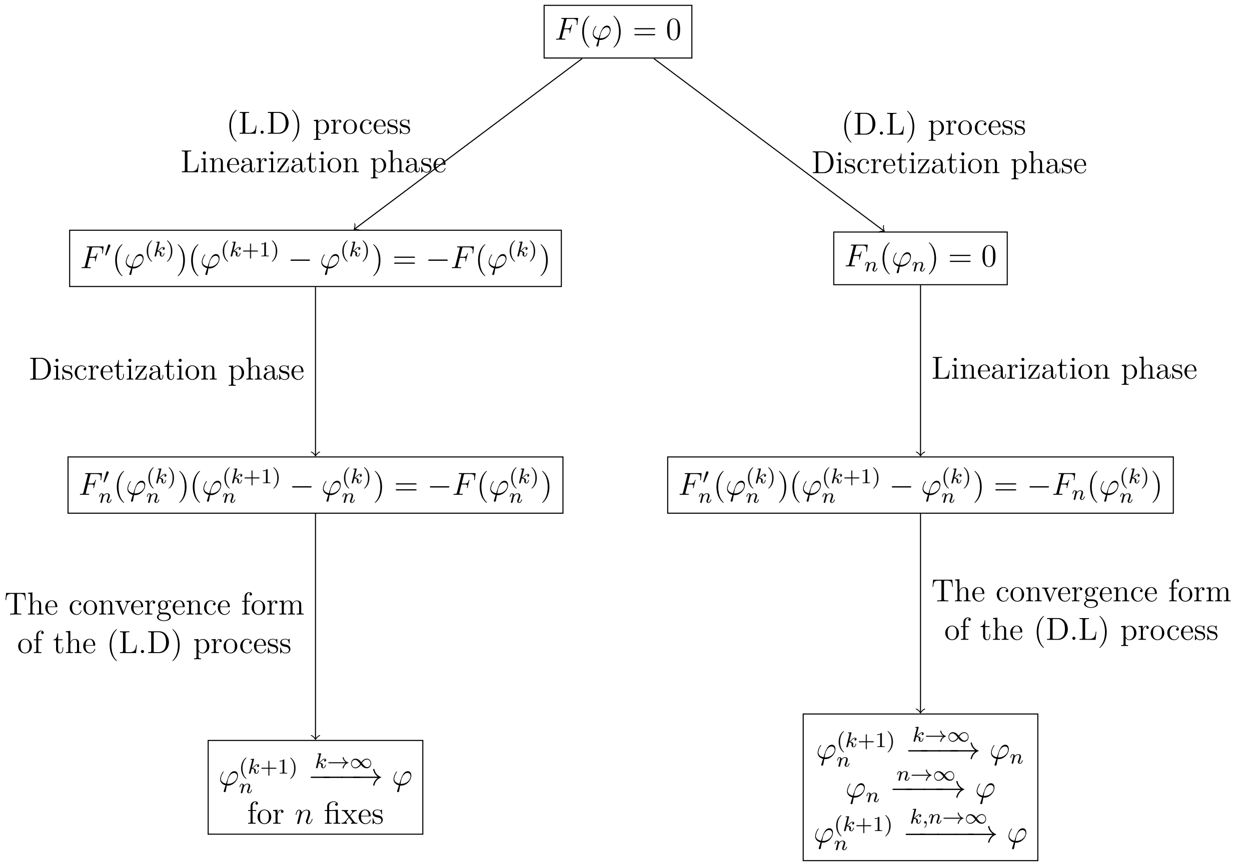

Figure 2.1: Diagram shows the main steps in developing two numerical procedures, the (L.D) new process and the (D.L) classical process.

For solving the fundamental nonlinear functional problem given as $F(\varphi) = 0$. It highlights the difference between the two techniques, demonstrating that the (L.D) method produces an approximate solution $\varphi_n^{(k)}$ that converges with fixed n and $m, (m \gg n)$ as the number of iterations k tends towards infinity. Conversely, the (D.L) classical process necessitates both k and n to approach infinity to achieve convergence. These findings indicate that the (L.D) new process is a preferred approach for solving various types of nonlinear functional equations, including the system of Fredholm nonlinear integral equation of the second kind.

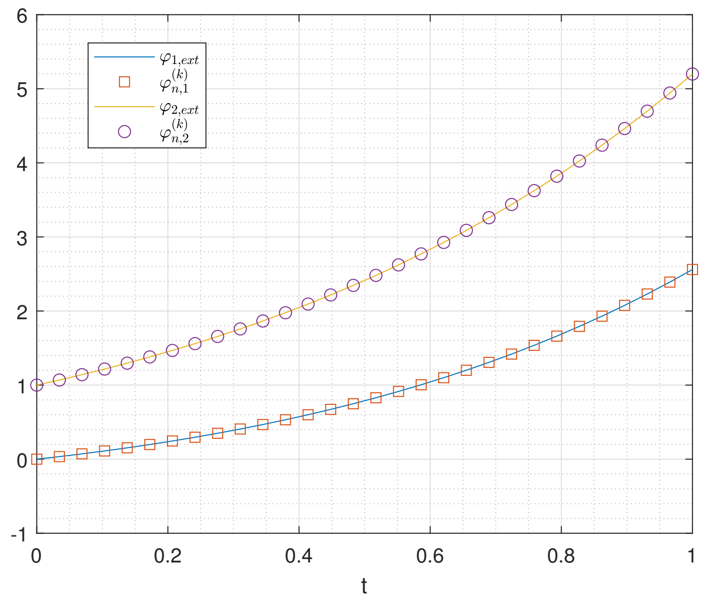

Figure 2.2: Exacts and Approximates solutions of Example 2.5.1, applying the (LN.DN) new process with $n = 20$ and $m = 180$.

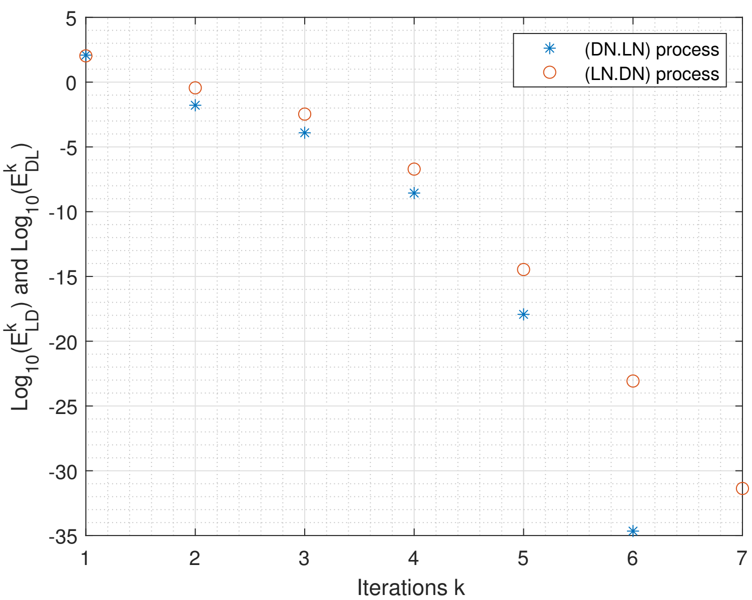

Figure 2.3: Graph of $log_{10}(E^k_{LD})$ and $log_{10}(E^k_{DL})$, the log_{10} of the error between two successive iterates approximations $\varphi_n^{(k+1)}$ and $\varphi_n^{(k)}$ of the first system. (Example 2.5.1)

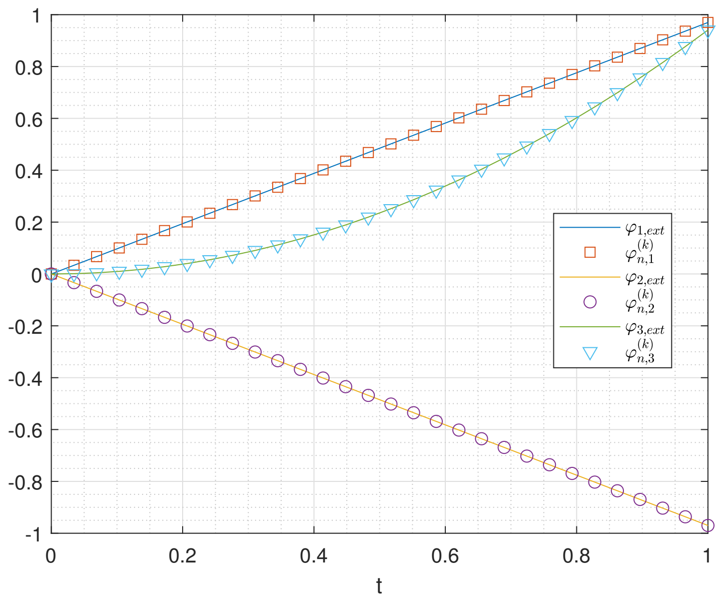

Figure 2.4: Exacts and Approximates solutions of Example 2.5.2, applying the (LN.DN) new process with $n = 20$ and $m = 180$.

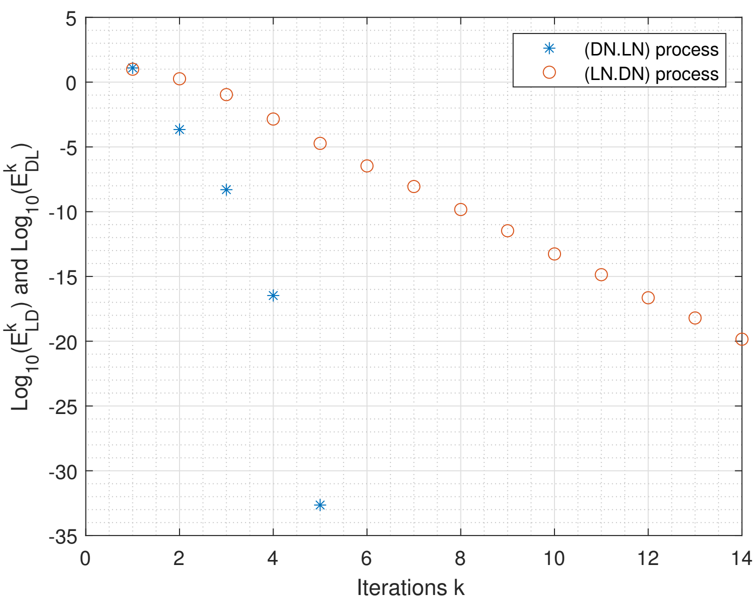

Figure 2.5: Graph of $log_{10}(E^k_{LD})$ and $log_{10}(E^k_{DL})$, the log_{10} of the error between two successive iterates approximations $\varphi_n^{(k+1)}$ and $\varphi_n^{(k)}$ of the second system. (Example 2.5.2).

Chapter 3

Efficient Numerical Scheme for Nonlinear Integral Equations: (L.D.D) Method

Contents

When solving functional nonlinear equations, it is important to determine whether to start with a linearization process or discretization. Previous researches, including Chapter 2 and recent papers [45, 22, 23, 31], has confirmed that linearizing before discretizing (L.D) is a better approach. The chapter proposes a new process called (L.D.D) or "outer-inner iteration" process that builds upon the developed process presented in two previous works by [22, 31]. The focus of this paper is to improve the precision and smoothness of the (L.D.D) process during the programming stage of the resolution procedure by using a different discretization method.

The proposed (L.D.D) process includes a linearization phase and a double discretization phase, where Kantoroviche's method is used for discretization. This method offers a different theoretical framework and faster convergence compared to the Sloan's method used in the previous work by [31].

Overall, the chapter aims to provide a more effective and efficient process for solving problems by improving the accuracy and smoothness of the solution during the programming stage.

This chapter is divided into three sections, starting with section 3.1, which provides a description of the (L.D.D) process and its convergence analysis. Section 3.2 demonstrates the application of Kantorovich's projection to solve nonlinear Fredholm integral equations. The efficacy of our proposed new process is examined through numerical examples in section 3.3.

3.1 The Description of the (L.D.D) process

Let $\mathcal{X}$ be a complex Banach space, where its norm is denoted by $\|\cdot\|$. The space $\mathcal{L}(\mathcal{X})$ defines the Banach algebra of bounded linear operators from $\mathcal{X}$ to itself, where its norm is given by

$$\forall A \in \mathcal{L}(\mathcal{X}), \qquad \|A\|_{\mathcal{L}(\mathcal{X})} = \sup\Big\{\|Ax\| : \ \|x\| \leq 1\Big\}.$$

In the complex Banach space $\mathcal{X}$, let $F : \mathcal{V} \subset \mathcal{X} \longrightarrow \mathcal{X}$ be a nonlinear Fréchet differentiable operator defined on a nonempty open set $\mathcal{V}$ of $\mathcal{X}$. In general, the nonlinear integro-differential equations are set as

$$\text{Find } \varphi \in \mathcal{V} : \quad F(\varphi) = 0_{\mathcal{X}}. \tag{3.1}$$

Let assume that the main problem (3.1), has a unique solution, i.e.

$$(H_1) \quad \exists! \, \varphi \in \mathcal{V} : \ F(\varphi) = 0_{\mathcal{X}}.$$

In order to solve equation (3.1), we can use a Newton-type method in a finite-dimensional space. We can linearize the equation and obtain the exact solution as the limit of a sequence $(\varphi^{(k)})_{k\geq 0}$, which is given by the following scheme:

$$F'(\varphi^{(k)})(\varphi^{(k+1)} - \varphi^{(k)}) = -F(\varphi^{(k)}), \quad \varphi^{(0)} \in \mathcal{V}, \quad k = 1, 2, \cdots \quad . \tag{3.2}$$

These iterated equations define the linearization phase. However, dealing with the exact formulation of $F'(\varphi^{(k)})^{-1}$ becomes a major challenge in each iteration as we are dealing with infinite-dimensional functional nonlinear equations. Therefore, we make an assumption that

$$(H_2) \quad F'(\varphi)^{-1} \text{ exist, and } \exists\, \eta > 0 \text{ such that } \|F'(\varphi)^{-1}\| \leq \eta < \infty.$$

In the book authored by [5], one can find information on the convergence of Newton's method. However, it is not possible to compute $F'\left(\varphi^{(k)}\right)^{-1}$ exactly in every iteration. Therefore, we introduce a discretization phase to the scheme (3.2) and define a new iterate scheme based on a double discretization phase, which we refer to as the (L.D.D) iterate scheme: Find $\varphi^{(k+1)}_{n,m} \in \mathcal{X}$, where $n, m \in \mathbb{N}^*$

$$F'_n(\pi_m\varphi^{(k)}_{n,m})(\varphi^{(k+1)}_{n,m} - \varphi^{(k)}_{n,m}) = -F(\pi_m\varphi^{(k)}_{n,m}), \quad \varphi^{(0)}_{n,m} \in \mathcal{V}, \quad k = 1, 2, \cdots \quad . \tag{3.3}$$

The first discretization phase is applied on the operator $F'(x)$, which is approximated by $F'_n(x)$, and the second phase discretization is defined by involving the operator projection $(\pi_m)_{m\in\mathbb{N}^*}$ defined from $\mathcal{X}$ into itself, on $F(\cdot)$ and $F'_n(\cdot)$ where, this projection satisfies the condition:

$$\forall v \in \mathcal{X}, \quad \pi_m v \longrightarrow v, \quad m \longrightarrow \infty.$$

We will now demonstrate that the sequence $(\varphi_{n,m}^{(k)})_{k\geq 0}$, defined by problem (3.3), converges to the solution φ of equation (3.1) as k tends to infinity, with fixed integers n and m. Let $B_R(\varphi)$ denote the ball centered at φ with radius $R > 0$. We make the assumption that

$$(H_3) \quad F' : \mathcal{V} \rightarrow \mathcal{L}(\mathcal{X}) \,\text{is}\, \lambda - \text{Lipschitz over}\, B_R(\varphi).$$

We define the constant r such that,

$$r = \min\left\{R, \frac{1}{2\eta\lambda}\right\}.$$

Additionally, we assume that for a sufficiently large n, the discretization process satisfies the following condition

$$(H_4) \quad \forall\, x \in \mathcal{V} \subset \mathcal{X}, \quad \|F'_n(x) - F'(x)\| \leq \delta_n \longrightarrow 0, \;\text{ as }\; n \longrightarrow \infty.$$

The fulfillment of the condition (H_4) is sufficient to guarantee the convergence of the scheme (3.3). However, this condition is not satisfied by Sloan's method, as shown in [3] (page 187), which was established for an (L.D.D) scheme in [31]. Nevertheless, we will demonstrate in the next section that Kantorovich's method can satisfy condition (H_4) and lead to convergence.

3.1.1 Convergence Analysis of The (L.D.D) Process

Before delving into the convergence analysis, we establish a series of lemmas that will be utilized in the proof of our convergence theorem.

