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Presented by :
DJAGHOUT Manal

Entitled

Discretization of some hyperbolic problems

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Intitulée

Discrétisation de quelques problèmes hyperboliques.

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2020/2021.

Discretization of some hyperbolic problems

DJAGHOUT Manal

Doctoral thesis in mathematics

University of 8 May 1945-Guelma

I dedicate this humble work to :

My Dear parents, who always stood beside me.

To my wonderful sister who has no parallel in the world Hayat. My brothers Rami, Fouzi and Abd elghani.

All my family,

all my friends ...

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First of all, I thank God Almighty, who helped me to complete this work and pursue my academic career that God has decreed.

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I am pleased to extend my sincere thanks to my distinguished professors in the members of the jury, Professor **ELLAGGOUNE Fateh** the president and Professors **BADI Sabrina, MAOUNI Messaoud** and **KHEMIS Rabah** the examiners, for their kindness to accept the discussion of this thesis. I also take the opportunity to extend my sincere thanks to all those who helped me in carrying out this research without forgetting the members of the **LMAM** laboratory.

الملخص

الهدف من هذه الأطروحة هو الاهتمام بالدراسة النظرية والرقمية للمعادلة التفاضلية من النوع الزائدي.

لقد قمنا في العمل الأول بتوسيع طريقة روث للتقدير الزمني، وطريقة العناصر المحدودة للتقدير المكاني لمعادلة تلغراف بمصطلح غير محلي مرتبط بالشروط الأولية والشروط الحدودية، وعليه فان الفكرة الرئيسية لهذا العمل هي إعطاء مخططات شبه منفصلة ومنفصلة تمامًا واستخراج تقديرات مسبقة وتقديرات خطأ مسبقة في المساحات المناسبة. أما من الناحية العددية فإن وجود معاملات غير محلية في المعادلة يسبب صعوبات في حل نظام المعادلات غير الخطية التي تم الحصول عليها، لذلك كان علينا معالجة هذه الصعوبات من خلال تطبيق طريقة عددية مخصصة لحل هذا النوع من المشاكل ، وفي نهاية هذا العمل نقدم مثالًا رقميًا لدعم تقديراتنا النظرية.

الغرض من العمل الثاني هو دراسة نفس المعادلة التفاضلية الزائدية غير المحلية من خلال الجمع بين طريقة العناصر المحدودة المختلطة مع طريقة روث، حيث يتم اشتقاق تقديرات مسبقة وتقديرات الخطأ لكل من المخططات شبه المنفصلة والمنفصلة تمامًا في المساحات التي تناسب هذا العمل وننهي عملنا بتجربة عددية تثبت نتائجنا النظرية.

الكلمات المفتاحية: طريقة روث، طريقة العناصر المحدودة، طريقة العناصر المحدودة المختلطة H^1 غلركين، معادلة التلغراف، المصطلح الغير محلي، التقديرات المسبقة.

Abstract

The aim of this thesis to interest in the theoretical and numerical study of a differential equation of the hyperbolic type.

In the first work we extend the Rothe method for to time discretization and finite element method for the spatial discretization of telegraph equation with nonlocal term associated with initial conditions and boundary conditions.
The main idea in this work is to give semi discrete and fully discrete schemes and extract a priori estimates and a priori error estimates for in suitable spaces. As for the numerical aspect, the presence of non-local coefficients in the equation causes difficulties to solve a system of nonlinear equations obtained. Therefore, we had to address these difficulties by applying a dedicated numerical method to solve this type of problem, and at the end of this work we provide a numerical example to support our theoretical estimates.

The purpose of the second work is to study the same non-local hyperbolic differential equation by combining the H^1-Galerkin mixed finite element method with the Rothe method.
A priori estimates and error estimates are derived for both semi discrete and fully discrete schemes in spaces that fit this work and we finish our work with a numerical experiment that proves our theoretical results.

Key-words : Rothe's method, finite element method, H^1-Galerkin mixed finite element method, telegraph equation, nonlocal term and a priori estimate.

Résumé

Cette thèse a pour objectif de s'intéresser à l'étude théorique et numérique d'une équation différentielle de type hyperbolique.

Dans le premier travail, nous étendons la méthode de Rothe pour la discrétisation temporelle et la méthode des éléments finis pour la discrétisation spatiale d'une équation télégraphique avec un terme non local associé aux conditions initiales et aux conditions aux limites.
L'idée principale de ce travail est de donner des schémas semi discrets et totalement discrets et d'extraire des estimations a priori et des estimations d'erreur a priori dans des espaces appropriés. Quant à l'aspect numérique, la présence de coefficients non locaux dans l'équation pose des difficultés pour résoudre un système d'équations non linéaires obtenu. Par conséquent, nous avons dû résoudre ces difficultés en appliquant la méthode numérique dédiée pour résoudre ce type de problème, et fin de ce travail nous fournissons un exemple numérique pour étayer nos estimations théoriques.

L' objectif du second travail est d'étudier la même équation différentielle hyperbolique non locale en combinant la méthode des éléments finis mixtes H^1-Galerkin avec la méthode de Rothe.
Les estimations a priori et les estimations d'erreur sont dérivées à la fois pour des schémas semi discrets et totalement discrets dans les espaces qui correspondent à ce travail et nous terminons notre recherche par une expérience numérique qui prouve nos résultats théoriques.

Mots-Clés : Méthode de Rothe, méthode des éléments finis, la méthode des éléments finis mixtes H^1-Galerkin, équation de télégraphe, terme non local, estimation a priori.

Contents

Introduction

The motivation for choosing to study partial differential equations with nonlocal coefficient is the interest in describing how important these equations in the modeling of many physical, biological and mechanical phenomena.
The problems that contain partial differential equations with nonlocal coefficient include different applications in many problems, including population activity such that describe the population density subject to evolution (see [18]), and also conduction heat transfer process where a phase transition occurs, and in moisture transport such as swelling grains or polymers (see [8, 10, 50, 52] and references cited therein),... etc. The presence of non-local coefficient in the problems leads to improving the advantages of qualitative and quantitative analysis of these problems.

In the recent period, great emphasis has been placed on how to apply basic functional methods to boundary problems accompanied by linear or non-linear differential equations in order to improve the theoretical and practical results of studying these problems, especially in Banach or Hilbert spaces, in contrast to the classical methods.

The topic of the thesis is the interest in the numerical study of the partial differential equation of the type of hyperbolic with the term nonlocal, and it is also accompanied by initial conditions and boundary conditions by using various approximations methods.

These approximate methods have many advantages, including :

-Create algorithms for numerical solutions.

-Finding approximate solutions to the difficulty in obtaining an accurate solution.

-Prove the existence of the solution and its uniqueness…etc.

Among the most commonly used approximate methods for solving problems involving linear or nonlinear differential equations, we mention: The finite element method, H^1-Galerkin mixed finite element method and mixed finite element method…

The aim of our research, in the first work, we combine the Rothe method and the finite element method to study a problem containing the acoustic telegraph equation with nonlocal term, constant coefficients accompanied by boundary conditions, and initial conditions which this equation is used to model effects of the mixture propagation and wave propagation by introducing a term that explains the effects of finite velocity on standard heat or standard mass transport (see [22, 40]). The nonlocal term a in our equation is the diffusion that depends on a nonlocal quantity $\int_\Omega u(x,t)dx$ and assumed to depend on the entire population in the domain Ω. Recent years have seen an increasing interest in studying nonlocal problems, of these problems it is possible to refer to [14, 19, 26]. It also has extended applications which you can see in [51].

In the second work, we are expanding the H^1-Galerkin mixed finite element method to the spatial discretization and applying the Rothe method for the time discretization in order to study the same problem in the first work. These methods enable us to obtain an approximate solution and, facilitate finding the optimal a priori estimates.

The H^1-Galerki finite element mixed method is a developed method for mixed finite element, this proposed method has the advantage of not subjecting the choice of the finite element spaces to the LBB consistency conditions, and we refer to some references interested in studying problems by using the H^1-Galerkin mixed finite element method [16, 17, 43, 44, 56].

In ref [15], Che et al. proposed H^1-Galerkin mixed finite element method for the following nonlinear viscoelasticity-type equation based on H^1-Galerkin method, and expanded mixed element method.

$$u_{tt} - \nabla.(a(x,u)\nabla u_t + b(x,u)\nabla u) = f(x,t),\ \ (x,t) \in \Omega \times J,$$

$$u(x,t) = 0,\ \ (x,t) \in \partial\Omega \times J,$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x)\ \ x \in \Omega,$$

where $J = (0;T]$ with $0 < T < \infty$. the given data $u0(x)$, $u1(x)$, $a(x,u)$, $b(x,u)$, and f are sufficiently smooth.

