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THÈSE

EN VUE DE L'OBTENTION DU DIPLOME DE DOCTORAT EN 3ème CYCLE

Domaine : Mathématiques Filière : Mathématiques
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Présentée par

TAIR Boutheina

Intitulée

Etude des équations intégro-différentielles linéaire de seconde espèce : Approche analytique et numérique

Soutenue le : ………………. Devant le Jury composé de :

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By: Boutheina TAIR

Entitled:

Study of linear integro-differential equations of the second kind: Analytical and numerical approach

Under the direction of: Pr. Hmaza GUEBBAI

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To my parent

Thanks

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Abstract

The objective of this thesis is to deal with linear integro-differential Fredholm equations especially as these kinds of equations play an important role in modeling of different problems in various fields such as physics and biology. Therefore, we are going to study them in the analytically and numerically sense. We study these equations in Banach space $C^1[a,b]$ with two cases of continuous and weakly singular kernels. In the continuous case we construct three methods based on the Nyström, Collocation and Kantorovich methods in order to find the best approximation of our solution. In the weakly singular case, we construct two methods which are b-spline collocation and product integration in an essential reason which is a good precision and acceleration in the calculations..

We analysis our equation analytically in Sobolev spaces $W^{1,p}[a,b],\ p \in [1,+\infty[$. We give a sufficient condition that shows the existence and uniqueness of the solution in these mathematical spaces. We have constructed in both spaces: $W^{1,1}[a,b]$ and $H^1[a,b]$ two projection methods based on Galerkin and Kantorovich.

Keywords: Fredholm integral equation, Integro-differential equation, System of integral equations, Projection methods, Iterative Methods, Nystöm Method

Mathematics Subject Classification: 45B05, 47G20, 45F05, 65R10, 65F10,64R20

Résumé

L'objectif de cette thèse est de traiter les équations linéaires intégro-différentielles de Fredholm analytiquement et numériquement, surtout que ce genre d'équations, jouent un rôle important pour modéliser différents problèmes dans plusieurs domaines, en particulier la physique et la biologie.
Nous allons étudier ces équations dans l'espace de Banach $C^1[a,b]$, dans les cas respectivement des noyaux continues et faiblement singulières.
Dans le but de chercher la meilleur approximation, nous allons construire dans le cas continue, trois méthodes numériques, basées sur les méthodes Nyström, Collocation et Kantorovich.
Dans le cas faiblement singulière, nous adaptons le principe de comparaison en appliquant deux méthodes, la méthode des intégrations de produits et la méthodes b-spline collocation, afin déterminer la bonne précision concernant l'erreur et le temps de calcul.
Nous traitons notre équation analytiquement dans les espaces de Sobolev $W^{1,p}[a,b],\ p \in [1,+\infty[$ et nous proposons une condition suffisante, pour montrer l'existence et l'unicité de la solution dans ce types d'espace. Enfin, nous adaptions notre techniques aux espaces : $W^{1,1}[a,b]$ et $H^1[a,b]$ en associant deux méthodes de projection à savoir Galerkin et Kantorovich.

Mots clé : Equation intégrale de Fredholm, Equation Integro-differential, Système des équations integrales, Méthodes de projection, Méthodes Itératives, Méthode de Nystöm

Mathematics Subject Classification :45B05, 47G20, 45F05, 65R10, 65F10,64R20

الملخص

الهدف من هذه الاطروحة هوالتعامل مع معادلات فريدهولم التكاملية التفاضلية الخطية و التي لها أهمية الكبيرة و ذلك من خلال استخدامها في نمذجة العديد من المشاكل في مختلف المجالات العلمية من أهمها الفيزياء و البيولوجيا بحيث نقوم بدراستها بشكل تحليلي والحس العددي.

ندرس هذه المعادلات في فضاء باناخ $C^1[a,b]$ مع حالتين مختلفتين لنواتين نواتين مستمرتين ونواتين ضعيفتين. في الحالة المستمرة، نبني حلول رقمية مرتكزة على ثلاث طرق هي : نيشتروم, طريقة التجميع و كونتوروفيتش من أجل إيجاد أفضل حل تقريبي ممكن. في حالة النواتين ضعيفتين ، نحن نبني طريقتين وهما التجميع بالاعتماد على الدوال b-spline وتكامل المنتج بهدف الحصول على أفضل حل رقمي من ناحية الدقة و السرعة في الحسابات.

ندرس معادلتنا تحليليًا في فضاءات سوبوليف و ذلك من خلال بناء فرضية تعيطنا وجود ووحدانية الحل في هذا النوع من الفضاءات الرياضية. ولكن نقوم ببناء حلين رقميين مختلفين في كل من الفضاءات : $W^{1,1}$ [a,b] و $H^{1,1}$ [a,b] معتمدين على كل من طريقتي الاسقاط : ڤالركين و كونتروفيتش.

الكلمات المفتاحية: معادلات تكاملية لفريدولم, معادلات تكاملية-تفاضلية, نظام المعادلات التكاملية, طريق الاسقاط, الطرق الترابطية, طريقة نيشتروم.

تصنيف مواضيع الرياضيات: 45B05, 47G20, 45F05, 65R10, 65F10, 64R20

Notations

$\mathbb{R}$: Set of real numbers.

$\mathbb{C}$: Set of complex numbers.

$C^0[a,b]$: The Banach space of continuous functions.

$C^1[a,b]$: The Banach space of continuously differentiable functions.

$W^{1,p}[a,b],\ p\ \in [1,+\infty[$: The Sobolev space, which vector space of functions that have weak derivative.

$H^1[a,b]$: The Sobolev space with $p=2$.

$L^P[a,b],\ p\ \in [1,+\infty[$: The vector space of classes of functions whose exponent power p is integrable in the Lebesgue sense.

T^{-1}: The inverse of operator T.

$re(T)$: The resolvant set of T.

$R(T,\lambda)$: The resolvant of operator T.

$sp(T)$: The spectrum set of T.

$\mathcal{B}(X,Y)$: The space of linear and bounded operator.

A_T: The block operator matrix.

$\langle\cdot,\cdot\rangle$: Scalar product.

$(\cdot,\cdot)$: Dual product

Contents

Introduction

An integro-differential equation can be defined as an hybrid of an integral equation and an ordinary equation or partial differential equation. In the simple sense, it is the relation between the integral and the one or more unknown function u derivative's which appear out side or underside the integral sign. Recently, integro-differential equations play an important role, as they are considered the best model to express many phenomena in many fields. Its first appearance was a qualitative leap in the field of applied mathematics and mathematical modeling. So let's go back to the early 1900s, when the first appearance of this equation was by Volterra, who used it to express the development of individuals over a period of time [81]. Volterra may have been the first to use it, but its wide diffusion, development and different forms are due to many researchers such as Abel, Lotka, Fredholm, Maltus and Verluslst[**?**]. Later, many mathematicians followed their approach, and many competed either to develop others forms of this equation or to devise methods that could help solve it.

Its first appearance was in biology as mentioned earlier, but this does not prevent us from converting a differential equation into an integro-differential equation or from appearing as a solution to a differential equation. Its appearance has not been limited to mathematics in all its branches, but has transcended many fields. The integro-differential equations are used to model several phenomena [89] such as the system of leaking aquifers [29], the earthquake [59]. Also, they are used to define some procedures in chemistry or to explain the RL model in electric circuit [55, 54]. The use of these equations are not limited to the above domains. It can be used in medicine to model many diseases [28]. The most famous of which is malaria [73], the process of coagulation [55], cancer chemotherapy [51], the sugar quantity in blood [56] and they are used to model the behaviour of corona virus [35, 85].

Throughout history, integro-differential equations have seen different forms, models and types, which makes it difficult to solve and to find a single numerical method for different types. So far, many research papers have been published with the aim of creating numerical methods for a specific type or improving existing methods in order to find numerical solutions closest to the exact solution. They invented and built many numerical methods, its main objective is to find the best approximate solution for this type of equations, and below we will present some of them: Block boundary value method [90, 11, 17, 86, 50], Jacobi and Gauss-Siedel Method [41], Best approximity point result [14], Operational matrix [37, 64], Modified Brensten-Kantorovich operator [12], Hybrid Legendre polynomial [43], Tau numerical method [30], Haar wavelet method [42], the variational iterative method [62, 45], Enbedded Pseudo Rung-Kutta method [76], . . .

Given the great importance of these equations, many researchers have played an important role in studying different models. The most well known integro-differential equation which was studied in different papers [86, 64, 65, 66] has the following form

$$\forall u \in X,\ \forall x \in [a,b],\ \lambda u'(x) = \int_a^x K(x,t,u(t))\,dt + f(x,u(x)), \quad u(x_0) = u_0, \tag{1}$$

which is called the Volterra non linear integro-differential equation and X is a Banach space. Another equation has a form different to (1) is well studied in [25, 60, 59]. It is given as

$$\forall u \in X\ \forall x \in [a,b],\ \lambda u(x) = \int_a^x K(x,t,u(t),u'(t))\,dt + f(x). \tag{2}$$

If x of (4) is fixed in $[a,b]$, we obtain the following non-linear Fredholm integro-differential equation

$$\forall u \in X,\ \lambda u(x) = \int_a^b K(x,t,u(t),u'(t))\,dt + f(x). \tag{3}$$

The linear form of (3) is

$$\forall u \in X,\ \forall x \in [a,b],\ \lambda u(x) = \int_a^b K(x,t)u'(t)\,dt + f(x). \tag{4}$$

We have another equation which related between (2) and (3) is known as Fredholm-Volterra integro-differential equation [77] and [78]

$$\forall u \in X,\ \lambda u(x) = \int_a^x K_1(x,t,u(t))\,dt + \int_a^b K_2(x,t,u'(t))\,dt + f(x). \tag{5}$$

The combination of the previous equations allows the birth of a new equation of the following form, to which we will give much attention and care in this thesis.

$$\forall u \in X,\ \lambda u(x) = \int_a^b K_1(x,t)u(t)\,dt + \int_a^b K_2(x,t)u'(t)\,dt + f(x), \tag{6}$$

where, in all above equations λ is a complex parameter and f is given function in X.

Maybe there is no problem in the field that can be modelled in the form of the above equation (6), but the mathematical importance of this equation appearance when we use the linearisation method to treat (2) and (3) like Newton-Kantorovich method. Therefore, our research can be a reference that facilitates the study of this type of equations because through this thesis, we are trying to deal with the different forms of this equation. It should be noted that we have searched a little and we find on the other side the biggest competitor of these equations, which are the differential equations. We find that any problem of the following form

$$\begin{cases} \lambda u''(x) + a(x)u'(x) + b(x)u(x) = g(x), \\ u(a) = \alpha_1, u(b) = \beta_1 \\ u'(a) = \alpha_2, u'(b) = \beta_2, \end{cases} \tag{7}$$

can be written in the previous form (6).

The use of differential equations is clearly shown in physical phenomena of solids in free vibration, in electricity through a circuit in mechanics through Newton's equation and many examples shown below

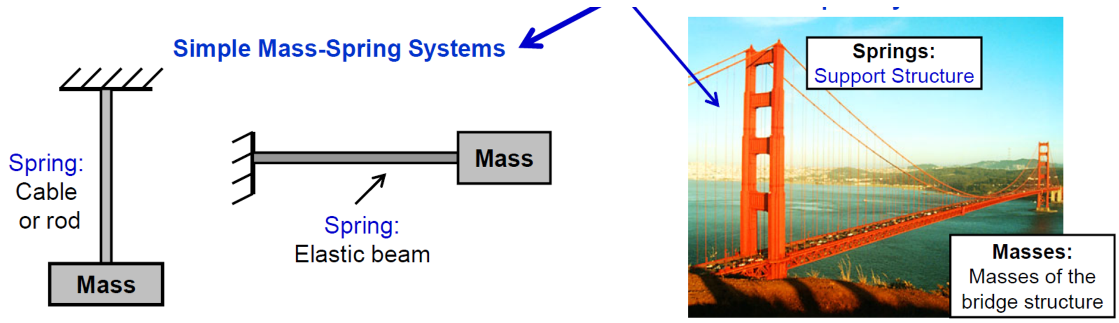


Figure 1: Physical demonstration of a simple mass-spring system: A mass attached to an elastic support

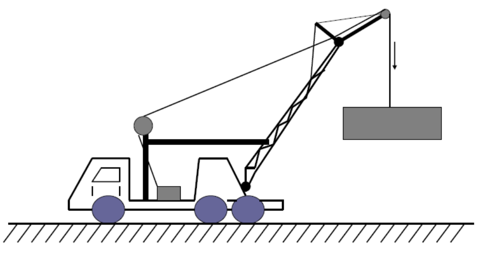


Figure 2: Physical demonstration of a simple mass-spring system: A mass attached to an elastic cable

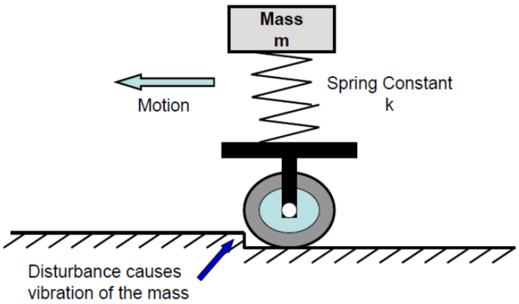


Figure 3: Physical demonstration of a simple mass-spring system in free vibration

The last examples have the next form:

$$u''(t) + \frac{k}{m}u(t) = 0,$$

where, $u(t)$ signified the instantaneous position of the mass, m is a mass and k is a spring constant.

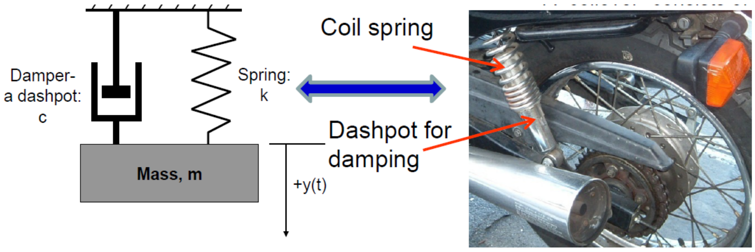


Figure 4: Physical model of damped vibration

This example has this form

$$u''(t) + \frac{c}{m}u'(t) + \frac{k}{m}u(t) = 0.$$

Therefore, it can be written in the form of the above equation (6), and by using the clear methods of this equation, you can get better numerical solutions than the existing numerical solutions. Here, we have clarified the importance of the equation (6) and we are going to study numerically and analytically. Now, we would like to review and discuss the focus of our thesis.

Our thesis contains four essential chapters. In **the first chapter**, we give some concepts and definitions that we are going to use in the thesis. In **the second chapter**, we study and to analyse the existence and uniqueness of the solution in the Banach space $X = C^1[a, b]$. We construct three different methods based on the Nyström, collocation and Kantorovich methods to approximate the solution. We introduce theorems that show the convergence of numerical solutions. At the end of the chapter, we compare the numerical results of the three approximated solutions and develop a new iterative system since the system obtained by the three methods is of very large size.

In the **third chapter**, we study analytically and numerically the equation in the Banach space $X = C^1[a, b]$ but this time the two kernels of our equation are weakly singular. So, we find sufficient conditions to prove the existence and uniqueness of the solution. For the numerical approach, we construct two different methods. The first one is based on the product integration method. The second is based on b-spline collocation methods. The difference between the two is in the size of the system and the speed of convergence. While the first method gives us a system of four blocks, the second method gives us a system of only one block. The consistency of the numerical methods is well explained and the comparison between the two methods is well illustrated in the numerical example.

Because, we are interested in the smallest details of our equation, here we study the uniqueness and presence of the solution in the most complex Banach spaces: Sobolev spaces. Especially since they are considered to be the best spaces that well express many real-life phenomena. We study the uniqueness of the solution and its existence in the space $W^{1,P}[a,b]$, but the numerical methods developed one only $W^{1,1}[a,b]$ and $H^1[a,b]$. Our numerical solution will be obtained by constructing two methods based on each of Kantrorovich and Galerkin. After having obtained our apical solution, we study its convergence through several theorems which explain and prove it in the sense of the norm of two spaces $W^{1,1}[a,b]$ and $H^1[a,b]$. All this will be expressed in **the last chapter** .

Chapter 1

Preliminaries

Contents

In this chapter, we shall begin by introducing the essential basic notions that we use during this thesis. We start by recalling some notions about the functional spaces. Then, we present some notions and theorems on the linear and bounded operators which helps us to introduce the classical process of integral equations defined on a bounded interval. We focus on the linear Fredholm integral equations and we propose some approximate methods applied to this type of equations. The presentation of the Fredholm integral equations allow us to well introduce the integro-differential equations concept. Note that, all theorems and properties will be presented without proofs.

1.1 Operator Concept

Let X and Y be Banach spaces with the norms $||.||_X$ and $||.||_Y$ respectively.

1.1.1 Linear and Bounded Operator

A function T which maps X into Y is called a linear operator [23], if

$$\forall\, v, w \in X \text{ and } \alpha, \gamma \in \mathbb{C} \text{ (or } \mathbb{R}), \quad T(\alpha v + \gamma w) = \alpha T(v) + \gamma T(w),$$

and bounded if there exists a positive constant β, such that

$$\forall v \in X,\ ||Tv||_Y \leq \beta ||v||_X.$$

The norm $||.||$ of the operator T is defined by

$$||T|| = \sup_{||v||_X \neq 0} \frac{||Tv||_Y}{||v||_X} = \sup_{||v||_X \leq 1} ||Tv||_Y = \sup_{||v||_X = 1} ||Tv||_Y.$$

Theorem 1.1.1. *[2] Let $T : (X, ||.||_X) \to (Y, |.||_Y)$ be a linear operator. Then, the following assertions are equivalent:*

1. *T is continuous on all X,*
2. *T is continuous in 0,*
3. *T is bounded.*

Theorem 1.1.2. *The linear space $\mathcal{B}(X, Y)$ of bounded linear operators from X into Y is a Banach space.*

Notation 1.1.1.

1. *We denoted by $\mathcal{B}(X)$ the Banach space of linear and bounded operators from X into itself.*
2. *The operator that sends each $v \in X$ to 0 will be denoted by $\mathcal{O}$ and the identity operator will be denoted by I.*

Definition 1.1.1. *[2] Let $(T_n)_{n\in\mathbb{N}} \in \mathcal{B}(X, Y)$ an approximations sequence of T. $(T_n)_{n\in\mathbb{N}}$ converges pointwise (or simply) to T, if only if*

$$\lim_{n\to+\infty} ||(T_n - T)v||_Y = 0, \quad \forall v \in X.$$

Definition 1.1.2. *[2] $(T_n)_{n\in\mathbb{N}}$ is called a uniform approximations sequence of $T \in \mathcal{B}(X, Y)$, if and only if*

$$\lim_{n\to+\infty} ||T_n - T|| = 0.$$

Theorem 1.1.3. ***(Banach Steinhaus)****[10]*
Let $(T_n)_{n\in\mathbb{N}}$ be a family of linear and continuous operators in X from Y. Assume that

$$\sup_{n\in\mathbb{N}} ||T_n v||_Y < \infty, \ \forall v \in X.$$

Then,

$$\sup_{n\in\mathbb{N}} ||T_n|| < \infty.$$

In other words, there is a positive constant β such that

$$\forall v \in X, \ ||T_n v||_Y \leq \beta ||v||_X, \ \forall n \in \mathbb{N}.$$

Corollary 1.1.1. *[10] Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of linear and bounded operators from X to Y, such that T_n converges pointwise to T. Then,*

1. *$\sup_{n\in\mathbb{N}} ||T_n|| < \infty$,*
2. *$T \in \mathcal{B}(X, Y)$,*
3. *$||T|| \leq \lim_{n\to+\infty} \inf ||T_n||$.*

Remarque 1.1.1.

- *The second result of the last corollary shows that $(T_n)_{n\in\mathbb{N}}$ converges to T uniformly on compact sets.*
- *In the American literature the **Banach Steinhaus** theorem's is referred to as the **Principle Uniform Boundess**, which expresses the result: We deduce a uniform estimate from point estimate.*

1.1.2 Compact Operator

In this section, we present the notion of a compact operator with some theorems concerning this type of operators.

Definition 1.1.3. *[34] Let $T \in \mathcal{B}(X,Y)$. It is called compact if it maps each bounded set in X into a relatively compact set in Y.*

Lemma 1.1.1. *[48] Let $(f_n)_{n\in\mathbb{N}}$ be an equicontinuous sequence in $C^0[a,b]$ and $f \in C^0[a,b]$ such that for each $x \in [a,b]$, $|f_n(x) - f(x)| \to 0$ as $n \to +\infty$. Then, $||f_n - f||_\infty \to 0$ as $n \to +\infty$.*

Theorem 1.1.4. *[84]*

1. *An operator $T \in \mathcal{B}(X,Y)$ is a compact operator if and only if for every bounded sequence $(v_n)_{n\in\mathbb{N}} \in X$, the sequence $(Tv_n)_{n\in\mathbb{N}}$ has a convergent subsequence in Y.*
2. *(**Arezlà-Ascoli theorem**)*
   Let D be a compact set and M a subset of $C^0(D)$ with $||.||_\infty$. M is pre-compact if it is
   (a) *Uniformly bounded, i.e $\sup_{f\in M} ||f||_\infty < \infty$.*
   (b) *Equicontinuous, i.e; $\forall \varepsilon > 0, x \in D, \exists\, \delta$ with,*
   $$|f(x) - f(y)| \leq \varepsilon, \quad \forall f \in M, \text{ and } y \in D, \text{ with } |x - y| \leq \delta.$$
3. *Compact linear operators are bounded.*
4. *Linear combinations of compact linear compact operators are compact.*

Definition 1.1.4. *[10] An operator $T \in \mathcal{B}(X,Y)$ is called the finite rank if dim $(Ran(T)) < +\infty$.*

Theorem 1.1.5. *[10]*

1. *Let $T : X \longrightarrow Y$ is a bounded linear operator of finite rank. Then, T is compact.*
2. *If $(T_n)_{n\in\mathbb{N}} \in \mathcal{B}(X)$ is a sequence of compact operators and $||T_n - T|| \to 0$ as $n \to \infty$. Then, T is compact.*

1.1.3 Spectral Theory

The spectral theory plays an important role in our study. So, in this part we give some definitions and theorems concerning this theory.

Definition 1.1.5. *[2]*

- *The kernel (or null space) of a linear operator T is*
$$Ker(T) = \Big\{ v \in X : Tv = 0 \Big\}.$$
- *The range of T is the image of X under T:*
$$Ran(T) = T(X) = \Big\{ w \in Y,\ w = Tv,\ \forall v \in X \Big\}.$$

Proposition 1.1.1. *[2]*

- *The operator $T \in \mathcal{B}(X,Y)$ is injective if $Ker(T) = \{0\}$.*
- *The operator $T \in \mathcal{B}(X,Y)$ is surjective if $Ran(T) = Y$.*
- *The operator is bijective if it is injective and surjective.*

Definition 1.1.6. *[2] The operator $T \in \mathcal{B}(X,Y)$ is inversible, if T is bijective and T^{-1} is bounded.*

Definition 1.1.7. *[2] Let $T \in \mathcal{B}(X)$ The resolvent set of T is defined by*

$$re(T) = \{\lambda \in \mathbb{C} : \ \lambda I - T \text{ is bijective } \},$$

and for $\lambda \in re(T)$, $R(T,\lambda) = (\lambda I - T)^{-1}$ is called the resolvent operator of T at λ.

Definition 1.1.8. *[2] Let $T \in \mathcal{B}(X)$. The spectrum of T is the set*

$$sp(T) = \mathbb{C} \setminus re(T), = \left\{ \lambda \in \mathbb{C} : \ \lambda I - T \text{ has no inverse } \right\}$$

and an element of $sp(T)$ will be called a spectral value of T. But, $\lambda \in sp(T)$ is called an eigenvalue, if the equation $\lambda v = Tv$ has a non null solution (eigenfunction) $v \in X$.

Theorem 1.1.6. *(**Neumann Expansion**)[13] If T is a bounded linear operator from X into X and $\lambda \in \mathbb{C}$ such that $||T|| < |\lambda|$. Then, $\lambda I - T$ has an inverse in $\mathcal{B}(X,Y)$ given by the following uniformly convergent series*

$$R(T,\lambda) = (\lambda I - T)^{-1} = \sum_{k=0}^{+\infty} \frac{T}{\lambda^{k+1}},$$

and

$$||(\lambda I - T)^{-1}|| < \frac{1}{|\lambda| - ||T||}.$$

Let $T \in \mathcal{B}(X)$. If λ is a given complex paramater. Then, for every $w \in X$, a solution $v \in X$ of the linear equation

$$(\lambda I - T)v = w,$$

will be uniquely determinate by w if and only if $\lambda \in re(T)$ and v is given by

$$v = R(T,\lambda)w.$$

1.1.4 Collectively Compact Operator Approximation

The notion of collectively compact operators is important to show the convergence of numerical solutions. For that, we introduce it in this section.

Definition 1.1.9. *[75] The sequence $(T_n)_{n\in\mathbb{N}} \in \mathcal{B}(X)$ is called a collectively compact approximation sequence of $T \in \mathcal{B}(X)$ if and only if $(T_n)_{n\in\mathbb{N}}$ converges pointwise to T and there is exists $n_0 \geq 0$, such that*

$$\bigcup_{n \geq n_0} \left\{ (T_n - T)v, \ v \in X, \ ||v||_X \leq 1 \right\}$$

is a relatively compact subset.

Theorem 1.1.7. *[48] Suppose that $(T_n)_{n\in\mathbb{N}}$ is a collectively compact sequence in $\mathcal{B}(X)$. Then, the following hold:*

1. *Each T_n is compact,*
2. *If $T_n \to T$ pointwise. Then, T is compact.*

Definition 1.1.10. *[2] The sequence $(T_n)_{n\in\mathbb{N}} \in \mathcal{B}(X)$ is ν-convergences to $T \in \mathcal{B}(X)$ if $||T_n|| < +\infty$ and*

$$\lim_{n\to+\infty} ||(T_n - T)T|| = 0, \qquad \lim_{n\to+\infty} ||(T_n - T)T_n|| = 0.$$

Theorem 1.1.8. *[75] Let $(T_n)_{n\in\mathbb{N}} \in \mathcal{B}(X,Y)$ be a sequence of operators pointwise convergent to $T \in \mathcal{B}(X,Y)$ and A is a compact operator. Then,*

$$\lim_{n\to+\infty} ||(T_n - T)A|| = 0.$$

Proof. Since A is a compact operator, the following set

$$\mathcal{K} = \left\{ Av, \quad ||v||_X \leq 1 \right\},$$

is relatively compact in the Banach space Y. By the **Banach-Steinhaus** theorem 1.1.3 T_n converges uniformly in Y and

$$\lim_{n\to+\infty} ||(T_n - T)A|| = \lim_{n\to+\infty} \sup_{||v||_X\leq 1} ||(T_n - T)Av||_Y = \lim_{n\to+\infty} \sup_{w\in\mathcal{K}} ||(T_n - T)w||_Y = 0.$$

□

1.1.5 Block Operator matrix

A block operator matrix is a matrix, where its elements are linear and bounded operators. Every linear and bounded operator can be written as a block operator matrix if the space in which the operator defined is decomposed in two or more components.

Definition 1.1.11. $\tilde{X}_N = \prod_{i=0}^{N} X_i$ *is the Banach product space, equipped with the following norm*

$$\forall\, V = (v_1, v_2, \dots v_n) \in \tilde{X}_n,\ ||V||_{\tilde{X}} = \sum_{i=0}^{N} ||v_i||_{X_i}.$$

Definition 1.1.12. A_T *is a block operator matrix with the following representation*

$$\begin{aligned} A_T :\quad \tilde{X}_N &\longmapsto \tilde{X}_N \\ V &\longmapsto A_T V = \left(\sum_{j=0}^{N} T_{1j} v_j, \sum_{j=0}^{N} T_{2j} v_j, \dots, \sum_{j=0}^{N} T_{Nj} v_j \right), 1 \leq j \leq n, \end{aligned}$$

where, $\{T_{ij}\}_{1\leq i,j\leq N}$ *are linear and bounded operator.*

Definition 1.1.13. $\mathcal{B}(\tilde{X}_N)$ *is the Banach space of linear and bounded operators defined in the product space* $\tilde{X}$ *into* $\tilde{X}$*, with the norm*

$$\forall\, A_T \in \mathcal{B}(\tilde{X}), |||A_T||| = \sup_{||V||_{\tilde{X}_N=1}} ||A_T V||_{\tilde{X}_N}.$$

Definition 1.1.14. *Let I be the identity operator and $\mathcal{O}$ the null operator of the Banach space X. We define the block identity operator as* $I_N : \tilde{X}_N \longrightarrow \tilde{X}_N$

$$I_N = \begin{pmatrix} I & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} \\ \mathcal{O} & I & \mathcal{O} & \dots & \mathcal{O} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & I \end{pmatrix}$$

Proposition 1.1.1. *The norm of the block operator matrix A_T is given by*

$$|||A_T||| = \max_{1\leq p\leq 2} \sum_{q=1}^{2} ||T_{qp}||.$$

Proof. We have $|||A_T||| = \sup_{||U||_{\tilde{X}_N}=1} ||A_T U||_{\tilde{X}_N}$ such that,

$$\begin{aligned}
||A_T U||_{\tilde{X}_N} &= \sum_{q=1}^{N} || \sum_{p=1}^{N} T_{qp} ||_X, \\
&\leq \sum_{q=1}^{N}\sum_{p=1}^{N} ||T_{qp} u_p||_X, \\
&\leq \sum_{q=1}^{N}\sum_{p=1}^{N} ||T_{qp}||\ ||u_p||_X, \\
&\leq \sum_{q=1}^{N} ||T_{qp}|| \sum_{p=1}^{N} ||u_p||_X, \\
&\leq \max_{1\leq p\leq N} \sum_{q=1}^{N} ||T_{qp}||\ ||U||_{\tilde{X}}.
\end{aligned}$$

This gives

$$|||A_T||| \leq \max_{1\leq p\leq N} \sum_{q=1}^{N} ||T_{qp}||.$$

Without loss of generality, assume that $\max_{1\leq p\leq N} \sum_{q=1}^{N} ||T_{qp}|| = \sum_{q=1}^{N} ||T_{q1}||$. Let $U_0 = (1, 0, \dots 0)$. We have

$$||A_T U_0|| = \sum_{q=1}^{N} ||T_{q1}|| = \max_{1\leq p\leq N} \sum_{q=1}^{N} ||T_{qp}||.$$

Then,

$$|||A_T||| = \max_{1\leq p\leq N} \sum_{q=1}^{N} ||T_{qp}||.$$

□

1.2 Cubic Spline Interpolation

Cubic spline interpolation is considered as a problem of drawing a smooth curve through a $n+1$ points. So, we can use a polynomial of degree $n \geq 1$ passing through all these points, but this is not available if n is large enough. For this reason, we use different polynomials in different regions. Then, the obtained interpolation function is piecewise polynomial. In addition, these approximations may be continuous but in general the derivatives can be non-continuous.

1.2.1 Smooth Cubic Spline

The essential purpose of constructing spline functions is to find an approximate derivable functions. So, let $[a, b]$ be an interval of $\mathbb{R}$, let the given function $f(x_i)$ for $i = 0, 1, \dots n$, such that $\{x_i\}_{i=0}^{n}$ are defined as $a = x_0 < x_1 < \cdots < x_n$ and let g be an interpolation function of f on the interval $[a, b]$. We divide the interval $[a, b]$ into n subintervals $I_i = [x_{i-1}, x_i]$ for $0 \leq i \leq n$. The first simple approximation g can be fined by employing the linear interpolating in each subintervals I_i. In this case, f is approximate by a straight line. This approximation g is refried to linear spline which is not be smooth. To fined a smooth approximation, we can use a polynomial of higher degree over each of the subintervals I_i, such as the piecewise cubic interpolation.

