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Et analyse numérique

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Intitulé

On some properties of divisor function

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On some properties of divisor function

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Résumé

Nous étudions la fonction qui représente le nombre de diviseurs positifs d'un entier $n \in \mathbb{N}$, notée $d(n)$. Tout d'abort, nous avons $d(1) = 1$, $d(2) = d(3) = 2$, $d(4) = 3$, ...ext. Ce travail présente aussi les plus importantes propriétées de cette fonction. De plus, en précisant quelques propriétées exigent l'utilisation des fonctions multiplicatives. En perspective, il existe plusieurs types des questions ouvertes en relation avec la fonction $d(n)$ comme les equations diophantiennes faisant intervenir la fonction d'Euler.

Mots clés. Fonctions arithmétiques, Fonction diviseur, équations Diophantiennes.

Abstract

Recall that the divisor function $d(n)$ counts the number of positive divisors of n. For instance, $d(1) = 1$, $d(2) = d(3) = 2$, $d(4) = 3$, and so on. In this work, we present the most important properties of the divisor function $d(n)$. By design, some of the properties require to use several multiplicative functions. For future research there are several types of open questions related to the divisor function as well as Diophantine equations and inequalities.

Keywords and phrases Arithmetic functions, divisor function, Diophantine equations.

Table of notations

Notation	**Explanation**
$\mathbb{Z}$	The set of integers
$\mathbb{N}$	The set of positive integers
$a \in A$	The element a belongs to the set A
$n\|m$	n divides m or m is divisible by n
$\gcd(m,n)$ or (m,n)	The greatest common divisor of m and n
$\pi(n)$	The number of primes $\leq n$
$d(n)$ or $\tau(n)$	Number of positive divisors of n
$\sigma(n)$	Sum of positive divisors of n
$\sigma_\alpha(n)$	Generalized sum of divisors functions
$\varphi(n)$	Euler's totient function
φ_s	The generalozed Euler's function
ψ_s	The related Euler's function
$\omega(n)$	The number of distinct prime factors of n
$\Omega(n)$	The total number of distinct prime factors of n
$\Lambda(n)$	Von Mangoldt function
$\lambda(n)$	Liouville function
$id(n)$	Identity function: $id(n)$; defined by $id(n) = n$ for all n.
$\mu(n)$	Moebius function
$\lfloor x \rfloor$	The largest positive integer $\leq x$
$\gamma(n)$	The kernel of n given by $\gamma(n) = \prod_{p\|n} p$.
$[m,n]$	The least common multiple of m and n.

Introduction

Recall that a real or complex valued function defined on the positive integers (or all integers) is called an arithmetic function or a number-theoretic function. A multiplicative function is an arithmetic function $f(m)$ such that $f(mn) = f(m)f(n)$ for all pairs of relatively prime positive integers m and n. If $f(m)$ is multiplicative, then it is easy to prove by induction on k that if $m_1, ..., m_k$ are pairwise relatively prime positive integers, then $f(m_1...m_k) = f(m_1)...f(m_k)$. For details, see [7],[6].

The work on the number of positive divisors was introduced since a long time. Note that in 1537 (see [1]), Girolamo Cardano claimed that if $n = p_1p_2...p_r$, where $p_1, p_2, ..., p_r$ are distinct primes, then the number of positive divisors of n is equal to $2 + 2 + 2^2 + 2^3 + ... + 2^{r-1}$, where we use $d(n)$ as the standard notation of this function throughout.

The goal of this work is to give important results about divisor functions. For example, we indicate when $d(n)$ is prime and composite, a power of 2, ... etc. Also, n is a square if and only if $d(n)$ is odd and we compute the product of the positive divisors of a given integer n. An explicit formula of $d(n)$ in terms of the prime powers that exactly divide n will be used to solve Diophantine equations and inequalities. In addition, we find all the natural numbers which have precisely a fixed number N of positive divisors from which we can determine the least natural number with this property. One of the most properties is to deal with iterating sequences involving the divisor function and give expressions defined by other arithmetic functions that limit the function d from above and from below. Many other problems related to the knowledge of consecutive numbers that have the same number of positive devisors [3], which is a great question given as: How many numbers n for which $d(n) = d(n+1) = ...d(n+a)$ $(a \geq 1)$?

As a conclusion, based on the Fundamental Theorem of Arithmetic, we give with elementary proofs several types of classical theorems involving the divisor function and some other multiplicative functions. Some other references, eg, see [2],[8] and [5].

Chapter 1

Basic arithmetic functions

First, we state the Fundamental Theorem of Arithmetic and then recall definitions of basic arithmetic functions and we illustrate an example for each function.

Theorem 1.1 (Fundamental Theorem of Arithmetic, see [6, page 25]) *Every positive integer n greater than 1 can be written uniquely as the product of primes:*

$$n = q_1^{\alpha_1} q_2^{\alpha_2} ... q_k^{\alpha_k} = \prod_{i=1}^{k} q_i^{\alpha_i}, \tag{1.1}$$

where $q_1, q_2, ..., q_k$ are distinct primes and $\alpha_1, \alpha_2, ..., \alpha_k$ are natural numbers. The equation (1.1) is often called the prime power decomposition of n, or the standard prime factorization of n.

Example 1.1 *Let $n = 2000$ and $m = 2022$. Then $n = 2^4 \cdot 5^3$ and $m = 2 \cdot 3 \cdot 337$.*

An important class of arithmetic functions are *multiplicative* functions defined as follows. For details, one can see [1],[6],[4].

Definition 1.1 *Let $f : \mathbb{N} \longrightarrow \mathbb{C}$ be an arithmetic function.*

1. *f is called multiplicative if $f \neq 0$ and*
$$f(m \cdot n) = f(m) \cdot f(n) \tag{1.2}$$
whenever $\gcd(m, n) = 1$.

2. *f is called additive if it satisfies*
$$f(m \cdot n) = f(m) + f(n) \tag{1.3}$$
whenever $\gcd(m, n) = 1$. *If the condition (1.2) (resp. (1.3)) holds without the restriction* $\gcd(m, n) = 1$*, then f is called completely (or totally) multiplicative resp. completely (or totally) additive.*

Remark 1.1 *We have the following property of all multiplicative functions. If f is multiplicative then $f(1) = 1$. In fact, since f is not identically zero, there exists $n \in \mathbb{N}$ such that $f(n) \neq 0$. We have $f(n) = f(n)f(1)$ as f is multiplicative. Hence $f(1) = 1$.*

We present the famous important arithmetic functions as follows. Some of which are multiplicative.

1. **Divisor function**[1]: $d(n)$, the number of positive divisors of n (including the trivial divisors $d = 1$ and $d = n$). As usual, the notation "$d|n$" as the range for a sum or product means that d ranges over the positive divisors of n. Thus, the number of divisors function is given by
$$d(n) = \sum_{d|n} 1. \tag{1.4}$$

[1] Another common notation for the divisor function is $\tau(n)$.

For example, the positive divisors of 15 are $1, 3, 5$, and 15. So $d(15) = 4$. Note that if p is prime, $d(p) = 2$.

2. **Sum of divisors function:** $\sigma(n)$, the sum over all positive divisors of n; i.e.,

$$\sigma(n) = \sum_{d|n} d.$$

Let us have the natural number $n \geq 2$ with its canonical representation $n = q_1^{\alpha_1}...q_k^{\alpha_k}$, where $q_1, ..., q_k$ are distinct primes and $\alpha_1, ..., \alpha_k$ are positive integers. We have

$$\sigma(n) = \prod_{i=1}^{k} \frac{q_i^{\alpha+1} - 1}{q_i - 1}. \tag{1.5}$$

3. **Generalized sum of divisors functions**[2]**:** $\sigma_\alpha(n)$, defined by $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$. Here α can be any real or complex parameter. This function generalizes the divisor function ($\alpha = 0$) and the sum of divisors function ($\alpha = 1$).

4. **Number of distinct prime factors:** The omega function $\omega(n)$ is defined as the number of distinct prime factors of n, where $\omega(1) = 0$, i.e., $\omega(n) = k$ if $n \geq 2$ and $n = q_1^{\alpha_1} q_2^{\alpha_2}...q_k^{\alpha_k}$. Or equivalently, $\omega(n) = \sum_{p|n} 1$.

5. **Identity function:** $id(n)$; defined by $id(n) = n$ for all n.

6. **Moebius function:** $\mu(n)$, defined by $\mu(1) = 1$, $\mu(n) = 0$ if n is not square-free (i.e., divisible by the square of a prime), and $\mu(n) = (-1)^k$

[2]Sometimes $\sigma_\alpha(n)$ is called the divisor power sum function.

if n is composed of k distinct prime factors (i.e., $n = q_1 q_2 ... q_k$, where $q_1, q_2, ..., q_k$ are distinct primes.

7. **Von Mangoldt function:** $\Lambda(n)$, defined by $\Lambda(n) = 0$ if n is not a prime power, and $\Lambda(p^m) = \log p$ for any prime power p^m.

8. **Total number of prime divisors:** $\Omega(n)$, defined in the same way as $\omega(n)$, except that prime divisors are counted with multiplicity. Thus, $\Omega(1) = 0$ and $\Omega(n) = \sum_{i=1}^{k} \alpha_i$ if $n \geq 2$ and $n = q_1^{\alpha_1} q_2^{\alpha_2} ... q_k^{\alpha_k}$, i.e., $\Omega(n) = \sum_{p^m|n} 1$ For square-free integers n, the functions $\omega(n)$ and $\Omega(n)$ are equal and are related to the Moebius function by $\mu(n) = (-1)^{\omega(n)}$. For all integers n, $\lambda(n) = (-1)^{\Omega(n)}$.

9. **Liouville function:** $\lambda(n)$, defined by $\lambda(1) = 1$ and $\lambda(n) = (-1)^k$ if n is composed of k not necessarily distinct prime factors (i.e., if $n = q_1^{\alpha_1} q_2^{\alpha_2} ... q_k^{\alpha_k}$, then $\lambda(n) = \prod_{i=1}^{k} (-1)^{\alpha_i}$. Thus, $\lambda(n) = (-1)^{\Omega(n)}$.

10. $\pi(x)$: The number of primes $\leq x$. For example, $\pi(5.3) = 3$.

11. **Euler's phi function:** $\varphi(n)$ is defined as

$$\varphi(n) = \sum_{\substack{1 \leq k < n \\ \gcd(k,n)=1}} 1.$$

Definition 1.2 *The Euler phi function $\varphi(n)$ is the arithmetic function that counts the number of integers in the set $1, 2, ..., n-1$ that are relatively prime to n.*

Let us take the first numbers. We have

- $\varphi(1) = 1$. The only number n such that, $\gcd(n, 1) = 1$ is 1 itself.

- $\varphi(2) = 1$. The only number n such that, $\gcd(n, 2) = 1$ is 1.
- $\varphi(3) = 2$. The only number n such that, $\gcd(n, 3) = 1$ are $1, 2$.
- $\varphi(4) = 2$. The only number n such that, $\gcd(n, 4) = 1$ are $1, 3$.

Theorem 1.2 (see [1]) *The Euler phi function is multiplicative. Moreover,*

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right). \tag{1.6}$$

Theorem 1.3 (see [1]) *Let n be a positive integer. Then*

$$\sum_{d|n} \varphi(d) = n. \tag{1.7}$$

Example 1.2 *Let $n = 28$ and $d \mid 28$. Let C_d denote the class of those positive integers $\leq n$, where $(m, n) = d$. Since 28 has six positive factors $1, 2, 4, 7, 14$, and 28, there are six such classes:*

- $C_1 = \{1, 3, 5, 9, 11, 13, 15, 17, 19, 23, 25, 27\}$,
- $C_2 = \{2, 6, 10, 18, 22, 26\}$,
- $C_4 = \{4, 8, 12, 16, 20, 24\}$,
- $C_7 = \{7, 21\}$,
- $C_{14} = \{14\}$,
- $C_{28} = \{28\}$.

In fact, these classes contain $12 = \varphi(28) = \varphi(28/1)$, $6 = \varphi(14) = \varphi(28/2)$, $6 = \varphi(7) = \varphi(28/4)$, $2 = \varphi(4) = \varphi(28/7)$, $1 = \varphi(2) = \varphi(28/14)$, *and* $1 = \varphi(1) = \varphi(28/28)$ *elements, respectively. Also, they form a partitioning of the set of positive integers* ≤ 28*. Therefore, the sum of the numbers of elements in the various classes must equal* 28*; that is,* $12+6+6+2+1+1 = 28$*. In other words,*

$$\varphi(28) + \varphi(14) + +\varphi(7) + \varphi(4) + \varphi(2) + \varphi(1) = 28,$$

that is $\sum\limits_{d|28} \varphi(d) = 28$.

Chapter 2

On the behaviour of $d(n)$

In the following table, we give $d(n)$ for $1 \leq n \leq 12$.

n	1	2	3	4	5	6	7	8	9	10	11	12
$d(n)$	1	2	2	3	2	4	2	4	3	4	2	6

Table 1. The number of divisors for $1 \leq n \leq 12$.

Next, we compute the number of positive divisors of a prime power. We have.