In ref [42], PANI. studied and analyzed H^1-Galerkin mixed finite element method for the following parabolic partial differential equations with nonselfadjoint elliptic parts.

$$p_t - (ap_x)_x + b_x + cp = f(x,t),\ \ (x,t) \in (0,1) \times J,$$

$$p(0,t) = p(1,t),\ \ t \in \bar{J},$$

$$p(x,0) = p_0,\ \ x \in (0,1),$$

where $J = (0;T]$ with $T < 1$. The coefficients a, b, c are smooth functions depends of x and a is bounded below by $a_0 \geq 0$.

For the Rothe method, it is an efficient tool that can be cited in the the discretization of linear or non-linear evolution equations.

The Rothe method (or the method of lines) is the One of the most popular methods for solving partial equation; this method is used in the time discretization of evolution equations where the derivatives with respect to one variable are replaced by the corresponding difference quotients that finally lead to systems of differential equations for functions of the remaining variables. Rothe's method was presented by E.Rothe in 1930, it has been adopted and developed by many authors for example O.A. Ladyzenskaja [33, 34] and K. Rektorys[48, 49] for solving second order linear and quasilinear parabolic problems. Recently Rothe's method has been studied linear and quasilinear hyperbolic equations we can see in [9, 28, 53, 2, 6, 27]. It also applied to different types of problems some of them mentioned [3, 28, 32, 36, 38, 39, 41].

The discretization scheme of the Rothe method is given as follows :
We subdivide the interval $[0, T]$ into n subintervals of length $\tau = \frac{T}{n}$ and denote by u_h^i the values of u_h at $t = i\tau$, for $i = 1, ..., n$. We define the first and second finite differences as $\delta p_h^i = \dfrac{p_h^i - p_h^{i-1}}{\tau}$ and $\delta^2 p_h^i = \dfrac{\delta p_h^i - \delta p_h^{i-1}}{\tau}$ for all $t = t_i$.

-We obtain a system formed of n equations in x where the unknown is $u^i(x)$ so we approximate the problem posed at any point $t = t_i$, for all $i = \overline{1, n}$ by a new discrete problem.
-We determine the functions u^n solutions of the system obtained.
-We construct the Rothe functions defined by

$$u^n = u^{i-1} + (t - t_{i-1})\delta u^i \quad \forall t \in [t_{i-1}, t_i], \ 1 \leq i \leq n,$$

$$\delta u^n = \delta u^{i-1} + (t - t_{i-1})\delta^2 u^i \quad \forall t \in [t_{i-1}, t_i], \; 1 \le i \le n,$$

and the auxiliary functions

$$\bar{u}^n = \begin{cases} u^i & t \in [t_{i-1}, t_i], \\ \\ u^0 & t \in [-\tau, 0], \end{cases}$$

Content of the thesis

This thesis is composed of three chapters.

In the first chapter, we give fundamental notions of functional analysis, those of Sobolev spaces and Bochner spaces. afterwards, we also determine definitions, properties and fundamental theorems for this work: Cauchy-Lipschitz, Lax-Milgram Young inequality, Gronwall inequality, ect.
We give as well an overview of finite elements and mixed finite elements.

In the second chapter, we treat the telegraph equation with nonlocal term by combining Rothe's method for time discretization and finite elements method to spatial discretization. We start with a general introduction to this chapter, we give a definition of the weak formulation and the basic assumptions for this work.
Next, we derive the discretization scheme, the a priori estimates and the error estimates based on the time discretization.
Thereafter, for the spatial discretization we deduce the fully discrete scheme and we derive the a priori estimates and the error estimates. Finally, we describe a numerical experiment to support the theoretical result of this work.

The third chapter is devoted to studying the nonlocal hyperbolic equation based on the H^1-Galerkin mixed finite element method, for spatial discretization and on the Roth method for time discretization. First, we give a general introduction to this chapter ; we give to a clear and precise formulation of the mentioned problem, followed by the variational formulation. After that, the use the H^1-Galerkin mixed finite element method for spatial discretization, which allows us to extract the semi discrete scheme and derive the optimal error estimates. We then divide the time domain and give the fully discretization scheme; the a priori estimates and the error estimates according to the Rothe method. Finally, we finish with a digital experiment that describes our theoretical results.

We end this thesis with a conclusion, perspectives and bibliography.

1

Preliminaries

In this chapter, we determine fundamental notions of functional analysis, those of Sobolev spaces and Bochner spaces. After which, we also recall definitions, properties and fundamental theorems for this work: Cauchy-Lipschitz, Lax-Milgram Young inequality, Gronwall inequality, ect. We also give an overview of finite elements and mixed finite elements.

Most of these reminders are mostly drawn from books [1, 4, 7, 20, 31, 54, 55].

1.1 Functional spaces

Let Ω is a bounded domain of $\mathbb{R}^d$ with $d \geq 1$ and T a strictly positive real number :

1.1.1 Spaces $L^p(\Omega)$

Definition 1.1.1. *[7] For $1 \leq p < \infty$, we denote by $L^p(\Omega)$ the space of measurable functions u from Ω in $\mathbb{R}$ such that*

$$\int_{\Omega} \|u(x)\|^p dx < \infty, \tag{1.1}$$

with the norm

$$\|u\|_{L^p(\Omega)} = \Big(\int_{\Omega} \|u(x)\|^p dx\Big)^{\frac{1}{p}}. \tag{1.2}$$

Remark 1.1.2. *If $p = 2$, $L^2(\Omega)$ is the space of measurable functions with square integrable over Ω for the scalar product defined as*

$$(u, v) = \int_{\Omega} u(x)v(x)dx, \tag{1.3}$$

with the norm

$$\|u\|_{L^2(\Omega)} = \Big(\int_{\Omega} |u|^2 dx\Big)^{\frac{1}{2}}. \tag{1.4}$$

Remark 1.1.3. *if $p = \infty$, we define the space L^∞ as the space of all functions u that are essentially bounded over Ω, with the corresponding norm*

$$\|u\|_{L^\infty(\Omega)} = essup_{x\in\Omega}|u(x)| = \inf\Big\{C \geq 0 : |u(x)| \leq C \ pp \ over \ \Omega\Big\}. \tag{1.5}$$

Definition 1.1.4. *We say that u a function of $L^2(\Omega)$ is weakly derivable in $L^2(\Omega)$, if there exists $w_i \in L^2(\Omega)$, $\forall i = 1, ..., N$, such that*

$$\int_\Omega u \frac{\partial v}{\partial x_i} = -\int_\Omega w_i v, \quad \forall v \in C_c^\infty(\Omega), \tag{1.6}$$

where $\frac{\partial u}{\partial x_i} = w_i$ and $C_c^\infty(\Omega)$ is the space of compactly supported class C^∞ functions over Ω

1.1.2 Sobolev spaces

Definition 1.1.5. *[1] For an integer $m \geq 0$, the space $H^m(\Omega)$ is the Sobolev space of the order m, built on $L^2(\Omega)$:*

$$H^m(\Omega) = \{u : u \in L^2(\Omega) \text{ and, } \forall \alpha \in \mathbb{N}^n \text{such that } |\alpha| \leq m, D^\alpha u \in L^2(\Omega)\}, \tag{1.7}$$

here D^α is the weak partial derivative.

$H^m(\Omega)$ is a Hilbert space for the scalar product

$$(u, v)_{H^m(\Omega)} = \sum_{|\alpha| \leq m} (D^\alpha u, D^\alpha v), \tag{1.8}$$

and for the norm

$$\|u\|_{H^m(\Omega)} = \Big(\sum_{|\alpha| \leq m} \|D^\alpha u\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}}. \tag{1.9}$$

Proposition 1.1.6. *if $m = 1$, the Hilbert space $H^1(\Omega)$ equipped with the scalar product*

$$(u, v)_{H^1(\Omega)} = (u(x), v(x)) + (\nabla u(x), \nabla v(x)), \tag{1.10}$$

and the norm

$$\|u\|_{H^1(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \sum_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}. \tag{1.11}$$

Definition 1.1.7. *We pose*

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \ on \, \partial\Omega\}, \tag{1.12}$$

it is the adherence of $C_c^{\infty}(\Omega)$ *in* $H^1(\Omega)$

Remark 1.1.8. *The space* $H^{-1}(\Omega)$ *is the dual of space* $H_0^1(\Omega)$.