In cubic spline interpolation, we try to find an approximation S for any given function f, verifying the following conditions

1. $S(x) \in C^2[a, b]$,

2. $S(x_i) = f(x_i)$, for $i = 0, 1 \dots n$,

3. On each interval I_i, $S(x)$ is a polynomial of degree 3:

$$S(x) = \begin{cases} S_1(x), & x_0 \leq x \leq x_1, \\ S_2(x), & x_1 < x \leq x_2, \\ \vdots & \\ S_n(x), & x_{n-1} < x \leq x_n, \end{cases}$$

where each S_i has the following form

$$S_i(x) = a_i + b_i x + c_i x^2 + d_i x^3, \quad i = 0, 1 \dots n.$$

To determine the cubic spline $S(x)$, we need to calculate the coefficients a_i, b_i, c_i, d_i, by assuming that

$$\begin{aligned} S_i(x_{i-1}) &= f(y_{i-1}), \ i = 1, \dots n, \\ S_i(x) &= f(x_i), \ i = 1, \dots n, \\ S_i'(x_{i-1}) &= S_{i+1}'(x_i), \ i = 1, \dots n-1, \\ S_i''(x_i) &= S_{i+1}''(x_i), \ i = 1, \dots n-1. \end{aligned}$$

We have $4n - 2$ conditions. But we must calculate $4n$ coefficients of the spline polynomial. Then, we need to add two conditions and there are three types of these conditions like

1. $S_1(x_0) = f(x_0)$ and $S_n'(x_n) = f'(x_n)$,
2. $S_1''(x_0) = f''(x_0)$ and $S_n''(x_n) = f''(x_n)$,
3. $S_1(x_0) = S_n(x_0), S_1'(x_0) = S'_n(x_n)$ and $S''_1(x_0) = S_n(x_n)$.

There are many methods to calculate the coefficients of the polynomial spline. But, we choose the following explicit form of cubic spline based on the moments $M_i = S''(x_i)$

$$\begin{aligned} S_i(x) &= M_{i-1}\frac{(x_i - x)^3}{6h} + M_i\frac{(x - x_{i-1})^3}{6h} + \left(f(x_{i-1}) - \frac{M_{i-1}h^2}{6}\right)\left(\frac{x_i - x}{h}\right) \\ &+ \left(f(x_i) - \frac{M_i h^2}{6}\right)\left(\frac{x - x_{i-1}}{h}\right). \end{aligned}$$

For the left and right limit of the points x_i, we have

$$S_i'(x_i^-) = \frac{h}{6}M_{i-1} + \frac{h}{3}M_i + \frac{f(x_i) - f(x_{i-1})}{h},$$
$$S_i'(x_i^+) = -\frac{h}{3}M_i - \frac{h}{6}M_{i+1} + \frac{f(x_{i+1}) - f(x_i)}{h}.$$

The continuity of $S_i'(x)$ at x_i yields

$$M_{i-1} + 4M_i + M_{i+1} = 6\left(\frac{f(x_{i+1}) - f(x_i)}{h^2} - \frac{f(x_i) - f(x_{i-1})}{h^2}\right), \ i = 1, 2, \dots n-1.$$

But, for many applications, it is more convenient to work with the slopes $m_i = S'(x_i)$ rather than the moments M_i. Here, we present another interpolation in each segment $[x_{i-1}, x_i]$ as

$$\begin{aligned} S_i(x) &= m_{i-1}\frac{(x_i-x)^2(x-x_{i-1})}{h^2} - m_i\frac{(x-x_{i-1})^2(x_i-x)}{h^2} \\ &+ f(x_{i-1})\frac{(x_i-x)^2[2(x-x_{i-1})+h]}{h^2} + f(x_i)\frac{(x-x_{i-1})^2[2(x_i-x)+h]}{h^2}, \\ S_i'(x) &= m_{i-1}\frac{(x_i-x)(2x_{i-1}+x_i-3x)}{h^2} - m_i\frac{(x-x_{i-1})(2x_i+x_{i-1}-3x)}{h^2} \\ &+ \frac{f(x_i)-f(x_{i-1})}{h^3}6(x_i-x)(x-x_{i-1}). \end{aligned}$$

The limit values of second derivative at x_i are

$$\begin{aligned} S_i''(x_i^-) &= \frac{2}{h}m_{i-1} + \frac{4}{h}m_i - 6\frac{f(x_i)-f(x_{i-1})}{h^2}, \\ S_i''(x_i^+) &= \frac{-4}{h}m_i - \frac{2}{h}m_{i+1} + 6\frac{f(x_{i+1}-f(x_i)}{h^2}. \end{aligned}$$

For finding the results, we require solving this system

$$m_{i-1} + 4m_i + m_{i+1} = 3\frac{f(x_{i+1})-f(x_i)}{h}, \quad i = 1, 2, \ldots, n-1,$$

1.2.2 B-spline functions

In many applications it is necessary to express the approximations functions as a linear combination of approximate basis functions.This is case for differential or integral equations.

A simple presentation of spline in terms of independent basis functions can be given as:

$$\forall n \geq 1, \quad \forall x \in [a,b], \sum_{i=0}^{m-n-1} \alpha_i B_{i,n}(x).$$

where the $m-1$ b-splines of degree n are fined by the following recurrence formula:

$$\forall x \in [a,b], B_{i,n}(x) = \frac{x-x_i}{x_{i+n}-x_i}B_{i,n-1}(x) + \frac{x_{i+n+1}-x}{x_{i+n+1}-x_{i+1}}B_{i+1,n-1}(x),$$

with

$$B_{i,0}(x) = \begin{cases} 1, & \text{if } x_i \leq x \leq x_{i+1}, \\ 0, & \text{else.} \end{cases}$$

For $n = 1$, we get the following basis:

$$B_{i,1}(x) = \begin{cases} \dfrac{x-x_{i-1}}{x_i-x_{i-1}}, & \text{if } x_{i-1} \leq x \leq x_i, \\ \dfrac{x_{i+1}-x}{x_{i+1}-x_i}, & \text{if } x_i \leq x \leq x_{i+1}, \\ 0, & \text{else.} \end{cases} \tag{1.1}$$

It is clear that the latter (1.1) is a linear piecewice function.

For $n = 2$, we have

$$B_{i,2}(x) = \begin{cases} \dfrac{(x-x_{i-1})^2}{(x_{i+1}-x_{i-1})(x_i-x_{i-1})}, & x_{i-1} \leq x < x_i, \\ \dfrac{(x-x_{i-1})(x_{i+1}-x)}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)} + \dfrac{(x_{i+2}-x_i)(x-x_i)}{(x_{i+2}-x_i)(x_{i+1}-x_i)}, & x_i \leq x < x_{i+1}, \\ \dfrac{(x_{i+2}-x)^2}{(x_{i+2}-x_i)(x_{i+2}-x_{i+1})}, & x_{i+1} \leq x < x_{i+2}, \\ 0, & \text{else.} \end{cases} \tag{1.2}$$

which is called the quadrature b-spline basis.

For $n = 3$, we get

$$B_{i,3}(x) = \begin{cases} \dfrac{(x-x_{i-1})^3}{(x_{i+2}-x_{i-1})(x_{i+1}-x_{i-1})(x_i-x_{i-1})}, & x_{i-1} \leq x < x_i, \\ \dfrac{(x-x_{i-1})^2(x_{i+1}-x)}{(x_{i+2}-x_{i-1})(x_{i+1}-x_{i-1})(x_{i+1}-x_i)} + \dfrac{(x-x_{i-1})(x_{i+2}-x_i)(x-x_i)}{(x_{i+2}-x_i)(x_{i+2}-x_i)(x_{i+1}-x_i)} & \\ + \dfrac{(x-x_{i-1})(x_{i+2}-x)^2}{(x_{i+2}-x_{i-1})(x_{i+2}-x_i)(x_{i+2}-x_{i+1})}, & x_i \leq x < x_{i+1}, \\ \dfrac{(x_{i+3}-x)(x-x_i)^2}{(x_{i+3}-x_i)(x_{i+2}-x_i)(x_{i+1}-x_i)} + \dfrac{(x-x_{i-1})(x_{i+2}-x_i)^2}{(x_{i+2}-x_{i-1})(x_{i+2}-x_i)(x_{i+2}-x_{i+1})} & \\ + \dfrac{(x_{i+3}-x)(x-x_i)(x_{i+2}-x)}{(x_{i+3}-x_i)(x_{i+2}-x_i)(x_{i+2}-x_{i+1})}, & x_i \leq x < x_{i+1}, \\ \dfrac{(x_{i+3}-x)^3}{(x_{i+3}-x_i)(x_{i+3}-x_{i+1})(x_{i+3}-x_{i+2})}, & \\ 0, & \text{else.} \end{cases} \tag{1.3}$$

(1.3) is a cubic b-spline basis.

1.3 Continuity Modules

In the thesis, we need to use different formulas of continuity modulus. In this section, we present a list of these formulas.
We start by defining the continuity module in $X = C^0[a, b]$

$$\forall\, h > 0, \forall\, v \in X, \kappa_{\infty,0}(g, h) = \sup_{|x-y|\leq h} |v(x) - v(y)|.$$

For all functions define on the square $[a, b]^2$:

$$\forall\, x \in [a, b],\ \forall\, h > 0, \forall g \in C^0([a, b]^2, \mathbb{R}), \kappa_{\infty,0}(g, h)(x) = \sup_{|y_1-y_2|\leq\delta} |g(x, y_1) - g(x, y_2)|,$$

and the continuity module $\bar{\kappa}_{\infty,1}$ as

$$\forall\, x \in [a, b],\ \forall h > 0,\ \frac{\partial g}{\partial x} \in C^0([a, b]^2, \mathbb{R}), \bar{\kappa}_{\infty,1}(g, h)(x) = \bar{\kappa}_{\infty,0}(g, h)(x) + \bar{\kappa}_{\infty,0}(\partial_x g, h)(x),$$

where $\partial_x g$ is the partial derivative of g respect to x.

We present the continuity module in the Banach space $\tilde{X}_2 = X \times X$ as:

$$\forall h > 0, \forall\, V = (v_1, v_2) \in \tilde{X}_2,\ \kappa_{\infty,1}(V,h) = \kappa_{\infty,0}(v_1,h) + \kappa_{\infty,0}(v_2,h).$$

We introduce the continuity modulus $\omega_0(h,.)$ in $L^1[a,b]$ by

$$\forall g \in L^1[a,b],\ \kappa_{1,0}(h,v) = \sup_{x\in[0,h]} \int_a^b |g(x+y) - g(y)|\, dy,$$

and the continuity modulus $\kappa_{1,1}(h,.)$ in $W^{1,1}[a,b]$ by

$$\forall g \in W^{1,1}[a,b],\ \kappa_{1,1}(h,g) = \omega_0(h,v) + \omega_0(h,g').$$

We give the generalization continuity modulus for any continuous function $G(x,t) \in L^1([a,b]^2, \mathbb{R})$ on the square $[a,b]^2$: for any $t \in [a,b]$

$$\bar{\kappa}_{1,0}(h,G)(t) = \sup_{x\in[0,h]} \int_a^b |G(x+y,t) - G(y,t)|\, dy.$$

We introduce the generalization continuity modulus for $\dfrac{\partial G}{\partial x}(x,t) \in L^1([a,b]^2, \mathbb{R})$

$$\forall t\ \in [a,b]\ \bar{\kappa}_{1,1}(h,G)(t) = \bar{\kappa}_{1,0}(h,G)(t) + \bar{\kappa}_{1,0}(h,\partial_x G)(t),$$

where $\partial_x G = \dfrac{\partial G}{\partial x}(x,.)$, and we define

$$||\bar{\kappa}_{1,1}(h,G)||_\infty = \max_{a\le t\le b} |\bar{\kappa}_{1,1}(h,G)(t)|.$$

1.4 Fredholm Integral Equations

Many problems and equations in physics can be transformed into Fredholm integral equations like the initial and boundary value problems. Every Fredholm integral equation contains a function obtained by the unknown function u by integration and of the form $\int_a^b K(x,t)u(t)\, dt$, where K is called the kernel and is assumed known. In general, the Fredholm integral equation contains the known function called the free term. Usually, they are complex-valued functions of real variables.

This type of integral equations has three kinds and each kind refers to the localisation of the unknown function u. First kind has the unknown function present under integral sign only

$$f(x) = \int_a^b K(x,t)u(t)\, dt.$$

The second and third kind have the unknown function outside the integral sign. Thus, the following equation is of the second kind

$$\lambda u(x) = \int_a^b K(x,t)u(t)\, dt + f(x), \tag{1.4}$$

where λ is a numerically parameter generally complex. In particular applications λ is composed of physical quantities.

In this part, we set $X = C^0[a,b]$ with the norm

$$\forall v \in X, ||v||_X = \max_{a \leq x \leq b} |v(x)|.$$

We define the linear integral operator T by

$$\begin{aligned} T: \quad X &\longrightarrow X \\ u &\longmapsto Tu(x) = \int_a^b K(x,t)u(t)\, dt, \quad a \leq x \leq b. \end{aligned}$$

Then, the integral equation of the second kind (1.4) is rewritten as

$$(\lambda I - T)u = f. \tag{1.5}$$

Recalling that

$$||T|| = \sup_{||u||_X \leq 1} ||Tu||_X,$$

the norm of operator T is given as

$$||T|| = \max_{a \leq x \leq b} \int_a^b |K(x,t)|\, dt,$$

because for each $u \in X$ with $||u||_X \leq 1$

$$|Tu(x)| \leq \max_{a \leq x \leq b} \int_a^b |K(x,t)|\, dt, \; \forall x \in [a,b],$$

and so

$$||T|| \leq \max_{a \leq x \leq b} \int_a^b |K(x,t)|\, dt.$$

Since K is continuous, there exists $x_0 \in [a,b]$ such that

$$\int_a^b |K(x_0,t)|\, dt = \max_{a \leq x \leq b} \int_a^b |K(x,t)|\, dt.$$

We set

$$\sup_{||u||_X \leq 1} ||Tu||_X = ||Tu_0||_X.$$

Then, for all $\varepsilon > 0$, we choose $u_0 \in X$ by

$$u_0(t) = \frac{K(x_0,t)}{|K(x_0,t)| + \varepsilon}, \quad t \in [a,b].$$

It is clear that $||u_0||_X \leq 1$. So that

$$|Tu_0(x_0)| = \int_a^b \frac{|K(x_0,t)|^2}{|K(x_0,t)| + \varepsilon}\, dt \geq \int_a^b \frac{|K(x_0,t)|^2 - \varepsilon^2}{|K(x_0,t)| + \varepsilon}\, dt.$$

Hence,

$$||T|| \geq \int_a^b |K(x_0,t)|\, dt - \varepsilon(b-a).$$

Since this holds for all $\varepsilon > 0$, we have

$$||T|| \geq \int_a^b |K(x_0,t)|\, dt = \max_{a\leq x\leq b} \int_a^b |K(x,t)|\, dt.$$

We say that if $||T|| < |\lambda|$ the Fredholm integral equation (1.5) has a unique solution in X.

1.4.1 Approximation Methods For Fredholm Integral Equation with Continuous Kernel

In all most common situation where an exact solution to a problem cannot be found directly but its existence and uniqueness are assured as in the previous case, the approximation techniques become important. In what follows, we discus some methods which can deal with successfully.

Projection Approximation Methods

We are searching for an approximation solution of the integral Fredholm equation (1.5) based on the projection method principle. We need to define the sequence of finite rank $(P_n)_{n\geq 1} \in \mathcal{B}(X)$. We denote by X_n the sequence of finite dimensional subspaces of the Banach space X which have the basis $\{e_i\}_{i=0}^n$, by X_n^* the dual subspaces of X_n with the basis $\{e_i^*\}_{i=0}^n$ and by $(.,.)$ the duality bracket, such that

$$(e_i, e_j^*) = \begin{cases} 1 & \text{if} \quad i = j; \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

The sequence of finite rank $(P_n)_{n\geq 1} \in \mathcal{B}(X)$ is given as

$$\begin{aligned} P_n : \quad X_n &\longrightarrow X \\ u &\longmapsto P_n u = \sum_{i=0}^{n} (u, e_i^*) e_i. \end{aligned} \tag{1.6}$$

This approximation operator is used to construct several projection methods like Galerkin and Kantorovich methods which will take a large part in our study.

Lemma 1.4.1. *Let $T \in \mathcal{B}(X)$ and $(P_n)_{n\geq 1} \in \mathcal{B}(X)$ be a sequence of finite rank presented by (1.6). If $(P_n)_{n\geq 1}$ converges pointwise to the identity operator and T is a compact, then*

$$\lim_{n\to\infty} ||(I - P_n)T|| = 0,$$

Proof. See theorem 1.1.8 proof. □

Theorem 1.4.1. *Let $T \in \mathcal{B}(X)$. Assume that $(\lambda I - T)^{-1}$ exists, is bounded and*

$$\lim_{n\to\infty} ||(I - P_n)T|| = 0.$$

Then, for all $n \geq 1, (\lambda I - P_n T)^{-1}$ exists from X to X. Moreover, it is uniformly bounded:

$$\sup_{n\geq n_0} ||(\lambda I - T)^{-1}|| < \infty.$$

Proof. For $n \geq 1$, we have

$$\begin{aligned} \lambda I - P_n T &= (\lambda I - T) + (T - P_n T), \\ &= (\lambda I - P_n T)[I + (\lambda I - P_n T)^{-1}(T - P_n T)]. \end{aligned}$$

Now fix $n_0 \in \mathbb{N}$ such that

$$\varepsilon_n = \sup_{n \geq n_0} ||(\lambda I - T)^{-1}|| < \frac{1}{(\lambda I - T)^{-1}}.$$

Then,

$$||(\lambda I - T)^{-1}(T - P_n T)|| \leq ||(\lambda I - T)^{-1}|| \, ||(T - P_n T)|| < 1,$$

which gives us by Neumann's theorem 1.1.6, the existence of these inverse

$$[I + (\lambda I - T)^{-1}(T - P_n T)]^{-1},$$

which is uniformly bounded

$$||[I + (\lambda I - T)^{-1}(T - P_n T)]^{-1}|| \leq \frac{1}{1 + \varepsilon_n ||(\lambda I - T)^{-1}||}.$$

All of this proved that the inverse $(\lambda I - P_n A)^{-1}$ exists and can be written as:

$$(\lambda I - P_n T)^{-1} = [I + (\lambda I - T)^{-1}(T - P_n T)]^{-1}(\lambda I - T)^{-1},$$

with

$$||(\lambda I - P_n T)^{-1}|| \leq \frac{||(\lambda I - T)^{-1}||}{1 + \varepsilon_n ||(\lambda I - T)^{-1}||} = C_n.$$

□

Galerkin Method

The word Galerkin is used in different ways by researchers in books and research papers to define Fredholm's integral equation approximation solution. For example, in Atkinson [5] and Nair [48] books, Galerkin method is a name of a numerical treatment used to find an approximation sequence $(u_n)_{n\geq 1}$ solution of

$$(\lambda I - P_n T)u_n = P_n f. \tag{1.7}$$

However, in Ahues in [2], the principle of Galerkin method is employed to determine the sequence $(u_n)_{n\geq 1}$ solution of

$$(\lambda I - P_n T P_n)u_n = f. \tag{1.8}$$

We focus on the definition (1.7), for this reason we give the following theorem

Theorem 1.4.2. *Let $(P_n)_{n\geq 1}$ be a sequence of finite rank projection operator defined by (1.6). The sequence $(u_n)_{n\geq 1}$ given by*

$$u_n = \sum_{i=0}^{n} \alpha_i e_i,$$

is a solution of (1.7) if and only if we get the following system

$$\lambda \alpha_j = \sum_{i=0}^{n} \alpha_i (Te_i, e_j^*) + (f, e_j^*), \quad j = 0, 1, \ldots n. \tag{1.9}$$

Proof. Assume that

$$u_n = \sum_{i=0}^{n} \alpha_i e_i.$$

From the definition of $(P_n)_{n\geq 1}$ in (1.6), the equation (1.5) is equivalent to the following

$$\lambda u_n = \sum_{i=0}^{n} (Tu_n, e_i^*) e_i + \sum_{i=0}^{n} (f, e_i^*) e_i.$$

Multiplying by e_j^*, we get

$$\lambda (u_n, e_j^*) = (Tu_n, e_j^*) + (f, e_j^*).$$

From the equation (1.10), we get

$$\lambda \left(\sum_{i=0}^{n} \alpha_i e_i, e_j^* \right) = \sum_{i=0}^{n} \left(T \sum_{i=0}^{n} \alpha_i e_i, e_i^* \right) e_i + \sum_{i=0}^{n} (f, e_i^*) e_i,$$

which is equivalent to

$$\lambda \sum_{i=0}^{n} \alpha_i (e_i, e_j^*) = \sum_{i=0}^{n} \alpha_i (Te_i, e_j^*) + (f, e_j^*).$$

Finally,

$$\lambda \alpha_j = \sum_{i=0}^{n} \alpha_i (Te_i, e_i^*) + (f, e_j^*).$$

□

Theorem 1.4.3. *Let u_n and u be solutions of* (1.7) *and* (1.5) *respectively. Then, we obtain the two-side error estimate*

$$\frac{|\lambda|}{||\lambda I - P_n T||} ||u - P_n u||_X \leq ||u - u_n||_X \leq |\lambda| \; ||(\lambda I - P_n T)^{-1}|| \; ||u - P_n u||_X,$$

and if $(P_n)_{n\geq 1}$ converges pointwise to the identity operator, then

$$\lim_{n\to\infty} ||u - u_n||_X = 0.$$

Proof. Applying the projection operator P_n on the equation (1.5), we obtain

$$\lambda P_n u - P_n T u + \lambda u - \lambda u = P_n f,$$

which equivalent to

$$(\lambda I - P_n T) u = P_n f + \lambda (u - P_n u).$$

Substracting $(\lambda I - P_n T) u = P_n f$ to get

$$(\lambda I - P_n T)(u - u_n) = \lambda (u - P_n u). \tag{1.10}$$

Then,

$$||u - u_n||_X \leq |\lambda| \, ||(\lambda I - P_n T)^{-1}|| \, ||u - P_n u||_X. \tag{1.11}$$

From (1.10), we have

$$|\lambda| \, ||u - P_n u||_X = ||(\lambda I - P_n T)|| \, ||u - u_n||_X. \tag{1.12}$$

Moreover,

$$\begin{aligned} ||(\lambda I - P_n T)|| &\leq ||\lambda I - T|| + ||(I - P_n)T||, \\ &\leq ||\lambda I - T|| + \varepsilon_n. \end{aligned} \tag{1.13}$$

Substituting (1.13) in (1.12), we obtain

$$|\lambda| \, ||u - P_n u||_X \leq \left(||\lambda I - T|| + \varepsilon_n \right) ||u - u_n||_X. \tag{1.14}$$

Hence,

$$\frac{|\lambda|}{||\lambda I - T|| + \varepsilon_n} \, ||u - P_n u||_X \leq ||u - u_n||_X.$$

From (1.11) and (1.14), we get the result. Moreover, we have

$$\lim_{n \to \infty} ||u - P_n u||_X = 0.$$

This implies that u_n converges to u in the sense of the norm of Banach space X. □

We shall consider two special case of Galerkin method, depending on the framework space. The first one is collocation method if $X = C^0[a, b]$ and the second one is the orthogonal Galerkin method if applying this method in Hilbert space $X = L^2[a, b]$.

(i) **Collocation Method**

We take $X = C^0[a, b]$, and we define Δ_n, for all $n \geq 1$ the uniform discretization of interval $[a, b]$ as

$$\Delta_n = \left\{ n \geq 1, a = x_0 < x_1 < \cdots < x_{n-1} < x_n, \; h = x_{j+1} - x_j, \; 0 \leq j \leq n \right\}. \tag{1.15}$$

Now, we return to the essential condition of the basis's choice (1.6). For this reason, $\{e_i\}_{i=0}^{n}$ are the following haat functions

$$e_i(x) = \begin{cases} 1 + \dfrac{|x - x_i|}{h}, & \text{if } x \in [x_{i-1}, x_{i+1}], \\ 0, & \end{cases}$$

$$e_0(x) = \begin{cases} 1 + \dfrac{x_1 - x}{h}, & x \in [x_0, x_1], \\ 0, & \end{cases}$$

$$e_n(x) = \begin{cases} 1 + \dfrac{x - x_{n-1}}{h}, & x \in [x_{n-1}, x_n], \\ 0, & \end{cases}$$

with the duality bracket is defined as

$$(u, e_i^*) = u(x_i), \quad u \in X, \ i = 0, 1, \dots n.$$

Then, the sequence $\{P_n\}_{n\geq 1}$ verifies the following interpolation condition:

$$P_n u(x_i) = u(x_i), \quad i = 0, 1, \dots n.$$

It is clear that $\{P_n\}_{n\geq 1}$ is an interpolation projection based on nodes $x_0, x_1, \dots, x_n$.
The system (1.7), is equivalent to

$$\lambda X = MX + F,$$

where $X = (\alpha_0, \alpha_1, \dots \alpha_n)^t \in \mathbb{R}^{n+1}$, $F = (f(x_0), f(x_1), \dots, f(x_n))^t \in \mathbb{R}^{n+1}$ and M is a matrix of size $(n+1) \times (n+1)$, such that its elements are given by

$$\begin{aligned} (Te_i, e_j^*) &= \frac{1}{h}\int_{x_{i-1}}^{x_i} K(x_j, t)(t - x_{i-1})\, dt + \int_{x_i}^{x_{i+1}} K(x_j, t)(x_{i+1} - t)\, dt, && 0 \leq j \leq n, 1 \leq i \leq n-1, \\ (Te_0, e_j^*) &= \frac{1}{h}\int_{x_0}^{x_1} K(x_j, t)(x_1 - t)\, dt, && 0 \leq j \leq n, \\ (Te_n, e_j^*) &= \frac{1}{h}\int_{x_{n-1}}^{x_n} K(x_j, t)(t - x_{n-1})\, dt, && 0 \leq j \leq n. \end{aligned}$$

(ii) **Orthogonal Galerkin Method**

Assuming that $X = L^2[a, b]$ with an inner product $\langle ., . \rangle$. We recall that the duality bracket is the inner product in this Hilbert space.
For each $n \geq 1$, let X_n be a finite dimensional subspace of X. Let $\{e_i\}_{i=0}^n$ be a basis of X_n checks the condition (1.6), that means it is an orthonormal basis with respect to the inner product $\langle ., . \rangle$.
Then, the sequence $(P_n)_{n\geq 1}$ is an orthogonal projection. Now, it has the above presentation

$$\forall u \in X, \quad P_n u = \sum_{i=0}^{n} \langle u, e_i \rangle e_i,$$

and the system (1.10) has this form

$$\lambda \alpha_j = \sum_{i=0}^{n} \alpha_i \, \langle Te_i, e_j \rangle + \langle f, e_j \rangle, \quad j = 0, 1, \dots n. \tag{1.16}$$

System (1.16) can be written as

$$\lambda X = MX + F,$$

where $X = (\alpha_0, \alpha_1, \ldots \alpha_n)^t$ is a vector of $\mathbb{R}^{n+1}$, M is a matrix with size $(n+1) \times (n+1)$ such that each element m_{ij} are given by

$$m_{ij} = \int_a^b \int_a^b K(x,t) e_j(t) e_i(x) \, dt \, dx, \quad 0 \leq i, j \leq n,$$

and each element $F_i, i = 0, 1 \ldots n$ of the vector F is given by

$$F_i = \int_a^b f(x) e_i(x) \, dx.$$

Kantorovich Method

In Galerkin method the convergence of the approximate solution to the exact solution is guaranteed by the convergence $P_n f \to f$. To avoid this situation which is considered as an inconvenience, we propose the Kantrorovich method where this approximation procedure was first studied by Schock. For this purpose, we assume that $f \in Ran(T)$.

Recalling the Fredholm integral equation

$$\forall u \in X, \quad \lambda u = Tu + f,$$

and applying the linear integral operator T to both sides of equation, we obtain

$$\lambda v = Tv + f,$$

with $v = Tu$. Now, we call u_n^k a Kantorovich approximation of u and it is defined by

$$u_n^K = \frac{1}{\lambda}\Big(v_n^G + f\Big),$$

where, v_n^G is a Galekin approximation defined in the last section and verifying this approximation equation

$$\lambda v_n^G = P_n T v_n^G + P_n T f.$$

To illustrate the relation between two methods, we observe

$$\begin{aligned} \lambda u_n^K &= v_n^G + f, \\ &= \frac{1}{\lambda}\Big(P_n T v_n^G + P_n T f\Big) + f, \\ &= \frac{1}{\lambda} P_n T (v_n^G + f) + f. \end{aligned}$$

Recall that $v_n^G = T u_n^K$. Then, the Kantorovich approximation equation has the following form:

$$\lambda u_n^K = P_n T u_n^K + f. \tag{1.17}$$

Theorem 1.4.4. *Let u_n^K be an approximation solution and u be an exact solution. Then, we have*

$$\frac{||(T - P_nT)||}{||\lambda I - T|| + ||(T - P_nT)||} \quad \leq \quad ||u - u_n^K||_X \leq ||(\lambda I - P_nT)^{-1}|| \; ||(T - P_nT)||.$$

Moreover, assuming that P_n converges pointwise to the identity operator. We get

$$\lim_{n\to\infty} ||u - u_n^K||_X = 0.$$

Proof. For n large, we have

$$(\lambda I - P_nT)(u - u_n^K) \quad = \quad (T - P_nT)u, \tag{1.18}$$

$$\Big[(\lambda I - T) + (P_nT - T)\Big](u - u_n^K) \quad = \quad (T - P_nT)u. \tag{1.19}$$

From (1.18), we have

$$||u - u_n^K||_X \quad \leq \quad ||(\lambda I - P_nT)^{-1}|| \; ||(T - P_nT)||,$$

and from (1.19), it follows

$$\frac{||(T - P_nT)||}{||\lambda I - T|| + ||(T - P_nT)||} \quad \leq \quad ||u - u_n^K||_X.$$

Then, we obtain the result. □

Nyström Method

In this part, we describe another approximation method called the Nyström method which is highly applicable for the numerical solution of the Fredholm integral equation of the second kind (1.4) in the Banach space $X = C^0[a, b]$.

We recall Δ_n, for all $n \geq 1$ the uniform discrtization of interval $[a, b]$ (1.15). This method is based on the next numerical integration scheme

$$\int_a^b g(x)\, dx \approx \sum_{i=0}^{n} \omega_i g(x_i), \tag{1.20}$$

where $\{w_i\}_{i=0}^n$ are called by the weights under the condition

$$\sup_{n\geq 1} \sum_{i=0}^{n} |\omega_i| < +\infty.$$

The Nyström approximation u_n is the solution of the following numerical equation

$$\forall x \in [a, b], \qquad \lambda u_n(x) = \sum_{i=0}^{n} \omega_i K(x, x_i) u_n(x_i) + f(x). \tag{1.21}$$

We select the collocation points x_j, $j = 0, 1, \ldots, n$ to obtain the solution at this knots. We get the following linear algebraic system

$$\lambda u_n(x_j) = \sum_{i=0}^{n} \omega_i K(x_j, x_i) u_n(x_i) + f(x_j), \qquad j = 0, 1, \ldots n.$$

We return to the equation (1.21) which has this simple formula

$$\forall x \in [a,b], \qquad \lambda u_n(x) = T_n u_n(x) + f(x),$$

with T_n is linear and bounded numerical operator of finite rank. It is defined by

$$\forall x \in [a,b], \qquad T_n u(x) = \sum_{i=0}^{n} \omega_i K(x, x_i) u(x_i).$$

Despite the wide application of Nyström's method on integral equations due to its simplicity, but only a number of researchers who developed the error analysis of this numerical treatment during the 1940s-1970s year. At the beginning, the aim of the researchers was to demonstrate the stability and convergence of the method. Later, the goal changed to the creation of an error analysis in a more abstract framework that contains all possible situations and what we mean by them the error domain, the convergence rate and the functional space. The concept that generalises all possible cases is the concept of collectively compact approximation where the Nyström approximation operator can be verified by.
Now, we present the most important theorem that helped for the creation collectively compact operator concept approximation.