Lemma 3.1.1. *If the hypotheses $(H_1) - (H_3)$ are satisfied, then for all $x \in B_r(\varphi)$, $F'(x)$ is invertible such that*

$$\left\|F'(x)^{-1}\right\| \leq 2\eta.$$

Proof. Let $x \in \mathcal{V} \subset \mathcal{X}$, then

$$F'(x) = F'(\varphi)\Big(I - F'(\varphi)^{-1}\left[F'(\varphi) - F'(x)\right]\Big),$$

thus

$$\left\|F'(\varphi)^{-1}\left[F'(\varphi) - F'(x)\right]\right\| \leq \left\|F'(\varphi)^{-1}\right\| \, \|F'(\varphi) - F'(x)\|.$$

Now, according to hypothesis (H_3), for all $x \in B_r(\psi)$, $\|F'(\varphi) - F'(x)\| \leq \lambda\, r$. Hence

$$\left\|F'(\varphi)^{-1}\Big(F'(\varphi) - F'(x)\Big)\right\| \leq \eta\lambda r \leq \frac{1}{2}.$$

So, using the Neummann Expansion Theorem 1.1.5, we conclude that $F'(x)$ is invertible, such that

$$F'(x)^{-1} = \left(I - F'(\varphi)^{-1}\left[F'(\varphi) - F'(x)\right]\right)^{-1} F'(\varphi)^{-1}.$$

In addition, we find that

$$\begin{aligned}
\left\|F'(x)^{-1}\right\| &= \left\|\left(I - F'(\varphi)^{-1}\left[F'(\varphi) - F'(x)\right]\right)^{-1} F'(\varphi)^{-1}\right\| \\
&\leq \left\|F'(\varphi)^{-1}\right\| \left\|\left(I - F'(\varphi)^{-1}\left[F'(\varphi) - F'(x)\right]\right)^{-1}\right\| \\
&\leq \eta \sum_{p=0}^{\infty} \left\|F'(\varphi)^{-1}\left(F'(\varphi) - F'(x)\right)\right\|^{p} \\
&\leq \eta \sum_{p=0}^{\infty} \left(\frac{1}{2}\right)^{p} = 2\eta,
\end{aligned}$$

which completes the proof.

□

Lemma 3.1.2. *Assume that $(H_1) - (H_4)$ holds, then for all $x \in B_r(\varphi)$ and for n large enough, the operator $F_n'(x)$ is invertible such that,*

$$\sup_{x \in B_r(\varphi)} \left\|I - F_n'(x)^{-1}F'(x)\right\| \leq \rho_n,$$

$$\sup_{x \in B_r(\varphi)} \left\|F_n'(x)^{-1}\right\| \leq 2\eta\left(1 + \rho_n\right),$$

where $\rho_n \to 0$, as $n \to \infty$.

Proof. For all $x \in B_r(\varphi)$, and for $n \in \mathbb{N}^*$, then by Lemma 3.1.1

$$F_n'(x) = F'(x)\left(I - F'(x)^{-1}\left[F'(x) - F_n'(x)\right]\right).$$

So, according to the hypothesis (H_4), such that $\delta_n < \dfrac{1}{2\eta}$, we find that

$$\left\|F'(x)^{-1}\left(F'(x) - F_n'(x)\right)\right\| \leq 2\eta\delta_n < 1.$$

Now, using the Neummann Expansion Theorem 1.1.5, we conclude that $F_n'(x)$ is invertible, and

$$\left\| F_n'(x)^{-1} \right\| \leq \frac{2\eta}{1-2\eta\delta_n}.$$

On the other hand, we notice that

$$I - F_n'(x)^{-1}F'(x) = F_n'(x)^{-1}\left(F_n'(x) - F'(x)\right).$$

So, we define the sequence $\rho_n = \dfrac{2\eta\delta_n}{1-2\eta\delta_n}$. Thus,

$$\sup_{x\in B_r(\psi)} \left\| I - F_n'(x)^{-1}F'(x) \right\| \leq \rho_n.$$

Similarly, we find the estimation,

$$\left\| F_n'(x)^{-1} \right\| \leq 2\eta + 2\eta\rho_n.$$

This completes the proof.

□

Lemma 3.1.3. *Let $L(\cdot) : \mathcal{X} \to \mathcal{X}$ be a α-Lipschitz operator. For all $x \in \mathcal{X}$, if $L(x)^{-1} \in \mathcal{L}(\mathcal{X})$ such that $\|L(x)^{-1}\| \leq \mu$, then,*

$$L^{-1}(\cdot) \text{ is } (\mu^2\alpha) - \text{Lipschitz.}$$

Proof. For all $x, y \in \mathcal{X}$, we have,

$$L(x)^{-1} - L(y)^{-1} = L(x)^{-1}\left(L(y) - L(x)\right)L(y)^{-1},$$

hence

$$\begin{aligned}\left\| L(x)^{-1} - L(y)^{-1} \right\| &\leq \left\| L(x)^{-1} \right\| \|L(y) - L(x)\| \left\| L(y)^{-1} \right\| \\ &\leq \mu^2\alpha \|x - y\|.\end{aligned}$$

□

Proposition 3.1.1. *Assume that $(H_1) - (H_4)$ holds. Then, for n large enough*

$$F_n'(\cdot)^{-1} \text{ is } (2+\lambda)(2\eta(1+\rho_n))^2 - \text{Lipschitz for all } x \in B_r(\varphi).$$

Proof. For all $x, y \in B_r(\varphi)$, then

$$F'_n(x) - F'_n(y) = (F'_n(x) - F'(x)) + F'(x) + (F'(y) - F'_n(y)) - F'(y).$$

So, using (H_3) and (H_4), we find that

$$\begin{aligned}
\|F'_n(x) - F'_n(y)\| &\leq \|F'_n(x) - F'(x)\| + \|F'_n(y) - F'(y)\| + \|F'(x) - F'(y)\| \\
&\leq \delta_n + \delta_n + \lambda \|x - y\| \\
&\leq 2\,\delta_n + \lambda \|x - y\|\,.
\end{aligned}$$

Now, it is clear that, for n large enough such that $x \neq y$, we choose $\delta_n \leq \|x - y\|$. Thus

$$\|F'_n(x) - F'_n(y)\| \leq (2 + \lambda) \|x - y\|\,.$$

So, we conclude $F'_n(x)$ is $(2+\lambda)$- Lipschitz, and for all $x \in B_r(\varphi)$,

$$\left\|F'_n(x)^{-1}\right\| \leq 2\eta(1 + \rho_n),$$

and according to Lemma 3.1.2 we have

$$\left\|F'_n(x)^{-1}\right\| \leq 2\eta\,(1 + \rho_n)\,,$$

and also according to Lemma 3.1.3, we find that

$$F'_n(\cdot)^{-1} \text{ is } (2\eta(1 + \rho_n))^2 (2 + \lambda) - \text{Lipschitz}.$$

Thus, we have completed the proof.

□

Let the constants η, R, λ, ρ_n be defined in previous lemmas. We define also the constant ℓ such that,

$$\forall x, y \in \mathcal{V}: \quad \|F(x) - F(y)\| \leq \ell \|x - y\|.$$

The next theorem, is the principal result of our new strategy.

Theorem 3.1.1. *If the initial function $\varphi^{(0)}_{n,m} \in B_{\omega_n}(\varphi)$, for $n, m \in \mathbb{N}^*$. Then the sequence $(\varphi^{(k)}_{n,m})_{k\geq 0}$ defined by the scheme (3.3), converges to φ the solution of equation (3.1), such that*

$$\left\|\varphi^{(k)}_{n,m} - \varphi\right\| \leq \omega_n \left(\frac{1 + \rho_n}{2}\right)^k \xrightarrow{k\to\infty} 0,$$

where,

$$\omega_n = \min\left\{\frac{r}{2}, \frac{1 - \rho_n - 4\eta\ell(1 + \rho_n)}{2\left[\lambda\eta(1 + \rho_n) + \ell(2 + \lambda)(2\eta(1 + \rho_n))^2\right]}\right\}.$$

Proof. Let $n, m \in \mathbb{N}^*$. If $\varphi_{n,m}^{(0)} \in B_{\omega_n}(\varphi)$, then according to Lemma 3.1.2, the operator $F'_n(\varphi_{n,m}^{(0)})$ is invertible. Now by induction we assume that $\varphi_{n,m}^{(k)} \in B_{\omega_n}(\varphi)$. So, we have according to scheme (3.3),

$$\begin{aligned}
\varphi_{n,m}^{(k+1)} - \varphi_{n,m}^{(k)} &= - F'_n(\pi_m\varphi_{n,m}^{(k)})^{-1}F(\pi_m\varphi_{n,m}^{(k)}) \\
&= - F'_n(\pi_m\varphi_{n,m}^{(k)})^{-1}F\left(\varphi_{n,m}^{(k)}\right) - F'_n(\pi_m\varphi_{n,m}^{(k)})^{-1}\left[F(\pi_m\varphi_{n,m}^{(k)}) - F(\varphi_{n,m}^{(k)})\right] \\
&= - \left[F'_n(\pi_m\varphi_{n,m}^{(k)})^{-1} - F'_n(\varphi_{n,m}^{(k)})^{-1}\right]F(\varphi_{n,m}^{(k)}) - F'_n(\varphi_{n,m}^{(k)})^{-1}F(\varphi_{n,m}^{(k)}) \\
&\quad - F'_n(\pi_m\varphi_{n,m}^{(k)})^{-1}\left[F(\pi_m\varphi_{n,m}^{(k)}) - F(\varphi_{n,m}^{(k)})\right].
\end{aligned}$$

Hence,

$$\begin{aligned}
\varphi_{n,m}^{(k+1)} - \varphi = \varphi_{n,m}^{(k)} - \varphi - F'_n(\varphi_{n,m}^{(k)})^{-1}\left[F(\varphi_{n,m}^{(k)}) - F(\varphi)\right] \\
- \left[F'_n(\pi_m\varphi_{n,m}^{(k)})^{-1} - F'_n(\varphi_{n,m}^{(k)})^{-1}\right]\left[F(\varphi_{n,m}^{(k)}) - F\left(\varphi\right)\right] \\
- F'_n(\pi_m\varphi_{n,m}^{(k)})^{-1}\left[F(\pi_m\varphi_{n,m}^{(k)}) - F(\varphi_{n,m}^{(k)})\right].
\end{aligned}$$

So,

$$\begin{aligned}
\left\|\varphi_{n,m}^{(k+1)} - \varphi\right\| \leq \left\|\varphi_{n,m}^{(k)} - \varphi - F'_n(\varphi_{n,m}^{(k)})^{-1}\left[F(\varphi_{n,m}^{(k)}) - F(\varphi)\right]\right\| \\
+ \left\|\left[F'_n(\pi_m\varphi_{n,m}^{(k)})^{-1} - F'_n(\varphi_{n,m}^{(k)})^{-1}\right]\left[F(\varphi_{n,m}^{(k)}) - F(\varphi)\right]\right\| \\
+ \left\|F'_n(\pi_m\varphi_{n,m}^{(k)})^{-1}\left[F(\pi_m\varphi_{n,m}^{(k)}) - F(\varphi_{n,m}^{(k)})\right]\right\|.
\end{aligned}$$

In this step, we estimate each part of this inequality separately. For the first part, we use the integral form of Lagrange's mean value formula (see [59]) as follows:

$$\begin{aligned}
\varphi_{n,m}^{(k)} - \varphi - F'_n(\varphi_{n,m}^{(k)})^{-1}\left[F(\varphi_{n,m}^{(k)}) - F(\varphi)\right] &= \int_0^1 \left[I - F'_n(\varphi_{n,m}^{(k)})^{-1}F'((1-x)\varphi_{n,m}^{(k)} + x\varphi)\right](\varphi_{n,m}^{(k)} - \varphi)dx \\
&= \int_0^1 \left[I - F'_n(\varphi_{n,m}^{(k)})^{-1}F'(\varphi_{n,m}^{(k)})\right](\varphi_{n,m}^{(k)} - \varphi)dx \\
&\quad - \int_0^1 F'_n(\varphi_{n,m}^{(k)})^{-1}\left[F'((1-x)\varphi_{n,m}^{(k)} + x\varphi) - F'(\varphi_{n,m}^{(k)})\right](\varphi_{n,m}^{(k)} - \varphi)dx,
\end{aligned}$$

and

$$\begin{aligned}
\left\|\varphi_{n,m}^{(k)} - \varphi - F'_n(\varphi_{n,m}^{(k)})^{-1}\left[F(\varphi_{n,m}^{(k)}) - F(\varphi)\right]\right\| \leq \left\|I - F'_n(\varphi_{n,m}^{(k)})^{-1}F'(\varphi_{n,m}^{(k)})\right\|\left\|\varphi_{n,m}^{(k)} - \varphi\right\| \\
+ \left\|F'_n(\varphi_{n,m}^{(k)})^{-1}\right\|\left\|\varphi_{n,m}^{(k)} - \varphi\right\|\int_0^1 \left\|F'((1-x)\varphi_{n,m}^{(k)} + x\varphi) - F'(\varphi_{n,m}^{(k)})\right\|dx.
\end{aligned}$$

Now, since $\varphi_{n,m}^{(k)} \in B_{\omega_n}(\varphi)$, and according to Lemme 3.1.2, we find that

$$\left\| I - F_n'(\varphi_{n,m}^{(k)})^{-1} F'(\varphi_{n,m}^{(k)}) \right\| \leq \rho_n,$$

and given that the ball $B_{\omega_n}(\varphi)$ is a convex set, so for $x \in [0,1]$, $(1-x)\varphi_{n,m}^{(k)} + x\varphi \in B_{\omega_n}(\varphi)$. We use (H_3) to get

$$\left\| F'((1-x)\varphi_{n,m}^{(k)} + x\varphi) - F'(\varphi_{n,m}^{(k)}) \right\| \leq \lambda x \left\| \varphi_{n,m}^{(k)} - \varphi \right\|,$$

hence

$$\int_0^1 \left\| F'((1-x)\varphi_{n,m}^{(k)} + x\varphi) - F'(\varphi_{n,m}^{(k)}) \right\| dx \leq \frac{1}{2}\lambda \left\| \varphi_{n,m}^{(k)} - \varphi \right\|.$$

So, we gather the first estimation as:

$$\begin{aligned} \left\| \varphi_{n,m}^{(k)} - \varphi - F_n'(\varphi_{n,m}^{(k)})^{-1} \left[F(\varphi_{n,m}^{(k)}) - F(\varphi) \right] \right\| \leq & \rho_n \left\| \varphi_{n,m}^{(k)} - \varphi \right\| \\ & + \left(2\eta(1+\rho_n) \left\| \varphi_{n,m}^{(k)} - \varphi \right\| \right) \frac{1}{2}\lambda \left\| \varphi_{n,m}^{(k)} - \varphi \right\|. \end{aligned}$$

For the second part, as F is a Fréchet differentiable operator, there exist $\ell > 0$ such that

$$\left\| F(\varphi_{n,m}^{(k)}) - F(\varphi) \right\| \leq \ell \left\| \varphi_{n,m}^{(k)} - \varphi \right\|,$$

and according to Proposition 3.1.1 we have

$$\left\| \left[F_n'(\pi_m \varphi_{n,m}^{(k)})^{-1} - F_n'(\varphi_{n,m}^{(k)})^{-1} \right] \right\| \leq (2+\lambda)(2\eta(1+\rho_n))^2 \left\| \pi_m \varphi_{n,m}^{(k)} - \varphi_{n,m}^{(k)} \right\|,$$

hence

$$\begin{gathered} \left\| \left[F_n'(\pi_m \varphi_{n,m}^{(k)})^{-1} - F_n'(\varphi_{n,m}^{(k)})^{-1} \right] \left[F(\varphi_{n,m}^{(k)}) - F(\varphi) \right] \right\| \leq \\ \ell(2+\lambda)(2\eta(1+\rho_n))^2 \left\| \varphi_{n,m}^{(k)} - \varphi \right\| \left\| \pi_m \varphi_{n,m}^{(k)} - \varphi_{n,m}^{(k)} \right\|. \end{gathered}$$

For the third part, as

$$\left\| F(\pi_m \varphi_{n,m}^{(k)}) - F(\varphi_{n,m}^{(k)}) \right\| \leq \ell \left\| \pi_m \varphi_{n,m}^{(k)} - \varphi_{n,m}^{(k)} \right\|,$$

and $\varphi_{n,m}^{(k)} \in B_{\omega_n}(\varphi)$, so using the convergence of $(\pi_m)_{m\in}$ to the identity operator, we find that

$$\pi_m \varphi_{n,m}^{(k)} \xrightarrow{m\to\infty} \varphi_{n,m}^{(k)} \Leftrightarrow \forall \varepsilon > 0, \exists m_0 \in \mathbb{N}^*, \forall m > m_0, \left\| \pi_m \varphi_{n,m}^{(k)} - \varphi_{n,m}^{(k)} \right\| \leq \varepsilon.$$

Now, we choose m large enough, where if we put $\varepsilon = \left\| \varphi_{n,m}^{(k)} - \varphi \right\| < \frac{r}{2}$, we get

$$\left\| \pi_m \varphi_{n,m}^{(k)} - \varphi \right\| \leq \left\| \pi_m \varphi_{n,m}^{(k)} - \varphi_{n,m}^{(k)} \right\| + \left\| \varphi_{n,m}^{(k)} - \varphi \right\| \leq 2 \left\| \varphi_{n,m}^{(k)} - \varphi \right\| \leq r.$$

So, we conclude that $\pi_m \varphi_{n,m}^{(k)} \in B_{\omega_n}(\varphi)$, and according to Lemma 3.1.2

$$\left\| F_n'(\pi_m \varphi_{n,m}^{(k)})^{-1} \right\| \leq 2\eta(1+\rho_n).$$

Then,

$$\left\| F_n'(\pi_m \varphi_{n,m}^{(k)})^{-1} \left[F(\pi_m \varphi_{n,m}^{(k)}) - F(\varphi_{n,m}^{(k)}) \right] \right\| \leq 2\eta(1+\rho_n)\ell \left\| \pi_m \varphi_{n,m}^{(k)} - \varphi_{n,m}^{(k)} \right\|.$$