The duality product between $H_0^1(\Omega)$ *and* $H^{-1}(\Omega)$ *is as follows*

$$\langle u, \varphi \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} = \int_{\Omega} \varphi(x) u(x) dx, \ \forall \varphi \in H_0^1(\Omega). \tag{1.13}$$

Definition 1.1.9. *[1] For* $1 \le p < \infty$, *and* $m \in \mathbb{N}$ *we define the sobolev space* $W^{m,p}(\Omega)$ *as follows*

$$W^{m,p}(\Omega) = \left\{u \in L^p(\Omega) ; D^{\alpha} u \in L^p(\Omega), \forall \alpha \in \mathbb{N} \text{ such that } |\alpha| \le m\right\}, \tag{1.14}$$

with the norm

$$\|u\|_{m,p} = \sum_{|\alpha| \le m} \|D^{\alpha} u\|_{L^p(\Omega)}. \tag{1.15}$$

Remark 1.1.10. *If* $p = 2$, *we have*

$$W^{m,p}(\Omega) = H^m(\Omega). \tag{1.16}$$

1.1.3 Bochner spaces

We recall the following spaces :

- $C^0(0, T; L^2(\Omega))$ $\left(\textit{also denoted } C(0, T; L^2(\Omega))\right)$ *is the space of defined and continuous* *functions on* $[0,T]$ *with values in* $L^2(\Omega)$*, it is a Banach space for the norm*

$$\|u\|_{C^0(0,T;L^2(\Omega))} = \max_{0\le t\le T} \|u(t)\|_{L^2(\Omega)}. \tag{1.17}$$

• $L^2(0,T;H_0^1(\Omega))$ *is the space of functions with integrable square over* $[0,T]$ *with values in* $H_0^1(\Omega)$*, it is a Hilbert space for the scalar product*

$$(u,v)_{L^2(0,T;H_0^1(\Omega))} = \int_0^T \big(u(t),v(t)\big)_{H_0^1(\Omega)} dt, \tag{1.18}$$

and the norm

$$\|u\|^2_{L^2(0,T;H_0^1(\Omega))} = \int_0^T \|u(t)\|^2_{H_0^1(\Omega)} dt. \tag{1.19}$$

• $L^\infty(0,T;H_0^1(\Omega))$*is the space of functions essentially bounded on* $[0,T]$ *with values in* $H_0^1(\Omega)$*, it is a Banach space for the norm*

$$\|u\|_{L^\infty(0,T;H_0^1(\Omega))} = \sup_{[0,T]} \|u\|_{H_0^1(\Omega)}. \tag{1.20}$$

• $L^2(0,T;L^2(\Omega))$ *is a Hilbert space for the scalar product*

$$(u,v)_{L^2(0,T;L^2(\Omega))} = \int_0^T \big(u(t),v(t)\big)_{L^2(\Omega)} dt. \tag{1.21}$$

and the norm

$$\|u\|^2_{L^2(0,T;L^2(\Omega))} = \int_0^T \|u(t)\|^2_{L^2(\Omega)} dt. \tag{1.22}$$

Let X *be a Banach space, we also mention the norms in the discrete form for some Bochner spaces*

$$\|u\|_{L^\infty(0,T,\tau;X)} = \max_{1\le m\le J} \|u^m\|_X. \tag{1.23}$$

$$\|u\|^2_{L^2(0,T,\tau;X)} = \tau \sum_{m=1}^{J} \|u^m\|^2_X, \tag{1.24}$$

where τ *is the discretization step of the interval* $[0,T]$

1.2 Some properties and fundamental theorems

Lemma 1.2.1. *(***Young's inequality***)*

If $a,b \geq 0$ *and* $1 \leq p,q \leq \infty$ *such that* $\frac{1}{p}+\frac{1}{q}=1$*, then*

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q. \tag{1.25}$$

Lemma 1.2.2. *(***The** ϵ**-inequality***)*

For all $\epsilon \geq 0$ *and if* $a,b \geq 0$ *, then*

$$2ab \leq \epsilon a^2 + \frac{1}{\epsilon}b^2, \tag{1.26}$$

or in other words

$$2ab \leq \epsilon a^2 + C_\epsilon b^2. \tag{1.27}$$

Remark 1.2.3. $C_\epsilon = C(\epsilon^{-1})$ *with* ϵ *is small.*

Lemma 1.2.4. *[14](***Abel's summation principle***)*

$$2a(a-b) = a^2 - b^2 + (a-b)^2, \ \forall a,b. \tag{1.28}$$

Lemma 1.2.5. *(***Poincaré inequality***)*

For all $u \in H_0^1(\Omega)$, there exists a constant $C(\Omega)$, such that

$$\|u\|_{L^2(\Omega)} \leq C(\Omega)\Big(\sum_{i=1}^{n}\Big\|\frac{\partial u}{\partial x_i}\Big\|^2\Big)^{\frac{1}{2}}. \tag{1.29}$$

Theorem 1.2.6. *[7](***Cauchy-Schwarz inequality***)*

For all $u, v \in L^2(\Omega)$, we have

- **Continuous form**

$$\Big|\int_\Omega u(x)v(x)dx\Big| \leq \Big(\int_\Omega |u(x)|^2\Big)^{\frac{1}{2}}\Big(\int_\Omega |v(x)|^2\Big)^{\frac{1}{2}}. \tag{1.30}$$

- **Discrete form**

$$\Big|\sum_{i=1}^{N} u_i v_i dx\Big| \leq \Big(\sum_{i=1}^{N}|u_i|^2\Big)^{\frac{1}{2}}\Big(\sum_{i=1}^{N}|v_i|^2\Big)^{\frac{1}{2}}. \tag{1.31}$$

Theorem 1.2.7. *(***Hölder inequality***)*

It is a generalization of Cauchy-Schwarz inequality.

For $1 \leq p, q < \infty$, $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$ such that $\frac{1}{p}+\frac{1}{q}=1$, we have

- **Continuous form**

$$\Big|\int_\Omega u(x)v(x)dx\Big| \leq \Big(\int_\Omega |u(x)|^p\Big)^{\frac{1}{p}}\Big(\int_\Omega |v(x)|^q\Big)^{\frac{1}{q}}. \tag{1.32}$$

- **Discrete form**

$$\Big|\sum_{i=1}^{N} u_i v_i\Big| \leq \Big(\sum_{i=1}^{N}|u_i|^p\Big)^{\frac{1}{p}}\Big(\sum_{i=1}^{N}|v_i|^q\Big)^{\frac{1}{q}}. \tag{1.33}$$

Theorem 1.2.8. *(***Triangular inequality***)*

$$\Big(\int_\Omega \big(u(x)+v(x)\big)^2 dx\Big)^{\frac{1}{2}} \leq \Big(\int_\Omega (u(x))^2 dx\Big)^{\frac{1}{2}} + \Big(\int_\Omega (v(x))^2 dx\Big)^{\frac{1}{2}}. \tag{1.34}$$

Lemma 1.2.9. *[14](***Gronwall inequality, continuous form***)*

Let u, v, w be real valued functions on $I = [a, +\infty]$, we assume u and w are continuous functions, if $u(t) \geq 0$, v is nondecreasing and if $w(t)$ satisfies the inequality

$$w(t) \leq v(t) + \int_a^t u(s)w(s)ds, \ \ \forall t \in [a,b], \tag{1.35}$$

then

$$w(t) \leq v(t)\exp\Big(\int_a^t u(s)ds\Big), \ \ \forall t \in [a,b]. \tag{1.36}$$

Lemma 1.2.10. *[14](***Gronwall inequality, discrete form***)*

Let u_i, w_n be nonnegative real numbers and $v_n \geq v_{n-1}$, if

$$w_n \leq v_n + \sum_{i=0}^{n-1} u_i w_i, \ \ \forall n \geq 0, \tag{1.37}$$

then

$$w_n \leq v_n \exp\Big(\sum_{i=0}^{n-1} u_i\Big), \ \ \forall n \geq 0. \tag{1.38}$$

Theorem 1.2.11. *[1](***Green's formula***)*

Let Ω be a regular bounded open of class C^2, and $n(x)$ its exterior normal. Let $u \in H^2(\Omega)$ and $v \in H^2(\Omega)$, we have

$$\int_\Omega \Delta u v dx = -\int_\Omega \nabla u \nabla v dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} v d\sigma. \tag{1.39}$$

Definition 1.2.12. (Continuity of a bilinear form $a(.,.)$)

We say that $a(.,.)$ is a continuous bilinear form if it satisfies

$$\exists M > 0 \ \ \forall u, v \in V \ : \ |a(u,v)| \leq M\|u\|\|v\|. \tag{1.40}$$

Definition 1.2.13. (Coercivity of a bilinear form $a(.,.)$)

we say that $a(.,.)$ is a coercive bilinear form if it satisfies

$$\exists \alpha > 0 \ \ \forall u \in V \ : \ |a(u,u)| \geq \alpha \|u\|^2. \tag{1.41}$$

Theorem 1.2.14. *[7](**Lax-Milgram**)*

Let V be a Hilbert space, and let $a(.,.) : V \times V \to \mathbb{R}$ a continuous and coercive bilinear form and $L(.) : V \to \mathbb{R}$ a continuous linear form. Then, there exists a unique $u \in V$ such that

$$\forall v \in V, \ \ a(u,v) = l(v). \tag{1.42}$$

1.3 The finite element method

1.3.1 Introduction

The finite element method is one of the numerical tools that depend on the variational formulation (and therefore on the weak solutions), meaning that this method proposes creating a discrete algorithm based on weak formulas, as it allows us to search on an approximate solution to a partial differential problem on a compact domain with boundary conditions or inside the compact.