The next theorem shows the pointwise convergence of $(T_n)_{n\geq 1}$ (1.22) to T (1.5).

Theorem 1.4.5. *[5, 48] Let $(T_n)_{n\geq 1}$ be the Nyström approximation of the integral operator T. Then,*

1. $\bigcup_{n=1}^{\infty} \left\{ T_n u,\ u \in X,\ ||u||_X \leq 1 \right\}$ *is equicontinuous,*
2. $(T_n)_{n\geq 1}$ *is pointwise approximation of T.*

Proof. see [5, 48]. □

Corollary 1.4.1. *Let T be a compact operator and T_n defined by (1.5) and (1.22) respectively. Then,*

$$\lim_{n\to\infty} ||(T - T_n)T|| = 0 \quad \text{and} \quad \lim_{n\to\infty} ||(T - T_n)T|| = 0.$$

Proof. By theorem.1.4.5 T_n is a pointwise convergence of the compact operator T and the set

$$M = \left\{ T_n u,\ u \in X,\ ||u||_X \leq 1,\ n \in \mathbb{N} \right\},$$

is equicontinuous. Hence, by theorem 1.1.8, we obtain the result. □

1.4.2 Approximation Methods For Fredholm Integral Equation with Weakly-Singular Kernel

So far in this thesis, we have been interested in the numerical methods to solve Fredholm integral equations under the assumption that the kernel is a smooth function. Among cases the domain kernel's definition is infinite or the kernel has a singularity with its domain of definition. The integral equation is said to be singular.
Let $X = C^0[a,b]$, the Fredholm integral equation with weakly singular kernel has the following form

$$\forall u \in X, \lambda u(x) = \int_a^b p(|x-t|) h(x,t) u(t)\, dt + f(x), \quad a \leq x \leq b, \tag{1.22}$$

where u is an unknown function. We assume that $f \in X$, $H(x,t) \in C^0([a,b]^2, \mathbb{R})$ and the function p verifies the following hypothesis

$$(\mathfrak{H}) \left\| \begin{array}{l} p \in L^1(0, b-a), \\ \\ p(0) = 0. \end{array} \right.$$

In general, p has the following form

1. Logarithmic form
$$p(|x-t|) = \log|x-t|.$$

2. Algebric form
$$p(|x-t|) = |x-t|^{-\alpha}, \quad 0 < \alpha < 1.$$

Product Trapezoidal Rule

Let $n \geq 1$, we recall Δ_n the disceritization of the interval $[a,b]$ defined by (1.15).
$\forall u \in X$ and $x_{i-1} \leq t \leq x_i$, we define

$$[H(x,t)u(t)]_n = \frac{1}{h}\Big[(x_i - t)H(x, x_{i-1})u(x_{i-1}) + (t - x_{i-1})H(x, x_i)u(x_i)\Big], \ \forall \ i = 0, 1, \ldots n.$$

Let T_n be a numerical approximation of the integral

$$\forall u \in X, \ A_n u(x) = \sum_{i=0}^{n} \omega_i(x) H(x, x_i) u(x_i), \ \forall x \in [a,b],$$

with the weights $\{\omega_j\}_{j=0}^{n}$ defined by

$$\begin{cases} w_0(x) &= \dfrac{1}{h}\displaystyle\int_{x_0}^{x_1} p(|x-t|)(x_1 - t)\, dt, \\ w_n(x) &= \dfrac{1}{h}\displaystyle\int_{x_{n-1}}^{x_n} p(|x-t|)(t - x_{n-1})\, dt, \\ \omega_i(x) &= \dfrac{1}{h}\displaystyle\int_{x_i}^{x_{i+1}} p(|x-t|)(x_{i+1} - t))\, dt + \dfrac{1}{h}\displaystyle\int_{x_{i-1}}^{x_i} p(|x-t|)(t - x_{i-1})\, dt, \ \forall i = 1, \ldots n-1. \end{cases}$$

Then, the approximation equation of (1.22) has the form

$$\lambda u_n(x) = \sum_{i=0}^{n} \omega_i(x) H(x, x_i) u(x_i) + f(x), \quad \forall x \in [a,b]. \tag{1.23}$$

By choosing the collocation point x_j, we get the following algebraic system

$$\lambda u_n(x_j) = \sum_{i=0}^{n} \omega_i(x_j) H(x_j, x_i) u(x_i) + f(x_j), \quad j = 0, 1, \ldots n.$$

We define forall $x \in [a,b]$ the error $e_n(x)$ by

$$e_n(x) = \int_a^b p(|x-t|) H(x,t) u(t)\, dt - \sum_{i=0}^{n} \omega_i(x) H(x, x_i) u(x_i). \tag{1.24}$$

Theorem 1.4.6. *Let u_n be a solution of (1.23) and u the exact solution of (1.22). Then, $u_n \to u$ as $n \to \infty$*

Proof. We have

$$
\begin{aligned}
|e_n(x)| &\leq \int_a^b |p(|x-t|)| \left| H(x,t)u(t) - [H(x,t)u(t)]_n \right| dt, \\
&\leq \int_a^b |p(|x-t|)| \left| H(x,t)u(t) - \frac{1}{h}\big[(x_i - t)H(x,t)u(x_{i-1}) + (t - x_{i-1})H(x,t)u(x_i)\big] \right. \\
&+ \left. \frac{1}{h}\big[(x_i - t)(H(x,t) - H(x,x_{i-1}))\, u(x_{i-1}) + (t - x_{i-1})(H(x,t) - H(x,x_i))\, u(x_i)\big] \right| dt, \\
&\leq \int_a^b \int_a^b |p(|x-t|)|\, |H(x,t)|\kappa_{\infty,0}(u,h) + \kappa_{\infty,0}(H,h)(x)||u||_X.
\end{aligned}
$$

Then,

$$
||e_n||_X \leq \left[\max_{a \leq x,t \leq b} |H(x,t)|\kappa_{\infty,0}(u,h) + \max_{a \leq x \leq b} \kappa_{\infty,0}(H,h)(x)\; ||u||_X \right] ||p||_{L^1(0,b-a)}.
$$

From (1.22) and (1.24)

$$
||u - u_n||_\infty \leq ||e_n||_X,
$$

and when $n \to \infty$, we get $||e_n||_X$, where, $||e_n||_X = \sup_{a \leq x \leq b} |e_n(x)|$. □

Chapter 2

Analytical and Numerical Study of Linear Fredholm Integro-differential Equation

Contents

In this chapter, we begin the analysis of the integro-differential equation. We make some assumptions on the kernels to reformulate our equation to an equivalent system of integral equations. In section 2.1, we study the existence and uniqueness of the solution. In section 2.2, we search for a numerical solution based on Nyström's method, we follow some steps to show the convergence of the approximate solution among these steps the ν-convergence of the Nyström operator. In section 2.3, we apply two projection methods: Collocation and Kantorovich. In the convergence analysis, we show the convergent of the projection operator to identity operator. Afterwards, we have cried an iterative system of the three methods because the application of the last methods leads to a resolution of a very large algebraic system.

2.1 Analytical study

Let $C^1[a,b]$ be the Banach space of the continuous and differential functions with usual norm given by

$$\forall x \in [a,b], \quad ||v||_{C^1[a,b]} = \sup_{a\leq x\leq b} |v(x)| + \sup_{a\leq x\leq b} |v'(x)|.$$

We recall that, our linear Fredholm integro-differential equation is given as

$$\forall x \in [a,b],\ \lambda u(x) = \int_a^b K_1(x,t)u(t)\,dt + \int_a^b K_2(x,t)u'(t)\,dt + f(x). \tag{2.1}$$

Our problem in this section is to study the solution existence and uniqueness of linear Fredholm integro-differential equation (2.1) in the Banach space X. We need some conditions on the two kernels $\{K_p\}_{1\le p\le 2}$, that allow us to transform our integro-differential equation into a system of integral equations. Assuming that

$$(H)\left\|\ \frac{\partial K_p}{\partial x}(x,t) \in C^0([a,b]^2,\mathbb{R}), \quad p=1,2.\right.$$

We obtain the possibility of defining the derivative u' by

$$\forall x \in [a,b],\ \lambda u'(x) = \int_a^b \frac{\partial K_1}{\partial x}(x,t)u(t)\,dt + \int_a^b \frac{\partial K_2}{\partial x}(x,t)u'(t)\,dt + f'(x).$$

Now, we set $u_p = u^{(p-1)} \in C^0[a,b]$ for $p=1,2$, then the equation (2.1) is equivalent to the following integral equations system

$$\forall x \in [a,b] \begin{cases} \lambda u_1(x) = & \int_a^b K_1(x,t)u_1(t)\,dt + \int_a^b K_2(x,t)u_2(t)\,dt + f(x), \\ \lambda u_2(x) = & \int_a^b \frac{\partial K_1}{\partial x}(x,t)u_1(t)\,dt + \int_a^b \frac{\partial K_2}{\partial x}(x,t)u_2(t)\,dt + f'(x). \end{cases} \tag{2.2}$$

Its solution (u_1,u_2) must be found in the product space $\tilde{X}_2 = X \times X$ with the norm described by (1.1.11) where, $X = C^0[a,b]$.

Using the definitions and notations in 1.1.5, we define a family of linear operators $\{T_{qp}\}_{1\le q,p\le 2}$ from the system (4.8). For $p=1,2$, we have

$$\begin{aligned} \forall x \in [a,b],\ \forall v \in X, \quad T_{1p}v(x) &= \int_a^b K_p(x,t)v(t)\,dt, \\ \forall x \in [a,b], \forall v \in X, \quad T_{2p}v(x) &= \int_a^b \frac{\partial K_p}{\partial x}(x,t)v(t)\,dt, \end{aligned}$$

such that $\{T_{qp}\}_{1\le q,p\le 2} \subset BL(X)$. We represent A_T the block operator matrix in the following way

$$\begin{aligned} A_T :\quad & \tilde{X}_2 \longrightarrow \tilde{X}_2 \\ & V \longmapsto A_T V(x) = \Big(T_{11}v_1(x) + T_{12}v_2(x), T_{21}v_1(x) + T_{22}v_2(x)\Big), \end{aligned} \tag{2.3}$$

Finally, the system (2.2) yields the next simplified version

$$\lambda U = A_T U + F. \tag{2.4}$$

The equation (2.4) has a unique solution if $(\lambda I_2 - A_T)^{-1}$ exists and is bounded as shows in the following theorem.

Theorem 2.1.1. *If $|\lambda| > |||A_T|||$ then (2.4) has a unique solution U in $\tilde{X}_2$.*

Proof. We write,

$$(\lambda I_2 - A_T)^{-1} = \lambda(I_2 - \frac{1}{\lambda}A_T).$$

Then, we use Neumann's theorem 1.1.6 the fact $\frac{1}{|\lambda|}|||A_T||| < 1$, to show that $(\lambda I_2 - A_T)^{-1}$ exists and

$$|||(\lambda I_2 - A_T)^{-1}||| \leq \frac{1}{|\lambda| - |||A_T|||}.$$

□

In the rest of our work, we need the compactness of the block matrix A_T. We state next two theorems to show that the block matrix A_T is compact.

Theorem 2.1.2. *All operators $(T_{qp})_{1\leq q,p\leq 2}$ are compact in X.*

Proof. We prove that the subsets $T_{qp}(B_X) = \{T_{qp}(u_p),\ u_p \in B_X\}_{qp}$ are relatively compact for all $q, p = 1, 2$, where $B_X = \{v \in X, ||v||_X \leq 1\}$. Firstly, we have

$$|T_{1p}(u_p)| \quad \leq (b-a) \max_{a\leq x\leq b} \int_a^b |K_p(x,t)|\, dt\, ||u_p||_{C^0[a,b]}, \quad p = 1, 2,$$

$$|T_{2p}(u_p)| \quad \leq (b-a) \max_{a\leq x\leq b} \int_a^b \left|\frac{\partial K_p}{\partial x}(x,t)\right| dt\, ||u_p||_{C^0[a,b]}, \quad p = 1, 2.$$

This proves that the sets $T_{qp}(B_X)$ are bounded. Now let $x, y \in [a,b]$ and $u_p \in X$. Then,

$$\begin{aligned} |T_{1p}u_p(x) - T_{1p}u_p(y)| &\leq (b-a) \max_{a\leq t\leq b} |K_p(x,t) - K_p(y,t)|\, ||u_p||_X, \\ |T_{2p}u_p(x) - T_{2p}u_p(y)| &\leq (b-a) \max_{a\leq t\leq b} \left|\frac{\partial K_p}{\partial x}(x,t) - \frac{\partial K_p}{\partial x}(y,t)\right| ||u_p||_X. \end{aligned}$$

Since $K_p(.,.)$ and $\frac{\partial K_p}{\partial x}(.,.)$ are uniformly continuous, the sets $T_{qp}(B_X)$ are equicontinuous subsets of X. By Arzela-Ascoli 2 $\overline{T_{qp}(B_X)}$ are compact. Finally, we get that the operators $\{T_{qp}\}_{1\leq q,p\leq 2}$ are compact in X. □

Theorem 2.1.3. *The operator matrix A_T is compact in $\tilde{X}_2$.*

Proof. Let the bounded set $D = D_1 \times D_2 \subset \tilde{X}_2$, such that $\forall (z_1, z_2) \in D, \exists\, C > 0,\ ||z_1||_X + ||z_2||_X < C$. We need to prove that the set

$$A_T(D) = \prod_{q=1}^{2} \left\{ \sum_{p=1}^{2} T_{qp}(z_p), z_p \in D_p \right\}_q,$$

is relatively compact. For $q = 1, 2$, we have that the sets

$$\left\{ \sum_{p=1}^{2} T_{qp}(z_p), z_p \in D_p \right\}_q,$$

are relatively compact because the operators T_{qp} are compact operators and $\sum_{p=1}^{2} T_{qp}$ is also compact. Therefore,

$$\overline{\left\{ \sum_{p=1}^{2} T_{qp}(z_p), z_p \in D_p \right\}_q}$$

are a compact sets for $q = 1, 2$, then by Tikhonov theorem [79] the product

$$\prod_{q=1}^{2}\overline{\left\{\sum_{p=1}^{2} T_{qp}(z_p), z_p \in D_p\right\}_q}$$

is compact. On the other hand,

$$\begin{aligned}\prod_{q=1}^{2}\overline{\left\{\sum_{p=1}^{2} T_{qp}(z_p), z_p \in D_p\right\}_q} &= \overline{\prod_{q=1}^{2}\left\{\sum_{p=1}^{2} T_{qp}(z_p), z_p \in D_p\right\}_q} \\ &= \overline{A_T(D)},\end{aligned}$$

which proving that $A_T(D)$ is relatively compact. □

2.2 Nyström Method

We recall Δ_n, for all $n \geq 1$ the uniform discertization of interval $[a, b]$ given in (1.15) and apply the Nyström method. We get,

$$\forall x \in [a,b], \quad \begin{cases} \lambda u_{1,n}(x) &= \displaystyle\sum_{i=0}^{n} \omega_i K_1(x, x_i) u_{1,n}(x_i) + \sum_{i=0}^{n} \omega_i K_2(x, x_i) u_{2,n}(x_i) + f(x), \\ \lambda u_{2,n}(x) &= \displaystyle\sum_{i=0}^{n} \omega_i \frac{\partial K_1}{\partial x}(x, x_i) u_{1,n}(x_i) + \sum_{i=0}^{n} \omega_i \frac{\partial K_2}{\partial x}(x, x_i) u_{2,n}(x_i) + f'(x), \end{cases}$$

where the weights $\{w_i\}_{i=0}^{n}$ verifying

$$\sup_{n\geq 1} \sum_{i=0}^{n} |w_i| = W < +\infty.$$

We select the collocation points $x_j,\ j = 0, 1, \ldots n$, to obtain the following algebraic system for all

$$\begin{cases} \lambda u_{1,n}(x_j) &= \displaystyle\sum_{i=0}^{n} \omega_i K_1(x_j, x_i) u_{1,n}(x_i) + \sum_{i=0}^{n} \omega_i K_2(x_j, x_i) u_{2,n}(x_i) + f(x_j), \\ \lambda u_{2,n}(x_j) &= \displaystyle\sum_{i=0}^{n} \omega_i \frac{\partial K_1}{\partial x}(x_j, x_i) u_{1,n}(x_i) + \sum_{i=0}^{n} \omega_i \frac{\partial K_2}{\partial x}(x_j, x_i) u_{2,n}(x_i) + f'(x_j), \end{cases} \quad j = 0, 1 \ldots n$$

Now, we present the sequences approximation operators $\{T_{pq,n}\}_{1\leq p,q\leq 2}$, $\forall n \geq 1$ and $\forall x \in [a, b]$

$$\begin{aligned} \forall v \in X, \quad T_{1q,n} v(x) &= \sum_{i=1}^{n} \omega_i K_q(x, x_i) v(x_i), \quad q = 1, 2, \\ \forall v \in X, \quad T_{2q,n} v(x) &= \sum_{i=1}^{n} \omega_i \frac{\partial K_q}{\partial x}(x, x_i) v(x_i), \quad q = 1, 2, \end{aligned}$$

and the Nyström block operator matrix $A_{T_n} : \tilde{X}_2 \longrightarrow \tilde{X}_2$

$$A_{T_n} = \begin{pmatrix} T_{11,n} & T_{12,n} \\ T_{21,n} & T_{22,n} \end{pmatrix}. \tag{2.5}$$

Finally, the Nyström equation has this form

$$(\lambda I_2 - A_{T_n}) U_n^N = F, \tag{2.6}$$

where, $U_n^N = (u_{1,n}, u_{2,n})^t$ is the Nyström solution.

Convergence of the Nyström Method

The Nyström operator is known to be among operators who do not converge in the norm sense. In order to prove the convergence of the approximate solution to exact solution, we are going to show that our approximate operator is ν-convergence.

We explain what we have provided in this section in order to make it easier to follow the steps of convergence study. We start by proving that the Nyström block operator A_{T_n} is ν-convergent to A_T. Then, we present the theorem which show that $(\lambda I_2 - A_{T_n})^{-1}$ exists and is bounded. Finally, the approximate solution convergence is given in the last theorem of this part.

In the next theorem, we show the pointwise convergence of the Nyström block approximation.

Theorem 2.2.1. *Let A_{T_n} be a Nyström block operator matrix (2.5) and A_T (2.3) be a block operator matrix. Then,*

$$\forall\, U \in \tilde{X}_2, \quad \lim_{n\to\infty} ||(A_{T_n} - A_T)U||_{\tilde{X}_2} = 0.$$

Proof. First, by the convergence of quadrature rule, we get

$$\forall x \in [a,b], \lim_{n\to\infty} |T_{pq}u_q(x) - T_{pq,n}u_q(x)| = 0, \quad q,p = 1,2.$$

We start by proving that our approximation operators $\{T_{pq,n}\}_{1\le q,p\le 2}$ are equicontinous. For each $u_q \in C^0[a,b]$, $x,y \in [a,b]^2$ and for $q = 1,2$, we have

$$\begin{aligned} |T_{1q,n}u_q(x) - T_{1q,n}(y)| &\le W \max_{a\le t\le b} |K_q(x,t) - K_q(y,t)|, \\ |T_{2q,n}u_q(x) - T_{2q,n}(y)| &\le W \max_{a\le t\le b} \left| \frac{\partial K_q}{\partial x}(x,t) - \frac{\partial K_q}{\partial y}(y,t) \right|. \end{aligned}$$

By the uniform continuity of $K_q(.,.)$ and $\partial K_q(.,.)$, we deduce that $\{T_{pq,n}u_q\}_{1\le q,p\le 2}$ are equicontinuous. Then, using lemma 1.1.1, it follows that all $\{T_{pq,n}u_q\}_{1\le p,q\le 2}$ are uniformly converge to $\{T_{pq}v\}_{1\le p,q\le 2}$. Thus

$$\forall u_q \in X, \quad \lim_{n\to\infty} ||(T_{pq,n} - T_{pq})u_q||_X = 0, \quad p,q = 1,2.$$

Finally,

$$||(A_{T_n} - A_T)U||_{\tilde{X}} = \max_{1\le p\le 2} \sum_{q=1}^{2} ||(T_{pq,n} - T_{pq})u_q||_X,$$

which gives $||(A_{T_n} - A_T)U||_{\tilde{X}_2} \to 0$, when $n \to \infty$. □

Let define the numerical integration errors for the integrand $K_q(x,.)K_q(.,y)$, $\frac{\partial K_q}{\partial x}(x,.)\frac{\partial K_q}{\partial x}(.,y)$ and $\forall x,y \in [a,b],\ q = 1,2$ and $\forall n \ge 1$ as

$$\begin{aligned} \mathcal{E}_{1q,n}(x,y) &= \int_a^b K_q(x,t)K_q(t,y)\,dt - \sum_{i=1}^{n} \omega_i K_q(x,x_i)K_q(x_i,y), \\ \mathcal{E}_{2q,n}(x,y) &= \int_a^b \frac{\partial K_q}{\partial x}(x,t)\frac{\partial K_q}{\partial x}(t,y)\,dt - \sum_{i=1}^{n} \omega_i \frac{\partial K_q}{\partial x}(x,x_i)\frac{\partial K_q}{\partial x}(x_i,y). \end{aligned}$$

For the following theorem, we prove that the above errors are uniform convergence to 0.

Theorem 2.2.2. *Let $n \geq 1$. Then, the errors $\{\mathcal{E}_{pq,n}\}_{1\leq p,q\leq 2}$ are uniformly convergent to 0.*

Proof. We start by proving the equicontinity of errors (2.7). From the definition of equicontinuty

$$\forall (x_1, x_2) \in [a,b]^2,\ \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall (y_1, y_2) \in [a,b]^2,\ ||(x_1, x_2) - (y_1, y_2)||_{\mathbb{R}^2} < \delta,$$

we need to prove that

$$|\mathcal{E}_{pq,n}(x_1, y_1) - \mathcal{E}_{pq,n}(x_2, y_2)| \leq \varepsilon, \quad q, p = 1, 2.$$

We have, for $q = 1, 2$ and $p = 1$

$$|\mathcal{E}_{1q,n}(x_1, y_1) - \mathcal{E}_{1q,n}(x_2, y_2)| \leq |\mathcal{E}_{1q,n}(x_1, y_1) - \mathcal{E}_{1q,n}(x_1, y_2)| + |\mathcal{E}_{1q,n}(x_1, y_2) - \mathcal{E}_{1q,n}(x_2, y_2)|,$$

with,

$$|\mathcal{E}_{1q,n}(x_1, y_1) - \mathcal{E}_{1q,n}(x_1, y_2)| \leq (b - a + W) \max_{a\leq t\leq b} |K_q(x_1, t) - K_q(x_2, t)|,$$

$$|\mathcal{E}_{1q,n}(x_1, y_2) - \mathcal{E}_{1q,n}(x_2, y_2)| \leq (b - a + W) \max_{a\leq t\leq b} |K_q(t, y_1) - K_q(t, y_2)|.$$

In the same way, for $p = 2$ we have

$$|\mathcal{E}_{2q,n}(x_1, y_1) - \mathcal{E}_{2q,n}(x_2, y_2)| \leq |\mathcal{E}_{2q,n}(x_1, y_1) - \mathcal{E}_{2q,n}(x_1, y_2)| + |\mathcal{E}_{2q,n}(x_1, y_2) + \mathcal{E}_{2q,n}(x_2, y_2)|,$$

with

$$|\mathcal{E}_{2q,n}(x_1, y_1) - \mathcal{E}_{1q,n}(x_1, y_2)| \leq (b - a + W) \max_{a\leq t\leq b} \left| \frac{\partial K_q}{\partial x}(x_1, t) - \frac{\partial K_q}{\partial x}(x_2, t) \right|,$$

$$|\mathcal{E}_{2q,n}(x_1, y_2) - \mathcal{E}_{1q,n}(x_2, y_2)| \leq (b - a + W) \max_{a\leq t\leq b} \left| \frac{\partial K_q}{\partial x}(t, y_1) - \frac{\partial K_q}{\partial x}(t, y_2) \right|,$$

The Hein theorem's [48] proved that K_q and $\dfrac{\partial K_q}{\partial x}$ are uniform continuous in the compact set $[a,b]^2$. Then,

$$|\mathcal{E}_{1q,n}(x_1, y_1) - \mathcal{E}_{1q,n}(x_2, y_2)| \leq 2\,\varepsilon,$$

$$|\mathcal{E}_{2q,n}(x_1, y_1) - \mathcal{E}_{2q,n}(x_2, y_2)| \leq 2\,\varepsilon.$$

By Lemma 1.1.1, we get that $\{\mathcal{E}_{pq,n}\}_{1\leq p,q\leq 2}$ are uniformly convergent to 0. □

Theorem 2.2.3. *Let A_T and A_{T_n} be the block operator matrix present by (2.3) and (2.5) respectively. Then,*

$$\lim_{n\to+\infty} |||(A_T - A_{T_n})A_{T_n}||| = 0, \quad \textit{and} \quad \lim_{n\to+\infty} |||(A_T - A_{T_n})A_{T_n}||| = 0. \tag{2.7}$$

Proof. For n large enough, we have

$$(A_T - A_{T_n})A_T = \begin{pmatrix} T_{11} - T_{11,n} & T_{12} - T_{12,n} \\ T_{21} - T_{21,n} & T_{22} - T_{22,n} \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix},$$

$$(A_T - A_{T_n})A_{T_n} = \begin{pmatrix} T_{11} - T_{11,n} & T_{12} - T_{12,n} \\ T_{21} - T_{21,n} & T_{22} - T_{22,n} \end{pmatrix} \begin{pmatrix} T_{11,n} & T_{12,n} \\ T_{21,n} & T_{22,n} \end{pmatrix}.$$

Using Hadamart product (product element by element), we get

$$(A_T - A_{T_n})A_T = \begin{pmatrix} (T_{11} - T_{11,n})T_{11} & (T_{12} - T_{12,n})T_{12} \\ (T_{21} - T_{21,n})T_{21} & (T_{22} - T_{22,n})T_{22} \end{pmatrix},$$

$$(A_T - A_{T_n})A_{T_n} = \begin{pmatrix} (T_{11} - T_{11,n})T_{11,n} & (T_{12} - T_{12,n})T_{12,n} \\ (T_{21} - T_{21,n})T_{21,n} & (T_{22} - T_{22,n})T_{22,n} \end{pmatrix}.$$

By the norm of block operator matrix, we obtain

$$|||(A_T - A_{T_n})A_T||| = \max_{1\le q\le 2} \sum_{p=0}^{2} ||(T_{pq} - T_{pq,n})T_{pq}||,$$

$$|||(A_T - A_{T_n})A_{T_n}||| = \max_{1\le q\le 2} \sum_{p=0}^{2} ||(T_{pq} - T_{pq,n})T_{pq,n}||,$$

where $\forall\ z^{(q-1)} \in C^0[a,b]$ and $q = 1, 2$

$$(T_{1q} - T_{1q,n})T_{1q}z^{(q-1)}(x) = \int_a^b \mathcal{E}_{1q,n}(x,y)\ z^{(q-1)}(y)\ dy,$$

$$(T_{2q} - T_{2q,n})T_{2q}z^{(q-1)}(x) = \int_a^b \mathcal{E}_{2q,n}(x,y)\ z^{(q-1)}(y)\ dy,$$

$$(T_{1q} - T_{1q,n})T_{1q,n}z^{(q-1)}(x) = \sum_{i=0}^{n} w_i\ \mathcal{E}_{1q,n}(x,x_i)\ z^{(q-1)}(x_i),$$

$$(T_{2q} - T_{2q,n})T_{2q,n}z^{(q-1)}(x) = \sum_{i=0}^{n} w_i\ \mathcal{E}_{2q,n}(x,x_i)\ z^{(q-1)}(x_i),$$

$$||(T_{1q} - T_{1q,n})T_{1q}|| \le (b-a) \max_{a\le x,y\le b} |\mathcal{E}_{1q,n}(x,y)|,$$

$$||(T_{1q} - T_{1q,n})T_{1q,n}|| \le W \max_{a\le x,y\le b} |\mathcal{E}_{1q,n}(x,y)|,$$

$$||(T_{2q} - T_{2q,n})T_{2q}|| \le (b-a) \max_{a\le x,y\le b} |\mathcal{E}_{2q,n}(x,y)|,$$

$$||(T_{2q} - T_{2q,n})T_{2q,n}|| \le W \max_{a\le x,y\le b} |\mathcal{E}_{2q,n}(x,y)|.$$

Finally, by theorem 2.2.2, it follows that $\max_{a\le a,y\le b} |\mathcal{E}_{pq,n}(x,y)| \to 0$ as $n \to \infty$. □

Theorem 2.2.4. *Let A_{T_n} be a Nyström block operator. Then, $(\lambda I_2 - A_{T_n})^{-1}$ exists and bounded.*

Proof. For large n, we have

$$\begin{aligned}
\Big[I_2 + (\lambda I_2 - A_T)^{-1}A_{T_n}\Big](\lambda I_2 - A_{T_n}) &= (\lambda I_2 - A_T)^{-1}\Big[\lambda I_2 - A_T + A_{T_n}\Big], \\
&= (\lambda I_2 - A_T)^{-1}\Big[\lambda(\lambda I_2 - A_{T_n}) - \lambda(A_T - A_{T_n}) \\
&+ (A_T - A_{T_n})A_T\Big], \\
&= (\lambda I_2 - A_T)^{-1}\Big[\lambda(\lambda I_2 - A_T) + (A_T - A_{T_n})A_T\Big], \\
&= \lambda I_2 + (\lambda I_2 - A_T)^{-1}(A_T - A_{T_n})A_T.
\end{aligned}$$

Then,

$$\frac{1}{\lambda}\Big[I_2 + (\lambda I_2 - A_T)^{-1}A_{T_n}\Big](\lambda I_2 - A_{T_n}) = I_2 + \frac{1}{\lambda}(\lambda I_2 - A_T)^{-1}(A_T - A_{T_n})A_T.$$

In the other side, we have

$$|||\frac{1}{\lambda}(\lambda I_2 - A_T)^{-1}(A_T - A_{T_n})A_T||| \leq \frac{1}{|\lambda|}|||(\lambda I_2 - A_T)^{-1}|||\ |||(A_T - A_{T_n})A_T|||.$$

But, in the previous theorem $|||(A_T - A_{T_n})A_T||| \to 0$ when $n \to \infty$, we get

$$|||\frac{1}{\lambda}(\lambda I_2 - A_T)^{-1}(A_T - A_{T_n})A_T||| < 1,$$

which proves that by Neumann's theorem $\left(I_2 + \frac{1}{\lambda}(\lambda I_2 - A_T)^{-1}(A_T - A_{T_n})A_T\right)^{-1}$ exists and is bounded. Then, the inverse of $(\lambda I_2 - A_{T_n})$ has the following form:

$$(\lambda I_2 - A_{T_n})^{-1} = \left(\lambda I_2 + (\lambda I_2 - A_T)^{-1}(A_T - A_{T_n})A_T\right)^{-1}\left(I_2 + (\lambda I_2 - A_T)^{-1}A_{T_n}\right),$$

and

$$\begin{aligned}
|||(\lambda I_2 - A_{T_n})^{-1}||| &\leq |||(\lambda I_2 + (\lambda I_2 - A_T)^{-1}(A_T - A_{T_n})A_T\Big)^{-1}|||\ |||I_2 + (\lambda I_2 - A_T)^{-1}A_{T_n}|||, \\
&\leq \frac{|||I_2 + (\lambda I_2 - A_T)^{-1}A_{T_n}|||}{|\lambda| - |||(\lambda I_2 - A_T)^{-1}(A_T - A_{T_n})A_T|||}.
\end{aligned}$$

□

Finally, we give a majoration of the error in the next theorem

Theorem 2.2.5. *Let $U_n^N = (u_{1,n}, u_{2,n})^t$ be the Nyström solution of (2.6) and $U = (u_1, u_2)^t$ be the exact solution of (2.4). Then*

$$\lim_{n\to\infty} ||U_n - U||_{\tilde{X}_2} = 0.$$

Proof. From to equation (2.6) and (2.4), we have

$$\lambda(U_n - U) = A_{T_n}U_n^N - A_T = A_{T_n}(U - U_n^N) + (A_{T_n} - A_T)U.$$

Then,

$$U_n^N - U = (\lambda I_2 - A_{T_n})^{-1}\Big[(A_{T_n} - A_T)U\Big].$$

We get

$$||U_n^N - U||_{\tilde{X}_2} \leq ||(\lambda I_2 - A_{T_n})^{-1}||\ ||(A_{T_n} - A_T)U||_{\tilde{X}_2},$$

and when $n \to \infty$, we get the result. □

2.3 Projection Approximation

In this section, we give the basic concept of projection method see [5, 2, 48]. In order to illustrate the general framework of Collocation and Kantorovich methods. First, we recall Δ_n (1.15) the uniform discretization of the interval $[a,b]$. We denote by $\tilde{X}_{2,n}$ the sequence of finite dimension subspace of the Banach space $\tilde{X}_2$. The principle of both methods is based on the application of projection operators $\{P_n\}_{n\geq 1}$ defined by

$$\begin{aligned} P_n: \quad \tilde{X}_2 &\longrightarrow \tilde{X}_{2,n} \\ V &\longmapsto P_nV(x) = \Big(P_{1,n}v_1(x), P_{2,n}v_2(x)\Big), \end{aligned} \tag{2.8}$$

such that $\{P_{q,n}v\}_{n\geq 1}$ for $q = 1, 2$ are sequences of projection interpolation. This can be written in the following linear combination of piecewise linear functions.

$$\forall n \geq 1, \forall x \in [a,b], \quad P_{q,n}v_q(x) = \sum_{i=0}^{n} \alpha_i e_i(x),$$

where $\{\alpha_i\}_{i=0}^n$ are unknown coefficient to be determined and $\{e_i\}_{i=0}^n$ are the hat functions. These are given by (**??**), such that $\{P_n\}_{n\geq 1}$ satisfies the following condition interpolation

$$P_{q,n}v_q(x_i) = v_q(x_i), \quad i = 0, 1 \ldots n.$$

We start by showing that our projection is pointwise convergent to the identity operand in the space $\tilde{X}_2$.