Proposition 2.1 *Let p be prime and α a positive integer. Then $d(p^\alpha) = \alpha + 1$.*

Proof. The divisors of p^α are $1, p, p^2, ..., p^{\alpha-1}$ and p^α. Consequently, p^α has exactly $\alpha + 1$ divisors, so that $d(p^\alpha) = \alpha + 1$. ■

When n has two distinct prime powers. That is, $n = p^\alpha q^\beta$, where p, q are two distinct primes and $\alpha, \beta \geq 1$. The positive divisors of n are given below

$$\begin{array}{ccccc}
1 & p & p^2 & \cdots & p^\alpha \\
q & pq & p^2q & \cdots & p^\alpha q \\
q^2 & pq^2 & p^2q^2 & \cdots & p^\alpha q^2 \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
q^\beta & pq^\beta & p^2q^\beta & \cdots & p^\alpha q^\beta
\end{array}$$

Thus, there are $(\alpha+1)(\beta+1)$ positive divisors of n.

2.1 Explicit formulas of $d(n)$

We find the formula for $d(n)$ in terms of the prime factorization of n. That is, in terms of the prime powers dividing n.

Proposition 2.2 *Assume that* $n = q_1^{\alpha_1} q_2^{\alpha_2} ... q_k^{\alpha_k}$, *where* $q_1, q_2, ..., q_k$ *are distinct primes and* $\alpha_1, \alpha_2, ..., \alpha_k \geq 1$. *Then*

$$d(n) = (\alpha_1 + 1)(\alpha_2 + 1) ... (\alpha_k + 1). \tag{2.1}$$

Proof. Let $n = q_1^{\alpha_1} q_2^{\alpha_2} ... q_k^{\alpha_k}$. By (1.4), we obtain

$$d(n) = \sum_{0 \leq x_1 < a_1} \sum_{0 \leq x_2 < a_2} ... \sum_{0 \leq x_k < a_k} 1 = \prod_{i=1}^{k} (\alpha_i + 1).$$

The proof is finished. ■

Remark 2.1 *As a conclusion, we have the following properties:*

- $d(n) = 1$ *if and only if* $n = 1$,
- $d(n) = 2$ *if and only if* $n = p$ *(*p *is a prime),*
- n *is a square-free* $\Rightarrow d(n)$ *is a power of* 2,
- $d(n) = 2^a$ *with* $a \geq 2 \not\Rightarrow n$ *is a square-free,*
- $d(n) > 2$ *if and only if* n *is composite.*

Corollary 2.1 *Let* n *be a fixed positive integer with* $n \geq 2$. *The equation* $d(x) = n$ *has infinitely many integer solutions.*

Proof. We put $x = p^{n-1}$, where p is a prime number. Then clearly, $d(x) = n$.

■

Proposition 2.3 *For any positive integer n we have*

$$d(n) = \sum_{k=1}^{n} \left(\left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor \right). \tag{2.2}$$

Proof. Note that

$$\left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor = \begin{cases} 1, \text{ if } k \mid n \\ 0, \text{ if } k \nmid n \end{cases}$$

Hence

$$\sum_{k=1}^{n} \left(\left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor \right) = \sum_{k \mid n}^{n} 1 = d(n).$$

The proof is finished. ■

As an application, for $n = 6$ we see that

$$\begin{aligned} \sum_{k=1}^{6} \left(\left\lfloor \frac{6}{k} \right\rfloor - \left\lfloor \frac{5}{k} \right\rfloor \right) &= \left\lfloor \frac{6}{1} \right\rfloor - \left\lfloor \frac{5}{1} \right\rfloor + \left\lfloor \frac{6}{2} \right\rfloor - \left\lfloor \frac{5}{2} \right\rfloor + \left\lfloor \frac{6}{3} \right\rfloor - \left\lfloor \frac{5}{3} \right\rfloor + \left\lfloor \frac{6}{4} \right\rfloor - \left\lfloor \frac{5}{4} \right\rfloor \\ &\quad + \left\lfloor \frac{6}{5} \right\rfloor - \left\lfloor \frac{5}{5} \right\rfloor + \left\lfloor \frac{6}{6} \right\rfloor - \left\lfloor \frac{5}{6} \right\rfloor \\ &= \left\lfloor \frac{6}{1} \right\rfloor - \left\lfloor \frac{5}{1} \right\rfloor + \left\lfloor \frac{6}{2} \right\rfloor - \left\lfloor \frac{5}{2} \right\rfloor + \left\lfloor \frac{6}{3} \right\rfloor - \left\lfloor \frac{5}{3} \right\rfloor + \left\lfloor \frac{6}{6} \right\rfloor - \left\lfloor \frac{5}{6} \right\rfloor \\ &= 1 + 1 + 1 + 1 = 4 = d(6). \end{aligned}$$

Next, there is an explicit formula of $d(n)$ using the integer part of the logarithm of n in basis 2.

Theorem 2.1 *Let n be an even perfect number. Then $d(n) = [\log_2 n] + 2$, where $\log_2 n$ stands for the logarithm of n in basis 2.*

Proof. Since n is an even perfect number, there exists a positive integer k such that

$$n = 2^{k-1}\left(2^k - 1\right) = 2^{k-1}p,$$

where $p = 2^k - 1$ is Mersenne prime. Then $d(n) = d\left(2^{k-1}\right).d(p) = 2k$. On the other hand, taking the logarithm in basis 2 on both sides of the first equation, we obtain successively

$$\begin{aligned}
\log_2 n &= (k-1)\log_2 2 + \log_2\left(2^k - 1\right) \\
&= (k-1) + \log_2\left(2^k\left(1 - \frac{1}{2^k}\right)\right) \\
&= (k-1) + k\log_2 2 + \log_2\left(1 - \frac{1}{2^k}\right) \\
&= 2k - 1 + \log_2\left(1 - \frac{1}{2^k}\right) \\
&= 2k - 1 + \alpha_k,
\end{aligned}$$

where it is easy to see that the expression α_k satisfies $-1 < \alpha_k < 0$. But since $d(n) = 2k$, this last equation can be written as $d(n) = \log_2 n + 1 - \alpha_k$. Since $0 < -\alpha_k < 1$, we may therefore write, $\log_2 n + 1 + 0 < d(n) < \log_2 n + 1 + 1$, that is, $\log_2 n + 1 < d(n) < \log_2 n + 2$, which means of course that

$$d(n) = [\log_2 n + 2] = [\log_2 n] + 2,$$

This completes the proof. ■

2.2 Basic properties of $d(n)$

A natural question, when is $d(n)$ odd? The answer is given by the following theorem:

Theorem 2.2 *$d(n)$ is odd if and only if n is a perfect square.*

Proof. Assume that $n = \prod_{i=1}^{k} q_i^{\alpha_i}$, then $d(n) = \prod_{i=1}^{k} (\alpha_i + 1)$, in which case it is clear that $d(n)$ is odd if and only if α_i is even for every $i \geq 1$. We put $\alpha_i = 2b_i$, where $b_i \geq 1$. Thus, $n = \left(\prod_{i=1}^{k} q_i^{b_i}\right)^2$, which is a perfect square. Conversely, if n is a perfect square, that is, $n = m^2$, $m \in \mathbb{N}$. Assume that $m = \prod_{i=1}^{k} q_i^{e_i}$, we therefore obtain that $n = \prod_{i=1}^{k} q_i^{2e_i}$, and the uniqueness of the canonical representation of n then implies that $\alpha_i = 2e_i$ for $i = 1, 2, ..., k$. Thus, $d(n)$ is odd. ■

Theorem 2.3 *Let n be a positive integer. Then*

$$\prod_{d|n} d = n^{\frac{d(n)}{2}}. \tag{2.3}$$

Proof. Note that if d runs through the set of divisors of n, then n/d does also. Therefore, we have

$$\left(\prod_{d|n} d\right)^2 = \prod_{d|n} d \cdot \prod_{d|n} \frac{n}{d} = \prod_{d|n} n = n^{d(n)}.$$

As required. Note also that in the case when $d(n)$ is odd, the formula still holds because by proposition 2.2, n is then a perfect square. The proof is finished. ■

Corollary 2.2 *For every $n \geq 3$, we have $d(n(n+1)) \geq 6$.*

Proof. First, we show $(n, n+1) = 1$, that is, n and $n+1$ are coprime. Let $d = (n, n+1)$, then d divided the difference $(n+1) - n = 1$ and so $d = 1$. Therefore, $d(n(n+1)) = d(n) \cdot d(n+1)$. There are two cases to consider.

- If n is prime, i.e., $n+1$ is composite, then $d\left(n\left(n+1\right)\right)=2d\left(n+1\right)\geq 2\cdot 3=6$.

- If n is composite, then $d\left(n\right)\geq 3$ and so $d\left(n\right)d\left(n+1\right)\geq 3d\left(n+1\right)\geq 3\cdot 2=6$.

This completes the proof. ■

Remark 2.2 *Let n be a positive integer, and put $n=2^{a}.m$, where m is odd. That is,* $\gcd\left(2,m\right)=1$. *Then*

$$\frac{d\left(2n\right)}{d\left(n\right)}=\frac{d\left(2^{a+1}\right)d\left(m\right)}{d\left(2^{a}\right)d\left(m\right)}=\frac{a+2}{a+1}.$$

We conclude that $d\left(2n\right)/d\left(n\right)$ is a positive integer if and only if $a+1$ divides $a+2$. This statement is only true for $a=0$, and hence n is odd.

Proposition 2.4 *Let $n\geq 1$, and consider the functions $f_{1}\left(n\right)$and $f_{2}\left(n\right)$ which stand respectively for the product of the odd divisors of n and for the product of the even divisors of n. Then*

$$f_{1}\left(n\right)=m^{\frac{d(m)}{2}} \tag{2.4}$$

and

$$f_{2}\left(n\right)=\left(2^{a(a+1)}\cdot m^{a}\right)^{\frac{d(m)}{2}}=\left(2n\right)^{\frac{ad(m)}{2}}, \tag{2.5}$$

where m and a are defined implicitly by $n=2^{a}\cdot m$ with m is odd.

Proof. By Theorem 2.3, we have $\prod\limits_{d|n}d=n^{\frac{d(n)}{2}}$. Defining m and a by $n=2^{a}\cdot m$, with m is odd. The relation $f_{1}\left(n\right)=m^{\frac{d(m)}{2}}$ is immediate. To establish the relation

$$f_{2}\left(n\right)=\left(2^{a(a+1)}\cdot m^{a}\right)^{\frac{d(m)}{2}}=\left(2n\right)^{\frac{ad(m)}{2}},$$

we first observe that $f_1(n) \cdot f_2(n) = n^{\frac{d(n)}{2}}$, so that $f_2(n) = \frac{n^{\frac{d(n)}{2}}}{f_1(n)}$. Substituting in this last equation, we easily obtain for the desired equality. ■

Theorem 2.4 *Let n be a positive integer. Then*

$$\sum_{k=1}^{n} d(k) = \sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the largest positive integer $\leq x$.

Example 2.1 *Let $n = 12$. Then*

$$\sum_{k=1}^{12} d(k) = 1 + 2 + 2 + 3 + 2 + ... + 6 = 35,$$

and

$$\begin{aligned} \sum_{k=1}^{12} \left\lfloor \frac{12}{k} \right\rfloor &= \left\lfloor \frac{12}{1} \right\rfloor + \left\lfloor \frac{12}{2} \right\rfloor + \left\lfloor \frac{12}{3} \right\rfloor + \left\lfloor \frac{12}{4} \right\rfloor + \left\lfloor \frac{12}{5} \right\rfloor + \left\lfloor \frac{12}{6} \right\rfloor + \\ &\quad + \left\lfloor \frac{12}{7} \right\rfloor + \left\lfloor \frac{12}{8} \right\rfloor + \left\lfloor \frac{12}{9} \right\rfloor + \left\lfloor \frac{12}{10} \right\rfloor + \left\lfloor \frac{12}{11} \right\rfloor + \left\lfloor \frac{12}{12} \right\rfloor \\ &= 12 + 6 + 4 + 3 + 2 + 2 + 1 + 1 + 1 + 1 + 1 + 1 = 35. \end{aligned}$$

As required.

Proposition 2.5 *Let $d_1(n)$ be the number of odd divisors of n. Then d_1 is a multiplicative function .*

Proof. Let m, n be positive integers such that $(m, n) = 1$. If m and n are both odd, then since d is multiplicative, $d_1(m \cdot n) = d(m \cdot n) = d(m) \cdot d(n) = d_1(m) \cdot d_1(n)$, and so the result is proved in this case.

Now, assume that one of these two integers is even, say m, then there exists a positive integer α such that $m = 2^{\alpha} \cdot r$ with r is odd (note that$(r, n) = 1$). We then have, $d_1 (m \cdot n) = d_1 (r \cdot n) = d (r \cdot n) = d (r) \cdot d (n) = d_1 (2^{\alpha} \cdot r) \cdot d_1 (n) = d_1 (m) \cdot d_1 (n)$.

This completes the proof. ■

Theorem 2.5 *For any positive integer n, we have*

$$d (n) \leq 2\sqrt{n}. \tag{2.6}$$

Proof. Let $d_1 < d_2 < ... < d_k$ be the divisors of n not exceeding $\sqrt{n}$. The remaining divisors are

$$\frac{n}{d_1}, \frac{n}{d_2}, ..., \frac{n}{d_k}.$$

It follows that $d (n) \leq 2k \leq 2\sqrt{n}$. ■

Theorem 2.6 *Let n be a positive integer. Then*

$$\left(\sum_{d|n} d (d) \right)^2 = \sum_{d|n} d^3 (d) .$$

Example 2.2 *If $n = 6 = 2.3$, then* $\left(\sum_{d|6} d (d) \right)^2 = (1 + 2 + 2 + 4)^2 = 9^2 = \sum_{d|6} d^3 (d) = 1^3 + 2^3 + 2^3 + 4^3$.