The finite element method changes the space of test functions of infinite dimension for the variational formulation by a space of approximate test functions of finite dimension. And then talking about the existence and uniqueness of the solution, stability and convergence of numerical methods, as well as estimating the error between the exact solution and approximate solution. To clarify more on this method we see [20, 35]

1.3.2 General principle of the finite element method: resolution of a matrix system

The general approach of the finite element method is as follows. Let a bounded open domain Ω of $\mathbb{R}^n$ $(n \geq 1)$, with a boundary $\partial\Omega$. the variational formulation of partial differential equation (PDE) is generally taken as follows : Find $u \in V$ such that

$$a(u, v) = l(v), \ \forall v \in V. \tag{1.43}$$

In order to find the approximate solution of u we are using internal approximative, as shown below:
Let Υ_h be a partition of Ω made of a finite number of elements $\mathcal{T}$, such that

$$\overline{\Omega} = \cup_{\mathcal{T} \in \Upsilon_h} \overline{\mathcal{T}} \tag{1.44}$$

$$\mathcal{T} \cap \mathcal{L} = \emptyset \ \text{if} \ \mathcal{T} \neq \mathcal{L}. \tag{1.45}$$

We note $h_{\mathcal{T}} := diam\,\mathcal{T}$ the diameter of $\mathcal{T}$ and $h := \max_{\mathcal{T} \in \Upsilon_h} h_{\mathcal{T}}$ the step of the mesh. Thanks to which we will create an approximation space $V_h \subset V$ of finite dimension. So that V_h will be the set of continuous functions on Ω and affine on each a mesh.
For example *in* **the first dimension** *we choose Ω the interval $]a, b[$, We divide this interval into N+1 subintervals of length $h_i = x_{i+1} - x_i$ for $1 \leq i \leq N+1$, takes $h = \max_{1 \leq i \leq N+1} h_i$. The approximation space V_h can be determined as follows*

$$V_h = \left\{ \Phi_h \in C^0([a, b]) \ \text{such that} \ \Phi_h|_{[x_i, x_{i+1}]} \ \text{is linear} \forall \, i = \overline{1, N+1} \right\}. \tag{1.46}$$

and let $\{\Phi_i(x)\}_{i=1}^{N+1}$ be the basic functions for the space V_h, taken as follows

$$\Phi_i(x) = \begin{cases} \frac{x-x_{i-1}}{x_i-x_{i-1}}, & \text{if } x \in [x_{i-1}, x_i], \\ \frac{x_{i+1}-x}{x_{i+1}-x_i}, & \text{if } x \in [x_i, x_{i+1}], \\ 0, & \text{if not.} \end{cases}$$

and

$$\Phi_i(x_k) = \delta_{ik} = \begin{cases} 1, & \text{if } k = i, \\ 0, & \text{if not.} \end{cases}$$

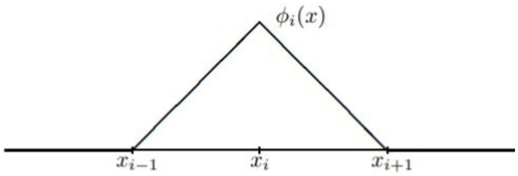

Fig 1 *: basis function for dimension two.*

For the second dimension, *we choose* Ω *as a rectangle (example :* $\Omega =]0,1[\times]0,1[$*) and we take* Υ_h *is a triangulation made of triangles* $\mathscr{T}$ *such that no nodes of every triangle lies in the interior of a side of another triangle.*

The bellow figure is a uniform mesh of $\Omega =]0,1[\times]0,1[$

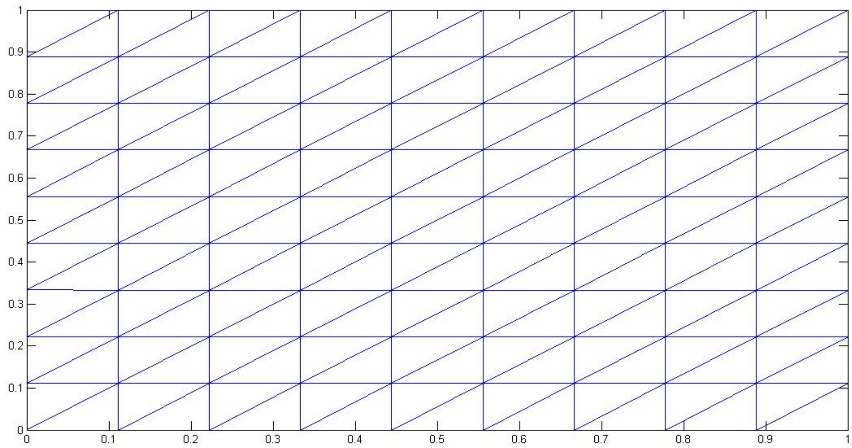

Fig 2 : mesh of the Ω.

Then the approximation space V_h *written as follows*

$$V_h = \left\{\Phi_h \in C^0(\bar{\Omega}) \text{ tel que } \Phi_h|_{\mathcal{T}} \text{ is polynomial of degree one} \forall \mathcal{T} \in \Upsilon_h\right\},$$

where its base is determined by the functions $\{\Phi_j\}_{j=1}^N$ *which it satisfies*

$$\phi_j(x_k, y_k) = \delta_{jk} = \begin{cases} 1, & \text{if } k = j, \\ 0, & \text{if not.} \end{cases},$$

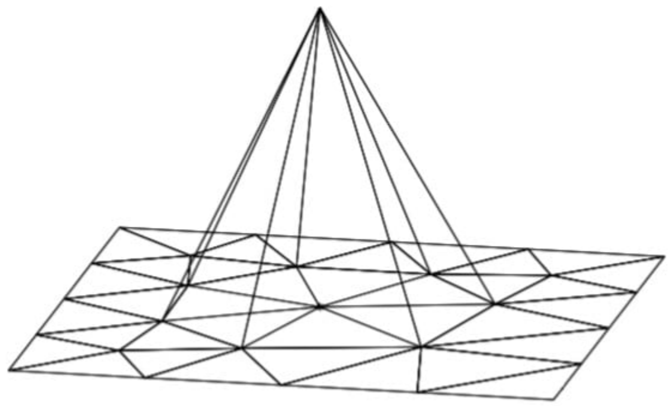

Fig 3 : *basis function for dimension two.*

The approximate problem in the general case is written as follows : Find $u_h \in V_h$ *such that*

$$a(u_h, v_h) = l(v_h),\ \forall v_h \in V_h. \tag{1.47}$$

Let $\{\phi_j\}_{j=1}^N$ *be basic functions for the space* V_h*. We write the approximation* u_h *of* u *as*

$$u_h = \sum_{j=1}^{N} \alpha_j \phi_j \in V_h. \tag{1.48}$$

Therefore the problem (1.43) *becomes : Find* $\overline{\alpha} = [\alpha_1, \alpha_2, ..., \alpha_N]$ *such that*

$$\sum_{j=1}^{N} \alpha_j a(\phi_j, v_h) = l(v_h),\ v_h \in V_h. \tag{1.49}$$

Take into account the linearities of $a(.,.)$ *and* $l(.)$ *: Find* $\overline{\alpha} = [\alpha_1, \alpha_2, ..., \alpha_N]$ *such that*

$$\sum_{j=1}^{N} \alpha_j a(\phi_j, \phi_k) = l(\phi_k),\ \forall k = 1, ..., N. \tag{1.50}$$

Hence obtaining the linear system must be solved

$$\begin{pmatrix} a(\phi_1,\phi_1) & a(\phi_1,\phi_2) & \cdots & a(\phi_1,\phi_N) \\ a(\phi_2,\phi_1) & a(\phi_2,\phi_2) & \cdots & a(\phi_2,\phi_N) \\ \vdots & \vdots & \ddots & \vdots \\ a(\phi_N,\phi_1) & a(\phi_N,\phi_2) & \cdots & a(\phi_N,\phi_N) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix} = \begin{pmatrix} l(\phi_1) \\ l(\phi_2) \\ \vdots \\ l(\phi_N). \end{pmatrix} \tag{1.49}$$

1.3.3 Convergence

Lemma 1.3.1. *(***Lemma of Cea.***)*

Let the exact solution u and be the approximate solution u_h. We have the following error

$$\|u-u_h\| \leq \frac{M}{\alpha} \inf_{v_h\in V_h} \|u-v_h\|. \tag{1.50}$$

$$\|u-u_h\| \leq \frac{M}{\alpha} \inf_{v_h\in V_h} d(u,v_h). \tag{1.51}$$

Here M and α are constants that satisfy (1.40) *and* (1.41) *respectively, and d is the distance induced by the norm $\|.\|$.*

1.4 The mixed finite element method

1.4.1 Introduction

Among the numerical methods proposed for solving partial differential equations is the mixed finite element method, and it is considered one of the preferred methods over traditional methods because one of its advantages is the preservation of physical quantities such as the amount of mass, temperature and Movement quantity ...