In the following theorem, we show the convergence of P_nU to U in $\tilde{X}_2$. and then, the convergent of A_T to P_nA_T.

Theorem 2.3.1. *Let $U = (u_1, u_2)^t \in \tilde{X}_2$ and $\{P_n\}_{n\geq 1}$ be a projection operators sequence given by (2.8). Then,*

$$||(I_2 - P_n)U||_{\tilde{X}_2} \leq \kappa_{\infty,1}(U, h),$$

and when $h \to 0$, we get $||(I_2 - P_n)U||_{\tilde{X}_2} \to 0$.

Proof. For $n \geq 1$, it is clear that

$$||(I_2 - P_n)U||_{\tilde{X}_2} = \max_{a\leq x\leq b}|u_1(x) - P_{1,n}u_1(x)| + \max_{a\leq x\leq b}|u_2(x) - P_{2,n}u_2(x)|.$$

We have, for all $(u_1, u_2) \in \tilde{X}_2$

$$\begin{aligned}
|u_1(x) - P_{1,n}u_1(x)| &= |u(x) - \sum_{i=0}^{n} u_1(x_i)e_i(x)| \\
&= \left|\sum_{i=0}^{n} e_i(x)[u_1(x) - u_1(x_i)]\right| \\
&\leq \sum_{i=0}^{n} |e_i(x)|\,|u_1(x) - u_1(x_i)| \\
&\leq \kappa_{\infty,0}(u_1, h), \qquad (2.9)
\end{aligned}$$

such that, $|\sum_{i=0}^{n} e_i(x)| = 1$. In the same way, we get

$$|u_2(x) - P_{2,n}u_2(x)| \quad \leq \quad \kappa_{\infty,0}(u_2, h). \qquad (2.10)$$

Substituting (2.9) and (2.10) in (2.9), we obtain

$$||(I_2 - P_n)U||_{\tilde{X}_2} \leq \kappa_{\infty,0}(u_1, h) + \kappa_{\infty,0}(u_2, h).$$

□

Lemma 2.3.1. *Let A_T be bounded and compact block operator matrix and $\{P_n\}_{n\geq 1}$ converges pointwise to the identity operator. Then,*

$$\lim_{n\to\infty} |||(I_2 - P_n)A_T||| = 0.$$

Proof. Since A_T is compact

$$\mathcal{M} = \{A_T U, ||U||_{\tilde{X}_2} \leq 1\}$$

is relatively compact sets in the Banach space $\tilde{X}$ and by the Banach Steinhaus theorem $\{P_n\}_{n\in\mathbb{N}}$ converges uniformly to I_2 in $\mathcal{M}$, i.e

$$\lim_{n\to\infty} |||(I_2 - P_n)A_T||| = \lim_{n\to\infty} \sup_{||U||_{\tilde{X}}} ||(I_2 - P_n)A_T U||_{\tilde{X}_2} = \lim_{n\to\infty} \sup_{Z\in\mathcal{M}} ||(I_2 - P_n)Z||_{\tilde{X}_2}.$$

□

In the numerical treatment, we need to verify the existence of $(\lambda I_2 - P_n A_T)^{-1}$ to demonstrate the convergence of the approximate solution u_n to our solution. We introduce the following theorem to ensure this existence.

Theorem 2.3.2. *Let $P_n A_T$ be a projection approximation of block matrix operator and A_T given by (2.3). Then, $(\lambda I_2 - P_n A_T)^{-1}$ exists and bounded.*

Proof. We have

$$(\lambda I_2 - P_n A_T) = (\lambda I_2 - A_T)\big[I_2 - (\lambda I_2 - A_T)^{-1}(A_T - P_n A_T)\big].$$

Using lemma2.3, we get

$$\lim_{n\to\infty} |||(I_2 - P_n)A_T||| = 0.$$

This shows

$$|||(\lambda I_2 - A_T)^{-1}(A_T - P_n A_T)||| \leq |||(\lambda I_2 - A_T)^{-1}|||\ |||(I_2 - P_n)A_T||| \leq \varepsilon_n,$$

which gives

$$|||(\lambda I_2 - A_T)^{-1}(A_T - P_n A_T)||| < 1,$$

so that

$$\Big(I_2 - (\lambda I_2 - A_T)^{-1}(A_T - P_n A_T)\Big)^{-1},$$

exists and

$$\left|\left|\left|\Big(I_2 - (\lambda I_2 - A_T)^{-1}((I_2 - P_n)A_T)\Big)^{-1}\right|\right|\right| \leq \frac{1}{1-\varepsilon_n}.$$

Finally, $(\lambda I_2 - P_n A_T)^{-1}$ exists and $|||(\lambda I_2 - P_n A_T)^{-1}||| \leq C$, where

$$C = \frac{|||\big(\lambda I_2 - A_T\big)^{-1}|||}{1-\varepsilon_n}.$$

□

Now, we present two projection methods. The first is Collocation method which is introduced by Atkinson [5] and Nair [48] for searching the sequence $U_n^C \in \tilde{X}_{2,n} \subset \tilde{X}_2$ solution of

$$\lambda U_n^C = P_n A_T U_n^C + P_n F, \tag{2.11}$$

whereas the second one is for finding the sequence $U_n^{Kan} \in \tilde{X}_{2,n} \subset \tilde{X}_2$ solution of

$$\lambda U_n^{Kan} = P_n A_T U_n^{Kan} + F. \tag{2.12}$$

It's the Kantorovich method.

2.3.1 Collocation Method

The main idea of collocation method is to find a sequence U_n^C solution of

$$\lambda U_n^C = P_n A_T U_n^C + P_n F, \tag{2.13}$$

where $P_n F = (P_n f, P_n f')^t$, and $U_n = (u_{1,n}^C, u_{2,n}^C)^t$ is Collocation solution are given by

$$\forall x \in [a,b], \quad \begin{cases} u_{1,n}^C(x) &= \sum_{i=0}^{n} \alpha_i e_i(x), \\ u_{2,n}^C(x) &= \sum_{i=0}^{n} \beta_i e_i(x). \end{cases}$$

where $\{\alpha_i\}_{0\leq i\leq n}$ and $\{\beta_i\}_{0\leq i\leq n}$ are coefficients to be determined.

The next theorem shows that the approximate solution U_n^C converges to the exact solution U.

Theorem 2.3.3. *Let $U = (u_1, u_2)^t$ the solution of (2.4) and U_n^C is the Collocation approximation of (2.13). Then,*

$$||U - U_n^C||_{\tilde{X},2} \leq C\Big[\kappa_{\infty,1}(f,h) + |||(I_2 - P_n)A_T|||\Big].$$

Proof. We have, for n large enough

$$U - U_n^C = (\lambda I_2 - P_n A_T)^{-1}\Big[F - P_n F + (A_T - P_n A_T)U\Big].$$

Then,

$$\begin{aligned}
||U - U_n^C||_{\tilde{X}_2} &\leq |||(\lambda I_2 - P_n A_T)^{-1}|||\Big[||(I_2 - P_n)F||_{\tilde{X}_2} + ||(A_T - P_n A_T)U||_{\tilde{X}_2}\Big],\\
&\leq C\Big[||(I_2 - P_n)F||_{\tilde{X}_2} + |||(I_2 - P_n)A_T|||\Big],\\
&\leq C\Big[\kappa_{\infty,1}(f,h) + |||(I_2 - P_n)A_T|||\Big].
\end{aligned}$$

Also, we have

$$||U - U_n^C||_{\tilde{X}_2} = ||u_n - u||_{C^1[a,b]}.$$

Then, when $n \to \infty$, we obtain $u_{1,n}^C = u_n^C \to u$ in $C^1[a,b]$. □

System Approximation

The system (2.13) can be written as

$$\forall x \in [a,b], \quad \begin{cases} \lambda u_{1,n}^C(x) = P_n T_{11} u_{1,n}^C(x) + P_n T_{12} u_{2,n}^C(x) + P_n f(x), \\ \\ \lambda u_{2,n}^C(x) = P_n T_{21} u_{1,n}^C(x) + P_n T_{22} u_{2,n}^C(x) + P_n f'(x). \end{cases} \tag{2.14}$$

These are equivalent to

$$\begin{cases} \sum_{j=0}^{n} \alpha_j e_j(x) = \sum_{j=0}^{n} \Big[T_{11} u_{1,n}^C(x_j) + T_{12} u_{2,n}^C(x_j) + f(x_j)\Big] e_j(x), \\ \\ \sum_{j=0}^{n} \beta_j e_j(x) = \sum_{j=0}^{n} \Big[T_{21} u_{1,n}^C(x_j) + T_{22} u_{2,n}^C(x_j) + f'(x_j)\Big] e_j(x). \end{cases}$$

This leads to the resolution of following block matrix system form:

$$\begin{cases} \alpha_j = \sum_{i=0}^{n} A_{11}(j,i)\alpha_i + \sum_{i=0}^{n} A_{12}(j,i)\beta_i + f_j \\ \\ \beta_j = \sum_{i=0}^{n} A_{21}(j,i)\alpha_i + \sum_{i=0}^{n} A_{22}(j,i)\beta_i + f'_j, \end{cases} \qquad j = 0, 1, \ldots n$$

where $f_j = f(x_j)$, $f'_j = f'(x_j)$ and for $p = 1, 2$

$$A_{1p}(j,i) = \frac{1}{h}\int_{x_{i-1}}^{x_i} K_p(x_j,t)(t-x_{i-1})\,dt + \int_{x_i}^{x_{i+1}} K_p(x_j,t)(x_{i+1}-t)\,dt,\ 0\le j\le n, 1\le i\le n-1,$$

$$A_{1p}(j,0) = \frac{1}{h}\int_{x_0}^{x_1} K_p(x_j,t)(x_1-t)\,dt, \quad 0\le j\le n,$$

$$A_{1p}(j,n) = \frac{1}{h}\int_{x_{n-1}}^{x_n} K_p(x_j,t)(t-x_{n-1})\,dt, \quad 0\le j\le n,$$

$$A_{2p}(j,i) = \frac{1}{h}\int_{x_{i-1}}^{x_i} \frac{\partial K_p}{\partial x}(x_j,t)(t-x_{i-1})\,dt + \int_{x_i}^{x_{i+1}} \frac{\partial K_p}{\partial x}(x_j,t)(x_{i+1}-t)\,dt,\ 0\le j\le n, 1\le i\le n-1,$$

$$A_{2p}(j,0) = \frac{1}{h}\int_{x_0}^{x_1} \frac{\partial K_p}{\partial x}(x_j,t)(x_1-t)\,dt, \quad 0\le j\le n,$$

$$A_{2p}(j,n) = \frac{1}{h}\int_{x_{n-1}}^{x_n} \frac{\partial K_p}{\partial x}(x_j,t)(t-x_{n-1})\,dt, \quad 0\le j\le n, 1\le i\le n-1.$$

2.3.2 Kantorovich Method

In the collocation method, the convergence of the approximate sequence is guaranteed by the following assumption $P_n f \to f$, when $n\to\infty$. We can cancel this hypothesis, if we assume that $(f,f')\in Ran(A_T)$. For this raison, we apply on both sides of the system (2.4) the block operator matrix A_T to get

$$\lambda A_T U = A_T A_T U + A_T F, \tag{2.15}$$

We denote

$$V = A_T U = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22}\end{pmatrix}\begin{pmatrix} u \\ u'\end{pmatrix},$$

and (2.15) is equivalent to

$$\lambda V = A_T V + A_T F. \tag{2.16}$$

If $V=(v_1,v_2)^t$ is the solution of (2.16), then the system (2.4) can be rewritten as

$$U^{Kan} = \frac{1}{\lambda}\Big(V+F\Big).$$

We compute an approximate solution in the following form:

$$U_n^{Kan} = \frac{1}{\lambda}\Big(V_n^C + F\Big), \tag{2.17}$$

such that $V_n^C = (v_{1,n}^C, v_{2,n}^C)^t$ is the Collocation solution of

$$\lambda V_n^C = P_n A_T V_n^C + P_n A_T F,$$

represented in the following way

$$\forall x \in [a,b], \quad \begin{cases} v_{1,n}^C(x) = \sum_{i=0}^{n} \overline{\alpha_i} e_i(x), \\ \\ v_{2,n}^C(x) = \sum_{i=0}^{n} \overline{\beta_i} e_i(x). \end{cases}$$

Therefore, we can define Kantorovich's approximation as

$$\lambda U_n^{Kan} = P_n A_T U_n^{Kan} + F, \tag{2.18}$$

with $U_n^{Kan} = (u_{1,n}^{Kan}, u_{2,n}^{Kan})^t$. We notice that from the two equations (2.18) and (2.17)

$$||U_n^{Kan} - U||_{\tilde{X}_2} \leq \frac{1}{|\lambda|} ||V_n^C - V_n||_{\tilde{X}_2}.$$

Now, we state a theorem which shows the Kantorovich approximation (2.17) convergence to the exact solution.

Theorem 2.3.4. *Let $U = (u_1, u_2)^t$ be solution of (2.4) and $U_n^{Kan} = (u_{1,n}^{Kan}, u_{2,n}^{Kan})^t$ is the approximate solution given by (2.18). Then,*

$$||U - U_n^{Kan}||_{\tilde{X}_2} \leq C \, |||(I_2 - P_n)A_T|||.$$

Proof. For n large enough, we have

$$\begin{aligned} U - U_n^{Kan} &= (\lambda I_2 - P_n A_T)^{-1} F - (\lambda I_2 - A_T)^{-1} F \\ &= (\lambda I_2 - P_n A_T)^{-1} \Big[F - (\lambda I_2 - P_n A_T)(\lambda I_2 - A_T)^{-1} F \Big] \\ &= (\lambda I_2 - P_n A_T)^{-1} \Big[(\lambda I_2 - A_T) U - (\lambda I_2 - P_n A_T) U \Big] \\ &= (\lambda I_2 - P_n A_T)^{-1} \Big[A_T U - P_n A_T U \Big]. \end{aligned}$$

Then

$$\begin{aligned} ||U - U_n^{Kan}||_{\tilde{X}} &\leq C \, ||(I_2 - P_n)A_T U||_{\tilde{X}_2} \\ &\leq C \, |||(I_2 - P_n)A_T|||. \end{aligned}$$

□

This theorem shows the convergence of Kantorovich approximation solution to the exact solution when $n \to \infty$ in $C^1[a,b]$ because

$$||U - U_n^{Kan}||_{\tilde{X}_2} = ||u - u_n^{Kan}||_{C^1[a,b]}.$$

System Approximation

Applying the Kantorovich method, we get the following block matrix system

$$\begin{cases} \overline{\alpha_j} = \sum_{i=0}^{n} A_{11}(j,i)\overline{\alpha_i} + \sum_{i=0}^{n} A_{12}(j,i)\overline{\beta_i} + G_1(j), \\ \\ \overline{\beta_j} = \sum_{i=0}^{n} A_{21}(j,i)\overline{\alpha_i} + \sum_{i=0}^{n} A_{22}(j,i)\overline{\beta_i} + G_2(j), \end{cases}$$

such that for $j = 0, 1, \dots n$

$$\begin{cases} u^K_{n,1}(x_j) = \dfrac{\overline{\alpha_j} + f(x_j)}{\lambda} = u^K_n(x_j), \\ \\ u^K_{n,2}(x_j) = \dfrac{\overline{\beta_j} + f'(x_j)}{\lambda} = u'^K_n(x_j), \end{cases}$$

for $p = 1, 2$ and $0 \le j \le n$, we have

$$A_{1p}(j,i) = \frac{1}{h}\int_{x_{i-1}}^{x_i} K_p(x_j,t)(t - x_{i-1})\,dt + \int_{x_i}^{x_{i+1}} K_p(x_j,t)(x_{i+1} - t)\,dt,\ 1 \le i \le n-1,$$

$$A_{1p}(j,0) = \frac{1}{h}\int_{x_0}^{x_1} K_p(x_j,t)(x_1 - t)\,dt,$$

$$A_{1p}(j,n) = \frac{1}{h}\int_{x_{n-1}}^{x_n} K_p(x_j,t)(t - x_{n-1})\,dt,$$

$$A_{2p}(j,i) = \frac{1}{h}\int_{x_{i-1}}^{x_i} \frac{\partial K_p}{\partial x}(x_j,t)(t - x_{i-1})\,d,t + \int_{x_i}^{x_{i+1}} \frac{\partial K_p}{\partial x}(x_j,t)(x_{i+1} - t)\,dt,\ 1 \le i \le n-1,$$

$$A_{2p}(j,0) = \frac{1}{h}\int_{x_0}^{x_1} \frac{\partial K_q}{\partial x}(x_j,t)(x_1 - t)\,dt,$$

$$A_{2p}(j,n) = \frac{1}{h}\int_{x_{n-1}}^{x_n} \frac{\partial K_q}{\partial x}(x_j,t)(t - x_{n-1})\,dt.$$

$$G_1(j) = \int_a^b K_1(x_j,t)f(t)\,dt + \int_a^b K_2(x_j,t)f'(t)\,dt, \quad j = 0, 1, \dots n,$$

$$G_2(j) = \int_a^b \frac{\partial K_1}{\partial x}(x_j,t)f(t)\,dt + \int_a^b \frac{\partial K_1}{\partial x}(x_j,t)f'(t)\,dt, \quad j = 0, 1, \dots n.$$

2.4 Projection block iterative scheme

From the previous results, we understand that if we have n too large, we get a good precision on the solution and a good error convergence to zero. But, it poses problem because the size of the block matrix $P_n A_T$ becomes large. Therefore, the conditioning of the matrix can be affected. An iterative scheme for solving the two projection schemes presented by (2.11) and (2.12) is necessary. In [11], authors proposed a numerical method for solving 2×2 block systems and may be applied to solve this system especially if the coefficient matrices are ill-conditioned. In this section, we use the block iterative scheme in order to avoid calculating the inverse of the whole matrix $(\lambda I_2 - P_n A_T)^{-1}$.

So, for $p = 1, 2$, $|\lambda| > ||T_{pp}||$, then $(\lambda I - T_{pp})^{-1}$ exists and bounded. We can present the matrix $(\lambda I_2 - A_T)$ in the following form

$$\lambda I_2 - A_T = \begin{pmatrix} \lambda I - T_{11} & \mathcal{O} \\ \mathcal{O} & \lambda I - T_{22} \end{pmatrix} \left[\begin{pmatrix} I & \mathcal{O} \\ \mathcal{O} & I \end{pmatrix} - \begin{pmatrix} \mathcal{O} & (\lambda I - T_{11})^{-1} T_{12} \\ (\lambda I - T_{22})^{-1} T_{21} & \mathcal{O} \end{pmatrix} \right],$$

The main idea of this scheme comes from $(\lambda I_2 - A_T)$ introduced by (2.19) and used to construct Collocation block iterative scheme

$$\begin{pmatrix} \lambda I - P_n T_{11} & \mathcal{O} \\ \mathcal{O} & \lambda I - P_n T_{12} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} P_n f \\ P_n f' \end{pmatrix},$$

where

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} I & -(\lambda I - P_n T_{11})^{-1} P_n T_{12} \\ -(\lambda I - P_n T_{22})^{-1} P_n T_{21} & I \end{pmatrix} \begin{pmatrix} u_n^G \\ u_n'^G \end{pmatrix}.$$

Therefore, we write the Collocation scheme in the following form for all $n \geq 1$

$$\begin{cases} y_1 = (\lambda I_2 - P_n T_{11})^{-1} P_n f, \\ y_2 = (\lambda I_2 - P_n T_{22})^{-1} P_n f'. \end{cases}$$

$$\begin{cases} u_{1,n}^{C,0} &= y_1, \\ u_{2,n}^{C,0} &= y_2, \\ u_{1,n}^{C,k} &= (\lambda I_2 - P_n T_{11})^{-1} P_n T_{12} u_{2,n}^{G,k-1} + y_1, \quad \forall\, n, k \geq 1, \\ u_{2,n}^{C,k} &= (\lambda I_2 - P_n T_{22})^{-1} P_n T_{21} u_{1,n}^{G,k-1} + y_2, \quad \forall\, n, k \geq 1. \end{cases}$$

With the same principle given above, we build the Kantrovich iterative scheme:

$$\begin{cases} z_1 = (\lambda I_2 - P_n T_{11})^{-1} (P_n T_{11} f + P_n T_{12} f'), \\ z_2 = (\lambda I_2 - P_n T_{22})^{-1} (P_n T_{21} f + P_n T_{22} f'). \end{cases}$$

$$\begin{cases} u_{1,n}^{Kan,0} &= z_1, \\ \\ u_{2,n}^{Kan,0} &= z_2, \\ \\ u_{1,n}^{Kan,k} &= (\lambda I_2 - P_n T_{11})^{-1} P_n T_{12} u_{2,n}^{kan,k-1} + z_1, \quad \forall\, n, k \geq 1, \\ \\ u_{2,n}^{Kan,k} &= (\lambda I_2 - P_n T_{22})^{-1} P_n T_{21} u_{1,n}^{kan,k-1} + z_2, \quad \forall\, n, k \geq 1. \end{cases}$$

In the next theorem, we prove that the iterative scheme of Kantorovich and Collocation methods converges to the exact solution.

Theorem 2.4.1. *Let* $U_n^k = (u_{1,n}^k, u_{2,n}^k)^t$, $\forall\ k, n \geq 1$ *the iterative solution of projection method and* $U = (u_1, u_2)^t$ *solution of* (2.4)*. Then,*

$$\forall n \geq 1, \quad \lim_{k\to\infty} ||U - U_n^k||_{\tilde{X}_2} \leq ||U - U_n||_{\tilde{X}_2}.$$

Proof. We have

$$||U - U_n^k||_{\tilde{X}_2} \leq ||U - U_n||_{\tilde{X}_2} + ||U_n - U_n^k||_{\tilde{X}_2},$$

such that

$$U_n - U_n^k = \begin{pmatrix} \mathcal{O} & (\lambda I - P_n T_{11})^{-1} P_n T_{12} \\ (\lambda I - P_n T_{22})^{-1} p_n T_{21} & \mathcal{O} \end{pmatrix} \begin{pmatrix} u_{1,n} - u_{1,n}^{k-1} \\ u_{2,n} - u_{2,n}^{k-1} \end{pmatrix}.$$

By recurrence, we get

$$U_n - U_n^k = \begin{pmatrix} \mathcal{O} & (\lambda I - P_n T_{11})^{-1} P_n T_{12} \\ (\lambda I - P_n T_{22})^{-1} P_n T_{21} & \mathcal{O} \end{pmatrix}^k \begin{pmatrix} u_{1,n} - u_{1,n}^{0} \\ u_{2,n} - u_{2,n}^{0} \end{pmatrix}.$$

We denote $M_{P_n T_{qp}}$ the block matrix given as

$$M_{P_n T_{qp}} = \begin{pmatrix} \mathcal{O} & (\lambda I - P_n T_{11})^{-1} P_n T_{12} \\ (\lambda I - P_n T_{22})^{-1} P_n T_{21} & \mathcal{O} \end{pmatrix}.$$

Then,

$$U_n - U_n^k = M_{P_n T_{pq}}^k \Big(U_n - U_n^0 \Big), \tag{2.19}$$

where

$$U_n - U_n^0 = M_{P_n T_{qp}} \Big(U_n - U + U \Big). \tag{2.20}$$

Substituting (2.20) in (2.19), we obtain

$$||U_n - U_n^k||_{\tilde{X}_2} \leq |||M_{P_n T_{qp}}|||^{k+1} \Big[||U_n - U||_{\tilde{X}_2} + ||U||_{\tilde{X}_2} \Big], \tag{2.21}$$

Then, by the inequality (2.21)

$$\begin{aligned} ||U-U_n^k||_{\tilde{X}_2} &\leq ||U-U_n||_{\tilde{X}_2}+||U_n-U_n^k||_{\tilde{X}_2}, \\ &\leq \left(1+|||M_{P_nT_{qp}}|||^{k+1}\right)||U-U_n||_{\tilde{X}_2}+||M_{P_nT_{qp}}||^{k+1}||U||_{\tilde{X}_2}. \end{aligned}$$

Now, we will prove that $|||M_{P_nT_{qp}}|||<1$. By usingthe norm given in (1.1),we get

$$|||M_{P_nT_{qp}}|||=\max\left(||(\lambda I-P_nT_{22})^{-1}P_nT_{21}||,||(\lambda I-P_nT_{11})^{-1}P_nT_{12}||\right).$$

But,

$$||(\lambda I-P_nT_{22})^{-1}P_nT_{21}||\leq ||(\lambda I-P_nT_{22})^{-1}||\ ||P_nT_{21}||.$$

Also, we have

$$\begin{aligned} ||(\lambda I-P_nT_{22})^{-1}|| &\leq ||(I-(\lambda I-P_nT_{22})^{-1}(T_{22}-P_nT_{22})||\ ||(\lambda I-T_{22})^{-1}||, \\ &\leq \frac{||(\lambda I-T_{22})^{-1}||}{1-||(\lambda I-T_{22})^{-1}||\ ||(I-P_n)T_{22}||}. \end{aligned}$$

Theorem 2.1.1 shows that $\lim_{n\to\infty}||(I-P_n)T_{22}||=0$, we get

$$||(\lambda I-P_nT_{22})^{-1}|| \leq ||(\lambda I-T_{22})^{-1}||.$$

Then,

$$||(\lambda I-P_nT_{22})^{-1}P_nT_{21}||\leq ||(\lambda I-T_{22})^{-1}||\ ||T_{21}||<1.$$

In the same way, we get

$$||(\lambda I-P_nT_{11})^{-1}P_nT_{12}||\leq ||(\lambda I-T_{11})^{-1}||\ ||T_{12}||<1.$$

So, $|||M_{P_nT_{pq}}|||<1$ which proves that

$$\forall n\geq 1,\ \lim_{k\to\infty}||U-U_n^k||_{\tilde{X}_2}\leq ||U-U_n||_{\tilde{X}_2}.$$

We prove in theorem 2.3.3 and 2.3.4,

$$\lim_{n\to\infty}||U-U_n||_{\tilde{X}_2},$$

which gives

$$\lim_{n\to\infty}\left(\lim_{k\to\infty}||U-U_n^k||_{\tilde{X}_2}\right)=0.$$

□

2.5 Numerical Tests

In order to show the effectiveness of the suggested methods, we apply these approaches to the following numerical examples. We use the following estimate to calculate the error between the exact solution and the approximate solution at points x_i

$$err = \max_{0\le i\le n} |u(x_i) - u_n(x_i)| + \max_{0\le i\le n} |u'(x_i) - u'_n(x_i)|,$$

where the points $x_i = a + ih$, for $h = \dfrac{b-a}{n}$ and $i = 0, 1, \ldots n$.

2.5.1 Test 01

We consider the following integro-differential equation

$$\forall x \in [0,1], \quad \lambda u(x) = \int_0^1 \frac{u(t)}{e^x + e^t}\, dt + \int_0^1 \frac{u'(t)}{1 + x + e^{2t}}\, dt + f(x). \tag{2.22}$$

We choose $\lambda = 2$ and the exact solution $u(x) = e^x$ to get:

$$f(x) = \log(e^x + 1) - \log\left(e^1 + e^x\right) + 2e^x - \arctan\left(\frac{e^1}{\sqrt{x+1}}\right) - \frac{\arctan\left(\frac{1}{\sqrt{x+1}}\right)}{\sqrt{x+1}}.$$

Table 2.1: The error between the exact and approximation solution of equation (2.22)

n	Nyström	Collocation	Kantorovich
10	1.6368e-04	6.6852e-04	1.0445e-05
100	1.6355e-06	7.1682e-06	1.1061e-07

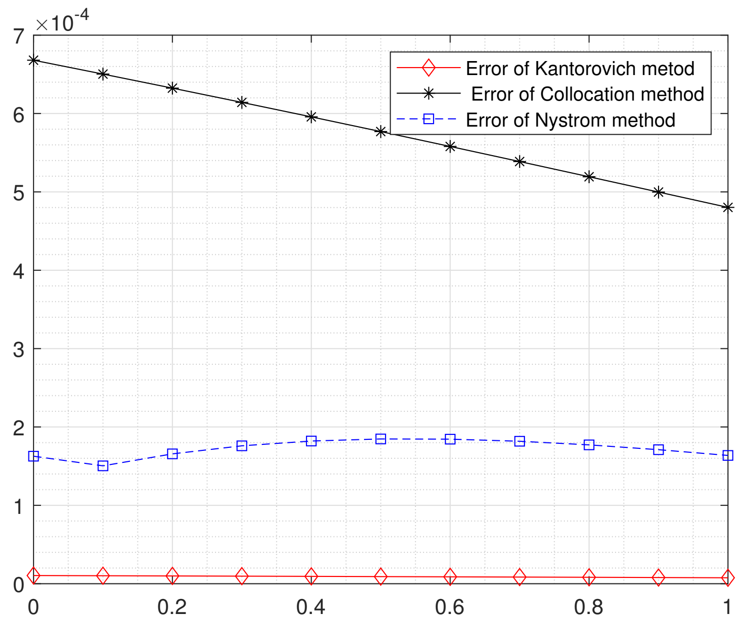


Figure 2.1: Errors of Nyström, Galerkin and Kantorovich Solutions of (2.22) with n=10.