From the proof we use the following lemma, which is prove by induction.

Lemma 2.1 *We have*

$$\left(\sum_{i=1}^{n} j \right)^2 = \sum_{i=1}^{n} j^3 = \left(\frac{n (n + 1)}{2} \right)^2 .$$

Proof of Theorem 2.6. Let $n = q_1^{\alpha_1} q_2^{\alpha_2} ... q_k^{\alpha_k}$ be the representation of n as product of distinct prime powers. Then

$$\begin{aligned}
\sum_{d|n} d^3(d) &= \sum_{d_1|q_1^{\alpha_1},...,d_k|q_k^{\alpha_k}} d^3(d_1)\, d^3(d_2)\, ... d^3(d_k) \\
&= \prod_{j=1}^{k} \sum_{d_j|q_j^{\alpha_j}} d^3(d_j) = \prod_{j=1}^{k} \left(1^3 + 2^3 + ... + (\alpha_j+1)^3\right) \\
&= \left(\prod_{j=1}^{k} (1 + 2 + ... + (\alpha_j + 1)) \right)^2 = \left(\sum_{d|n} d(d) \right)^2 .
\end{aligned}$$

The proof is finished. ■

Example 2.3 *We identify all natural numbers having exactly* 14 *divisors. Since* $n > 1$, *we have that* $n = q_1^{\alpha_1} q_2^{\alpha_2} ... q_r^{\alpha_r}$ *and* $d(n) = (\alpha_1 + 1)(\alpha_2 + 1) ... (\alpha_r + 1) = 14 = 2 \cdot 7$. *Then, either* $r = 2$ *with* $\alpha_1 = 1$ *and* $\alpha_2 = 6$ *or* $r = 1$ *with* $\alpha_1 = 13$. *It follows that the positive numbers with exactly* 14 *divisors are of two kinds;*

- *The numbers* $p \cdot q^6$,*where* p *and* q *are distinct prime numbers.*
- *The numbers* p^{13}, *where* p *is an arbitrary prime number.*

Example 2.4 *We can easily find the smallest positive integer* x *such that* $d(x) = 9$, $d(x) = 10$ *and* $d(x) = 15$. *In fact, we have* $x =$, $x =$ *and* $x =$, *respectively.*

- We find all the natural numbers which have precisely 10 divisors. Indeed, if $d(n) = 10$, then by (2.1) we have $(a_1 + 1)(a_2 + 1) ... (a_k + 1) = 10$. We may, of course, assume that $a_1 \leq a_2 \leq ... \leq a_k$. Since there are two ways of presenting 10 as the product of natural numbers ≥ 2 written in the order of their magnitude, namely $10 = 2 \cdot 5$ and $10 = 10$, then either $k = 2$, $a_1 = 1$ and $a_2 = 4$ or $k = 1$ and $a = 9$. It follows that the natural numbers which have precisely 10 divisors are either the numbers $p \cdot q^4$, where p, $q \neq p$ are arbitrary primes or the numbers p^9, where p is an arbitrary prime.

- Now, we find the least natural number n for which $d(n) = 10$. In view of the above problem, consider the numbers 2^9, $2 \cdot 3^4$ and $3 \cdot 2^4$. Clearly, $n = 3 \cdot 2^4$ is the least natural number n for which $d(n) = 10$.

- In general, it is easy to prove that for given prime numbers p, q with $q > p$ the least natural number that has precisely pq divisors is the number $2^{q-1} \cdot 3^{p-1}$. Also, we can prove that for given prime numbers $q_1, q_2, ..., q_s$ with $q_1 < q_2 < ... < q_s$ the least natural number that has precisely $q_1 q_2 ... q_s$ divisors is the number $2^{q_s - 1} \cdot 3^{q_{s-1}-1} ... p_s^{q_1 - 1}$, where p_i is the i-th prime number.

- A. Schinzel proved that for all natural numbers h and m there exists a natural number $n > h$ such that

$$\frac{d(n)}{d(n \pm i)} > m, \text{ for } i = 1, 2, ..., h.$$

Theorem 2.7 *The number of pairs of positive integers with least common multiple equal to the positive integer n is $d(n^2)$.*

Theorem 2.8 $\sum_{k|n} d(k) \mu\left(\frac{n}{k}\right) = 1$*, for any positive integer n.*

For example, for $n = 6$ we have

$$\sum_{k|n} d(k)\,\mu\left(\frac{n}{k}\right) = d(1)\,\mu(6) + d(2)\,\mu(3) + d(3)\,\mu(2) + d(6)\,\mu(1) = 1.$$

Proposition 2.6 *n is prime if and only if $d(n) = 2$ and $d(n)$ is prime if and only if $n = p^{q-1}$, where p and q are prime numbers.*

Proof. Clearly, if n is prime then n has only two positive divisors, namely 1 and p. Thus, $d(n) = 2$. Conversely, if $d(n) = 2$ with $n = q_1^{\alpha_1}...q_k^{\alpha_k}$. By (2.1), $(\alpha_1 + 1)...(\alpha_k + 1) = 2$ from which it follows that $k = 1$ and $\alpha_1 + 1 = 2$. Hence, $n = q_1$ is prime.

Now, assume that $d(n) = q$ is prime. Equivalently, n has one prime factors, say p. Otherwise, $d(n)$ is composite. Thus, $n = p^{d(n)-1} = p^{q-1}$. ■

Theorem 2.9 *For all positive integers m and n, we have*

$$d(mn) \leq d(m)\,d(n). \tag{2.7}$$

Proof. The proof holds immediately from the prime factorization of m and n. In fact, let $m = \prod_{i=1}^{r} p_i^{\alpha_i}$, $n = \prod_{i=1}^{r} p_i^{\beta_i}$ $(\alpha_i, \beta_i \geq 0)$be the canonical factorizations of m and n. (Here some α_i or β_i can take the values 0, too). Then

$$d(mn) = \prod_{i=1}^{r} (\alpha_i + \beta_i + 1) \geq \prod_{i=1}^{r} (\beta_i + 1)$$

with equality only if $\alpha_i = 0$ for all i. Thus,

$$d(mn) \geq d(n)$$

for all m, n, with equality only for $m = 1$. Since

$$\prod_{i=1}^{r}(\alpha_i+\beta_i+1) \leq \prod_{i=1}^{r}(\alpha_i+1)\prod_{i=1}^{r}(\beta_i+1),$$

we get the relation

$$d(mn) \leq d(m)\,d(n)$$

with equality only for $(n,m) = 1$. ■

Proposition 2.7 *If $n \mid m$, then*

$$\frac{d(mn)}{d(m)} \leq \frac{d(n^2)}{d(n)}. \tag{2.8}$$

Proof. Let $m = \prod p^{\alpha} \cdot \prod q^{\beta}$ and $n = \prod p^{\alpha'}$ $(\alpha' \leq \alpha)$ be the prime factorizations of m and n. Then

$$\frac{d(mn)}{d(m)} = \frac{\prod(\alpha'+\alpha+1)\prod(\beta+1)}{\prod(\alpha+1)\prod(\beta+1)} = \prod\frac{\alpha'+\alpha+1}{\alpha+1}$$

Now, it is clear that $\frac{\alpha'+\alpha+1}{\alpha+1} \leq \frac{2\alpha'+1}{\alpha'+1} \Leftrightarrow \alpha' \leq \alpha$. This immediately implies relation (2.8). ■

We can remark several properties of these functions for two natural non-zero numbers m and n.

Theorem 2.10 *For all $m, n \in \mathbb{N}^*$, we have*

$$d(mn) \leq d(m)\,n. \tag{2.9}$$

For all $m, n \in \mathbb{N}^*$, we also have

$$\frac{d(m)}{m} \leq \frac{d(n)}{n}.$$

Proof. We will show that $d(m) \leq m$, for all $m \in \mathbb{N}^*$. From the inequality (2.6), $d(m) \leq 2\sqrt{m}$ but $m \geq 2\sqrt{m}$ for $m \geq 4$, therefore $d(m) \leq m$, $m \geq 4$. For $m \in \{1, 2, 3\}$ it is easy to see that the inequality is true. From the inequality (2.7), $d(m)\, d(n) \geq d(mn)$, for all $m \in \mathbb{N}^*$, but $d(n) \leq n$, so $d(m)$ for all $m \in \mathbb{N}^*$. Since $n \mid m$, we have $m = nd$, and from the inequality (2.9), we obtain $d(nd) \leq d(n)\, d$, which is equivalent with $nd(m) \leq nd(n) = md(n)$. ∎

Corollary 2.3 *For all $m, n \in \mathbb{N}^*$, we have*

$$\frac{d(mn)}{mn} \leq \frac{d(m) + d(n)}{m+n} \tag{2.10}$$

and

$$d(mn) \leq \frac{m^2 d(n) + n^2 d(m)}{m+n}. \tag{2.11}$$

Proof. Applying the inequality (2.9), we deduce that

$$\begin{aligned}(m+n)\, d(mn) &= md(mn) + nd(mn) \leq mnd(m) + mnd(n) \\ &= mn\,(d(m) + d(n)),\end{aligned}$$

which means that

$$(m+n)\, d(mn) = md(mn) + nd(mn) \leq m^2 d(n) + n^2 d(m).$$

This proves (2.10). Similarly, we prove the inequality (2.11). ∎

Theorem 2.11 *For all $m, n \in \mathbb{N}^*$, we have*

$$d((m,n))\, d([m,n]) = d(m) \cdot d(n), \tag{12}$$

where (m, n) is the greatest common divisor of m and n and $[m, n]$ is the least common multiple of m and n.

Proof. Let m and n be tow natural non-zero numbers. We will factorize the numbers m and n in prime factors, that is, $m = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k} \cdot q_1^{\beta_1} q_2^{\beta_2} ... q_s^{\beta_s}$, $n = p_1^{\gamma_1} p_2^{\gamma_2} ... p_k^{\gamma_k} \cdot r_1^{\delta_1} r_2^{\delta_2} ... r_t^{\delta_t}$, $q_j \neq r_l$, for all $j \in \{1, ..., s\}$ and for all $l \in \{1, ..., t\}$, therefore

$$d(m) = \prod_{i=1}^{k} (\alpha_i + 1) \prod_{j=1}^{s} (\beta_j + 1)$$

and

$$d(n) = \prod_{i=1}^{k} (\gamma_i + 1) \prod_{l=1}^{t} (\delta_l + 1).$$

We obtain

$$d((m, n)) = \prod_{i=1}^{k} (\min\{\alpha_i, \gamma_i\} + 1),$$

and

$$d([m, n]) = \prod_{i=1}^{k} (\max\{\alpha_i, \gamma_i\} + 1) \prod_{j=1}^{s} (\beta_j + 1) \prod_{l=1}^{t} (\delta_l + 1),$$

which means that $d((m, n))\, d([m, n]) = d(m) \cdot d(n)$ for all $m, n \in \mathbb{N}^*$. ■

Theorem 2.12 *For all $m, n \in \mathbb{N}^*$ we have*

$$d^2(mn) \geq d\left(m^2\right) d\left(n^2\right). \tag{13}$$

Proof. We consider $m = \prod\limits_{i=1}^{k} p_i^{\alpha i} \prod\limits_{j=1}^{s} q_j^{\beta j}$ and $n = \prod\limits_{i=1}^{k} p_i^{\gamma i} \prod\limits_{l=1}^{t} r_l^{\delta l}$, which means that $mn = \prod\limits_{i=1}^{k} p_i^{\alpha i + \gamma i} \cdot \prod\limits_{j=1}^{s} q_j^{\beta j} \cdot \prod\limits_{l=1}^{t} r_l^{\delta l}$. Hence $d(m) = \prod\limits_{i=1}^{k} (\alpha_i + 1) \prod\limits_{j=1}^{s} (\beta_j + 1)$ and $d(n) = \prod\limits_{i=1}^{k} (\gamma_i + 1) \prod\limits_{l=1}^{t} (\delta_l + 1)$. Therefore,

$$d(mn) = \prod_{i=1}^{k} (\alpha_i + \gamma_i + 1) \prod_{j=1}^{s} (\beta_j + 1) \prod_{l=1}^{t} (\delta_l + 1),$$

and so

$$d(m)\,d(n) = d(mn)\cdot\prod_{i=1}^{k}\frac{(\alpha_i+1)(\gamma_i+1)}{\alpha_i+\gamma_i+1} = d(mn)\cdot\prod_{i=1}^{k}\left(1+\frac{\alpha_i\cdot\gamma_i}{\alpha_i+\gamma_i+1}\right)\geq d(mn).$$

Since $d(m^2) = \prod_{i=1}^{k}(2\alpha_i+1)\prod_{j=1}^{s}(2\beta_j+1)$ and $d(n^2) = \prod_{i=1}^{k}(2\gamma_i+1)\prod_{l=1}^{t}(2\delta_l+1)$, we obtain the equality

$$d\left(m^2\right)\tau\left(n^2\right) = \prod_{i=1}^{k}(2\alpha_i+1)\prod_{j=1}^{s}(2\beta_j+1)\prod_{i=1}^{k}(2\gamma_i+1)\prod_{l=1}^{t}(2\delta_l+1)$$

