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Filière : **Mathématique**

Présentée par

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Intitulée

Existence and stability for a system of nonlinear damped wave equations

Soutenue le : 04/06/2023…………..... Devant le Jury composé de :

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Abstract

The present thesis is devoted to study the existence, uniqueness and asymptotic behaviour in time of solution for damped systems. This work consists of four chapters. In chapter 1, we recall some fundamental inequalities. In chapter 2, we consider a very important problem from the point of view of application in science and engineering. A system of three wave equations having a different damping effects in an unbounded domain with strong external forces. Using the Faedo-Galerkin method and some energy estimations, we will prove the existence of global solution in $\mathbb{R}^n$ owing to the weighted function. By imposing a new appropriate conditions, which are not used in the literature, with the help of some special estimations and generalized Poincaré's inequality, we obtain an unusual decay rate for the energy function. In chapter 3, we will be concerned with a problem for m-nonlinear viscoelastic wave equations, under suitable conditions we show the effect of weak and strong damping terms on decay rate for systems of nonlinear m-wave equations in viscoelasticity. In chapter 4, we consider Petrovsky-Petrovsky coupled system with nonlinear strong damping. We prove, under some appropriate assumptions, that this system is stable. Furthermore, we use the multiplier method and some general weighted integral inequalities to obtain decay properties of solution.

Keywords and phrases: Viscoelastic wave equation, Strong nonlinear system, Global solution, Faedo-Galerkin approximation, Decay rate, Blow up, Strong damping, Petrovsky-Petrovsky.

AMS Subject Classification: 35L05, 58J45, 35L80, 35B40, 35L20, 58G16, 35B40, 35L70.

Résumé

La présente thèse est consacrée à l'étude de l'existence, l'unicité et le comportement asymptotique en temps de la solution pour quelques systèmes amortis. Cette these se compose de quatre chapitres. Au chapitre 1, nous rappelons quelques résultats et inégalités fondamentales. Dans le chapitre 2, nous considérons un problème trés important du point de vue de l'application en sciences et en ingénierie. Un système de trois équations d'onde ayant des effets d'amortissement différents dans un domaine illimité avec une force externe. En utilisant la méthode de Faedo-Galerkin et quelques estimations d'énergie, nous prouverons l'existence d'une solution globale dans $\mathbb{R}^n$ grace à la fonction pondérée. En imposant de nouvelles conditions appropriées, qui ne sont pas utilisées dans la littérature, à l'aide de quelques estimations spéciales et de l'inégalité de Poincaré généralisée, nous obtenons un taux de décroissance inhabituel pour la fonction énergétique. Dans le chapitre 3, nous traiterons un système de m equations d'onde en viscoélastique non linéaire avec un amortissement et des termes sources, dans des conditions appropriées, nous prouvons un résultat d'explosion/croissance des solutions. Dans le chapitre 4, on considère un système couplé d'équations de Petrovsky-Petrovsky avec des termes dissipatifs non linéaires. Nous prouvons, sous certaines hypothèses appropriées, que ce système est stable. De plus, nous utilisons la méthode du multiplicateur et certaines inégalités intégrales pondérées générales pour obtenir les propriétés de décroissance de la solution.

Mots-clés et phrases : Équation d'onde viscoélastique, Système non linéaire fort, Solution globale, Approximation de Faedo-Galerkin, Taux de décroissance, Blow up, Fort amortissement, Petrovsky-Petrovsky.

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Contents

Introduction

Stabilization of evolution problems

Problems of global existence and stability in time of Partial Differential Equations are subject, recently, of many works. In this thesis we are interested in the study of the global existence and the stabilization of some evolution equations. The purpose of the stabilization is to attenuate the vibrations by feedback, thus consists in guaranteeing the decrease of energy of the solutions to 0 in a more or less fast way by a mechanism of dissipation.

More precisely, the problem of stabilization consists in determining the asymptotic behavior of the energy by $E(t)$, to study its limits in order to determine if this limit is null or not and if this limit is null, to give an estimations of the decay rate of the energy to zero.

This problem has been studied by many authors for various systems. They are several type of stabilization,

(1) Strong stabilization: $E(t) \longrightarrow 0$, as $t \longrightarrow \infty$.

(2) Logarithmic stabilization: $E(t) \leq c(\log t)^{-\delta}, \forall t > 0, (c, \delta > 0)$.

(3) Polynomial stabilization: $E(t) \leq ct^{-\delta}, \forall t > 0, (c, \delta > 0)$.

(4) Exponential stabilization: $E(t) \leq ce^{-\delta t}, \forall t > 0, (c, \delta > 0)$.

For wave equation with dissipation of the form

$$u'' - \Delta_x u + g\left(u'\right) = 0,$$

stabilization problems have been investigated by many authors:

When $g : \mathbb{R} \to \mathbb{R}$ is continuous and increasing function such that $g(0) = 0$, global existence of solutions is known for all initial conditions (u_0, u_1) given in $H_0^1(\Omega) \times L^2(\Omega)$.This result is, for a consequence of the general theory of nonlinear semi-groups of contractions generated by a maximal monotone operator.

Moreover, if we impose on the control the condition, $\forall\lambda \neq 0$, $g(\lambda) \neq 0$, then strong asymptotic stability of solutions occurs in $H_0^1(\Omega) \times L^2(\Omega)$, *i.e.*,

$$\left(u, u^{'}\right) \to (0,0) \quad \text{strongly in } H_0^1(\Omega) \times L^2(\Omega),$$

Without speed of convergence. These results follow, from the invariance principle of Lasalle. If the solution goes to 0 as time goes to ∞, how to get energy decay rates?

Dafermos has written in 1978 "Another advantage of this approach is that it is so simple that it requires only quite weak assumptions on the dissipative mechanism. The corresponding drawback is that the deduced information is also weak, never yielding, for example, decay rates of solutions."

Many authors have worked since then on energy decay rates. First results were obtained for linear stabilization, then for polynomial stabilization (see A. Haraux [14], V. Komornik [19], and E. Zuazua [12]) and then extended to arbitrary growing feedbacks (close to 0). In the same time, geometrical aspects were considered.

By combining the multiplier method with the techniques of micro-local analysis, Lasiecka et *al* [11, 16], have investigated different dissipative systems of partial differential equations (with Dirichlet and Neumann boundary conditions) under general geometrical conditions with nonlinear feedback without any growth restrictions near the origin or at infinity. The computation of decay rates is reduced to solving an appropriate explicitly given ordinary differential equation of monotone type. More precisely, the following explicit decay estimate of the energy is obtained:

$$E(t) \leq h\left(\frac{t}{t_0} - 1\right), \forall t \geq 0.$$

where $t_0 > 0$ and h is the solution of the following differential equation:

$$h^{'}(t) + q(h(t)) = 0, t \geq 0 \text{ and } h(0) = E(0),$$

and the function q is determined entirely from the behavior at the origin of the nonlinear feedback by proving that E satisfies

$$(Id - q)^{-1}(E((m+1)t_0)) \leq E(mt_0), \ \forall m \in \mathbb{N}.$$

System of nonlinear wave equations

To enrich this topic, it is necessary to talk about previous works regarding the nonlinear coupled system of wave equations, from a qualitative and quantitative study.Let us beginning with the single wave equation treated in [22], where the aim goal was mainely on the system

$$\begin{cases} u_{tt} + \mu u_t - \Delta u - \omega \Delta u_t = u \ln |u|, \ (x,t) \in \Omega \times (0, \infty) \\ u(x,t) = 0, x \in \partial\Omega, t \geq 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \end{cases} \tag{0.0.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$ with a smooth boundary $\partial\Omega$. The author firstly constructed a local existence of weak solution by using contraction mapping principle and of course showed the global existence, decay rate and infinite time blow up of the solution with condition on initial energy.

In $m-$equations, paper in [7] considered a system

$$u_{itt} + \gamma u_{it} - \Delta u_i + u_i = \sum_{i,j=1, i \neq j}^{m} |u_j|^{p_j} |u_i|^{p_i} u_i, \ i = 1, 2, \ldots, m, \tag{0.0.2}$$

where the absence of global solutions with positive initial energy was investigated. Next, a nonexistence of global solutions for system of three semilinear hyperbolic equations was introduced in [5]. A coupled system semilinear hyperbolic equations was investigated by many authors and a different results were obtained with the nonlinearities in the form $f_1 = |u|^{p-1}|v|^{q+1}u, f_2 = |v|^{p-1}|u|^{q+1}v$.

In the case of non-bounded domain $\mathbb{R}^n$, we mention the paper recently published by T. Miyasita and Kh. Zennir in [35], where the considered equation as follows

$$u_{tt} + a u_t - \phi(x) \Delta \left(u + \omega u_t - \int_0^t g(t-s) u(s)\, ds \right) = u|u|^{p-1}, \tag{0.0.3}$$

with initial data

$$\begin{cases} u(x,0) = u_0(x), \\ u_t(x,0) = u_1(x). \end{cases} \tag{0.0.4}$$

The authors was successful in highlighting the existence of unique local solution and they continued to extend it to be global in time. The rate of the decay for solution was the main result by considering the relaxation function is strictly convex, for more results related to decay rate of solution of this type of problems, please see [18, 28, 34].

Regarding the study of the coupled system of two nonlinear wave equations, it is worth recalling some of the work recently published. Baowei Feng and *al.* considered in [**?**], a coupled system for viscoelastic wave equations with nonlinear sources in bounded domain ($(x,t) \in \Omega \times (0,\infty)$) with smooth boundary as follows

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + u_t = f_1(u,v) \\ \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(s)\,ds + v_t = f_2(u,v). \end{cases} \tag{0.0.5}$$

Here, the authors concerned with a system in $\mathbb{R}^n (n = 1, 2, 3)$. Under appropriate hypotheses, they established a general decay result by multiplication techniques to extends some existing results for a single equation to the case of a coupled system.

It is worth noting here that there are several studies in this field and we particularly refer to the generalization that Shun and *all.* made in studying a complicate non-linear case with degenerate damping term in [37]. The IBVP for a system of nonlinear viscoelastic wave equations in a bounded domain was considered in the problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + (|u|^k + |v|^q)|u_t|^{m-1}u_t = f_1(u,v), \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(s)\,ds + (|v|^\theta + |u|^\rho)|v_t|^{r-1}v_t = f_2(u,v), \\ u(x,t) = v(x,t) = 0, x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x) \\ u_t(x,0) = u_1(x), v_t(x,0) = v_1(x), \end{cases} \tag{0.0.6}$$

where Ω is a bounded domain with a smooth boundary. Given certain conditions on the kernel functions, degenerate damping and nonlinear source terms, they got a decay rate of the energy function for some initial data.

Chapter 1

Preliminary

1- Continuous function spaces

2- L^p Spaces

3- Sobolev Spaces

4- Semigroups of bounded linear operators

5- Lyapunov stability theory

6- P-Laplace operator

In this preliminary we shall introduce and state some necessary notations needed in the proof of our results, and some the basic results which concerning the semi-groupe theory and Layponov functionals and other theorems. The knowledge of all these notations and results are important for our study.

1.1 Continuous function spaces

We start this work by giving some useful notations and conventions.

Let $x = (x_1, x_2, ..., x_n)$ denote the generic point of an open set Ω of $\mathbb{R}^n$. Let u be a function defined from Ω to $\mathbb{R}^n$, we designate by $D_i u(x) = u_i(x) = \dfrac{\partial u(x)}{\partial x_i}$ the partial derivative of u with respect to $x_i\,(1 \leq i \leq n)$. Let's also define the gradient and the p-Laplacian from u, respectively as following

$$\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n})^{\,T} \quad \text{and} \quad |\nabla u|^2 = \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^2$$

$$\Delta_p u(x) = div\left(|\nabla u|^{p-2} \nabla u\right)(x).$$

Note by $C(\Omega)$ the space of continuous functions from Ω to $\mathbb{R}$, $C(\Omega, \mathbb{R}^m)$ the space of continuous functions from Ω to $\mathbb{R}^m$ and $C_b\left(\overline{\Omega}\right)$ the space of all continuous and bounded functions on $\overline{\Omega}$, it is equipped with the norm $\|.\|_\infty$;

$$\|u\|_\infty = \sup_{x \in \overline{\Omega}} |u(x)|$$

For $k \geq 1$ integer, $C^k(\Omega)$ is the space of functions u which are k times derivable and whose derivation of order k is continuous on Ω. $C_c^k(\Omega)$ is the set of functions of $C^k(\Omega)$ whose support is compact and contained in Ω.

We also define $C^k\left(\overline{\Omega}\right)$ as the set of restrictions to $\overline{\Omega}$ of elements from $C^k(\mathbb{R}^n)$ or as being the set of functions of $C^k(\Omega)$, such that for all $0 \leq j \leq k$, and for all $x_0 \in \partial\Omega$, the limit $\lim\limits_{x \to x_0} D_j u(x)$ exists and depends only on x_0.

$C_0^\infty(\Omega)$ or $\mathfrak{D}(\Omega)$, is the space of the infinitely differentiable functions, with compact supports called test function space.

The Hölder space $C^{k,\alpha}(\Omega)$, where Ω is an open subset of $\mathbb{R}^n$ and $k \geq 0$ an integer, $0 < \alpha \leq 1$, consists of those real or complex-valued k-times continuously differentiable functions f on Ω verifying

$$|f^{\beta}(x) - f^{\beta}(y)| \leq C\|x - y\|^{\alpha}$$

where $C > 0$, $|\beta| \leq k$.

1.2 L^p Spaces

Let Ω be an open set of $\mathbb{R}^n$, equipped with the Lebesgue measure dx. We denote by $L^1(\Omega)$ the space of integrable functions on Ω with values in $\mathbb{R}$, it is provided with the norm

$$\|u\|_{L^1} = \int_{\Omega} |u(x)|\, dx.$$

Let $p \in \mathbb{R}$ with $1 \leq p < +\infty$, we define the space $L^p(\Omega)$ by

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R},\ f \text{ measurable and } (\|f\|_{L^p})^p = \int_{\Omega} |f(x)|^p\, dx < +\infty \right\}$$

equipped with norm

$$\|u\|_{L^p} = \left(\int_{\Omega} |u(x)|^p\, dx \right)^{\frac{1}{p}}.$$

We also define the space $L^{\infty}(\Omega)$

$$L^{\infty}(\Omega) = \{ f : \Omega \to \mathbb{R},\ f \text{ measurable},\ \exists c > 0, \text{ so that } |f(x)| \leq c \quad \text{a.e. on } \Omega \},$$

it will be equipped with the essential-sup norm

$$\|u\|_{L^{\infty}} = \underset{x\in\Omega}{ess\sup} |u(x)| = \inf \{ c\ ;\ |u(x)| \leq c \quad \text{a.e. on } \Omega \}.$$

We say that a function $f : \Omega \to \mathbb{R}$ belongs to $L^p_{loc}(\Omega)$ if $\mathbf{1}_K f \in L^p(\Omega)$ for any compact $K \subset \Omega$.

Theorem 1. *(Dominated convergence Theorem)*
Let $\{f_n\}_{n\geq 1}$ be a series of functions of $L^1(\Omega)$ converging almost everywhere to a measurable function f. It is assumed that there exists $g \in L^1(\Omega)$ such that for all $n \geq 1$, we get

$$|f_n| \leq g \quad a.e \text{ on } \Omega.$$

Then $f \in L^1(\Omega)$ *and*

$$\lim_{n\to+\infty} \|f_n - f\|_{L^1} = 0, \text{ and } \int_\Omega f(x)\,dx = \lim_{n\to+\infty} \int_\Omega f_n(x)\,dx.$$

1.3 Sobolev spaces

Definition 1. *Let Ω be an open set of $\mathbb{R}$, and $1 \le i \le n$. A function $u \in L^1_{loc}(\Omega)$ has an i^{th} weak derivative in $L^1_{loc}(\Omega)$ if there exists $f_i \in L^1_{loc}(\Omega)$ such that for all $\varphi \in C_0^\infty(\Omega)$ we have*

$$\int_\Omega u(x)\,\partial_i\varphi(x)\,dx = -\int_\Omega f_i(x)\,\varphi(x)\,dx.$$

This leads to say that the i^{th} derivative within the meaning of distributions of u belongs to $L^1_{loc}(\Omega)$, we write

$$\partial_i u = \frac{\partial u}{\partial x_i} = f_i$$

1.3.1 $W^{1,p}(\Omega)$ spaces

Let Ω be a bounded or unbounded open set of $\mathbb{R}^n$, and $p \in \mathbb{R}$, $1 \le p \le +\infty$, the space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega);\ \text{such that } \partial_i u \in L^p(\Omega), 1 \le i \le n\}$$

where $\partial_i u$ is the i^{th} weak derivative of $u \in L^1_{loc}(\Omega)$.

For $1 \le p < +\infty$ we define the space $W_0^{1,p}(\Omega)$ as being the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$, and we write

$$W_0^{1,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W^{1,p}}.$$

Theorem 2. *(Poincaré's inequality)*

Assume Ω is a bounded open subset of $\mathbb{R}^n$, $u \in W_0^{1,p}(\Omega)$ for some $1 \le p < n$. Then we have the estimate

$$\|u\|_{L^q(\Omega)} \le C\|\nabla u\|_{L^p(\Omega)}$$

for each $q \in [1, p^*]$, where $p^* = \dfrac{np}{n-p}$ and the constant C depends only on q, p, n and Ω.

Remark 1. *In view of this Poincaré's inequality, if* Ω *is bounded, then on* $W_0^{1,p}(\Omega)$ *the norm* $\|u\|_{W^{1,p}(\Omega)}$ *is equivalent to* $\|\nabla u\|_{L^p(\Omega)}$.

Theorem 3. *(Rellich-Kondrachov compactness theorem) [10]*

Assume Ω *is a bounded open subset of* $\mathbb{R}^n$ *with* C^1 *boundary, and* $1 \leq p < n$. *Then*

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega)$$

for each $1 \leq q < p^*$.

1.3.2 $W^{m,p}(\Omega)$ Spaces

Let Ω be an open set of $\mathbb{R}^n, m \geq 2$ integer number and p real number such that $1 \leq p \leq +\infty$, we define the space $W^{m,p}(\Omega)$ as following

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega)\,,\ \text{such that } \partial^\alpha u \in L^p(\Omega)\,,\ \forall \alpha, |\alpha| \leq m\}$$

where $\alpha \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + ... + \alpha_n$ the length of α and $\partial^\alpha u = \partial_1^{\alpha_1}...\partial_n^{\alpha_n}$ is the weak derivative of a function $u \in L^1_{loc}(\Omega)$ in the sense of definition (1).

The space $W^{m,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{m,p}} = \|u\|_{L^p} + \sum_{0<|\alpha|\leq m} \|\partial^\alpha u\|_{L^p}\,.$$

For $p = 2$, the space $W^{m,2}(\Omega)$ is noted $H^m(\Omega)$.

1.4 Semigroups of bounded linear operators

The goal of this section is to prove Lumer-Phillips' theorem (see Theorems 1.4.3 and 1.4.6 of [8]) in a Hilbert space setting. For that purpose, we first recall the notion of m-dissipative operators.

Definition 2. *Let* $\mathcal{A} : D(\mathcal{A}) \subset X \longrightarrow X$ *be a (unbounded) linear operator.* $\mathcal{A}$ *is called dissipative if* $\Re(\mathcal{A}v, v)_x \leq 0, \forall v \in D(\mathcal{A})$. *The dissipative operator* $\mathcal{A}$ *is called* m*-dissipative if* $(\lambda I - \mathcal{A})$ *is surjective for some* $\lambda > 0$.

Theorem 4. *A linear operator $\mathcal{A}$ is dissipative if and only if*

$$\|(\lambda I - \mathcal{A})_x\|_X \geq \lambda \|x\|_X, \forall x \in D(\mathcal{A}), \lambda > 0, \tag{1.4.1}$$

Proof. Assume that $\mathcal{A}$ is dissipative and fix $x \in D(\mathcal{A})$ and $\lambda > 0$. Then

$$\lambda \|x\|_X^2 \leq \mathfrak{R}((\lambda - \mathcal{A})x, x)_X$$

and by Cauchy-Schwarz's inequality we conclude that

$$\lambda \|x\|_X^2 \leq \|(\lambda - \mathcal{A})x\|_X \|x\|_X,$$

that directly leads to (1.4.1). Conversely assume that (1.4.1) holds and fix $x \in D(\mathcal{A})$, then for all $\lambda > 0$, one has

$$\lambda^2 \|x\|_X^2 \leq \lambda \|x\|_X^2 - 2\lambda \mathfrak{R}(\mathcal{A}x, x)_x + \|\mathcal{A}x\|_X^2.$$

Dividing this inequality by 2λ, we get equivalently

$$\mathfrak{R}(\mathcal{A}x, x)_x \leq \frac{1}{2\lambda} \|\mathcal{A}x\|_X^2, \lambda > 0.$$

Passing to the limit as λ goes to infinity yields the dissipatedness of $\mathcal{A}$. Now we can prove some useful properties of m-dissipative operators. □

Theorem 5. *Let $\mathcal{A}$ be a m-dissipative operator. Then the next properties hold.*

1. *$\mathcal{A}$ is closed.*
2. *For all $\lambda > 0$, the operator $\lambda I - \mathcal{A}$ is an isomorphism from $D(\mathcal{A})$ onto X. Moreover $(\lambda I - \mathcal{A})^{-1}$is a linear bounded operator such that*

$$\left\|(\lambda I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda}.$$

3. *$D(\mathcal{A})$ is dense in X.*

Proof. Let us start with point 1. As $\mathcal{A}$ is a m-dissipative operator, there exists $\lambda_0 > 0$ such that $R(\lambda_0 I - \mathcal{A}) = X$, hence by (1.4.1) it follows that $\lambda_0 I - \mathcal{A}$ has a bounded inverse. As $(\lambda_0 I - \mathcal{A})^{-1}$ is bounded, it is also closed. Then $\lambda_0 I - \mathcal{A}$ is closed and therefore $\mathcal{A}$ as well. To prove point 2 it suffices to prove that $R(\lambda I - \mathcal{A}) = X$ for all $\lambda > 0$. For that purpose, we introduce the set

$$\Lambda = \{\lambda \in (0, \infty) \text{ such that } R(\lambda I - \mathcal{A}) = X\}.$$

First Λ is open. Indeed (1.4.1) implies that Λ is a subset of the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$. As $\rho(\mathcal{A})$ is open, for every $\lambda \in \Lambda$, there exists a neighborhood of λ included in $\rho(\mathcal{A})$. The intersection of this neighborhood with the real line is clearly included into Λ, which proves that Λ is open. Let us also show that Λ is closed. Let a sequence. $(\lambda_n)_n$ of elements of Λ such that

$$\lambda_n \longrightarrow \lambda > 0 \text{ as } n \longrightarrow \infty$$

Then for an arbitrary element $y \in X$, and any n, there exists $x_n \in D(\mathcal{A})$ such that

$$(\lambda_n I - \mathcal{A})_{x_n} = y \tag{1.4.2}$$

Owing to (1.4.1), it follows that

$$\|x_n\|_X \leq \lambda_n^{-1} \|y\|_X \tag{1.4.3}$$

and therefore the sequence $(x_n)_n$ is bounded. Now we apply (1.4.1) with $x_n - x_m$ and λ_m to obtain

$$\lambda_m \|x_n - x_m\|_X \leq \|\lambda_m (x_n - x_m) - \mathcal{A}(x_n - x_m)\|_X ,$$

and by using (1.4.2) we deduce that

$$\lambda_m \|x_n - x_m\|_X \leq |\lambda_m - \lambda_n| \|x_n\|_X .$$

and by (1.4.3), we deduce that there exists $x \in X$ such that x_n converges to x in X. But (1.4.2) then implies that $\mathcal{A}x_n$ converges to $\lambda x - y$ and since $\mathcal{A}$ is closed, we conclude that $x \in D(\mathcal{A})$ with $\lambda x - \mathcal{A}x = y$. This shows that λ belongs to Λ and the closeness of Λ is proved. In conclusion Λ is a closed, open and non empty subset of $(0, \infty)$ and therefore it coincides with $(0, \infty)$.