2.5.2 Test 02

We consider the following integro-differential equation

$$\forall x \in [-2,2], \lambda u(x) = \int_{-2}^{2} \frac{u(t)}{(x^2+t^3+8)^2+1} dt + \int_{-2}^{2} (x+4)^{|t|} u'(t)dt + f(x). \quad (2.23)$$

We choose $\lambda = 4$ and the exact solution $u(x) = x^2$, to obtain

$$f(x) = 4x^2 - \frac{1}{3}\Big(\arctan\big(x^2+16\big) - \arctan\big(x^2\big)\Big).$$

Table 2.2: The error between the exact and approximate solution of equation (**??**)

n	Nyström	Collocation	Kantorovich
10	0.2315	1.4478e-03	1.6786e-05
100	0.0034	1.7149e-05	1.8614e-07

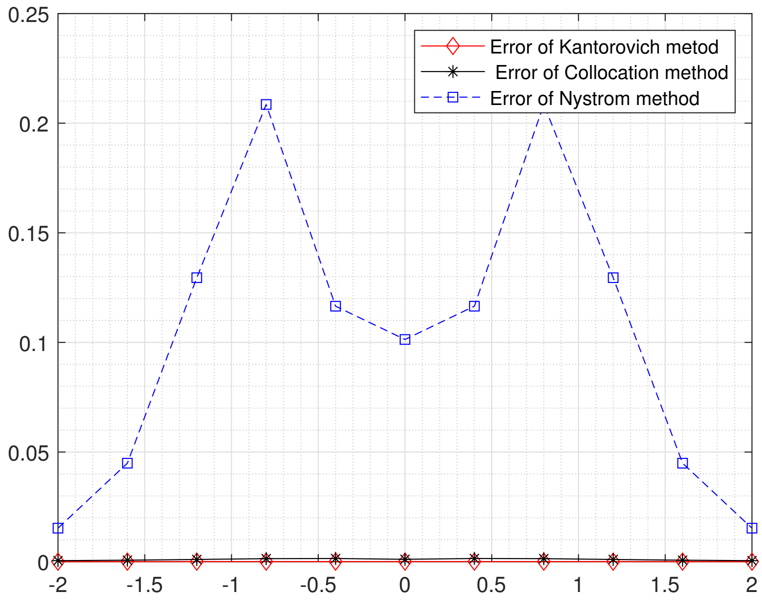


Figure 2.2: Errors of Nyström, Galerkin and Kantorovich Solutions of (2.23) with n=10.

2.5.3 Test 03

We give third equation in the following form

$$\forall x \in [1, 10], \ \lambda u(x) = \int_1^{10} \frac{u(t)}{x - t + t\log(t)}\, dt + \int_1^{10} \frac{u'(t)}{\sqrt{x + \log(t)}}\, dt + f(x). \tag{2.24}$$

We choose $\lambda = 20$ and the exact solution $u(x) = \log(x)$, we get

$$f(x) = 20\log(x) - 2\sqrt{(x + \log(10))} + 2\sqrt{x} - \arctan(10\log(10) - 10 + x) + \arctan(x - 1).$$

Table 2.3: The error between the exact and approximation solution of equation (2.22)

n	Nyström	Collocation	Kantorovich
50	2.3856e-04	6.7363e-05	8.6247e-06
200	1.5078e-05	4.2353e-06	5.3879e-07
500	2.4140e-06	6.7791e-07	8.6205e-08

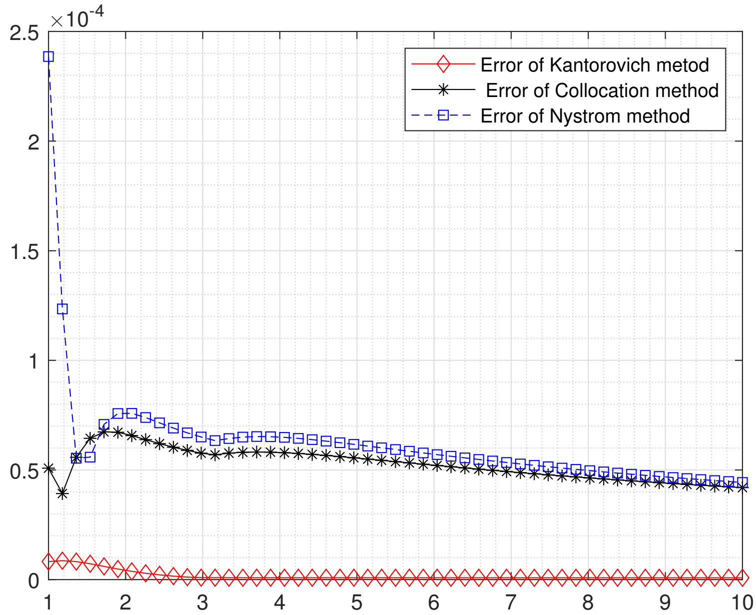


Figure 2.3: Error of Nyström, Galerkin and Kantorovich Solutions of (2.24) with n=50.

Chapter 3

Analytical and Numerical Study of Linear Fredholm Integro-differential Equation with Weakly Singular Kernels

Contents

Recall the aim of this work is to make a complete and thorough study of the integro-differential Fredholm equation. Therefore, we would like to focus on another form whose equation takes. Thus, our objective in this part is to study the possibility that both kernels of our equation are weakly singular. For this reason, we start in the section 3.1 with the construction of a sufficient condition that gives the existence and the uniqueness of the solution. Subsequently, we propose two different numerical methods which are: Product integration method in section 3.3 and Cubic collocation b-spline method in section 3.4. In order to find the best numerical solution and reduce the calculation time. We present some theorems to ensure the convergence of the approximate solution. In section 3.5, we give numerical examples to see the error behaviour of two approximate methods.
In this chapter, we study the following linear Fredhom integro-differential equation in the Banach space $C^1[a,b]$

$$\forall x \in [a,b], \quad \lambda u(x) = \int_a^b p_1(|x-t|)K_1(x,t)u(t)\,dt + \int_a^b p_2(|x-t|)K_2(x,t)u'(t)\,dt + f(x), \tag{3.1}$$

where $\lambda \neq 0$ is a real or complex parameter, the kernels K_i verify the next assumptions for $i = 1, 2$

$$(\mathcal{H}_1)\left\|\begin{array}{c}\dfrac{\partial K_i}{\partial x}(x,t)\in C^0([a,b]^2,\mathbb{R}),\\ \\ \exists M_i>0,\ \max\limits_{a\le x,t\le b}\left(|K_i(x,t)|,\left|\dfrac{\partial K_i}{\partial x}(x,t)\right|\right)\le M_i,\end{array}\right.$$

and the singularity part p_i verifies the following assumptions for $i = 1, 2$

$$(\mathcal{H}_2)\left\|\begin{array}{c}p_i\in W^{1,1}(0,b-a).\\ \\ p_i(0)=0,\\ \\ \lim\limits_{s\to 0^+}|p_i'(s)|=+\infty.\end{array}\right.$$

For $i = 1, 2$, p_i are chosen to take one of the following forms

1. $p_i(x) = |x-t|^{\alpha+1}, \quad \alpha \in]0,1[$,
2. $p_i(x) = |x-t|\log(|x-t|) - |x-t|$.

Under the assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, we can give the derivative u' implicitly as

$$\begin{aligned}\forall x\in[a,b],\ \lambda u'(x) &= \int_a^b p_1(|x-t|)\frac{\partial K_1}{\partial x}(x,t)u(t)\,dt+\int_a^b sign(x-t)p_1'(|x-t|)K_1(x,t)u(t)\,dt\\ &+\int_a^b p_2(|x-t|)\frac{\partial K_2}{\partial x}(x,t)u'(t)\,dt+\int_a^b sign(x-t)p_2'(|x-t|)K_2(x,t)u'(t)\,dt+f'(x),\end{aligned}\tag{3.2}$$

where

$$sign(x-t)=\begin{cases}1, & \text{if}\quad x>t,\\ -1, & \text{else.}\end{cases}$$

We cannot move on to the search for a numerical solution without proving the existence and uniqueness of the exact solution. This is done in the next section.

3.1 Analytical Study

The aim of this part, is to use the hypotheses mentioned above to construct a sufficient condition that ensures the solution existence and uniqueness of the equation (3.1). So, let us define T the linear integro-differential operator by

$$\begin{aligned}T:\quad C^1[a,b] &\longrightarrow C^1[a,b]\\ u &\longmapsto Tu(x)=\int_a^b p_1(|x-t|)K_1(x,t)u(t)\,dt+\int_a^b p_2(|x-t|)K_2(x,t)u'(t)\,dt.\end{aligned}$$

Thus, the equation (3.1) has an equivalent form, which is given by

$$(\lambda I - T)u = f,\tag{3.3}$$

where I is the identity operator of the Banach space $C^1[a,b]$.

To prove that the equation (3.1) has a unique solution, we need to prove that the inverse of $\lambda I - T$ exists and is bounded.

Theorem 3.1.1. *The linear Fredholm integro-differentail equation* (3.1) *has a unique solution* $u \in C^1[a,b]$ *if*

$$|\lambda| > 2\Big[M_1||p_1||_{W^{1,1}[0,b-a]} + M_2||p_2||_{W^{1,1}[0,b-a]}\Big].$$

Proof. We have,

$$\begin{aligned} |Tu(x)| &\leq \int_a^b |p_1(|x-t|)|\ |K_1(x,t)|\ |u(t)|\ dt + \int_a^b |p_2(|x-t|)|\ |K_2(x,t)|\ |u'(t)|\ dt, \\ &\leq M_1||p_1||_{L^1[0,b-a]}\ ||u||_{C^0[a,b]} + M_2||p_2||_{L^1[0,b-a]}\ ||u'||_{C^0[a,b]}. \end{aligned}$$

and

$$\begin{aligned} |(Tu)'(x)| &\leq \int_a^b |p_1(|x-t|)|\ |\frac{\partial K_1}{\partial x}(x,t)|\ |u(t)|\ dt + \int_a^b |p_1'(|x-t|)|\ |K_1(x,t)|\ |u(t)|\ dt \\ &+ \int_a^b |p_2(|x-t|)|\ |\frac{\partial K_2}{\partial x}(x,t)|\ |u'(t)|\ dt + \int_a^b |p_2'(|x-t|)|\ |K_2(x,t)||u'(t)|\ dt, \\ &\leq M_1||p_1||_{W^{1,1}[0,b-a]}\ ||u||_{C^0[a,b]} + M_2||p_2||_{W^{1,1}[0,b-a]}\ ||u'||_{C^0[a,b]}. \end{aligned}$$

Then,

$$||Tu||_{C^1[a,b]} \leq 2\Big[M_1||p_1||_{W^{1,1}[0,b-a]} + M_2||p_2||_{W^{1,1}[0,b-a]}\Big]||u||_{C^1[a,b]},$$

which gives ,

$$||T|| \leq 2\Big[M_1||p_1||_{W^{1,1}[0,b-a]} + M_2||p_2||_{W^{1,1}[0,b-a]}\Big].$$

Using the fact that

$$|\lambda| > 2\Big[M_1||p_1||_{W^{1,1}[0,b-a]} + M_2||p_2||_{W^{1,1}[0,b-a]}\Big],$$

we get

$$||T|| < |\lambda|.$$

Therefore, according to theorem 1.2.11, we have $sp(T) \subseteq \overline{B(0,||T||)}$, where B is the ball center 0 and with radius $||T||$. Therefore,

$$\Big\{\lambda \in \mathbb{C}^*(\mathbb{R}^*), |\lambda| > 2\big[M_1||p_1||_{W^{1,1}[0,b-a]} + M_2||p_2||_{W^{1,1}[0,b-a]}\big]\Big\} \subseteq re(T).$$

By Neumann's theorem [5, 2, 89], $(\lambda I - T)^{-1}$ and

$$||(\lambda I - T)^{-1}|| \leq \frac{1}{|\lambda| - 2\big[M_1||p_1||_{W^{1,1}[0,b-a]} + M_2||p_2||_{W^{1,1}[0,b-a]}\big]}.$$

This proves that the equation (3.1) has a unique solution. □

In the numerical treatment, we start by the construction an approximate method based on product integration.

3.2 From the equation to the system

In the analysis of convergence of the product integration method, we need to prove some proprieties on the linear operator T. So, we transform our operator T to a matrix of block linear operators. We put $X = C^0[a,b]$ and $\tilde{X}_2 = X \times X$.
Let defined the projection operator P and the injection J by

$$P: \quad \tilde{X}_2 \longrightarrow C^1[a,b]$$
$$(v_1, v_2) \longmapsto P(v_1, v_2) = Pv_1,$$

$$J: \quad Y \longmapsto \tilde{X}$$
$$v \longmapsto J(v) = (v_1, v_2),$$

We denote the function $u^{(p-1)}$ by u_p for $p = 1, 2$. Then,

$$T = PA_T J,$$

with A_T is a block operator matrix defined on $\tilde{X}_2$ into itself by

$$A_T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix},$$

such that, for $r = 1, 2$

$$\forall x \in [a,b],\ \forall v \in X,\ T_{1r}v(x) = \int_a^b p_r(|x-t|)K_r(x,t)v(t)\,dt,$$

$$\forall x \in [a,b],\ \forall v \in X,\ T_{2r}v(x) = \int_a^b p_r(|x-t|)\frac{\partial K_r}{\partial x}(x,t)v(t)\,dt + \int_a^b sign(x-t)p'(|r-t|)K_r(x,t)v(t)\,dt,$$

and the norm of A_T is defined by

$$|||A_T||| = \max_{1\le r\le 2}\sum_{l=1}^{2}||T_{lr}||.$$

Finally, the equation (3.3) has an equivalent form is given by

$$\lambda U = A_T U + F, \tag{3.4}$$

where $U = (u_1, u_2)$ et $F = (f_1, f_2)$.

3.3 Product Integration Method

Let $n \ge 1$. We recall that Δ_n is a discretization of the interval $[a,b]$, which is given by (1.15). By using the product trapezoidal rule 1, we get for all $t \in [x_{j-1}, x_j]$, $u_r \in X$ and $r = 1, 2$,

$$[K_r(x,t)u_r(t)]_n = \frac{1}{h}(x_j - t)K_r(x, x_{j-1})u_r(x_{j-1}) + \frac{1}{h}(t - x_{j-1})K_r(x, x_{j-1})u_r(x_j),$$

$$\left[\frac{\partial K_r}{\partial x}(x,t)u_r(t)\right]_n = \frac{1}{h}(x_j - t)\frac{\partial K_r}{\partial x}(x, x_{j-1})u_r(x_{j-1}) + \frac{1}{h}(t - x_{j-1})\frac{\partial K_r}{\partial x}(x, x_{j-1})u_r(x_j).$$

Then, we can introduce the approximate operators for all $v \in X$, $x \in [a,b]$ and $r = 1,2$ as

$$\begin{aligned} T_{1r,n}v(x) &= \sum_{j=0}^{n} w_{r,j}(x)K_r(x,x_j)v(x_j), \\ T_{2r,n}v(x) &= \sum_{j=0}^{n} w_{r,j}(x)\frac{\partial K_r}{\partial x}(x,x_j)v(x_j) + \sum_{j=0}^{n} \bar{w}_{r,j}(x) sign(x-x_j)K_r(x,x_j)v(x_j), \end{aligned}$$

with,

$$\begin{cases} w_{r,0} &= \dfrac{1}{h}\displaystyle\int_{x_0}^{x_1} (x_1 - t)p_r(|x-t|)\,dt, \\ w_{r,n} &= \dfrac{1}{h}\displaystyle\int_{x_{n-1}}^{x_n} (t - x_{n-1})p_r(|x-t|)\,dt, \\ w_{r,j} &= \dfrac{1}{h}\displaystyle\int_{x_j}^{x_{j+1}} (x_j - t)p_r(|x-t|)\,dt + \dfrac{1}{h}\displaystyle\int_{x_{j-1}}^{x_j} (t - x_{j-1})p_r(|x-t|)\,dt,\ 1 \le j \le n-1, \end{cases}$$

and

$$\begin{cases} \bar{w}_{r,0} &= \dfrac{1}{h}\displaystyle\int_{x_0}^{x_1} (x_1 - t)p'_r(|x-t|)\,dt, \\ \bar{w}_{r,n} &= \dfrac{1}{h}\displaystyle\int_{x_{n-1}}^{x_n} (t - x_{n-1})p'_r(|x-t|)\,dt, \\ \bar{w}_{r,j} &= \dfrac{1}{h}\displaystyle\int_{x_j}^{x_{j+1}} (x_j - t)p'_r(|x-t|)\,dt + \dfrac{1}{h}\displaystyle\int_{x_{j-1}}^{x_j} (t - x_{j-1})p_r(|x-t|)\,dt,\ 1 \le j \le n-1. \end{cases}$$

So, we get an approximation system of the following form

$$\begin{cases} \lambda u_{1,n}(x) &= \sum\limits_{j=0}^{n} w_{r,j}(x)K_1(x,x_j)u_{1,n}(x_j) + \sum\limits_{j=0}^{n} w_{r,j}(x)K_2(x,x_j)u_{2,n}(x_j), \\ \lambda u_{2,n}(x) &= \sum\limits_{j=0}^{n} w_{r,j}(x)\dfrac{\partial K_1}{\partial x}(x,x_j)u_{1,n}(x_j) + \sum\limits_{j=0}^{n} \bar{w}_{r,j}(x) sign(x-x_j)K_1(x,x_j)u_{1,n}(x_j) + f_1(x) \\ & \quad + \sum\limits_{j=0}^{n} w_{r,j}(x)\dfrac{\partial K_1}{\partial x}(x,x_j)u_{1,n}(x_j) + \sum\limits_{j=0}^{n} \bar{w}_{r,j}(x) sign(x-x_j)K_2(x,x_j)u_{2,n}(x_j) + f_2(x), \end{cases}$$

By using the collocation points $x = x_i$ for $i = 0,1,\dots n$, we get the algebraic system:

$$\begin{cases} \lambda u_{1,n}(x_i) &= \sum\limits_{j=0}^{n} w_{r,j}(x_i)K_1(x_i,x_j)u_{1,n}(x_j) + \sum\limits_{j=0}^{n} w_{r,j}(x_i)K_2(x_i,x_j)u_{2,n}(x_j) + f_1(x_i), \\ \lambda u_{2,n}(x_i) &= \sum\limits_{j=0}^{n} w_{r,j}(x_i)\dfrac{\partial K_1}{\partial x}(x_i,x_j)u_{1,n}(x_j) + \sum\limits_{j=0}^{n} \bar{w}_{r,j}(x_i) sign(x_i-x_j)K_1(x_i,x_j)u_{1,n}(x_j) \\ & \quad + \sum\limits_{j=0}^{n} w_{r,j}(x_i)\dfrac{\partial K_1}{\partial x}(x_i,x_j)u_{1,n}(x_j) + \sum\limits_{j=0}^{n} \bar{w}_{r,j}(x_i) sign(x-x_j)K_2(x_i,x_j)u_{2,n}(x_j) + f_2(x_i), \end{cases}$$

which is equivalent to the next matrix equation

$$\lambda U_n = A_{T_n} U_n + F, \tag{3.5}$$

where $U_n = (u_{1,n}, u_{2,n})$ and

$$A_{T_n} = \begin{pmatrix} T_{11,n} & T_{12,n} \\ T_{21,n} & T_{22,n} \end{pmatrix}.$$

Convergence Analysis

The convergence analysis of the product integration method is based on essential step. So, we start by presenting a theorem and a lemma. We follow all the steps that are contained in it.

To show the following convergence

$$||U - U_n||_{\tilde{X}_2} \to 0, \text{ when } n \to \infty,$$

we will demonstrate that A_{T_n} verifies the following steps in order to apply theorems 1.1.8 and 1.1.7.

Let A_T be a block operator matrix defined on the Banach space $\tilde{X}_2$ into itself and $\{A_{T_n}\}_{n\geq 1}$ be a sequence of an approximation operator A_T. Then,

(i) The operators A_T and A_{T_n} are linear (clear),

(ii) The block matrix operator A_{T_n}, for $n \geq 1$, verifies this pointwise convergence

$$\forall U \in \tilde{X}_2, \quad \lim_{n\to\infty} ||A_T U - A_{T_n} U||_{\tilde{X}_2} = 0.$$

(iii) The set $\{A_{T_n}, n \geq 1\}$ is collectively compact, which means that the set

$$S = \left\{ A_{T_n} U, n \geq 1, ||U||_{\tilde{X}_2} \leq 1 \right\},$$

has a compact closure in $\tilde{X}_2$.

We present the following theorem and on which we are going to use the above mentioned steps.

Theorem 3.3.1. *Let $\{T_{lr,n}\}_{1\leq l,r\leq 2}$ be an approximation operator of $\{T_{lr}\}_{1\leq l,r\leq 2}$. Then, the sets*

$$\left\{ T_{lr,n} u_r,\ u_r \in C^0[a,b], ||u_r||_{C^0[a,b]} \leq 1,\ n \geq 1 \right\}_{1\leq l,r\leq 2},$$

are pointwise bounded and equicontinuous.

Proof. Let $u_r \in X$ and $x \in [a,b]$. For $r = 1, 2$, we have

$$|T_{1r,n} u_r(x)| \leq M_r ||p_r||_{L^1[0,b-a]} ||u_r||_X,$$

$$|T_{2r,n} u_r(x)| \leq M_r ||p'_r||_{L^1[0,b-a]} ||u_r||_X,$$

which gives,

$$||T_{lr,n}|| \leq M_r ||p_r||_{L^1[0,b-a]},$$

$$||T_{lr,n}|| \leq M_r ||p'_r||_{L^1[0,b-a]}.$$

This implies the uniform boundedness of $T_{lr,.}$. From the Banach-Steinaus theorem 1.1.3, $T_{lr,n}$ are point-wise bounded. Let prove now the equicontinuous. For every $\varepsilon > 0$ there exist, $\delta > 0$ such that for $x, y \in [a,b]$, $|x-t| < \delta$, we have

$$\begin{aligned} |T_{1r,n}u_r(x) - T_{1r,n}u_r(y)| &\leq \int_a^b |p_r(|x-t|)[K_r(x,t)u_r(t)]_n|\,dt - \int_a^b |p_r(|y-t|)[K_r(y,t)u_r(t)]_n\,dt, \\ &\leq \int_a^b |p_r(|x-t|) - p_r(|y-t|)|\,|[K_r(x,t)u_r(t)]_n|\,dt \\ &+ \int_a^b |p_r(|x-t|)|[\{K_r(x,t) - K_r(y,t)\}u_r(t)]_n|\,dt, \end{aligned}$$

for $n \geq 1$, $r = 1, 2$, we get

$$\begin{aligned} \int_a^b |p_r(|x-t|) - p_r(|y-t|)|\,|[K_r(x,t)u_r(t)]_n|\,dt &\leq M_r\kappa_1(p_r, h), \\ \int_a^b |p_r(|x-t|)|[\{K_r(x,t) - K_r(y,t)\}u_r(t)]_n|\,dt &\leq \max_{a\leq t\leq b}|K_r(y,t) - K_r(x,t)|\,||p_r||_{L^1[a,b]}. \end{aligned}$$

From the uniform continuity of K_r, for $r = 1, 2$. With the same process, we can prove that

$$|T_{2r,n}u_r(x) - T_{2r,n}u_r(y)| \leq \varepsilon.$$

Then, we get that the sets

$$\Big\{T_{lr,n}u_r,\ u_r \in X, ||u_r||_X \leq 1,\ n \geq 1\Big\}_{1\leq l,r\leq 2},$$

are equicontinuous. □

Lemma 3.3.1. *For $n \geq 1$, the set $\Big\{A_{T_n}, n \geq 1\Big\}$ is collectively compact.*

Proof. We show that the closure of

$$S = \Big\{A_{T_n}U,\ n \geq 1, ||U||_{\tilde{X}_2} \leq 1\Big\},$$

is compact. It is easy to show that

$$S = \prod_{l=1}^{2}\Big\{\sum_{r=1}^{2}T_{lr,n}u_r,\ n \geq 1,\ ||u_r||_X \leq 1\Big\}_{lr},$$

and by using theorem 3.3.1 and the Arzelà-Ascoli theorem, we deduce that

$$\overline{\Big\{\sum_{r=1}^{2}T_{lr,n}u_r,\ n \geq 1,\ ||u_r||_X \leq 1\Big\}_{lr}},$$

is compact. Also, we have

$$\overline{\prod_{l=1}^{2}\Big\{\sum_{r=1}^{2}T_{lr,n}u_r,\ n \geq 1,\ ||u_r||_X \leq 1\Big\}_{lr}} = \prod_{l=1}^{2}\overline{\Big\{\sum_{r=1}^{2}T_{lr,n}u_r,\ n \geq 1,\ ||u_r||_X \leq 1\Big\}_{lr}}.$$

Finally, the set S is a collectively compact. □

Theorem 3.3.2. *Let A_T be a block operator matrix and A_{T_n} be a block approximation operator. Then,*

$$\forall\, U\ \in \tilde{X}_2,\ \lim_{n\to\infty}||(A_T - A_{T_n})U||_{\tilde{X}_2} = 0.$$

Proof. For n large enough, we have

$$|T_{1r}u_r(x) - T_{1r,n}u_r(x)| \leq \int_a^b |p_r(x-t)|\ |[K_r(x,t)u_r(t)]_n - K_r(x,t)u_r(t)|\ dt,$$

such that for $n \geq$, $r = 1, 2$ and $j = 0, \dots n$

$$\begin{aligned}
\Big|[K_r(x,t)u_r(t)]_n - K_r(x,t)u_r(t)\Big| &= \Big|\frac{1}{h}(x_j - t)\Big[K_r(x,x_{j-1})u_r(x_{j-1}) - K_r(x,t)u_r(t)\Big] \\
&+ \frac{1}{h}(t - x_{j-1})\Big[K_r(x,x_j)u_r(x_j) - K_r(x,t)u_r(t)\Big]\Big|, \\
&\leq 2\Big[M_r\kappa_{\infty,0}(u_r,h) + ||\kappa_{\infty,0}(K_r,h)||_\infty ||u_r||_X\Big].
\end{aligned}$$

On the other side, we have

$$\begin{aligned}
|T_{2r}u_r(x) - T_{2r,n}u_r(x)| &\leq \int_a^b |p_r'(x-t)|\ |[K_r(x,t)u_r(t)]_n - K_r(x,t)u_r(t)|\ dt \\
&+ \int_a^b |p_r(x-t)|\ |[\frac{\partial K_r}{\partial x}(x,t)u_r(t)]_n - \frac{\partial K_r}{\partial x}(x,t)u_r(t)|\ dt,
\end{aligned}$$

such that

$$\begin{aligned}
\Big|[\frac{\partial K_r}{\partial x}(x,t)u_r(t)]_n - \frac{\partial K_r}{\partial x}(x,t)u_r(t)\Big| &\leq 2\Big[M_r\kappa_{\infty,0}(u_r,h) + ||\kappa_{\infty,0}(\partial x K_r,h)||_\infty\ ||u_r||_X\Big]||p_r||_{L^1[0,b-a]} \\
&+ 2\Big[M_r\kappa_{\infty,0}(u_r,h) + ||\kappa_{\infty,0}(K_r,h)||_\infty\ ||u_r||_X\Big]||p_r'||_{L^1[0,b-a]}, \\
&\leq \Big[4M_r\kappa_{\infty,0}(u_r,h) + 2||\kappa_{\infty,1}(K_r,h)||_\infty\ ||u_r||_X\Big]||p_r||_{W^{1,1}[0,b-a]}.
\end{aligned}$$

Finally,

$$\begin{aligned}
||(A_T - A_{T_n})U||_{\tilde{X}_2} &\leq 2\Big[M_1\Big(1 + 2||p_1||_{W^{1,1}[0,b-a]}\Big) + M_2\Big(1 + 2||p_2||_{W^{1,1}[0,b-a]}\Big)\Big]\kappa_{\infty,1}(u,h) \\
&+ 2\Big[||\kappa_{\infty,1}(K_1,h)||_\infty\Big(1 + 2||p_1||_{W^{1,1}[0,b-a]}\Big) \\
&+ ||\kappa_{\infty,1}(K_2,h)||_\infty\Big(1 + 2||p_2||_{W^{1,1}[0,b-a]}\Big)\Big]||U||_{\tilde{X}_2}.
\end{aligned}$$

When $n \to \infty$, we have $h \to 0$ and we get the result. □

From lemma 3.3.1 and theorem 3.3.2, we obtain that our block approximation matrix verifies all of properties (i)-(iii), then by lemma 3.3.1, we get

$$|||(A_{T_n} - A_T)A_T||| \to 0, \quad |||(A_{T_n} - A_T)A_{T_n}||| \to 0, \quad \text{when } n \to +\infty.$$

Theorem 3.3.3. *Assume that*

$$\lim_{n\to\infty} ||(A_T - A_{T_n})U||_{\tilde{X}_2} = 0,$$

and

$$|||(A_{T_n} - A_T)A_T||| \to 0, \quad |||(A_{T_n} - A_T)A_{T_n}||| \to 0, \quad \textit{when } n \to +\infty.$$

Then, for sufficiently large n, the approximate inverse $(\lambda I_2 - A_{T_n})^{-1}$ exists and is uniformly bounded. We also have

$$|||(\lambda I_2 - A_{T_n})^{-1}||| \leq \frac{1 + |||(\lambda I_2 - A_T)^{-1}||| \; |||A_{T_n}|||}{|\lambda| - |||(\lambda I_2 - A_T)^{-1}||| \; ||||(A_T - A_{T_n})A_{T_n}|||} < \infty.$$

Proof. See theorem 11.4.4 in [5]. □

Theorem 3.3.4. *Let $U = (u_1, u_2)$ be the solution of the system (3.4) and $U_n = (u_{1,n}, u_{2,n})$ is the solution of (3.5). Then,*

$$||U - U_n||_{\tilde{X}_2} = 0.$$

Proof. For n large enough, we have

$$\begin{aligned} \lambda(U - U_n) &= A_T U - A_{T_n} U_n, \\ &= A_T U - A_{T_n} U + A_{T_n} U - A_{T_n} U_n, \\ &= (A_T - A_{T_n})U + A_{T_n}(U - U_n). \end{aligned}$$

Then,

$$U - U_n = (\lambda I_2 - A_{T_n})^{-1} \Big[(A_T - A_{T_n})U \Big],$$

which gives

$$||U - U_n||_{\tilde{X}_2} \leq |||(\lambda I_2 - A_{T_n})^{-1}||| \; ||(A_T - A_{T_n})U||_{\tilde{X}_2},$$

and when $n \to \infty$, we get the result. □

3.4 Collocation cubic b-spline method

The purpose of creating a new method for solving this equation is not only to improve the approximate solution but also to try to build a fast method in solving it. In the latter method, we obtain an algebraic system of size $2n + 2$ equations and when n is large enough then the system size will be large and thus the solution spends more time in computing. The tool we use to solve this problem is to apply one of the projection methods but with a choice of differentiable and continuous basis. So we take the cubic b-splines which are functions of class C^2. The advantage of this method is that it eliminates the number of equations by half. Instead of solving a system of $2n+2$ equations, we solve a system of $n+1$ algebraic equations.
So we apply the collocation processes and we need to give the definition of cubic b-spline functions (see [7, 18, 22, 32, 44]).