But $d^2(mn) = \prod_{i=1}^{k}(\alpha_i+\gamma_i+1)^2\prod_{j=1}^{s}(\beta_j+1)^2\prod_{l=1}^{t}(\delta_l+1)^2$. It is easy to see the equality

$$\begin{aligned} d^2(mn) &= d\left(m^2\right)d\left(n^2\right)\cdot\prod_{i=1}^{k}\left(1+\frac{(\alpha_i+\gamma_i)^2}{(2\alpha_i+1)(2\gamma_i+1)}\right)\cdot\prod_{j=1}^{s}\left(1+\frac{\beta_j^2}{(2\beta_j+1)}\right)\times \\ &\quad \prod_{l=1}^{t}\left(1+\frac{\delta_l^2}{(2\delta_l+1)}\right). \end{aligned}$$

Since $1+\frac{(\alpha_i+\gamma_i)^2}{(2\alpha_i+1)(2\gamma_i+1)}\geq 1$, for all $i=\overline{1,k}$, $1+\frac{\beta_j^2}{(2\beta_j+1)}\geq 1$ for all $j=\overline{1,s}$, $1+\frac{\delta_l^2}{(2\delta_l+1)}\geq 1$ for all $l=\overline{1,t}$, we obtain $d^2(mn)\geq d(m^2)\,d(n^2)$. ■

Theorem 2.13 *Let m and n be two natural non- zero numbers, then $d(mn)\leq n\sqrt{m}+m\sqrt{n}$.*

Proof. We apply the inequality (2.6) for m and n, we have $nd(m)\leq 2n\sqrt{m}$ and $md(n)\leq 2m\sqrt{n}$. By adding the inequalities, we obtain

$$nd(m)+md(n)\leq 2n\sqrt{m}+2m\sqrt{n}, \tag{2.12}$$

but using the inequality (2.10), we have $d(mn)\leq d(m)\,n$ and $d(mn)\leq d(n)\,m$, for all m and $n\in\mathbb{N}^*$, we deduce

$$2d(mn)\leq d(m)\,n+d(n)\,m \tag{2.13}$$

so, from the inequalities (2.12) and (2.13), we obtain the inequality

$$d(mn) \leq n\sqrt{m} + m\sqrt{n}.$$

■

Proposition 2.8 *Let $\omega(n)$ denotes the number of distinct prime divisors of n, and let $\Omega(n)$ denotes the total number of prime divisors of n. We have $2^{\omega(n)} \leq d(n) \leq 2^{\Omega(n)}$.*

Proof. By definition we have $d(n) = \prod_{i=1}^{k}(\alpha_i + 1)$ since $\omega(n) = k$. Moreover, we have $\Omega(n) = \sum_{i=1}^{k}\alpha_i$. For $1 \leq i \leq k$, we also have $\alpha_i \geq 1$, and so $\alpha_i + 1 \geq 2$. By multiplication, we get $\prod_{i=1}^{k}(\alpha_i + 1) \geq 2^k$. Hence,

$$d(n) \geq 2^{\omega(n)}. \tag{2.14}$$

On the other hand, $\alpha_i + 1 \leq 2^{\alpha_i}$ for $1 \leq i \leq k$ and by multiplication once again, we obtain

$$\prod_{i=1}^{k}(\alpha_i + 1) \leq \prod_{i=1}^{k}2^{\alpha_i},$$

and so $\prod_{i=1}^{k}(\alpha_i + 1) \leq 2^{\sum_{i=1}^{k}\alpha_i}$. That is,

$$d(n) \leq 2^{\Omega(n)}. \tag{2.15}$$

Combining (2.14) and (2.15), we get the desired result. ■

The equation $2d(n^2) = 3d(n)$.

Proposition 2.9 *Let $n = q_1^{\alpha_1}q_2^{\alpha_2}...q_k^{\alpha_k}$ be the prime factorization of $n > 1$ ($n = 1$ is not a solution). First, for every $n \geq 1$, we prove that*

$$\frac{d(n^2)}{d(n)} \geq \left(\frac{3}{2}\right)^{\omega(n)}. \tag{2.16}$$

Proof. In fact, by definition we have $d(n) = \prod_{i=1}^{k}(\alpha_i + 1)$ and $d(n^2) = \prod_{i=1}^{k}(2\alpha_i + 1)$. Since $2(2\alpha_i + 1) \geq 3(\alpha_i + 1)$, we deduce that

$$\frac{d(n^2)}{d(n)} = \prod_{i=1}^{k}\frac{2\alpha_i + 1}{\alpha_i + 1} \geq \left(\frac{3}{2}\right)^k = \left(\frac{3}{2}\right)^{\omega(n)}.$$

We can have equality in (2.16) only for $\alpha_i = 1$ $(i = 1, ..., k)$, that is, for $n = q_1 q_2 ... q_k$, i.e., n is square-free.

Finally, the stated equation implies by (2.16) that $\frac{3}{2} \geq \left(\frac{3}{2}\right)^k$ which gives $k = 1$, i.e., the proposed equation has the solutions $n = p$ with $p \geq 2$ is prime. ■

Theorem 2.14 *Let m be a positive integer and define*

$$A_m := \{n \in \mathbb{N};\, m|d(n)\}.$$

Then A_m contains an infinite arithmetical progression.

Proof. Let $n_t = 2^m t + 2^{m-1}$, $t = 0, 1, ...$ and we prove that these numbers form an infinite arithmetical progression. In fact, the exponent of the number 2 in the factorization of the number $n_t = 2^m t + 2^{m-1}$ is $m - 1$. Hence $m|d(n)$. The proof is finished. ■

2.3 Some iteration using the divisor function

The goal of this subsection is to know some properties on the sequence

$$n, d(n), d(d(n) + a), d(d(d(n) + a) + a), ...$$

where a is a fixed nonnegative integer.

Proposition 2.10 *Let n be a positive integer. Define the sequence $n_1, n_1, ...,$by $n_1 = d(n)$ and $n_{k+1} = d(n_k)$ for $k = 1, 2...$Then there is a positive integer r such that $2 = n_r = n_{r+1} = n_{r+2} = ...$ Moreover, the place can be arbitrarily given.*

Proof. The proof holds immediately from the fact that every chain of natural numbers has a minimal element. In fact, we remark that if n is a natural number greater than 2, then $d(n) < n$. So the proof is finished since $d(2) = 2$. For the proof of the second part, we use the equality $d(2^{n-1}) = n$. ■

Notation 2.1 *We denote by d^s the arithmetic function given by $d^s(n) = (d(n))^s$, where $d(n)$ is the number of positive divisors of n. Moreover, for every $\ell, m \in \mathbb{N}$ we denote by $\delta^{s,\ell,m}(n)$ the sequence:*

$$\underbrace{d^s\left(d^s\left(\dots d^s\left(d^s(n)+\ell\right)+\ell\dots\right)+\ell\right)}_{m\text{-times}} = \begin{cases} d^s(n), \text{ for } m = 1 \\ d^s\left(d^s(n)+\ell\right), \text{ for } m = 2 \\ d^s\left(d^s\left(d^s(n)+\ell\right)+\ell\right), \text{ for } m = 3 \\ \dots \end{cases}$$

Theorem 2.15 *Let $n \geq 1$. There exists a positive integer m_0 such that for every $m \geq m_0$, one has*

$$\delta^{1,1,m}(n) \quad = 2 \text{ or } 3. \tag{2.17}$$

Proof. First, we will show that for every $n \geq 2$,

$$\delta^{1,1,2}(n) \leq \delta^{1,1,1}(n) = d(n). \tag{2.18}$$

Obviously the last inequality holds when n is prime. In the case when n is composite, we distinguish two cases.

1. Assume that $d(n)+1=p$, where $p\geq 5$ is prime (because if $p=2$ or $p=3$, then $d(n)=1$ or $d(n)=2$, respectively, meaning that $n=1$ or n is prime). Then

$$\delta^{1,1,2}(n)=2<p-1=d(n).$$

2. Assume that $d(n)+1$ is composite. We put $d(n)+1=ab$, with $1<a\leq b$, and consider three cases:

2.1. $a\neq 2$ and $b\neq 2$. We have

$$a^2(b-d(b))+b^2(a-d(a))>a+b,$$

because $a-d(a)$ and $b-d(b)$ are both positive. It follows that

$$\begin{aligned} a^2d(b)+b^2d(a) &< (a+b)(ab-1) \\ &= (a+b)\,d(n). \end{aligned} \tag{2.19}$$

Therefore,

$$\begin{aligned} \delta^{1,1,2}(n) &= d(ab) \\ &= \frac{ad(ab)+bd(ab)}{a+b} \\ &< \frac{a^2d(b)+b^2d(a)}{a+b}<d(n), \end{aligned} \tag{2.20}$$

where the right-hand side of (2.20) holds by (2.19) and the left hand side because $d(ab)\leq d(a)\,d(b)$, $d(a)<a$ and $d(b)<b$.

2.2. $a=2$ and $(2,b)=1$. Since $b\geq 3$, we obtain

$$\begin{aligned} d(ab) &= d(2b) \\ &= 2d(b) \\ &< 2b. \end{aligned}$$

Thus,

$$\begin{aligned}\delta^{1,1,2}(n) &= d(ab)\\ &\leq 2b-1=d(n).\end{aligned}$$

2.3. $a=2$ and $(2,b)\neq 1$ with $b\geq 2$. We can put $2b=2^N b'$, where $N\geq 2$, $b'\geq 1$ and $\left(2^N,b'\right)=1$. It follows that

$$\begin{aligned}d(ab) &= d\left(2^N b'\right)\\ &= (N+1)\,d(b')\\ &< 2^N b',\end{aligned}\tag{2.21}$$

because $t+1<2^t$ for every $t\geq 2$ and $d(b')\leq b'$. Therefore,

$$\begin{aligned}\delta^{1,1,2}(n) &= d(ab)\\ &\leq 2^N b'-1=d(n).\end{aligned}\tag{2.22}$$

This proves (2.18).

We are now ready to prove (2.17). For $n=1$, $d(1)+1=2$. Then for every $m\geq m_0=2$, we have

$$\delta^{1,1,m}(n) = 2.$$

It is the same when n is prime, where $m_0=1$. Assume that n is composite with $n\geq 4$, that is, $d(n)\geq 3$. Note that if $d(n)=3$, then $\delta^{1,1,m}(n)=3$ for every $m\geq 1$. If $d(n)\geq 4$, then by applying (2.18) repeatedly we obtain for every $m\geq 1$

$$2\leq \delta^{1,1,m}(n)\leq ...\leq \delta^{1,1,3}(n)\leq \delta^{1,1,2}(n)\leq \delta^{1,1,1}(n)=d(n),\tag{2.23}$$

noting that $d(t) \geq 2$ whenever $t \geq 2$. For every $i \geq 2$ we will prove that one of the following statements:

$$\begin{aligned} \cdot\ \delta^{1,1,i+1}(n) &= 2 \\ \cdot\ \delta^{1,1,i+1}(n) &= 3 \\ \cdot\ \delta^{1,1,i+1}(n) &< \delta^{1,1,i}(n) \\ \cdot\ \delta^{1,1,i+2}(n) &< \delta^{1,1,i+1}(n) \end{aligned} \tag{2.24}$$

holds. Let $i \geq 2$. There are two cases:

I. $\delta^{1,1,i}(n) + 1$ is prime. Then

$$\delta^{1,1,i+1}(n) = \delta^{1,1,i+2}(n) = ... = 2. \tag{2.25}$$

II. $\delta^{1,1,i}(n) + 1$ is composite. We also consider three cases as in the proof of (2.18).

II.1. $\delta^{1,1,i}(n) + 1 = xy$ with $x \neq 2$ and $y \neq 2$. Using (2.20), we get

$$d\left(\delta^{1,1,i}(n) + 1\right) = \delta^{1,1,i+1}(n) < \delta^{1,1,i}(n). \tag{2.26}$$

II.2. $\delta^{1,1,i}(n) + 1 = 2y$ with $(2, y) = 1$ and $y \geq 3$. In this case, assume that $y = q_1^{\alpha_1} q_2^{\alpha_2} ... q_r^{\alpha_r}$, where $q_1, q_2, ..., q_r$ are distinct prime numbers and $\alpha_1, \alpha_2, ..., \alpha_r$ are positive integers. Since $q_j \geq 3$, for $j = 1, 2, ..., r$,

$$\alpha_j + 1 < q_j^{\alpha_j}, \text{ for } j = 1, 2, ..., r,$$

and so

$$2(\alpha_j + 1) < 2q_j^{\alpha_j} - 1, \text{ for } j = 1, 2, ..., r.$$

Therefore,

$$d(2y) = 2\prod_{j=1}^{r}(\alpha_j + 1) < 2q_1^{\alpha_1} q_2^{\alpha_2} ... q_r^{\alpha_r} - 1 = 2y - 1.$$

It follows that

$$d\left(\delta^{1,1,i}(n)+1\right)=\delta^{1,1,i+1}(n)<\delta^{1,1,i}(n). \tag{2.27}$$

II.3. $\delta^{1,1,i}(n)+1=2^{M}y$ with $M\geq 2$, $\left(2^{M},y\right)=1$ and $y\geq 1$. As in (2.21) and (2.22), we have $d\left(\delta^{1,1,i}(n)+1\right)=\delta^{1,1,i+1}(n)\leq 2^{M}y-1$. If

$$\delta^{1,1,i+1}(n)<2^{M}y-1=\delta^{1,1,i}(n), \tag{2.28}$$

we obtain the desired inequality. Otherwise, $\delta^{1,1,i+1}(n)=2^{M}y-1$. In this case, we see that

$$d\left(\delta^{1,1,i+1}(n)+1\right)=\delta^{1,1,i+2}(n)=(M+1)\,d(y),$$

where by (2.23), $(M+1)\,d(y)\leq 2^{M}y-1$. If $(M+1)\,d(y)<2^{M}y-1$, we have

$$\delta^{1,1,i+2}(n)=(M+1)\,d(y)<2^{M}y-1=\delta^{1,1,i+1}(n), \tag{2.29}$$

which is the inequality we need. In the remaining case $(M+1)\,d(y)=2^{M}y-1$. Here we distinguish two cases, $y=1$ and $y\geq 2$.