Hence,

$$\begin{aligned}\left\| \varphi_{n,m}^{(k+1)} - \varphi \right\| \leq & \rho_n \left\| \varphi_{n,m}^{(k)} - \varphi \right\| + \left(\lambda\eta(1+\rho_n) \left\| \varphi_{n,m}^{(k)} - \varphi \right\| \right) \left\| \varphi_{n,m}^{(k)} - \varphi| \right\| \\ & + \ell(2+\lambda)(2\eta(1+\rho_n))^2 \left\| \varphi_{n,m}^{(k)} - \varphi \right\| \left\| \pi_m \varphi_{n,m}^{(k)} - \varphi_{n,m}^{(k)} \right\| + 2\eta\ell(1+\rho_n) \left\| \pi_m \varphi_{n,m}^{(k)} - \varphi_{n,m}^{(k)} \right\|,\end{aligned}$$

and with more simplification, we find that

$$\left\| \varphi_{n,m}^{(k+1)} - \varphi \right\| \leq \left([\rho_n + 2\eta\ell(1+\rho_n)] + \left[\lambda\eta(1+\rho_n) + \ell(2+\lambda)(2\eta(1+\rho_n))^2 \right] \left\| \varphi_{n,m}^{(k)} - \varphi \right\| \right) \left\| \varphi_{n,m}^{(k)} - \varphi \right\|.$$

Now, we define the constant ω_n such that

$$\omega_n = \min \left\{ \frac{r}{2}, \frac{1 - \rho_n - 4\eta\ell(1+\rho_n)}{2\left[\lambda\eta(1+\rho_n) + \ell(2+\lambda)(2\eta(1+\rho_n))^2 \right]} \right\}.$$

So, as $\varphi_{n,m}^{(k)} \in B_{\omega_n}(\psi)$, we obtain that

$$\left[\lambda\eta(1+\rho_n) + \ell(2+\lambda)(2\eta(1+\rho_n))^2 \right] \left\| \varphi_{n,m}^{(k)} - \varphi \right\| \leq \frac{1 - \rho_n - 4\eta\ell(1+\rho_n)}{2},$$

hence

$$\left\| \varphi_{n,m}^{(k+1)} - \varphi \right\| \leq \left(\frac{1+\rho_n}{2} \right) \left\| \varphi_{n,m}^{(k)} - \varphi \right\|.$$

As $\dfrac{1+\rho_n}{2} < 1$ that gives $\varphi_{n,m}^{(k+1)} \in B_{\omega_n}(\psi)$. Finally we get the desired result

$$\left\| \varphi_{n,m}^{(k)} - \varphi \right\| \leq \omega_n \left(\frac{1+\rho_n}{2} \right)^k \xrightarrow{k \to \infty} 0.$$

This completes the proof.

□

The next section will provide numerical evidence to support our theoretical findings by implementing the (L.D.D) scheme with double Kantorovich's discretization to solve nonlinear Fredholm integral equations. We chose Kantorovich's method because it guarantees a rapid convergence process and satisfies condition (H_4), which is not the case for the (L.D.D) scheme with double Sloan's method (see [3], pp. 187).

Remarque 3.1.1. *Theoretically, the (L.D.D) new scheme is batter than the (D.L) classical method in the sense that $\varphi_{n,m}^{(k)} \longrightarrow \varphi$ as $k \longrightarrow +\infty$ whatever n and m big enough for the (L.D.D) scheme, unlike in the (D.L) classical method, where $\varphi_n^{(k)} \rightarrow \varphi$ as $k \longrightarrow +\infty$ and $n \longrightarrow +\infty$.*

3.2 Application on Nonlinear Fredholm Integral Equations Using Kantorovich's Projection

In this section, we will show how we use the hypotheses $(H_1) - (H_4)$ of the Theorem 3.1.1 on the numerical examples provided. Let $\mathcal{X}$ be the Banach space of real continued functions defined $[0,1]$. Let $K : \mathcal{X} \rightarrow \mathcal{X}$ be a nonlinear integral compact operator defined as

$$K(\phi)(t) = \int_0^1 \kappa(t,s,\phi(s))\, ds, \quad \phi \in \mathcal{V}, \quad t \in [0,1],$$

and we denote by $D = K'$ the Fréchet derivative of K such that

$$\forall\, \phi \in \mathcal{X}, [D(\phi)v](t) = \int_0^1 \frac{\partial \kappa}{\partial \phi}(t,s,\phi(s))\, v(s)\, ds, \quad v \in \mathcal{X}, \quad t \in [0,1].$$

We set our problem as follows

$$\text{Find } \varphi \in \mathcal{V}, \quad \varphi(t) = \int_0^1 \kappa(t,s,\varphi(s))\, ds + g(t) \quad t \in [0,1],$$

for a given function $g \in \mathcal{X}$, and we can rewrite this problem as

$$\text{Find } \varphi \in \mathcal{X}, \quad \varphi = K(\varphi) + g. \tag{3.4}$$

By setting, for all $x \in \mathcal{X}$

$$F(x) = x - K(x) - g, \textit{ and } \quad F'(x)y = (I - D(x))y, \quad y \in \mathcal{X}.$$

Problem (3.4) can be written as

$$\text{Find } \varphi \in \mathcal{V} \subset \mathcal{X}, \quad F(\varphi) = 0_{\mathcal{X}},$$

to get our fundamental problem (3.1).

As previously described, the (L.D.D) process can be used to solve problems such as problem (3.4). The first step is to perform the linearization phase using the Newton scheme, resulting in the following linear operator equation:

$$\left(I - D(\varphi^{(k)})\right)\left(\varphi^{(k+1)} - \varphi^{(k)}\right) = -F\left(\varphi^{(k)}\right), \qquad \varphi^{(k)} \in \mathcal{V}, \quad k = 1, 2, \cdots .$$

Next, for $n, m \in \mathbb{N}^*$, we apply a Double discretization Kantorovich projection to get our discretized linear problem:

$$\left(I - \pi_n D(\pi_m \varphi^{(k)}_{n,m})\right)\left(\varphi^{(k+1)}_{n,m} - \varphi^{(k)}_{n,m}\right) = -F\left(\pi_m \varphi^{(k)}_{n,m}\right), \qquad \varphi^{(k)}_{n,m} \in \mathcal{V}, \quad k = 1, 2, \cdots ,$$

where it is can be writing also as

$$\begin{cases} \text{Find } \varphi^{(k+1)}_{n,m} \in \mathcal{X}, \\ \varphi^{(k+1)}_{n,m} - \pi_n D(\pi_m \psi^{(k)}_{n,m})\varphi^{(k+1)}_{n,m} = S^{(k)}_{n,m}, \\ S^{(k)}_{n,m} = K(\pi_m \varphi^{(k)}_{n,m}) - \pi_n D(\pi_m \varphi^{(k)}_{n,m})\varphi^{(k)}_{n,m} + g. \end{cases} \tag{3.5}$$

We suppose that

$$\begin{cases} (i) \ \text{Problem (3.4) has a unique solution } \varphi \in \mathcal{V}, \\ (ii) \ (I - D(\varphi)) \ \text{ is invertible, } \exists\, \eta > 0, \ \|(I - D(\varphi))^{-1}\| \leq \eta < \infty, \\ (iii) \ D : \mathcal{V} \to \mathcal{L}(\mathcal{X}) \text{ is } \lambda - \text{Lipschitz over } B_R(\varphi). \end{cases} \tag{3.6}$$

Given the assumptions made previously, we can ensure that the conditions $(H_1)-(H_3)$ hold. Now, we need to demonstrate that the remaining conditions are satisfied to apply our convergence theorem. Recall that the discretization process defined in problem (3.5) is based on the double approximation, as follows

$$\text{For all } n \gg 1, \quad D_n(\phi)v = \pi_n D(\phi)v = \sum_{p=1}^{n} < D(\phi)v, e^*_p > e_p, \quad \phi, v \in \mathcal{X}, \tag{3.7}$$

where the set $\{e_1, e_2, \cdots, e_n\}$ represents a sequentially arranged basis of the image space of $(\pi_n)_{n\in\mathbb{N}^*}$, while $\{e^*_1, e^*_2, \cdots, e^*_n\}$ denotes an associated adjoint basis.

As in Lemma 3.1.1, we can prove that, for all $\phi \in B_r(\varphi), (I - D(\phi))$ is invertible and

$$\left\|(I - D(\phi))^{-1}\right\| \leq 2\eta.$$

Proposition 3.2.1. *Assume that the hypotheses (3.6) hold, and we suppose also that*

1. $\forall \phi \in \mathcal{X}$, $\pi_n \phi \xrightarrow{n\to\infty} \phi$,
2. *The set*

$$\mathcal{Z} = \left\{D(\phi)v \ : \ \phi \in B_R(\varphi), \ v \in \mathcal{X}, \|v\| = 1\right\},$$

is relatively compact. Then

$$\sup_{\phi \in B_R(\varphi)} \|D_n(\phi) - D(\phi)\| \leq \delta_{n,m} \xrightarrow{n,m\to\infty} 0.$$

Proof. Since the set $\mathcal{Z}$ is relatively compact, the operator $(\pi_n)_{n\in\mathbb{N}}$ converges pointwise to the identity operator I, and therefore the pointwise convergence on relatively compact sets is also uniform convergence.

□

By noting that the Kantorovich projection method satisfies point 1) of Proposition 3.2.1 (see [3] pp.185) and the fact that the integral operators are compact operators, we can conclude that point 2) of the same proposition is also satisfied. Therefore, by using Proposition 3.2.1 and the hypotheses (3.6), and according to Lemma 3.1.2, we can establish that the operator $(I - D_n(\phi))^{-1}$ is invertible, and furthermore,

$$\sup_{\phi\in B_r(\varphi)} \left\|(I - D_n(\phi))^{-1}\right\| \leq 2\eta(1+\rho_n),$$

where r, ρ_n are parameters defined in Lemmas 3.1.1−3.1.2. At this point, we have satisfied all the hypotheses of Theorem 3.1.1. Thus, we have guaranteed that

$$\varphi_{n,m}^{(k)} \xrightarrow{k\to\infty} \varphi, \text{ for } n, m \text{ fixed in } \mathbb{N}^*.$$

3.2.1 The Method of Implementing the (L.D.D) Process

This subsection aims to provide a clear explanation of how we will develop scheme (3.5) to solve the problem effortlessly and obtain an approximate solution $\varphi_{n,m}^{(k)}$. First, let $n \in \mathbb{N}^*$, and considering the equidistant subdivision Δ_n of $[0,1]$ defined by:

$$\Delta_n = \left\{t_p = (p-1)h,\ h = \frac{1}{n-1}, p = 1, 2, \cdots, n\right\}.$$

We introduce the definitions of some notation that we employ to establish the structure of matrices and linear forms presented in the final statement of our problem as follows:

$$\widetilde{e}_n U = \sum_{p=1}^{n} U(p) e_p, \quad \forall U \in \mathbb{C}^n, \widetilde{e}_n = (e_1, e_2, \cdots, e_n),$$

$$\ll W, \widetilde{e}_n^* \gg (i,j) = \langle w_j, e_i^* \rangle = w_j(t_i), \quad t_i \in \Delta_n, \forall W = (w_1, w_2, \cdots, w_m) \in \mathcal{X}^{1\times m}. \quad (3.8)$$

Utilizing these notations, we can simplify the representation of our projection operator (3.7) to be expressed as:

$$\pi_n D(\phi) v = \widetilde{e}_n \ll D(\phi)v, \widetilde{e}_n^* \gg, \quad \phi, v \in \mathcal{X}. \quad (3.9)$$

Let us concentrate on the implementation of our (L.D.D) scheme. We remark that,

$$S_{n,m}^{(k)} = K(\pi_m \varphi_{n,m}^{(k)}) - D_n(\pi_m \varphi_{n,m}^{(k)})\varphi_{n,m}^{(k)} + g,$$

then,

$$(I-\pi_n)\varphi_{n,m}^{(k+1)} = (I-\pi_n)D_n(\pi_m\varphi_{n,m}^{(k)})\varphi_{n,m}^{(k+1)} + (I-\pi_n)S_{n,m}^{(k)}.$$

Applying the property of the projection operators $\pi_n^2 = \pi_n$, to get

$$\begin{aligned}(I-\pi_n)D_n(\cdot) =&(I-\pi_n)\pi_n D(\cdot)\\ =&\pi_n D(\cdot) - \pi_n^2 D(\cdot)\\ =&D_n(\cdot) - \pi_n D(\cdot)\\ =&0,\end{aligned}$$

to find

$$\begin{aligned}(I-\pi_n)\varphi_{n,m}^{(k+1)} =&(I-\pi_n)S_{n,m}^{(k)}\\ =&(I-\pi_n)(K(\pi_m\varphi_{n,m}^{(k)}) - D_n(\pi_m\varphi_{n,m}^{(k)})\varphi_{n,m}^{(k)} + g)\\ =&(I-\pi_n)(K(\pi_m\varphi_{n,m}^{(k)}) + g) - (I-\pi_n)D_n(\pi_m\varphi_{n,m}^{(k)})\\ =&(I-\pi_n)(K(\pi_m\varphi_{n,m}^{(k)}) + g),\end{aligned}$$

and we can write,

$$(I-\pi_n)\varphi_{n,m}^{(k+1)} = (I-\pi_n)\,S_{n,m}^{(k)} = (I-\pi_n)\,(K(\pi_m\varphi_{n,m}^{(k)}) + g).$$

So, we have

$$\begin{aligned}\varphi_{n,m}^{(k+1)} =&(I-\pi_n)\varphi_{n,m}^{(k+1)} + \pi_n\varphi_{n,m}^{(k+1)}\\ =&(I-\pi_n)(K(\pi_m\varphi_{n,m}^{(k)}) + g) + \pi_n\varphi_{n,m}^{(k+1)},\end{aligned}$$

and by applying the projection $(\pi_n)_{n\in\mathbb{N}^*}$ to the first equation of our scheme (3.5), and using the notation (3.9), we get

$$\begin{aligned}\pi_n\varphi_{n,m}^{(k+1)} =&\pi_n D(\pi_m\varphi_{n,m}^{(k)})\varphi_{n,m}^{(k+1)} + \pi_n S_{n,m}^{(k)}\\ =&\widetilde{e}_n \ll D(\pi_m\varphi_{n,m}^{(k)})\varphi_{n,m}^{(k+1)}, \widetilde{e}_n^* \gg + \widetilde{e}_n \ll S_{n,m}^{(k)}, \widetilde{e}_n^* \gg\\ =&\widetilde{e}_n \ll D(\pi_m\varphi_{n,m}^{(k)})\left((I-\pi_n)\varphi_{n,m}^{(k+1)} + \pi_n\varphi_{n,m}^{(k+1)}\right), \widetilde{e}_n^* \gg + \widetilde{e}_n \ll S_{n,m}^{(k)}, \widetilde{e}_n^* \gg\\ =&\widetilde{e}_n \ll D(\pi_m\varphi_{n,m}^{(k)})\left((I-\pi_n)(K(\pi_m\varphi_{n,m}^{(k)}) + g)\right), \widetilde{e}_n^* \gg + \widetilde{e}_n \ll D(\pi_m\varphi_{n,m}^{(k)})\pi_n\varphi_{n,m}^{(k+1)}, \widetilde{e}_n^* \gg\\ &+ \widetilde{e}_n \ll S_{n,m}^{(k)}, \widetilde{e}_n^* \gg\\ =&\widetilde{e}_n \ll D(\pi_m\varphi_{n,m}^{(k)})\left((I-\pi_n)(K(\pi_m\varphi_{n,m}^{(k)}) + g)\right), \widetilde{e}_n^* \gg\\ &+ \widetilde{e}_n \ll D(\pi_m\varphi_{n,m}^{(k)})\widetilde{e}_n \ll \varphi_{n,m}^{(k+1)}, \widetilde{e}_n^* \gg, \widetilde{e}_n^* \gg + \widetilde{e}_n \ll S_{n,m}^{(k)}, \widetilde{e}_n^* \gg\\ =&\widetilde{e}_n \ll D(\pi_m\varphi_{n,m}^{(k)})\left((I-\pi_n)(K(\pi_m\varphi_{n,m}^{(k)}) + g)\right), \widetilde{e}_n^* \gg\\ &+ \widetilde{e}_n \ll D(\pi_m\varphi_{n,m}^{(k)})\widetilde{e}_n, \widetilde{e}_n^* \gg\ll \varphi_{n,m}^{(k+1)}, \widetilde{e}_n^* \gg + \widetilde{e}_n \ll S_{n,m}^{(k)}, \widetilde{e}_n^* \gg .\end{aligned}$$

So,

$$\widetilde{e}_n \ll \varphi_{n,m}^{(k+1)}, \widetilde{e}_n^* \gg = \widetilde{e}_n \ll D(\pi_m \varphi_{n,m}^{(k)}) \left((I-\pi_n)(K(\pi_m \varphi_{n,m}^{(k)}) + g) \right), \widetilde{e}_n^* \gg + \widetilde{e}_n \ll S_{n,m}^{(k)}, \widetilde{e}_n^* \gg$$
$$+ \widetilde{e}_n \ll D(\pi_m \varphi_{n,m}^{(k)}) \widetilde{e}_n, \widetilde{e}_n^* \gg \ll \varphi_{n,m}^{(k+1)}, \widetilde{e}_n^* \gg,$$

as a result, this led to writing

$$\left(I - \ll D(\pi_m \varphi_{n,m}^{(k)}) \widetilde{e}_n, \widetilde{e}_n^* \gg \right) \ll \varphi_{n,m}^{(k+1)}, \widetilde{e}_n^* \gg = \ll D(\pi_m \varphi_{n,m}^{(k)}) \left((I-\pi_n)(K(\pi_m \varphi_{n,m}^{(k)}) + g) \right), \widetilde{e}_n^* \gg$$
$$+ \ll S_{n,m}^{(k)}, \widetilde{e}_n^* \gg .$$