Definition 3.4.1. *Let $\Delta_n = \{a = x_0 < x_1 < \ldots x_n = b\}$ be a uniform partition of interval $[a, b]$, with $x_i = a + ih$ and $h = \dfrac{b-a}{n}$. Let $\mathcal{B}_3(\Delta_n)$ be the space of cubic b-spline functions, given as follow:*

$$\mathcal{B}_3(\Delta_n) = \Big\{ S \in C^2[a, b] : S \mid_{[x_i, x_{i+1}]} \in P^3, \; i = 0, 1, \ldots n-1 \Big\},$$

where, $S \mid_{[x_i, x_{i+1}]}$ is the restriction of the spline function $S : [0, 1] \longrightarrow \mathbb{R}$ in each sub-interval $[x_i, x_{i+1}]$ and P_n^3 is the space of cubic polynomials. For $i = -1, 0, \ldots n, n+1$, we give the uniform cubic b-spline *from* (4.5)

$$B_i^3(x) = \frac{1}{h^3}\begin{cases} (x-x_{i-2})^3, & x \in [x_{i-2}, x_{i-1}], \\ (x-x_{i-2})^3 - 4(x-x_{i-1})^3, & x \in [x_{i-1}, x_i], \\ (x_{i+2}-x)^3 - 4(x_{i+1}-x)^3, & x \in [x_i, x_{i+1}], \\ (x_{i+2}-x)^3, & x \in [x_{i+1}, x_{i+2}], \\ 0, & \textit{otherwise}. \end{cases}$$

It is clear that $\{B_{-1}, B_0, B_1, \dots, B_{n-1}, B_n, B_{n+1}\}$ *forms a basis of* $\mathcal{B}_3(\Delta_n)$*. Then* $\mathcal{B}_3(\Delta_n)$ *is a finite dimensional linear sub-space of* $C^2[a,b]$ *with dimension* $n+3$*.*

We define $(P_n^3)_{n\in\mathbb{N}^*}$ the sequence of linear projection operator

$$\begin{aligned} P_n^3 : C^1[a,b] &\longrightarrow \mathcal{B}_3(\Delta_n) \\ v &\longmapsto P_n^3 v(x) = \sum_{i=-1}^{n+1} \alpha_i B_i^3(x). \end{aligned} \tag{3.6}$$

It satisfies the following interpolation condition for all $v \in C^1[a,b]$

$$\begin{cases} P_n^3 v(x_i) &= v(x_i), \quad i = 0, 1 \dots n, \\ (P_n^3 v)'(a) &= v'(a), \\ (P_n^3 v)'(b) &= v'(b). \end{cases}$$

We construct an approximation solution $u_n = P_n^3 u_n$, verifying the following equation

$$\lambda u_n = P_n^3 T u_n + P_n^3 f. \tag{3.7}$$

3.4.1 Convergence Analysis

To study the convergence of the method, we recall what is spline interpolation. We will go through steps, including the passage from writing as a linear combination to writing using spline interpolation. The proof that converge to the identity operator, and that our approximate operator converges in norm.Finally the use of the latter to prove that the exact solution converges in the space $C^1[a,b]$.

For the first time, it will be demonstrated that $P_n^3 u$ convergence to u in the sense of the norm $C^1[a,b]$ especially since a lot of researches like [8, 49, 63],[33, 52] and [74] have shown this convergence, provided that is u is defined in $C^4[a,b]$, $u \in C^0[-1,1]$ [80, 4] or it is necessary that u is periodic function [6]. Moreover, to prove this convergence, we recall the spline interpolation of u on the grid $\Delta_n, n \geq 1$ which denoted by $S_n(x)$(see [32, 33, 49, 52]) and defined as: In each interval $[x_{i-1}, x_i]$ the spline $S_n(x)$ has this formula which based on the moments $M_i = S''(x_i)$

$$\begin{aligned} S_n(x) &= M_{i-1}\frac{(x_i - x)^3}{6h} + M_i\frac{(x - x_{i-1})^3}{6h} + \left(u(x_{i-1}) - \frac{M_{i-1}h^2}{6}\right)\left(\frac{x_i - x}{h}\right) \\ &+ \left(u(x_i) - \frac{M_i h^2}{6}\right)\left(\frac{x - x_{i-1}}{h}\right). \end{aligned} \tag{3.8}$$

However, any cubic spline $S_n(x)$ constructed on segment $[a,b]$ is described on the linear combination of cubic b-spline [61, **?**, 80]. Then, $S_n(x)$ has the following form

$$\forall x \in [a,b], \; S_n(x) = P_n^3 u(x).$$

This implies that:

$$\|(I - P_n^3)u\|_{C^1[a,b]} = \|u - S_n\|_{C^1[a,b]}.$$

We return to the spline form (3.8) and for the left and right limit of the point x_i

$$\begin{aligned} S_n'(x_i^-) &= \frac{h}{6}M_{i-1} + \frac{h}{3}M_i + \frac{u(x_i) - u(x_{i-1})}{h}, \\ S_n'(x_i^+) &= -\frac{h}{3}M_i - \frac{h}{6}M_{i+1} + \frac{u(x_{i+1}) - u(x_i)}{h}. \end{aligned}$$

The continuity of $S_n'(x)$ at x_j yields for $i = 1, 2, \ldots n-1$

$$M_{i-1} + 4M_i + M_{i+1} = 6\left(\frac{u(x_{i+1}) - u(x_i)}{h^2} - \frac{u(x_i) - u(x_{i-1})}{h^2}\right). \tag{3.9}$$

But, in many applications, it is more convenient to work with the slopes $m_i = S_n'(x_i)$ rather than the moments M_i. Here, we present another formula of S_n in each segment $[x_{i-1}, x_i]$

$$\begin{aligned} S_n(x) &= m_{i-1}\frac{(x_i - x)^2(x - x_{i-1})}{h^2} - m_i\frac{(x - x_{i-1})^2(x_i - x)}{h^2} \\ &+ u(x_{i-1})\frac{(x_i - x)^2[2(x - x_{i-1}) + h]}{h^2} \\ &+ u(x_i)\frac{(x - x_{i-1})^2[2(x_i - x) + h]}{h^2}, \\ S_n'(x) &= m_{i-1}\frac{(x_i - x)(2x_{i-1} + x_i - 3x)}{h^2} - m_i\frac{(x - x_{i-1})(2x_i + x_{i-1} - 3x)}{h^2} \\ &+ \frac{u(x_i) - u(x_{i-1})}{h^3}6(x_i - x)(x - x_{i-1}). \end{aligned} \tag{3.10}$$

The limit values of second derivative at x_i are given as

$$\begin{aligned} S''(x_i^-) &= \frac{2}{h}m_{i-1} + \frac{4}{h}m_i - 6\frac{u(x_i) - u(x_{i-1})}{h^2}, \\ S''(x_i^+) &= \frac{-4}{h}m_i - \frac{2}{h}m_{i+1} + 6\frac{u(x_{i+1} - u(x_i)}{h^2}. \end{aligned}$$

For results, we require

$$m_{i-1} + 4m_i + m_{i+1} = 3\frac{u(x_{i+1}) - u(x_i)}{h}, \quad i = 1, 2, \ldots, n-1, \tag{3.11}$$

with the end conditions

$$\begin{cases} m_0 = u'(x_0), \\ m_n = u'(x_n). \end{cases}$$

Then, system (3.9) has the form $CM = D$ and system (3.11) has the form $Cm = E$, where

$$C = \begin{bmatrix} 2 & 1 & \dots & \dots & \dots & \dots & 0 \\ 1 & 4 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 4 & 1 & 0 & \dots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 1 & 4 & 1 \\ 0 & \dots & \dots & \dots & \dots & 1 & 2 \end{bmatrix},$$

$m = (m_0, m_1, \dots m_n)^t \in \mathbb{R}^{n+1}$, $M = (M_0, M_1, \dots, M_n)^t \in \mathbb{R}^{n+1}$, $D = (d_0, d_1, \dots d_n)^t \in \mathbb{R}^{n+1}$ such that d_i are given by

$$\begin{cases} d_0 &= 3\dfrac{u(x_1) - u(x_0)}{h}, \\ d_i &= 3\dfrac{u(x_i) - u(x_{i-1})}{h^2} + 3\dfrac{u(x_{i+1}) - u(x_i)}{h^2}, \quad i = 1, 2, \dots n-1, \\ d_n &= 3\dfrac{u(x_{n-1}) - u(x_n)}{h}, \end{cases}$$

and the vector $E = (e_0, e_1, \dots e_n)^t \in \mathbb{R}^{n+1}$ have the following elements

$$\begin{cases} e_0 &= \dfrac{6}{h}\left(\dfrac{u(x_1) - u(x_0)}{h} - u'(x_0)\right), \\ e_i &= 6\left(\dfrac{u(x_{i+1}) - u(x_i)}{h^2} - \dfrac{u(x_i) - u(x_{i-1})}{h^2}\right), \ i = 1, 2, \dots n-1, \\ e_n &= \dfrac{6}{h}\left(u'(x_n) - \dfrac{u(x_n) - u(x_{n-1})}{h}\right). \end{cases}$$

Theorem 3.4.1. *Let $P_n^3 u$ defined by (3.6) and u the exact solution of (3.1). Then,*

$$\|u - P_n^3 u\|_{C^1[a,b]} \leq c\, \kappa_1(u, h),$$

where c is a positive constant.

Proof. On each interval $[x_{i-1}, x_i]$ and we use the spline form (3.8), we get

$$
\begin{aligned}
P_n^3 u(x) - u(x) &= S_n(x) - u(x) \\
&= M_{i-1}\frac{(x_{i-1}-x)^3}{6h} + M_i\frac{(x-x_{i-1})^3}{6h} + \left(u(x_{i-1} - \frac{M_{i-1}h^2}{6}\right)\left(\frac{x_i - x}{h}\right) \\
&+ \left(u(x_i) - \frac{M_i h^2}{6}\right)\left(\frac{x-x_{i-1}}{h}\right) - f(x), \\
&= \frac{M_{i-1}}{6h}(x_i - x)^3 - \frac{M_{i-1}}{6h}h^2(x_i - x) + \frac{M_i}{6h}(x - x_i - 1)^3 - \frac{M_i}{6h}h^2(x - x_{i-1}) \\
&+ u(x_{i-1})\left(\frac{x_i - x}{h}\right) + u(x_i)\left(\frac{x - x_{i-1}}{h}\right) - f(x), \\
&= M_{i-1}\frac{(x_i - x)[(x_i - x)^2 - h^2}{6h} + M_i\frac{(x - x_{i-1})[(x - x_{i-1})^2 - h^2]}{6h} \\
&+ \left(\frac{u(x_i) - u(x_{i-1})}{2} - f(x)\right) - \left(u(x_i) - u(x_{i-1})\right)\frac{x_i - x_{i-1} - 2x}{2h}, \\
&= \frac{(x - x_i)(x - x_{i-1})}{6h}\Big[(2x_i - x_{i-1} - x)M_{i-1} + (x - 2x_{i-1} + x_i)M_i\Big] \\
&+ \left(\frac{u(x_i) + u(x_{i-1})}{2} - f(x)\right) - \left(u(x_i) - u(x_{i-1})\right)\frac{x_i + x_{i-1} - 2x}{2h}.
\end{aligned}
$$

We have

$$
\begin{aligned}
(x_i - x)(x - x_{i-1}) &\leq h^2, \\
(2x_i - x_{i-1} - x) &\leq 2h, \\
(x - 2x_{i-1} + x_i) &\leq 2h, \\
\mid x_i + x_{i-1} - 2x \mid &\leq h.
\end{aligned}
$$

Then, we get

$$
\begin{aligned}
\mid S_n(x) - u(x) \mid &\leq \frac{h^2}{3}\Big[\mid M_{i-1} \mid + \mid M_i \mid\Big] + \mid \frac{u(x_i) - u(x_{i-1})}{2} - u(x) \mid \\
&+ \frac{\mid u(x_i) + u(x_{i-1}) \mid}{2},
\end{aligned}
$$

this implies

$$
\|S_n - u\|_\infty \leq \frac{2h^2}{3}\|M\|_{\mathbb{R}^{n+1}} + \frac{3}{2}\,\kappa_0(u, h).
$$

On the other side, we have $M = C^{-1}D$, then $\|M\|_{\mathbb{R}^{n+1}} \leq ||| C^{-1} ||| \ \|D\|_{\mathbb{R}^{n+1}}$, where

$$
\begin{aligned}
\|D\|_{\mathbb{R}^{n+1}} &= \max_{0\leq i\leq n} \mid d_i \mid \\
&= \frac{6}{h^2}\max_{0\leq i\leq n} \mid \frac{u(x_{i+1}) - u(x_i)}{h} - \frac{u(x_i) - u(x_{i-1})}{h} \mid, \\
&\leq \frac{3}{h^2}\,\kappa_0(u, h).
\end{aligned}
$$

Finally,

$$\|S_n - u\|_\infty \le \left(2 \,|||\, C^{-1} \,|||\, + \frac{3}{2}\right)\kappa_0(u,h).$$

Now, from the second interpolation formula (3.10) in each segment $[x_{i-1}, x_i]$, we have

$$\begin{aligned} S_n'(x) - \frac{u(x_i) - u(x_{i-1})}{h} &= m_{i-1}\frac{(x_i - x)(2x_{i-1} + x_i - 3x)}{h^2} - m_i\frac{(x - x_{i-1})(2x_i + x_i - 3x}{h^2} \\ &+ \frac{u(x_i) - u(x_{i-1})}{h}6(x - x_{i-1})(x_i - x) - \frac{u(x_i) - u(x_{i-1})}{h}, \end{aligned}$$

with,

$$\begin{aligned} 6(x - x_{i-1})(x_i - x) &= \frac{6}{h^2}\left(\frac{x - x_{i-1}}{2}\right)^2 - \frac{1}{2}, \\ (x_i - x)(2x_{i-1} + x_i - 3x) &= \frac{3}{h^2}\left(x - \frac{x_{i-1} + x_i}{2}\right)^2 - \frac{1}{4} + \frac{1}{h}\left(x - \frac{x_i + x_{i-1}}{2}\right). \end{aligned}$$

So that

$$\begin{aligned} S_n'(x) - \frac{u(x_i) - u(x_{i-1})}{h} &= \left[\frac{3}{h^2}\left(x - \frac{x_{i-1} + x_i}{2}\right)^2 - \frac{1}{4}\right]\Bigg[(m_{i-1} - u'(x_{i-1}) \\ &+ (m_i - u'(x_i) - 2\left(\frac{u(x_i) - u(x_{i-1})}{h}\right)\Bigg] \\ &+ \frac{1}{h}\left(x - \frac{x_i + x_{i-1}}{2}\right)\Bigg[m_i - u'(x_i) + m_{i-1} \\ &- u'(x_{i-1} + u'(x_i) + u'(x_{i-1})\Bigg]. \end{aligned}$$

Then,

$$\begin{aligned} \mid S_n'(x) - \frac{u(x_i) - u(x_{i-1})}{h} \mid &\le \frac{1}{2}\Big[\mid m_{i-1} - u'(x_{i-1}) \mid + \mid m_i - u'(x_i) \mid + 2\,\kappa_0(u', h)\Big] \\ &+ \mid m_i - u'(x_i) \mid + \mid m_{i-1} - u'(x_{i-1}) \mid + \kappa_0(u', h). \end{aligned} \tag{3.12}$$

We denote $U = (u(x_0), u(x_1), \ldots u(x_n))$ and $U_1 = (u'(x_0), u'(x_1), \ldots u'(x_n))$. In other way, we have

$$\|m - U_1\|_{\mathbb{R}^{n+1}} \le \|\, m - \frac{1}{6}E \,\|_{\mathbb{R}^{n+1}} + \|\, \frac{E}{6} + U_1 \,\|_{\mathbb{R}^{n+1}}\ . \tag{3.13}$$

Also, we have

$$C\left(m - \frac{E}{6}\right) = \left(I_{n+1} - \frac{C}{6}\right)E,$$

where, I_{n+1} is the identity matrix of size $(n+1)\times(n+1)$. We have

$$\left(I_{n+1}-\frac{1}{6}C\right)E=\begin{bmatrix}\dfrac{e_0}{3}-\dfrac{e_1}{6}\\ \\ \dfrac{-e_0}{6}+\dfrac{e_1}{3}-\dfrac{e_2}{6}\\ \vdots\\ \vdots\\ \dfrac{-e_{n-2}}{6}+\dfrac{e_{n-1}}{3}-\dfrac{e_n}{6}\\ \dfrac{-e_{n-1}}{6}+\dfrac{e_n}{3}\end{bmatrix},$$

with the end and the first conditions, we get

$$\begin{cases} \mid \dfrac{e_0}{3}-\dfrac{e_1}{6}\mid & \leq 3\ \kappa_0(u',h),\\ \mid \dfrac{-e_{i-1}}{6}+\dfrac{e_i}{3}-\dfrac{e_{i+1}}{6}\mid & \leq 6\ \kappa_0(u',h), \quad i=1,2,\dots,n-1,\\ \mid \dfrac{-e_{n-1}}{6}+\dfrac{e_n}{3}\mid & \leq 3\ \kappa_0(u',h),\end{cases}$$

which gives $\|E\|_{\mathbb{R}^{n+1}}\leq 6\ \kappa_0(u',h)$. Then, from (3.13)

$$\|m-\frac{E}{6}\|_{\mathbb{R}^{n+1}}\leq 6\ |||\ C^{-1}\ |||\ \ \kappa_0(u',h), \tag{3.14}$$

Substituting (3.14) in (3.12)

$$\mid S_n'(x)-\frac{u(x_i)-u(x_{i-1})}{h}\mid\leq\left(11\ |||\ C^{-1}\ |||\ +1\right)\kappa_0(u',h).$$

and therefore,

$$\mid u'(x)-\frac{u(x_i)-u(x_{i-1})}{h}\mid\leq\kappa_0(u',h).$$

We get

$$\begin{aligned}\|S_n'-u'\|_\infty &\leq \max_{a\leq x\leq b}\mid S_n'(x)-\frac{u(x_i)-u(x_{i-1})}{h}\mid+\max_{a\leq x\leq b}\mid\frac{u(x_i)-u(x_{i-1})}{h}-u'(x)\mid,\\ &\leq \left(11\ \ |||\ C^{-1}\ |||\ +2\right)\kappa_0(u',h). \end{aligned}\tag{3.15}$$

Then, we obtain from (3.12) and (3.15):

$$\|u-P_n^3u\|_{C^1[a,b]}=\|u-S_n\|_{C^1[a,b]}\leq c\ \kappa_1(u,h),$$

where $c=\max\left(11\ |||\ C^{-1}\ |||\ +1, 2\ |||\ C^{-1}\ |||\ +\dfrac{3}{2}\right)$. □

Theorem 3.4.2. *Let P_n^3 be a projection operator given by (3.6) and T is a compact operator defined by* (3.3). *Then,*

$$\lim_{n\to\infty}\|(I-P_n^3)T\|=0. \tag{3.16}$$

Proof. Since T is compact the set

$$\mathcal{M} = \left\{ Tu, u \in C^1[a,b], \|u\|_{C^1[a,b]} \leq 1 \right\}$$

is relatively compact in the Banach space $C^1[a,b]$ and by Banach-Steinhaus theorem [5], P_n^3 converges uniformly to the identity operator I in $C^1[a,b]$

$$\lim_{n\to\infty} \|(I - P_n^3)T\| = \lim_{n\to\infty} \sup_{\|u\|_{C^1[a,b]}} \|(I - P_n^3)Tu\|_{C^1[a,b]} = \sup_{v\in\mathcal{M}} \|(I - P_n^3)v\|_{C^1[a,b]}.$$

□

Under above theorems, we get P_n^3 is pointwise convergent to the identity operator and $P_n^3 T$ convergence to T. Then, $(\lambda I - P_n^3 T)^{-1}$ exists and $\|(\lambda I - P_n^3 T)^{-1}\| < \infty$ (see [81]).

Theorem 3.4.3. *Let u_n be the solution of (3.12) and u be the exact solution of (3.1). Then,*

$$\lim_{n\to\infty} \|u - u_n\|_{C^1[a,b]} = 0$$

Proof. For n enough large, $u - u_n = (\lambda I - P_n^3 T)^{-1}\Big[(T - P_n^3 T)u + (f - P_n^3 f)\Big]$ and we obtain the following estimation

$$\|u - u_n\|_{C^1[a,b]} \leq \|(\lambda I - P_n^3 T)^{-1}\| \Big[\|(I - P_n^3)T\| + \|(I - P_n^3)f\|_{C^1[a,b]}\Big].$$

When $n \to \infty$, we get the result. □

3.4.2 System Approximation

From the equation (3.12), we have for all $x \in [a,b]$

$$\lambda u_n(x) = P_n^3 T u_n(x) + P_n^3 f(x).$$

We put $x = x_j$ for $j = 0, 1, \dots n$, then by the conditions interpolation of the sequence $(P_n^3)_{n\in\mathbb{N}}$ (3.4), we get

$$\lambda u_n(x_j) = T u_n(x_j) + f(x_j),$$

which is equivalent to the following algebraic system for $j = 0, 1, \dots n$

$$\begin{aligned} \sum_{i=-1}^{n+1} & \alpha_i \Big[\lambda B_i^3(x_j) - \int_a^b p_1(|x_j - t|) K_1(x_j,t) B_i^3(t)\, dt - \int_a^b p_2(|\, x_j - t\,|)\, K_2(x_j,t) {B_i^3}'(t)\, dt\Big] \\ = & \ f(x_j), \end{aligned} \tag{3.17}$$

where $\{\alpha_i\}_{i=-1}^{n+1}$ are unknowns coefficients to be determined.

But we have a problem with the system (3.17) which has $n+1$ equations with $n+3$ unknowns α_i. For this reason, we introduce $\overline{B^3}_i(x)$ be a modified Cubic b-splines basis [31]

$$\begin{cases} \overline{B^3}_0(x) = & B_0^3(x) + 2B_{-1}^3(x), & \text{for } j = 0, \\ \overline{B^3}_1(x) = & B_1^3(x) - B_{-1}^3(x), & \text{for } j = 1, \\ \overline{B^3}_j(x) = & B_j^3(x), & \text{for } j = 2, \dots n-2, \\ \overline{B^3}_{n-1}(x) = & B_{n-1}^3(x) - B_{n+1}^3(x), & \text{for } j = n-1, \\ \overline{B^3}_n(x) = & B_n^3(x) + 2B_{n+1}^3(x), & \text{for } j = n. \end{cases}$$

In the next table, we present the value of $\overline{B^3}_i(x_j)$ and its derivatives

Table 3.1: Value of modified b-cubic spline and their derivatives

x_j	$j=0$	$j=i-2$	$j=i-1$	$j=i$	$j=i+1$	$j=i+2$	$j=n$
$\overline{B^3}_i(x_j)$	6	0	1	4	1	0	6
$\overline{B^3}'_i(x_j)$	$\frac{-6}{h}$	0	$\frac{-3}{h}$	0	$\frac{3}{h}$	0	$\frac{6}{h}$

Finally, the solution u_n has a new representation

$$\forall x \in [a,b], \; u_n(x) = \sum_{i=0}^{n} \overline{\alpha}_i \overline{B}_i^3(x),$$

and the system (3.17) has a new form given by

$$\begin{aligned} \sum_{i=0}^{n} \quad & \overline{\alpha_i} \left[\lambda \overline{B_i}^3(x_j) - \int_a^b p_1(|x_j - t|) K_1(x_j,t) \overline{B_i}^3(t)\, dt - \int_a^b p_2(|\, x_j - t\,|)\; K_2(x_j,t) \overline{B_i^3}'(t)\, dt \right] \\ = \quad & f(x_j), \end{aligned}$$

where $\{\overline{\alpha}_i\}_{i=0}^n$ are a new coefficients to be deteminated.

3.5 Numerical tests

To illustrate the efficiency of our numerical processes and to compare between the both methods, we give two numerical examples. The first is defined as

3.5.1 Test 01

Consider the following integro-differential equation

$$7u(x) = \int_0^1 \sqrt{|x-t|} \frac{e^x u(t)}{t+1}\, dt + \int_0^1 |x-t|^{\frac{1}{3}} u'(t)\, dt + f(x), \tag{3.18}$$

where,

$$f(x) = 7x^2 + \frac{(9x+12)(1-x)^{\frac{4}{3}}}{14} + \frac{8x^{\frac{7}{2}}e^x}{105} - \frac{9x^{\frac{7}{3}}}{14} - e^x(1-x)^{\frac{3}{2}}\frac{8x^2+12x+15}{105}.$$

We have

$$P_1(|x-t|) = \sqrt{|x-t|},$$
$$P_2(|x-t|) = |x-t|^{\frac{1}{3}},$$

and

$$K_1(x,t) = \frac{e^x}{t+1},$$
$$K_2(|x-t|) = 1.$$

Moreover, $M_1 = e^1$ and $M_2 = 1$. In addition, $7 > e^1||P_1||_{W^{1,1}[0,b-a]} + ||P_2||_{W^{1,1}[0,b-a]}$, then the equation (4.9) has a unique solution.

The next table present the error discrete between the exact and approximate solution which given by

$$err_n = \max_{0\leq i\leq n}|u_n(x_i) - u(x_i)|.$$

Table 3.2: The error between the exact and approximation solution of equation (4.9)

n	Product integration	time(seconds)	Collocation cubic b-spline	time(seconds)
10	2.4205e-04	0.985059	4.7601e-05	0.620084
50	9.6603e-06	14.386013	8.5975e-07	7.456484
100	2.4147e-06	51.016760	1 .6386e-07	27.258581

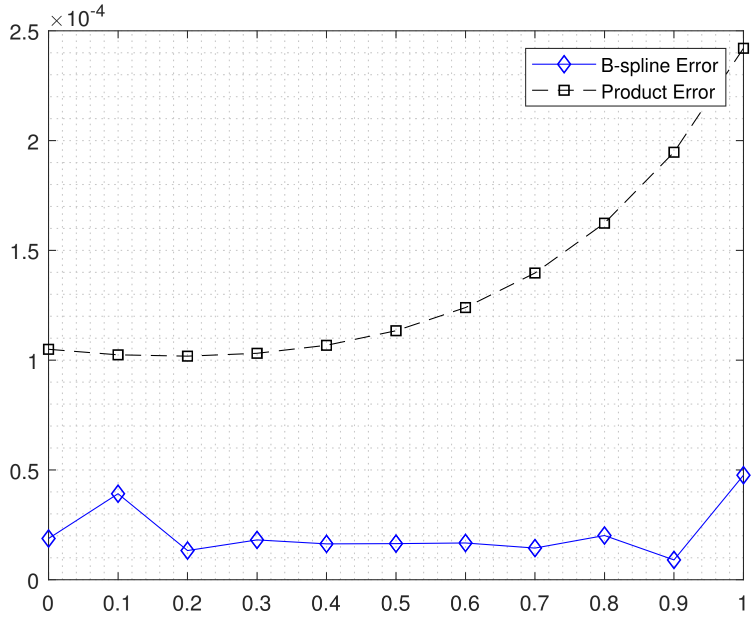


Figure 3.1: Errors of B-spline and Product integration method (4.9) n=10.

3.5.2 Test 02

Consider the following integro-differential equation

$$u(x) = \int_0^{10} |x-t|^{\frac{1}{3}} u(t)\, dt + \int_0^{10} \sqrt{|x-t|} u'(t)\, dt + f(x), \tag{3.19}$$

where,

$$f(x) = x - \frac{3}{28}(3x-10)(10-x)^{\frac{4}{3}} - \frac{2}{3}(10-x)^{\frac{3}{2}} + \frac{2}{3}x^{\frac{3}{2}} + \frac{9}{28}x^{\frac{7}{3}}.$$

We have

$$P_1(|x-t|) = |x-t|^{\frac{1}{3}},$$

$$P_2(|x-t|) = \sqrt{|x-t|},$$

and

$$K_1(x,t) = 1,$$

$$K_2(|x-t|) = 1.$$

Moreover, $M_1 = 1$ and $M_2 = 1$. In addition, $11 > ||P_1||_{W^{1,1}[0,b-a]} + ||P_2||_{W^{1,1}[0,b-a]}$, then the equation (4.9) has a unique solution.

Table 3.3: The error between the exact and approximation solution of equation (4.4)

n	Product integration	time(seconds)	Collocation cubic b-spline	time(seconds)
50	1.9077e-07	14.038054	5.5756e-11	7.198711
100	1.8224e-07	57.896836	2.2340e-11	33.793924
200	4.5985e-08	213.576772	8.7540e-12	116.348824

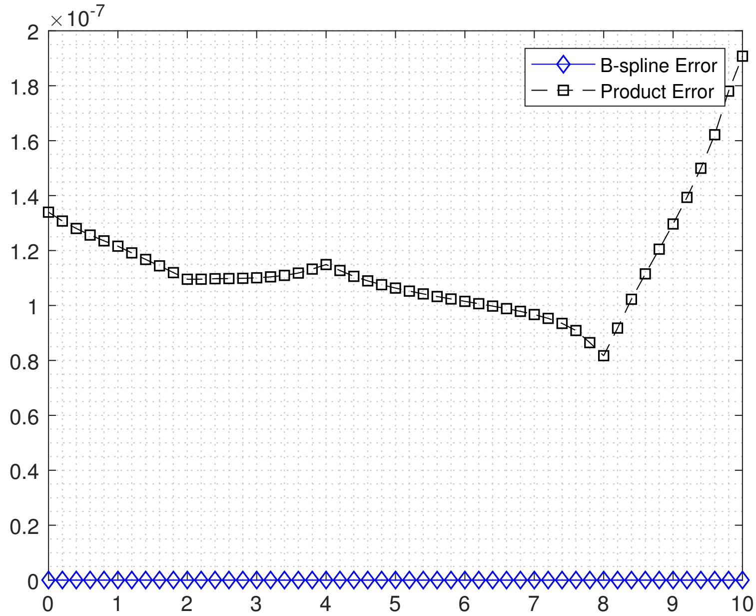


Figure 3.2: Error of B-spline and Product integration method (4.4) n=50.

Chapter 4

Analytical and Numerical Study of Linear Fredholm Integro-differential Equation In Sobolev Spaces

Contents

In this chapter, we study the existence and uniqueness of the solution of a linear integro-differential Fredholm equation (2.1) in the most complex and weak Banach spaces, which are Sobolev spaces. We suppose a theory that proves the existence of the solution under certain conditions, which allow us later to be in harmony with our two analytical and numerical studies. This theory is well detailed in section 4.1. In this chapter, we are going to construct two solutions, each based on two Projection methods: Galerkin and Kantorovich which are well explained in section 4.2. We present some theorems through which we will prove the convergence of both solutions to the exact solution, but only in the sense of the norm of two spaces $H^1[a,b]$ and $W^{1,1}[a,b]$. This what we present in sections 4.3 and 4.4. Finally, we give a numerical examples 4.5 to see the error computation of the Galerkin and Knatrovich method in the two Sobolev spaces.