This method allows to solve mixed problems whose unknowns are two functions representing on the one hand the state of the considered system, and on the other hand a Lagrange multiplier associated to a constraint that the state must satisfy. For more details can be found on the following references [23, 24, 30, 37, 47].

1.4.2 General principle of the mixed finite element method: resolution of a matrix system

Let V, W be two Hilbert spaces, the mixed variational problem writes as follows : Find $(u,w) \in V \times W$ such that

$$\begin{cases} a(u,v) + c(w,v) = l_V(v) & \forall v \in V \\ \\ c(u,\eta) = 0 & \forall \eta \in W. \end{cases} \tag{1.52}$$

Here $a(.,.)$, $c(.,.)$ two bilinear forms on $V \times V$, $V \times W$ respectively, and $l_V(.)$ linear form on V.

Theorem 1.4.1. *(Inf-sup stability) For $u \in V$, we have*

$$\gamma \leq C \inf_{0\neq\eta\in W} \sup_{0\neq u\in V} \frac{c(u,\eta)}{\|u\|_V \|\eta\|_W}. \tag{1.53}$$

This theorem implies that the mixed problem (1.51) *is well posed. In other words, it admits a unique solution $(u,w) \in V \times W$ which satisfy $\|u\|_V^2 + \|w\|_W^2 \leq \lambda \|f\|_V^2$, where λ is a positive constant.*

For internal approximative, we substitute the two spaces V and W which often have an infinite dimension by two subspaces V_h and W_h whose dimension is finite, we will *also try to get an approximate solution to the following mixed problem : Find* $(u_h, w_h) \in V_h \times W_h$ *such that*

$$\begin{cases} a(u_h, v_h) + c(w_h, v_h) = l_V(v_h) & \forall v_h \in V_h \\ \\ c(u_h, \eta_h) = 0 & \forall \eta_h \in W_h. \end{cases} \tag{1.54}$$

Therefore, we can write the approximate mixed problem in the following matrix form

$$\begin{pmatrix} A_h & C_h^T \\ C_h & 0 \end{pmatrix} \begin{pmatrix} u_h \\ w_h \end{pmatrix} = \begin{pmatrix} f_h \\ 0 \end{pmatrix} \tag{1.55}$$

where $A_h : V_h \to V_h$ *and* $C_h : V_h \to W_h$ *are operations defined as follows*

$$(A_h u_h, v_h)_V = a(u_h, v_h) \text{ and } (C_h \eta_h, v_h)_W = c(\eta_h, v_h) \ \ \forall u_h, v_h \in V_h, \forall \eta_h \in W_h,$$

and $(f_h, v_h)_V = l(v_h) \ \ \forall v_h \in V_h.$

1.4.3 Convergence

$c(..)$ *verify the inf-sup condition on the* $V_h \times W_h$ *i.e. the approximate mixed problem is well posed. We affirm the existence and uniqueness of the solution to this approximate mixed problem that achieves convergence with the solution of the mixed problem. This convergence is explained in the following theorem*

Theorem 1.4.2. *There exist a positive constant* C *independent of* h *such that*

$$\|u - u_h\|_V + \|w - w_h\|_W \leq \Big\{ \inf_{v_h \in V_h} \|u - v_h\|_V + \inf_{\eta_h \in W_h} \|w - \eta_h\|_W \Big\}. \tag{1.56}$$

2

Full Discretization to an Hyperbolic Equation with Nonlocal Coefficient

2.1 Introduction

In this chapter, we present a full discretization of the telegraph equation with nonlocal coefficient by combining the Rothe method to the time discretization and the finite element method for the spatial discretization. After that we derive a priori estimates and the optimal a priori error estimates for both semi discrete and fully discrete schemes.

The fully discrete scheme for our problem gives a system of nonlinear equations. We use Newton Raphson method to solve this system. It is known that the Newton Raphson iteration is the most popular for solving nonlinear algebraic equations because it is fast convergent in a small number of iteration. One of the main difficulties of using Newtons is the fully Jacobien matrix, this difficulty can be addressed by reformulate the system through the application of the technique used by Sudhakar [14].

2.2 Position of the problem

Let Ω *is a simply connected bounded domain of* $\mathbb{R}^k$, $k \geq 1$ *with Lipschitz continuous boundary* $\partial\Omega$*. Consider the following nonlocal hyperbolic problem.*

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \dfrac{\partial u}{\partial t} + a(l(u))(\mathscr{A} u) = f(x,t,u) & in\ Q = \Omega \times I \\ \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & in\ \Omega \\ \\ u = 0 & on\ \partial\Omega \times I. \end{cases} \tag{2.1}$$

Where a *is a function depends of* $l(u)$ *with*

$$l(u) = \int_\Omega u(x,t)\,dx. \tag{2.2}$$

We introduce the elliptic differential operator A *defined by*

$$\mathscr{A} u := -div(A(x)\nabla u) + b(x)u, \tag{2.3}$$

where $A(x)$ *is a symmetric matrix with entries that are uniformly bounded and measurable,* $b(x)$ *is a bounded function and we assume that* f, u_0, u_1 *and* $A(x)$ *are smooth enough functions.*

2.3 Definitions and assumptions

In this section, we present a definition of the weak formulation and we assume hypotheses necessary for this work to prove the existence and uniqueness of the weak solution.

Definition 2.3.1. *let* $(.,.)_{\mathscr{A}}$ *be the inner product of* V *defined by*

$$(u,v)_{\mathscr{A}} = (A\nabla u, \nabla v) + (bu, v) \quad \forall u, v \in V, \tag{2.4}$$

and the norms on V *is denoted* $\|.\|_{\mathscr{A}}$.

Definition 2.3.2. *A function* u *is called a weak solution of* (2.1) *if*

$$\begin{cases} 1)\ u : Q \to \mathbb{R} \ \textit{ and } \ u \in H^1\big(I, L^2(\Omega)\big) \cap L^2(I, V) \textit{such that,} \\ \\ \forall v \in H^1\big(I; L^2(\Omega)\big) \cap L^2(I, v) \ \textit{with}\ v(x,T) = 0. \\ \\ 2)\ -\displaystyle\int_I \big(\partial_t u, \partial_t v\big) - \big(u_1, v(.,0)\big) + \int_I \big(\partial_t u, v\big) + \int_I a(l(u))\big(u,v\big)_{\mathscr{A}} = \int_I \big(f,v\big) \\ \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x). \end{cases} \tag{2.5}$$

Along this work we shall always assume the following assumptions

$(H1)$ $u^0 \in V,\ \ u^1 \in L^2(\Omega)$

$(H2)$ $f : \Omega \times I \times \mathbb{R} \to \mathbb{R}$ *is Lipschitz continuous in the sense*

$$\|f(x,t,s) - f(x,t',s')\| \le C\{|t-t'|(|s| + |s'| + |s-s'|)\}, \tag{2.6}$$

and satisfies the condition of growth

$$\|f(x,t,\xi)\| \leq C(1+|\xi|), \quad \forall (x,t,\xi) \in \Omega \times I \times \mathbb{R}. \tag{2.7}$$

$(H3)$ $a:\mathbb{R}\to\mathbb{R}$ *is lipschitz continuous with the lipschitz constant* L_M, *this means*

$$|a\big(l(u)\big) - a\big(l(v)\big)| \leq L_M\|u-v\|, \quad \forall u,v \in V, \tag{2.8}$$

and satisfies

$$0 < m_a \leq a(s) \leq M_a < \infty, \quad \forall s \in \mathbb{R}. \tag{2.9}$$

$(H4)$ $A(x)$ *is symmetric matrix satisfies*

$$(A\xi,\xi) \geq C\|\xi\|^2. \tag{2.10}$$

and let $(.,.)_{\mathscr{A}}$ *be a bounded, coercive and symmetric bilinear form according to choose the coefficients* $A(x)$, *i.e.,*

$$|(u,v)_{\mathscr{A}}| \leq C\|u\|_{\mathscr{A}}\|V\|_{\mathscr{A}}, \quad (u,u)_{\mathscr{A}} \geq C\|u\|^2_{\mathscr{A}}, \quad \forall u,v \in V. \tag{2.11}$$

2.4 Time discretization

In this section, we create a scheme of discretization in time based on the Rothe method and also extract some a priori estimates and the optimal a priori error estimates.