Let us finish with point 3. Let $y \in X$ be such that

$$(y, x)_X = 0, x \in D(\mathcal{A}) \tag{1.4.4}$$

If we show that

$$(y, \mathcal{A}x)_X = 0, x \in D(\mathcal{A}) \tag{1.4.5}$$

then we will obtain that

$$(y, x - \mathcal{A}x)_X = 0, x \in D(\mathcal{A})$$

and since $R(I-\mathcal{A}) = X$, we deduce that $y = 0$.

It then remains to show (1.4.5). Let $x \in D(\mathcal{A})$ be fixed, then by point 2, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in D(\mathcal{A})$ and

$$x = x_n - \frac{1}{n}\mathcal{A}x_n, \forall n \in \mathbb{N}. \tag{1.4.6}$$

This implies that

$$\mathcal{A}x_n = n(x_n - x)$$

and from the regularity $x, x_n \in D(\mathcal{A})$, we deduce that x_n belongs to $D(\mathcal{A}^2)$ and that the next identity holds

$$\mathcal{A}x = \mathcal{A}\left(I - \frac{1}{n}\mathcal{A}\right)x_n.$$

or equivalently

$$\mathcal{A}x_n = \mathcal{A}\left(I - \frac{1}{n}\mathcal{A}\right)^{-1}\mathcal{A}x.$$

From point 2, we know that

$$\left\|\left(I - \frac{1}{n}\mathcal{A}\right)^{-1}\right\|_{\mathcal{L}(X)} \leq 1$$

and therefore

$$\|\mathcal{A}x_n\|_X \leq \|\mathcal{A}x\|_X .$$

Moreover as X is a Hilbert space, there exists a subsequence $(\mathcal{A}x_{nk})$ of $(\mathcal{A}x_n)_n$ and $z \in X$ such that $\mathcal{A}x_{nk}$ converges weakly to z This implies that the sequence of pairs $((x_{nk}, \mathcal{A}x_{nk}))_k$ converges weakly to(x, z) in $X \times X$.Hence by Mazur's Lemma there exists another sequence $((\widetilde{x}_l, z_l))_l$ made of convex combinations of $(x_{nj}, \mathcal{A}x_{nj})$ (that then guarantees that $z_l = \mathcal{A}\widetilde{x}_l$) such that $(\widetilde{x}_l, z_l) = (\widetilde{x}_l, \mathcal{A}\widetilde{x}_l)$ converges strongly to(x, z) in $X \times X$ as l goes to ∞. As $\mathcal{A}$ is closed, we deduce that $z = \mathcal{A}x$.

Finally by (1.4.6) and (1.4.4) we have

$$(y, \mathcal{A}x_{nk})_X = n_k(y, x_{nk} - x) = 0$$

and passing to the limit in k, we find that (1.4.5) holds.

Let us now go on with the notion of linear semigroups. □

Definition 3. *A one parameter family* $(S(t))_{t\geq 0}$ *of* $\mathcal{L}(X)$ *is a semigroup of bounded linear operators on* X *if*

1.
$$S(0) = Id_x,$$

2.
$$S(t+s) = S(t)\,S(s)\,,\ \forall t, s \geq 0.$$

The linear operator $\mathcal{A}$ *defined by:*

$$D(\mathcal{A}) = \left\{ z \in X;\ \lim_{t \longrightarrow 0^+} \frac{S(t)\,z - z}{t} \text{ exists} \right\}$$

and

$$\mathcal{A}z = \lim_{t \longrightarrow 0^+} \frac{S(t)\,z - z}{t}, \forall z \in D(\mathcal{A})$$

is called the infinitesimal generator of the semigroup $(S(t))_{t\geq 0}$ *and* $D(\mathcal{A})$ *is called the domain of* $\mathcal{A}$.

A semigroup $(S(t))_{t\geq 0}$ *of bounded linear operators is called a strongly continuous (or a* C_0-*semigroup)if*

$$\lim_{t \longrightarrow 0^+} S(t)\,z = z,\ \forall z \in X. \tag{1.4.7}$$

A strongly continuous $(S(t))_{t\geq 0}$ *on* X *satisfying*

$$\|S(t)\|_{\mathcal{L}(X)} \leq 1,\ \forall t \geq 0,$$

is called a C_0*-semigroup of contractions.*

Let us now prove some useful properties of C_0- semigroups of contractions.

Theorem 6. *Let* $(S(t))_{t\geq 0}$ *be a* C_0-*semigroup of contractions on* X*. Then*

1. *For all* $x \in X$*, the mapping* $t \longrightarrow S(t)x$ *is a continuous function from* $[0, \infty)$ *into* X.
2. *For all* $x \in X$ *and all* $t \geq 0$,
$$\lim_{h \longrightarrow 0} \frac{1}{h} \int_t^{t+h} S(s)\,xds = S(s)\,x. \tag{1.4.8}$$
3. *For all* $x \in X$ *and all* $t > 0$*, the element* $\int_0^t S(s)\,xds$ *belongs to* $D(\mathcal{A})$*, and*
$$\mathcal{A}\left(\int_0^t S(s)\,xds\right) = S(t)\,x - x \tag{1.4.9}$$

4. For all $x \in D(\mathcal{A})$ and all $t > 0$, the element $S(t)x$ belongs to $D(\mathcal{A})$, and the mapping $t \longrightarrow S(t)x$ is a continuous differentiable function from $(0, \infty)$ into X and

$$\frac{d}{dt}S(t)x = \mathcal{A}S(t)x = S(t)\mathcal{A}x, \ \forall t \geq 0. \tag{1.4.10}$$

5. For all $x \in D(\mathcal{A})$ and all $t > s \geq 0$, we have

$$S(t)x - S(s)x = \int_s^t S(u)\mathcal{A}xdu = \int_s^t \mathcal{A}S(u)xdu.$$

Proof. For point 1, by (1.4.7), the continuity property trivially holds at $t = 0$. Now fix $x \in X$ and take an arbitrary $t > 0$ then for $h \geq 0$, we may write

$$S(t+h)x - S(t)x = S(t)(S(h)x - x),$$

and consequently

$$\|S(t+h)x - S(t)x\|_X \leq \|S(h)x - x\|_X,$$

On the other hand for $h < 0$ such that $t + h > 0$, we have,

$$S(t+h)x - S(t)x = S(t+h)(x - S(-h)x).$$

In both cases, by (1.4.7) we find that $S(t+h)x - S(t)x$ goes to zero as h goes to zero. Point 2 directly follows from point 1.

To prove point 3, fix $x \in X$ and $h > 0$. then we clearly have

$$\begin{aligned} \frac{S(h)-I}{h}\int_0^t S(s)xds &= \frac{1}{h}\int_0^t (S(s+h)x - S(s)x)\,ds \\ &= \frac{1}{h}\int_0^{t+h} S(s)xds - \frac{1}{h}\int_0^t S(s)xds \end{aligned}$$

Hence by (1.4.8), we deduce that the right-hand side tends to $S(t)x - x$ as h goes to zero.By the definition of A this proves the assertions. For point 4, let $x \in D(\mathcal{A})$ and $t, h > 0$,then by the semigroup property

$$\frac{S(h)-I}{h}S(t)x = S(t)\left(\frac{S(h)-I}{h}\right)x.$$

Hence by the definition of $\mathcal{A}$ and the continuity of the semigroup, we get

$$\lim_{h \longrightarrow 0^+} \frac{S(h)-I}{h}S(t)x = S(t)\lim_{h \longrightarrow 0^+}\left(\frac{S(h)-I}{h}\right)x = S(t)\mathcal{A}x.$$

This shows that $S(t)x$ belongs to $D(\mathcal{A})$, that $\mathcal{A}S(t)x = S(t)\mathcal{A}x$ and that the right derivative of $S(t)x$ exists with

$$\frac{d^+}{dt}S(t)x = \mathcal{A}S(t)x = S(t)\mathcal{A}x$$

For the left derivative, for $0 < h < t$ we write

$$\begin{aligned}\frac{S(t)x - S(t-h)x}{h} - S(t)\mathcal{A}x &= S(t-h)\left(\frac{S(h)x - x}{h} - \mathcal{A}x\right) \\ &\quad + \left(S(t-h)\mathcal{A}x - S(t)\mathcal{A}x\right).\end{aligned}$$

□

1.5 Lyapunov Stability Theory

The investigation of stability for hereditary systems is often related to the construction of Lyapunov functionals. The general method of Lyapunov functionals construction which was proposed by V. Kolmanovskii and L. Shaikhet [**?**] and successfully used already for functional differential equations, for difference equations with discrete time, for difference equations with continuous time, is used here to investigate the stability of delay evolution equations, in particular, partial differential equations.

1.5.1 Notations and definitions

Let U and H be two real separable Hilbert spaces such that $U \subset H \equiv H^* \subset U^*$, where the injections are continuous and dense. Let $\|\|$, $||$ and $\|\|_*$ be the norms in U, H and H^*respectively, $((\cdot,\cdot))$ and $(\cdot,\cdot)$ be the scalar products in U and H respectively, and $\langle .,. \rangle$ the duality product between U and $U*$. We assume that

$$|u| \leq \beta \|u\|, u \in U \tag{1.5.1}$$

Let $C(-h, 0, H)$ be the Banach space of all continuous functions from $[-h, 0]$ to H, $x_t \in C(-h, 0, H)$ for each $t \in [0, \infty)$, be the function defined by $x_t(s) = x(t+s)$ for all $s \in [-h, 0]$. The space $C(-h, 0, U)$ is similarly defined. Let $A(t,\cdot) : U \to U^*$, $f_1(t,\cdot) : C(-h, 0, H) \to U*$ and

$f_2(t,\cdot) : C(-h,0,U) \to U*$ be three families of nonlinear operators defined for $t > 0, A(t,0) = 0, f_1(t,0) = 0, \quad f_2(t,0) = 0.$

Consider the equation

$$\frac{du(t)}{dt} = A\left(t, u(t)\right) + f_1(t, u_t) + f_1(t, u_t), t > 0 \tag{1.5.2}$$

$$u(s) = \psi(s), s \in [-h, 0]$$

Let us denote by $u(\cdot;\psi)$ the solution of Eq. (1.5.2) corresponding to the initial condition ψ.

Definition 4. *The trivial solution of Eq. (1.5.2) is said to be stable if for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$|u\left(t;\psi\right)| < \varepsilon \textit{ for all } t \geq 0, \textit{ if } |\psi|_{C_H} = \sup_{s\in[-h,0]} |\psi\left(s\right)| < \delta.$$

Definition 5. *The trivial solution of Eq. (1.5.2) is said to be exponentially stable if it is stable and there exists a positive constant* λ *such that for any* $\psi \in C(-h,0,U)$ *there exists* C*(which may depend on* ψ*) such that* $|u\left(t;\psi\right)| \leq Ce^{-\lambda t}$ *for* $t > 0$.

1.5.2 Lyapunov type stability theorem

Let us now prove a theorem which will be crucial in our stability investigation.

Theorem 7. *Assume that there exists a functional* $V(t, u_t)$ *such that the following conditions hold for some positive numbers* c_1, c_2 *and* λ*:*

$$|u\left(t;u_t\right)| \leq c_1 e^{\lambda t} \left|u(t)\right|^2, t \geq 0, \tag{1.5.3}$$

$$|u\left(0;u_0\right)| \leq c_2 \left|\psi\right|^2_{C_H}, \tag{1.5.4}$$

$$\frac{d}{dt} V\left(t, u_t\right) \leq 0, t \geq 0. \tag{1.5.5}$$

Then the trivial solution of Eq. (1.5.2) is exponentially stable.

Note that Theorem 7 implies that the stability investigation of Eq. (1.5.2) can be reduced to the construction of appropriate Lyapunov functionals. A formal procedure to construct Lyapunov functionals is described below.

1.5.3 Procedure of Lyapunov functionals construction

The procedure consists of four steps.

Step 1.

To transform Eq. (1.5.2) into the form

$$\frac{dz(t,u_t)}{dt} A_1\left(t,u(t)\right)+A_2\left(t,u_t\right) \tag{1.5.6}$$

where $z(t,\cdot)$ and $A_2(t,\cdot)$ are families of nonlinear operators, $z(t,0) = 0, A_2(t,0) = 0$,operator $A_1(t,\cdot)$ only depends on t and $u(t)$, but does not depend on the previous values $u(t+s), s < 0$.

Step 2.

Assume that the trivial solution of the auxiliary equation without memory

$$\frac{dy(t)}{dt} = A_1\left(t,y(t)\right) \tag{1.5.7}$$

is exponentially stable and therefore there exists a Lyapunov function $v(t,y(t))$, which satisfies the conditions of Theorem 7.

Step 3.

A Lyapunov functional $V(t,ut)$ for Eq. (1.5.6) is constructed in the form $V = V1+V2$, where $V_1(t,ut) = v(t,z(t,u_t))$. Here the argument y of the function $v(t,y)$ is replaced on the functional $z(t,x_t)$ from the left-hand part of Eq. (1.5.6).

Step 4.

Usually, the functional $V_1(t,u_t)$ almost satisfies the conditions of Theorem 7. In order to fully satisfy these conditions, it is necessary to calculate $\frac{d}{dt}V_1(t,u_t)$ and estimate it. Then, the additional functional $V_2(t,u_t)$ can be chosen in a standard way.

Note that the representation (1.5.6) is not unique. This fact allows, using different representations type of (1.5.6) or different ways of estimating $\frac{d}{dt}V_1(t,u_t)$, to construct different Lyapunov functionals and, as a result, to get different sufficient conditions of exponential stability.

1.6 P-Laplace operator

The study of eigenvalue problems is an important object of research in functional analysis. It is known that in the framework of the Ljusternik-Schnirelman theory one can find estimates for the number of critical points of functionals from which some results on eigensolutions for nonlinear differential equations are deduced.
A nonlinear operator equation can be formulated of the form

$$Au = \lambda Bu.$$

In the case of p-Laplace operator, the following nonlinear eigenvalue problem has been extensively investigated in the past thirty years

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u, & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6.1}$$

We are going to state the following definition and some famous results.

Definition 6. *We say that $u \in W_0^{1,p}(\Omega)$, $u \neq 0$, is an eigenfunction of the operator $-\triangle_p u$ if:*

$$\int_\Omega |\nabla u|^{p-2} \nabla u.\nabla \varphi dx = \lambda \int_\Omega |u|^{p-2} u.\varphi \; dx \tag{1.6.2}$$

for all $\varphi \in C_0^\infty(\Omega)$. The corresponding real number λ is called eigenvalue.

Let λ_1 defined by

$$\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx} \tag{1.6.3}$$

equivalent to

$$\lambda_1 = \inf \left\{ \int_\Omega |\nabla u|^p \, dx; \; \int_\Omega |u|^p \, dx = 1, u \in W_0^{1,p}(\Omega) \right\}.$$

λ_1 is the first eigenvalue of the p-Laplacian operator with null Dirichlet conditions at the edge.

Lemma 1. *λ_1 is isolated, i.e : there exists $\delta > 0$ such that in the interval $(\lambda_1, \lambda_1 + \delta)$, there is no other eigenvalues of (1.6.2).*

Lemma 2. *The first eigenvalue* λ_1 *is simple, i.e : if* u, v *are two eigenfunctions associated with* λ_1*, then, there exists* k *such that* $u = kv$*.*

Lemma 3. *Let* u *be an eigenfunction associated with the eigenvalue* λ_1*, then* u *does not change sign on* Ω*. Further if* $u \in C^{1,\alpha}(\Omega)$*, then* $u(x) \neq 0$ *,* $\forall x \in \overline{\Omega}$*.*

Definition 7. *Let* ω *be a part of a Banach space* X *and* $F : \omega \to \mathbb{R}$*. If* $u \in \omega$*, we say that* F *is Gâteaux differentiable (or G-differentiable) at* u*, if there exists* $l \in X'$ *such that in each direction* $z \in X$ *where* $F(u + tz)$ *exists for* $t > 0$ *small enough, the directional derivative* $F'_z(u)$ *exists and we have*

$$\lim_{t \to 0^+} \frac{F(u + tz) - F(u)}{t} = \langle l, z \rangle .$$

We write $F'(u) = l$*.*

Theorem 8. *Let* $\Omega \subset \mathbb{R}^n$ *an open set,* $n \geq 3$*. For* $p \in (1, +\infty)$*, we define a functional* $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ *by*

$$J(u) = \int_\Omega |\nabla u|^p \, dx$$

then J *is differentiable in* $W_0^{1,p}(\Omega)$ *and*

$$J'(u)(v) = p \int_\Omega |\nabla u|^{p-2} \nabla u . \nabla v dx, \forall v \in W_0^{1,p}(\Omega) .$$

Proof. We consider the function $\varphi : \mathbb{R}^n \to \mathbb{R}$, defined by $\varphi(x) = |x|^p$, it is a function of class C^1, and $\nabla \varphi = p |x|^{p-2} x$.

Then for all $x, y \in \mathbb{R}^n$,

$$\lim_{t \to 0} \frac{\varphi(x + ty) - \varphi(x)}{t} = p |x|^{p-2} x.y$$

as a consequence

$$\lim_{t \to 0} \frac{|\nabla u(x) + t \nabla v(x)|^p - |\nabla u(x)|^p}{t} = p |\nabla u(x)|^{p-2} \nabla u(x) . \nabla v(x) .$$

By Mean value theorem, for almost every $x \in \Omega$ and for $t > 0$, there exists a function θ that takes its values in $]0, 1[$ and we can write

$$\begin{aligned} & \left|\nabla u\left(x\right)+t\nabla v\left(x\right)\right|^{p}-\left|\nabla u\left(x\right)\right|^{p}-tp\left|\nabla u\left(x\right)\right|^{p-2}\nabla u\left(x\right).\nabla v\left(x\right) \\ = \; & tp\left|\nabla u\left(x\right)+\theta\left(t,x\right)t\nabla v\left(x\right)\right|^{p-2}\left(\nabla u\left(x\right)+\theta\left(t,x\right)t\nabla v\left(x\right)\right).\nabla v\left(x\right) \\ & -tp\left|\nabla u\left(x\right)\right|^{p-2}\nabla u\left(x\right).\nabla v\left(x\right). \end{aligned} \tag{1.6.4}$$

Dividing by t, we get for almost every x

$$\lim_{t\to 0}\frac{\left|\nabla\left(u+tv\right)\left(x\right)\right|^{p}-\left|\nabla u\left(x\right)\right|^{p}-tp\left|\nabla u\left(x\right)\right|^{p-2}\nabla u\left(x\right).\nabla v\left(x\right)}{t}=0.$$

On the other hand, one can see that the second member of the equality (4.2.13) devided by t is bounded by

$$h\left(x\right)=2\left|\nabla v\left(x\right)\right|\left(\left|\nabla u\left(x\right)\right|+\left|\nabla v\left(x\right)\right|\right)^{p-1}.$$

Then using the Holder inequality we have

$$\left|h\right|\leq C\left\|\nabla v\right\|_{p}\left(\left\|\nabla u\right\|_{p}^{p-1}+\left\|\nabla v\right\|_{p}^{p-1}\right).$$

One can apply the Dominated convergence theorem and conclude

$$J'\left(u\right)\left(v\right)=p\int_{\Omega}\left|\nabla u\right|^{p-2}\nabla u.\nabla v dx,\forall v\in W_{0}^{1,p}\left(\Omega\right),$$

then J is Gâteaux differentiable. □

Lemma 4. (*Comparison lemma*) *Let* $u,v\in W_{0}^{1,p}\left(\Omega\right)$ *satisfying*

$$\int_{\Omega}\left|\nabla u\right|^{p-2}\nabla u.\nabla\varphi dx\leq\int_{\Omega}\left|\nabla v\right|^{p-2}\nabla v.\nabla\varphi dx \tag{1.6.5}$$

for all $\varphi\in W_{0}^{1,p}\left(\Omega\right),\varphi\geq 0,$ *then* $u\leq v$ *a.e in* Ω.