At last, we are able to articulate our estimated solution in the form of:

$$\varphi_{n,m}^{(k+1)} = (I-\pi_n)(K(\pi_m \varphi_{n,m}^{(k)}) + g) + \widetilde{e}_n U_{n,m}^{(k+1)}, \tag{3.10}$$

where $U_{n,m}^{(k+1)} \in \mathbb{C}^n$ is a column vector we get it by solving the following linear system

$$\left(I_n - M_n^{(k)} \right) U_{n,m}^{(k+1)} = d_{n,m}^{(k)},$$

which for $i, j = 1, 2, \cdots, n$, we have

$$\begin{aligned} M_n^{(k)}(i,j) =& \ll D(\pi_m \varphi_{n,m}^{(k)}) \widetilde{e}_n, \widetilde{e}_n^* \gg (i,j) \\ =& \langle D(\pi_m \varphi_{n,m}^{(k)}) e_j, e_i^* \rangle \\ =& \int_0^1 \frac{\partial \kappa}{\partial \varphi} \left(t_i, s, \pi_m \varphi_{n,m}^{(k)}(s) \right) e_j(s) ds, \end{aligned}$$

$$\begin{aligned} d_{n,m}^{(k)}(i) &= \ll S_{n,m}^{(k)}, \widetilde{e}_n^* \gg (i) + \ll D(\pi_m \varphi_{n,m}^{(k)}) \left((I-\pi_n)(K(\pi_m \varphi_{n,m}^{(k)}) + g) \right), \widetilde{e}_n^* \gg (i) \\ &= \langle K(\pi_m \varphi_{n,m}^{(k)}), e_i^* \rangle - \langle D(\pi_m \varphi_{n,m}^{(k)}) \varphi_{n,m}^{(k)}, e_i^* \rangle + \langle g, e_i^* \rangle \\ &\quad + \langle D(\pi_m \varphi_{n,m}^{(k)})(I-\pi_n)(K(\pi_m \psi_{n,m}^{(k)}) + g), e_i^* \rangle \\ &= \int_0^1 \kappa \left(t_i, s, \pi_m \varphi_{n,m}^{(k)}(s) \right) ds - \int_0^1 \frac{\partial \kappa}{\partial \varphi} \left(t_i, s, \pi_m \varphi_{n,m}^{(k)}(s) \right) \varphi_{n,m}^{(k)}(s) ds + g(t_i) \\ &\quad + \int_0^1 \frac{\partial \kappa}{\partial \varphi} \left(t_i, s, \pi_m \varphi_{n,m}^{(k)}(s) \right) \left(\int_0^1 \kappa \left(s, x, \pi_m \varphi_{n,m}^{(k)}(x) \right) dx + g(s) \right) ds \\ &\quad - \int_0^1 \frac{\partial \kappa}{\partial \varphi} \left(t_i, s, \pi_m \varphi_{n,m}^{(k)}(s) \right) \sum_{p=1}^n \left(\int_0^1 \kappa \left(t_p, x, \pi_m \varphi_{n,m}^{(k)}(x) \right) dx + g(t_p) \right) e_p(s) ds. \end{aligned}$$

3.3 Numerical Examples

In this section, we demonstrate the effectiveness of our (L.D.D) scheme by solving three problems. The first problem involves a system of nonlinear Fredholm integro-differential equations, which was previously solved in [31] using Sloan's discretization. We will compare our results obtained using the (L.D.D) scheme with Kantorovich's discretization to

Algorithm 1: (L.D.D) Algorithm

Data: n, m, g, φ_{ext}
Result: $E_{n,m}$, plot $(\varphi_{n,m}^{(k+1)}, \varphi_{ext}, \log(e))$
Initialization:
$\varphi_{n,m}^{(0)} \leftarrow 0, M_n^{(0)} \leftarrow 0_{n\times n}, d_n^{(0)} \leftarrow 0_n, k \leftarrow 1, Tol \leftarrow 10^{-12}, E_{n,m}^k \leftarrow 1;$
while $E_{n,m}^k > Tol$ **do**
  for $i \leftarrow 1$ **to** n **do**
    for $j \leftarrow 1$ **to** n **do**
      Calculate and save $M_n^{(k)}(i,j)$;
      if $i{=}j$ **then**
        $A_n^{(k)} \leftarrow 1 - M_n^{(k)}(i,j)$;
      else
        $A_n^{(k)} \leftarrow -M_n^{(k)}(i,j)$;
      end
    end
    Calculate and save $d_n^{(k)}(i)$;
  end
  $X_n^{(k+1)} \leftarrow (A_n^{(k)})^{-1}.d_n^{(k)}$; `/* (Calculate and save the vector solution of the linear system) */`
  for $p \leftarrow 1$ **to** n **do**
    $\pi_n K(t) \leftarrow \pi_n K(t) + K(t_n(p))\,.\,e_p(t)$;
    $\pi_n g(t) \leftarrow \pi_n g(t) + g(t_n(p))\,.\,e_p(t)$;
    $e_n X_n^{(k+1)}(t) \leftarrow e_n X_n^{(k+1)}(t) + X_n^{(k+1)}(p)\,.\,e_p(t)$;
  end
  $\varphi_{n,m}^{(k+1)}(t) \leftarrow K(t) + g(t) - \pi_n K(t) - \pi_n g(t) + e_n X_n^{(k+1)}(t)$;
  $\varphi_{n,m}^{(k+1)}(t) \leftarrow \varphi_{n,m}^{(k)}(t)$; `/* (Save the previous iterate solution, k = 1,2...) */`
  $E_{n,m}^k(t) \leftarrow \max\limits_{t\in[0,1]} \left\|\varphi_{n,m}^{(k+1)}(t) - \varphi_{n,m}^{(k)}(t))\right\|$; `/* (Calculate the iterate error) */`
  $k \leftarrow k+1$; `/* (Increment the number of iterations k by 1) */`
end
$E_{n,m} \leftarrow \max\limits_{t\in[0,1]} \left\|\varphi_{n,m}^{(k+1)}(t) - \varphi_{ext}(t)\right\|$; `/* (Calculate the error between the exact solution and the last iterate solution) */`

those obtained in [31] and the (D.L)-Classical method. The second and third problems involve systems of two and three (respectively) nonlinear Fredholm integro-differential equations. We solve these problems using both our (L.D.D) scheme and the (D.L)-Classical process. We present the results of the numerical applications in separate tables and figures.

Let $\left(\varphi_{n,m,1}^{(k)}, \varphi_{n,m,2}^{(k)}, \cdots, \varphi_{n,m,N}^{(k)}\right) \in \mathcal{V} \subset \mathcal{X}, k \in \mathbb{N}^*$ the k order approximate solution of our equations system (1.17) according to the scheme (3.5) obtained bu apply the (L.D.D) method. We specify the stopping condition on the parameter k as:

$$E^k_{n,m} = \sum_{i=1}^{N} \max_{1\leq p\leq n} \left|\varphi^{(k+1)}_{n,m,i}(t_p) - \varphi^{(k)}_{n,m,i}(t_p)\right| \leq 10^{-09}, \quad t_p \in \Delta_n.$$

We denote the obtained error formula by

$$E_{n,m} = \sum_{i=1}^{N} \max_{1\leq p\leq n} \left|\varphi_{i,ext}(t_p) - \varphi^{(k)}_{n,m,i}(t_p)\right|, \quad t_p \in \Delta_n,$$

where, $\Psi_{ext} = (\varphi_{1,ext}, \varphi_{2,ext}, \cdots, \varphi_{N,ext}) \in \mathcal{V} \subset \mathcal{X}$ is the exact solution of the initial equations system (1.17). We pass directly to the numerical examples.

Example 3.3.1. *Consider the following nonlinear Fredholm integro-differential equation*

$$\varphi(t) = z\int_0^1 \left(\exp(t)(\varphi(s))^2 - \exp(-t)(\varphi'(s))^2\right) ds + g(t), \quad z \in \mathbb{R}, t \in [0,1], \tag{3.11}$$

with $\varphi \in C^1([0,1], \mathbb{R})$, and the function g is given by

$$g(t) = \sqrt{1+t} - \frac{z}{4}\left(6\exp(t) - \log(2)\exp(-t)\right), \quad t \in [0,1].$$

We derive equation (3.11) to obtain the following system of nonlinear integro-differential equations

$$\begin{cases} \varphi(t) = z\int_0^1 \left(\exp(t)(\varphi(s))^2 - \exp(-t)(\varphi'(s))^2\right) ds + g(t), \\ \varphi'(t) = z\int_0^1 \left(\exp(t)(\varphi(s))^2 + \exp(-t)(\varphi'(s))^2\right) ds + g'(t), \end{cases} \tag{3.12}$$

and the function g' is given by

$$g'(t) = \frac{1}{2\sqrt{1+t}} - \frac{z}{4}\left(6\exp(t) + \log(2)\exp(-t)\right), \quad t \in [0,1].$$

So, system (3.12) is similar to the following system

$$\begin{cases} \varphi_1(t) = z\int_0^1 \left(\exp(t)(\varphi_1(s))^2 - \exp(-t)(\varphi_2(s))^2\right) ds + g_1(t), \\ \varphi_2(t) = z\int_0^1 \left(\exp(t)(\varphi_1(s))^2 + \exp(-t)(\varphi_2(s))^2\right) ds + g_2(t), \end{cases} \tag{3.13}$$

where $\Psi_{ext} = \left(\sqrt{1+t}, \dfrac{1}{2\sqrt{1+t}}\right)$ is the exact solution of our system (3.13).

Example 3.3.2. *Consider the following nonlinear Fredholm integro-differential equation*

$$\varphi(t) = \frac{z}{20}\int_0^1 \cos\left(\exp(s) + \arccos\left(\frac{s+t}{3}\right) + \varphi(s) - \varphi'(s)\right) ds + g(t), \quad z \in \mathbb{R}, t \in [0,1], \tag{3.14}$$

with $\varphi \in C^1([0,1],\mathbb{R})$*, and the function* g *is given by*

$$g(t) = t\exp(t) - \frac{z}{60}\left(t + \frac{1}{2}\right), \quad t \in [0,1].$$

We derive equation (3.14) to obtain the following system of nonlinear integro-differential equations

$$\begin{cases} \varphi(t) = \dfrac{z}{20}\displaystyle\int_0^1 \cos\left(\exp(s) + \arccos\left(\frac{s+t}{3}\right) + \varphi(s) - \varphi'(s)\right) ds + g(t), \\ \varphi'(t) = \dfrac{z}{60}\displaystyle\int_0^1 \frac{1}{\sqrt{1-\left(\frac{s+t}{3}\right)^2}} \sin\left(\exp(s) + \arccos\left(\frac{s+t}{3}\right) + \psi(s) - \varphi'(s)\right) ds + g'(t), \end{cases} \tag{3.15}$$

and the function g' *is given by*

$$g'(t) = (1+t)\exp(t) - \frac{z}{60}, \quad t \in [0.1].$$

So, system (3.15) is similar to the following system

$$\begin{cases} \varphi_1(t) = \dfrac{z}{20}\displaystyle\int_0^1 \cos\left(\exp(s) + \arccos\left(\frac{s+t}{3}\right) + \varphi_1(s) - \psi_2(s)\right) ds + g_1(t), \\ \varphi_2(t) = \dfrac{z}{60}\displaystyle\int_0^1 \frac{1}{\sqrt{1-(\frac{s+t}{3})^2}} \sin\left(\exp(s) + \arccos\left(\frac{s+t}{3}\right) + \varphi_1(s) - \varphi_2(s)\right) ds + g_2(t), \end{cases} \tag{3.16}$$

where $\Psi_{ext} = (t\exp(t), (t+1)\exp(t))$ *is the exact solution of our system (3.16).*

Example 3.3.3. *Consider the following nonlinear Fredholm integro-differential equation*

$$\varphi(t) = \frac{z}{50}\int_0^1 \frac{\sin(\pi s)}{1+t}(2\pi\varphi(s) + \varphi'(s)^2 - \cos(\pi s)\varphi''(s))\, ds + g(t), \quad z \in \mathbb{R}, t \in [0,1], \tag{3.17}$$

with $\psi \in C^2([0,1],\mathbb{R})$*, and the function* g *is given by*

$$g(t) = \cos(\pi t) - \frac{z\pi}{25(1+t)}, \quad t \in [0,1].$$

We derive equation (3.17) two times to obtain the following system of nonlinear integro-differential equations

$$\begin{cases} \varphi(t) = \dfrac{z}{50}\displaystyle\int_0^1 \frac{\sin(\pi s)}{1+t}(2\pi\varphi(s) + \varphi'(s)^2 - \cos(\pi s)\varphi''(s))\, ds + g(t), \\ \varphi'(t) = -\dfrac{z}{50}\displaystyle\int_0^1 \frac{\sin(\pi s)}{(1+t)^2}(2\pi\varphi(s) + \psi'(s)^2 - \cos(\pi s)\varphi''(s))\, ds + g'(t), \\ \varphi''(t) = \dfrac{z}{25}\displaystyle\int_0^1 \frac{\sin(\pi s)}{(1+t)^3}(2\pi\varphi(s) + \psi'(s)^2 - \cos(\pi s)\varphi''(s))\, ds + g''(t), \end{cases} \tag{3.18}$$

where

$$g'(t) = -\pi \sin(\pi t) + \frac{z\pi}{25(1+t)^2},$$

and

$$g''(t) = -\pi^2 \cos(\pi t) - \frac{2z\pi}{25(1+t)^3},$$

and by the same notation technique always do, system (3.18) is similar to the following system

$$\begin{cases} \varphi_1(t) = \frac{z}{50}\int_0^1 \frac{\sin(\pi s)}{1+t}(2\pi\varphi_1(s) + \psi_2(s)^2 - \cos(\pi s)\varphi_3(s))\,ds + g_1(t), \\ \varphi_2(t) = -\frac{z}{50}\int_0^1 \frac{\sin(\pi s)}{(1+t)^2}(2\pi\varphi_1(s) + \varphi_2(s)^2 - \cos(\pi s)\varphi_3(s))\,ds + g_2(t), \\ \varphi_3(t) = \frac{z}{25}\int_0^1 \frac{\sin(\pi s)}{(1+t)^3}(2\pi\varphi_1(s) + \varphi_2(s)^2 - \cos(\pi s)\varphi_3(s))\,ds + g_3(t), \end{cases} \tag{3.19}$$

where $\Psi_{ext} = (\cos(\pi t), -\pi\sin(\pi t), -\pi^2\cos(\pi t))$ *is the exact solution of our system* (3.19).

Our technique described in subsection 1.4.1 has been used to transform all systems (3.13), (3.16) and (3.19) into the form of a nonlinear operator equation (1.17), which resembles the fundamental nonlinear functional problems (3.1) presented at the start of this chapter. Therefore, we can now use the (L.D.D) new scheme to solve all of these problems.

Example 3.3.4. *Consider the following nonlinear Fredholm integro-differential system of equations*

$$\begin{cases} \varphi_1(t) = z\int_0^1 e^{t-6s}\left(e^{3s}\varphi_2^2 + \varphi_3^2\right) + g_1(t), \\ \varphi_2(t) = z\int_0^1 e^{t-6s}\left(\varphi_3^2 + e^{4s}\varphi_1^2\right) + g_2(t), \\ \varphi_3(t) = z\int_0^1 e^{t-4s}\left(e^{2s}\varphi_1^2 + \varphi_2^2\right) + g_3(t), \end{cases} \tag{3.20}$$

where

$$\begin{cases} g_1(t) = e^t - ze^{t+1}, \\ g_2(t) = e^{2t} - 2ze^t, \\ g_3(t) = e^{3t} - 2ze^t, \end{cases}$$

and $\Psi_{ext} = (e^t, e^{2t}, e^{3t})$ *is the exact solution of our system* (3.20).

The error $E_{n,m}$ if $z = 0.1, n = 10$ and $m = 10$							
k	(L.D.D)-Sloan [31]	k	(L.D.D) Kantorovich	CPU time	k	(D.L) Classical	CPU time
$k = 2$	2.2003 e-01	$k = 2$	8.1548 e-01	5.2 e-02s	$k = 2$	9.2558 e-01	8.1 e-03s
$k = 6$	5.1157 e-03	$k = 3$	4.6688 e-02	1.1 e-01s	$k = 3$	5.6988 e-02	2.4 e-02s
$k = 10$	1.7150 e-04	$k = 4$	8.6227 e-05	1.7 e-01s	$k = 4$	5.6988 e-02	7.3 e-02s
$k = 14$	5.4942 e-05	$k = 5$	8.9757 e-08	2.4 e-01s	$k = 5$	5.6988 e-02	1.2 e-01s
$k = 18$	5.5931 e-05	$k = 6$	8.8003 e-08	3.1 e-01s	$k = 6$	5.6988 e-02	1.5 e-01s

Table 3.1: Numerical results of Example 3.3.1, where we compared between our (L.D.D) scheme applying Kantoroviche method and [31]-(L.D.D) scheme applying Sloan's method and the (D.L) classical method.