Assume that $y=1$. That is $M+1=2^{M}-1$. Obviously the last equality holds if and only if $M=2$. Hence $\delta^{1,1,i+1}(n)=3$ and by (2.28),

$$\delta^{1,1,i}(n)=\delta^{1,1,i+1}(n)=....=3. \tag{2.30}$$

But, when $M\geq 3$, we have

$$\delta^{1,1,i+2}(n)<\delta^{1,1,i+1}(n), \tag{2.31}$$

since $M+1<2^{M}-1$.

Assume that $y = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r} \geq 2$, where $p_1, p_2, \ldots, p_r$ are distinct prime numbers and $\alpha_1, \alpha_2, \ldots, \alpha_r$ are positive integers. Since $M + 1 < 2^M$ and $\alpha_j + 1 \leq q_j^{\alpha_j}$, for $j = 1, 2, \ldots, r$, it follows for $M = 2$ that

$$2^M y - 1 - (M+1)\, d(y) = 4 p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r} - 3 \prod_{j=1}^{r} (\alpha_j + 1) - 1 \geq 1, \qquad (2.32)$$

which is impossible. For $M \geq 3$, we also see that

$$2^M y - 1 - (M+1)\, d(y) \geq \left(2^M - (M+1)\right) \prod_{j=1}^{r} (\alpha_j + 1) - 1 \geq 7, \qquad (2.33)$$

which is impossible as well. Then one has $(M+1)\, d(y) < 2^M y - 1$ when $y \geq 2$. Hence by (2.29), (2.32) and (2.33),

$$\delta^{1,1,i+2}(n) < \delta^{1,1,i+1}(n). \qquad (2.34)$$

Combining (2.25)-(2.34), we obtain (2.24). Since there exists no infinite descending chain on the natural numbers, as every chain of natural numbers has a minimal element, we obtain $\delta^{1,1,m}(n) = 2$ or 3 for some $m \geq 1$. This completes the proof of Theorem 2.15. ■

Notation 2.2 *Let k be a positive integer and let W_k be the subset given by*

$$W_k = \{n \in \mathbb{N} \;;\; \omega(n) \geq k\},$$

where $\omega(n)$ denotes the number of distinct prime factors of n. In the following theorem we show with respect to the equation $\delta^{1,1,m}(n) = 2$ that the order m can be arbitrarily given for infinitely many $n \in W_k$. This is obtained by using Dirichlet's Theorem about primes in an arithmetic progression.

Theorem 2.16 *Let M_0 be a positive integer with $M_0 \geq 2$. There are infinitely many $n \in W_k$ such that for every $m \geq M_0$,*

$$\delta^{1,1,m}(n) = \delta^{1,1,M_0}(n). \tag{2.35}$$

Proof. We divide the proof into two parts:

(i) We put $m_0 = M_0 - 1$. It suffices to prove that there are infinitely many primes p such that

$$p = \delta^{1,1,m_0}(n) + 1, \tag{2.36}$$

where $n \in W_k$. Indeed, by Dirichlet's Theorem, the arithmetic progression $2^{k-1}t + 1$; $t \geq 1$ contains infinitely many primes. Let p, q_1, q_2 be distinct primes of the form $2^{k-1}t + 1$. Then 2^{k-1} divides both $p - 1$ and $q_1^a q_2^b - 1$ for every $a, b \geq 1$. Let $(l_1, l_2, ..., l_k)$ be an arbitrary $k-$tuple of distinct primes.

In the case when $m_0 = 1$, we put

$$n = l_1 l_2 ... l_{k-1} \times l_k^{\frac{p-1}{2^{k-1}} - 1} \in W_k.$$

It follows that $d(n) + 1 = p$. Thus, (2.35) is true for every $m \geq 2$.

In the case when $m_0 \geq 2$, we can put

$$\begin{cases} s_1 = \dfrac{q_1^{2^{k-1}-1} q_2^{s_2} - 1}{2^{k-1}} - 1 \\ s_2 = \dfrac{q_1^{2^{k-1}-1} q_2^{s_3} - 1}{2^{k-1}} - 1 \\ \vdots \\ s_{m_0-1} = \dfrac{q_1^{2^{k-1}-1} q_2^{s_{m_0}} - 1}{2^{k-1}} - 1 \\ s_{m_0} = \dfrac{p-1}{2^{k-1}} - 1. \end{cases}$$

Let $n = l_1 l_2 ... l_{k-1} \times l_k^{s_1} \in W_k$, or, equivalently,

$$n = l_1 l_2 ... l_{k-1} \times l_k^{\frac{q_1^{2^{k-1}-1} q_2^{\frac{q_1^{2^{k-1}-1} q_2^{\vdots^{\frac{q_1^{2^{k-1}-1} q_2^{\frac{q_1^{2^{k-1}-1} q_2^{\frac{p-1}{2^{k-1}}-1}-1}{2^{k-1}}-1}-1}{2^{k-1}}-1}}-1}{2^{k-1}}-1}-1}{2^{k-1}}-1}, \tag{2.37}$$

For $m_0 = 3$ we obtain:

$$\begin{cases} s_1 = \dfrac{q_1^{2^{k-1}-1} q_2^{s_2} - 1}{2^{k-1}} - 1 \\ s_2 = \dfrac{q_1^{2^{k-1}-1} q_2^{s_3} - 1}{2^{k-1}} - 1 \\ s_3 = \dfrac{p-1}{2^{k-1}} - 1. \end{cases}$$

Let $n = l_1 l_2 ... l_{k-1} \times l_k^{s_1} \in W_k$, then

$$n = l_1 l_2 ... l_{k-1} \times l_k^{\frac{q_1^{2^{k-1}-1} q_2^{\frac{q_1^{2^{k-1}-1} q_2^{\frac{p-1}{2^{k-1}}-1}-1}{2^{k-1}}-1}-1}{2^{k-1}}-1}, \tag{2.38}$$

where the first exponentiation contains $(m_0 - 1)$ fractions involving q_1 and q_2. This writing in the form of towering storeys allows to calculate successively $\delta^{1,1,1}(n)$, $\delta^{1,1,2}(n), ..., \delta^{1,1,M_0}(n)$. Indeed, for each such integer n, it follows from the definition of d that

$$\delta^{1,1,1}(n)+1 = q_1^{2^{k-1}-1} q_2^{\frac{q_1^{2^{k-1}-1} q_2^{\frac{q_1^{2^{k-1}-1} q_2^{\vdots^{\frac{q_1^{2^{k-1}-1} q_2^{\frac{q_1^{2^{k-1}-1} q_2^{\frac{p-1}{2^{k-1}}-1}-1}{2^{k-1}}-1}-1}{2^{k-1}}-1}}-1}{2^{k-1}}-1}-1}{2^{k-1}}-1},$$

where the first exponentiation of q_2 contains (m_0-2) fractions involving q_1 and q_2. By repeating these steps we can reach to the top of (2.37) as follows

$$\delta^{1,1,m_0-1}(n)+1 = q_1^{2^{k-1}-1} q_2^{\frac{p-1}{2^{k-1}}-1},$$

and so

$$\delta^{1,1,m_0}(n)+1 = p. \tag{2.39}$$

This ends the proof of Part (i).

(ii) Now, from (3.16) we obtain $\delta^{1,1,m_0+1}(n) = \delta^{1,1,M_0}(n) = 2$, and therefore (2.35) is true for every $m \geq M_0$. This completes the proof of Theorem 2.16. ■

Corollary 2.4 *For any positive integer m, there are infinitely many n such that $\delta^{1,1,m}(n) = 2$.*

Proof. Let $m \geq 1$. If $m = 1$, then $\delta^{1,1,1}(p) = d(p) = 2$ for any prime p. If $m \geq 2$, it follows from the proof of (2.36) that there exist infinitely many $n \in W_k$ such that $\delta^{1,1,m-1}(n)+1$ is prime. Therefore,

$$\delta^{1,1,m}(n) = d\left(\delta^{1,1,m-1}(n)+1\right) = 2.$$

This completes the proof. ■

Corollary 2.5 *For any positive integer m, there are infinitely many n such that $\delta^{1,1,m}(n) = 3$.*

Proof. Let $m \geq 1$, and let p be an odd prime. Assume that $m = 1$, and put $n = p^2$. Then $\delta^{1,1,m}(n) = 3$. Assume that $m \geq 2$, and define the positive integer

$$n = p^{p^{p^{\cdot^{\cdot^{\cdot^{p^{p^2-2}-2}}}}}-2-2},$$

which contains m exponentiations involving the prime number p. As in the proof of Theorem 2.16, we obtain

$$\delta^{1,1,1}(n) = p^{p^{\cdot^{\cdot^{\cdot^{p^{p^2-2}-2}}}}-2} - 1,$$

which contains $(m-1)$ exponentiations involving the prime number p. By this way, we successively compute $\delta^{1,1,2}(n), \delta^{1,1,3}(n), \ldots$. At the end, we obtain

$$\begin{cases} \delta^{1,1,m-2}(n) = p^{p^2-2} - 1, \\ \delta^{1,1,m-1}(n) = p^2 - 1, \\ \delta^{1,1,m}(n) = 3. \end{cases}$$

This completes the proof. ■

Remark 2.3 *It seems very likely that Proposition 2.15 can be generalized. Computation suggests that for every $n \geq 1$, there exists an order m such that*

$$\begin{cases} \delta^{3,1,m}(n) = 64 \text{ or } 512, \\ \delta^{9,1,m}(n) = 68719476736 \text{ or } 18014398509481984, \\ \delta^{10,1,m}(n) = 1048576 \text{ or } 61917364224. \end{cases} \tag{2.40}$$

See also Example 2.6 below.

Following the same idea as in the proof of Theorem 2.16, we want to present a property on the growth of $\delta^{1,\ell,m}(n)$ for an infinity of values of $n \in W_k$.

Proposition 2.11 *Let a, m_0, k, ℓ be positive integers with a odd and 2^k divides ℓ. There are infinitely many $n \in W_k$ such that $d\left(\delta^{1,\ell,m_0}(n) + a + \ell\right) = 2$.*

Proof. Since $\left(2^{k-1}, a\right) = 1$, by Dirichlet's Theorem, there are infinitely many primes of the form $2^{k-1}t + a$. Let b be the odd positive integer given by $b = \dfrac{\ell}{2^{k-1}} + 1$, and let p be a prime number of the form $2^{k-1}t + a$, where $\dfrac{p-a}{2^{k-1}} > b$. We study the following two cases.

In the first case, we assume that $m_0 = 1$. For

$$n = l_1 l_2 \ldots l_{k-1} \times l_k^{\frac{p-a}{2^{k-1}} - b} \in W_k,$$

where $l_1, l_2, \ldots, l_k$ are distinct primes, it follows that $d(n) = \delta^{1,\ell,1}(n) = p - a - \ell$. Thus, $d\left(\delta^{1,\ell,1}(n) + a + \ell\right) = 2$.

In the second case, we assume that $m_0 \geq 2$. Similarly, let q_1, q_2 be two distinct primes of the form $2^{k-1}t + b$, since $\left(2^{k-1}, b\right) = 1$. Consider the positive integer of the form

$$n = l_1 l_2 \ldots l_{k-1} \times l_k^{\dfrac{q_1^{2^{k-1}-1} q_2^{\dfrac{q_1^{2^{k-1}-1} q_2^{\vdots^{\dfrac{q_1^{2^{k-1}-1} q_2^{\dfrac{q_1^{2^{k-1}-1} q_2^{\frac{p-a}{2^{k-1}} - b}}{2^{k-1}} - b}}{2^{k-1}} - b}}}{2^{k-1}} - b}}{2^{k-1}} - b},$$

where the first exponentiation contains (m_0-1) fractions involving q_1 and q_2 and $l_1, l_2, ..., l_k$ are also distinct primes. Since $\ell = 2^{k-1}(b-1)$, it follows that

$$\delta^{1,\ell,1}(n)+\ell = q_1^{2^{k-1}-1} q_2^{\frac{q_1^{2^{k-1}-1} q_2^{\cdot^{\cdot^{\cdot^{\frac{q_1^{2^{k-1}-1} q_2^{\frac{q_1^{2^{k-1}-1} q_2^{\frac{p-a}{2^{k-1}}-b}}{2^{k-1}}-b}}{2^{k-1}}-b}}}}}{2^{k-1}}-b},$$

where the first exponentiation of q_2 contains (m_0-2) fractions involving q_1 and q_2. Repeating the process, as in the proof of Theorem 2.16 and Corollary 2.5, we obtain

$$\begin{cases} \delta^{1,\ell,m_0-1}(n)+\ell = q_1^{2^{k-1}-1} q_2^{\frac{p-a}{2^{k-1}}-b}, \\ \delta^{1,\ell,m_0}(n)+\ell = p-a, \\ d\left(\delta^{1,\ell,m_0}(n)+a+\ell\right) = 2. \end{cases}$$

This completes the proof. ■

Now, we present some examples to illustrate the results stated in Theorem 2.15, Remark 2.3 and Theorem 2.16, respectively.