Let $p \in [1,+\infty[$. We recall the linear Fredholm integro-differential equation

$$\forall x \in [a,b], \quad \lambda u(x) = \int_a^b K_1(x,t)u(t)\,dt + \int_a^b K_2(x,t)u'(t)\,dt + f(x), \tag{4.1}$$

in the Sobolev space $W^{1,p}$ is equipped with the following norm

$$||v||_{W^{1,p}[a,b]} = ||v||_{L^p[a,b]} + ||v'||_{L^p[a,b]}. \tag{4.2}$$

4.1 Analytical study in the Sobolev space $W^{1,p}[a,b]$

In this section, we are interested on the solution existence and uniqueness of (3.1) in the Sobolev space $W^{1,p}[a,b]$. We propose two different hypotheses on the kernels K_i, $i=1,2$ depending on $p \in [1,+\infty[$

($\mathcal{L}_1$) For $p \in]1,\infty[$, we assume that the kernels K_i, $i=1,2$ checked the next assumptions

$$(\mathcal{L}_1) \Big\| \; K_i(x,t) \in L^p([a,b]^2,\mathbb{R}) \text{ and } \frac{\partial K_i}{\partial x}(x,t) \in L^p([a,b]^2,\mathbb{R}),$$

($\mathcal{L}_2$) For $p=1$ suppose that

$$(\mathcal{L}_2) \Big\| \max_{a\le x\le b} \int_a^b |K_i(x,t)|\, dt < +\infty \text{ and } \max_{a\le x\le b} \int_a^b \left|\frac{\partial K_i}{\partial x}(x,t)\right| dt < +\infty \;.$$

Now, for all $p \in [1,+\infty[$ and $u \in W^{1,p}[a,b]$, we define the following linear operator

$$\forall x \in [a,b],\; Tu(x) = \int_a^b K_1(x,t)u(t)\, dt + \int_a^b K_2(x,t)u'(t)\, dt. \tag{4.3}$$

Proposition 4.1.1. *Let $p \in [1,+\infty[$, $f \in W^{1,p}[a,b]$ and K_i, $i=1,2$ verify one of hypotheses ($\mathcal{L}_1$) or ($\mathcal{L}_2$). Then, $Tu \in W^{1,p}[a,b]$ for all $u \in W^{1,p}$. In addition, the weak derivatives $(Tu)'$ is given by*

$$(Tu)'u(x) = \int_a^b \frac{\partial K_1}{\partial x}(x,t)u(t)\, dt + \int_a^b \frac{\partial K_2}{\partial x}(x,t)u'(t)\, dt, \quad p.p \text{ for all } x \in [a,b].$$

Proof. To prove $Tu \in W^{1,p}[a,b]$, it is necessary show that $Tu \in L^p[a,b]$ and $(Tu)' \in L^p[a,b]$, for $p \in [1,+\infty[$.
First, we verify that $Tu \in L^p[a,b]$ for all $p \in [1,+\infty[$

I. For $p \in]1,+\infty[$, we use the Hölder's Inequality [1]to deduce

$$\int_a^b |K_1(x,t)u(t)|\, dt \le \left(\int_a^b |K_1(x,t)|^q\, dt\right)^{\frac{1}{q}} \left(\int_a^b |u(t)|^p\, dt\right)^{\frac{1}{q}}, \tag{4.4}$$

$$\int_a^b |K_2(x,t)u'(t)\, dt| \le \left(\int_a^b |K_2(x,t)|^q\, dt\right)^{\frac{1}{q}} \left(\int_a^b |u'(t)|^p\, dt\right)^{\frac{1}{q}}. \tag{4.5}$$

By combining equalities (4.4) and (4.5),

$$\left|\int_a^b K_1(x,t)u(t)\, dt + \int_a^b K_2(x,t)u'(t)\, dt\right| \le \Big(|K_1(x,t)|^q\, dt\Big)^{\frac{1}{q}} \left(\int_a^b |u(t)|^p\, dt\right)^{\frac{1}{q}}$$
$$+\left(\int_a^b |K_2(x,t)|^q\, dt\right)^{\frac{1}{q}} \left(\int_a^b |u'(t)|^p\, dt\right)^{\frac{1}{q}}. \tag{4.6}$$

Then,

$$\left|\int_a^b K_1(x,t)u(t)\,dt+\int_a^b K_2(x,t)u'(t)\,dt\right|^p\le\left(|K_1(x,t)|^q\,dt\right)^{\frac{1}{q}}\left(\int_a^b |u(t)|^p\,dt\right)^{\frac{1}{q}}$$
$$+\left(\int_a^b |K_2(x,t)|^q\,dt\right)^{\frac{1}{q}}\left(\int_a^b |u'(t)|^p\,dt\right)^{\frac{1}{q}}\Bigg]^p.$$

Applying the previous results, we get

$$||Tu||^p_{L^p[a,b]}\le\int_a^b\left[\left(\int_a^b|K_1(x,t)|^q dt\right)^{\frac{1}{q}}+\left(\int_a^b|K_2(x,t)|^q dt\right)^{\frac{1}{q}}\right]^p dx\;||u||_{W^{1,p}[a,b]}.$$

Using the fact $(z+y)^p\le 2^{p-1}(z^p+y^p)$, we have

$$\begin{aligned}||Tu||^p_{L^p[a,b]}&\le 2^{p-1}\Bigg[\int_a^b\left(\int_a^b|K_1(x,t)|^q\,dt\right)^{\frac{p}{q}}dx\\&+\int_a^b\left(\int_a^b|K_2(x,t)|^q\,dt\right)^{\frac{p}{q}}dx\Bigg]\;||u||_{W^{1,p}[a,b]}<+\infty.\end{aligned}$$

II. For $p=1$, we have

$$||Tu||_{L^1[a,b]}\le\left(\max_{a\le x\le b}\int_a^b|K_1(x,t)|\,dt+\max_{a\le x\le b}\int_a^b|K_2(x,t)|\,dt\right)||u||_{W^{1,1}[a,b]}<+\infty.$$

This proves that $Tu\in L^p[a,b]$, for $p\in[1,+\infty[$.
Secondly, we show that there exists a function $g\in L^p[a,b]$, $p\in[1,+\infty[$ verifying

$$\int_a^b Tu(x)\phi'(x)\,dx=-\int_a^b g(x)\phi(x)\,dx,\quad\forall\phi\in C_c^\infty[a,b].$$

Let $\phi\in C_c^\infty[a,b]$. Then,

$$\begin{aligned}\int_a^b Tu(x)\phi'(x)\,dx&=\int_a^b\left[\int_a^b K_1(x,t)u(t)\,dt+\int_a^b K_2(x,t)u'(t)\,dt\right]\phi'(x)dx,\\&=\int_a^b\left[\int_a^b K_1(x,t)\phi'(x)\,dx\right]u(t)\,dt+\int_a^b\left[\int_a^b K_2(x,t)\phi'(x)\,dx\right]u'(t)\,dt,\\&=-\int_a^b\left[\int_a^b\frac{\partial K_1}{\partial x}(x,t)\phi(x)\,dx\right]u(t)\,dt-\int_a^b\left[\int_a^b\frac{\partial K_2}{\partial x}(x,t)\phi(x)\,dx\right]u'(t)\,dt,\\&=-\int_a^b\left[\int_a^b\frac{\partial K_1}{\partial x}(x,t)u(t)\,dt+\int_a^b\frac{\partial K_2}{\partial x}(x,t)u'(t)\,dt\right]\phi(x)\,dx,\\&=-\int_a^b(Tu)'(x)\phi(x)\,dx.\end{aligned}$$

Hence, $\forall\phi\in C_c^\infty[a,b]$, we get $g=(Tu)'$. Finally, let verify that $(Tu)'\in L^p[a,b]$.

I. For $p\in]1,+\infty[$, with the same last processes. We can prove that

$$||(Tu)'||^p_{L^p[a,b]}\le 2^{p-1}\left[\int_a^b\left(\int_a^b\left|\frac{\partial K_1}{\partial x}(x,t)\right|^q dt\right)^{\frac{p}{q}}dx+\int_a^b\left(\int_a^b\left|\frac{\partial K_2}{\partial x}(x,t)\right|^q dt\right)^{\frac{p}{q}}dx\right]||u||_{W^{1,p}[a,b]}.$$

II. For $p = 1$, we have

$$||(Tu)'||_{L^1[a,b]} \leq \left(\max_{a\leq x\leq b} \int_a^b \left|\frac{\partial K_1}{\partial x}(x,t)\right| dt + \max_{a\leq x\leq b} \int_a^b \left|\frac{\partial K_2}{\partial x}(x,t)\right| dt \right) ||u||_{W^{1,1}[a,b]}. \tag{4.7}$$

So, we have $Tu \in W^{1,p}[a,b]$, for all $p \in [1,+\infty[$, this completes the proof. □

Under proposition 4.1.1, the derivative u' is given implicitly as

$$\forall\, x \in [a,b], \quad \lambda u'(x) = \int_a^b \frac{\partial K_1}{\partial x}(x,t)u(t)\, dt + \int_a^b \frac{\partial K_2}{\partial x}(x,t)u'(t)\, dt + f'(x). \tag{4.8}$$

We introduce the following theorem to show that $T \in BL(W^{1,p}[a,b])$.

Theorem 4.1.1.

1. *If the kernels K_i, for $i = 1,2$ verify the assumption $(\mathcal{L}_1)$ and $p \in]1,+\infty[$, then $T \in BL(W^{1,p}[a,b])$ and*

$$||T|| \leq 2^{\frac{2p-2}{p}} \sum_{i=0}^{2} \left[\int_a^b \left(\int_a^b |K_i(x,t)|^q\, dt \right)^{\frac{p}{q}} dx + \int_a^b \left(\int_a^b \left|\frac{\partial K_i}{\partial x}(x,t)\right|^q dt \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}}.$$

2. *If the kernels K_i, for $i = 1,2$ verify the assumption $(\mathcal{L}_2)$, then $T \in BL(W^{1,1}[a,b])$ and*

$$||T|| \leq \sum_{i=0}^{2} \left(\max_{a\leq c\leq b} \int_a^b |K_i(x,t)|\, dt + \max_{a\leq x\leq b} \int_a^b \left|\frac{\partial K_i}{\partial x}(x,t)\right| dt \right).$$

Proof. We recall that the norm of linear operator T is

$$||T|| = \sup_{||u||_{W^{1,p}[a,b]}=1} ||Tu||_{W^{1,p}[a,b]}.$$

1. We start by prove that the operator T is bounded in $W^{1,p}[a,b]$, for $p \in]1,+\infty[$. By summation of (4.7) and (4.7),

$$\begin{aligned} ||Tu||^p_{L^p[a,b]} + ||(Tu)'||^p_{L^p[a,b]} \quad &\leq \quad 2^{p-1} \sum_{i=0}^{2} \left[\int_a^b \left(\int_a^b |K_i(x,t)|^q\, dt \right)^{\frac{p}{q}} dx \right. \\ &+ \left. \int_a^b \left(\int_a^b \left|\frac{\partial K_i}{\partial x}(x,t)\right|^q dt \right)^{\frac{p}{q}} dx \right] ||u||_{W^{1,p}[a,b]}. \end{aligned}$$

On the other side, we have

$$\left(||Tu||_{L^p[a,b]} + ||(Tu)'||_{L^p[a,b]} \right)^p \leq 2^{p-1} \left(||Tu||^p_{L^p[a,b]} + ||(Tu)'||^p_{L^p[a,b]} \right).$$

We get,

$$\begin{aligned} ||Tu||_{L^p[a,b]} + ||(Tu)'||_{L^p[a,b]} \quad &\leq \quad 2^{\frac{2p-2}{p}} \sum_{i=0}^{2} \left[\int_a^b \left(\int_a^b |K_i(x,t)|^q\, dt \right)^{\frac{p}{q}} dx \right. \\ &+ \left. \int_a^b \left(\int_a^b \left|\frac{\partial K_i}{\partial x}(x,t)\right|^q dt \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}} ||u||_{W^{1,p}[a,b]}. \end{aligned}$$

Finally,

$$||Tu||_{W^{1,p}[a,b]} \leq 2^{\frac{2p-2}{p}} \sum_{i=0}^{2} \left[\int_a^b \left(\int_a^b |K_i(x,t)|^q \, dt \right)^{\frac{p}{q}} dx + \int_a^b \left(\int_a^b \left| \frac{\partial K_i}{\partial x}(x,t) \right|^q dt \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}} ||u||_{W^{1,p}[a,b]}.$$

This last equality proves that $Tu \in BL(W^{1,p}[a,b])$ for $p \in]1,+\infty[$. Then,

$$||T|| \leq 2^{\frac{2p-2}{p}} \sum_{i=0}^{2} \left[\int_a^b \left(\int_a^b |K_i(x,t)|^q \, dt \right)^{\frac{p}{q}} dx + \int_a^b \left(\int_a^b \left| \frac{\partial K_i}{\partial x}(x,t) \right|^q dt \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}}.$$

2. Now, we prove that $T \in BL(W^{1,1}[a,b])$.
From (4.7) and (4.10), we find that $||Tu||_{W^{1,1}[a,b]}$ is increased by

$$||Tu||_{W^{1,1}[a,b]} \leq \sum_{i=0}^{2} \left(\max_{a \leq x \leq b} \int_a^b |K_i(x,t)| \, dt + \max_{a \leq x \leq b} \int_a^b \left| \frac{\partial K_i}{\partial x}(x,t) \right| dt \right) ||u||_{W^{1,1}[a,b]},$$

so that Tu is a linear and bounded operator in $W^{1,1}[a,b]$. Using the norm of linear operator, we get

$$||T|| \leq \sum_{i=0}^{2} \left(\max_{a \leq x \leq b} \int_a^b |K_i(x,t)| \, dt + \max_{a \leq x \leq b} \int_a^b \left| \frac{\partial K_i}{\partial x}(x,t) \right| dt \right).$$

□

In the next corollary, we give a sufficient condition to ensure the solution existence and uniqueness of equation (4.1).

Corollary 4.1.1. *Equation (4.1) has a unique solution in $W^{1,p}[a,b]$ if*

1. *For $p \in]1,+\infty[$, the kernels K_i for $i = 1,2$ check the assumption $(\mathcal{L}_1)$ and*

$$|\lambda| > 2^{\frac{2p-2}{p}} \sum_{i=0}^{2} \left[\int_a^b \left(\int_a^b |K_i(x,t)|^q \, dt \right)^{\frac{p}{q}} dx + \int_a^b \left(\int_a^b \left| \frac{\partial K_i}{\partial x}(x,t) \right|^q dt \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}}. \qquad (4.9)$$

2. *For $p = 1$, the kernels K_i for $i = 1,2$ verify the assumption $(\mathcal{L}_2)$ and*

$$|\lambda| > \sum_{i=0}^{2} \left(\max_{a \leq x \leq b} \int_a^b |K_i(x,t)| \, dt + \max_{a \leq x \leq b} \int_a^b \left| \frac{\partial K_i}{\partial x}(x,t) \right| dt \right). \qquad (4.10)$$

Proof.

1. To prove (4.9), we will show that

$$||T|| = 2^{\frac{2p-2}{p}} \sum_{i=0}^{2} \left[\int_a^b \left(\int_a^b |K_i(x,t)|^q \, dt \right)^{\frac{p}{q}} dx + \int_a^b \left(\int_a^b \left| \frac{\partial K_i}{\partial x}(x,t) \right|^q dt \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}}.$$

By using the sup definition, we have

$$\forall\, \varepsilon > 0, \exists\, u_1 \in W^{1,p}(\Omega);\ ||u_1||_{W^{1,p}[a,b]} = 1,\ ||Tu_1||_{W^{1,p}[a,b]} > ||T|| - \varepsilon,$$

since,

$$2^{\frac{2p-2}{p}} \sum_{i=0}^{2} \left[\int_a^b \left(\int_a^b |K_i(x,t)|^q \, dt \right)^{\frac{p}{q}} dx + \int_a^b \left(\int_a^b \left| \frac{\partial K_i}{\partial x}(x,t) \right|^q dt \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}} ||u_1||_{W^{1,p}[a,b]} > ||A|| - \varepsilon,$$

For any arbitrary ε, we get

$$2^{\frac{2p-2}{p}} \sum_{i=0}^{2} \left[\int_a^b \left(\int_a^b |K_i(x,t)|^q \, dt \right)^{\frac{p}{q}} dx + \int_a^b \left(\int_a^b \left| \frac{\partial K_i}{\partial x}(x,t) \right|^q dt \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}} \geq ||T||,$$

and

$$||T|| = 2^{\frac{2p-2}{p}} \sum_{i=0}^{2} \left[\int_a^b \left(\int_a^b |K_i(x,t)|^q \, dt \right)^{\frac{p}{q}} dx + \int_a^b \left(\int_a^b \left| \frac{\partial K_i}{\partial x}(x,t) \right|^q dt \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}} < |\lambda|.$$

By Neumann's theorem 1.7 $(\lambda I - T)^{-1}$ exists and

$$||(\lambda I - T)^{-1}|| < \frac{1}{|\lambda| - ||T||},$$

where I is the identity operator defined in $W^{1,p}[a,b]$ in itself. Then our equation has a unique solution.

2. To show the equality (4.10), we will prove that

$$||T|| = \sum_{i=0}^{2} \left(\max_{a \leq x \leq b} \int_a^b |K_i(x,t)| \, dt + \max_{a \leq x \leq b} \int_a^b \left| \frac{\partial K_i}{\partial x}(x,t) \right| dt \right).$$

Also, by sup characterisation we have

$$\forall\, \varepsilon > 0\ , \exists\, u_1 \in W^{1,1}(\Omega);\ ||u_1||_{W^{1,1}[a,b]} = 1,\ ||Tu_1||_{W^{1,1}[a,b]} > ||A|| - \varepsilon,$$

such that

$$\sum_{i=0}^{2} \left(\max_{a \leq x \leq b} \int_a^b |K_i(x,t)| \, dt + \max_{a \leq x \leq b} \int_a^b \left| \frac{\partial K_i}{\partial x}(x,t) \right| dt \right) ||u_1||_{W^{1,1}[a,b]} \geq ||Tu_1||_{W^{1,1}[a,b]} > ||T|| - \varepsilon. \tag{4.11}$$

Since ε is arbitrary and from the equality (4.11), we get

$$||T|| = \sum_{i=0}^{2} \left(\max_{a \leq x \leq b} \int_a^b |K_i(x,t)| \, dt + \max_{a \leq x \leq b} \int_a^b \left| \frac{\partial K_i}{\partial x}(x,t) \right| dt \right) < |\lambda|.$$

Using Neumann's theorem 1.7, we get $(\lambda I - T)^{-1}$ exists and bounded, such that this time I is the identity operator of $W^{1,1}[a,b]$. Then, the equation (4.1) has a unique solution in $W^{1,1}[a,b]$.

□

In the rest of our thesis, we present different constructions of numerical solutions and we show that these approaches converge to the exact solution. So, in the convergence analysis we need the compactness of the operator A, which is proved in the coming theorem.

Theorem 4.1.2. *Let $p \in [1, +\infty[$. Then, T is a compact operator.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence defined in $W^{1,p}[a,b]$, since $||u_n||_{W^{1,p}[a,b]} < \infty$. $\{Tu_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,p}[a,b]$. Therefore, it has a sub sequences $\{Tu_{n_k}\}_{k\in\mathbb{N}}$ converges in $W^{1,p}[a,b]$. Consequently $\{Tu_n\}_{n\in\mathbb{N}}$ is precompact. Thus, T is compact. $\square$

4.2 Projection Approximation

In this section, we present the basic idea of projection methods 1.4.1 in the Sobolev space $W^{1,p}[a,b]$. We put $X = L^p[a,b]$ and we denote $\tilde{X}_2$ by $X \times X$. Let $\{X_n\}_{n\geq 1}$ be a sequence of finite dimensional subspace of $L^p[a,b]$ and $\{\tilde{X}_{2,n}\}_{n\geq 1}$ be a sequence of finite dimensional subspace of $\tilde{X}$. Here, the construction of an approximation solution in the Sobolev space is based on the following projection operator $\{P_n\}_{n\geq 1}$ which is given by

$$\begin{aligned} P_n : \quad \tilde{X}_2 &\longrightarrow \tilde{X}_{2,n} \\ V &\longmapsto P_n V = (P_{1,n}v_1, P_{2,n}v_2), \end{aligned} \tag{4.12}$$

where $\{P_{r,n}\}_{n\geq 1}$, for $r = 1,2$ are sequences have the following form

$$\forall v \in X, \quad P_{r,n}v = \sum_{i=0}^{n} (v, e_i^*)e_i, \; r = 1,2,$$

such that, $(.,.)$ is the duality product in Banach space, $\{e_i\}_{i=0}^n$ is an ordered basis of X_n and $\{e_i^*\}_{i=0}^n$ are linear continuous functional, since

To illustrate our numerical process in $W^{1,p}[a,b]$ by using the Galerkin 1.4.1 and Kantorovich 1.4.1 approximation methods, we need to present our operator T in the equivalent form. For this reason, we set $u^{(r-1)} = u_r$ for $r = 1,2$ and we define the following block operator matrix A_T which is given as

$$\begin{aligned} A_T : \qquad \tilde{X}_2 \qquad &\longrightarrow \tilde{X}_2 \\ U = (u_1, u_2) &\longmapsto A_T U = (T_{11}u_1 + T_{12}u_2, T_{21}u_1 + T_{22}u_2), \end{aligned}$$

with the operators $\{T_{lr}\}_{1\leq l,r\leq 2}$ are defined for all $x \in [a,b]$ and $r = 1,2$ by

$$\begin{aligned} \forall\, v \in X,\; T_{1r}v(x) &= \int_a^b K_r(x,t)v(t)\,dt, \\ \forall\, v \in X,\; T_{2r}v(x) &= \int_a^b \frac{\partial K_r}{\partial x}(x,t)v(t)\,dt. \end{aligned}$$

We rewrite equations (4.1) and (4.8) in this simple version:

$$(\lambda I_2 - A_T)U = F,$$

where I_2 is the identity operator of $\tilde{X}_2$ and $F = (f_1, f_2)$.

Define now, two matrix approximations $A^G_{T_n} = P_n A_T P_n$ and $A^K_{T_n} = P_n A_T$. The first is called block Galerkin approximation matrix and the second one is block Kantorovich matrix. Using the above approximation operators Galerkin and Kantorovich, we get two approximation equations.

1. The first one is the Galerkin equation, which has the following form

$$(\lambda I_2 - A^G_{T_n})U_n = F. \tag{4.13}$$

From the equation (4.13), we have

$$\begin{aligned}
\lambda U &= P_n A_T P_n U + F,\\
&= P_n A_T (P_{1,n} u_{1,n}, P_{2_n} u_{2,n}) + (f_1, f_2),\\
&= P_n (T_{11} P_{1,n} u_{1,n} + T_{12} P_{2,n} u_{2,n}, T_{21} P_{1,n} u_{1,n} + T_{22} P_{2,n} u_{2,n}) + (f_1, f_2)\\
&= (P_{1,n} T_{11} P_{1,n} u_{1,n} + P_{2,n} T_{12} P_{2,n} u_{2,n}, P_{1,n} T_{21} P_{1,n} u_{1,n} + P_{2,n} T_{22} P_{2,n} u_{2,n}) + (f_1, f_2).
\end{aligned}$$

Thus, the unique solution $U_n = (u_{1,n}, u_{2,n})$ of Galekin equation (4.13) is given by

$$\begin{cases}
u_{1,n} = \dfrac{1}{\lambda}\left[\sum\limits_{i=0}^{n}\sum\limits_{j=0}^{n}(u_{1,n}, e_j^*)\ (T_{11}e_j, e_i^*)e_i + \sum\limits_{i=0}^{n}\sum\limits_{j=0}^{n}(u_{2,n}, e_j^*)\ (T_{12}e_j, e_i^*)e_i + f_1\right],\\[2em]
u_{2,n} = \dfrac{1}{\lambda}\left[\sum\limits_{i=0}^{n}\sum\limits_{j=0}^{n}\langle u_{1,n}, e_j^*)\ (T_{21}e_j, e_i^*)e_i + \sum\limits_{i=0}^{n}\sum\limits_{j=0}^{n}(u_{2,n}, e_j^*)\ (T_{22}e_j, e_i^*)e_i + f_2\right],
\end{cases}$$

such that, $\overline{X}_1(i) = (u_{1,n}, e_j^*)$ and $\overline{X}_2(i) = (u_{2,n}, e_j^*)$ are unknowns that we will find by solving the following algebraic system

$$\lambda \begin{pmatrix}\overline{X}_1\\ \overline{X}_2\end{pmatrix} = \begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\begin{pmatrix}\overline{X}_1\\ \overline{X}_2\end{pmatrix} + \begin{pmatrix}\overline{Y}_1\\ \overline{Y}_2\end{pmatrix}, \tag{4.14}$$

with

$$\begin{aligned}
A_{n,lr}(i,j) &= (T_{lr}e_i, e_j^*), 1 \le l, r \le 2, \quad 0 \le i, j \le n,\\
\overline{Y}_1(i) &= (f_1, e_i^*), \quad 0 \le i \le n,\\
\overline{Y}_2(i) &= (f_2, e_i^*), \quad 0 \le i \le n.
\end{aligned}$$

2. The second one is Kantorovich approximation

$$(\lambda I_2 - A^K_{T_n})U_n = F, \tag{4.15}$$

From the equation (4.15), we have

$$\begin{aligned}
\lambda U_n &= P_n A_T U + F,\\
&= P_n (T_{11} u_{1,n} + T_{12} u_{2,n}, T_{21} u_{1,n} + T_{22} u_{2,n}) + (f_1, f_2),\\
&= (P_{1,n} T_{11} u_{1,n} + P_{2,n} T_{12} u_{2,n}, P_{1,n} T_{21} u_{1,n} + P_{2,n} T_{22} u_{2,n}) + (f_1, f_2).
\end{aligned}$$

Then, the unique solution of Kantorovich system has the following form

$$\begin{cases} u_{1,n} = \frac{1}{\lambda}\Big[\sum_{i=0}^{n}(T_{11}u_{1,n}, e_i^*)e_i + \sum_{i=0}^{n}(T_{12}u_{2,n}, e_i^*)e_i + f_1\Big], \\ u'_n = \frac{1}{\lambda}\Big[\sum_{i=0}^{n}(T_{21}u_{1,n}, e_i^*)e_i + \sum_{i=0}^{n}(T_{22}u_{2,n}, e_i^*)e_i + f_2\Big]. \end{cases}$$

The unknowns of the previous system are $X_1(i) = (T_{11}u_{1,n} + T_{12}u_{2,n}, e_i^*)$ and $X_2(i) = (T_{21}u_{1,n} + T_{22}u_{2,n}, e_i^*)$ for $i = 0, 1, \dots n$. To find them we multiply the system by the block matrix A_T and apply e_j^* [48]. We find this n+1-dimensional linear system with the two unknown vectors X_1 and X_2

$$\lambda \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \begin{pmatrix} A_{n,11} & A_{n,12} \\ A_{n,21} & A_{n,22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \tag{4.16}$$

where

$$\begin{aligned} A_{n,lr}(i,j) &= (T_{lr}e_i, e_j^*), \quad 1 \le l, r \le 2,\ 0 \le i, j \le n, \\ Y_1(i) &= (T_{11}f + T_{21}f', e_i^*), \quad 0 \le i \le n, \\ Y_2(i) &= (T_{21}f + T_{22}f', e_i^*), \quad 0 \le i \le n. \end{aligned}$$

4.3 Study of Galerkin and Kantorovich approximation in $W^{1,1}[a,b]$

In this section, we set $X = L^1[a,b]$ and $\tilde{X}_2 = X \times X$. Let $n \ge 1$, we recall Δ_n the uniform partition of $[a,b]$ (1.15).Let $\tilde{X}_{n,2} = X_n \times X_n$, where X_n be a subspace of $L^1[a,b]$ has a basis $\{e_i\}_{i=1}^n$ such that

$$e_i(x) = \begin{cases} 1 & \text{if} \quad x \in [x_{i-1}, x_i], \\ 0 & \text{else} \end{cases}.$$

We define the duality product as

$$\forall v \in X,\ (v, e_j^*) = \frac{1}{h}\int_{x_{j-1}}^{x_j} v(x)\,dx.$$

Lemma 4.3.1. *Let $P_n : \tilde{X}_2 \to \tilde{X}_{2,n}$ be a sequence of projection operator. Then,*

1. *$P_n \in BL(\tilde{X}_2, \tilde{X}_{2,n})$,*
2. *For all $U \in \tilde{X}_2$, we have*

$$||(I_2 - P_n)U||_{\tilde{X}_2} \le 2\,\kappa_{1,1}(h,u) \underset{h\to 0}{\to} 0,$$

which means that $(P_n)_{n\ge1}$ is pointwise convergent to I_2.

Proof. It is clear that $\{P_n\}_{n\ge1}$ is linear. We need to prove that it is also bounded. Since

$$\begin{aligned} ||P_{1,n}u_1||_X &\le \frac{1}{h}\sum_{i=0}^{n}\int_a^b\int_{x_{i-1}}^{x_i} |u_1(y)||e_i(x)|\,dy\ \ dx, \\ &\le \frac{1}{h}\sum_{i=0}^{n}\int_{x_{i-1}}^{x_i}\int_{x_{i-1}}^{x_i} |u_1(y)|\,dy\ \ dx \le (b-a)||u_1||_X, \end{aligned}$$

$||P_nU||_{\tilde{X}_2} \leq (b-a)||U||_{\tilde{X}_2}$. This implies that $\{P_n\}_{n\geq 1}$ is bounded and $||P_n|| \leq (b-a)$. We have, for $n \geq 1$

$$||U - P_nU||_{\tilde{X}_2} = ||(I - P_{1,n})u_1||_X + ||(I - P_{2,n})u_2||_X. \tag{4.17}$$

But,

$$\begin{aligned}
||(I - P_{1,n})u_1||_X &= \int_a^b |u_1(x) - P_{1,n}u_1(x)|\, dx, \\
&= \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |u_1(x) - P_{1,n}u_1(x)|\, dx \\
&\leq \frac{1}{h}\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \int_{x_{i-1}}^{x_i} |u_1(x) - u_1(t)|\, dt\, dx, \\
&\leq \frac{2}{h}\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \int_{t}^{x_i} |u_1(x) - u_1(t)|\, dx\, dt, \\
&\leq \frac{2}{h}\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \int_{0}^{h} |u_1(x+t) - u_1(x)|\, dx\, dt, \\
&\leq \frac{2}{h} \int_{a}^{b} \int_{0}^{h} |u_1(x+t) - u_1(x)|\, dx\, dt, \\
&\leq \frac{2}{h} \int_{0}^{h} \int_{a}^{b} |u_1(x+t) - u_1(x)|\, dt\, dx, \\
&\leq 2\, \kappa_{1,0}(h, u_1).
\end{aligned} \tag{4.18}$$

In the same way,

$$||(I - P_{2,n})u_2||_X \leq 2\, \kappa_{1,0}(h, u_2). \tag{4.19}$$

Substituting (4.18) and (4.19) in (4.17), we obtain

$$||U - P_nU||_{\tilde{X}_2} \leq 4\, \kappa_{1,1}(h, U),$$

so that for $h = \dfrac{b-a}{n}$, when $n \to +\infty$ we get $h \to 0$ and

$$\lim_{n\to\infty} ||U - P_nU||_{\tilde{X}_2} = 0.$$

□

4.3.1 Kantorovich Approximaion

In this section, we study the convergence of Kantorovich in the space $W^{1,1}[a,b]$.