We divide the interval I *into* n *subintervals of length* $\tau = \frac{T}{n}$ *and denote* $u^i = u(t_i,x)$, $t_i = i\tau$, $i = 0,1,...,n$. *Let* u^{-1} *be defined as* $u^{-1}(x) = u^0(x) - \tau u^1(x)$, *the reccurent approx-*

imation scheme for $i = 1, ..., n$ *becomes*

$$\begin{cases} \text{Find } u^i \cong u(., t_i) \in V \;\; i = 1, 2, ..., n \text{ such that the equation,} \\ \\ \left(\delta^2 u^i, v\right) + \left(\delta u^i, v\right) + a(l(u^i))\left(u^i, v\right)_{\mathscr{A}} = \left(f^i, v\right). \end{cases} \tag{2.12}$$

We define the Rothe's functions by a piecewise linear interpolation with respect to the time t,

$$u^n = u^{i-1} + (t - t_{i-1})\delta u^i \quad \forall t \in [t_{i-1}, t_i], \;\; 1 \le i \le n, \tag{2.13}$$

$$\delta u^n = \delta u^{i-1} + (t - t_{i-1})\delta^2 u^i \quad \forall t \in [t_{i-1}, t_i], \;\; 1 \le i \le n, \tag{2.14}$$

together with the step function

$$\bar{u}^n = \begin{cases} u^i & t \in [t_{i-1}, t_i], \\ \\ u^0 & t \in [-\tau, 0]. \end{cases} \tag{2.15}$$

We denote by $\bar{f}^n$ *the function*

$$\bar{f}^n = \begin{cases} f^i & t \in [t_{i-1}, t_i], \quad . \\ \\ 0 & t = 0. \end{cases} \tag{2.16}$$

Then, the problem (2.12) *can be redefined as follows :*

$$\begin{cases} \forall v \in H^1\left(I; L^2(\Omega)\right) \cap L^2(I, v) \text{ with } v(x, T) = 0, \\ \\ \left(\partial_t \delta u^n, v\right) + \left(\partial_t u^n, v\right) + a(l(\bar{u}^n))\left(\bar{u}^n, v\right)_{\mathscr{A}} = \left(\bar{f}^n, v\right). \end{cases} \tag{2.17}$$

By integrating the above equation over I, we get

$$\begin{cases} \forall v \in H^1\big(I; L^2(\Omega)\big) \cap L^2(I, v) \text{ with } v(x,T) = 0, \\ \\ -\int_I \big(\delta u^n, \partial_t v\big) - \big(\delta u^n(0), v(.,0)\big) + \int_I \big(\partial_t u^n, v\big) + \int_I a(l(\bar{u}^n))\big(\bar{u}^n, v\big)_{\mathscr{A}} = \int_I \big(\bar{f}^n, v\big). \end{cases} \tag{2.18}$$

2.4.1 A priori estimates

Lemma 2.4.1. *For $1 \le i \le s \le n$, the estimates*

$$\|\delta u^s\|^2 + \sum_{i=1}^{s} \|\delta u^i - \delta u^{i-1}\|^2 + \sum_{i=1}^{s} \tau \|\delta u^i\|^2 + m_a \|u^s\|^2_{\mathscr{A}} + m_a \sum_{i=1}^{s} \|u^i - u^{i-1}\|^2_{\mathscr{A}} \le C\tau. \tag{2.19}$$

Proof.

Choose $v = \delta u^i$ in the equation (2.12), we get

$$(\delta u^i - \delta u^{i-1}, \delta u^i) + \tau \|\delta u^i\|^2 + m_a (u^i, u^i - u^{i-1})_{\mathscr{A}} \le \tau \|f^i\| \|\delta u^i\|.$$

Using Young, we obtain

$$\|\delta u^i\|^2 - \|\delta u^{i-1}\|^2 + \|\delta u^i - \delta u^{i-1}\|^2 + \tau \|\delta u^i\|^2 + m_a\big(\|u^i\|^2_{\mathscr{A}} - \|u^{i-1}\|^2_{\mathscr{A}} + \|u^i - u^{i-1}\|^2_{\mathscr{A}}\big)$$

$$\le \tau \|f^i\| \|\delta u^i\|.$$

Taking summation from $i = 1$ to s to get

$$\|\delta u^s\|^2-\|\delta u^0\|^2+\sum_{i=1}^{s}\|\delta u^i-\delta u^{i-1}\|^2+\sum_{i=1}^{s}\tau\|\delta u^i\|^2+m_a\|u^s\|_{\mathscr{A}}^2-m_a\|u^0\|_{\mathscr{A}}^2$$

$$+m_a\sum_{i=1}^{s}\|u^i-u^{i-1}\|_{\mathscr{A}}^2\leq\sum_{i=1}^{s}\tau\|f^i\|\|\delta u^i\|.$$

Applying the Abel's summing formula, we obtain

$$\begin{aligned}\|\delta u^s\|^2+\sum_{i=1}^{s}\|\delta u^i-\delta u^{i-1}\|^2+\sum_{i=1}^{s}\tau\|\delta u^i\|^2+m_a\|u^s\|_{\mathscr{A}}^2+m_a\sum_{i=1}^{s}\|u^i-u^{i-1}\|_{\mathscr{A}}^2\\ \leq C+\epsilon\sum_{i=1}^{s}\tau\|f^i\|^2+C_\epsilon\sum_{i=1}^{s}\tau\|\delta u^i\|^2,\\ \leq\epsilon\Big(1+\sum_{i=1}^{s}\sum_{r=1}^{i-1}\tau^2\|\delta u^r\|^2\Big)+C_\epsilon\sum_{i=1}^{s}\tau\|\delta u^i\|^2.\end{aligned}\tag{2.20}$$

Using the Gronwall's lemma (see, e.g.[45]*) inequality and choosing* $\epsilon=\tau$ *to get*

$$\|\delta u^s\|^2+\sum_{i=1}^{s}\|\delta u^i-\delta u^{i-1}\|^2+\sum_{i=1}^{s}\tau\|\delta u^i\|^2+m_a\|u^s\|_{\mathscr{A}}^2+m_a\sum_{i=1}^{s}\|u^i-u^{i-1}\|_{\mathscr{A}}^2\leq C\tau.\ \blacksquare$$

Corollary 2.4.2. *There exists a positive constant* C *such that*

$$\|\partial_t u^n\|_{L^2(I;L^2(\Omega))}^2\leq C,\|u^n\|_{L^2(I;V)}^2\leq C,\tag{2.21}$$

$$\|u^n-\bar{u}^n\|_{L^2(I;V)}^2\leq\frac{C}{n},\tag{2.22}$$

$$\|u^n-\bar{u}^n\|_{L^2(I;L^2(\Omega))}^2\leq\frac{C}{n^2},\|u^n-\bar{u}_\tau^n\|_{L^2(I;L^2(\Omega))}^2\leq\frac{C}{n^2},\tag{2.23}$$

$$\|\delta u^n - \partial_t u^n\|^2_{L^2(I;L^2(\Omega))} \leq \frac{C}{n}, \tag{2.24}$$

where $\bar{u}^n_\tau = \bar{u}^n(., t-\tau)$.

Proof.

All the estimates of this corollary are a result of Lemma 2.4.1 *and the definitions of* u^n, δu^n *and* $\bar{u}^n$. *The estimates* $(2.21)_1$ *and* (2.23) *are a result of* $(2.19)_3$ *and the definition of* $\bar{u}^n_\tau$, *whereas* $(2.21)_2$ *is a result of* $(2.19)_4$. *The estimates* $(2.19)_5$ *and* $(2.19)_2$ *imply* (2.22) *and* (2.24), *respectively.*■

2.4.2 A priori error estimates

We denote by $e_u = u - u^n$ *and* $e_f = f - \bar{f}^n$.