The error $E_{n,m}$ if $z = 10$ at $k = 3$								
	(L.D.D)-Kantorovich				(D.L)-Classical			
n	$m = 50$	CPU time	$m = 100$	CPU time	$m = 50$	CPU time	$m = 100$	CPU time
5	6.689 e-08	9.3 e-02s	1.622 e-08	1.1 e-01s	1.0146 e-02	1.3 e-02s	1.0146 e-02	1.3 e-02s
20	4.158 e-09	5.9 e-01s	8.529 e-10	6.6 e-01s	1.0096 e-02	4.5 e-02s	1.0095 e-02	5.2 e-02s
50	1.183 e-09	2.6 e-00s	1.857 e-10	2.8 e-00s	1.0094 e-02	2.5 e-01s	1.0093 e-02	2.0 e-01s
100	1.035 e-09	9.2 e-00s	5.979 e-11	9.6 e-00s	1.0094 e-02	5.5 e-01s	1.0093 e-02	5.8 e-01s

Table 3.2: Numerical results of Example 3.3.2, where we compared between our (L.D.D) new scheme and the (D.L) classical method.

The error $E_{n,m}$ if $z = 0.1, n = 15$ at $k = 5$				
m	(L.D.D)-Kantrovich	CPU time	(D.L)-Classical	CPU time
5	1.037871850698868 e-04	2.92s	7.7395458721862 e-02	1.65s
10	4.615256983570827 e-06	3.05s	7.7273067743482 e-02	1.72s
15	1.176507064125706 e-06	3.09s	7.7055432555865 e-02	1.79s
30	5.045708066493429 e-07	3.12s	7.6925458071442 e-02	1.85s

Table 3.3: Numerical results of Example 3.3.3, where we compared between our (L.D.D) new scheme and the (D.L) classical method.

The error $E_{n,m}$ if $z = 0.01, n = 10$ at $k = 5$				
m	(L.D.D)-Kantrovich	CPU time	(D.L)-Classical	CPU time
5	1.927880563266093 e-04	10.33s	2.5386618746219 e-01	8.23s
10	7.280855369742653 e-05	11.95s	9.1208345678932 e-02	8.42s
15	4.856819527887524 e-06	12.35s	9.1208345678932 e-02	9.12s
30	1.818007136808689 e-06	14.41s	9.1208345678932 e-02	9.43s

Table 3.4: Numerical results of Example 3.3.4, where we compared between our (L.D.D) new scheme and the (D.L) classical method.

The results of Example 3.3.1 using the (L.D.D)-Kantorovich's, (L.D.D)-Sloan's, and (D.L)-Classical methods are presented in Table 3.1 and Figure 3.1. The results confirm that our process is more efficient based on the obtained error value, and its approximate solution converges in 6 iterations ($k = 6$), which is faster than the (L.D.D)-Sloan's method that converges in 18 iterations ($k = 18$) and faster than the (D.L)-Classical method, which does not converge for small n.

Table 3.2 shows that our (L.D.D)-Kantorovich's method yields better results as we increase n and m, with only 3 iterations required for convergence ($k = 3$) (see Figure 3.3). In contrast, the (D.L)-Classical process does not converge for small n.

Tables 3.3 and 3.4 show that the best approximate solution can be obtained with a fixed n (at most 10 or 15 respectively), and the higher the value of m (at most 30), the better the approximate solution obtained (see Figure 3.4 and 3.6). These results are in accordance with the theoretical part presented in this research. However, we also compared the execution time of the (L.D.D)-Kantorovich's and (D.L)-Classical methods in all four tables and found similar results. The computations were performed using the Matlab software on a machine with an Intel(R) Core(TM) i7-8665U CPU @1.90GHz 2.11GHz and 32 GB RAM.

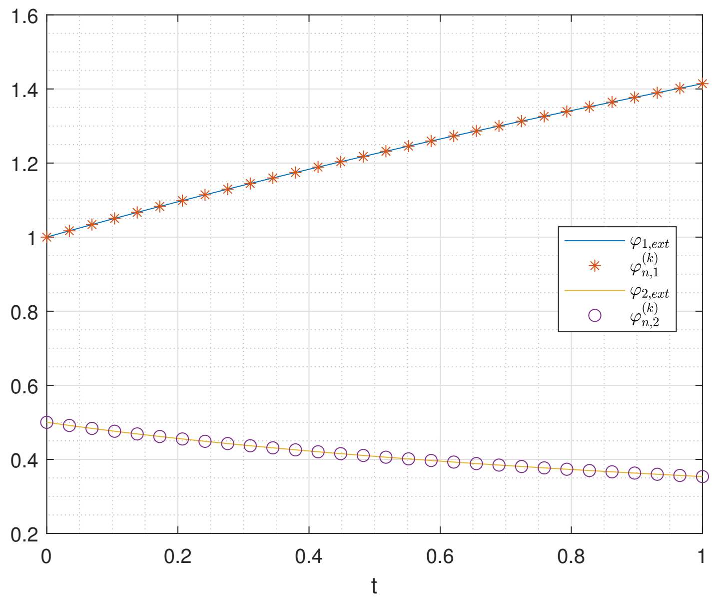

Figure 3.1: Exacts and approximate solutions of Example 3.3.1, applying (L.D.D) Method.

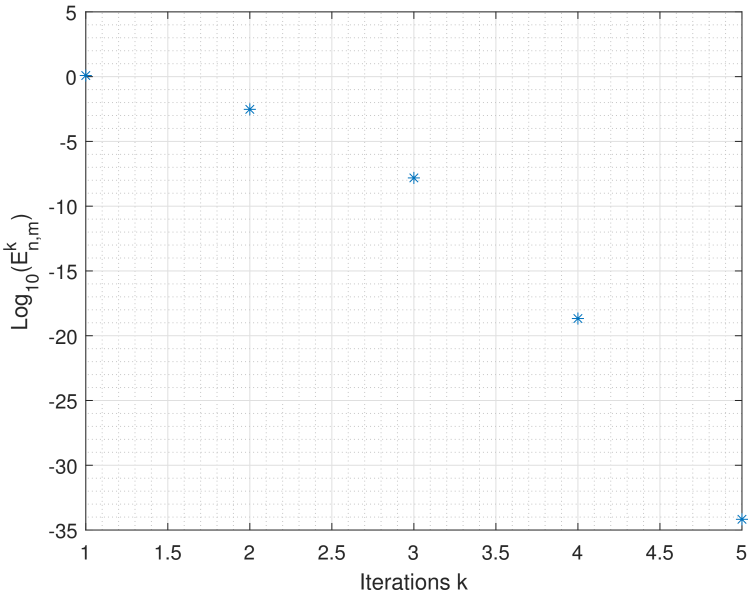

Figure 3.2: Graph of $log_{10}(E_{n,m}^k)$, the log_{10} of the error between two successive iterates approximations $\varphi_n^{(k+1)}$ and $\varphi_n^{(k)}$ of the first system. (Example 3.3.1)

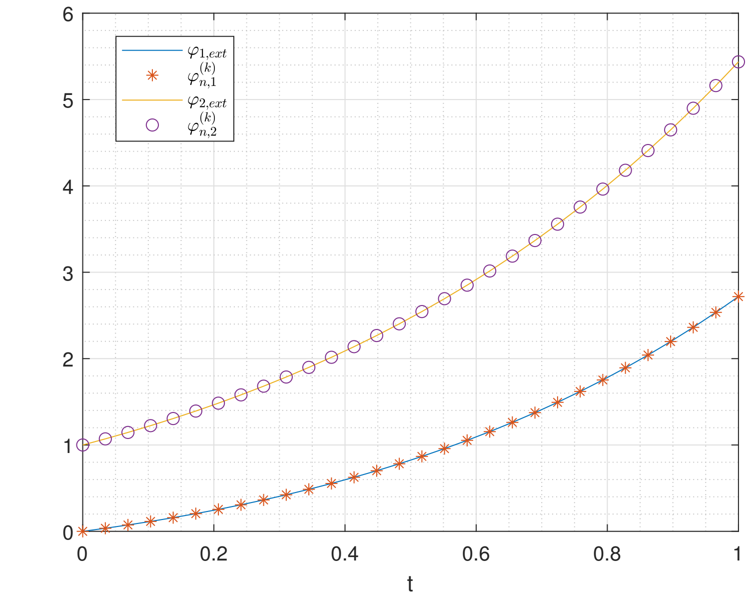

Figure 3.3: Exacts and approximate solutions of Example 3.3.2, applying (L.D.D) Method.

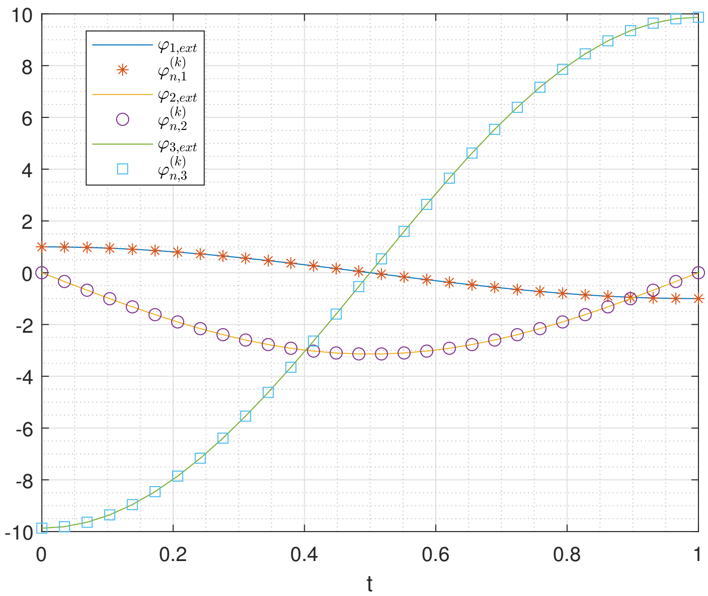

Figure 3.4: Exacts and approximate solutions of Example 3.3.3, applying (L.D.D) Method.

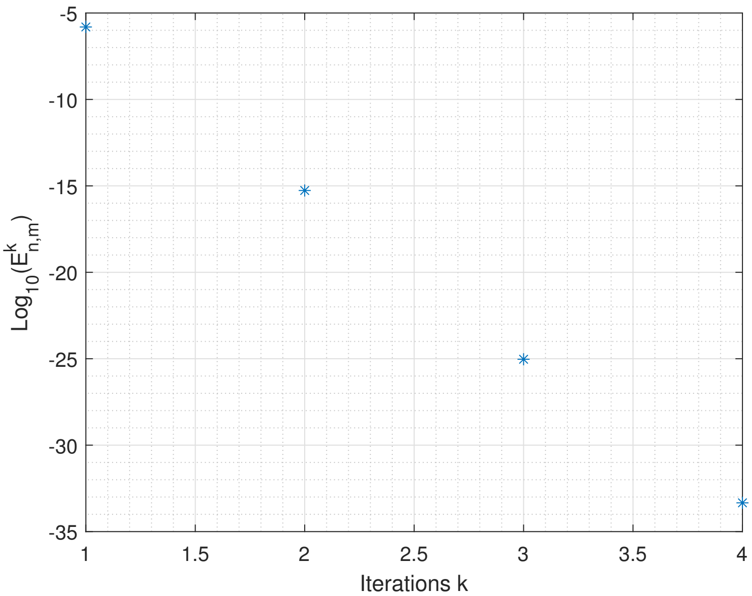

Figure 3.5: Graph of $log_{10}(E^k_{n,m})$, the log_{10} of the error between two successive iterates approximations $\varphi_n^{(k+1)}$ and $\varphi_n^{(k)}$ of the third system. (Example 3.3.3)

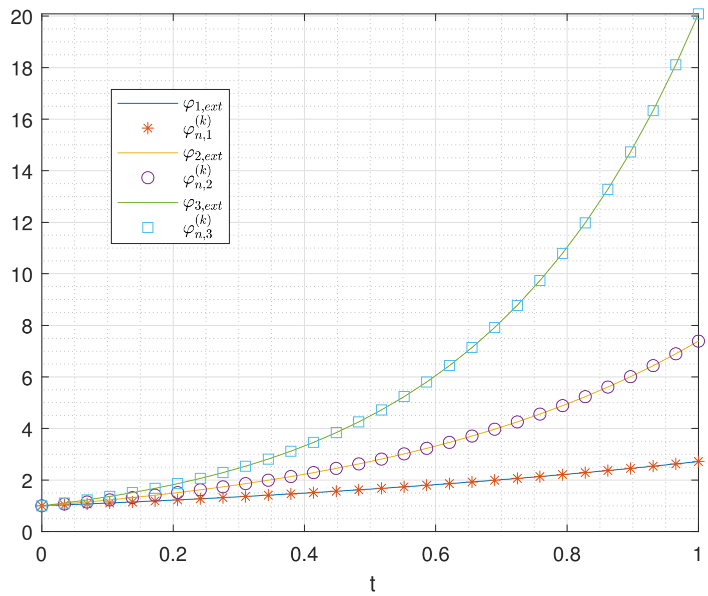

Figure 3.6: Exacts and approximate solutions of Example 3.3.4, applying (L.D.D) Method.

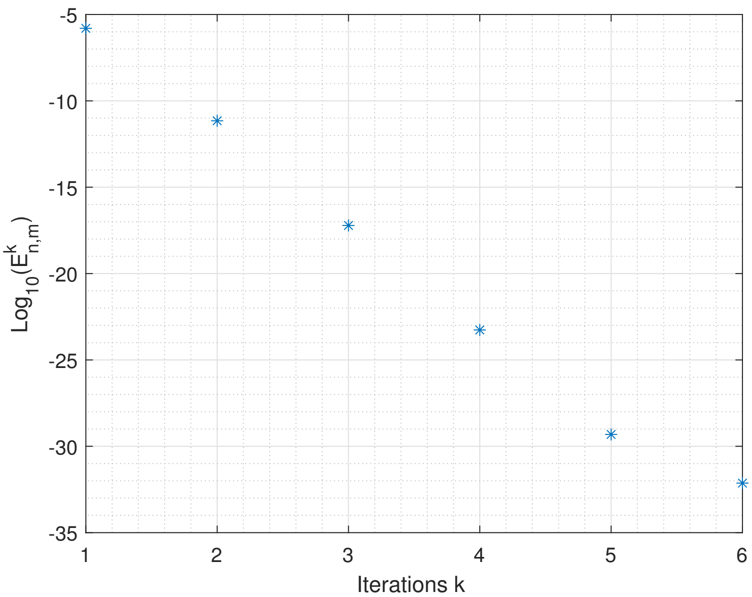

Figure 3.7: Graph of $log_{10}(E^k_{n,m})$, the log_{10} of the error between two successive iterates approximations $\varphi_n^{(k+1)}$ and $\varphi_n^{(k)}$ of the fourth system. (Example 3.3.4)

Chapter 4

Solving System of Weakly Singular Nonlinear Integral Equations Using the (LN.DK) Method

Contents

In prior chapters 2, 3 and recent papers [22, 23, 31], we and the other authors have investigated the Discretization-Linearization (D.L) and Linearization-Discretization (L.D) schemes. Our research has shown that the latter method is superior to the former because the approximate solution generated by the modern (L.D) strategy converges to the exact solution of the main problem under certain conditions in the discretization process, while the classical (D.L) process yields only an approximate vector [45, 46].

In this chapter, we will apply the modern (L.D) strategy to approximate solutions for a system of integro-differential nonlinear Fredholm equations of the second kind with a weakly singular kernel. In the first step of our new (LN.DK) process, we will use the classical iterative **N**ewton method for **L**inearization and then apply the **K**antorovich projection method for **D**iscretization. This will result in a discretized linear scheme defined by integrals with a weakly singular kernel that we will estimate using the product integration rule, and by using the Nyström method for the remaining regular integrals.

In this chapter, we begin by presenting our problem statement which we aim to solve in section 4.1. Next, in section 4.2, we provide a detailed description of the (LN.DK) strategy that we propose to tackle the problem. Moving on, we demonstrate the compactness of the operators with weak singular kernels in section 4.3. We then proceed to analyze the convergence of the (LN.DK) process in section 4.4, followed by presenting numerical examples in section 4.5 to conclude the chapter.

4.1 The Main Problem

We consider the regular function κ as

$$\begin{aligned} \kappa : [0,1]^2 \times \mathbb{R}^2 &\longrightarrow \mathbb{R}, \\ (t,s,\varphi(s),\varphi'(s)) &\longrightarrow \kappa(t,s,\varphi(s),\varphi'(s)), \end{aligned}$$

and the kernel g verify the same conditions defined in ([9], [50]) and others as follows

$$(\mathcal{H}_1) : \left\| \begin{array}{l} (i) \lim\limits_{s\to 0^+} g'(s) = +\infty, \\ (ii)\, g \in W^{1.1}([0,1],\mathbb{R}), \\ (iii)\, g'(t) \geq 0 \text{ for all } t \in]0,1], \\ (iv)\, g' \text{ is a strictly decreasing function on }]0,1]. \end{array} \right.$$

The conditions suggest that the singularity in our problem arises from the derivative of the function g, denoted as g'. We define the Banach space,

$$W^{1.1}([0,1],\mathbb{R}) = \left\{ p \in L^1([0,1],\mathbb{R}) : p' \in L^1([0,1],\mathbb{R}), p' \text{ is the weak derivative of } p \right\},$$

under the following norm [2]:

$$\|p\|_{W^{1.1}} = \|p\|_{L^1} + \|p'\|_{L^1} = \int_0^1 |p(s)|ds + \int_0^1 |p'(s)|ds.$$

In this work, we trait the following nonlinear type of equations

$$\varphi(t) = \int_0^1 g(|t-s|)\kappa(t,s,\varphi(s),\varphi'(s))ds + f(t). \quad t \in [0,1], \tag{4.1}$$

with a given function $f \in C^1([0,1],\mathbb{R})$ and φ is the exact solution of problem (4.1) in the same space.