Theorem 4.3.1. *Let A_T be a block compact operator matrix and $A^K_{T_n}$ be a block Kantorovich approximation. Then,*

$$||(A_T - A^K_{T_n})U||_{\tilde{X}_2} \leq 2 \max\Big(||\bar{\kappa}_{1,1}(h, K_1)||_\infty, ||\bar{\kappa}_{1,1}(h, K_2)||_\infty \Big),$$

and we have $\lim_{n\to\infty} |||(A_T - A^K_{T_n})||| = 0.$

Proof. By using the norm of block operator matrix and the equality (4.17) in lemma.4.3.1, we get for $||U||_{\tilde{X}_2} = 1$

$$\begin{aligned} ||(A_T - A^K_{T_n})U||_{\tilde{X}_2} &\leq \max_{1\leq r\leq 2}\sum_{l=1}^{2}||(T_{lr} - \pi T_{lr})u_r||_X \\ &\leq 2 \max_{1\leq r\leq 2}\sum_{l=1}^{2}\kappa_{1,0}(h, T_{lr}u_r), \end{aligned} \tag{4.20}$$

where, for $r = 1, 2$

$$\begin{aligned} \kappa_{1,0}(h, T_{1r}u_r) &= \sup_{x\in[0,h]}\int_a^b |T_{1r}u_r(x+y) - T_{lr}u_r(y)|\, dy, \\ &\leq \sup_{x\in[0,h]}\int_a^b\int_a^b |K_r(x+y,t) - K_r(y,t)|\, |u_r(t)|\, dt\, dy \\ &\leq \sup_{x\in[0,h]}\int_a^b \sup_{t\in[a,b]} |K_r(x+y,t) - K_r(y,t)|\, dy\, ||u_r||_X, \\ &\leq \sup_{t\in[a,b]}\int_a^b \sup_{x\in[0,h]} |K_r(x+y,t) - K_r(y,t)|\, dy\, ||u_r||_X, \\ &\leq \sup_{t\in[a,b]} \bar{\kappa}_{1,0}(h, K_r)(t). \end{aligned}$$

In same way, we get

$$\kappa_{1,0}(h, T_{2r}u_r) \leq \sup_{t\in[a,b]} \bar{\kappa}_{1,0}(h, \partial_x K_r)(t), \tag{4.21}$$

where, $\partial_x K_r$ is a partial derivative of K_r respect to x.
Substitute (4.21) and (4.21) in (4.20) to get

$$\begin{aligned} ||(A_T - A^K_{T_n})U||_{\tilde{X}_2} &\leq 2\max\Big(\bar{\kappa}_{1,1}(h,K_1)(t), \bar{\kappa}_{1,1}(h,K_2)(t)\Big), \\ &\leq 2\Big(||\bar{\kappa}_{1,1}(h,K_1)||_\infty + ||\bar{\kappa}_{1,1}(h,K_2)||_\infty\Big). \end{aligned}$$

Let us prove that $\lim_{n\to\infty}|||(A_T - A^K_{T_n})||| = 0$. Let A_T be a compact bloc matrix operator. The set $\{A_T V, ||V||_{\tilde{X}_2} \leq 1\}$, is relatively compact. Using the Banach-Steinhaus theorem, the pointwise convergence of P_n to I_2 is uniform and we obtain

$$\lim_{n\to+\infty}|||(A_T - A^K_{T_n})||| = |||(I_2 - P_n)A_T||| = \lim_{n\to+\infty}\sup_{V\in\tilde{X}_2}||(I_2 - P_n)V||_{\tilde{X}_2} = 0.$$

□

Theorem 4.3.2. *Let u be the solution of (4.1) and let u_n be the Kantorovich approximation. Then,*

$$\lim_{n\to 0}||u - u_n||_{W^{1,1}[a,b]} = 0. \tag{4.22}$$

Proof. We have that $A^K_{T_n}$ converge to A_T in the operator norm. Then, $(\lambda I_2 - A^K_{T_n})^{-1}$ exists and is bounded (see [2, 5]). We can write, for n large

$$U - U_n = (\lambda I_2 - A^K_{T_n})^{-1}\Big((A_T - A^K_{T_n})(u, u')\Big),$$

we set $C_2 = |||(\lambda I_2 - A^K_{T_n})^{-1}|||$ to obtain

$$\begin{aligned} ||u-u_n||_{W^{1,1}[a,b]} &= ||U-U_n||_{\tilde{X}_2} \le C_2||(A_T - A^K_{T_n})(u,u')||_{\tilde{X}_2}, \\ &\le 2\,C_2 \max\Big(||\bar{\kappa}_{1,1}(h,K_1)||_\infty, ||\bar{\kappa}_{1,1}(h,K_2)||_\infty\Big). \end{aligned}$$

Then,

$$\lim_{n\to\infty} ||u_n - u||_{W^{1,1}[a,b]} = 0.$$

□

Kantorovich System

Now, we introduce the elements of the system (4.16)

$$A_{1r}(i,j) = \frac{1}{h}\int_{x_{i-1}}^{x_i}\int_{x_{j-1}}^{x_j} K_r(x,t)\,dt\,dx, \quad 1\le i,j\le n,\ r=1,2,$$

$$A_{2r}(i,j) = \frac{1}{h}\int_{x_{i-1}}^{x_i}\int_{x_{j-1}}^{x_j} \frac{\partial K_r}{\partial x}(x,t)\,dt\,dx, \quad 1\le i,j\le n,\ r=1,2,$$

$$Y_1(i) = \frac{1}{h}\int_{x_{i-1}}^{x_i}\left[\int_a^b K_1(x,t)f(t)\,dt + \int_a^b K_2(x,t)f'(t)\,dt\right]dx,\ 1\le i\le n,$$

$$Y_2(i) = \frac{1}{h}\int_{x_{i-1}}^{x_i}\left[\int_a^b \frac{\partial K_1}{\partial x}(x,t)f(t)\,dt + \int_a^b \frac{\partial K_2}{\partial x}(x,t)f'(t)\,dt\right]dx, 1\le i\le n.$$

4.3.2 Galerkin Approximation

In this part, we present some theorems to show the convergence of the Galerkin solution.

Theorem 4.3.3. *Let A_T be a block compact operator matrix and $A^G_{T_n}$ be a block Galerkin approximation. Then,*

$$\lim_{n\to\infty} |||(A_T - A^G_{T_n})A_T||| = 0, \quad \textit{and} \quad \lim_{n\to\infty} |||(A_T - A_{T_n})A^G_{T_n}||| = 0. \tag{4.23}$$

Proof. For n large enough, we have

$$(A^G_{T_n} - A_T)A^G_{T_n} = (A^K_{T_n} - A_T)A^G_{T_n},$$

and

$$(A^G_{T_n} - A_T)A = (A^K_{T_n} - A_T)A^K_{T_n} + A_T(A^K_{T_n} - A_T)$$

$$\begin{aligned} |||(A^G_{T_n} - A_T)A^G_{T_n}||| &\le |||(A^K_{T_n} - A_T)|||\ ||P_n||^2\ |||A_T|||, \\ |||(A^G_{T_n} - A_T)A_T||| &\le (||P_n||+1)\ |||A_T|||\ |||A^K_{T_n} - A_T|||. \end{aligned}$$

By using theorem 4.3.1, we get

$$|||(A^G_{T_n} - A_T)A^G_{T_n}||| \le 2(b-a)^2\ |||A_T|||\max\Big(||\bar{\kappa}_{1,1}(h,K_1)||_\infty, ||\bar{\kappa}_{1,1}(h,K_2)||_\infty\Big),$$

$$|||(A^G_{T_n} - A_T)A_T||| \le 2\,(b-a+1)|||A_T|||\max\Big(||\bar{\kappa}_{1,1}(h,K_1)||_\infty, ||\bar{\kappa}_{1,1}(h,K_2)||_\infty\Big),$$

□

and when $n \to \infty$, we get the result and we have that $(\lambda I_2 - A_T^G)^{-1}$ exists and is bounded (see [2]).

Theorem 4.3.4. *Let u be the solution of (4.1) and let u_n be the Galerkin approximation. Then,*

$$\lim_{n \to 0} ||u - u_n||_{W^{1,1}[a,b]} = 0.$$

Proof. For n large enough, we have

$$U - U_n = (\lambda I_2 - A_{T_n}^G)^{-1} \Big[A_{T_n}^K (P_n U - U) + (A_{T_n}^K - A_T) U \Big],$$

$$||U - U_n||_{\tilde{X}_2} \le C_1 \Big[|||A_{T_n}^K||| \; ||P_n U - U||_{\tilde{X}_2} + ||(A_{T_n}^K - A_T) U||_{\tilde{X}_2} \Big],$$

By theorem 4.3.1 and lemma4.3.1, we get

$$||U - U_n||_{\tilde{X}_2} \le 2\, C_1 \Big[|||A_{T_n}^K||| \; \kappa_{1,1}(h,u) + \max \Big(\bar{\kappa}_{1,1}(h, K_1)(t), \bar{\kappa}_{1,1}(h, K_2)(t) \Big) \Big].$$

Then,

$$\lim_{n \to \infty} ||u_n - u||_{W^{1,1}[a,b]} = 0.$$

□

Galerkin System

The elements of system (4.14) are presented as

$$A_{1r}(i,j) = \frac{1}{h} \int_{x_{i-1}}^{x_i} \int_{x_{j-1}}^{x_j} K_r(x,t)\, dt\, dx, \quad 1 \le i,j \le n, \; r = 1,2,$$

$$A_{2r}(i,j) = \frac{1}{h} \int_{x_{i-1}}^{x_i} \int_{x_{j-1}}^{x_j} \frac{\partial K_r}{\partial x}(x,t)\, dt\, dx, \quad 1 \le i,j \le n, \; r = 1,2,$$

$$\bar{Y}_1(i) = \frac{1}{h} \int_{x_{i-1}}^{x_i} f(x)\, dx, \quad 1 \le i \le n,$$

$$\bar{Y}_2(i) = \frac{1}{h} \int_{x_{i-1}}^{x_i} f'(x)\, dx, \quad 1 \le i \le n.$$

4.4 Study of Galerkin and Kantorovich method in $H^1[a,b]$

Let $X = L^2[a,b]$, and X_n be a subspace of $L^2[a,b]$ has the following orthonormal basis

$$\forall x \in [a,b], \; e_i(x) = \sqrt{2i+1}\, L_i\Big(\frac{2x - b - a}{b - a}\Big),$$

such that $L_i(t)$ is the Legendre polynomial defined on $[-1,1]$ by the following recursive formula

$$\begin{cases} L_0(t) &= 1, \\ L_1(t) &= t, \\ L_{i+1}(t) &= \dfrac{2i+1}{i+1} L_i(t) - \dfrac{i}{i+1} L_{i-1}(t). \end{cases}$$

Since for all $v \in X$ and $j = 0, \dots n$

$$\langle v, e_j \rangle = \int_a^b u(x) e_j(x)\, dx,$$

the subsequence operator π_n be an orthogonal projection in X. Then,

Lemma 4.4.1. *Let $P_n : \bar{X} \to \tilde{X}_{2,n}$ be a sequence of orthogonal projection, then*

$$\lim_{n\to\infty} ||(I_2 - P_n)U||_{\tilde{X}_2} = 0,$$

i.e P_n is pointwise convergent to I_2.

Proof. Let $\mathcal{P}$ be a polynomial of degree n. Then, we write p_n as a linear combination of Legendre Polynomials for any coefficients α_i

$$\forall x \in [-1,1], \quad \mathcal{P}(x) = \sum_{j=0}^{n} \alpha_j L_j(x).$$

Let us prove that

$$\langle u_1 - P_{1,n} u, \sum_{j=0}^{n} \alpha_j L_j \rangle = \sum_{j=0}^{n} \alpha_i \langle u_1 - P_{1,n} u_1, L_j \rangle.$$

we have

$$\begin{aligned} \langle u_1 - P_{1,n} u_1, L_j \rangle &= \langle u_1, L_j \rangle - \sum_{i=0}^{n} \langle P_{1,n} u_1, L_i \rangle L_j, \\ &= \langle u_1, L_j \rangle - \langle u_1, L_j \rangle \langle L_j, L_j \rangle = 0. \end{aligned}$$

This leads to

$$\langle u_1 - P_{1,n} u_1, \sum_{j=0}^{n} \alpha_j e_j \rangle = 0.$$

By Pythagorian theorem, we get

$$||\langle u_{1,n} - P_{1,n} u_1, \sum_{j=0}^{n} \alpha_j L_j \rangle||^2_{L^2[-1,1]} = ||u_1 - P_{1,n} u_1||^2_{L^2[-1,1]} + ||P_{1,n} u_1 - \sum_{j=0}^{n} \alpha_j L_j||^2_{L^2[-1,1]}.$$

Then,

$$||u_1 - P_{1,n} u_1||^2_{L^2[-1,1]} \leq \left|\left| \langle u_1 - P_{1,n} u_1, \sum_{j=0}^{n} \alpha_j L_j \rangle \right|\right|^2_{L^2[-1,1]}.$$

From the density of $C^0[-1,1]$ in $L^2[-1,1]$, we get for any function $u_1 \in L^2[-1,1]$ there is a function $v \in C^0[-1,1]$, such that $||u_1 - v||_{L^2[-1,1]} \leq \frac{\varepsilon}{2}$. In other way, by Weierstrass theorem for any $v \in C^0[-1,1]$, there is exist a polynomial $\mathcal{P}$ such that $||v - \mathcal{P}||_{C^0[-1,1]} \leq \frac{\varepsilon}{2}$. So,

$$||v - \mathcal{P}||_{L^2[-1,1]} \leq \sqrt{2}||v - \mathcal{P}||_{C^0[-1,1]} \leq \frac{\varepsilon}{2}.$$

Finally,

$$\begin{aligned} ||u_1 - P_{1,n}u_1||^2_{L^2[-1,1]} \leq ||u_1 - P_{1,n}u_1, \sum_{j=0}^{n} \alpha_j L_j||^2_{L^2[-1,1]}|| &\leq ||u_1 - v||_{L^2[-1,1]} + ||u_1 - v||^2_{L^2[-1,1]}, \\ &\leq \varepsilon. \end{aligned}$$

In the same way,

$$||u_2 - P_{2,n}u_2||^2_{L^2[-1,1]} \leq \varepsilon,$$

which gives

$$\lim_{n\to\infty} ||(I_2 - P_n)U||_{\tilde{X}_2} = 0.$$

□

4.4.1 Kantorovich Approximation

Theorem 4.4.1. *If $\{P_n\}_{n\geq 1}$ is pointwise convergent to I_2 in $\tilde{X}_2$, then*

$$\lim_{n\to+\infty} ||A^K_{T_n} - A_T|| = 0.$$

Proof. Fist, we prove that A_T is compact. Let U_n be a sequence of $\tilde{X}_2$, such that $||U||_{\tilde{X}_2} < \infty$. We have

$$||A_T U||_{\tilde{X}_2} \leq C||U||_{\tilde{X}_2}.$$

Then, A_T is bounded, so that it has a convergent sub sequence $\{A_T U_n\}_{n\geq 1}$, which proves that A_T is compact.
Since A_T is compact, then $\mathcal{M} = \{A_T U, ||U||_{\tilde{X}_2} \leq 1\}$ are relatively compact sets in the Banach space $\tilde{X}_2$ and by the Banach Steinhaus theorem's ([5, 2]) $\{P_n\}_{n\in\mathbb{N}}$ converges uniformly to I_2 in $\mathcal{M}$, i.e

$$\begin{aligned} \lim_{n\to\infty} |||A^K_{T_n} - A_T||| &= \lim_{n\to\infty} \sup_{||U||_{\tilde{X}_2}} ||A^K_{T_n}U - A_T U||_{\tilde{X}_2}, \\ &= \lim_{n\to\infty} \sup_{Z\in\mathcal{M}} ||(I_2 - P_n)Z||_{\tilde{X}_2}. \end{aligned}$$

□

Theorem 4.4.2. *Let u the exact solution and u_n is a Kantorovich solution. Then,*

$$\lim_{n\to 0} ||u - u_n||_{H^1[a,b]} = 0.$$

Proof. For n sufficiently large, $U - U_n = (\lambda I_2 - A^P_{T_n})^{-1}(A^P_{T_n} - A_T)U$. Then,

$$||u - u_n||_{H^1[a,b]} = ||U - U_n||_{\tilde{X}_2} \leq C_2||(A^P_{T_n} - A_T)U||_{\tilde{X}_2}.$$

By the last theorem, when $n \to \infty$, we get $||u - u_n||_{H^1[a,b]} \to 0$. □

Kantorovich System

From the definition of the basis $(L_i)_{i=0}^n$, the elements of the system (4.16) are given by

$$A_{1r}(i,j) = \int_a^b \int_a^b K_r(x,t)e_i(x)e_j(t)\,dt\,dx, \quad 1 \le i,j \le n,\ r=1,2,$$

$$A_{2r}(i,j) = \frac{1}{h}\int_a^b \int_a^b \frac{\partial K_r}{\partial x}(x,t)e_i(x)e_j(t)\,dt\,dx,\ 1 \le i,j \le n,\ r=1,2,$$

$$Y_1(i) = \int_a^b \left[\int_a^b K_1(x,t)f(t)e_i(x)\,dt + \int_a^b K_2(x,t)f'(t)\,dt\right]e_i(x)\,dx, \quad 1 \le i \le n,$$

$$Y_2(i) = \int_a^b \left[\int_a^b \frac{\partial K_1}{\partial x}(x,t)f(t)\,dt + \int_a^b \frac{\partial K_2}{\partial x}(x,t)f'(t)\,dt\right]e_i(x)\,dx,\ 1 \le i \le n.$$

4.4.2 Galerkin Approximation

Theorem 4.4.3. *Let A_T be a block compact operator matrix, $A^K_{T_n}$ and $A^G_{T_n}$ are the block Kantorovich and Galerikn approximation matrix respectively. If $\lim_{n\to\infty} |||A^K_{T_n} - A||| = 0$. Then,*

$$\lim_{n\to\infty} |||(A^G_{T_n} - A_T)A_T||| = 0 \quad \textit{and} \quad \lim_{n\to\infty} ||(A^G_{T_n} - A_T)A^G_{T_n}|| = 0.$$

This means $A^G_{T_n}$ ν-convergence ($A^G_{T_n} \xrightarrow{\nu} A_{T_n}$).

Proof. For n large enough, we have

$$(A^G_{T_n} - A_T)A^G_{T_n} = (A^K_{T_n} - A_T)A^G_{T_n},$$

and

$$(A^G_{T_n} - A_T)A = (A^K_{T_n} - A_T)A^K_{T_n} + A_T(A^K_{T_n} - A_T).$$

We have,

$$\begin{aligned} |||(A^G_{T_n} - A_T)A^G_{T_n}||| &\le |||(A^K_{T_n} - A_T)|||\ ||P_n||^2\ ||A_T||, \\ |||(A^G_{T_n} - A_T)A_T||| &\le (||P_n|| + 1)\ ||A_T||\ ||A^K_{T_n} - A_T||. \end{aligned}$$

These inequalities prove that $A^G_{T_n} \xrightarrow{\nu} A_T$. □

Theorem 4.4.4. *Let u be the solution of (4.1) and let u_n be the Galerkin approximation. Then,*

$$\lim_{n\to 0} ||u - u_n||_{H^1[a,b]} = 0. \tag{4.24}$$

Proof. For n enough large,

$$U - U_n = (\lambda I_2 - A^G_{T_n})^{-1}\left[A^K_{T_n}(P_nU - U) + (A^K_{T_n} - A_T)U\right],$$

$$\begin{aligned} ||U - U_n||_{\tilde{X}} &\le C_1\Big[|||A^K_{T_n}|||\ ||P_nU - U||_{\tilde{X}_2} \\ &+ ||(A^K_{T_n} - A_T)U||_{\tilde{X}_2} + ||P_nF - F||_{\tilde{X}_2}\Big], \end{aligned}$$

Then, by theorem 4.4.1

$$\lim_{n\to\infty}||u-u_n||_{H^1[a,b]}=\lim_{n\to\infty}||U-U_n||_{\tilde{X}_2}=0.$$

□

Galerkin System

By the definition of basis $\{e_i\}_{i=1}^{n}$ and the duality product (4.3), the elements of system (4.14) are represented as

$$A_{1r}(i,j) = \int_a^b\int_a^b K_r(x,t)e_j(t)e_i(x)\,dt\,dx, \quad 1\le i,j\le n,\ r=1,2,$$

$$A_{2r}(i,j) = \int_a^b\int_a^b \frac{\partial K_r}{\partial x}(x,t)e_j(t)e_i(x)\,dt\,dx, \quad 1\le i,j\le n,\ r=1,2,$$

$$\bar{Y}_1(i) = \int_a^b f(x)e_i(x)\,dx, \quad 1\le i\le n,$$

$$\bar{Y}_2(i) = \int_a^b f'(x)e_i(x)\,dx, \quad 1\le i\le n.$$

4.5 Numerical Examples

4.5.1 Numerical tests in $W^{1,1}[0,1]$

To treat our method in this Soblev space, we give this numerical example

Test 01

We consider the following integro-differential equation

$$\lambda u(x)=\int_0^1\Big(|x-t|\log(|x-t|)-|x-t|\Big)u(t)\,dt+\int_0^1\sqrt{|x-t|}u'(t)\,dt+f(x), \tag{4.25}$$

where, $\lambda=9$ and

$$f(x)=\frac{(x-1)^2}{144}\Big(25x^2+38x+45-\log(1-x)(36+24x+12x^2)\Big)$$
$$-\frac{4}{15}(2x+3)(1-x)^{\frac{3}{2}}+9x^2-\frac{8}{15}x^{\frac{5}{2}}-\frac{4}{144}x^4(12\log(x)-25).$$

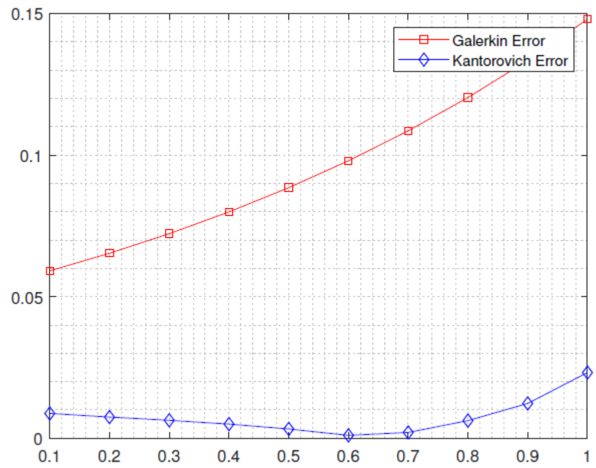
The exact solution $u(x)=x^2$, in the next table, we present the error between the exact and approximate solution, with the error

$$err_n=||u-u_n||_{W^{1,1}[0,1]}$$

n	Galekin	Kantorovich
10	0.0750	0.0041
50	0.0150	8.747e-04
100	0.0075	4.4641e-04

Table 4.1: Numerical results for equation (4.25)

Figure 4.1: Errors between exact and approximate solution of (4.25)



Test 02

We consider the following integro-differential equation

$$\lambda u(x) = \int_0^1 \sqrt{|e^x - e^t|} u(t)\, dt + \int_0^1 |e^x - e^t|^{\frac{1}{3}} u'(t)\, dt + f(x), \tag{4.26}$$

where, $\lambda = 11$ and

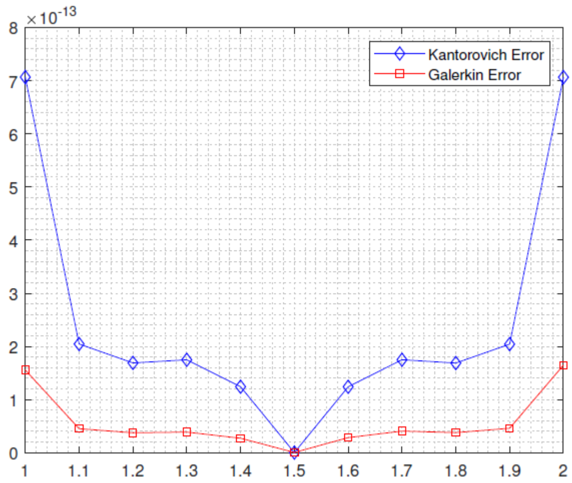
$$f(x) = 7e^M x - \frac{2}{3}(e^1 - e^x)^{\frac{3}{2}} - \frac{3}{4}(e^1 - e^x)^{\frac{4}{3}} - \frac{2}{3}(e^x - 1)^{\frac{3}{2}} - \frac{3}{4}(e^x - 1)^{\frac{4}{3}}.$$

The exact solution $u(x) = e^x$, in the next table, we present the error between exact and approximate solution.

n	Galekin	Kantorovich
10	0.0864	0.0219
50	0.0172	0.0047
100	0.0086	0.0038

Table 4.2: Numerical results for equation (4.26)

Figure 4.2: Errors between exact and approximate solution of (4.26)



4.5.2 Numerical tests in $H^1[a,b]$

To observe the error behaviour in the Sobolev space $H^1[1,2]$, we give the following numerical example

Test 01

We consider the following integro-differential equation

$$\forall x \in [1,2],\ \lambda u(x) = \int_1^2 \frac{u(t)}{e^x + e^t}\, dt + \int_1^2 \frac{u'(t)}{1 + x + e^{2t}}\, dt + f(x), \tag{4.27}$$

where, $\lambda = 2$, $u(x) = e^x$ and

$$f(x) = \log\left(\frac{e^1 + e^x}{e^2 + e^x}\right) + 2e^x + \arctan\left(\frac{e^1}{\sqrt{x+1}}\right) - \frac{\arctan\left(\frac{e^2}{\sqrt{x+1}}\right)}{\sqrt{x+1}}.$$

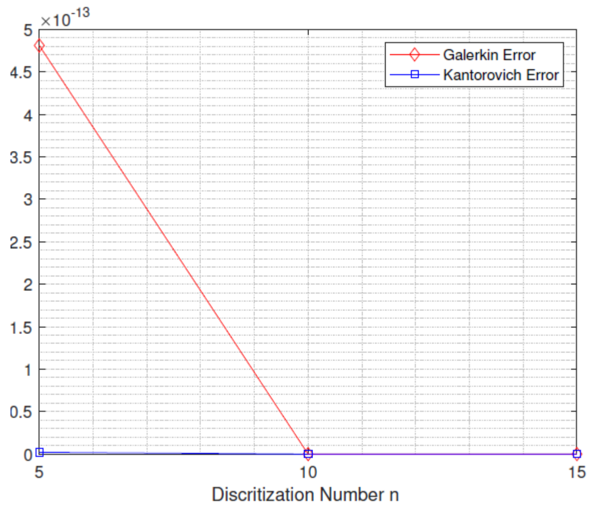
The next table shows the numerical error between the our approximation solution and the exact solution which we define it as

$$err_n = ||u - u_n||_{H^1[1,2]}.$$

n	Galekin	Kantorovich
2	0.0287	1.9624e-04
5	3.7704e-06	1.6025e-07
10	6.6715e-14	1.8394e-13
15	1.7161e-22	9.7426e-25

Table 4.3: Numerical results for equation (4.27)

Figure 4.3: Errors between exact and approximate solution of (4.27)



Test 02

We consider the following integro-differential equation

$$\forall x \in [0,1],\ \lambda u(x) = \int_0^1 \frac{u(t)}{\sqrt{x+t+2}}\,dt + \int_0^1 \frac{u'(t)}{\sqrt{x+t^4+1}}\,dt + f(x), \tag{4.28}$$

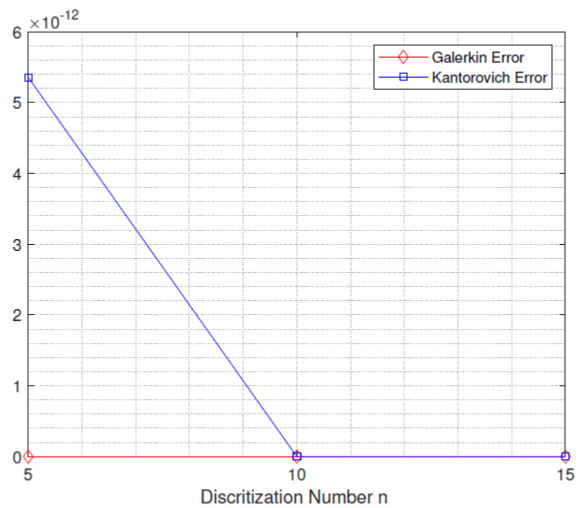
where, $\lambda = 4$, $u(x) = x^2$ and

$$f(x) = 4x^2 - \frac{\arctan\left(\frac{1}{\sqrt{x+1}}\right)}{\sqrt{x+1}} - \frac{2}{3}\left(\sqrt{x+3} - \sqrt{x+2}\right).$$

In the following table, we present the error between the exact and approximate solution in the sense norm of $H^1[0,1]$.

n	Galekin	Kantorovich
2	2.75151e-37	3.60996e-7
5	2.42575e-36	5.35642e-12
10	5.579869e-35	6.79e-20
15	3.8871e-12	5.6832e-22

Figure 4.4: Errors between exact and approximate solution of (4.28)



Conclusion

Integro-differential equations play an important role in many scientific areas. Through this thesis, we have tried to pay attention to study one specific type, which is the linear Fredholm integro-differential equation. We have previously explained its great importance. We tried to study the different forms that the equation can take.

At first, we were interested on the equation defined in the Banach space $C^1[a,b]$. On this basis, we constructed a sufficient condition to ensure the existence and the uniqueness of the solution in the Banach $C^1[a,b]$. After that, we built three different numerical methods: Nyström, Collocation and Kantorovich. Our aim, is to compare them in terms of behaviour and convergence analysis. In addition, to search for the best error that converges to zero. We have provided several numerical examples which show that the Kantorovich method is the best of the three methods.

In the second part of this thesis, we assumed that the kernels K_i for $i = 1, 2$ are weakly singular. From this, we were able to construct a sufficient condition that shows the existence and the uniqueness of the solution, which made the numerical and analytical studies coherent. Because the product integration method is the most common and widely used method for solving this type of equations, we applied it and developed a theory that explains its convergence in $C^1[a,b]$. We did not stop there, we also built another method based on B-Spline functions, through which we obtained a system that contains a block instead of a set of blocks as in the previous methods. We also studied the convergence of the new numerical solution in the Banach space $C^1[a,b]$. The advantage that distinguishes the B-spline method from Product integration is that we can gain in accuracy and shorter time, as the examples show that this method consumes half the time that the first one does.

In the last part, we changed the space from $C^1[a,b]$ to the larger spaces $W^{1,p}[a,b], p \in [1,+\infty[$. We were able to show the existence and uniqueness of the solution in $W^{1,p}[a,b]$. Then, we constructed two methods based on the Kantrorovich and Galerkin methods. We have studied the convergence behaviour of both methods through some steps, the most important of which is to show the ν-convergent of the Kantorovich and Gakerkin operators. Our objective in this section is to answer on the following question: Is the Kantorovich method still the best one in changing the space? The answer is well illustrated in the numerical examples.

As perspective, we would try to apply other projection methods like the Kulkurni method [39], convolution and Fourier series .... Although with this method, it is necessary to solve a system of the same size as Galerkin's method, the solution obtained converges faster than the projection methods proposed in this thesis. We will not only focus on the linear Fredholm integro-differential equation of the form (6). Also, we are going to study the linear and non linear Volterra integro-differential equations.

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