2.3.1 Examples

Example 2.5 *With respect to the result of Theorem 2.15, the following table shows the first value of m for which $\delta^{1,1,m}(n) = 2$ or 3, for different values* *of* n.

n	m	$\delta^{1,1,m}(n)$	n	m	$\delta^{1,1,m}(n)$
1	2	2	$2^{2^{10}}$	3	2
2	1	2	$2^{2^{50}}$	4	2
2^2	1	3	$2^{2^{100}}$	4	3
2^{2^2}	3	2	$2^{2^{150}}$	4	2
2^{2^3}	3	2	$2^{2^{200}}$	3	3
2^{2^4}	3	2	$2^{2^{220}}$	4	3
2^{2^5}	3	2	$2^{2^{250}}$	4	2
2^{2^6}	3	3	$2^{2^{300}}$	4	3
2^{2^7}	3	3	$2^{2^{350}}$	3	2
2^{2^8}	3	3	$2^{2^{380}}$	3	2

In the following example we give two positive integers n and n' which have the same distinct prime factors and satisfying the first equation of (2.40), and this after 4-fold iterations. That is, $set(n) = set(n')$, $\delta^{3,1,m}(n) = 64$ and $\delta^{3,1,m}(n') = 512$ for every $m \geq 4$.

Example 2.6 *Let* $(q_1, q_2, ..., q_{13})$ *be an arbitrary* 13*-tuple of distinct primes. We put*

$$\begin{cases} n = q_1^{25} \times q_2^{22} \times q_3^{15} \times q_4^{70} \times q_5^{11} \times q_6^{15} \times q_7^{13} \times q_8 \times q_9 \times q_{10} \times q_{11} \times q_{12}^{15} \times q_{13}^{10}, \\ n' = q_1^{2000} \times q_2^{302} \times q_3^{105} \times q_4^{700} \times q_5^{101} \times q_6^{15} \times q_7^{13} \times q_8 \times q_9 \times q_{10} \times q_{11} \times q_{12}^{1500} \times q_{13}^{999}. \end{cases}$$

It is clear that n *and* n' *have the same distinct prime factors, where* $|set(n)| = |set(n')| = 13$. *By computation, we see that* $\delta^{3,1,m}(n) = 64$ *and* $\delta^{3,1,m}(n') = 512$ *for every* $m \geq 4$.

$$\begin{cases} \delta^{3,1,1}(n) = 135\,964\,112\,015\,285\,579\,850\,731\,807\,751\,719\,092\,224 \\ \delta^{3,1,2}(n) = 56\,623\,104 \\ \delta^{3,1,3}(n) = 4096 \\ \delta^{3,1,4}(n) = 64, \end{cases}$$

and

$$\begin{cases} \delta^{3,1,1}(n') = 2^{42}3^{9}5^{9}7^{3}17^{3}19^{3}23^{3}29^{3}53^{3}79^{3}101^{3}701^{3} \\ \delta^{3,1,2}(n') = 2097\,152 \\ \delta^{3,1,3}(n') = 1728 \\ \delta^{3,1,4}(n') = 512. \end{cases}$$

As an application of Theorem 2.16, the following example gives the smallest positive integer $x \in W_3$ in the form (2.37) such that $\delta^{1,1,4}(x) = 2$.

Example 2.7 *In view of Theorem 2.15, assume that $k = 3$ and $M_0 = 4$. Then the positive integer*

$$x = 5 \times 3 \times 2^{511\,528\,924\,107\,551\,574\,707\,030} \in W_3$$

is the smallest number that satisfies (2.36) and (2.37). Indeed, we see that

$$\delta^{1,1,1}(x) + 1 = 2046\,115\,696\,430\,206\,298\,828\,125 = 5^{30}13^3.$$

Therefore, we have

$$\begin{cases} \delta^{1,1,2}(x) + 1 = 125 = 5^3, \\ \delta^{1,1,3}(x) + 1 = 5, \\ \delta^{1,1,4}(x) = 2. \end{cases}$$

On the other hand, we can write

$$x = 5 \times 3 \times 2^{\frac{13^{2^{3-1}-1} \times 5^{\frac{5^{2^{3-1}-1} \times 13^{\frac{5-1}{2^{3-1}}-1} - 1}{2^{3-1}}-1} - 1}{2^{3-1}}-1}.$$

Thus we have shown that x is the smallest one, since the numbers 5 and 13 are chosen to be the smallest distinct primes satisfying the properties of p, q_1 and q_2 which are stated in the proof of Theorem 2.16.

Chapter 3

On relations between $d(n)$ and some other functions

At first, the relation between the multiplicative functions $nd(n)$ and $\varphi(n)$ is given by the following theorem, where $\varphi(n)$ is the Euler's function.

Theorem 3.1 *We have*

$$\underline{\lim}\frac{\varphi(n)}{nd(n)} = 0, \tag{3.1}$$

and

$$\overline{\lim}\frac{\varphi(n)}{nd(n)} = 1/2. \tag{3.2}$$

Proof. At first, we have $F(n) = \frac{\varphi(n)}{nd(n)} = \frac{1}{d(n)}\prod_{p|n}\left(1-\frac{1}{p}\right)$. We put $n = 2^m$ with $m \in \mathbb{N}$, we also have

$$0 < F(n) = F(2^m) = \frac{1-\frac{1}{2}}{m+1} \to 0$$

as $m \to \infty$. This proves (3.1). On the other hand, for each integer $n \geq 1$, we have $F(n) \leq 1/2$. In fact, the inequality (3.2) holds for every $n \geq 2$ because $\varphi(n) \leq n$ and $d(n) \geq 2$. ■

Theorem 3.2 *The number 2 divides $\sigma(n) - d(m)$ for all positive integers n, where m is the largest odd divisor of n.*

Proof. Assume that $n = 2^{\alpha} q_1^{\alpha_1} q_2^{\alpha_2} ... q_r^{\alpha_r}$ where the q_i, for $1 \leq i \leq r$ are odd. That is, $m = q_1^{\alpha_1} q_2^{\alpha_2} ... q_r^{\alpha_r}$ is the largest odd divisor of n. Therefore,

$$\sigma(n) - d(m) = \left(2^{\alpha+1} - 1\right)(1 + q_1 + ... + q_1^{\alpha_1}) ... (1 + q_r + ... + q_r^{\alpha_r}) - \prod_{i=1}^{r} (\alpha_i + 1)$$

The result follows since $2^{\alpha+1} - 1$ is odd and $1 + q_1 + q_2^2 + ... + q_i^{\alpha_i}$ is odd whenever α_i is even and even whenever α_i is odd. The proof is finished. ■

A relation between the divisor function, Euler's function and the sum of positive divisors is proved by Liouville in 1857. In fact, he proved that for any positive integer n, one has

$$\sum_{d|n} d \cdot \varphi(d) \cdot \sigma\left(\frac{n}{d}\right) = \sum_{d|n} d^2.$$

For example, for $n = 6$ we have

$$\begin{aligned} \sum_{d|n} d \cdot \varphi(d) \cdot \sigma\left(\frac{n}{d}\right) &= 1 \cdot \varphi(1) \cdot \sigma(6) + 2 \cdot \varphi(2) \cdot \sigma(3) + 3 \cdot \varphi(3) \cdot \sigma(2) + 6 \cdot \varphi(6) \cdot \sigma(1) \\ &= 50 = \sum_{d|n} d^2. \end{aligned}$$

3.1 Diophantine equations involving the divisor function

Now, recall that a *Diophantine equation* is an equation of the form:

$$f(x_1, x_2, ..., x_k) = b$$

that we want to solve in integers or nonnegative integers. This means that the values of the variables $x_1, x_2, ..., x_k$ will be integers or nonnegative integers. Usually the function $f(x_1, x_2, ..., x_k)$ is a polynomial with integer coefficients or a real-valued function whose domain is the set $\mathbb{N}$. Let us start with some simple Diophantine equations involving the divisor function and Euler's function.

1. On the equation $d(n) = \varphi(n)$. Here, we give a comparison between the value $d(n)$ and Euler's function at the same point n.

Theorem 3.3 $\{1, 3, 8, 10, 24, 30\}$ *are the only solutions of* $d(n) = \varphi(n)$. *Moreover, we have* $\varphi(n) > d(n)$ *for* $n > 30$.

Proof. Clearly, $n = 1$ is a solution. Next, let $n > 1$ with $n = \prod p^{\alpha}$ (for simplicity we do not use indices), where p is prime and $\alpha \geq 1$. Then

$$\frac{\varphi(p^{\alpha})}{d(p^{\alpha})} = \frac{p^{\alpha-1}(p-1)}{\alpha+1}.$$

For $p \geq 3$ we see that $p^{\alpha-1} \cdot (p-1) \geq 3^{\alpha-1} \cdot 2 \geq \alpha + 1$ for all α (which can be proved easily by induction on α) with equality only for $\alpha = 1$ and $p = 3$. One gets

$$\varphi(n) \geq d(n) \text{ for all odd,} \tag{3.3}$$

with equality for $n \in \{1, 3\}$.

Let now be n even, i.e, $n = 2^{\alpha} \cdot m$ with m is odd and $\alpha \geq 1$. For $\alpha \geq 3$ one can write

$$\varphi(n) = \varphi(2^{\alpha}) \cdot \varphi(m) \geq 2^{\alpha-1} d(m)$$

on base of (3.3). But $2^{\alpha-1} \geq \alpha + 1$, with equality for $\alpha = 3$, so

$$\varphi(n) \geq d(n) \text{ for } n \text{ is even and } 8 \mid n. \tag{3.4}$$

In the above inequality we must have $m = 1$ or $m = 5$, so in (3.4) we can have equality only for $n = 1 \cdot 8 = 8$, $n = 3 \cdot 8 = 24$. We have to study the remaining cases $\alpha = 1$ and $\alpha = 2$. For $\alpha = 1$ one obtains the equation

$$\varphi(m) = 2d(m), \ m = \text{odd}, \tag{3.5}$$

while for $\alpha = 3$ we have

$$2\varphi(m) = 3d(m), \ m = \text{odd}. \tag{3.6}$$

Let $m = \prod_{p \geq 3} p^{\beta}$. Then (3.5) becomes

$$\prod_{p \geq 3} \frac{p^{\beta-1}(p-1)}{\beta+1} = 2$$

with equality only for $\beta = 1$, thus $m = 5$ or $m = 3 \cdot 5$ are the single possibilities. From here, as solutions we get $n = 2 \cdot 5 = 10$ and $n = 2 \cdot 3 \cdot 5 = 30$. In the same manner, (3.6) becomes

$$\prod_{p \geq 3} \frac{p^{\beta-1}(p-1)}{\beta+1} = \frac{3}{2}.$$

But, $\dfrac{2 \cdot 3^{\beta-1}}{\beta+1} \geq 1$ and $\dfrac{4 \cdot 5^{\beta-1}}{\beta+1} > \dfrac{3}{2}$. Thus, we cannot have equality. Therefore, this case doe not provide solutions. By summing, all solutions of the initial equations are:

$$n \in \{1, 3, 8, 10, 24, 30\}.$$

As a consequence, we can write $\varphi(n) > d(n)$ for $n > 30$. ■

2. The equation $d(n) + \varphi(n) = n + 1$.

Proposition 3.1 *The only solutions of $d(n) + \varphi(n) = n + 1$ are $1, 4$ and p with p is prime.*

Proof. One can remark that $n = p$ with p is prime and $n = 1$ are solutions. Let $n = p^{\alpha}$ be a prime power such that $d(n) + \varphi(n) = n + 1$. Then $\alpha = p^{\alpha-1}$, or equivalently, $\alpha = p$ and $\alpha - 1 = 1$. Hence, $n = 4$.

Now, let $n > 1$ be composite with $n \neq 4$ and let $d \neq 1$ be a divisor of n. Then $\gcd(d, n) \neq 1$. Therefore, clearly $\varphi(n) \leq n - d\,(n)$ from the definitions of d and φ (which is the number of couples (i, n) such that $\gcd(i, n) = 1$, $i < n$), and so $d(n) + \varphi(n) < n + 1$. Therefore, n cannot be a solution. ■

3. On the equation $d\,(n) + \varphi\,(n) = n$.

Theorem 3.4 *The equation $\varphi\,(n) + d\,(n) = n$ has the only solutions $n = 8$ and $n = 9$.*

Proof. Case 1. Let n be an even number. Then it is well-known that $\varphi\,(n) \leq n/2$. Using the relation $d\,(n) < 2\sqrt{n}$ (see, (2.6)), we get

$$\varphi\,(n) + d\,(n) \leq \frac{n}{2} + 2\sqrt{n} \leq n$$

if $2\sqrt{n} \leq n/2$, i.e., $n \geq 16$. Now, for $n < 16$ and even, a simple verification shows that $\varphi\,(n) + d\,(n) < n$ holds true with a strict inequality, except for $n = 8$, when there is equality. Therefore, the only even solution except for $n = 4$ is $n = 8$.