In order to obtain additional information about the exact solution φ, we differentiate both sides of equation (4.1) as shown below

$$\varphi'(t) = \int_0^1 S_{ts}\, g'(|t-s|)\kappa(t,s,\varphi(s),\varphi'(s))\, ds + \int_0^1 g(|t-s|)\frac{\partial \kappa}{\partial t}(t,s,\varphi(s),\varphi'(s))\, ds + f'(t), \tag{4.2}$$

where

$$S_{ts} = sign(t-s) = \begin{cases} 1 & t > s, \\ -1 & t < s, \\ 0 & t = s. \end{cases}$$

To construct our system of nonlinear integro-differential equations, we employ the same technique outlined in subsection 1.4.1. So, we set the ensuing notations for all $i = 1, 2$, $f_i = f^{(i-1)}$, and $\varphi_i = \varphi^{(i-1)}$ to get the following system

$$\begin{cases} \varphi_1(t) = \int_0^1 g(|t-s|)\kappa(t, s, \varphi_1(s), \varphi_2(s))ds + f_1(t), \\ \varphi_2(t) = \int_0^1 S_{ts}\, g'(|t-s|)\kappa(t, s, \varphi_1(s), \varphi_2(s))\, ds \, + \int_0^1 g(|t-s|)\frac{\partial \kappa}{\partial t}(t, s, \varphi_1(s), \varphi_2(s))\, ds \, + f_2(t). \end{cases} \tag{4.3}$$

We define the product Banach space $\mathcal{X} = C^1([0,1], \mathbb{R}) \times C^0([0,1], \mathbb{R})$ provided with the norm

$$\forall \varPhi = (\phi_1, \phi_2) \in \mathcal{X} : \quad \|\varPhi\|_{\mathcal{X}} = \|\phi_1\|_\infty + \|\phi_2\|_\infty = \sup_{t\in[0,1]} |\phi_1(t)| + \sup_{t\in[0,1]} |\phi_2(t)| \,.$$

The system (4.3) can be rewritten as

$$\begin{cases} \varphi_1(t) = K_1(\varphi)(t) + f_1(t), \\ \varphi_2(t) = K_2(\varphi)(t) + f_2(t) = K_2^s(\varphi)(t) + K_2^r(\varphi)(t) + f_2(t), \end{cases} \tag{4.4}$$

where $\varphi = (\varphi_1, \varphi_2) \in \mathcal{X}$ and operators K_1, K_2^s, and K_2^r presented by

$$\begin{cases} K_1(\varphi)(t) = \int_0^1 g(|t-s|)\kappa(t, s, \varphi_1(s), \varphi_2(s))ds, \\ K_2^s(\varphi)(t) = \int_0^1 S_{ts}\, g'(|t-s|)\kappa(t, s, \varphi_1(s), \varphi_2(s))\, ds, \\ K_2^r(\varphi)(t) = \int_0^1 g(|t-s|)\frac{\partial \kappa}{\partial t}(t, s, \varphi_1(s), \varphi_2(s))\, ds. \end{cases} \tag{4.5}$$

So, the main problem we trait is

$$\text{Find } \varphi \in \mathcal{X} : \quad \varphi = K(\varphi) + F, \tag{4.6}$$

with

$$K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} K_1 \\ K_2^s + K_2^r \end{pmatrix}, \text{ and } F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

For all $j = 1, 2$, let $T_{1j}(\cdot) = \dfrac{\partial K_1}{\partial \varphi_j}(\cdot), T^s_{2j}(\cdot) = \dfrac{\partial K^s_2}{\partial \varphi_j}(\cdot)$, and $T^r_{2j}(\cdot) = \dfrac{\partial K^r_2}{\partial \varphi_j}(\cdot)$ be the Fréchet derivative linear operators of the operators K_1, K^s_2, and K^r_2 (respectively) associated to φ_j, such as

$$\begin{cases} \left[T_{1j}(\varphi).v\right](t) = \displaystyle\int_0^1 g(|t-s|)\frac{\partial \kappa}{\partial \varphi_j}(t, s, \varphi_1(s), \varphi_2(s))\, v(s)ds, \\ \left[T^s_{2j}(\varphi).v\right](t) = \displaystyle\int_0^1 S_{ts}\, g'(|t-s|)\frac{\partial \kappa}{\partial \varphi_j}(t, s, \varphi_1(s), \varphi_2(s))\, v(s)ds, \\ \left[T^r_{2j}(\varphi).v\right](t) = \displaystyle\int_0^1 g(|t-s|)\frac{\partial^2 \kappa}{\partial t \partial \varphi_j}(t, s, \varphi_1(s), \varphi_2(s))\, v(s)ds, \end{cases} \tag{4.7}$$

for $v \in C([0,1], \mathbb{R})$ and $t \in [0,1]$.

Let I_{22} be the 2×2 block identity operator of the space $\mathcal{L}(\mathcal{X})$ of all linear bounded operators defined from $\mathcal{X}$ to $\mathcal{X}$, and $M_T(\cdot) : \mathcal{X} \longrightarrow \mathcal{X}$ is the Fréchet derivative operator of the operator K defined for all $\phi \in \mathcal{X}$ as

$$M_T(\phi)h = \begin{pmatrix} T_{11}(\phi) & T_{12}(\phi) \\ T^s_{21}(\phi) + T^r_{21}(\phi) & T^s_{22}(\phi) + T^r_{22}(\phi) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \quad h \in \mathcal{X}. \tag{4.8}$$

We set $B_\nu(\varphi)$ as the ball of center φ and radius $\nu > 0$, and we assume that

$$(\mathcal{H}_2) : \left\| \begin{array}{l} (i)\;\; \text{Problem (4.6) has a unique solution } \varphi \in \mathcal{X}, \\ (ii)\; I_{22} - M_T(\varphi) \text{ is invertible, and } \exists \mu > 0, \|(I_{22} - M_T(\varphi))^{-1}\| \leq \mu < +\infty, \\ (iii) \exists \ell > 0, \text{ such that } M_T(\varphi) : \mathcal{X} \longrightarrow \mathcal{L}(\mathcal{X}) \text{ is } \ell-\text{Lipschitz over } B_\nu(\varphi). \end{array} \right.$$

After defining all necessary notation and setting up the required conditions, we proceed to describe our strategy for solving the main problem (4.6).

4.2 The description of The (LN.DK) Strategy

The first step to solve problem (4.6) is the linearization process using Newton's method to get the following linear scheme

$$\left(I_{22} - M_T(\varphi^{(k)})\right)\left(\varphi^{(k+1)} - \varphi^{(k)}\right) = -\varphi^{(k)} + K(\varphi^{(k)}) + F, \quad \varphi^{(0)} \in \mathcal{X},\, k = 1, 2, \cdots \quad . \tag{4.9}$$

Due to the challenge of obtaining the operator $\left(I_{22} - M_T(\varphi^{(k)})\right)^{-1}$ in every iteration k, it becomes necessary to discretize our scheme (4.9). To achieve this, we utilize a family of bounded projections of finite rank in $\mathcal{X}$, defined for $n \in \mathbb{N}^*$ as follows

$$\forall X = (X_1, X_2) \in \mathcal{X},\, \pi_n X = (\pi_n X_1, \pi_n X_2) = \left(\sum_{p=1}^n < X_1, e^*_p > e_p, \sum_{p=1}^n < X_2, e^*_p > e_p\right) \tag{4.10}$$

where $\widetilde{e}_n = (e_1, e_2, \cdots, e_n)$ is an ordered basis of the image space of $(\pi_n)_{n\in\mathbb{N}^*}$ and $\widetilde{e}^*_n = (e^*_1, e^*_2, \cdots, e^*_n)$ is an adjoint basis of the previous one. For $i, j = 1, 2$, $\phi \in \mathcal{X}$, and for $v \in C([0,1],\mathbb{R})$, we construct the approach operators

$$T^c_{n,ij}(\phi)v = (\pi_n T^c_{ij}(\phi))v = \sum_{p=1}^{n} < T^c_{ij}(\phi)v, e^*_p > e_p, \quad c \in \{\ , s, r\}\,. \tag{4.11}$$

Using the definition (4.8) presented earlier, we can express the approximate operator $M_{T_n}(\cdot)$ of the operator $M_T(\cdot)$ for n sufficiently large and $\phi \in \mathcal{X}$ as follows

$$M_{T_n}(\phi)h = \pi_n(M_T(\phi)h) = \begin{pmatrix} T_{n,11}(\phi) & T_{n,12}(\phi) \\ T^s_{n,21}(\phi) + T^r_{n,21}(\phi) & T^s_{n,22}(\phi) + T^r_{n,22}(\phi) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \quad h \in \mathcal{X}.$$

We can now express the discretized scheme of the linear problem (4.9) as follows

$$\left(I_{22} - M_{T_n}(\varphi^{(k)}_n)\right)\left(\varphi^{(k+1)}_n - \varphi^{(k)}_n\right) = -\varphi^{(k)}_n + K(\varphi^{(k)}_n) + F, \quad \varphi^{(0)}_n \in \mathcal{X},\ k = 1, 2, \cdots. \tag{4.12}$$

The discretized solution of the exact solution $\varphi = (\varphi_1, \varphi_2) \in \mathcal{X}$ for our main problem (4.6) is represented by $\varphi^{(k)}_n = (\varphi^{(k)}_{n,1}, \varphi^{(k)}_{n,2}) \in \mathcal{X}$. Therefore, the linear problem that we solve in a discretized manner can be formulated as follows

$$\begin{cases} \text{Find } \varphi^{(k+1)}_n \in \mathcal{X}, \\ \varphi^{(k+1)}_n - M_{T_n}(\varphi^{(k)}_n)\varphi^{(k+1)}_n = b^{(k)}_n, \\ b^{(k)}_n = K(\varphi^{(k)}_n) - M_{T_n}(\varphi^{(k)}_n)\varphi^{(k)}_n + F. \end{cases} \tag{4.13}$$

As we have explained in subsection 3.2.1 of the previous Chapter 3, the equalities presented below are written based on scheme (4.13)

$$(I - \pi_n)\,\varphi^{(k+1)}_n = (I - \pi_n)\,b^{(k)}_n = (I - \pi_n)\left(K(\varphi^{(k)}_n) + F\right),$$

and for all $i, j = 1, 2$, we can simplify the representation of our projection operator (4.11) to be expressed for $v \in C([0.1],\mathbb{R})$ as

$$T^c_{n,ij}(x)v = (\pi_n T^c_{ij}(x))v = \sum_{p=1}^{n} < T^c_{ij}(x)v, e^*_p > e_p = \widetilde{e}_n \ll T^c_{ij}(x)v, \widetilde{e}_n \gg, \quad c \in \{\ , s, r\}\,.$$

where the bracket $\ll \cdot, \cdot \gg$ is well defined in (3.8).

Finally, we can give the expression of our approximate solution as follows

$$\varphi^{(k+1)}_n = (I - \pi_n)\left(K(\varphi^{(k)}_n) + F\right) + E_n Y^{(k+1)}_n, \tag{4.14}$$

where $E_n = (\widetilde{e}_n, \widetilde{e}_n)$ and $Y^{(k+1)}_n \in \mathbb{C}^{n\times 2}$ is the column vectors represent the solution of the following linear system

$$\underbrace{\left(I_2 - A_n^{(k)}\right)}_{2.n\times 2.n}\underbrace{Y_n^{(k+1)}}_{2.n\times 1} = \underbrace{B_n^{(k)}}_{2.n\times 1}, \tag{4.15}$$

where, for all $i,j = 1,2$, and all $l,p = 1,2,\cdots,n$, we set

$$[A_n^{(k)}]_{ij}(l,p) = \ll M_T(\varphi_n^{(k)})(i,j)\widetilde{e}_n, \widetilde{e}_n^* \gg (l,p) = \langle M_T(\varphi_n^{(k)})(i,j)e_p, e_l^*\rangle, \tag{4.16}$$

$$[B_n^{(k)}]_i(l) = \ll K_i(\varphi_n^{(k)}), \widetilde{e}_n^* \gg (l) - \ll (M_T(\varphi_n^{(k)})\varphi_n^{(k)})(i), \widetilde{e}_n^* \gg (l) \tag{4.17}$$

$$+ \ll f_i, \widetilde{e}_n^* \gg (l) + \ll M_T(\varphi_n^{(k)})(I-\pi_n)(K(\varphi_n^{(k)}) + F)(i), \widetilde{e}_n^* \gg (l) \tag{4.18}$$

$$= \langle K_i(\varphi_n^{(k)}), e_l^*\rangle - \langle (M_T(\varphi_n^{(k)})\varphi_n^{(k)})(i), e_l^*\rangle + \langle f_i, e_l^*\rangle \tag{4.19}$$

$$+ \langle (M_T(\varphi_n^{(k)})(I-\pi_n)(K(\varphi_n^{(k)}) + F))(i), e_l^*\rangle, \tag{4.20}$$

where $\langle \cdot,\cdot\rangle$ represent the duality brackets between the space $\mathcal{X}$ and its dual space $\mathcal{X}^*$.

4.3 Compactness of Operators with Weak Singular Kernels

For all $j = 1,2$, and $\theta = (\theta_1,\theta_2) \in \mathcal{X}$, we consider a function N_j taking the same form as the weak singular kernel of operator $T_{2j}^s(\cdot)$ as following

$$N_j : \left\{[0,1]^2 \times \mathbb{R}^2, t \neq s\right\} \longrightarrow \mathbb{R} \tag{4.21}$$

$$(t,s,\theta(s)) \longrightarrow S_{ts}\, g'(|t-s|)\frac{\partial \kappa}{\partial \theta_j}(t,s,\theta(s)). \tag{4.22}$$

We denote

$$\left\|\frac{\partial \kappa}{\partial \theta_j}\right\|_\infty = \sup_{(t,s,\theta(s))\in[0,1]^2\times\mathbb{R}^2}\left|\frac{\partial \kappa}{\partial \theta_j}(t,s,\theta(s))\right|.$$

To facilitate our discussion on the compactness of $T_{2j}^s(\cdot)$ on the Banach space $C([0,1],\mathbb{R})$, we will present some results that will be relevant for our analysis. Our first step is to introduce the function $G : [0,1] \longrightarrow \mathbb{R}$, which will have significant technical importance throughout this section

$$G(t) = \int_0^t p'(s)\, ds. \tag{4.23}$$

We further define for all $\eta > 0$ and for all $j = 1,2$,

$$\varepsilon(N_j,\eta) = \sup\left\{\int_0^1 |N_j(t,s,\theta(s)) - N_j(\tau,s,\theta(s))|\, ds, \quad (t,\tau)\in[0,1]^2,\ |t-\tau|\leq\eta\right\}.$$

The oscillation of a function $h \in C([0,1],\mathbb{R})$ with respect to $\eta > 0$ is defined by

$$\omega_\infty(h,\eta) = \sup\left\{|h(t)-h(\tau)|, \quad (t,\tau) \in [0,1]^2,\ |t-\tau| \leq \eta\right\}.$$

This definition can be generalized in a partial way to any continuous function $Q \in C([0,1]^2 \times \mathbb{R}^2, \mathbb{R})$, $\forall s \in [0,1], \forall \theta = (\theta_1, \theta_2) \in \mathcal{X}$ as follows

$$\omega_\infty(Q,\eta)(s,\theta(s)) = \sup\left\{|Q(t,s,\theta(s)) - Q(\tau,s,\theta(s))|, \quad (t,\tau) \in [0,1]^2, |t-\tau| \leq \eta\right\}.$$

Lemma 4.3.1. *Let $\eta > 0$ and $Q \in C([0,1]^2 \times \mathbb{R}^2, \mathbb{R})$. Then the function*

$$(s,\theta(s)) \in [0,1] \times \mathbb{R}^2 \longrightarrow \omega_\infty(Q,\eta)(s,\theta(s)) \in \mathbb{R},$$

is continues on $[0,1] \times \mathbb{R}^2$, and

$$\lim_{s \to 0^+} \|\omega_\infty(Q,\eta)\|_\infty = 0.$$

Proof. This follows properties of uniform continuity of Q on $[0,1]^2 \times \mathbb{R}^2$. □

Lemma 4.3.2. *Let g' verified $(\mathcal{H}_1)(ii) - (iii)$. Then*

$$\sup_{t \in [0,1]} \int_0^1 g'(|t-s|)\, ds = 2\int_0^{1/2} g'(s)\, ds < +\infty.$$

Proof. Let $y : [0,1] \longmapsto \mathbb{R}$ defined by

$$y(t) = \int_0^1 g'(|t-s|)\, ds = G(t) + G(1-t),$$

where G is the function defined by (4.23). The function y has an axial symmetry with respect to $\frac{1}{2}$ and

$$y'(t) = g'(t) - g'(1-t) \begin{cases} > 0 & \text{if } 0 < t < \frac{1}{2}, \\ < 0 & \text{if } \frac{1}{2} < t < 1, \end{cases}$$

for all $t \in [0,1]$. So

$$\sup_{t \in [0,1]} \int_0^1 g'(|t-s|)\, ds = \max_{t \in [0,1]} y(t) = y(1/2) = 2\int_0^{1/2} g'(s)\, ds,$$

and according to $(\mathcal{H}_1)(ii)$ this last integral is finite.