Proof. This proof is based on the arguments presented in [10]. We start by defining the function $J:W_{0}^{1,p}\left(\Omega\right)\to\mathbb{R}$ by the formula

$$J\left(u\right)=\frac{1}{p}\int_{\Omega}\left|\nabla u\right|^{p}dx. \tag{1.6.6}$$

It is clear that the functional J is Gâteaux differentiable and continuous and its derivative at $u \in W_0^{1,p}(\Omega)$ is the function $J'(u) \in W_0^{-1,p}(\Omega)$, i.e

$$J'(u)(\varphi) = \int_\Omega |\nabla u|^{p-2} \nabla u.\nabla \varphi dx, \ \forall \varphi \in W_0^{1,p}(\Omega). \tag{1.6.7}$$

$J'(u)$ is continuous and bounded. We will show that $J'(u)$ is strictly monotonic in $W_0^{1,p}(\Omega)$. Indeed, for all $u, v \in W_0^{1,p}(\Omega), u \neq v$ without loss of generality, we can suppose that

$$\int_\Omega |\nabla u|^p dx \geq \int_\Omega |\nabla v|^p dx.$$

Using the Cauchy inequality we have

$$\nabla u.\nabla v \leq |\nabla u| |\nabla v| \leq \frac{1}{2}\left(|\nabla u|^2 + |\nabla v|^2\right). \tag{1.6.8}$$

From formula (1.6.8) we deduce

$$\int_\Omega |\nabla u|^p dx - \int_\Omega |\nabla u|^{p-2} \nabla u.\nabla v dx \geq \frac{1}{2}\int_\Omega |\nabla u|^{p-2}\left(|\nabla u|^2 - |\nabla v|^2\right) dx \tag{1.6.9}$$

$$\int_\Omega |\nabla v|^p dx - \int_\Omega |\nabla v|^{p-2} \nabla v.\nabla u dx \geq \frac{1}{2}\int_\Omega |\nabla v|^{p-2}\left(|\nabla v|^2 - |\nabla u|^2\right) dx. \tag{1.6.10}$$

If $|\nabla u| \geq |\nabla v|$, by using (1.6.6)-(1.6.8) we get

$$\mathrm{I}_1(u) = J'(u)(u) - J'(u)(v) - J'(v)(u) + J'(v)(v)$$

$$= \left(\int_\Omega |\nabla u|^p dx - \int_\Omega |\nabla u|^{p-2} \nabla u.\nabla v dx\right)$$

$$- \left(\int_\Omega |\nabla v|^{p-2} \nabla v.\nabla u dx - \int_\Omega |\nabla v|^p dx\right)$$

$$\geq \int_\Omega \tfrac{1}{2} |\nabla u|^{p-2}\left(|\nabla u|^2 - |\nabla v|^2\right) dx$$

$$- \tfrac{1}{2}\int_\Omega |\nabla v|^{p-2}\left(|\nabla u|^2 - |\nabla v|^2\right) dx$$

$$= \frac{1}{2}\int\limits_{\Omega}\left(|\nabla u|^{p-2}-|\nabla v|^{p-2}\right)\left(|\nabla u|^{2}-|\nabla v|^{2}\right)dx$$

$$\geq \frac{1}{2}\int\limits_{\Omega}\left(|\nabla u|^{p-2}-|\nabla v|^{p-2}\right)\left(|\nabla u|^{2}-|\nabla v|^{2}\right)dx.$$

If $|\nabla v| \geq |\nabla u|$, by changing the role of u and v in (1.6.6)-(1.6.8) we have

$$\begin{aligned}
I_2(v) &= J'(v)(v) - J'(v)(u) - J'(u)(v) + J'(u)(u) \\
&= \left(\int\limits_{\Omega}|\nabla v|^{p}\,dx - \int\limits_{\Omega}|\nabla v|^{p-2}\,\nabla v.\nabla u dx\right) \\
&- \left(\int\limits_{\Omega}|\nabla u|^{p-2}\,\nabla u.\nabla v dx - \int\limits_{\Omega}|\nabla u|^{p}\,dx\right) \\
&\geq \frac{1}{2}\int\limits_{\Omega}|\nabla v|^{p-2}\left(|\nabla v|^{2}-|\nabla u|^{2}\right)dx \qquad (1.6.11) \\
&-\frac{1}{2}\int\limits_{\Omega}|\nabla u|^{p-2}\left(|\nabla v|^{2}-|\nabla u|^{2}\right)dx \\
&= \frac{1}{2}\int\limits_{\Omega}\left(|\nabla v|^{p-2}-|\nabla u|^{p-2}\right)\left(|\nabla v|^{2}-|\nabla u|^{2}\right)dx \\
&\geq \frac{1}{2}\int\limits_{\Omega}\left(|\nabla v|^{p-2}-|\nabla u|^{p-2}\right)\left(|\nabla v|^{2}-|\nabla u|^{2}\right)dx.
\end{aligned}$$

From (1.6.9)-(1.6.10), we have

$$\left(J'(u) - J'(v)\right)(u-v) = I_1 = I_2 \geq 0, \forall u, v \in W_0^{1,p}(\Omega).$$

In addition, if $u \neq v$ and $\left(J'(u) - J'(v)\right)(u-v) = 0$, then we have

$$\int\limits_{\Omega}\left(|\nabla u|^{p-2}-|\nabla v|^{p-2}\right)\left(|\nabla u|^{2}-|\nabla v|^{2}\right)dx = 0.$$

If $|\nabla u| = |\nabla v|$ in Ω, we deduce that

$$\begin{aligned} \left(J'(u) - J'(v)\right)(u-v) &= J'(u)(u-v) - J'(v)(u-v) \\ &= \int_\Omega |\nabla u|^{p-2} |\nabla u - \nabla v|^2 \, dx = 0, \end{aligned} \tag{1.6.12}$$

i.e. $u - v$ is a constant. Given $u = v = 0$ on $\partial\Omega$ we are getting $u = v$, which is contrary with $u \neq v$. Then $\left(J'(u) - J'(v)\right)(u-v) > 0$ and $J'(u)$ is strictly monotonic in $W_0^{-1,p}(\Omega)$. Let u, v two functions such that (1.6.7) is satisfied, let's take $\varphi = (u-v)^+$ the positive part of $u - v$ as a test function in (1.6.7), we get

$$\left(J'(u) - J'(v)\right)(\varphi) = \int_\Omega |\nabla u|^{p-2} \nabla u.\nabla\varphi dx - \int_\Omega |\nabla v|^{p-2} \nabla v.\nabla\varphi dx \leq 0. \tag{1.6.13}$$

Relationships (1.6) and (1.6) imply that $u \leq v$. □

Chapter 2

System of three nonlinear wave equations depending on the relaxation functions

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The main aim of this work is to study the decay rate of a system of three semilinear wave equations with strong external forces in $\mathbb{R}^n$, including damping terms of memory type with past history which is very important problem from the point of view of application in sciences and engineering. We work in a weighted phase spaces where the problem is well defined and deduce a decay result depending on the relaxation functions. Using the Faedo-Galerkin method and some energy estimates, we prove the existence of global solution owing to to the weighted function. By imposing a new appropriate conditions, which are not used in the literature, with the help of some special estimates and generalized Poincaré's inequality, we obtain an unusual decay rate for the energy function. It is a generalization of similar results in [35] and [34] for a single equation and [39] for coupled system to the case of a system of three equations. The work is relevant in the sense that the problem is more complex than what can be found in the literature. However, the techniques involved in order to study this generalization is a combination of the techniques used in [35] in order to deal with the memory and weighted spaces with standard techniques in order to deal with coupled system with nonlinearities.

2.1 Introduction and preliminaries

We consider, for $x \in \mathbb{R}^n,\ t > 0$, the following system

$$\begin{cases} \theta(u_{tt} + \alpha u_t) - \beta\Delta u_t = \Delta u - \int_0^t \varpi_1(t-s)\Delta u(s)\,ds + \theta h_1(u,v,w) \\ \theta(v_{tt} + \alpha v_t) - \beta\Delta v_t = \Delta v - \int_0^t \varpi_2(t-s)\Delta v(s)\,ds + \theta h_2(u,v,w) \\ \theta(w_{tt} + \alpha w_t) - \beta\Delta w_t = \Delta w - \int_0^t \varpi_3(t-s)\Delta w(s)\,ds + \theta h_3(u,v,w) \\ u(x,0) = u_0(x), v(x,0) = v_0(x), w(x,0) = w_0(x) \\ u_t(x,0) = u_1(x), v_t(x,0) = v_1(x), w_t(x,0) = w_1(x), \end{cases} \tag{2.1.1}$$

where $\alpha \in \mathbb{R}$, $\beta > 0$, $n \geq 3$, the functions $h_i(.,.,.) \in (\mathbb{R}^3, \mathbb{R}), i = 1,2,3$ are given by

$$h_1(\xi_1,\xi_2,\xi_3) = (p+1)\Big[d|\xi_1+\xi_2+\xi_3|^{(p-1)}(\xi_1+\xi_2+\xi_3) + e|\xi_1|^{(p-3)/2}\xi_1|\xi_2|^{(p+1)/2}\Big],$$

$$h_2(\xi_1,\xi_2,\xi_3) = (p+1)\Big[d|\xi_1+\xi_2+\xi_3|^{(p-1)}(\xi_1+\xi_2+\xi_3) + e|\xi_2|^{(p-3)/2}\xi_2|\xi_3|^{(p+1)/2}\Big],$$

$$h_3(\xi_1,\xi_2,\xi_3) = (p+1)\Big[d|\xi_1+\xi_2+\xi_3|^{(p-1)}(\xi_1+\xi_2+\xi_3) + e|\xi_3|^{(p-3)/2}\xi_3|\xi_1|^{(p+1)/2}\Big],$$

with $d, e > 0, p > 3$. The function $\theta(x) > 0$ for all $x \in \mathbb{R}^n$ is a density such that

$$\theta \in L^\tau(\mathbb{R}^n) \quad \text{with} \quad \tau = \frac{2n}{2n - rn + 2r} \quad \text{for} \quad 2 \leq r \leq \frac{2n}{n-2}. \tag{2.1.2}$$

It is note hard to see that there exists a function $\mathcal{G} \in C^1(\mathbb{R}^3, \mathbb{R})$ such that

$$uh_1(u,v,w) + vh_2(u,v,w) + wh_3(u,v,w) = (p+1)\mathcal{G}(u,v,w),\ \forall (u,v,w) \in \mathbb{R}^3. \tag{2.1.3}$$

satisfies

$$(p+1)\mathcal{G}(u,v,w) = |u+v+w|^{p+1} + 2|uv|^{(p+1)/2} + 2|vw|^{(p+1)/2} + 2|wu|^{(p+1)/2}. \tag{2.1.4}$$

We define the function spaces $\mathcal{H}$ as the closure of $C_0^\infty(\mathbb{R}^n)$, as in [30], we have

$$\mathcal{H} = \{v \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) \mid \nabla v \in L^2(\mathbb{R}^n)^n\},$$

with respect to the norm $\|v\|_{\mathcal{H}} = (v,v)_{\mathcal{H}}^{1/2}$ for the inner product

$$(v,w)_{\mathcal{H}} = \int_{\mathbb{R}^n} \nabla v \cdot \nabla w \, dx,$$

and $L^2_\theta(\mathbb{R}^n)$ as that to the norm $\|v\|_{L^2_\theta} = (v,v)_{L^2_\theta}^{1/2}$ for

$$(v,w)_{L^2_\theta} = \int_{\mathbb{R}^n} \theta v w \, dx.$$

For general $r \in [1, +\infty)$

$$\|v\|_{L^r_\theta} = \left(\int_{\mathbb{R}^n} \theta \, |v|^r \, dx \right)^{\frac{1}{r}}.$$

is the norm of the weighted space $L^r_\theta(\mathbb{R}^n)$.

The main aim of this work is to consider an important problem from the point of view of application in sciences and engineering, namely, a system of three wave equations having a different damping effects in an unbounded domain with strong external forces including damping terms of memory type with past history. Using the Faedo-Galerkin method and some energy estimates, we proved the existence of global solution in $\mathbb{R}^n$ owing to the weighted function. By imposing a new appropriate condition, which not be used in the literature, with the help of some special estimates and generalized Poincaré's inequality, we obtained an unusual decay rate for the energy function. The work brings new contributions to the prior literature mainly in what concerns new decay rate estimates of the energy. The following references in connection to our system for a single equation [24] and [25]. The work [24] was the pioneer in the literature for the single equation, source of inspiration of several works, while the work [25] is a recent generalization of [24] by introducing less dissipative effects.

To enrich our topic, it is necessary to reviewer previous works regarding the nonlinear coupled system of wave equations, from a qualitative and quantitative study. Let us beginning with the single wave equation treated in [22], where the aim goal was mainely on the system

$$\begin{cases} u_{tt} + \mu u_t - \Delta u - \omega \Delta u_t = u \ln |u|, \ (x,t) \in \Omega \times (0,\infty) \\ u(x,t) = 0, x \in \partial\Omega, t \geq 0 \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \end{cases} \tag{2.1.5}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$ with a smooth boundary $\partial\Omega$. The author firstly constructed a local existence of weak solution by using contraction mapping principle and of course showed the global existence, decay rate and infinite time blow up of the solution with condition on initial energy.

Next, a nonexistence of global solutions for system of three semilinear hyperbolic equations was introduced in [5]. A coupled system for semilinear hyperbolic equations was investigated by many authors and a different results were obtained with the nonlinearities in the form $f_1 = |u|^{p-1}|v|^{q+1}u$, $f_2 = |v|^{p-1}|u|^{q+1}v$. (Please, see [4], [23], [38], [39], ...)

In the case of non-bounded domain $\mathbb{R}^n$, we mention the paper recently published by T. Miyasita and Kh. Zennir in [35], where the considered equation as follows

$$u_{tt} + au_t - \phi(x)\Delta\left(u + \omega u_t - \int_0^t g(t-s)u(s)\,ds\right) = u|u|^{p-1}, \tag{2.1.6}$$

with initial data

$$\begin{cases} u(x,0) = u_0(x), \\ u_t(x,0) = u_1(x). \end{cases} \tag{2.1.7}$$

The authors was successful in highlighting the existence of unique local solution and they continued to extend it to be global in time. The rate of the decay for solution was the main result by considering the relaxation function is strictly convex, for more results related to decay rate of solution of this type of problems, please see [13], [28], [18], [34], ...

Regarding the study of the coupled system of two nonlinear wave equations, it is worth recalling some of the work recently published. Baowei Feng et *al.* considered in [**?**], a coupled system for viscoelastic wave equations with nonlinear sources in bounded domain ($(x,t) \in \Omega \times (0,\infty)$) with smooth boundary as follows

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + u_t = f_1(u,v) \\ \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(s)\,ds + v_t = f_2(u,v). \end{cases} \tag{2.1.8}$$

Here, the authors concerned with a system in $\mathbb{R}^n (n = 1,2,3)$. Under appropriate hypotheses, they established a general decay result by multiplication techniques to extends some existing results for a single equation to the case of a coupled system.

It is worth noting here that there are several studies in this field and we particularly refer to the generalization that Shun et *al.* made in studying a complicate non-linear case with degenerate damping term in [37]. The IBVP for a system of nonlinear viscoelastic wave equations in a bounded domain was considered in the problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + (|u|^k + |v|^q)|u_t|^{m-1}u_t = f_1(u,v) \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(s)\,ds + (|v|^\theta + |u|^\rho)|v_t|^{r-1}v_t = f_2(u,v) \\ u(x,t) = v(x,t) = 0, x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x), v(x,0) = v_0(x) \\ u_t(x,0) = u_1(x), v_t(x,0) = v_1(x), \end{cases} \tag{2.1.9}$$

where Ω is a bounded domain with a smooth boundary. Given certain conditions on the kernel functions, degenerate damping and nonlinear source terms, they got a decay rate of the energy function for some initial data.

The lack of existence (Blow up) is considered one of the most important qualitative studies that must be spoken of, given its importance in terms of application in various applied sciences. Concerning the nonexistence of solution for a more degenerate case for coupled system of wave equations with different damping, we mention the papers [32], [31], [17], [40], ...

In $m-$equations, paper in [7] considered a system

$$u_{itt} + \gamma u_{it} - \Delta u_i + u_i = \sum_{i,j=1, i\neq j}^{m} |u_j|^{p_j}|u_i|^{p_i}u_i, \ i = 1, 2, \dots, m, \tag{2.1.10}$$

where the absence of global solutions with positive initial energy was investigated.

We introduce a very useful Sobolev embedding and generalized Poincaré inequalities.

Lemma 5. *[35] Let θ satisfy (2.1.2). For a positive constants $C_\tau > 0$ and $C_P > 0$ depending only on θ and n, we have*

$$\|v\|_{\frac{2n}{n-2}} \leq C_\tau \|v\|_{\mathcal{H}},$$

and

$$\|v\|_{L^2_\theta} \leq C_P \|v\|_{\mathcal{H}},$$

for $v \in \mathcal{H}$.

Lemma 6. *[29] Let θ satisfy (2.1.2), then the estimates*

$$\|v\|_{L^r_\theta} \leq C_r \|v\|_{\mathcal{H}},$$

and

$$C_r = C_\tau \|\theta\|_\tau^{\frac{1}{r}},$$

hold for $v \in \mathcal{H}$. Here $\tau = 2n/(2n - rn + 2r)$ for $1 \leq r \leq 2n/(n-2)$.

We assume that the kernel functions $\varpi_1, \varpi_2, \varpi_3 \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfying

$$1 - \overline{\varpi_1} = l > 0 \quad \text{for} \quad \overline{\varpi_1} = \int_0^{+\infty} \varpi_1(s)\, ds,\ \varpi_1'(t) \leq 0, \tag{2.1.11}$$

$$1 - \overline{\varpi_2} = m > 0 \quad \text{for} \quad \overline{\varpi_2} = \int_0^{+\infty} \varpi_2(s)\, ds,\ \varpi_2'(t) \leq 0, \tag{2.1.12}$$

$$1 - \overline{\varpi_3} = \nu > 0 \quad \text{for} \quad \overline{\varpi_3} = \int_0^{+\infty} \varpi_3(s)\, ds,\ \varpi_3'(t) \leq 0, \tag{2.1.13}$$

we mean by $\mathbb{R}^+$ the set $\{\tau \mid \tau \geq 0\}$. Noting by

$$\varpi(t) = \max_{t\geq 0} \Big\{\varpi_1(t), \varpi_2(t), \varpi_3(t)\Big\}, \tag{2.1.14}$$

and

$$\varpi_0(t) = \min_{t\geq 0} \Big\{ \int_0^t \varpi_1(s) ds, \int_0^t \varpi_2(s) ds, \int_0^t \varpi_3(s) ds\Big\}. \tag{2.1.15}$$

We assume that there is a function $\chi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$\varpi_i'(t) + \chi(\varpi_i(t)) \leq 0, \quad \chi(0) = 0, \quad \chi'(0) > 0 \quad \text{and} \quad \chi''(\xi) \geq 0,\ i = 1, 2, 3, \tag{2.1.16}$$

for any $\xi \geq 0$.

Holder and Young's inequalities give

$$\begin{aligned} \|uv\|^{(p+1)/2}_{L^{(p+1)/2}_\theta} &\leq \Big(\|u\|^2_{L^{(p+1)}_\theta} + \|v\|^2_{L^{(p+1)}_\theta}\Big)^{(p+1)/2} \\ &\leq \left(l\|u\|^2_{\mathcal{H}} + m\|v\|^2_{\mathcal{H}}\right)^{(p+1)/2}, \end{aligned} \tag{2.1.17}$$

and

$$\|vw\|^{(p+1)/2}_{L^{(p+1)/2}_\theta} \leq \left(m\|v\|^2_{\mathcal{H}} + \nu\|w\|^2_{\mathcal{H}}\right)^{(p+1)/2}, \tag{2.1.18}$$

and

$$\|wu\|^{(p+1)/2}_{L^{(p+1)/2}_\theta} \leq \left(\nu\|w\|^2_{\mathcal{H}} + l\|u\|^2_{\mathcal{H}}\right)^{(p+1)/2}. \tag{2.1.19}$$

Thanks to Minkowski's inequality to give

$$\begin{aligned} \|u+v+w\|^{(p+1)}_{L^{(p+1)}_\theta} &\leq c\left(\|u\|^2_{L^{(p+1)}_\theta} + \|v\|^2_{L^{(p+1)}_\theta} + \|w\|^2_{L^{(p+1)}_\theta}\right)^{(p+1)/2} \\ &\leq c\left(\|u\|^2_{\mathcal{H}} + \|v\|^2_{\mathcal{H}} + \|w\|^2_{\mathcal{H}}\right)^{(p+1)/2}. \end{aligned}$$

Then there exist $\eta > 0$ such that

$$\begin{aligned} &\|u+v+w\|^{(p+1)}_{L^{(p+1)}_\theta} + 2\,\|uv\|^{(p+1)/2}_{L^{(p+1)/2}_\theta} + 2\,\|vw\|^{(p+1)/2}_{L^{(p+1)/2}_\theta} + 2\,\|wu\|^{(p+1)/2}_{L^{(p+1)/2}_\theta} \\ &\leq \eta\left(l\|u\|^2_{\mathcal{H}} + m\|v\|^2_{\mathcal{H}} + \nu\|w\|^2_{\mathcal{H}}\right)^{(p+1)/2}. \end{aligned} \tag{2.1.20}$$

We need to define positive constants λ_0 and $\mathcal{E}_0$ by

$$\lambda_0 \equiv \eta^{-1/(p-1)} \quad \text{and} \quad \mathcal{E}_0 = \left(\frac{1}{2} - \frac{1}{p+1}\right)\eta^{-2/(p-1)}. \tag{2.1.21}$$

The main aim of the present section is to obtain a novel decay rate of solution from the convexity property of the function χ given in Theorem 11.
We denote an eigenpair $\{(\lambda_i, e_i)\}_{i\in\mathbb{N}} \subset \mathbb{R}\times\mathcal{H}$ of

$$-\Theta(x)\Delta e_i = \lambda_i e_i \quad x \in \mathbb{R}^n,$$

for any $i \in \mathbb{N}$, $(\Theta(x))^{-1} \equiv \theta(x)$. Then

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots \uparrow +\infty,$$

holds and $\{e_i\}$ is a complete orthonormal system in $\mathcal{H}$.

Definition 8. *The triplet functions (u, v, w) is said a weak solution to (2.1.1) on $[0, T]$ if satisfies* *for* $x \in \mathbb{R}^n$,

$$\begin{cases} \int_{\mathbb{R}^n} \theta(x)u_{tt}\varphi dx + \alpha \int_{\mathbb{R}^n} \theta(x)u_t\varphi dx - \beta \int_{\mathbb{R}^n} \Delta u_t\varphi dx = \int_{\mathbb{R}^n} \Delta u - \int_0^t \varpi_1(t-s)\Delta u(s)\, ds\varphi dx \\ \qquad + \int_{\mathbb{R}^n} \theta(x)h_1(u,v,w)\varphi dx, \\ \int_{\mathbb{R}^n} \theta(x)v_{tt}\psi dx + \alpha \int_{\mathbb{R}^n} \theta(x)v_t\psi dx - \beta \int_{\mathbb{R}^n} \Delta v_t\psi dx = \int_{\mathbb{R}^n} \Delta v - \int_0^t \varpi_2(t-s)\Delta v(s)\, ds\psi dx \\ \qquad + \int_{\mathbb{R}^n} \theta(x)h_2(u,v,w)\psi dx, \\ \int_{\mathbb{R}^n} \theta(x)w_{tt}\Psi dx + \alpha \int_{\mathbb{R}^n} \theta(x)w_t\Psi dx - \beta \int_{\mathbb{R}^n} \Delta w_t\Psi dx = \int_{\mathbb{R}^n} \Delta w - \int_0^t \varpi_3(t-s)\Delta w(s)\, ds\Psi dx \\ \qquad + \int_{\mathbb{R}^n} \theta(x)h_3(u,v,w)\Psi dx, \end{cases} \tag{2.1.22}$$

for all test functions $\varphi, \psi, \Psi \in \mathcal{H}$ for almost all $t \in [0,T]$.

2.2 Main results

The next theorem is concerned on the local solution (in time $[0,T]$).

Theorem 9. *(Local existence) Assume that*

$$1 < p \leq \frac{n+2}{n-2} \quad \text{and that} \quad n \geq 3. \tag{2.2.1}$$

Let $(u_0, v_0, w_0) \in \mathcal{H}^3$ *and* $(u_1, v_1, w_3) \in L^2_\theta(\mathbb{R}^n) \times L^2_\theta(\mathbb{R}^n) \times L^2_\theta(\mathbb{R}^n)$. *Under the assumptions (2.1.2)-(2.1.4) and (2.1.11)-(2.1.16), suppose that*

$$\alpha + \lambda_1\beta > 0. \tag{2.2.2}$$

Then (2.1.1) admits a unique local solution (u, v, w) *such that*

$$\in \mathcal{X}_T^3,\ \mathcal{X}_T \equiv C\big([0,T];\mathcal{H}\big) \cap C^1\big([0,T];L^2_\theta(\mathbb{R}^n)\big),$$

for sufficiently small $T > 0$.