Theorem 2.4.3. *Under the assumptions* $(H1)-(H4)$, *we have*

$$\|e_u\|^2_{C(I;L^2(\Omega))} + m_a\|e_u\|^2_{L^2(I;V)} \leq C(\tau^2+\tau). \tag{2.25}$$

Proof.

Subtracting (2.18) *from* (2.5) *and using* $v = e_u(t)$ *as a test function we obtain*

$$|e_u|^2 + m_a\int_I \|e_u\|^2 \leq \epsilon\|e_f\|^2_{L^2(I;L^2(\Omega))} + C_\epsilon\int_I |e_u|^2 + \frac{m_a}{2}\int_I \|u^n - \bar{u}^n\|^2_{\mathscr{A}} + \frac{m_a}{2}\int_I \|e_u\|^2_{\mathscr{A}} + C\tau. \tag{2.26}$$

Now we consider

$$\begin{aligned} |e_f|^2_{L^2(I;L^2(\Omega))} &\leq \int_I \Big(|f(t,u) - f(t,u^n)|^2 + |f(t,u^n) + f(t,\bar{u}^n_\tau)|^2 + |f(t,\bar{u}^n_\tau) - f(t_i,\bar{u}^n_\tau)|^2\Big) \\ &\leq \int_I \Big(|e_u| + |u^n - \bar{u}^n_\tau|^2 + \tau^2\Big). \end{aligned} \tag{2.27}$$

Substituting from (2.27) *in* (2.26), *using Corollary2.4.2, choosing e sufficiently small and* *applying Gronwalls lemma the proof completes.*■

2.5 Full Discretization

This section contains a fully discrete scheme by using the finite element method for spatial discretization and also the numerical method used to solve nonlinear equations obtained from the fully discrete scheme of a problem. In addition, we give some a priori estimates and the optimal a priori error estimates.

At each time t_i, $0 \le i \le n$, *we consider a triangulation* Υ_h^i *made of triangles* T^i *such that no nodes of every triangle lies in the interior of a side of another triangle. Let* V_h^i *be the discrete space of* V^i *defined by*

$$V_h^i = \left\{\Phi_h \in C^0(\bar{\Omega}) \textit{ tel que } \Phi_h|_{T^i} \textit{ is polynomial of degree one} \forall T^i \in \Upsilon_h^i\right\}.$$

Let $\{p_j\}_{j=1}^N$ *be interior nodes of* Υ_h^i *et* $\{\Phi_j(x)\}_{j=1}^N$ *be the basic functions for the space* V_h^i *such that any function will be the pyramid form in* V_h^i *and wich takes the value* 1 *at* $\{p_j\}_{j=1}^N$ *and vanishes at exterior nodes. We can write the solution* u_h^i *as*

$$u_h^i(t) = \sum_{j=1}^N \alpha_j^i \Phi_j(x) \in V_h^i.$$

Then, the fully discrete scheme for problem (2.1) *reads as*

$$\begin{cases} \textit{Find } u_h^i \in V_h^i(\Omega) \textit{ such that :} \\ \\ u_h(0) = u_h^0,\ u_{ht}(0) = u_h^1 \textit{ and } u_h^{-1} = u_h^0 - \tau u_h^1, \\ \\ \textit{and, } \ \forall v \in V_h^i, \\ \\ \big(\delta^2 u_h^i, v_h\big) + \big(\delta u_h^i, v_h\big) + a\big(l(u_h^i)\big)\big(u_h^i, v_h\big)_{\mathscr{A}} = \big(f^i, v_h\big). \end{cases} \tag{2.28}$$

We introduce the orthogonal projection operator $\Pi_h^i \ : H_0^1(\Omega) \longrightarrow V_h^i(\Omega)$ *such that*

$$(\nabla w, \nabla v) = (\nabla \Pi_h^i w, \nabla v) \ \ \forall w \in H_0^1(\Omega), v \in V_h^i(\Omega). \tag{2.29}$$

From fully discrete weak formulation of (2.26)*, we have*

$$\begin{cases} \textit{Find } u_h^i \in V_h^i(\Omega) \textit{ such that :} \\ \\ u_h(0) = u_h^0,\ u_{ht}(0) = u_h^1 \textit{ and } u_h^{-1} = u_h^0 - \tau u_h^1, \\ \\ \textit{and, } \ \forall v \in V_h^i, \\ \\ \big(\frac{u_h^i - \Pi_h^i u_h^{i-1}}{\tau} - \frac{u_h^{i-1} - \Pi_h^{i-1} u_h^{i-2}}{\tau}, v_h\big) + \tau\big(\frac{u_h^i - \Pi_h^i u_h^{i-1}}{\tau}, v_h\big) \\ \\ \qquad\qquad + \tau a\big(l(u_h^i)\big)\big(u_h^i, v_h\big)_{\mathscr{A}} = \tau\big(f^i, v_h\big). \end{cases} \tag{2.30}$$

This implies,

$$\begin{cases} \textit{Find } u_h^i \in V_h^i(\Omega) \textit{ such that :} \\ \\ u_h(0) = u_h^0,\ u_{ht}(0) = u_h^1 \textit{ and } u_h^{-1} = u_h^0 - \tau u_h^1, \\ \\ \textit{and, } \ \forall v \in V_h^i, \\ \\ (1+\tau)\big(u_h^i, v_h\big) + \tau^2 a\big(l(u_h^i)\big)\big(u_h^i, v_h\big)_{\mathscr{A}} \\ \\ \qquad\qquad = \tau^2\big(f^i, v_h\big) + \big((1+\tau)\Pi_h^i u_h^{i-1} + \big(u^{i-1} - \Pi_h^{i-1} u_h^{i-2}\big), v_h\big). \end{cases} \tag{2.31}$$

2.5.1 Numerical method

The problem (2.31) *give as a system of nonlinear algebraic equations by using finite element, then can be given this system as follows :*

$$F_j(\bar{\alpha}^i) = F_j(u_h^i) = 0 \quad 1 \le j \le N, \tag{2.32}$$

where $\bar{\alpha}^i = [\alpha_1^i, \alpha_2^i, ..., \alpha_N^i]$, *and*

$$F_j(u_h^i) = (1+\tau)\big(u_h^i, v_h\big) + \tau^2 a\big(l(u_h^i)\big)\big(u_h^i, v_h\big)_{\mathscr{A}} - \tau^2\big(f^i, v_h\big) - \big((1+\tau)\Pi_h^i u_h^{i-1}$$

$$+\big(u^{i-1} - \Pi_h^{i-1} u_h^{i-2}\big), v_h\big).$$

We use Newton-Raphson method to solve (2.31), *but the presence of nonlocal term in the equation destroys the sparsity of Newton-Raphson method.*

We compute the Jacobian matrix J To get the value of α_j^i by Newton's method, every element of the Jacobian matrix takes the form

$$\frac{\partial F_j(u_h^i)}{\partial \alpha_j^i} = (1+\tau)\big(\phi_j,\phi_l\big) + \tau^2\Big(\int_\Omega \phi_j\Big) a'\big(l(u_h^i)\big)\big(u_h^i,\phi_l\big)_{\mathscr{A}} + \tau^2 a\big(l(u_h^i)\big)\big(\phi_j,\phi_l\big)_{\mathscr{A}}$$
$$-\tau^2\big(f'(u_h^i)\phi_j,\phi_l\big). \tag{2.33}$$

In order to ensure the sparsity of the Jacobian matrix we modify the scheme (2.31) *according to the technic used by Chaudhary in* [14]. *Then the problem* (2.31) *can be rewritten as follows*

Find $d \in \mathbb{R}$, and $u_h^i \in V_h^i$ such that

$$l(u_h^i) - d = 0. \tag{2.34}$$

$$(1+\tau)\big(u_h^i, v_h\big) + \tau^2 a\big(l(u_h^i)\big)\big(u_h^i, v_h\big)_{\mathscr{A}} - \tau^2\big(f^i, v_h\big) - \big((1+\tau)\Pi_h^i u_h^{i-1}$$
$$+\big(u^{i-1} - \Pi_h^{i-1} u_h^{i-2}\big), v_h\big) = 0 \quad \forall v_h \in V_h^i. \tag{2.35}$$