□

Then we define

$$\alpha = 2\int_0^{1/2} g'(s)\, ds. \tag{4.24}$$

Lemma 4.3.3. *For all $\eta > 0$ small enough*

$$\varepsilon(g', \eta) \leq 4 \int_0^\eta g'(s)\, ds \xrightarrow{s \longrightarrow 0^+} 0. \tag{4.25}$$

Proof. For $\eta > 0$, and $(t, \tau) \in [0, 1]^2$ such as $0 < |t - \tau| \leq \eta$, let's define ξ as

$$\begin{aligned} \xi : [0, 1] &\longmapsto \mathbb{R} \\ s &\longmapsto |g'(|t - s|) - g'(|\tau - s|)|. \end{aligned}$$

Let's put $\zeta = \dfrac{t + \tau}{2}$, and we can write

$$\int_0^1 \xi(s)\, ds = \int_0^t \xi(s)\, ds + 2 \int_t^\zeta \xi(s)\, ds + \int_\tau^1 \xi(s)\, ds.$$

By examining the restriction of ξ to the interval $[0, 1]$, we can observe that it has axial symmetry with respect to ζ. Consequently, we can simplify our analysis and obtain the following equalities

$$\begin{aligned} \int_0^t \xi(s)\, ds &= \int_0^t (g'(t - s) - g'(\tau - s))\, ds = G(t) + G(\tau - t) - G(\tau), \\ \int_t^\zeta \xi(s)\, ds &= \int_t^\zeta (g'(s - t) - g'(\tau - s))\, ds = G((\tau - t)/2) + G((\tau - t)/2) - G(\tau - t), \\ \int_\tau^1 \xi(s)\, ds &= \int_t^1 (g'(s - \tau) - g'(s - t))\, ds = G(1 - \tau) - G(1 - t) + G(\tau - t), \end{aligned}$$

where G is the function defined by (4.23). Thereby

$$\begin{aligned} \varepsilon(g', \eta) =& 4 \int_0^{(\tau - t)/2} g'(s)\, ds - \int_t^\tau g'(s)\, ds - \int_{1-\tau}^{1-t} g'(s)\, ds \\ \leq& 4 \int_0^{(\tau - t)/2} g'(s)\, ds \\ \leq& 4 \int_0^{\eta/2} g'(s)\, ds \\ \leq& 4 \int_0^\eta g'(s)\, ds, \end{aligned}$$

and that what we wanted to demonstrate. The assertion (4.25) follows from the integrability properties of g'.

□

Lemma 4.3.4. *Let g' verified $(\mathcal{H}_1)$. Then for all $j = 1, 2$, we have*

$$\lim_{s \to 0^+} \varepsilon(N_j, \eta) = 0.$$

Proof. For $\eta > 0$ quite small and $(t, \tau, \theta) \in [0,1]^2 \times \mathcal{X}$ such that $|t-\tau| \leq \eta$, for all $j = 1, 2$

$$\begin{aligned}\int_0^1 |N_j(t,s,\theta(s)) - N_j(\tau,s,\theta(s))|ds &= \int_0^1 |S_{ts}\, g'(|t-s|)\frac{\partial \kappa}{\partial \theta_j}(t,s,\theta(s)) - S_{\tau s}\, g'(|\tau-s|)\frac{\partial \kappa}{\partial \theta_j}(\tau,s,\theta(s))|ds \\ &\leq \int_0^1 \left|\frac{\partial \kappa}{\partial \theta_j}(t,s,\theta(s))\right| |g'(|t-s|) - g'(|\tau-s|)|\, ds \\ &\quad + \int_0^1 \left|S_{ts}\frac{\partial \kappa}{\partial \theta_j}(t,s,\theta(s)) - S_{\tau s}\frac{\partial \kappa}{\partial \theta_j}(\tau,s,\theta(s))\right| |g'(|\tau-s|)|\, ds \\ &\leq \sup_{s\in[0,1]} \left|\frac{\partial \kappa}{\partial \theta_j}(t,s,\theta(s))\right| \int_0^1 |g'(|t-s|) - g'(|\tau-s|)|\, ds \\ &\quad + \sup_{s\in[0,1]} \left|S_{ts}\frac{\partial \kappa}{\partial \theta_j}(t,s,\theta(s)) - S_{\tau s}\frac{\partial \kappa}{\partial \theta_j}(\tau,s,\theta(s))\right| \int_0^1 g'(|\tau-s|)ds.\end{aligned}$$

Thus, according to the lemma (4.3.2)

$$\varepsilon(N_j, \eta) \leq \left\|\frac{\partial \kappa}{\partial \theta_j}\right\|_\infty \varepsilon(g', \eta) + \alpha \left\|\omega_\infty\left(S_{..}\frac{\partial \kappa}{\partial \theta_j}, \eta\right)\right\|_\infty,$$

where α is defined by (4.24). Finlay, using lemmas (4.3.1) – (4.3.3) to get that

$$\lim_{s\to 0^+} \varepsilon(N_j, \eta) = 0.$$

□

Theorem 4.3.1. *Let $T_{2j}^s(\cdot)$ defined by (4.7) – (4.21) and (4.22), for all $j = 1, 2$, and let $(\mathcal{H}_1)$ holds. Then $T_{2j}^s(\cdot)$ is a linear compact operator on $C([0,1], \mathbb{R})$.*

Proof. Let $v \in C([0,1], \mathbb{R})$ and $\epsilon > 0$. So, according to lemma (4.3.4), there exists $\eta > 0$ such that $\varepsilon(N_j, \eta) < \dfrac{\epsilon}{\|v\|_\infty}$. Let $t \in [0,1]$. Then, for $\tau \in [0,1]$ such as $|t-\tau| < \eta$, for all $j = 1, 2$, and for all $\theta \in \mathcal{X}$,

$$\begin{aligned}\left|[T_{2j}^s(\theta).v](t) - [T_{2j}^s(\theta).v](\tau)\right| &\leq \int_0^1 \left|S_{ts}\, g'(|t-s|)\frac{\partial \kappa}{\partial \theta_j}(t,s,\theta(s)) - S_{\tau s}\, g'(|\tau-s|)\frac{\partial \kappa}{\partial \theta_j}(\tau,s,\theta(s))\right| |v(s)|ds \\ &\leq \|v\|_\infty\, \varepsilon(N_j, \eta) < \epsilon.\end{aligned}$$

Also, for all $v \in C([0,1], \mathbb{R})$ and for $\theta \in \mathcal{X}$, we have $T_{2j}^s(\theta).v$ is continuous at all points of $[0,1]$, which proves that $C([0,1], \mathbb{R})$ is invariant by $T_{2j}^s(\theta)$.

Let $v \in C([0,1], \mathbb{R})$ such that $\|v\|_\infty \leq 1$. Then for $t \in [0,1]$,

$$\left|[T^s_{2j}(\theta).v](t)\right| \leq \int_0^1 \left|\frac{\partial\kappa}{\partial\theta_j}(t,s,\theta(s))\right| g'(|t-s|)|v(s)|ds$$
$$\leq \|v\|_\infty \sup_{s\in[0,1]} \left|\frac{\partial\kappa}{\partial\theta_j}(t,s,\theta(s))\right| \int_0^1 g'(|t-s|)ds.$$

So. according the lemma (4.3.2),

$$\left\|T^s_{2j}(\theta)\right\|_\infty \leq 2\left|\frac{\partial\kappa}{\partial\theta_j}(t,s,\theta(s))\right| \int_0^{1/2} g'(|t-s|)ds = \alpha\left\|\frac{\partial\kappa}{\partial\theta_j}\right\|_\infty.$$

Thereby, for all $j=1,2$, for all $\theta\in\mathcal{X}$, $T^s_{2j}(\theta)$ is bounded from $C([0,1],\mathbb{R})$ into itself.

The compactness is deduced by considering the truncated function g'_n defined by

$$g'_n(s) = \begin{cases} g'(1/n) & if \quad 0\leq s<1/n, \\ g'(s) & else. \end{cases}$$

For all $j=1,2$, for all $\theta\in\mathcal{X}$, we define the integral operator $T^s_{n,2j}(\cdot)$ as

$$[T^s_{n,2j}(\theta).v](t) = \int_0^1 S_{ts}\, g'_n(|t-s|)\frac{\partial\kappa}{\partial\theta_j}(t,s,\theta(s))\, v(s)\, ds,$$

and it is compact from $C([0,1],\mathbb{R})$ into itself, because its kernel is a continuous function on $[0,1]^2\times\mathcal{X}$. If $v\in C([0,1],\mathbb{R})$ and $\theta\in\mathcal{X}$, then

$$\begin{aligned}\left[\left(T^s_{2j}(\theta)-T^s_{n,2j}(\theta)\right).v\right](t) &= \int_{t-1/n}^{t} [g'(|t-s|)-g'(1/n)]\, S_{ts}\frac{\partial\kappa}{\partial\theta_j}(t,s,\theta(s))\, v(s)\, ds \\ &\quad+ \int_t^{t+1/n} [g'(|t-s|)-g'(1/n)]\, S_{ts}\frac{\partial\kappa}{\partial\theta_j}(t,s,\theta(s))\, v(s)\, ds \\ &= \int_0^{1/n} [g'(s)-g'(1/n)]\, S_{t(t-s)}\frac{\partial\kappa}{\partial\theta_j}(t,t-s,\theta(s))\, v(t-s)\, ds \\ &\quad+ \int_0^{1/n} [g'(s)-g'(1/n)]\, S_{t(t+s)}\frac{\partial\kappa}{\partial\theta_j}(t,t+s,\theta(s))\, v(t+s)\, ds.\end{aligned}$$

So. for all $v\in C([0,1],\mathbb{R})$ and $\theta\in\mathcal{X}$

$$\left\|T^s_{2j}(\theta)-T^s_{n,2j}(\theta)\right\|_\infty \leq 4\left\|\frac{\partial\kappa}{\partial\theta_j}\right\|_\infty \int_0^{1/n} g'(s)\, ds \xrightarrow{n\to+\infty} 0.$$

Finally, for all $j=1,2$, the operators $T^s_{2j}(\cdot)$ are the uniform limits of a sequences of compacts operators.

□

4.4 Convergence Analysis of The (LN.DK) Process

Now, we can consider that all the operators of the matrix $M_T(\cdot)$ are a compact operators. In this work, we suppose that $(\pi_n)_{n\in\mathbb{N}^*}$ is punctually convergent to the identity operator in the Banach space $\mathcal{X}$ on which $M_T(\cdot)$ is defined. So, the approximate operators matrix $M_{T_n}(\cdot)$ converge uniformly to the operators matrix $M_T(\cdot)$ (see ([3]) Theorem 4.1, pp.186); ie, For n large enough

$$(\mathcal{H}_3): \ \exists \gamma_n > 0, \forall \phi \in \mathcal{X}, \quad \|M_{T_n}(\phi) - M_T(\phi)\| \leq \gamma_n \xrightarrow{n\to\infty} 0.$$

Lemma 4.4.1. *Assume that* $(\mathcal{H}_2) - (\mathcal{H}_3)$ *holds, and let* $r = \min\left\{\nu, \frac{1}{2\ell\mu}\right\}$. *Then for all* $\phi \in B_r(\varphi)$ *the operator* $(I_{22} - M_T(\phi))$ *is invertible such that*

$$\left\|(I_{22} - M_T(\phi))^{-1}\right\| \leq 2\mu.$$

The proof of this lemma can be constructed in a similar manner as that of Lemma 2.2.3.

Lemma 4.4.2. *We assume that the aforementioned conditions in* $(\mathcal{H}_2)$ *are fulfilled. Then for* n *big enough, and all* $\phi \in B_r(\varphi)$, *the approximate operator* $(I_{22} - M(\phi))$ *is invertible, and there exists* $\epsilon_n \in]0,1]$ *such that,*

$$\sup_{\phi\in B_r(\varphi)} \left\|I_{22} - (I_{22} - M_{T_n}(\phi))^{-1}(I_{22} - M_T(\phi))\right\| \leq \epsilon_n,$$

$$\sup_{\phi\in B_r(\varphi)} \left\|(I_{22} - M_{T_n}(\phi))^{-1}\right\| \leq 2\mu(1+\epsilon_n).$$

The proof of this lemma can be constructed in a way similar to how Lemma 2.2.4 was proved.

Theorem 4.4.1. *(Convergence Theorem)*
If the initial function $\varphi_n^{(0)} \in B_{\varrho_n}(\varphi)$, *for* $n \in \mathbb{N}^*$. *Then the sequence* $(\varphi_n^{(k)})_{k\geq 0}$ *defined by the scheme* (4.13), *converges to* φ *the exact solution of problem* (4.1), *such that*

$$\left\|\varphi_n^{(k)} - \varphi\right\|_{\mathcal{X}} \leq \varrho_n \left(\frac{1+\epsilon_n}{2}\right)^k \xrightarrow{k\to\infty} 0,$$

where

$$\varrho_n = \min\left\{r, \left(\frac{1-\epsilon_n}{2\ell\mu(1+\epsilon_n)}\right)\right\}.$$

To prove this theorem, we can follow the same steps as in the proof of Theorem 2.3.1, but with the values of μ, ℓ, and ϵ_n set to η, λ_R, and δ_n, respectively.

4.4.1 Integrals Approximation

In this part, we will define wish approximate methods we are using to estimate integrals represented in the interpolation formula of our approximate iterate solution in (4.14) and in the linear system characterized by (4.15). For integrals that have a weak singular kernel, we must use the product integration rule to remove the singularity (*see*[6], [50]). For integrals with a regular kernel, we use the Nyström method (*see*[7], [44]).

First, we define for $N \gg n$ in $\mathbb{N}^*$ an equidistance subdivision Ω_N as follows:

$$\Omega_N = \left\{ t_r = (r-1)h, h = \frac{1}{N-1}, r = 1, 2, \cdots, N \right\}, \tag{4.26}$$

and the Nyström approximate integration formula on the previous subdivision Ω_N is given by

$$\int_0^1 G(t, s, \theta(s))ds = \sum_{r=1}^{N} q_r G(t, t_r, \theta(t_r)),$$

where q_r are reals and there exists $M < +\infty$, such that

$$\max_{N \geq 1} \sum_{r=1}^{N} |q_r| < M.$$

What is more, the integral product method consists of interpolating the regular terms of the kernel κ and his derivative $\dfrac{\partial \kappa}{\partial t}$ on the subdivision Ω_N using the piecewise linear functions in every subinterval $[t_r, t_{r+1}]$. So, for all $i = 1, 2, \cdots, n$ or $i = 1, 2, \cdots, N$ depend wish situation we have in our equations (4.20) and for all $\phi \in \mathcal{X}$, we have

$$\begin{aligned} P_{N,1}\left[\kappa\right](t_i, s, \phi(s)) \simeq &\left(\frac{s-t_r}{h}\right)\kappa(t_i, t_{r+1}, \phi(t_{r+1})) \\ &+ \left(\frac{t_{r+1}-s}{h}\right)\kappa(t_i, t_r, \phi(t_r)), \quad s \in [t_r, t_{r+1}], \end{aligned}$$

$$\begin{aligned} P_{N,1}\left[\frac{\partial \kappa}{\partial t}\right](t_i, s, \phi(s)) \simeq &\left(\frac{s-t_r}{h}\right)\frac{\partial \kappa}{\partial t}(t_i, t_{r+1}, \phi(t_{r+1})) \\ &+ \left(\frac{t_{r+1}-s}{h}\right)\frac{\partial \kappa}{\partial t}(t_i, t_r, \phi(t_r)), \quad s \in [t_r, t_{r+1}]. \end{aligned}$$

Then, for all $i = 1, 2, \cdots, n$ or $i = 1, 2, \cdots, N$, we approximate integrals with weak singular kernel $p(\cdot)G(\cdot, \cdot, \phi(\cdot))$ as follows

$$\int_0^1 p(|t_i - s|)G(t_i, s, \phi(s))ds = \sum_{r=1}^{N} \vartheta_r G(t_i, t_r, \phi(t_r)),$$

where ϑ_r are given by:

$$\vartheta_1 = \frac{1}{h}\int_0^{t_1}(t_1-s)sign(t_i-s)p(|t_i-s|)ds,$$
$$\vartheta_r = \frac{1}{h}\left(\int_{t_{r-1}}^{t_r}(s-t_{r-1})sign(t_i-s)p(|t_i-s|)ds + \int_{t_r}^{t_{r+1}}(t_{r+1}-s)sign(t_i-s)p(|t_i-s|)ds\right),$$
$$\vartheta_N = \frac{1}{h}\int_{t_{N-1}}^{1}(s-t_{N-1})sign(t_i-s)p(|t_i-s|)ds.$$

4.5 Numerical Examples

In this section, we will show the efficacy of our (LN.DK) new process to solve systems of nonlinear Fredholm integro-differential weak singular equations by applying it to solve two examples. All results got are represented in the Tables 4.1 and 4.2 below.