Remark 2. *The constant λ_1 introduced in (2.2.2) being the first eigenvalue of the operator $-\Delta$.*

We will show now the global solution in time established in Theorem 10. Let us introduce the potential energy $J : \mathcal{H}^3 \to \mathbb{R}$ defined by

$$\begin{aligned} J(u,v,w) &= \left(1 - \int_0^t \varpi_1(s)\, ds\right) \|u\|^2_{\mathcal{H}} + (\varpi_1 \circ u) \\ &+ \left(1 - \int_0^t \varpi_2(s)\, ds\right) \|v\|^2_{\mathcal{H}} + (\varpi_2 \circ v) \\ &+ \left(1 - \int_0^t \varpi_3(s)\, ds\right) \|w\|^2_{\mathcal{H}} + (\varpi_3 \circ w)\,. \end{aligned} \tag{2.2.3}$$

The modified energy is defined by

$$\mathcal{E}(t) = \frac{1}{2}\Big(\|u_t\|^2_{L^2_\theta} + \|v_t\|^2_{L^2_\theta} + \|w_t\|^2_{L^2_\theta} \Big) + \frac{1}{2} J(u,v,w) - \int_{\mathbb{R}^n} \theta(x)\mathcal{G}(u,v,w)dx, \tag{2.2.4}$$

here

$$(\varpi_j \circ w)(t) = \int_0^t \varpi_j(t-s) \|w(t) - w(s)\|^2_{\mathcal{H}}\, ds,$$

for any $w \in L^2(\mathbb{R}^n), j = 1,2,3$.

Theorem 10. *(Global existence) Let (2.1.2)-(2.1.4) and (2.1.11)-(2.1.16) hold. Under (2.2.1), (2.2.2) and for sufficiently small $(u_0,u_1),(v_0,v_1),(w_0,w_1) \in \mathcal{H} \times L^2_\theta(\mathbb{R}^n)$, problem (3.1.1) admits a unique global solution (u,v,w) such that*

$$(u,v,w) \in \mathcal{X}^3,\ \ \mathcal{X} \equiv C\big([0,+\infty);\mathcal{H}\big) \cap C^1\big([0,+\infty); L^2_\theta(\mathbb{R}^n)\big). \tag{2.2.5}$$

The non-classical decay rate for solution is given in the next Theorem

Theorem 11. *(Decay of solution) Let (2.1.2)-(2.1.19) and (2.1.11)-(2.1.16) hold. Under conditions (2.2.1), (2.2.2) and*

$$\gamma = \eta\Big(\frac{2(p+1)}{p-1}\mathcal{E}(0)\Big)^{(p-1)/2} < 1, \tag{2.2.6}$$

there exists $t_0 > 0$ depending only on $\varpi_1, \varpi_2, \varpi_3$, α, β, λ_1 and $\mathcal{X}'(0)$ such that

$$0 \le \mathcal{E}(t) < \mathcal{E}(t_0) \exp\left(-\int_{t_0}^t \frac{\varpi(s)}{1-\varpi_0(t)}\right), \tag{2.2.7}$$

holds for all $t \ge t_0$.

In particular, by the positively of ϖ in (2.1.14), we have, as in [33],

$$0 \leq \mathcal{E}(t) < \mathcal{E}(t_0) \exp\left(-\int_{t_0}^{t} \varpi(s)\, ds\right),$$

for a single wave equation. Condition (2.1.16) is imposed to make a different from [33] and [34], it leads $(\varpi' + \nu\varpi) \circ u$, here $\nu \in \mathbb{R}$.

The next, Lemma will play an important role in the sequel.

Lemma 7. *For $(u, v, w) \in \mathcal{X}_T^3$, the functional $\mathcal{E}(t)$ associated with problem (2.1.1) is a decreasing energy.*

Proof. For $0 \leq t_1 < t_2 \leq T$, we have

$$\begin{aligned}
&\mathcal{E}(t_2) - \mathcal{E}(t_1) \\
= &\int_{t_1}^{t_2} \frac{d}{dt} E(t)\, dt \\
= &-\int_{t_1}^{t_2} \left(\alpha \|u_t\|_{L_\theta^2}^2 + \beta \|u_t\|_{\mathcal{H}}^2 + \frac{1}{2}\varpi_1(t) \|u\|_{\mathcal{H}}^2 - \frac{1}{2}(\varpi_1' \circ u)\right) dt \\
- &\int_{t_1}^{t_2} \left(\alpha \|v_t\|_{L_\theta^2}^2 + \beta \|v_t\|_{\mathcal{H}}^2 + \frac{1}{2}\varpi_2(t) \|v\|_{\mathcal{H}}^2 - \frac{1}{2}(\varpi_2' \circ v)\right) dt \\
- &\int_{t_1}^{t_2} \left(\alpha \|w_t\|_{L_\theta^2}^2 + \beta \|w_t\|_{\mathcal{H}}^2 + \frac{1}{2}\varpi_3(t) \|w\|_{\mathcal{H}}^2 - \frac{1}{2}(\varpi_3' \circ w)\right) dt \\
\leq &\ 0,
\end{aligned}$$

owing to (2.1.11)-(2.1.16). □

The inner product is given as

$$(v, w)_* = \beta \int_{\mathbb{R}^n} \nabla v \cdot \nabla w\, dx + \alpha \int_{\mathbb{R}^n} \theta v w\, dx,$$

and the associated norm is given by

$$\|v\|_* = \sqrt{(v, v)_*},$$

$\forall v, w \in \mathcal{H}$. By (2.2.2), we get

$$(v, v)_* = \beta \int_{\mathbb{R}^n} |\nabla v|^2\, dx + \alpha \int_{\mathbb{R}^n} \theta v^2\, dx \geq (\beta\lambda_1 + a) \int_{\mathbb{R}^n} \theta v^2\, dx \geq 0.$$

The following lemma yields.

Lemma 8. *Let θ satisfy (2.1.2). Under condition (2.2.2), we get*

$$\sqrt{\beta}\,\|v\|_{\mathcal{H}} \le \|v\|_* \le \sqrt{\beta + C_P^2}\,\|v\|_{\mathcal{H}},$$

for $v \in \mathcal{H}$.

2.3 Proofs

We sketch here the outline of the proof for local solution by a standard procedure(See [13], [34]).

Proof. (Of Theorem 9.) Let $(u_0,u_1),(v_0,v_1),(w_0,w_1) \in \mathcal{H} \times L^2_\theta(\mathbb{R}^n)$. For any $(u,v,w) \in \mathcal{X}_T^3$, we can obtain a weak solution of the related system

$$\begin{cases} \theta(x)z_{tt} + \alpha\theta(x)z_t - \Delta\left(z + \beta z_t\right) = -\int_0^t \varpi_1(t-s)\Delta u(s)\,ds + \theta(x)h_1(u,v,w) \\ \theta(x)y_{tt} + \alpha\theta(x)y_t - \Delta\left(y + \beta y_t\right) = -\int_0^t \varpi_2(t-s)\Delta v(s)\,ds + \theta(x)h_2(u,v,w) \\ \theta(x)\zeta_{tt} + \alpha\theta(x)\zeta_t - \Delta\left(\zeta + \beta \zeta_t\right) = -\int_0^t \varpi_3(t-s)\Delta w(s)\,ds + \theta(x)h_3(u,v,w) \\ z(x,0) = u_0(x), y(x,0) = v_0(x), \zeta(x,0) = w_0(x) \\ z_t(x,0) = u_1(x), y_t(x,0) = v_1(x), \zeta_t(x,0) = w_1(x). \end{cases} \tag{2.3.1}$$

We reduces problem (2.3.1) to Cauchy problem for system of ODE by using the Faedo-Galerkin approximation. We then find a solution map $\top : (u,v,w) \mapsto (z,y,\zeta)$ from $\mathcal{X}_T^3$ to $\mathcal{X}_T^3$. We are now ready to show that $\top$ is a contraction mapping in an appropriate subset of $\mathcal{X}_T^3$ for a small $T > 0$. Hence $\top$ has a fixed point $\top(u,v,w) = (u,v,w)$, which gives a unique solution in $\mathcal{X}_T^3$. □

We will show the global solution. By using conditions on functions $\varpi_1, \varpi_2, \varpi_3$, we have

$$\begin{aligned} \mathcal{E}(t) &\ge \frac{1}{2}J(u,v,w) - \int_{\mathbb{R}^n} \theta(x)\mathcal{G}(u,v,w)dx \\ &\ge \frac{1}{2}J(u,v,w) - \frac{1}{p+1}\|u+v+w\|_{L_\theta^{(p+1)}}^{(p+1)} \\ &- \frac{2}{p+1}\Big(\|uv\|_{L_\theta^{(p+1)/2}}^{(p+1)/2} + \|vw\|_{L_\theta^{(p+1)/2}}^{(p+1)/2} + \|wu\|_{L_\theta^{(p+1)/2}}^{(p+1)/2} \Big) \\ &\ge \frac{1}{2}J(u,v,w) - \frac{\eta}{p+1}\Big[l\,\|u\|_{\mathcal{H}}^2 + m\,\|v\|_{\mathcal{H}}^2 + \nu\,\|w\|_{\mathcal{H}}^2 \Big]^{(p+1)/2} \\ &\ge \frac{1}{2}J(u,v,w) - \frac{\eta}{p+1}\Big(J(u,v,w) \Big)^{(p+1)/2} \\ &= G\left(\varsigma\right), \end{aligned} \tag{2.3.2}$$

here $\varsigma^2 = J(u, v, w)$, for $t \in [0, T)$, where

$$G(\xi) = \frac{1}{2}\xi^2 - \frac{\eta}{p+1}\xi^{(p+1)}.$$

Noting that $\mathcal{E}_0 = G(\lambda_0)$, given in (2.1.21). Then

$$\begin{cases} G'(\xi) \geq 0 \quad in \quad \xi \in [0, \lambda_0] \\ G'(\xi) < 0 \quad in \quad \xi > \lambda_0. \end{cases} \tag{2.3.3}$$

Moreover, $\lim\limits_{\xi \to +\infty} G(\xi) \to -\infty$. Then, we have the following lemma

Lemma 9. *Let* $0 \leq \mathcal{E}(0) < \mathcal{E}_0$.

(i) If $\|u_0\|^2_{\mathcal{H}} + \|v_0\|^2_{\mathcal{H}} + \|w_0\|^2_{\mathcal{H}} < \lambda_0^2$, *then local solution of (2.1.1) satisfies*

$$J(u, v, w) < \lambda_0^2, \ \forall t \in [0, T).$$

(ii) If $\|u_0\|^2_{\mathcal{H}} + \|v_0\|^2_{\mathcal{H}} + \|w_0\|^2_{\mathcal{H}} > \lambda_0^2$, *then local solution of (2.1.1) satisfies*

$$\|u\|^2_{\mathcal{H}} + \|v\|^2_{\mathcal{H}} + \|w\|^2_{\mathcal{H}} > \lambda_1^2, \ \forall t \in [0, T), \lambda_1 > \lambda_0.$$

Proof. Since $0 \leq \mathcal{E}(0) < \mathcal{E}_0 = G(\lambda_0)$, there exist ξ_1 and ξ_2 such that $G(\xi_1) = G(\xi_2) = \mathcal{E}(0)$ with $0 < \xi_1 < \lambda_0 < \xi_2$.

The case (i)**.** By (2.3.2), we have

$$G(J(u_0, v_0, w_0)) \leq \mathcal{E}(0) = G(\xi_1),$$

which implies that $J(u_0, v_0, w_0) \leq \xi_1^2$. Then we claim that $J(u, v, w) \leq \xi_1^2, \ \forall t \in [0, T)$. Moreover, there exists $t_0 \in (0, T)$ such that

$$\xi_1^2 < J(u(t_0), v(t_0), w(t_0)) < \xi_2^2.$$

Then

$$G(J(u(t_0), v(t_0), w(t_0))) > \mathcal{E}(0) \geq \mathcal{E}(t_0),$$

by Lemma 7, which contradicts (2.3.2). Hence we have

$$J(u, v, w) \leq \xi_1^2 < \lambda_0^2, \ \forall t \in [0, T).$$

The case (ii)**.** We can now show that $\|u_0\|^2_{\mathcal{H}} + \|v_0\|^2_{\mathcal{H}} + \|w_0\|^2_{\mathcal{H}} \geq \xi_2^2$ and that $\|u\|^2_{\mathcal{H}} + \|v\|^2_{\mathcal{H}} + \|w\|^2_{\mathcal{H}} \geq \xi_2^2 > \lambda_0^2$ in the same way as (i). □

Proof. (*Of Theorem 10.*) Let $(u_0, u_1), (v_0, v_1), (w_0, w_1) \in \mathcal{H} \times L^2_\theta(\mathbb{R}^n)$ satisfy both $0 \leq \mathcal{E}(0) < \mathcal{E}_0$ and $\|u_0\|^2_{\mathcal{H}} + \|v_0\|^2_{\mathcal{H}} + \|w_0\|^2_{\mathcal{H}} < \lambda_0^2$. By Lemma 7 and Lemma 9, we have

$$\begin{aligned}
&\frac{1}{2}\Big(\|u_t\|^2_{L^2_\theta} + \|v_t\|^2_{L^2_\theta} + \|w_t\|^2_{L^2_\theta} \Big) + l\,\|u\|^2_{\mathcal{H}} + m\,\|v\|^2_{\mathcal{H}} + \nu\,\|w\|^2_{\mathcal{H}} \\
&\leq \frac{1}{2}\Big(\|u_t\|^2_{L^2_\theta} + \|v_t\|^2_{L^2_\theta} + \|w_t\|^2_{L^2_\theta} \Big) + \left(1 - \int_0^t \varpi_1(s)\,ds\right) \|u\|^2_{\mathcal{H}} + (\varpi_1 \circ u) \\
&+ \left(1 - \int_0^t \varpi_2(s)\,ds\right) \|u\|^2_{\mathcal{H}} + (\varpi_2 \circ v) + \left(1 - \int_0^t \varpi_3(s)\,ds\right) \|w\|^2_{\mathcal{H}} + (\varpi_3 \circ w) \\
&\leq 2\mathcal{E}(t) + \frac{2\eta}{p+1}\Big[l\,\|u\|^2_{\mathcal{H}} + m\,\|u\|^2_{\mathcal{H}} + \nu\,\|w\|^2_{\mathcal{H}} \Big]^{(p+1)/2} \\
&\leq 2\mathcal{E}(0) + \frac{2\eta}{p+1}\Big(J(u,v,w) \Big)^{(p+1)/2} \\
&\leq 2\mathcal{E}_0 + \frac{2\eta}{p+1}\lambda_0^{p+1} \\
&= \eta^{-2/(p-1)}.
\end{aligned} \tag{2.3.4}$$

This completes the proof. □

Let

$$\begin{aligned}
\Lambda(u,v,w) \;=\; & \frac{1}{2}\left(1 - \int_0^t \varpi_1(s)\,ds\right) \|u\|^2_{\mathcal{H}} + \frac{1}{2}(\varpi_1 \circ u) \\
+\; & \frac{1}{2}\left(1 - \int_0^t \varpi_2(s)\,ds\right) \|v\|^2_{\mathcal{H}} + \frac{1}{2}(\varpi_2 \circ v) \\
+\; & \frac{1}{2}\left(1 - \int_0^t \varpi_3(s)\,ds\right) \|w\|^2_{\mathcal{H}} + \frac{1}{2}(\varpi_3 \circ w) - \int_{\mathbb{R}^n} \theta(x)\mathcal{G}(u,v,w)dx,
\end{aligned} \tag{2.3.5}$$

$$\begin{aligned}
\Pi(u,v,w) \;=\; & \left(1 - \int_0^t \varpi_1(s)\,ds\right) \|u\|^2_{\mathcal{H}} + (\varpi_1 \circ u) \\
+\; & \left(1 - \int_0^t \varpi_2(s)\,ds\right) \|v\|^2_{\mathcal{H}} + (\varpi_2 \circ v) \\
+\; & \left(1 - \int_0^t \varpi_3(s)\,ds\right) \|w\|^2_{\mathcal{H}} + (\varpi_3 \circ w) - (p+1)\int_{\mathbb{R}^n} \theta(x)\mathcal{G}(u,v,w)dx.
\end{aligned} \tag{2.3.6}$$

Lemma 10. *Let (u, v, w) be the solution of problem (2.1.1). If*

$$\|u_0\|^2_{\mathcal{H}} + \|v_0\|^2_{\mathcal{H}} + \|w_0\|^2_{\mathcal{H}} - (p+1)\int_{\mathbb{R}^n} \theta(x)\mathcal{G}(u_0,v_0,w_0)dx > 0. \tag{2.3.7}$$

Then under condition (2.2.6), the functional $\Pi(u,v,w) > 0, \;\forall t > 0$.

Proof. By (2.3.7) and continuity, there exists a time $t_1 > 0$ such that

$$\Pi(u, v, w) \geq 0, \forall t < t_1.$$

Let

$$Y = \{(u, v, w) \mid \Pi(u(t_0), v(t_0), w(t_0)) = 0,\ \Pi(u, v, w) > 0, \forall t \in [0, t_0)\}. \tag{2.3.8}$$

Then, by (2.3.5), (2.3.6), we have for all $(u, v, w) \in Y$,

$$\begin{aligned}
&\Lambda(u, v, w)\\
&= \frac{p-1}{2(p+1)}\Big[\Big(1 - \int_0^t \varpi_1(s)\,ds\Big)\|u\|_{\mathcal{H}}^2 + \Big(1 - \int_0^t \varpi_2(s)\,ds\Big)\|v\|_{\mathcal{H}}^2 + \Big(1 - \int_0^t \varpi_3(s)\,ds\Big)\|w\|_{\mathcal{H}}^2\Big]\\
&+ \frac{p-1}{2(p+1)}\Big[(\varpi_1 \circ u) + (\varpi_2 \circ v) + (\varpi_3 \circ w)\Big] + \frac{1}{p+1}\Pi(u, v, w)\\
&\geq \frac{p-1}{2(p+1)}\Big[l\,\|u\|_{\mathcal{H}}^2 + m\,\|v\|_{\mathcal{H}}^2 + \nu\,\|w\|_{\mathcal{H}}^2 + (\varpi_1 \circ u) + (\varpi_2 \circ v) + (\varpi_3 \circ w)\Big].
\end{aligned}$$

Owing to (2.2.4), it follows for $(u, v, w) \in Y$

$$l\,\|u\|_{\mathcal{H}}^2 + m\,\|v\|_{\mathcal{H}}^2 + \nu\,\|w\|_{\mathcal{H}}^2 \leq \frac{2(p+1)}{p-1}\Lambda(u, v, w) \leq \frac{2(p+1)}{p-1}\mathcal{E}(t) \leq \frac{2(p+1)}{p-1}\mathcal{E}(0). \tag{2.3.9}$$

By (2.1.20), (2.2.6) we have

$$\begin{aligned}
(p+1)\int_{\mathbb{R}^n} \mathcal{G}(u(t_0), v(t_0), w(t_0)) \;&\leq\; \eta\,\big(l\|u(t_0)\|_{\mathcal{H}}^2 + m\|v(t_0)\|_{\mathcal{H}}^2 + \nu\,\|w(t_0)\|_{\mathcal{H}}^2\big)^{(p+1)/2}\\
&\leq\; \eta\Big(\frac{2(p+1)}{p-1}E(0)\Big)^{(p-1)/2}(l\,\|u(t_0)\|_{\mathcal{H}}^2 + m\,\|v(t_0)\|_{\mathcal{H}}^2 + \nu\,\|w(t_0)\|_{\mathcal{H}}^2)\\
&\leq\; \gamma(l\,\|u(t_0)\|_{\mathcal{H}}^2 + m\,\|v(t_0)\|_{\mathcal{H}}^2 + \nu\,\|w(t_0)\|_{\mathcal{H}}^2)\\
&<\; \Big(1 - \int_0^{t_0} \varpi_1(s)ds\Big)\|u(t_0)\|_{\mathcal{H}}^2 + \Big(1 - \int_0^{t_0} \varpi_2(s)ds\Big)\|v(t_0)\|_{\mathcal{H}}^2\\
&+\; \Big(1 - \int_0^{t_0} \varpi_3(s)ds\Big)\|w(t_0)\|_{\mathcal{H}}^2\\
&<\; \Big(1 - \int_0^{t_0} \varpi_1(s)ds\Big)\|u(t_0)\|_{\mathcal{H}}^2 + \Big(1 - \int_0^{t_0} \varpi_2(s)ds\Big)\|v(t_0)\|_{\mathcal{H}}^2\\
&+\; \Big(1 - \int_0^{t_0} \varpi_3(s)ds\Big)\|w(t_0)\|_{\mathcal{H}}^2\\
&+\; (\varpi_1 \circ u) + (\varpi_2 \circ v) + (\varpi_3 \circ w),
\end{aligned} \tag{2.3.10}$$

hence $\Pi(u(t_0), v(t_0), w(t_0)) > 0$ on Y, which contradicts the definition of Y since $\Pi(u(t_0), v(t_0), w(t_0)) = 0$. Thus $\Pi(u, v, w) > 0,\ \forall t > 0$. □

We are ready to prove the decay rate.

Proof. (*Of Theorem 11.*) By (2.1.20) and (2.3.9), we have for $t \geq 0$

$$0 < l \|u\|_{\mathcal{H}}^2 + m \|v\|_{\mathcal{H}}^2 + \nu \|w\|_{\mathcal{H}}^2 \leq \frac{2(p+1)}{p-1}\mathcal{E}(t). \tag{2.3.11}$$

Let

$$I(t) = \frac{\varpi(t)}{1-\varpi_0(t)},$$

where ϖ and ϖ_0 defined in (2.1.14) and (2.1.15).