Case 2. Let n be odd and not a prime. Suppose that $\varphi\,(n) + d\,(n) = n$ holds for $n \geq 3$. Since $\varphi\,(n)$ is even, $d\,(n)$ should be an odd number. But, from Theorem 2.2, it is immediate that n must be a perfect square, i.e., $n = m^2$. As $\varphi\,(m^2) = m\varphi\,(m)$, the equality becomes

$$m\varphi\,(m) + d\left(m^2\right) = m^2. \tag{3.7}$$

Equality (3.7) implies that m should be a divisor of $d(m^2)$, i.e.,

$$d\left(m^2\right) = k \cdot m, \tag{3.8}$$

for certain, $k \geq 1$. As $d(N) < 2\sqrt{N}$, we get $d(m^2) < 2m$, implying that one must have $k = 1$ in (3.8). Equation $d(m^2) = m$ can be also written as $(2\alpha_1 + 1) \ldots (2\alpha_s + 1) = q_1^{\alpha_1} \ldots q_s^{\alpha_s}$, where $m = q_1^{\alpha_1} \ldots q_s^{\alpha_s}$ is the prime factorization of m.

Now, as m is odd, let $q_1$be the least odd prime factor of m, with $q_1 \geq 3$.Since the inequality $3^{\alpha_1} \geq 2\alpha_1 + 1$ holds true, with equality only for$\alpha_1 = 1$, and as $5^{\alpha_2} \geq 2\alpha_2 + 1$, etc., we must have $m = 3$. This finally gives $n = m^2 = 9$, as the single odd solution of the equation. This finishes the proof of Theorem 3.4. ■

4. The equation $\Lambda(n)(d(n) - 1) = \dfrac{d(n)\ln n}{2}$.

Theorem 3.5 *The only solution of the equation* $\Lambda(n)(d(n) - 1) = \frac{d(n)\ln n}{2}$ *are* $n = 1$ *and* $n = p$ *with* p *is prime.*

Proof. Let Λ be the von Mangoldt function, i.e., $\Lambda(n) = \ln p$ for $n = p^\alpha$ (p is prime and $\alpha \geq 1$); $\Lambda(n) = 0$ in other cases. The following identity is well known

$$\sum_{i|n} \Lambda(i) = \ln n. \tag{3.9}$$

The identity

$$2\sum_{i|n} \Lambda\left(\frac{n}{i}\right) d(i) = d(n) \ln n \tag{3.10}$$

can be proved via similar arguments. Now, we see that

$$\frac{d(n)\ln n}{2} = \Lambda(n) + \sum_{in} \Lambda\left(\frac{n}{i}\right) d(i) \leq \Lambda(n) + d(n)(\ln n - \Lambda(n))$$

and on base of property(3.10), as well as $i \mid n \Rightarrow d(i) \leq d(n)$ one gets

$$\Lambda(n)(d(n)-1) \leq \frac{d(n)\ln n}{2}$$

with equality for $n = 1$ and $n = prime$; which provide the most general solutions of proposed equation. ■

5. The equations $\varphi(d(n)) = d(\varphi(n))$ and $d(\gamma(n)) = \gamma(d(n))$.

Proposition 3.2 *The equation* $\varphi(d(n)) = d(\varphi(n))$ *has infinitely many solutions.*

Proof. Consider the numbers $n = 2^k$, where $k \geq 1$. For such a number to be a solution of $\varphi(d(n)) = d(\varphi(n))$, we must have $\varphi(k+1) = d\left(2^{k-1}\right) = k$, which is solvable only when $k+1$ is a prime number. Thus, $k = p-1$, with p is prime. Then for $n = 2^{p-1}$ with p is prime, we have $\varphi(d(n)) = d(\varphi(n))$. ■

Theorem 3.6 $d(\gamma(n)) = \gamma(d(n))$ *if and only if* $n = p^\alpha$ *with* p *is prime and* α *a positive integer such that* $\alpha+1$ *is a power of* 2*, that is,* $n = p^{2^x-1}$*with* $x \geq 1$.

Proof. Let us first show that the condition is sufficient. So, let $n = p^\alpha$, where p is prime and $\alpha = 2^k - 1$ for a certain positive integer k. We then have

$$d(\gamma(p^\alpha)) = d(p) = 2,$$

while

$$\gamma(d(p^\alpha)) = \gamma(\alpha+1) = \gamma\left(2^k\right) = 2.$$

To prove that the condition is necessary, we proceed by contradiction. Two situations may occur:

(i) $n = p^{\alpha}$ with $\alpha \neq 2^k - 1$ for each integer $k \geq 1$. In this case, we have $d(\gamma(p^{\alpha})) = d(p) = 2$, while $\gamma(d(p^{\alpha})) = \gamma(\alpha+1) > 2$.

(ii) $\omega(n) \geq 2$. Here, we have $n = q_1^{\alpha_1} q_2^{\alpha_2} ... q_r^{\alpha r}$ for certain prime numbers $q_1 < q_2 < ... < q_r$ and certain positive integers $\alpha_1, \alpha_2, ..., \alpha_r$.It follows that $d(\gamma(n)) = 2^r$, while $\gamma(d(n)) = \gamma((\alpha_1+1)(\alpha_2+1)...(\alpha_r+1))$ is either equal to 2 or else divisible by a prime number > 2, and therefore, in both cases, $\gamma(d(n))$ cannot be equal to 2^r with $r \geq 2$.

The proof is finished. ■

3.2 Some Diophantine inequations involving the divisor function

A *Diophantine inequality* is an inequality whose solutions are required to be integers of natural numbers. Let us consider the following Diophantine inequalities.

Proposition 3.3 *Let $n \geq 1$ and let $\sigma(n) = \sum_{d|n} d$. Then $d(n) \geq \dfrac{n}{\varphi(n)}$.*

Proof. Let k be the number of distinct prime factors of n. Suppose that $n = q_1^{\alpha_1} q_2^{\alpha_2} ... q_k^{\alpha_k}$, where $q_1, q_2, ..., q_k$ are distinct primes and $\alpha_1, \alpha_2, ..., \alpha_k$. By definition, we have

$$\begin{aligned} \varphi(n) \cdot d(n) &= n\left(1-\frac{1}{q_1}\right)\left(1-\frac{1}{q_2}\right)...\left(1-\frac{1}{q_k}\right)\cdot(\alpha_1+1)(\alpha_2+1)...(\alpha_k+1) \\ &\geq n\left(\frac{1}{2}\right)^k \cdot 2^k = n. \end{aligned}$$

This gives the result. ■

Proposition 3.4 *For any $n \geq 2$ we have*

$$d(n) \leq \frac{\sigma(n)}{\sqrt{n}}.$$

Proof. Let $d_1, d_2, ..., d_k$ be the divisors of n. That is, $k = d(n)$. They can be rewritten as

$$\frac{n}{d_1}, \frac{n}{d_2}, ..., \frac{n}{d_k},$$

and so

$$\sigma(n)^2 = n(d_1 + d_2 + ... + d_k)\left(\frac{1}{d_1} + \frac{1}{d_2} + ... + \frac{1}{d_k}\right) \geq n \cdot d(n).$$

The result follows. ■

Proposition 3.5 *For each integer $n \geq 2$, we have*

$$\sigma(n) \geq \varphi(n) + d(n),$$

with equality if and only if n is prime.

Proof. First of all, it is clear that $\varphi(n) \leq n - (d(n) - 1) = n + 1 - d(n)$, since for each $d \mid n$, $d > 1$, we have $(d, n) > 1$. It follows that $\sigma(n) > n \geq \varphi(n) + d(n) - 1$, and hence $\sigma(n) \geq \varphi(n) + d(n)$. ■

Proposition 3.6 *For any positive integer n, we have $d(n) \leq \frac{\sigma_2(n)}{n}$, where $\sigma_2(n) = \sum_{d|n} d^2$.*

For the proof, we need the following lemma. See for example, [1, page 4].

Lemma 3.1 (Arithmetic–Geometric means inequality) *Let $a_1, a_2, ..., a_k$ be positive real numbers. Then $(a_1 a_2 ... a_n)^{\frac{1}{n}} \leq \dfrac{a_1 + a_2 + ... + a_n}{n}$, with equality if and only if $a_1 = a_2 = ... = a_n$.*

Proof. From Theorem 2.2 and Proposition 3.6, we deduce that

$$n = \left(\prod_{d|n} d^2 \right)^{\frac{1}{d(n)}} \leq \frac{1}{d(n)} \sum_{d|n} d^2 = \frac{\sigma_2(n)}{d(n)},$$

and the result follows. ■

Proposition 3.7 *For each $n \in \mathbb{N}$, let $\sigma_2(n) = \sum_{d|n} d^2$. Then*

$$\frac{\sigma^2(n)}{d(n)} \leq \sigma_2(n) \leq \sigma^2(n), \ (n = 1, 2, ...)$$

Proof. First of all it is clear that $\sigma_2(n) = \sum_{d|n} d^2 \leq \left(\sum_{d|n} d \right)^2 = \sigma^2(n)$, which proves the second inequality. On the other hand, using the Cauchy-Schwarz inequality, we obtain, $\sigma^2(n) = \left(\sum_{d|n} d \cdot 1 \right)^2 \leq \sum_{d|n} d^2 \cdot \sum_{d|n} 1^2 = \sigma_2(n) \cdot d(n)$, hence the first inequality. ■

Theorem 3.7 *Let $f : \mathbb{N} \longrightarrow \mathbb{R}_+$ and let $F(n) = \sum_{d|n} f(d)$. Then*

$$\prod_{d|n} f(d) \leq \left(\frac{F(n)}{d(n)} \right)^{d(n)}, \ n = 1, 2, ...$$

In particular, we have

$$\prod_{d|n} \varphi(d) \leq \left(\frac{n}{d(n)} \right)^{d(n)}, \ n = 1, 2, ...$$

Proof. This follows from the fact that the geometric mean does not exceed the arithmetic mean (see Lemma 3.1). Indeed, since $\left[\prod_{d|n} f(d)\right]^{\frac{1}{d(n)}} \leq \frac{1}{d(n)} \prod_{d|n} f(d)$, the result follows. For the second part, we take $f(n) = \varphi(n)$ so that $F(n) = \sum_{d|n} \varphi(d) = n$, (see Theorem 1.3). ■

Remark 3.1 *By a second method, we show that*

$$d(n) \geq \frac{n}{\varphi(n)}, (n = 1, 2, ...).$$

Let $n \geq 1$. We have for $d \mid n$, $\varphi(n) \geq \varphi(d)$. Therefore, $\sum_{d|n} \varphi(n) \geq \sum_{d|n} \varphi(d) = n$. Thus, $\varphi(n) \cdot d(n) \geq n$.

3.3 Inequalities defined by arithmetic functions and integer-valued polynomials

We can compare between an arithmetic function formed by the production of certain multiplicative functions (for example, d, φ and σ) and a an integer-valued polynomial whose leading coefficient is positive. As we see in the following theorem.

Theorem 3.8 *Let s and n be positive integers with $n \geq 2$. Then,*

$$\varphi_s(n)^{d(n)} \psi_s(n) \sigma(n) \geq n^{3s+1} + n^{3s} - n^{2s+1} - n^{2s} - n^{s+1} - n^s + n + 1. \quad (3.11)$$

Proof. Firstly, for $s = 1$, we note that

$$\varphi(n)^{d(n)} \psi(n) \sigma(n) - \left(n^4 - 2n^2 + 1\right) = \begin{cases} 0, \text{ for } n = p, \\ 111, \text{ for } n = 4. \end{cases}$$

Next, it suffices to show that if $\varphi(n)^{d(n)}\psi(n)\sigma(n) \geq n^4 - 2n^2 + 1$ for some $n \geq 3$, then it is also true for $p \cdot n$ with $p \geq 2$ is prime. Indeed, for each such integer n and for any prime $p \geq 2$ we distinguish two cases.