Let $\varphi_n^{(k)} = (\varphi_{n,1}^{(k)}, \varphi_{n,2}^{(k)}) \in \mathcal{X}$ be the k order approximative solution obtained by solving the discretized linear problem (4.13) applying our new process and $\varphi = (\varphi_1, \varphi_2) \in \mathcal{X}$ our exact solution of the main problem (4.6). The stopping condition on the parameter k is defined as:

$$E_{n,N}^k = \max_{1\le p\le N}\left(\left|\varphi_{n,1}^{(k+1)}(t_p) - \varphi_{n,1}^{(k)}(t_p)\right| + \left|\varphi_{n,2}^{(k+1)}(t_p) - \varphi_{n,2}^{(k)}(t_p)\right|\right) \le 10^{-12}.$$

The estimate error obtained by applying our process is denoted by:

$$E_{n,N} = \max_{1\le p\le N}\left(\left|\varphi_1(t_p) - \varphi_{n,1}^{(k)}(t_p)\right| + \left|\varphi_2(t_p) - \varphi_{n,2}^{(k)}(t_p)\right|\right),$$

where for all $p = 1, 2, \cdots, N$, the points t_p are taking from the subdivision Ω_N presented in (4.26).

Example 4.5.1. *We take the following nonlinear integro-differential equation:*

$$\varphi(t) = \frac{1}{20}\int_0^1 \sqrt{|t-s|}\sin\left(e^s + \arcsin\left(\frac{s+t}{3}\right) + \varphi(s) - \varphi'(s)\right)ds + f(t), \quad t\in[0,1], \tag{4.27}$$

and the function f and the exact solution φ are given by:

$$f(t) = te^t - \frac{(7t+3)(1-t)^{\frac{3}{2}} + 7t^{\frac{5}{2}}}{450}, \qquad \varphi(t) = te^t.$$

To get more information about our solution, we derive equation (4.27) and according the same notation defined before in the construction of our main system (4.3), we get the following system of nonlinear integro-differential equations:

(LN.DK) Algorithm
Data: n, N, g, φ
Result: $E_{n,N}$, plot $(\varphi_n^{(k+1)}, \varphi, \log\left(E^k_{n,N}\right))$
Initialization:
$\varphi_n^{(0)} \longleftarrow 0, M_n^{(0)} \longleftarrow 0_{n\times n}, B_n^{(0)} \longleftarrow 0_n, k \longleftarrow 1, Tol \longleftarrow 10^{-12}, E^k_{n,N} \longleftarrow 1$
$h_n \longleftarrow 1/(n-1)$
$T_n \longleftarrow (0 : h_n : 1)^t$
while $E^k_{n,N} > Tol$ **do**
  for $i \longleftarrow 1$ **to** n **do**
    for $j \longleftarrow 1$ **to** n **do**
      Calculate and save $M_n^{(k)}(i,j)$
      if i=j **then**
        $A_n^{(k)} \longleftarrow 1 - M_n^{(k)}(i,j);$
      else
        $A_n^{(k)} \longleftarrow -M_n^{(k)}(i,j);$
      end
    end
    Calculate and save $B_n^{(k)}(i);$
  end
  $X_n^{(k+1)} \longleftarrow (A_n^{(k)})^{-1}.B_n^{(k)}$ (Calculate and save the vector solution of the linear system)
  $\pi_n K(t) \longleftarrow 0_{1\times 2}, \pi_n g \longleftarrow 0_{1\times 2}, e_n X_n^{(k+1)}(t) \longleftarrow 0_{1\times 2}$
  for $i \longleftarrow 1$ **to** 2 **do**
    for $p \longleftarrow 1$ **to** n **do**
      Calculate the vector basis $e_p(t)$
      $\pi_n K_i(t) \longleftarrow \pi_n K_i(t) + K_i(t_n(p))\,.\,e_p(t)$
      $\pi_n g_i(t) \longleftarrow \pi_n g_i(t) + g_i(t_n(p))\,.\,e_p(t)$
      $e_n X_{n,i}^{(k+1)}(t) \longleftarrow e_n X_{n,i}^{(k+1)}(t) + X_{n,i}^{(k+1)}(p)\,.\,e_p(t)$
    end
  end
  for $i \longleftarrow 1$ **to** 2 **do**
    $\varphi_{n,i}^{(k+1)}(t) \longleftarrow K_i(t) + g_i(t) - \pi_n K_i(t) - \pi_n g_i(t) + e_n X_{n,i}^{(k+1)}(t)$
    $\varphi_{n,i}^{(k+1)}(t) \longleftarrow \varphi_{n,i}^{(k)}(t)$ (Save the previous iterate solution, k = 1,2 $\cdots$)
  end
  $E^k_{n,N}(t) \longleftarrow \max\limits_{t\in[0,1]} \left(\left\|\varphi_{n,1}^{(k+1)}(t) - \varphi_{n,1}^{(k)}(t))\right\| + \left\|\varphi_{n,2}^{(k+1)}(t) - \varphi_{n,2}^{(k)}(t))\right\|\right)$(Calculate the iterate error)
  $k \longleftarrow k+1$ (Increment the number of iterations k by 1)
end
$E_{n,N} \longleftarrow \max\limits_{t\in[0,1]} \left(\left\|\varphi_{n,1}^{(k+1)}(t) - \varphi_1(t)\right\| + \left\|\varphi_{n,2}^{(k+1)}(t) - \varphi_2(t)\right\|\right)$ (Calculate the error between the exact solution and the last iterate solution).

$$\begin{cases} \varphi_1(t) = & \frac{1}{20}\int_0^1 \sqrt{|t-s|}\sin\left(e^s+\arcsin\left(\frac{s+t}{3}\right)+\varphi_1(s)-\varphi_2(s)\right)ds+f_1(t) \\ \varphi_2(t) = & \frac{1}{20}\int_0^1 \frac{sign(t-s)}{2\sqrt{|t-s|}}\sin\left(e^s+\arcsin\left(\frac{s+t}{3}\right)+\varphi_1(s)-\varphi_2(s)\right)ds \\ & +\frac{1}{60}\int_0^1 \frac{\sqrt{|t-s|}}{\sqrt{1-(\frac{s+t}{3})^2}}\cos\left(e^s+\arcsin\left(\frac{s+t}{3}\right)+\varphi_1(s)-\varphi_2(s)\right)ds+f_2(t). \end{cases}$$

We solve this system with our (LN.DK) new strategy and the results obtained are presented in Table 4.1.

Example 4.5.2. *Consider the following nonlinear integro-differential equation:*

$$\varphi(t) = \frac{1}{10}\int_0^1 \sqrt{|t-s|}\frac{ts^2\left(1+e^{-\sin(4s)}+e^{-4\cos(4s)}\right)}{1+e^{-\varphi(s)}+e^{-\varphi'(s)}}ds+f(t), \quad t\in[0,1], \tag{4.28}$$

where the function f and the exact solution φ are given by:

$$f(t) = \sin(4t) - \frac{(1-t)^{3/2}\left(8t^3+12t^2+15t\right)+8t^{9/2}}{525}, \qquad \varphi(t) = \sin(4t).$$

Using the same technique do before to construct the following system of equations:

$$\begin{cases} \varphi_1(t) = & \frac{1}{10}\int_0^1 \sqrt{|t-s|}\frac{ts^2\left(1+e^{-\sin(4s)}+e^{-4\cos(4s)}\right)}{1+e^{-\varphi_1(s)}+e^{-\varphi_2(s)}}ds+f_1(t) \\ \varphi_2(t) = & \frac{1}{10}\int_0^1 \frac{sign(t-s)}{2\sqrt{|t-s|}}\frac{s^2\left(1+e^{-\sin(4s)}+e^{-4\cos(4s)}\right)}{1+e^{-\varphi_1(s)}+e^{-\varphi_2(s)}}ds \\ & +\frac{1}{10}\int_0^1 \sqrt{|t-s|}\frac{s^2\left(1+e^{-\sin(4s)}+e^{-4\cos(4s)}\right)}{1+e^{-\varphi_1(s)}+e^{-\varphi_2(s)}}ds+f_2(t). \end{cases}$$

We solve this system with our (LN.DK) new strategy and the results obtained are presented in Table 4.2.

Results obtained on the both examples and represented in Tables 4.1 and 4.2 are confirm the efficacy of our new strategy to solve this genre of weak singular problems. By choosing $n = 5$ (the number of vectors basis taken for the discretization phase using the Kantorovich projection) and by increasing N (the number of nodes in the subdivision defined for the approximation of integrals in our problem) we remark that the estimated error $E_{n,N}$ is rapidly declining towards 0. However, in Figures 4.1 and 4.3 we see clearly that our estimate solutions obtained using the (LN.DK) new process converge to the exact solutions, and the logarithm of the iterate stopping condition $Log_{10}(E^k_{n,N})$ in Figures 4.2 and 4.4, prove that the (LN.DK) modern strategy has a linear convergence, which confirm that our method is reasonable. The Algorithm using for programming the resolution process of both numerical examples is given by the (LN.DK) Algorithm.

N	The error $E_{n,N}$	CPU time
10	6.3633 e-04	4.2122 e-02 s
30	1.1846 e-04	8.3219 e-02 s
50	5.5029 e-05	1.4618 e-01 s
100	1.9571 e-05	1.8753 e-01 s
500	1.7862 e-06	4.2764 e-00 s
1000	6.3642 e-07	7.1795 e-00 s

Table 4.1: Numerical results of Example 4.5.1 by fixing $n = 5$ and increasing N to 1000.

N	The error $E_{n,N}$	CPU time
10	1.8124 e-03	1.3581 e-01 s
30	4.2009 e-04	1.7794 e-00 s
50	2.0258 e-04	2.0515 e01 s
100	7.3898 e-05	1.5653 e02 s
500	7.0205 e-06	3.9812 e03 s
1000	2.5260 e-06	1.2471 e04 s

Table 4.2: Numerical results of Example 4.5.1 by fixing $n = 5$ and increasing N to 1000.

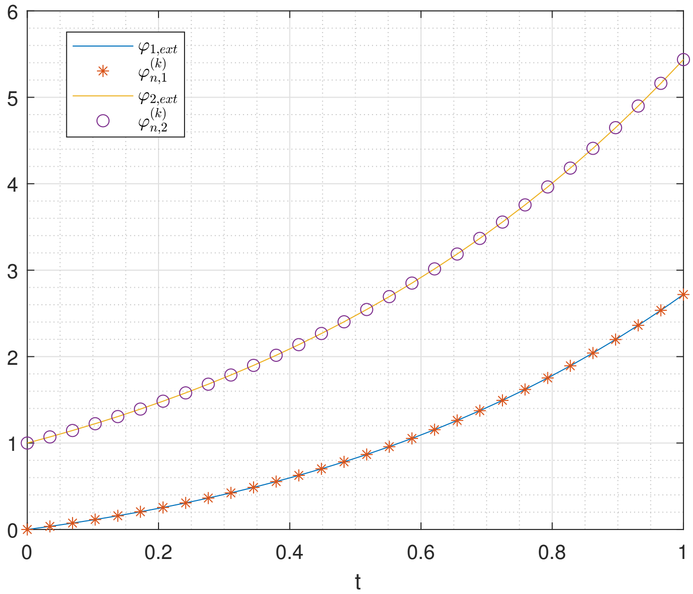

Figure 4.1: Exacts and approximates solutions of Example 4.5.1, applying (L.D) Method.

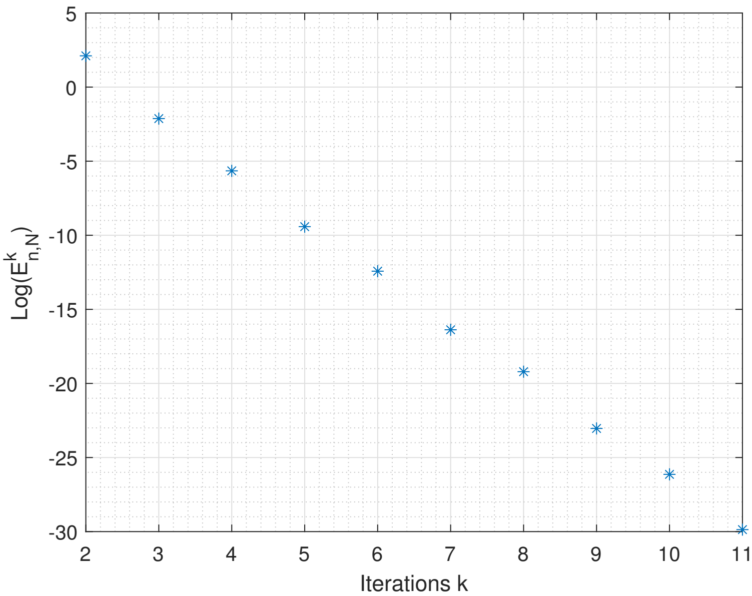

Figure 4.2: Graph of $Log_{10}(E^k_{n,N})$, the error between two successive iterates approximations $\varphi_n^{(k+1)}$ and $\varphi_n^{(k)}$ of the first system. (Example 4.5.1)

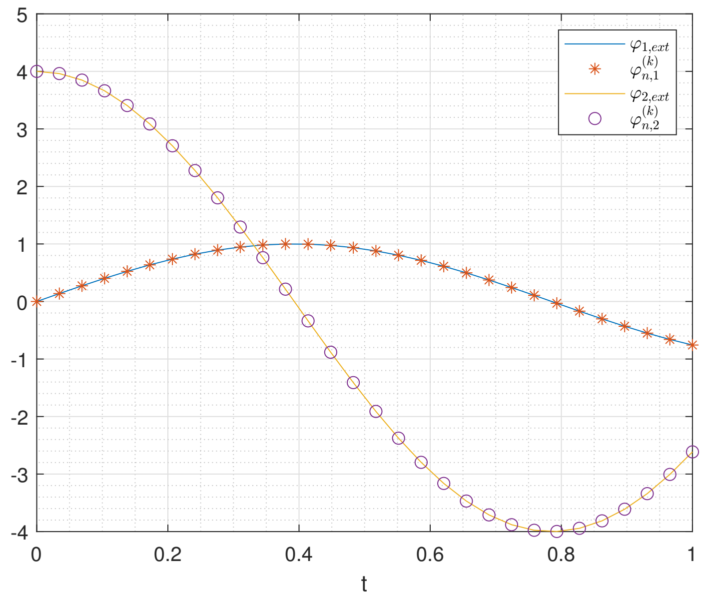

Figure 4.3: Exacts and approximates solutions of Example 4.5.2, applying (L.D) Method.

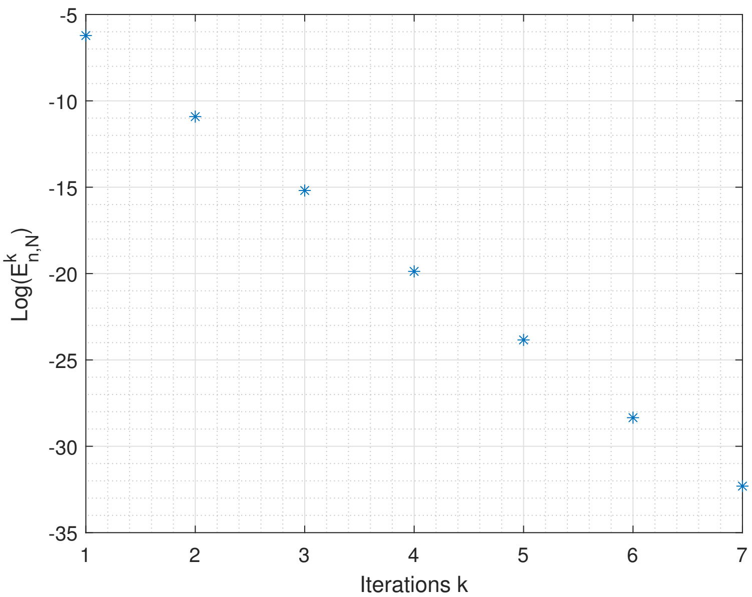

Figure 4.4: Graph of $Log_{10}(E^k_{n,N})$, the error between two successive iterates approximations $\varphi_n^{(k+1)}$ and $\varphi_n^{(k)}$ of the second system. (Example 4.5.2)

Conclusion

As systems of nonlinear integral equations are essential in many scientific fields, we focused our thesis on studying a specific type: the system of nonlinear Fredholm integral equations of the second kind, which we previously emphasized as highly significant. Our aim was to examine the various forms that these systems can assume.

In the first, we have constructed a Linearization-Discretization process for solving a system of nonlinear Fredholm integral equations defined in an infinite dimensional context. As well as, we have proposed the necessary conditions which guarantee the convergence analysis of this new process. However, the numerical tests show that our new process should be preferred to the classical method. The reason for this behavior is obviously that the sequence φ_n^k constructed by using the (LN.DN) new process converges to the exact solution φ. On the contrary, the sequence ϑ_n^k constructed using the (DN.LN) classical process converges to ϑ_n, which is just the solution of the discretized problem obtained by the Nyström method.

Secondly, we presents a new numerical method for solving nonlinear functional equations, specifically second-order systems of nonlinear Fredholm integro-differential equations. The (L.D.D) method involves a two-step process of linearization and double discretization, and has proven to be effective and accurate in several numerical experiments.

Lastly, the fourth chapter presents a novel algorithm (LN.DK) for solving a particular type of nonlinear integro-differential Fredholm equations with a weak singular kernel. The algorithm combines Newton's iterative process for linearization and Kantorovich's projection method for discretization. The discretized linear scheme is then approximated using the product integration method for weak singular terms and the Nyström method for regular integrals. The convergence process of the (LN.DK) algorithm is subject to predefined and necessary conditions. The practical examples demonstrate the effectiveness of this algorithm in solving systems of weakly singular nonlinear integro-differential equations.

Our goal for future works is to utilize our new Linearisation-Discretization process to address second-kind nonlinear Fredholm integral equations defined over a large interval. The Newton method's fast convergence rate for the linearisation process and the Kantorovich projection's high accuracy in the discretisation process will enable us to handle the challenging data capacity imposed by the equation's large interval.

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