Noting that $\lim_{t\to+\infty} \varpi(t) = 0$ by (2.1.11)-(2.1.15), we have

$$\lim_{t\to+\infty} I(t) = 0, \quad I(t) > 0, \ \forall t \geq 0.$$

Then we take $t_0 > 0$ such that

$$0 < \frac{1}{2}I(t) < \min\left\{2\left(\beta\lambda_1 + a\right), \chi'(0)\right\},$$

with (2.1.16) for all $t > t_0$. Due to (2.2.4), we have

$$\begin{aligned}
\mathcal{E}(t) &\leq \frac{1}{2}\Big(\|u_t\|_{L^2_\theta}^2 + \|v_t\|_{L^2_\theta}^2 + \|w_t\|_{L^2_\theta}^2 \Big) + \frac{1}{2}[(\varpi_1 \circ u) + (\varpi_2 \circ v) + (\varpi_3 \circ w)] \\
&+ \frac{1}{2}\left(1 - \int_0^t \varpi_1(s)\,ds\right) \|u\|_{\mathcal{H}}^2 + \frac{1}{2}\left(1 - \int_0^t \varpi_2(s)\,ds\right) \|v\|_{\mathcal{H}}^2 + \frac{1}{2}\left(1 - \int_0^t \varpi_3(s)\,ds\right) \|w\|_{\mathcal{H}}^2 \\
&\leq \frac{1}{2}\Big(\|u_t\|_{L^2_\theta}^2 + \|v_t\|_{L^2_\theta}^2 + \|w_t\|_{L^2_\theta}^2 \Big) + \frac{1}{2}[(\varpi_1 \circ u) + (\varpi_2 \circ v) + (\varpi_3 \circ w)] \\
&+ \frac{1}{2}(1-\varpi_0(t))[\|u\|_{\mathcal{H}}^2 + \|v\|_{\mathcal{H}}^2 + \|w\|_{\mathcal{H}}^2].
\end{aligned}$$

Then by definition of $I(t)$, we have

$$\begin{aligned}
I(t)\mathcal{E}(t) &\leq \frac{1}{2}I(t)\Big(\|u_t\|_{L^2_\theta}^2 + \|v_t\|_{L^2_\theta}^2 + \|w_t\|_{L^2_\theta}^2 \Big) + \frac{1}{2}\varpi(t)[\|u\|_{\mathcal{H}}^2 + \|v\|_{\mathcal{H}}^2 + \|w\|_{\mathcal{H}}^2] \\
&+ \frac{1}{2}I(t)[(\varpi_1 \circ u) + (\varpi_2 \circ v) + (\varpi_3 \circ w)],
\end{aligned} \tag{2.3.12}$$

and Lemma 7, we have for all $t_1, t_2 \geq 0$

$$\begin{aligned}
&\mathcal{E}(t_2) - \mathcal{E}(t_1) \\
&\leq -\int_{t_1}^{t_2} \left(\alpha \|w_t\|_{L^2_\theta}^2 + \alpha \|u_t\|_{L^2_\theta}^2 + \beta \|u_t\|_{\mathcal{H}}^2 + \frac{1}{2}\varpi(t)[\|u\|_{\mathcal{H}}^2 + \|v\|_{\mathcal{H}}^2 + \|w\|_{\mathcal{H}}^2]\right) dt \\
&- \int_{t_1}^{t_2} \left(\alpha \|v_t\|_{L^2_\theta}^2 + \beta \|v_t\|_{\mathcal{H}}^2 + \beta \|w_t\|_{\mathcal{H}}^2 - \frac{1}{2}(\varpi_1' \circ u) - \frac{1}{2}(\varpi_2' \circ v) - \frac{1}{2}(\varpi_3' \circ w)\right) dt,
\end{aligned}$$

then, by generalized Poincaré's inequalities, we get

$$\begin{aligned}\mathcal{E}'(t) &\leq -(\beta\lambda_1+\alpha)\left[\|u_t\|^2_{L^2_\theta}+\|v_t\|^2_{L^2_\theta}+\|w_t\|^2_{L^2_\theta}\right]\\ &- \frac{1}{2}\varpi(t)[\|u\|^2_{\mathcal{H}}+\|v\|^2_{\mathcal{H}}+\|w\|^2_{\mathcal{H}}]\\ &+ \frac{1}{2}[(\varpi_1'\circ u)+(\varpi_2'\circ v)+(\varpi_3'\circ w)],\end{aligned}$$

Finally, $\forall t \geq t_0$, we have

$$\begin{aligned}&\mathcal{E}'(t)+I(t)\mathcal{E}(t)\\ &\leq \left\{\frac{1}{2}I(t)-(\beta\lambda_1+\alpha)\right\}\left(\|u_t\|^2_{L^2_\theta}+\|v_t\|^2_{L^2_\theta}+\|w_t\|^2_{L^2_\theta}\right)\\ &+ \frac{1}{2}[(\varpi_1'\circ u)+(\varpi_2'\circ v)+(\varpi_3'\circ w)]+\frac{1}{2}I(t)((\varpi_1\circ u)+(\varpi_2\circ v)+(\varpi_3\circ w))\\ &\leq \frac{1}{2}\int_0^t\{\varpi_1'(t-\tau)+I(t)\varpi_2(t-\tau)\}\|u(t)-u(\tau)\|^2_{\mathcal{H}}\,d\tau\\ &+ \frac{1}{2}\int_0^t\{\varpi_2'(t-\tau)+I(t)\varpi_2(t-\tau)\}\|v(t)-v(\tau)\|^2_{\mathcal{H}}\,d\tau\\ &+ \frac{1}{2}\int_0^t\{\varpi_3'(t-\tau)+I(t)\varpi_3(t-\tau)\}\|w(t)-w(\tau)\|^2_{\mathcal{H}}\,d\tau\\ &\leq \frac{1}{2}\int_0^t\{\varpi_1'(\tau)+I(t)\varpi_1(\tau)\}\|u(t)-u(t-\tau)\|^2_{\mathcal{H}}\,d\tau\\ &+ \frac{1}{2}\int_0^t\{\varpi_2'(\tau)+I(t)\varpi_2(\tau)\}\|v(t)-v(t-\tau)\|^2_{\mathcal{H}}\,d\tau\\ &+ \frac{1}{2}\int_0^t\{\varpi_3'(\tau)+I(t)\varpi_3(\tau)\}\|w(t)-w(t-\tau)\|^2_{\mathcal{H}}\,d\tau\\ &\leq \frac{1}{2}\int_0^t\left\{-\chi\Big(\varpi_1(\tau)\Big)+\chi'(0)\varpi_1(\tau)\right\}\|u(t)-u(t-\tau)\|^2_{\mathcal{H}}\,d\tau\\ &+ \frac{1}{2}\int_0^t\left\{-\chi\Big(\varpi_2(\tau)\Big)+\chi'(0)\varpi_2(\tau)\right\}\|v(t)-v(t-\tau)\|^2_{\mathcal{H}}\,d\tau\\ &+ \frac{1}{2}\int_0^t\left\{-\chi\Big(\varpi_3(\tau)\Big)+\chi'(0)\varpi_3(\tau)\right\}\|w(t)-w(t-\tau)\|^2_{\mathcal{H}}\,d\tau\\ &\leq 0,\end{aligned}$$

by the convexity of χ and (2.1.16), we have

$$\chi(\xi)\geq\chi(0)+\chi'(0)\xi=\chi'(0)\xi.$$

Then

$$\mathcal{E}(t)\leq\mathcal{E}(t_0)\exp\left(-\int_{t_0}^t I(s)ds\right),$$

which completes the proof. $\square$

Chapter 3

Systems of m-nonlinear viscoelastic wave equations

The chapter discusses the effect of weak and strong damping terms on decay rate for systems of nonlinear m- wave equations in viscoelasticity. The factors that allowed system (3.1.1) to coexist for a long time are the strong nonlinearities in the sources. We showed, under a novel condition on the kernel function in (3.1.14), a new scenario for energy decay in (3.2.7) by using an appropriate energy estimates. This result extend our last result in [35], [39] for system of m-equations inspired from the paper [7].

3.1 Introduction and position of problem

We consider, for $x \in \mathbb{R}^n,\ t > 0,\ j = 1, 2, \dots, m$, the following system of m equations

$$\begin{cases} \left(|u_{jt}|^{\kappa-2}u_{jt}\right)_t + au_{jt} - \Theta(x)\Delta\left(u_j + \omega u_{jt} - \int_0^t \varpi_j(t-s)u_j(s)\,ds\right) = f_j(u_1, u_2, \dots, u_m) \\ u_j(x, 0) = u_{j0}(x), \\ u_{jt}(x, 0) = u_{j1}(x), \end{cases} \tag{3.1.1}$$

where $a \in \mathbb{R}$, $\omega > 0$, $n \geq 3, \kappa \geq 2$.

Various non-linear sources have been combined as follows, we combine all two consecutive equations together and of course the last equation with the first one, which get the whole system closely linked by the strong nonlinear sources. The functions $f_j(u_1, u_2, \dots, u_m) \in (\mathbb{R}^m, \mathbb{R})$ are given for $j = 1, 2, \dots, m-1$, by

$$f_j(u_1, u_2, \dots, u_m) = (p+1)\Big[d\Big|\sum_{i=1}^{m} u_i\Big|^{(p-1)} \sum_{i=1}^{m} u_i + e|u_j|^{(p-3)/2}u_j|u_{j+1}|^{(p+1)/2}\Big],$$

and

$$f_m(u_1, u_2, \dots, u_m) = (p+1)\Big[d\Big|\sum_{i=1}^{m} u_i\Big|^{(p-1)} \sum_{i=1}^{m} u_i + e|u_m|^{(p-3)/2}u_m|u_1|^{(p+1)/2}\Big],$$

with $d, e >,\ p > 3$.

There exists a function $\mathcal{F} \in C^1(\mathbb{R}^3, \mathbb{R})$ such that

$$\sum_{j=1}^{m} u_j f_j(u_1, u_2, \dots, u_m) = (p+1)\mathcal{F}(u_1, u_2, \dots, u_m),\ \forall (u_1, u_2, \dots, u_m) \in \mathbb{R}^m. \tag{3.1.2}$$

satisfies

$$(p+1)\mathcal{F}(u_1, u_2, \ldots, u_m) = \Big|\sum_{j=1}^{m} u_j\Big|^{p+1} + 2\Big|\sum_{j=1}^{m-1} u_j u_{j+1}\Big|^{(p+1)/2} + 2|u_m u_1|^{(p+1)/2}. \qquad (3.1.3)$$

In order to use Poincare's inequality which is a key in calculus for the PDEs, we will study the problem (3.1.1) in the presence of a density function θ to find a generalized formula for Poincare's inequality that can be used in unbounded domain $\mathbb{R}^n$. The function $\Theta(x) > 0$ for all $x \in \mathbb{R}^n$ is a density and $(\Theta)^{-1} = 1/\Theta(x) \equiv \theta(x)$ such that

$$\theta \in L^{\tau}(\mathbb{R}^n) \quad \text{with} \quad \tau = \frac{2n}{2n - rn + 2r} \quad \text{for} \quad 2 \leq r \leq \frac{2n}{n-2}. \qquad (3.1.4)$$

We define a new spaces related to the nature of our system, taking into account the boundless of spaces $\mathbb{R}^n$. The function spaces $\mathcal{H}$ is defined as the closure of $C_0^{\infty}(\mathbb{R}^n)$, as in [30], we have

$$\mathcal{H} = \{v \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) \mid \nabla v \in L^2(\mathbb{R}^n)^n\}.$$

with respect to the norm $\|v\|_{\mathcal{H}} = (v, v)_{\mathcal{H}}^{1/2}$ for the inner product

$$(v, w)_{\mathcal{H}} = \int_{\mathbb{R}^n} \nabla v \cdot \nabla w \, dx,$$

and $L^2_\theta(\mathbb{R}^n)$ as that to the norm $\|v\|_{L^2_\theta} = (v, v)_{L^2_\theta}^{1/2}$ for

$$(v, w)_{L^2_\theta} = \int_{\mathbb{R}^n} \theta v w \, dx.$$

For general $r \in [1, +\infty)$

$$\|v\|_{L^r_\theta} = \left(\int_{\mathbb{R}^n} \theta \, |v|^r \, dx\right)^{\frac{1}{r}}.$$

is the norm of the weighted space $L^r_\theta(\mathbb{R}^n)$.

The main aim of this work is to consider an important problem from the point of view of application in sciences and engineering, namely, a system of m wave equations having a different damping effects in an unbounded domain with strong external forces including damping terms of memory type with past history. Using the Faedo-Galerkin method and some energy estimates, we proved the existence of global solution in $\mathbb{R}^n$ owing to the weighted function. By imposing a new appropriate condition, which not be used in the literature, with the help of some special estimates and generalized Poincaré's inequality, we obtained an unusual decay rate for the energy function. The work brings new contributions to the prior literature mainly in what concerns new decay rate estimates of the energy. The following references in connection to our system for a single equation [24] and [25]. The work [24] was the pioneer in the literature for the single equation, source of inspiration of several works, while the work [25] is a recent generalization of [24] by introducing less dissipative effects.

With regard to the study of this type of systems without viscoelasticity, with the existence of both weak damping u_t and strong damping Δu_t, under condition (3.2.2), here we mention the work recently published in one equation in [22]

$$\begin{cases} u_{tt} + \mu u_t - \Delta u - \omega \Delta u_t = u \ln |u|, \ (x,t) \in \Omega \times (0,\infty) \\ u(x,t) = 0, x \in \partial\Omega, t \geq 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \end{cases} \tag{3.1.5}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$ with a smooth boundary $\partial\Omega$. The aim goal was mainely on the local existence of weak solution by using contraction mapping principle and of course the authors showed the global existence, decay rate and infinite time blow up of the solution with certain conditions on initial energy.

In the case of non-bounded domain $\mathbb{R}^n$, we mention the paper recently published by T. Miyasita and Kh. Zennir in [35], where the considered equation as follows

$$u_{tt} + a u_t - \phi(x)\Delta\left(u + \omega u_t - \int_0^t g(t-s)u(s)\,ds\right) = u|u|^{p-1}, \tag{3.1.6}$$

with initial data

$$\begin{cases} u(x,0) = u_0(x), \\ u_t(x,0) = u_1(x). \end{cases} \tag{3.1.7}$$

The authors was successful in highlighting the existence of unique local solution and they continued to extend it to be global in time. The rate of the decay for solution was the main result, for more results related to decay rate of solution of this type of problems, please see [13], [28], [18], [34], ...

Regarding the study of the coupled system of two nonlinear wave equations, it is worth recalling the work by Baowei Feng and *al.* which was considered in [**?**], a coupled system for viscoelastic wave equations with nonlinear sources in bounded domain with smooth boundary as follows

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + u_t = f_1(u,v) \\ \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(s)\,ds + v_t = f_2(u,v). \end{cases} \tag{3.1.8}$$

Under appropriate hypotheses, they established a general decay result by multiplication techniques to extends some existing results for a single equation to the case of a coupled system.
It is worth noting that there are several studies in this field and we particularly refer to the generalization that Shun and *al.* made in studying a complicate non-linear case with degenerate damping term in [37]. The IBVP for a system of nonlinear viscoelastic wave equations in a bounded domain was considered in the problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + (|u|^k + |v|^q)|u_t|^{m-1}u_t = f_1(u,v), \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(s)\,ds + (|v|^\theta + |u|^\rho)|v_t|^{r-1}v_t = f_2(u,v), \\ u(x,t) = v(x,t) = 0, x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x) \\ u_t(x,0) = u_1(x), v_t(x,0) = v_1(x), \end{cases} \tag{3.1.9}$$

where Ω is a bounded domain with a smooth boundary. Given certain conditions on the kernel functions, degenerate damping and nonlinear source terms, they got a decay rate of the energy function for some initial data.
In $n-$equations, paper in [7] considered a system

$$u_{itt} + \gamma u_{it} - \Delta u_i + u_i = \sum_{i,j=1, i\neq j}^{m} |u_j|^{p_j}|u_i|^{p_i}u_i, \;\; i = 1, 2, \ldots, m, \tag{3.1.10}$$

where the absence of global solutions with positive initial energy was investigated. Next, a nonexistence of global solutions for system of three semilinear hyperbolic equations was introduced in [5]. A coupled system of semilinear hyperbolic equations was investigated by many authors and a different results were obtained with the nonlinearities in the form $f_1 = |u|^{p-1}|v|^{q+1}u, f_2 = |v|^{p-1}|u|^{q+1}v$. (Please, see [4], [23], [38], ...)
We introduce a very useful Sobolev embedding and generalized Poincaré inequalities.

Lemma 11. *[35] Let θ satisfy (3.1.4). For positive constants $C_\tau > 0$ and $C_P > 0$ depending only on θ and n, we have*

$$\|v\|_{\frac{2n}{n-2}} \leq C_\tau \|v\|_{\mathcal{H}},$$

and

$$\|v\|_{L^2_\theta} \leq C_P \|v\|_{\mathcal{H}},$$

for $v \in \mathcal{H}$.

Lemma 12. *[29] Let θ satisfy (3.1.4), then the estimates*

$$\|v\|_{L^r_\theta} \leq C_r \|v\|_{\mathcal{H}},$$

and

$$C_r = C_\tau \|\theta\|_\tau^{\frac{1}{r}},$$

hold for $v \in \mathcal{H}$. Here $\tau = 2n/(2n - rn + 2r)$ for $1 \leq r \leq 2n/(n-2)$.

In the fifties and seventies of the last century, the linear theory of viscoelasticity was developed extensively and at the present, it has become widely used to represent this nucleus using several improvements to the nature of decreasing the kernel function. We assume that the kernel functions $\varpi_j \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfying

$$1 - \overline{\varpi_j} = \rho_j > 0 \quad \text{for} \quad \overline{\varpi_j} = \int_0^{+\infty} \varpi_j(s)\, ds, \ \varpi_j'(t) \leq 0, \tag{3.1.11}$$

we mean by $\mathbb{R}^+$ the set $\{\tau \mid \tau \geq 0\}$. Noting by

$$\mu(t) = \max_{t \geq 0} \Big\{\varpi_1(t), \varpi_2(t), \ldots, \varpi_m(t)\Big\}, \tag{3.1.12}$$

and

$$\mu_0(t) = \min_{t \geq 0} \Big\{ \int_0^t \varpi_1(s)ds, \int_0^t \varpi_2(s)ds, \ldots, \int_0^t \varpi_m(s)ds\Big\}. \tag{3.1.13}$$

We assume that there is a function $\chi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$, such that the novel properties

$$\varpi_j'(t) + \chi(\varpi_j(t)) \leq 0, \quad \chi(0) = 0, \quad \chi'(0) > 0 \quad \text{and} \quad \chi''(\xi) \geq 0, \ i = 1, 2, \ldots, m, \tag{3.1.14}$$

satisfied for any $\xi \geq 0$.

Holder and Young's inequalities give

$$\begin{aligned} \|u_i u_j\|_{L_\theta^{(p+1)/2}}^{(p+1)/2} &\leq \left(\|u_i\|_{L_\theta^{(p+1)}}^2 + \|u_j\|_{L_\theta^{(p+1)}}^2\right)^{(p+1)/2} \\ &\leq \left(\rho_i \|u_i\|_{\mathcal{H}}^2 + \rho_j \|u_j\|_{\mathcal{H}}^2\right)^{(p+1)/2}, \end{aligned} \tag{3.1.15}$$

Thanks to Minkowski's inequality to give

$$\begin{aligned} \left\|\sum_{j=1}^m u_j\right\|_{L_\theta^{(p+1)}}^{(p+1)} &\leq c\left(\sum_{j=1}^m \|u_j\|_{L_\theta^{(p+1)}}^2\right)^{(p+1)/2} \\ &\leq c\left(\sum_{j=1}^m \|u_j\|_{\mathcal{H}}^2\right)^{(p+1)/2}. \end{aligned}$$

Then there exist $\eta > 0$ such that

$$\begin{aligned} &\left\|\sum_{j=1}^m u_j\right\|_{L_\theta^{(p+1)}}^{(p+1)} + 2\left\|\sum_{j=1}^{m-1} u_j u_{j+1}\right\|_{L_\theta^{(p+1)/2}}^{(p+1)/2} + 2\|u_m u_1\|_{L_\theta^{(p+1)/2}}^{(p+1)/2} \\ &\leq \eta\left(\sum_{j=1}^m \rho_j \|u_j\|_{\mathcal{H}}^2\right)^{(p+1)/2}. \end{aligned} \tag{3.1.16}$$

We need to define positive constants λ_0 and $\mathcal{E}_0$ by

$$\lambda_0 \equiv \eta^{-1/(p-1)} \quad \text{and} \quad \mathcal{E}_0 = \left(\frac{1}{2} - \frac{1}{p+1}\right)\eta^{-2/(p-1)}. \tag{3.1.17}$$

The mainely aim of the present paper is to obtain a novel decay rate of solution from the convexity property of the function χ given in Theorem 14.

We denote an eigenpair $\{(\lambda_i, e_i)\}_{i\in\mathbb{N}} \subset \mathbb{R} \times \mathcal{H}$ of

$$-\Theta(x)\Delta e_i = \lambda_i e_i \quad x \in \mathbb{R}^n,$$

for any $i \in \mathbb{N}$. Then

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots \uparrow +\infty,$$

holds and $\{e_i\}$ is a complete orthonormal system in $\mathcal{H}$.

Definition 9. *The vectors $(u_1, u_2, \ldots, u_m)$ is said a weak solution to (3.1.1) on $[0,T]$ if satisfies for $x \in \mathbb{R}^n$*

$$\begin{aligned}\int_{\mathbb{R}^n} \left(|u_{jt}|^{\kappa-2}u_{jt}\right)_t \varphi_j dx + a\int_{\mathbb{R}^n} u_{jt}\varphi_j dx \quad &- \quad \int_{\mathbb{R}^n} \Theta(x)\Delta\left(u_j + \omega u_{jt} - \int_0^t \varpi_j(t-s)u_j(s)\,ds\right)\varphi_j dx \\ &= \quad \int_{\mathbb{R}^n} f_j(u_1, u_2, \ldots, u_m)\varphi_j dx, \qquad (3.1.18)\end{aligned}$$

for all test functions $\varphi_j \in \mathcal{H}, j = 1, 2, \ldots, m$ for almost all $t \in [0,T]$.

3.2 Statement of Main results

The local solution (in time $[0,T]$) is given in next Theorem.

Theorem 12. *(Local existence) Assume that*

$$1 < p \leq \frac{n+2}{n-2} \quad \text{and} \quad n \geq 3. \qquad (3.2.1)$$

Let $(u_{10}, u_{20}, \ldots u_{m0}) \in \mathcal{H}^m$ and $(u_1, u_1, \ldots, u_m) \in [L^\kappa_\theta(\mathbb{R}^n)]^m$. Under the assumptions (3.1.4)-(17) and (3.1.11)-(3.1.14), suppose that

$$a + \lambda_1\omega > 0. \qquad (3.2.2)$$

Then (3.1.1) admits a unique local solution $(u_1, u_2, \ldots, u_m)$ such that

$$(u_1, u_2, \ldots, u_m) \in \mathcal{X}_T^m,\ \mathcal{X}_T \equiv C\big([0,T];\mathcal{H}\big) \cap C^1\big([0,T]; L^\kappa_\theta(\mathbb{R}^n)\big),$$

for sufficiently small $T > 0$.