1. When p does not divide n. Since $\varphi(n) \geq 2$ and $d(n) \geq 2$, it follows that

$$\begin{aligned}
\varphi(pn)^{d(pn)}\psi(pn)\sigma(pn) &= (p-1)^{2d(n)}\varphi(n)^{2d(n)}(1+p)^2\psi(n)\sigma(n) \\
&= (p-1)^{2d(n)}\varphi(n)^{d(n)}(1+p)^2\left[\varphi(n)^{d(n)}\psi(n)\sigma(n)\right] \\
&\geq (p-1)^{2d(n)}\varphi(n)^{d(n)}(1+p)^2\left(n^4-2n^2+1\right) \\
&\geq (p-1)^4 2^2(1+p)^2\left(n^4-2n^2+1\right) \\
&= n^4\left(4p^6-8p^5-4p^4+16p^3-4p^2-8p+4\right)+ \\
&\quad n^2\left(-8p^6+16p^5+8p^4-32p^3+8p^2+16p-8\right)+ \\
&\quad 4p^6-8p^5-4p^4+16p^3-4p^2-8p+4.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\varphi(pn)^{d(pn)}\psi(pn)\sigma(pn)-\left((pn)^4-2(pn)^2+1\right) \qquad (3.12)\\
\geq\ & n^4\left(4p^6-8p^5-5p^4+16p^3-4p^2-8p+4\right)+ \\
& n^2\left(-8p^6+16p^5+8p^4-32p^3+10p^2+16p-8\right)+ \\
& 4p^6-8p^5-4p^4+16p^3-4p^2-8p+3.
\end{aligned}$$

Using the graph of the function $x \mapsto 4x^6-8x^5-4x^4+16x^3-4x^2-8x+3$, we have

$$4p^6-8p^5-4p^4+16p^3-4p^2-8p+3>0. \qquad (3.13)$$

In fact, we see that

$$4p^6-8p^5-4p^4+16p^3-4p^2-8p = 4p^4\left(p^2-2p-1\right)+4p\left(4p^2-p-2\right),$$

where $p^2 - 2p - 1 > 0$ holds for every $p \geq 3$ and $4p^2 - p - 2 > 0$ holds for every $p \geq 2$. This proves (3.13) for every $p \geq 2$, since its value at $p = 2$ is 35. Moreover, from the graph of the function:

$$x \mapsto \frac{8x^6 - 16x^5 - 8x^4 + 32x^3 - 10x^2 - 16x + 8}{4x^6 - 8x^5 - 5x^4 + 16x^3 - 4x^2 - 8x + 4},$$

by using the same manner as those of the proof of (3.13) we can prove that

$$0 < \frac{8p^6 - 16p^5 - 8p^4 + 32p^3 - 10p^2 - 16p + 8}{4p^6 - 8p^5 - 5p^4 + 16p^3 - 4p^2 - 8p + 4} \leq 3.2.$$

Since $n \geq 2$, then

$$n^2 > \frac{-\left(-8p^6 + 16p^5 + 8p^4 - 32p^3 + 10p^2 + 16p - 8\right)}{4p^6 - 8p^5 - 5p^4 + 16p^3 - 4p^2 - 8p + 4} > 0. \tag{3.14}$$

Setting

$$\begin{aligned} A &= 4p^6 - 8p^5 - 4p^4 + 16p^3 - 4p^2 - 8p + 3, \\ B &= -8p^6 + 16p^5 + 8p^4 - 32p^3 + 10p^2 + 16p - 8, \\ C &= 4p^6 - 8p^5 - 5p^4 + 16p^3 - 4p^2 - 8p + 4. \end{aligned}$$

Since $A > 0$ and $n^2 > \dfrac{-B}{C}$, it follows from the inequality (3.12) that

$$\varphi(pn)^{d(pn)} \psi(pn) \sigma(pn) - \left((pn)^4 - 2(pn)^2 + 1\right) > n^4 C + n^2 B + A > 0.$$

2. When p divides n. Since $\psi(pn) = p\psi(n)$, $\varphi(pn) = p\varphi(n)$, $\sigma(pn) >$

$p\sigma(n)$ and $d(pn) \geq d(n) + 1$, then

$$\begin{aligned}
\varphi(pn)^{d(pn)}\psi(np)\sigma(np) &= (p\varphi(n))^{d(pn)}\psi(pn)\sigma(pn) \\
&> p^{d(pn)+2}\varphi(n)^{d(pn)}\psi(n)\sigma(n) \\
&\geq p^{d(n)+3}\varphi(n)^{d(n)+1}\psi(n)\sigma(n) \\
&= p^{d(n)+3}\varphi(n)\left[\varphi(n)^{d(n)}\psi(n)\sigma(n)\right] \\
&\geq p^{d(n)+3}\varphi(n)\left(n^4 - 2n^2 + 1\right) \\
&\geq 2n^4p^5 - 4n^2p^5 + 2p^5.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \varphi(pn)^{d(pn)}\psi(pn)\sigma(pn) - \left((pn)^4 - 2(pn)^2 + 1\right) \\
\geq\ & 2n^4p^5 - n^4p^4 - 4n^2p^5 + 2n^2p^2 + 2p^5 - 1 \\
=\ & n^4\left(2p^5 - p^4\right) + n^2\left(-4p^5 + 2p^2\right) + 2p^5 - 1. \qquad (3.15)
\end{aligned}$$

Since $p \geq 2$, then $2p^5 - 1 > 0$. Using the graph of the function $x \mapsto \dfrac{4x^5 - 2x^2}{2x^5 - x^4}$ and the proof of (3.13), we can also prove that

$$0 < \frac{4p^5 - 2p^2}{2p^5 - p^4} \leq \frac{5}{2}.$$

Since $n \geq 2$, then

$$n^2 > \frac{-\left(-4p^5 + 2p^2\right)}{2p^5 - p^4} > 0. \qquad (3.16)$$

It follows from (3.15),(3.16) that

$$\varphi(pn)^{d(pn)}\psi(pn)\sigma(pn) - \left((pn)^4 - 2(pn)^2 + 1\right) > 0.$$

Hence, for $s = 1$, we have proved that the inequality $\varphi(n)^{d(n)}\psi(n)\sigma(n) \geq n^4 - 2n^2 + 1$ is true for every $n \geq 2$.

Now, assume for some $s \geq 1$ that the desired inequality holds for any composite positive integer n. We distinguish two cases:

Case 1. Suppose that n is not the square of a prime number. Then

$$\begin{aligned}
& \varphi_{s+1}(n)^{d(n)} \psi_{s+1}(n) \sigma(n) \\
= & \left(n^{s+1} \prod_{p|n}\left(1-\frac{1}{p^{s+1}}\right)\right)^{d(n)} n^{s+1} \prod_{p|n}\left(1+\frac{1}{p^{s+1}}\right) \sigma(n) \\
= & n^{d(n)}\left(n^{s} \prod_{p|n}\left(1-\frac{1}{p^{s+1}}\right)\right)^{d(n)} n^{s+1} \prod_{p|n}\left(1+\frac{1}{p^{s+1}}\right) \sigma(n) \\
\geq & n^{d(n)}\left(n^{s} \prod_{p|n}\left(1-\frac{1}{p^{s}}\right)\right)^{d(n)} n^{s} \prod_{p|n}\left(1+\frac{1}{p^{s}}\right) \sigma(n) \\
= & n^{d(n)}\left[\varphi_{s}(n)^{d(n)} \psi_{s}(n) \sigma(n)\right] \\
\geq & n^{d(n)}\left(n^{3s+1}+n^{3s}-n^{2s+1}-n^{2s}-n^{s+1}-n^{s}+n+1\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\varphi_{s+1}(n)^{\tau(n)} \psi_{s+1}(n) \sigma(n) & \geq n^{4}\left(\begin{array}{c} n^{3s+1}+n^{3s}-n^{2s+1}-n^{2s}-n^{s+1}-n^{s}+ \\ n+1 \end{array}\right) \quad (3.17) \\
& = n^{3s+5}+n^{3s+4}-n^{2s+5}-n^{2s+4}-n^{s+5}-n^{s+4}+n^{5}+n^{4},
\end{aligned}$$

where (3.17) holds because n is not of the form p^2 with p is prime, and therefore $\tau(n) \geq 4$. Since $n \geq 6$, it follows that

$$\begin{aligned}
& \varphi_{s+1}(n)^{d(n)} \psi_{s+1}(n) \sigma(n) - \left(n^{3s+4} + n^{3s+3} - n^{2s+3} - n^{2s+2} - n^{s+2} - n^{s+1} + n + 1\right) \\
\geq\ & n^{3s+5} - n^{3s+3} - n^{2s+4} + n^{2s+3} - n^{2s+5} + n^{2s+2} - n^{s+5} - n^{s+4} + n^{s+2} + \\
& n^{s+1} + n^5 + n^4 - n - 1 \\
\geq\ & 6^{3s+5} - 6^{3s+3} - 2^{2s+4} + 6^{2s+3} - 6^{2s+5} + 6^{2s+2} - 6^{s+5} - 6^{s+4} + 6^{s+2} + \\
& 6^{s+1} + 6^5 + 6^4 - 6 - 1 \\
=\ & 7560 \times 6^{3s} - 8820 \times 6^{2s} - 9030 \times 6^s + 9065 \\
\geq\ & 1270\,325.
\end{aligned}$$

Note that when n is prime, the inequality (3.11) becomes

$$\begin{aligned}
\varphi_s(n)^{d(n)} \psi_s(n) \sigma(n) &= (n^s - 1)^2 (n^s + 1)(n + 1) \\
&= n^{3s+1} + n^{3s} - n^{2s+1} - n^{2s} - n^{s+1} - n^s + n + 1.
\end{aligned}$$

Case 2. Suppose that $n = p^2$ for some prime number $p \geq 2$. We also have

$$\begin{aligned}
\varphi_s(n)^{d(n)} \psi_s(n) \sigma(n) &= \left(p^{2s} - p^s\right)^3 \left(p^{2s} + p^s\right)\left(1 + p + p^2\right) \\
&= p^{8s+2} + p^{8s+1} + p^{8s} - 2p^{7s+2} - 2p^{7s+1} - 2p^{7s} + \\
&\quad 2p^{5s+2} + 2p^{5s+1} + 2p^{5s} - p^{4s+2} - p^{4s+1} - p^{4s}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \varphi_s(n)^{d(n)} \psi_s(n) \sigma(n) - \left(n^{3s+1} + n^{3s} - n^{2s+1} - n^{2s} - n^{s+1} - n^s + n + 1\right) \\
\geq\ & p^{8s+2} + p^{8s+1} + p^{8s} - 2p^{7s+2} - 2p^{7s+1} - 2p^{7s} - p^{6s+2} - p^{6s} + 2p^{5s+2} + \\
& 2p^{5s+1} + 2p^{5s} + p^{2s+2} - p^{4s+1} + p^{2s} - p^2 - 1 \\
\geq\ & 7 \times 2^{8s} - 14 \times 2^{7s} - 5 \times 2^{6s} + 14 \times 2^{5s} - 2 \times 2^{4s} + 5 \times 2^{2s} - 5 \\
\geq\ & 111,
\end{aligned}$$

since $p \geq 2$. Hence, (3.11) is true for $n = p^2$ with p is prime.

Thus, our assertion is proved by induction on s. This completes the proof of Theorem 3.8. ■

Remark 3.2 *In the case when $n = p^2$ with p is prime, then (3.17) becomes*

$$\varphi_{s+1}(n)^{d(n)}\psi_{s+1}(n)\sigma(n) \geq n^{3s+4}+n^{3s+3}-n^{2s+3}-n^{2s+4}-n^{s+4}-n^{s+3}+n^4+n^3,$$

since $\tau(n) = 3$. Hence,

$$\begin{aligned} & \varphi_{s+1}(n)^{d(n)}\psi_{s+1}(n)\sigma(n) - \left(n^{3s+4}+n^{3s+3}-n^{2s+3}-n^{2s+2}-n^{s+2}-n^{s+1}+n+1\right) \\ \geq \; & -n^{2s+4}+n^{2s+2}-n^{s+4}-n^{s+3}+n^{s+2}+n^{s+1}+n^4+n^3-n-1, \end{aligned} \tag{3.18}$$

where the leading coefficient of (3.18) is negative. Therefore, in this case, the inequality (3.11) can not be easily deduced for the power $s+1$.

Chapter 4

Conclusion and open questions

Number Theory is a field where the problems to solve are *very easy* to formalize and to understand, but *very hard* to prove. Let us consider the following Diophantine equation involving the divisor function:

$$d\,(n+a) = d\,(n+b)\,,$$

which is derived from [3], where a, b are two distinct nonnegative integers. We conjecture that the above equation has infinitely many solutions. In particular, for $a = 2022$ and $b = 2021$ we have

$$d\,(n+2022) = d\,(n+2021)\,. \tag{4.1}$$

Using the following program in Maple:

$>$ **for** n **from** 1 to 1000 **do if** tau(n+202) = tau(n+201) **then** print(n); **end if**; **end do**:

It seems that (4.1) has infinitely many solutions, where the first terms are

$$20, 33, 34, 56, 64, 72, 80, 81, 88, 101, 105, 112, 113, 141, \ldots.$$

In the year 1940 the tables of the function $d(n)$ for $n \leq 10000$ were published by Gaisher [2]. As we check in the table, the equalities $d(n) = d(n+1) = d(n+2) = d(n+3) = 8$ hold for $n = 3655$, 4503, 5943, 6853, 7256, 8393, 9367. Also, for $n = 40311$ we see that

$$d(n) = d(n+1) = d(n+2) = d(n+3) = d(n+4). \tag{4.2}$$

The proof of (4.2) follows immediately from the factorization into primes of the numbers $40311 = 3^3 \cdot 1493$, $40312 = 2^3 \cdot 5039$, $40313 = 7 \cdot 13 \cdot 443$, $40314 = 2 \cdot 3 \cdot 6719$ and $40315 = 5 \cdot 11 \cdot 733$. In fact, these numbers have 8 divisors. A similar situation occurs for $n = 99655$. Note also that for $n \leq 10000$ we have $d(n) \leq 64$ and the maximum value $d(n) = 64$ is taken only for the numbers $n = 7560$ and 9240.

Erdös and Mirsky [2] formulated a conjecture which states that there are infinitely many natural numbers n for which $d(n) = d(n+1)$. For example, we have $d(2) = d(3)$, $d(14) = d(15)$, $d(33) = d(34) = d(35) = 4$, $d(242) = d(243) = d(244) = d(245) = 6$, ...etc. In 1981, Spiro [3] proved that $d(n) = d(n+5040)$ has infinitely many solutions. In fact, this conjecture was proved by [2]. If $n = 40311$, then

$$d(n) = d(n+1) = d(n+2) = d(n+3) = d(n+4).$$

In addition, we do not know whether there exists an infinite sequence of increasing natural numbers n_k $(k = 1, 2, ...)$ such that

$$\lim_{k\to\infty} \frac{d(n_k+1)}{d(n_k)} = 1.$$

An important general Diophantine equation related to the divisor function is given by

$$f(d(n)) = d(g(n)),$$

where f and g are two multiplicative functions. Does the above equation has infinitely many solutions?

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