Remark 3. *The constant λ_1 introduced in (3.2.2) being the first eigenvalue of the operator $-\Delta$.*

We will show now the global solution in time established in Theorem 13. Let us introduce the potential energy $J : \mathcal{H}^m \to \mathbb{R}$ defined by

$$J(u_1, u_2, \ldots, u_m) = \sum_{j=1}^m \left(1 - \int_0^t \varpi_j(s)\,ds\right)\|u_j\|^2_{\mathcal{H}} + (\varpi_j \circ u_j)\,. \qquad (3.2.3)$$

The modified energy is defined by

$$\mathcal{E}(t) = \frac{\kappa-1}{\kappa}\sum_{j=1}^m \|u_{jt}\|^\kappa_{L^\kappa_\theta} + \frac{1}{2}J(u_1, u_2, \ldots, u_m) - \int_{\mathbb{R}^n} \theta(x)\mathcal{F}(u_1, u_2, \ldots, u_m)dx, \qquad (3.2.4)$$

here

$$(\varpi_j \circ w)(t) = \int_0^t \varpi_j(t-s) \|w(t) - w(s)\|_{\mathcal{H}}^2 \, ds,$$

for any $w \in L^2(\mathbb{R}^n), j = 1, 2, \ldots, m$.

Theorem 13. *(Global existence) Let (3.1.4)-(17) and (3.1.11)-(3.1.14) hold. Under (3.2.1), (3.2.2) and for sufficiently small* $(u_{10}, u_{11}), (u_{20}, u_{21}), \ldots, (u_{m0}, u_{m1}) \in \mathcal{H} \times L^\kappa_\theta(\mathbb{R}^n)$*, problem (3.1.1) admits a unique global solution* $(u_1, u_2, \ldots, u_m)$ *such that*

$$(u_1, u_2, \ldots, u_m) \in \mathcal{X}^m, \;\; \mathcal{X} \equiv C\big([0, +\infty); \mathcal{H}\big) \cap C^1\big([0, +\infty); L^\kappa_\theta(\mathbb{R}^n)\big). \tag{3.2.5}$$

The nonclassical decay rate for solution is given in the next Theorem, where the existing results are a special case.

Theorem 14. *(Decay of solution) Let (3.1.4)-(17) and (3.1.11)-(3.1.14) hold. Under conditions (3.2.1), (3.2.2) and*

$$\gamma = \eta\Big(\frac{2(p+1)}{p-1}\mathcal{E}(0)\Big)^{(p-1)/2} < 1, \tag{3.2.6}$$

there exists $t_0 > 0$ *depending only on* ϖ_j*,* a*,* ω*,* λ_1 *and* $\mathcal{X}'(0)$ *such that*

$$0 \le \mathcal{E}(t) < \mathcal{E}(t_0) \exp\left(-\int_{t_0}^t \frac{\mu(s)}{1 - \mu_0(t)}\right), \tag{3.2.7}$$

holds for all $t \ge t_0$*.*

In particular, by the positivity of μ in (3.1.12), we have, as in [33],

$$0 \le \mathcal{E}(t) < \mathcal{E}(t_0) \exp\left(-\int_{t_0}^t \mu(s)\, ds\right),$$

for a single wave equation. Condition (3.1.14) is imposed to make a different from [33] and [34], it leads $(\mu' + \nu\mu) \circ u$, here $\nu \in \mathbb{R}$.

The next, Lemma will play an important role in the sequel.

Lemma 13. *For* $(u_1, u_2, \ldots, u_m) \in \mathcal{X}_T^m$*, the functional* $\mathcal{E}(t)$ *associated with problem (3.1.1) is a decreasing energy.*

Proof. For $0 \leq t_1 < t_2 \leq T$, we have

$$\begin{aligned}
&\mathcal{E}(t_2) - \mathcal{E}(t_1) \\
&= \int_{t_1}^{t_2} \frac{d}{dt} E(t)\, dt \\
&= -\sum_{j=1}^{m} \int_{t_1}^{t_2} \left(a \|u_{jt}\|_{L^2_\theta}^2 + \omega \|u_{jt}\|_{\mathcal{H}}^2 + \frac{1}{2}\varpi_j(t) \|u_j\|_{\mathcal{H}}^2 - \frac{1}{2}(\varpi_j' \circ u_j) \right) dt \\
&\leq 0,
\end{aligned}$$

owing to (3.1.11)-(3.1.14). □

We define an inner product as

$$(v, w)_* = \omega \int_{\mathbb{R}^n} \nabla v \cdot \nabla w\, dx + a \int_{\mathbb{R}^n} \theta v w\, dx,$$

and the associated norm is given by

$$\|v\|_* = \sqrt{(v, v)_*},$$

$\forall v, w \in \mathcal{H}$. By (3.2.2), we get

$$(v, v)_* = \omega \int_{\mathbb{R}^n} |\nabla v|^2\, dx + a \int_{\mathbb{R}^n} \theta v^2\, dx \geq (\omega\lambda_1 + a) \int_{\mathbb{R}^n} \theta v^2\, dx \geq 0.$$

The following Lemma yields.

Lemma 14. *Let θ satisfy (3.1.4). Under condition (3.2.2), we get*

$$\sqrt{\omega}\, \|v\|_{\mathcal{H}} \leq \|v\|_* \leq \sqrt{\omega + C_P^2}\, \|v\|_{\mathcal{H}},$$

for $v \in \mathcal{H}$.

3.3 Proofs

3.3.1 Proof of existence results

We sketch here the outline of the proof for local solution by a standard procedure (See [13], [34]).

Proof. (Of Theorem 12.) Let $(u_{10}, u_{11}), (u_{20}, u_{21}), \ldots, (u_{m0}, u_{m1}) \in \mathcal{H} \times L^{\kappa}_{\theta}(\mathbb{R}^n)$. For any $(u_1, u_2, \ldots, u_m) \in \mathcal{X}^m_T$, we can obtain a weak solution of the related system

$$\begin{cases} \left(|z_{jt}|^{\kappa-2} z_{jt}\right)_t + a z_{jt} - \Theta(x)\Delta\left(z_j + \omega z_{jt}\right) = -\Theta(x)\Delta \int_0^t \varpi_j(t-s) u_j(s)\, ds + f_j(u_1, u_2, \ldots, u_m) \\ z_j(x,0) = u_{j0}(x) \\ z_{jt}(x,0) = u_{j1}(x). \end{cases} \tag{3.3.1}$$

We reduces problem (3.3.1) to Cauchy problem for system of ODE by using the Faedo-Galerkin approximation. We then find a solution map

$$\top : (u_1, u_2, \ldots, u_m) \mapsto (z_1, z_2, \ldots, z_m)$$

from $\mathcal{X}^m_T$ to $\mathcal{X}^m_T$. We are now ready to show that $\top$ is a contraction mapping in an appropriate subset of $\mathcal{X}^m_T$ for a small $T > 0$. Hence $\top$ has a fixed point

$$\top(u_1, u_2, \ldots, u_m) = (u_1, u_2, \ldots, u_m),$$

which gives a unique solution in $\mathcal{X}^m_T$. □

We will show the global solution. For this end, by using conditions on functions ϖ_j, we have

$$\begin{aligned} \mathcal{E}(t) &\geq \frac{1}{2} J(u_1, u_2, \ldots, u_m) - \int_{\mathbb{R}^n} \theta(x) \mathcal{F}(u_1, u_2, \ldots, u_m) dx \\ &\geq \frac{1}{2} J(u_1, u_2, \ldots, u_m) - \frac{1}{p+1} \left\| \sum_{j=1}^m u_j \right\|^{(p+1)}_{L^{(p+1)}_{\theta}} - \frac{2}{p+1} \Big(\Big\| \sum_{j=1}^{m-1} u_j u_{j+1} \Big\|^{(p+1)/2}_{L^{(p+1)/2}_{\theta}} + \| u_m u_1 \|^{(p+1)/2}_{L^{(p+1)/2}_{\theta}} \Big) \\ &\geq \frac{1}{2} J(u_1, u_2, \ldots, u_m) - \frac{\eta}{p+1} \Big[\sum_{j=1}^m \rho_j \| u_j \|^2_{\mathcal{H}} \Big]^{(p+1)/2} \\ &\geq \frac{1}{2} J(u_1, u_2, \ldots, u_m) - \frac{\eta}{p+1} \Big(J(u_1, u_2, \ldots, u_m) \Big)^{(p+1)/2} \\ &= G(\beta), \end{aligned} \tag{3.3.2}$$

here $\beta^2 = J(u_1, u_2, \ldots, u_m)$, for $t \in [0, T)$, where

$$G(\xi) = \frac{1}{2} \xi^2 - \frac{\eta}{p+1} \xi^{(p+1)}.$$

Noting that $\mathcal{E}_0 = G(\lambda_0)$, given in (3.1.17). Then

$$\begin{cases} G'(\xi) \geq 0 \quad in \quad \xi \in [0, \lambda_0] \\ \\ G'(\xi) < 0 \quad in \quad \xi \leq \lambda_0. \end{cases} \tag{3.3.3}$$

Moreover, $\lim\limits_{\xi \to +\infty} G(\xi) \to -\infty$. Then, we have the following Lemma

Lemma 15. *Let* $0 \leq \mathcal{E}(0) < \mathcal{E}_0$.
(i) If $\sum\limits_{j=1}^{m} \|u_{j0}\|_{\mathcal{H}}^2 < \lambda_0^2$, *then local solution of (3.1.1) satisfies*

$$J(u_1, u_2, \ldots, u_m) < \lambda_0^2, \ \forall t \in [0, T).$$

(ii) If $\sum\limits_{j=1}^{m} \|u_{j0}\|_{\mathcal{H}}^2 > \lambda_0^2$, *then local solution of (3.1.1) satisfies*

$$\sum_{j=1}^{m} \|u_j\|_{\mathcal{H}}^2 > \lambda_1^2, \ \forall t \in [0, T), \lambda_1 > \lambda_0.$$

Proof. Since $0 \leq \mathcal{E}(0) < \mathcal{E}_0 = G(\lambda_0)$, there exist ξ_1 and ξ_2 such that $G(\xi_1) = G(\xi_2) = \mathcal{E}(0)$ with $0 < \xi_1 < \lambda_0 < \xi_2$.

The case (i)**.** By (3.3.2), we have

$$G(J(u_{10}, u_{20}, \ldots u_{m0})) \leq \mathcal{E}(0) = G(\xi_1),$$

which implies that $J(u_{10}, u_{20}, \ldots u_{m0}) \leq \xi_1^2$. Then we claim that $J(u_1, u_2, \ldots, u_m) \leq \xi_1^2, \ \forall t \in [0, T)$. Moreover, there exists $t_0 \in (0, T)$ such that

$$\xi_1^2 < J(u_1(t_0), u_2(t_0), \ldots, u_m(t_0)) < \xi_2^2.$$

Then

$$G(J(u_1(t_0), u_2(t_0), \ldots, u_m(t_0)) > \mathcal{E}(0) \geq \mathcal{E}(t_0),$$

by Lemma 13, which contradicts (3.3.2). Hence we have

$$J(u_1, u_2, \ldots, u_m) \leq \xi_1^2 < \lambda_0^2, \ \forall t \in [0, T).$$

The case (ii)**.** We can now show that $\sum\limits_{j=1}^{m} \|u_{j0}\|_{\mathcal{H}}^2 \geq \xi_2^2$ and that $\sum\limits_{j=1}^{m} \|u_j\|_{\mathcal{H}}^2 \geq \xi_2^2 > \lambda_0^2$ in the same way as (i). □

Proof. (*Of Theorem 13.*) Let $(u_0, u_1), (u_{20}, u_{21}), \ldots, (u_{m0}, u_{m1}) \in \mathcal{H} \times L^{\kappa}_{\theta}(\mathbb{R}^n)$ satisfy both $0 \leq \mathcal{E}(0) < \mathcal{E}_0$ and $\sum\limits_{j=1}^{m} \|u_{j0}\|^2_{\mathcal{H}} < \lambda_0^2$. By Lemma 13 and Lemma 15, we have

$$
\begin{aligned}
&\frac{2(\kappa-1)}{\kappa} \sum_{j=1}^{m} \|u_{jt}\|^{\kappa}_{L^{\kappa}_{\theta}} + \sum_{j=1}^{m} \rho_j \|u_j\|^2_{\mathcal{H}} \\
&\leq \frac{2(\kappa-1)}{\kappa} \sum_{j=1}^{m} \|u_{jt}\|^{\kappa}_{L^{\kappa}_{\theta}} + \sum_{j=1}^{m} \Big[\Big(1 - \int_0^t \varpi_j(s)\,ds\Big) \|u_j\|^2_{\mathcal{H}} + (\varpi_j \circ u_j) \Big] \\
&\leq 2\mathcal{E}(t) + \frac{2\eta}{p+1} \Big(\sum_{j=1}^{m} \rho_j \|u_j\|^2_{\mathcal{H}} \Big)^{(p+1)/2} \\
&\leq 2\mathcal{E}(0) + \frac{2\eta}{p+1} \Big(J(u_1, u_2, \ldots, u_m) \Big)^{(p+1)/2} \\
&\leq 2\mathcal{E}_0 + \frac{2\eta}{p+1} \lambda_0^{p+1} \\
&= \eta^{-2/(p-1)}.
\end{aligned} \tag{3.3.4}
$$

This completes the proof. □

3.3.2 Proof of Decay results

Let

$$
\begin{aligned}
\Lambda(u_1, u_2, \ldots, u_m) &= \frac{1}{2} \sum_{j=1}^{m} \Big[\Big(1 - \int_0^t \varpi_j(s)\,ds\Big) \|u_j\|^2_{\mathcal{H}} + (\varpi_j \circ u_j) \Big] \\
&- \int_{\mathbb{R}^n} \theta(x) \mathcal{F}(u_1, u_2, \ldots, u_m) dx,
\end{aligned}
$$

$$
\begin{aligned}
\Pi(u_1, u_2, \ldots, u_m) &= \sum_{j=1}^{m} \Big[\Big(1 - \int_0^t \varpi_j(s)\,ds\Big) \|u_j\|^2_{\mathcal{H}} + (\varpi_j \circ u_j) \Big] \\
&- (p+1) \int_{\mathbb{R}^n} \theta(x) \mathcal{F}(u_1, u_2, \ldots, u_m) dx.
\end{aligned}
$$

Lemma 16. *Let $(u_1, u_2, \ldots, u_m)$ be the solution of problem (3.1.1). If*

$$
\sum_{j=1}^{m} \|u_{j0}\|^2_{\mathcal{H}} - (p+1) \int_{\mathbb{R}^n} \theta(x) \mathcal{F}(u_1, u_2, \ldots, u_m) dx > 0. \tag{3.3.5}
$$

Then under condition (3.2.6), the functional $\Pi(u_1, u_2, \ldots, u_m) > 0, \ \forall t > 0$.

Proof. By (3.3.5) and continuity, there exists a time $t_1 > 0$ such that

$$\Pi(u_1, u_2, \ldots, u_m) \geq 0, \forall t < t_1.$$

Let

$$Y = \{(u_1, u_2, \ldots, u_m) \mid \Pi(u_1(t_0), u_2(t_0), \ldots, u_m(t_0)) = 0,\ \Pi(u_1, u_2, \ldots, u_m) > 0, \forall t \in [0, t_0)\} \tag{3.3.6}$$

Then, by (3.3.5), we have for all $(u_1, u_2, \ldots, u_m) \in Y$,

$$\begin{aligned}
&\Lambda(u_1, u_2, \ldots, u_m) \\
&= \frac{p-1}{2(p+1)} \sum_{j=1}^{m} \left(1 - \int_0^t \varpi_j(s)\, ds\right) \|u_j\|_{\mathcal{H}}^2 + \frac{p-1}{2(p+1)} \sum_{j=1}^{m} (\varpi_j \circ u_j) + \frac{1}{p+1} \Pi(u_1, u_2, \ldots, u_m) \\
&\geq \frac{p-1}{2(p+1)} \sum_{j=1}^{m} \Big[\rho_j \|u_j\|_{\mathcal{H}}^2 + (\varpi_j \circ u_j)\Big].
\end{aligned}$$

Owing to (3.2.4), it follows for $(u_1, u_2, \ldots, u_m) \in Y$

$$\rho_j \|u_j\|_{\mathcal{H}}^2 \leq \frac{2(p+1)}{p-1} \Lambda(u_1, u_2, \ldots, u_m) \leq \frac{2(p+1)}{p-1} \mathcal{E}(t) \leq \frac{2(p+1)}{p-1} \mathcal{E}(0). \tag{3.3.7}$$

By (3.1.16), (3.2.6) we have

$$\begin{aligned}
(p+1) \int_{\mathbb{R}^n} \mathcal{F}(u_1(t_0), u_2(t_0), \ldots, u_m(t_0)) &\leq \eta \sum_{j=1}^{m} \left(\rho_j \|u_j(t_0)\|_{\mathcal{H}}^2\right)^{(p+1)/2} \\
&\leq \eta \Big(\frac{2(p+1)}{p-1} E(0)\Big)^{(p-1)/2} \sum_{j=1}^{m} \rho_j \|u_j(t_0)\|_{\mathcal{H}}^2 \\
&\leq \gamma \sum_{j=1}^{m} \rho_j \|u_j(t_0)\|_{\mathcal{H}}^2 \\
&< \sum_{j=1}^{m} \Big(1 - \int_0^{t_0} \varpi_j(s) ds\Big) \|u_j(t_0)\|_{\mathcal{H}}^2 \\
&< \sum_{j=1}^{m} \Big(1 - \int_0^{t_0} \varpi_j(s) ds\Big) \|u_j(t_0)\|_{\mathcal{H}}^2 \\
&+ \sum_{j=1}^{m} (\varpi_j \circ u_j(t_0)),
\end{aligned} \tag{3.3.8}$$

hence $\Pi(u_1(t_0), u_2(t_0), \ldots, u_m(t_0)) > 0$ on Y, which contradicts the definition of Y since $\Pi(u_1(t_0), u_2(t_0), \ldots, u_m$
0. Thus $\Pi(u_1, u_2, \ldots, u_m) > 0,\ \forall t > 0.$ □

We are ready to prove the decay rate.

Proof. (*Of Theorem 14.*) By (3.1.16) and (3.3.7), we have for $t \geq 0$

$$0 < \sum_{j=1}^{m} \rho_j \left\| u_j \right\|_{\mathcal{H}}^2 \leq \frac{2(p+1)}{p-1} \mathcal{E}(t). \tag{3.3.9}$$

Let

$$I(t) = \frac{\mu(t)}{1-\mu_0(t)},$$

where μ and μ_0 defined in (3.1.12) and (3.1.13).

Noting that $\lim_{t \to +\infty} \mu(t) = 0$ by (3.1.11)-(3.1.13), we have

$$\lim_{t \to +\infty} I(t) = 0, \quad I(t) > 0, \ \forall t \geq 0.$$

Then we take $t_0 > 0$ such that

$$0 < \frac{2(\kappa-1)}{\kappa} I(t) < \min \left\{ 2\left(\omega \lambda_1 + a\right), \chi'(0) \right\}, \tag{3.3.10}$$

with (3.1.14) for all $t > t_0$. Due to (3.2.4), we have

$$\begin{aligned} \mathcal{E}(t) &\leq \frac{(\kappa-1)}{\kappa} \sum_{j=1}^{m} \left\| u_{jt} \right\|_{L_\theta^\kappa}^\kappa + \frac{1}{2} \sum_{j=1}^{m} \left(\varpi_j \circ u_j \right) + \frac{1}{2} \sum_{j=1}^{m} \left(1 - \int_0^t \varpi_j(s)\, ds \right) \left\| u_j \right\|_{\mathcal{H}}^2 \\ &\leq \frac{(\kappa-1)}{\kappa} \sum_{j=1}^{m} \left\| u_{jt} \right\|_{L_\theta^\kappa}^\kappa + \frac{1}{2} \sum_{j=1}^{m} \left(\varpi_j \circ u_j \right) + \frac{1}{2} (1 - \mu_0(t)) \sum_{j=1}^{m} \left\| u_j \right\|_{\mathcal{H}}^2 . \end{aligned}$$

Then by definition of $I(t)$, we have

$$I(t)\mathcal{E}(t) \leq \frac{(\kappa-1)}{\kappa} I(t) \sum_{j=1}^{m} \left\| u_{jt} \right\|_{L_\theta^\kappa}^\kappa + \frac{1}{2} \mu(t) \sum_{j=1}^{m} \left\| u_j \right\|_{\mathcal{H}}^2 + \frac{1}{2} I(t) \sum_{j=1}^{m} \left(\varpi_j \circ u_j \right), \tag{3.3.11}$$

and Lemma 13, we have for all $t_1, t_2 \geq 0$

$$\begin{aligned} &\mathcal{E}(t_2) - \mathcal{E}(t_1) \\ &\leq -\int_{t_1}^{t_2} \left(a \sum_{j=1}^{m} \left\| u_{jt} \right\|_{L_\theta^2}^2 + \omega \sum_{j=1}^{m} \left\| u_{jt} \right\|_{\mathcal{H}}^2 + \frac{1}{2} \mu(t) \sum_{j=1}^{m} \left\| u_j \right\|_{\mathcal{H}}^2 \right) dt \\ &+ \int_{t_1}^{t_2} \frac{1}{2} \sum_{j=1}^{m} (\varpi_j' \circ u_j)\, dt \end{aligned}$$

then, by generalized Poincaré's inequalities, we get

$$\mathcal{E}'(t) \leq -(\omega\lambda_1 + a)\sum_{j=1}^{m} \|u_{jt}\|^2_{L^2_\theta} - \frac{1}{2}\mu(t)\sum_{j=1}^{m}\|u_j\|^2_{\mathcal{H}} + \frac{1}{2}\sum_{j=1}^{m}(\varpi_j' \circ u_j),$$

Finally, by (3.3.10), $\forall t \geq t_0$, we have

$$\begin{aligned}
&\mathcal{E}'(t) + I(t)\mathcal{E}(t) \\
&\leq \left\{\frac{(\kappa-1)}{\kappa}I(t) - (\omega\lambda_1 + a)\right\}\sum_{j=1}^{m}\|u_{jt}\|^2_{L^2_\theta} \\
&+ \frac{1}{2}\sum_{j=1}^{m}(\varpi_j' \circ u_j) + \frac{1}{2}I(t)\sum_{j=1}^{m}(\varpi_j \circ u_j) \\
&\leq \frac{1}{2}\sum_{j=1}^{m}\int_0^t \left\{\varpi_j'(t-\tau) + I(t)\varpi_j(t-\tau)\right\}\|u_j(t) - u_j(\tau)\|^2_{\mathcal{H}}\, d\tau \\
&\leq \frac{1}{2}\sum_{j=1}^{m}\int_0^t \left\{\varpi_j'(\tau) + I(t)\varpi_j(\tau)\right\}\|u_j(t) - u_j(t-\tau)\|^2_{\mathcal{H}}\, d\tau \\
&\leq \frac{1}{2}\sum_{j=1}^{m}\int_0^t \left\{-\chi\Big(\varpi_j(\tau)\Big) + \chi'(0)\varpi_j(\tau)\right\}\|u_j(t) - u_j(t-\tau)\|^2_{\mathcal{H}}\, d\tau \\
&\leq 0,
\end{aligned}$$

by the convexity of χ and (3.1.14), we have

$$\chi(\xi) \geq \chi(0) + \chi'(0)\xi = \chi'(0)\xi.$$

Then

$$\mathcal{E}(t) \leq \mathcal{E}(t_0)\exp\left(-\int_{t_0}^{t} I(s)ds\right),$$

which completes the proof. □

Chapter 4

Existence and general decay estimates for a Petrovsky-Petrovsky coupled system with nonlinear strong damping

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In this chapter, we consider a coupled system of Petrovsky-Petrovsky equations with a nonlinear dissipative terms. We prove, under some appropriate assumptions, that this system is stable. Furthermore, we use the multiplier method and some general weighted integral inequalities to obtain decay properties of solution.

4.1 Introduction and preliminaries

In the present section, we consider problem

$$\begin{cases} u_1'' + \alpha u_2 + \Delta_x^2 u_1 - \mu(\Delta u_1'(x,t)) = 0, & in \quad \Omega \times \mathbb{R}^+, \\ u_2'' + \alpha u_1 + \Delta_x^2 u_2 - \mu(\Delta u_2'(x,t)) = 0, & in \quad \Omega \times \mathbb{R}^+, \\ u_i = \Delta u_i = 0 & on \quad \Gamma \times \mathbb{R}^+, \\ (u_1(0,x), u_2(0,x)) = (u_{10}(x), u_{20}(x)) & on \quad \Omega, \\ (u_1'(0,x), u_2'(0,x)) = (u_{11}(x), u_{21}(x)) & on \quad \Omega, \end{cases} \tag{4.1.1}$$

The constant α

$$\alpha \leq \frac{1}{2C_s} \tag{4.1.2}$$

where $C_s > 0$ depending only on the geometry of Ω is the constant such that

$\|\nabla z\|^2 \leq C_s \|\nabla \Delta z\|^2$ The problem of stabilization of weakly coupled systems have been studied by several authors. Under certain conditions imposed on the subset where the damping term is effective, Kapitonov [21] showed uniform stabilization of the solutions of a pair of hyperbolic systems coupled in velocities. In [3], the authors developed an approach to prove that, for $\alpha \in \mathbb{R}^+$ with α small enough,

$$\begin{cases} u_{tt} - \Delta u + \alpha v + u_t = 0 & in \quad \Omega \times \mathbb{R}^+, \\ v_{tt} - \Delta v + \alpha u = 0 & in \quad \Omega \times \mathbb{R}^+, \end{cases} \tag{4.1.3}$$

is not exponentially stable and the asymptotic behavior of solutions is at least of polynomial type $\frac{1}{t^m}$ with decay rate m depending on the smoothness of initial data.

In [2], Beniani et *al.* considered the Petrowsky-Petrowsky system

$$\begin{cases} u_{tt}+\phi(x)\left(\Delta^2 u-\int_{-\infty}^{t}\mu(t-s)\Delta^2 u(s)ds\right)+\alpha v=0 & \mathbb{R}^n\times\mathbb{R}^+ \\ v_{tt}+\phi(x)\Delta^2 v+\alpha u=0 & \mathbb{R}^n\times\mathbb{R}^+ \\ u=v=\Delta u=\Delta v=0 & \Gamma\times\mathbb{R}^+ \\ (u_0,v_0)\in\mathcal{D}^{2,2}(\mathbb{R}^n),\ (u_1,v_1)\in L_g^2(\mathbb{R}^n), \end{cases} \tag{4.1.4}$$

In this work, the authors proved, under suitable conditions, that the system is polynomial stable. Author [36] proved the existence of global solution, as well as, a general stability result for the following system

$$\begin{cases} u_{tt}+\Delta^2 u-g(\Delta u'(s))=0 & \Omega\times\mathbb{R}^+, \\ u=\Delta u=0 & \Gamma\times\mathbb{R}^+, \\ u(0)=u_0, \quad u'(0)=u_1 & \Omega. \end{cases} \tag{4.1.5}$$

Here, we assume that the function $\mu\in C(\mathbb{R},\mathbb{R})$ is a non-decreasing such that there exist constants $\varepsilon,c_1,c_2,\tau>0$ and a convex increasing function $H\in(\mathbb{R}_+,\mathbb{R}_+)$ of class $C^1(\mathbb{R}_+)\cap C^2(\mathbb{R}_+^*)$, linear on $[0,\varepsilon]$ or $H^{'}(0)=0$ and $H''>0$ on $]0,\varepsilon]$, such that

$$c_1\left|s\right|\le\left|\mu(s)\right|\le c_2\left|s\right|,\ \text{if}\ \left|s\right|>\varepsilon, \tag{4.1.6}$$

$$\left|s\right|^2+\left|\mu(s)\right|^2\le H^{-1}(s\mu(s)),\ \text{if}\ \left|s\right|\le\varepsilon, \tag{4.1.7}$$

$$\left|\mu'(s)\right|\le\tau. \tag{4.1.8}$$

Lemma 17. *For any function $u\in H_0^1(\Omega)\cap H^2(\Omega)$, we have*

$$\|\nabla u\|\le c\left\|\Delta u\right\|_{H^{-1}(\Omega)}\le c\|\Delta u\|, \tag{4.1.9}$$

where $H^{-1}(\Omega)=(H_0^1(\Omega))^{'}$.

Now we define the energy associated to the solution of the system (4.1.1) by

$$\mathcal{E}(t):=\frac{1}{2}\sum_{i=1}^{2}\left\|\nabla u_i^{'}\right\|_2^2+\frac{1}{2}\sum_{i=1}^{2}\left\|\nabla\Delta u_i\right\|_2^2+2\alpha\int_{\Omega}\nabla u_1.\nabla u_2 dx. \tag{4.1.10}$$

$$2\alpha\int_{\Omega}\nabla u_1.\nabla u_2\,dx\ge-\alpha C_s\int_{\Omega}\sum_{i=1}^{2}|\nabla\Delta u_i|^2+|\,dx$$

we deduce that

$$\mathcal{E}(t) \geq \frac{1}{2}\sum_{i=1}^{2}\left\|\nabla u_i^{'}\right\|_2^2 + \left(\frac{1}{2} - \alpha C_s\right)\sum_{i=1}^{2}\left\|\nabla \Delta u_i\right\|_2^2. \tag{4.1.11}$$

Note that E is the natural energy for system (4.1.1), given the structure of the damping term. The energy E is a non-increasing function of the time variable t and we have for almost every $t \geq 0$,

We first state a useful Lemmas

Lemma 18. (*Sobolev-Poincaré inequality*). *Let q be a number with $2 \leq q \leq +\infty \quad (n = 1, 2)$ or $2 \leq q \leq \frac{2n}{n-2}\ (n \geq 3)$ then there is a constant $c_* = c(\Omega, q)$ such that*

$$\|u\|_q \leq c_* \|\nabla u\|_2, \quad u \in H_0^1(\Omega). \tag{4.1.12}$$

Lemma 19. *[9] Let $\mathcal{E} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a non-increasing function and $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a convex and increasing function such that $\psi(0) = 0$, assume that*

$$\int_s^T \psi(\mathcal{E}(t)) \leq \mathcal{E}(S), \quad 0 \leq s < T, \tag{4.1.13}$$

Then $\mathcal{E}$ satisfies the following estimate

$$\mathcal{E}(t) \leq \psi^{-1}\left(H(t) + \psi(\mathcal{E}(0))\right), \quad \forall t \geq 0. \tag{4.1.14}$$

Where $\psi(t) = \int_t^1 \frac{1}{\psi(s)} ds$ for $t > 0$, $H(t) = 0$ for $0 \leq t \leq \frac{\mathcal{E}(0)}{\psi(\mathcal{E}(0))}$ and

$$H^{-1}(t) = t + \frac{\psi^{-1}(t + \psi(\mathcal{E}(0))}{\psi\left(\psi^{-1}(t + \psi(\mathcal{E}(0))\right)}, \quad \forall t \geq \frac{\mathcal{E}(0)}{\psi(\mathcal{E}(0))}$$

Remark 4. *Let us denote by H^* the conjugate function of the differentiable convex function H, i.e.,*

$$H^* = \sup_{s \in \mathbb{R}^+}(st - H(t))$$

*Then H^* is the Legendre transform of H, which is given by (see Arnold [**?** , p. 61-62])*

$$H^*(s) = s(H^{'})^{-1}(s) - H\left((H^{'})^{-1}(s)\right), \text{ if } s \in \left]0, H^{'}(r)\right],$$

and H^* satisfies the generalized Young inequality

$$ST \leq H^*(S) + H(T), \text{ if } S \in \left]0, H^{'}(r)\right], T \in \left]0, r\right]. \tag{4.1.15}$$

Lemma 20. *Let* (u_1, u_2) *be the solution of* (4.1.1). *Then*

$$\mathcal{E}'(t) = -\int_{\Omega} \sum_{i=1}^{2} \Delta u_i' \mu(\Delta u_i')\, dx \leq 0. \tag{4.1.16}$$

Proof. Multiplying first equation of (4.1.1) by $-\Delta u_1'$ and second equation by $-\Delta u_2'$ respectively, summing the obtained results follows the conclusion of inequality (4.1.16). □

4.2 Main results and proof

Introduce three real Hilbert spaces $\mathcal{H}, \mathcal{V}$ and $\mathcal{W}$ by

$$\mathcal{H} = H_0^1(\Omega), \|v\|_{\mathcal{H}}^2 = \int_{\Omega} |\nabla v|^2 dx,$$

$$\mathcal{V} = \left\{v \in H^3(\Omega) : v = \Delta v = 0 \text{ on } \Gamma\right\}, \|v\|_{\mathcal{V}}^2 = \int_{\Omega} |\nabla \Delta v|^2 dx,$$

and

$$\mathcal{W} = \left\{v \in H^5(\Omega) : v = \Delta v = \Delta^2 v = 0 \text{ on } \Gamma\right\}, \|v\|_{\mathcal{W}}^2 = \int_{\Omega} \left|\nabla \Delta^2 v\right|^2 dx,$$

Identifying $\mathcal{H}$ with its dual $\mathcal{H}'$ we have

$$\mathcal{W} \subset \mathcal{V} \subset \mathcal{H}.$$

with dense and compact imbedding. Our main results is the following

Theorem 15. *Let* $(u_{10}, u_{11}), (u_{20}, u_{21}) \in \mathcal{W} \times \mathcal{V}$, *assume that (4.1.6)-(4.1.8) hold. Then the solution of the problem (4.1.1) satisfies*

$$(u_1', u_2') \in L^\infty(\mathbb{R}_+; V), \quad (u_1'', u_2'') \in L^\infty(\mathbb{R}_+; \mathcal{H})$$

and

$$(u_1, u_2) \in L^\infty(\mathbb{R}_+; W)$$

Theorem 16. *Let $(u_{10}, u_{11}), (u_{20}, u_{21}) \in \mathcal{W} \times \mathcal{V}$, assume that (4.1.6)-(4.1.8) hold. Then the energy of solution of the problem (4.1.1), for some constants ω, ε_0, satisfies the following decay property*

$$\mathcal{E}(t) \leq \psi^{-1}(H(t) + \psi(\mathcal{E}(0))), \quad \forall t \geq 0. \tag{4.2.1}$$

Where $\psi(t) = \int_t^1 \frac{1}{\omega\psi(s)} ds$ for $t > 0$, $H(t) = 0$ for $0 \leq t \leq \frac{\mathcal{E}(0)}{\omega\psi(\mathcal{E}(0))}$ and

$$H^{-1}(t) = t + \frac{\psi^{-1}(t + \psi(\mathcal{E}(0))}{\omega\psi(\psi^{-1}(t + \psi(\mathcal{E}(0)))}, \quad \forall t \geq \frac{\mathcal{E}(0)}{\psi(\mathcal{E}(0))}$$

$$\varphi(s) = \begin{cases} s & \text{if } H \text{ is linear on } [0, \varepsilon_1] \\ sH'(\varepsilon_0 s) & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0, \varepsilon_1] \end{cases}$$

Proof. (**Of Theorem 15**) We use the Faedo-Galerkin method to prove the existence of global solution

Step 1. Approximate solutions.

We will use the Faedo-Galerkin method to prove the existence of a global solution. Let $T > 0$ be fixed and let $\{w_j\}, j \in \mathbb{N}$ be a basis of $\mathcal{H}$, V and W, i.e. the space generated by $\mathcal{B}_k = \{w_1, w_2, \ldots, w_k\}$ is dense in $\mathcal{H}$, V and W.

We construct approximate solutions u_k, $k = 1, 2, 3, \ldots$, in the form

$$u_1^k(t) = \sum_{j=1}^{k} c_{jk}(t) w_j(x), \quad u_2^k(t) := \sum_{i=0}^{k} h_{jk}(t) w_j(x),$$

where c_{jk} and h_{jk} is determined by the ordinary differential equations.

For any v in $\mathcal{B}_k$, $(u_1^k(t), u_2^k(t))$ satisfies the approximate equation

$$\begin{cases} \int_\Omega (u_1^{k''}(t) + \alpha u_2^k + \Delta^2 u_1^k - \mu(\Delta u_1^{k'})) v\, dx = 0, \\ \int_\Omega (u_2^{k''}(t) + \alpha u_1^k + \Delta^2 u_2^k - \mu(\Delta u_2^{k'})) v\, dx = 0, \end{cases} \tag{4.2.2}$$

with initial conditions

$$u_1^k(0) = u_1^{0,k} = \sum_{j=1}^{k} \langle u_1^0, w_j \rangle w_j \to u_1^0, \text{ in } W \text{ as } k \to +\infty, \tag{4.2.3}$$

$$u_2^k(0) = u_2^{0,k} = \sum_{j=1}^{k} \langle u_2^0, w_j \rangle w_j \to u_2^0, \text{ in } W \text{ as } k \to +\infty, \tag{4.2.4}$$

and

$$u_1^{k'}(0) = u_1^{1,k} = \sum_{j=1}^{k} \langle u_1^1, w_j \rangle w_j \to u_1^1, \text{ in } V \text{ as } k \to +\infty. \tag{4.2.5}$$

$$u_2^{k'}(0) = u_2^{1,k} = \sum_{j=1}^{k} \langle u_2^1, w_j \rangle w_j \to u_2^1, \text{ in } V \text{ as } k \to +\infty. \tag{4.2.6}$$

The standard theory of ODE guarantees that the system (4.2.2)-(4.2.6) has an unique solution in $[0, t_k)$, with $0 < t_k < T$, by Zorn Lemma since the nonlinear terms in (4.2.2) are locally Lipschitz continuous. Note that $u_1^k(t)$ and $u_2^k(t)$ are $\mathcal{C}^2$ functions.
In the next step, we obtain a priori estimates for the solution of system (4.2.2)-(4.2.6), so that it can be extended outside $[0, t_k)$ to obtain one solution defined for all $T > 0$, using a standard compactness argument for the limiting procedure.

Step 2. The first estimate

Setting $v = -2\Delta(u_1^k)'$ in $(4.2.2)_1$ and $v = -2\Delta(u_2^k)'$ in $(4.2.2)_2$, adding the resulting equations, we have

$$\sum_{i=1}^{2} \frac{d}{dt}\Big[\|\nabla u_i^{k'}\|^2 + \|\nabla\Delta u_i^k\|^2 + 2\alpha \int_\Omega \nabla u_1^k \nabla u_2^k \, dx\Big] + 2\int_\Omega \Delta u_i^{k'} \mu(\Delta u_i^{k'})\, dx = 0.$$

Integrating in $[0, t]$, $t < t_k$ and using (4.2.3) and (4.2.6), we obtain

$$\begin{aligned}
\sum_{i=1}^{2} \|\nabla u_i^{k'}(t)\|^2 \; &+ \; \|\nabla\Delta u_i^k(t)\|^2 + 2\int_0^t \int_\Omega \Delta u_i^{k'}(s)\mu(\Delta u_i^{k'}(s))\, dx\, ds + 2\alpha \int_0^t \int_\Omega \nabla u_1^k \nabla u_2^k \, dx ds \\
&\le \; \sum_{i=1}^{2} \Big(\|\nabla u_i^{1,k}\|^2 + \|\nabla\Delta u_i^{0,k}\|^2\Big) + +2\alpha \int_\Omega \nabla u_1^{0,k}.\nabla u_2^{0,k}\, dx \qquad (4.2.7)\\
&\le \; \sum_{i=1}^{2} \Big(\|\nabla u_i^{1,k}\|^2 + \|\nabla\Delta u_i^{0,k}\|^2\Big) + \alpha \sum_{i=1}^{2} \|\nabla u_i^{0,k}\|^2,
\end{aligned}$$

using (4.2.3)-(4.2.6), we obtain

$$\sum_{i=1}^{2} \|\nabla u_i^{k'}(t)\|^2 + (1 - 2\alpha C_s)\|\nabla\Delta u_i^k(t)\|^2 + 2\int_0^t \int_\Omega \Delta u_i^{k'}(s)\mu(\Delta u_i^{k'}(s))\, dx\, ds \le C_1 \tag{4.2.8}$$

where C_1 is a positive constant depending only on $\|u_i^1\|_{\mathcal{V}}$ and $\|u_i^0\|_{\mathcal{W}}$.

This estimate imply that the solution u_k exists globally in $[0,+\infty)$. Estimate (4.2.7) implies

$$u_i^k \text{ is bounded in } L^\infty(0,T;V), \tag{4.2.9}$$

$$(u_i^k)' \text{ is bounded in } L^\infty(0,T;\mathcal{H}), \tag{4.2.10}$$

$$\Delta(u_i^k)'\mu(\Delta(u_i^k)') \text{ is bounded in } L^1(\Omega\times(0,T)), \tag{4.2.11}$$

From (4.1.6), (4.1.7) and (4.2.11), it follows that

$$\mu(\Delta(u_i^k)') \text{ is bounded in } L^2(\Omega\times(0,T)).$$

As in Komornik [20], we consider the following partition of Ω,

$$\Omega_1=\{x\in\Omega : |\Delta u_i^{k'}|>\varepsilon\},\quad \Omega_2=\{x\in\Omega : |\Delta u_i^{k'}|\le\varepsilon\}$$

Using (4.1.6) and (4.2.11), we have

$$\begin{aligned}\int_0^T\int_{\Omega_1}|\mu(\Delta u_i^{k'}(s))|^2\,dx\,ds &\le c_2\int_0^T\int_{\Omega_1>\varepsilon}\Delta u_i^{k'}(s)\mu(\Delta u_i^{k'}(s))\,dx\,ds\\ &\le C\end{aligned}$$

exploit Jensen's inequality and the concavity of H^{-1}, we obtain

$$\begin{aligned}\int_{\Omega_2}|\mu(\Delta u_i^{k'}(t))|^2\,dx &\le \int_{\Omega_2}H^{-1}(\Delta u_i^{k'}(t)\mu(\Delta u_i^{k'}(t)))\,dx\,ds\\ &\le H^{-1}\Big(\tfrac{1}{|\Omega_2|}\int_{\Omega_2}\Delta u_i^{k'}(t)\mu(\Delta u_i^{k'}(t))\,dx\Big)\end{aligned}$$

using Remark 4, we have

$$\begin{aligned}\int_0^T\int_{\Omega_2}|\mu(\Delta u_i^{k'}(s))|^2\,dx\,dt &\le H^*(1)+\tfrac{1}{|\Omega_2|}\int_0^T\int_{\Omega_2}\Delta u_i^{k'}(s)\mu(\Delta u_i^{k'}(s))\,dx\,dt\\ &\le C\end{aligned}$$

Step 3. The second estimate

First, we estimate $(u_i^k)''(0)$. Differentiating (4.2.2) with respect to x, setting $v=\nabla(u_1^k)''(t)$ in $(4.2.2)_1$ and $v=\nabla(u_2^k)''(t)$ in $(4.2.2)_2$, adding the resulting equations, by choosing $t=0$, we obtain

$$\sum_{i=1}^{2}\|\nabla u_k''(0)\|^2+\Big(\nabla u_k''(0),\nabla\Delta^2u_k^0-\nabla\Big(\mu(\Delta u_1^k)\Big)\Big)+\alpha\nabla u_1^{0,k}.\nabla(u_2^k)''(0)+\alpha\nabla u_2^{0,k}.\nabla(u_1^k)''(0)=0.$$

Using Cauchy-Schwartz inequality and (4.1.8), we have

$$\begin{aligned} \|\nabla (u_i^k)''(0)\| &\leq \|\nabla \Delta^2 u_i^{0,k}\| + \|\nabla \Delta u_i^{1,k} \mu'(\Delta u_i^{1,k})\| \\ &\leq \|\nabla \Delta^2 u_i^{0,k}\| + \tau \|\nabla \Delta u_i^{1,k}\|. \end{aligned} \tag{4.2.12}$$

By (4.2.3) and (4.2.6), we get

$$(u_i^k)''(0) \quad \text{is bounded in} \quad \mathcal{H}. \tag{4.2.13}$$

The Third estimate.

Differentiating (4.2.2) with respect to t get

$$\sum_{i=1}^{2} \int_\Omega \Big((u_i^k)'''(t) + \Delta^2 (u_i^k)' \Big) v \, dx - \int_\Omega \Delta (u_i^k)'' \mu'(\Delta (u_i^k)') v \, dx + \alpha (u_i^k)' v = 0.$$

Taking $v = 2(\Delta u_i^k)''$, owing to the Green formula, we obtain

$$\sum_{i=1}^{2} \frac{d}{dt} \Big[\|\nabla (u_i^k)''\|^2 + \|\nabla \Delta (u_i^k)'\|^2 + 2\alpha \int_\Omega \nabla (u_1^k)' . \nabla (u_2^k)' \, dx \Big] + 2 \int_\Omega |\Delta (u_i^k)''|^2 \mu'(\Delta (u_i^k)') \, dx = 0.$$

By integration over $(0,t)$, we get

$$\begin{aligned} &\textstyle\sum_{i=1}^2 \|\nabla (u_i^k)''(t)\|^2 + \|\nabla \Delta (u_i^k)'(t)\|^2 + 2\alpha \int_\Omega \nabla (u_1^k)' . (u_2^k)' \, dx \\ &+2 \textstyle\sum_{i=1}^2 \int_0^t \int_\Omega (\Delta (u_i^k)''(s))^2 \mu'(\Delta (u_i^k)'(s) \, dx \, ds \\ &\leq \textstyle\sum_{i=1}^2 \|\nabla (u_i^k)''(0)\|^2 + \|\nabla \Delta u_i^{k,1}\|^2 + \alpha \|\nabla u_i^{k,1}\|^2. \end{aligned}$$

using (4.2.5) and (4.2.13), we have

$$\sum_{i=1}^{2} \|\nabla (u_i^k)''(t)\|^2 + (1 - 2\alpha C_s) \|\nabla \Delta (u_i^k)'(t)\|^2 + 2 \int_0^t \int_\Omega \Delta (u_i^k)''(s) \mu(\Delta (u_i^k)''(s)) \, dx \, ds \leq C_2 \tag{4.2.14}$$

By (4.2.3) and (4.2.13), we deduce that

$$(u_i^k)' \quad \text{is bounded in} \quad L^\infty(0,T;V) \tag{4.2.15}$$

and

$$(u_i^k)'' \quad \text{is bounded in} \quad L^\infty(0,T;\mathcal{H}) \tag{4.2.16}$$

By (4.2.15) we deduce that

$$(u_i^k)' \quad \text{is bounded in} \quad L^2(0,T;V).$$

Applying Rellich compactness Theorem given in **[6]**, we deduce that

$$(u_i^k)' \quad \text{is precompact in} \quad L^2(0,T;L^2(\Omega)), \tag{4.2.17}$$

Step 4. The fourth estimate

Differentiating (4.2.2) with respect to x, taking $v = \nabla\Delta^2(u_1^k)'$ in the first equation and $v = \nabla\Delta^2(u_2^k)'$ in the second equation in(4.2.2), add the resulting equations, we obtain that

$$\|\nabla\Delta^2 u_1^k\|^2 = \int_\Omega \nabla\Delta^2 u_1^k(-\nabla(u_1^k)'' - \alpha\nabla u_2^k + \nabla\Delta(u_1^k)'\mu'(\Delta(u_1^k)'))\,dx \tag{4.2.18}$$

and

$$\|\nabla\Delta^2 u_2^k\|^2 = \int_\Omega \nabla\Delta^2 u_2^k(-\nabla(u_1^k)'' - \alpha\nabla u_2^k + \nabla\Delta(u_2^k)'\mu'(\Delta(u_2^k)'))\,dx \tag{4.2.19}$$

Using Cauchy-Schwarz inequality, we have

$$\|\nabla\Delta^2 u_1^k\| \leq 2\Big(\int_\Omega \{|\nabla(u_1^k)''|^2 + \alpha^2|\nabla u_2^k|^2 + |\nabla\Delta(u_1^k)'\mu'(\Delta u_1^k)|^2\}\,dx\Big)^{\frac{1}{2}}. \tag{4.2.20}$$

and

$$\|\nabla\Delta^2 u_2^k\| \leq 2\Big(\int_\Omega \{|\nabla(u_2^k)''|^2 + \alpha^2|\nabla u_1^k|^2 + |\nabla\Delta(u_2^k)'\mu'(\Delta u_2^k)|^2\}\,dx\Big)^{\frac{1}{2}}. \tag{4.2.21}$$

Using (4.1.8), (4.2.15) and (4.2.16), we obtain

$$\sum_{i=1}^{2} \|\nabla\Delta^2 u_i^k\| \leq C_3,$$

for some C_3 independent of k, then

$$u_i^k \quad \text{are bounded in} \qquad L^\infty(0,T;W) \tag{4.2.22}$$

Step 5. Passage to the limit.

Applying Dunford-Petit Theorem, we conclude from (4.2.9), (4.2.12), (4.2.15) and (4.2.16), replacing the sequence u^k, with a subsequence if needed, that

$$u_i^k \rightharpoonup u_i, \text{ weak-star in } L^\infty(0,T;W) \tag{4.2.23}$$

$$(u_i^k)' \rightharpoonup u_i', \text{ weak-star in } L^\infty(0,T;V) \tag{4.2.24}$$

$$(u_i^k)'' \rightharpoonup u_i'', \text{ weak-star in } L^\infty(0,T;\mathcal{H}) \tag{4.2.25}$$

$$(u_i^k)' \longrightarrow u_i', \text{ almost everywhere in } \mathcal{A}, \tag{4.2.26}$$

$$\mu(\Delta(u_i^k)') \rightharpoonup \phi_i, \text{ weak-star in } L^2(\mathcal{A}) \tag{4.2.27}$$

where $\mathcal{A} = \Omega \times [0,T]$. It follows at once from (4.2.23) and (4.2.25), that for each fixed $v \in L^2([0,T] \times L^2(\Omega))$

$$\begin{aligned} &\int_0^T \int_\Omega \Big((u_1^k)''(x,t) + \Delta^2 u_1^k(x,t) + \alpha u_2^k(x,t)\Big) v\, dx\, dt \\ &\longrightarrow \int_0^T \int_\Omega \Big(u_1''(x,t) + \Delta^2 u_1(x,t) + \alpha u_2(x,t)\Big) v\, dx\, dt. \end{aligned} \tag{4.2.28}$$

and

$$\begin{aligned} &\int_0^T \int_\Omega \Big((u_2^k)''(x,t) + \Delta^2 u_2^k(x,t) + \alpha u_1^k(x,t)\Big) v\, dx\, dt \\ &\longrightarrow \int_0^T \int_\Omega \Big(u_2''(x,t) + \Delta^2 u_2(x,t) + \alpha u_1(x,t)\Big) v\, dx\, dt. \end{aligned} \tag{4.2.29}$$

As $(u_i^k)'$ is bounded in $L^\infty(0,T;V)$ and embedding of V in $\mathcal{H}$ is compact, we have

$$(u_i^k)' \longrightarrow u_i', \text{ strong in } L^2(0,T;\mathcal{H}). \tag{4.2.30}$$

It remains to show that

$$\int_0^T \int_\Omega \mu(\Delta(u_i^k)')\, v\, dx\, dt \longrightarrow \int_0^T \int_\Omega \mu(\Delta u_i')\, v\, dx\, dt. \tag{4.2.31}$$

To deal with (4.2.31), we need the next Lemma

Lemma 21. *For each $T > 0$, $\mu(\Delta u_i') \in L^1(\mathcal{A})$, $\|\mu(\Delta u_i')\|_{L^1(\mathcal{A})} \leq K$, where K is a constant independent of t and $\mu(\Delta(u_i^k)') \to \mu(\Delta u_i')$ in $L^1(\mathcal{A})$.*

Proof. We claim that

$$\mu(\Delta u_i') \in L^1(\mathcal{A}).$$

Indeed, since μ is continuous, we deduce from (4.2.26)

$$\mu(\Delta(u_i^k)') \longrightarrow \mu(\Delta u_i') \quad \text{almost everywhere in} \quad \mathcal{A}. \tag{4.2.32}$$

$$\Delta(u_i^k)'\mu(\Delta(u_i^k)') \longrightarrow \Delta u_i'\mu(\Delta u_i') \quad \text{almost everywhere in} \quad \mathcal{A}.$$

Hence, by (4.2.11) and Fatou's Lemma, we have

$$\int_0^T \int_\Omega \Delta u_i'(x,t)\mu(\Delta u_i'(x,t))\, dx\, dt \leq K_1, \text{ for } T > 0. \tag{4.2.33}$$

Now, we can estimate $\int_0^T\int_\Omega |\Delta\mu(u_i'(x,t))|\,dx\,dt$. By Cauchy-Schwartz inequality, we have

$$\int_0^T\int_\Omega |\mu(u_i'(x,t))|\,dx\,dt \le c|\mathcal{A}|^{1/2}\Big(\int_0^T\int_\Omega |\mu(u_i'(x,t))|^2\,dx\,dt\Big)^{1/2}.$$

Using (4.1.6), (4.1.7) and (4.2.33), we obtain

$$\begin{aligned}\int_0^T\int_\Omega |\mu(u_i'(x,t))|^2\,dx\,dt &\le \int_0^T\int_{|\Delta u_i'|>\varepsilon} \Delta u_i'\mu(\Delta u_i')\,dx\,dt + \int_0^T\int_{|\Delta u_i'|\le\varepsilon} H^{-1}(\Delta u_i'\mu(\Delta u_i'))\,dx\,dt\\ &\le c\int_0^T\int_\Omega \Delta u_i'\mu(\Delta u_i')\,dx\,dt + cH^{-1}\Big(\int_{\mathcal{A}} \Delta u_i'\mu(\Delta u_i')\,dx\,dt\Big)\\ &\le c\int_0^T\int_\Omega \Delta u_i'\mu(\Delta u_i')\,dx\,dt + c'H^*(1) + c''\int_\Omega \Delta u_i'\mu(\Delta u_i')\,dx\,dt \qquad (4.2.34)\\ &\le cK_1 + c'H^*(1), \quad \text{for } T>0.\end{aligned}$$

Then

$$\int_0^T\int_{\mathcal{A}} |\mu(u_i'(x,t))|\,dx\,d \le K, \quad \text{for } T>0.$$

Let $E\subset\Omega\times[0,T]$ and set

$$E_1 = \Big\{(x,t)\in E : |\mu(\Delta(u_i^k)'(x,t))| \le \frac{1}{\sqrt{|E|}}\Big\}, \quad E_2 = E\backslash E_1,$$

where $|E|$ is the measure of E. If $M(r) = \inf\{|s| : s\in\mathbb{R} \text{ and } |\mu(s)|\ge r\}$

$$\int_E |\mu(\Delta(u_i^k)')|\,dx\,dt \le c\sqrt{|E|} + \Big(M\Big(\frac{1}{\sqrt{|E|}}\Big)\Big)^{-1}\int_{E_2} |\Delta(u_i^k)'\mu(\Delta(u_i^k)')|\,dx\,dt.$$

By applying (4.2.11) we deduce that

$$\sup_k \int_E \mu(\Delta(u_i^k)')\ dx\ dt \longrightarrow 0, \text{ when } |E|\longrightarrow 0.$$

From Vitali's convergence Theorem, we deduce that

$$\mu(\Delta(u_i^k)') \to \mu(\Delta u_i') \quad \text{in } L^1(\mathcal{A}).$$

This completes the proof. □

Then (4.2.27) implies that

$$\mu(\Delta(u_i^k)') \rightharpoonup \mu(\Delta u_i'), \text{ weak-star in } L^2([0,T]\times\Omega).$$

We deduce, for all $v \in L^2([0,T] \times L^2(\Omega)$, that

$$\int_0^T \int_\Omega \mu(\Delta(u_i^k)')v\,dx\,dt \longrightarrow \int_0^T \int_\Omega \mu(\Delta u_i')v\,dx\,dt.$$

Finally we have shown that, for all $v \in L^2([0,T] \times L^2(\Omega))$:

$$\begin{cases} \int_\Omega (u_1''(t) + \alpha u_2 + \Delta^2 u_1 - \mu(\Delta u_1'))v\,dx = 0, \\ \int_\Omega (u_2''(t) + \alpha u_1 + \Delta^2 u_2 - \mu(\Delta u_2'))v\,dx = 0, \end{cases} \tag{4.2.35}$$

Therefore, (u_1, u_2) is a solution for problem (4.1.1). The proof of Theorem 15 is now completed

□

Here, we establish the decay estimate for solution in Theorem 16. For this end, we use method of multipliers and prepare a several Lemmas

Lemma 22. *We have*

$$\begin{aligned} 2\int_S^T \varphi(\mathcal{E})dt \;=\; & \left[\frac{\varphi(\mathcal{E})}{\mathcal{E}} \int_\Omega \sum_{i=1}^2 u_i^{'} \Delta u_i dx\right]_S^T - \int_S^T \left(\frac{\varphi(\mathcal{E})}{\mathcal{E}}\right)' \int_\Omega \sum_{i=1}^2 u_i^{'} \Delta u_i dxdt \\ & + \int_S^T \frac{\varphi(\mathcal{E})}{\mathcal{E}} \int_\Omega \sum_{i=1}^2 \left(2\left|\nabla u_i^{'}\right|^2 - \Delta u_i \mu(\Delta u_i^{'})\right) + 2\alpha \nabla u_1 . \nabla u_2 dxdt. \end{aligned} \tag{4.2.36}$$

for all $0 \leq S < T < +\infty$.

Proof. Multiplying (4.1.1)$_1$ by $-\frac{\varphi(\mathcal{E})}{\mathcal{E}}\Delta u_1$ and (4.1.1)$_2$ by $-\frac{\varphi(\mathcal{E})}{\mathcal{E}}\Delta u_2$ respectively, summing the obtained results, we have

$$\begin{aligned}
0 &= \int_S^T \frac{\varphi(\mathcal{E})}{E} \int_\Omega \sum_{i=1}^{2} \left(-\Delta u_i \left(u_i'' + \Delta^2 u_i - \mu(\Delta u_i')\right) - \alpha u_2.\Delta u_1 - \alpha u_1.\Delta u_2\right) dxdt \\
&= \int_S^T \frac{\varphi(\mathcal{E})}{\mathcal{E}} \left[-\int_\Omega \sum_{i=1}^{2} \left(u_i''\Delta u_i + u_i'\Delta u_i'\right) dx\right] dt + \int_S^T \frac{\varphi(\mathcal{E})}{E} \left[\int_\Omega \sum_{i=1}^{2} u_i'\Delta u_i' dx\right] dt \\
&\quad + \int_S^T \frac{\varphi(\mathcal{E})}{E} \int_\Omega \sum_{i=1}^{2} (-\Delta u_i)\, \Delta^2 u_i dxdt - \int_S^T \frac{\varphi(\mathcal{E})}{\mathcal{E}} \int_\Omega \sum_{i=1}^{2} (-\Delta u_i)\, \mu\left(\Delta u_i'\right) dxdt \\
&\quad -\alpha \int_S^T \frac{\varphi(\mathcal{E})}{\mathcal{E}} \int_\Omega (\Delta u_1.u_2 + \Delta u_2.u_1)\, dxdt \\
&= \left[\frac{\varphi(\mathcal{E})}{\mathcal{E}} \int_\Omega -\sum_{i=1}^{2} u_i'\Delta u_i dx\right]_S^T + \int_S^T \left(\frac{\varphi(\mathcal{E})}{\mathcal{E}}\right)' \int_\Omega \sum_{i=1}^{2} u_i'\Delta u_i dxdt \\
&\quad + \sum_{i=1}^{2} \int_S^T \frac{\varphi(\mathcal{E})}{\mathcal{E}} \int_\Omega -|\nabla u_i'|^2 + |\nabla\Delta u_i|^2 + (\Delta u_i)\, \mu\left(\Delta u_i'\right) dxdt + 2\alpha \int_\Omega \nabla u_1.\nabla u_2 dxdt.
\end{aligned}$$

Using the definition of the energy, hence (4.2.36) follows. □

Lemma 23. *We have*

$$A \int_S^T \varphi(\mathcal{E})dt \leq c\varphi(\mathcal{E}\,(S)) + \int_S^T \frac{\varphi(\mathcal{E})}{\mathcal{E}} \int_\Omega 2\left|\nabla u_i'\right|^2 + |\Delta u_i||\mu\left(\Delta u_i'\right)|dxdt \tag{4.2.37}$$

for all $0 \leq S < T < +\infty$.

Proof. Using the obvious estimates

$$\left\|u_i'\right\|_{L^2(\Omega)} \leq c\left\|\nabla u_i'\right\|_{L^2(\Omega)} \tag{4.2.38}$$

and

$$\|\Delta u_i\|_{L^2(\Omega)} \leq c\,\|\nabla\Delta u_i\|_{L^2(\Omega)} \tag{4.2.39}$$

Since E is non-increasing, we find that

$$\begin{aligned}
-\left[\tfrac{\varphi(\mathcal{E})}{\mathcal{E}} \int_\Omega \nabla u_i'\nabla u_i dx\right]_S^T \leq\; & \tfrac{\varphi(\mathcal{E}(S))}{\mathcal{E}(S)} \int_\Omega \nabla u_i'(S)\nabla u_i(S)dx - \tfrac{\varphi(\mathcal{E}(T))}{\mathcal{E}(T)} \int_\Omega \nabla u_i'(T)\nabla u_i(T)dx \\
&\leq c\varphi(\mathcal{E}(S)).
\end{aligned} \tag{4.2.40}$$

Furthermore, using (4.2.38) and (4.2.39) again,

$$\begin{aligned}\left|\int_S^T \left(\frac{\varphi(\mathcal{E})}{\mathcal{E}}\right)' \int_\Omega u_i' \Delta u_i dxdt\right| &= c\int_S^T \left|\left(\frac{\varphi(\mathcal{E})}{\mathcal{E}}\right)'\right| \mathcal{E}(t)dt \\ &\leq \varphi(\mathcal{E}(S)),\end{aligned} \tag{4.2.41}$$

Using Poincaré and Young's inequalities and the energy inequality from Lemma 19, we obtain

$$2\alpha \int_S^T \frac{\varphi(\mathcal{E})}{\mathcal{E}} \int_\Omega \nabla u_1.\nabla u_2 dxdt \leq c\int_S^T \varphi(\mathcal{E}(S))dt,$$

Using these two estimates, (4.2.37) follows from (4.2.36). □

Proof. (**Of Theorem 16**) 1. H is linear on $[0, \varepsilon_1]$:

we have $c_1 |s| \leq |\mu(s)| \leq c_2 |s|$, for all $s_¸ \in \mathbb{R}$, and then, using (4.1.6) and (4.1.7) and noting that $s \mapsto \frac{\varphi(s)}{s}$ is non-increasing,

$$\begin{aligned}\int_S^T \frac{\varphi(\mathcal{E})}{\mathcal{E}} \int_\Omega |\nabla u_i'|^2 dxdt &\leq c\int_S^T \frac{\varphi(\mathcal{E})}{\mathcal{E}} \int_\Omega \Delta u_i'.\mu(\Delta u_i')dxdt \\ &\leq c\varphi(\mathcal{E}(S)).\end{aligned} \tag{4.2.42}$$

Using Poincaré amd Young's inequalities and the energy inequality from Lemma 19, we obtain, for all $\varepsilon > 0$,

$$\begin{aligned}\int_S^T \frac{\varphi(\mathcal{E})}{\mathcal{E}} \int_\Omega |\Delta u_i \mu(\Delta u_i')| dxdt &\leq \varepsilon \int_S^T \frac{\varphi(\mathcal{E})}{\mathcal{E}} \int_\Omega \Delta u_i^2 dxdt + c_\varepsilon \int_S^T \frac{\varphi(\mathcal{E})}{\mathcal{E}} \int_\Omega \mu^2(\Delta u_i')dxdt \\ &\leq \varepsilon \int_S^T \frac{\varphi(\mathcal{E})}{\mathcal{E}} \int_\Omega \Delta u_i^2 dxdt + c_\varepsilon \int_S^T \frac{\varphi(\mathcal{E})}{\mathcal{E}} \int_\Omega \Delta u_i' \mu(\Delta u_i')dxdt \\ &\leq \varepsilon \int_S^T \varphi(\mathcal{E})dt + c_\varepsilon \varphi(\mathcal{E}(S)).\end{aligned} \tag{4.2.43}$$

Inserting these two inequalities into (4.2.37), choosing $\varepsilon > 0$ small enough, we deduce that

$$\int_S^T \varphi(\mathcal{E})dt \leq c\varphi(\mathcal{E}(S))$$

Choosing $\varphi(s) = s$. Then, for some $\omega > 0$

$$\int_S^{+\infty} \mathcal{E}(t)dt \leq \frac{1}{\omega}\mathcal{E}(S) \quad \forall S > 0$$

Using Lemma 19, we deduce from (4.1.14) that

$$\mathcal{E}(t) \leq C\mathcal{E}(0)e^{-wt}, \forall t \geq 0.$$

2. $H'(0) = 0$ and $H'' > 0$ on $]0, \varepsilon_1]$ for all $t \geq 0$ we denote by

$$\Omega_1 = \{x \in \Omega : |\Delta u'| \geq \varepsilon_1\}, \Omega_2 = \{x \in \Omega : |\Delta u'| \leq \varepsilon_1\}.$$

Using (4.1.6) and the fact that $s \mapsto \frac{\varphi(s)}{s}$ is non-decreasing, we obtain

$$c\int_S^T \frac{\varphi(\mathcal{E})}{\mathcal{E}} \int_{\Omega_1} |\Delta u_i'|^2 + \mu^2(\Delta u_i')dxdt \leq c\int_S^T \frac{\varphi(\mathcal{E})}{\mathcal{E}} \int_{\Omega} \Delta u_i'.\mu(\Delta u_i')dxdt \leq c\varphi(\mathcal{E}(S))$$

On the other hand, since H is convex and increasing, H^{-1} is concave and increasing. Therefore, (4.1.7) and the reversed Jensens inequality for concave function imply that

$$\begin{aligned} \int_S^T \tfrac{\varphi(\mathcal{E})}{\mathcal{E}} \int_{\Omega_2} |\Delta u_i'|^2 + \mu^2(\Delta u_i')dxdt &\leq \int_S^T \tfrac{\varphi(\mathcal{E})}{\mathcal{E}} \int_{\Omega_2} H^{-1}(\Delta u_i'.\mu(\Delta u_i'))dxdt \\ &\leq \int_S^T \tfrac{\varphi(\mathcal{E})}{\mathcal{E}} |\Omega| H^{-1}\left(\tfrac{1}{|\Omega|}\int_{\Omega} \Delta u_i'.\mu(\Delta u_i')dx\right) dt. \end{aligned} \tag{4.2.44}$$

Using remark 4, due to our choice $\varphi(s) = sH'(\varepsilon_0 s)$, we have

$$H^*(\frac{\varphi(s)}{s}) = \varepsilon_0 H'(\varepsilon_0 s) - H(\varepsilon_0 s) \leq \varepsilon_0\varphi(s). \tag{4.2.45}$$

Making use of (4.2.44) and (4.2.45) we have

$$\begin{aligned} \int_S^T \tfrac{\varphi(\mathcal{E})}{\mathcal{E}} \int_{\Omega_2} |\Delta u_i'|^2 + \mu^2(\Delta u_i')dxdt &\leq c\int_S^T H^*(\tfrac{\varphi(\mathcal{E})}{\mathcal{E}})dt + c\int_S^T u_i'.\mu(\Delta u_i')dxdt \\ &\leq c\int_S^T \varphi(\mathcal{E})dt + c\mathcal{E}(S) \end{aligned} \tag{4.2.46}$$

Then, choosing $\varepsilon_0 > 0$ small enough and using (4.2.37), we obtain in both cases

$$\begin{aligned} \int_S^T \varphi(\mathcal{E})dt &\leq c\left(\mathcal{E}(S) + \varphi(\mathcal{E}(S))\right) \\ &\leq c\left(1 + \tfrac{\varphi(\mathcal{E}(S))}{\mathcal{E}(S)}\right)\mathcal{E}(S) \\ &\leq c\mathcal{E}(S) \quad \forall S \geq 0. \end{aligned} \tag{4.2.47}$$

Using Lemma 19 in the particular case where $\psi(s) = \omega\varphi(s)$, we deduce from (4.1.14) our estimate (4.2.1).

This complete the proof of Theorem 16. □

4.3 Conclusion

We proved the existence of a weak solution and its decay to zero as time goes to infinity for a system of coupled evolutionary second order in time PDEs (4.1.1). The problem is defined on a bounded domain Ω, we used the Poincaré inequality and the Rellich-Kondrachov theorem on compact embedding. The preblem is supplemented with homogeneous Dirichlet-type boundary conditions on both functions u_1, u_2 and their Laplacians, as well as with the initial conditions. The existence of solutions is proved by means of the Galerkin method while the decay is obtained by a variant of a method of multipliers, developed in eighties/nineties by mathematicians such as A. Haraux [15], Martinez [27], V. Komornik [20, 21], Nakao [26]. The argument of the work follows very closely the argument of the following article [1]. Although the problem considered in the paper [1] is different than in the present research.

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