Take $v_h = \phi_j$, and reformulate the equations (2.34 − 2.35) *as follows*

$$F_j(u_h^i, d) = (1+\tau)\big(u_h^i, \phi_l\big) + \tau^2 a\big(d\big)\big(u_h^i, \phi_l\big)_{\mathscr{A}} - \tau^2\big(f^i, \phi_l\big)$$
$$-\big((1+\tau)\Pi_h^i u_h^{i-1} + \big(u^{i-1} - \Pi_h^{i-1} u_h^{i-2}\big), \phi_l\big). \tag{2.36}$$

$$F_{N+1}^i = l(u_h^i) - d. \tag{2.37}$$

This implies

$$J\begin{bmatrix} \bar{\alpha}^i \\ \beta \end{bmatrix} = \begin{bmatrix} A & b \\ c & \delta_{11} \\ & \end{bmatrix}\begin{bmatrix} \bar{\alpha}^i \\ \beta \end{bmatrix} = \begin{bmatrix} \bar{F}^i \\ F_{N+1}^i, \end{bmatrix}. \tag{2.38}$$

where $A = A_{N\times N}$, $b = b_{N\times 1}$ *and* $c = c_{1\times N}$ *take the form*

$$\begin{aligned} A_{jl} &= (1+\tau)\big(\phi_j, \phi_l\big) + \tau^2 a\big(d\big)\big(\phi_j, \phi_l\big)_{\mathscr{A}} - \tau^2\big(f'(u_h^i)\phi_j, \phi_l\big), \\ b_{j1} &= \tau^2 a'\big(d\big)\big(u_h^i, \phi_l\big)_{\mathscr{A}}, \\ c_{1l} &= \Big(\int_\Omega \phi_j\Big), \\ \delta_{11} &= -1, \end{aligned}$$

and $\bar{\alpha}^i = [\alpha_1^i, \alpha_2^i, ..., \alpha_N^i]^T$, $\bar{F}^i = [F_1^i, F_2^i, ..., F_N^i]^T$.

The matrix system (2.38) *can be solved by using the Sherman-Morrison Woodbury formula or block elimination with one-refinement algorithm in* [25, 26].

2.5.2 A priori estimates

Lemma 2.5.1. *The estimates*

$$\|\nabla\Pi_h^i u^i\| \le C. \tag{2.39}$$

$$\|\Pi_h^i u^i\|_{\mathscr{A}} \le c. \tag{2.40}$$

Proof. *For* $w = u^i$ *in* (2.29), *we have*

$$(\nabla u^i, \nabla v_h) = (\nabla\Pi_h^i u^i, \nabla v_h).$$

Choosing $v_h = \Pi_h^i u^i$, *to get*

$$\begin{aligned}\|\nabla\Pi_h^i u^i\|^2 &= (\nabla u^i, \nabla\Pi_h^i u^i)\\ &\le \|\nabla u^i\|\|\nabla\Pi_h^i u^i\|.\end{aligned}$$

Thus,

$$\begin{aligned}\|\nabla\Pi_h^i u^i\| &= \|\nabla u^i\|\\ &\le C.\end{aligned}$$

Further

$$\begin{aligned}
\|\Pi_h^i u^i\|_{\mathscr{A}}^2 &= (\Pi_h^i u^i, \Pi_h^i u^i)_{\mathscr{A}} \\
&= (A\nabla\Pi_h^i u^i, \nabla\Pi_h^i u^i) + (a(x)\Pi_h^i u^i, \Pi_h^i u^i) \\
&\leq C(\|\nabla\Pi_h^i u^i\|^2 + \|\Pi_h^i u^i\|^2).
\end{aligned}$$

Using Poincaré inequality, we obtain

$$\begin{aligned}
\|\Pi_h^i u^i\|_{\mathscr{A}} &\leq C(\|\nabla\Pi_h^i u^i\|) \\
&\leq c.
\end{aligned}$$

where c *and* C *are some positive constants.*

Lemma 2.5.2. *Let* $u_h^0 \in V_h^0$ *and* $u_h^1 \in V_h^0$ *and for* $1 \leq i \leq s \leq n$*, then the solution* $u_h^i \in V_h^i$ *of the problem* (2.28) *satisfied*

$$\|\delta u_h^s\|_{L^2(0,T;L^2(\Omega))}^2 + m_a\|u_h^s\|_{L^2(0,T;V)}^2 \leq C \tag{2.41}$$

Proof. *We use the same proof in lemma*2.4.1 *to obtain the existence of* u_h^i *and a priori estimates.*

$$\|\delta u_h^s\|^2 + \sum_{i=1}^{s}\|\delta u_h^i - \delta u_h^{i-1}\|^2 + \sum_{i=1}^{s}\tau\|\delta u_h^i\|^2 + m_a\|u_h^s\|_{\mathscr{A}}^2 + m_a\sum_{i=1}^{s}\|u_h^i - u_h^{i-1}\|_{\mathscr{A}}^2 \leq C\tau.$$

This means

$$\|\delta u_h^s\|^2 + m_a\|u_h^s\|_{\mathscr{A}}^2 \leq C.$$

We integrate from 0 *to* T, *to obtain*

$$\|\delta u_h^s\|^2_{L^2(0,T;L^2(\Omega))} + m_a \|u_h^s\|^2_{L^2(0,T;V)} \leq C. \ \blacksquare$$

2.5.3 A priori error estimates

We introduce the orthogonal projection to get an optimal convergence between u^i, u_h^i. *Therefor, we can take the error as follows*

$$\begin{aligned} e^i = u^i - u_h^i &= u^i - \Pi_h^i u^i + \Pi_h^i u^i - u_h^i \\ &= \rho_h^i + \theta_h^i. \end{aligned} \tag{2.42}$$

Theorem 2.5.3. ([46]) *: There exists a positive constant C, independent of h such that*

$$\|v - \Pi_h^i v\|_j \leq C h_j^i \|v\|_i \quad \forall v \in H^i \cap H_0^1 \quad j = 0,1; i = 1,2 \tag{2.43}$$

$$\|v_t - \Pi_h^i v_t\|_j \leq C h_j^i \|v_t\|_i \quad \forall v \in H^i \cap H_0^1 \quad j = 0,1; i = 1,2. \tag{2.44}$$

$$\|v_{tt} - \Pi_h^i v_{tt}\|_j \leq C h_j^i \|v_{tt}\|_i \quad \forall v \in H^i \cap H_0^1 \quad j = 0,1; i = 1,2. \tag{2.45}$$

Theorem 2.5.4. *We assume that* $\frac{\min(b(x))}{2} \geq \frac{16 e^3 c^2 L_{M_a}}{m_a^2}$ *where c is given in Eq.*(2.38). *Then,* *there exists a positive constant C such that*

$$\|u^i - u_h^i\|_{L^2(0,T,\tau,H^1(\Omega))} \leq C(h+h^2). \tag{2.46}$$

Proof. *From equations (2.12),(2.28), we have*

$$\begin{aligned}
\left(\delta^2\theta_h^i, v_h\right) + \left(\delta\theta_h^i, v_h\right) + a_h^i\left(\theta_h^i, v_h\right)_{\mathscr{A}} &= \left(\delta^2\Pi_h^i u^i, v_h\right) + \left(\delta\Pi_h^i u^i, v_h\right) + a_h^i\left(\Pi_h^i u^i, v_h\right)_{\mathscr{A}} \\
&\quad -\left(\delta^2 u_h^i, v_h\right) - \left(\delta u_h^i, v_h\right) - a_h^i\left(u_h^i, v_h\right)_{\mathscr{A}} \\
&= -\left(f^i, v_h\right) + \left(\delta^2\Pi_h^i u^i, v_h\right) + \left(\delta\Pi_h^i u^i, v_h\right) \\
&\quad + a_h^i\left(\Pi_h^i u^i, v_h\right)_{\mathscr{A}} \\
&= -\left(\delta^2 u^i, v_h\right) - \left(\delta u^i, v_h\right) - a^i\left(u^i, v_h\right)_{\mathscr{A}} \\
&\quad + \left(\delta^2\Pi_h^i u^i, v_h\right) + \left(\delta\Pi_h^i u^i, v_h\right) + a_h^i\left(\Pi_h^i u^i, v_h\right)_{\mathscr{A}} \\
&\quad + a^i\left(\Pi_h^i u^i, v_h\right) - a^i\left(\Pi_h^i u^i, v_h\right) \\
&= -\left(\delta^2(u^i - \Pi_h^i u^i), v_h\right) - \left(\delta(u^i - \Pi_h^i u^i), v_h\right) \\
&\quad - a^i\left((u^i - \Pi_h^i u^i), v_h\right)_{\mathscr{A}} + (a_h^i - a^i)\left(\Pi_h^i u_h^i, v_h\right)_{\mathscr{A}}.
\end{aligned}$$

Thus,

$$\left(\delta^2\theta_h^i, v_h\right) + \left(\delta\theta_h^i, v_h\right) + a_h^i\left(\theta_h^i, v_h\right)_{\mathscr{A}} = -\left(\delta^2\rho_h^i, v_h\right) - \left(\delta\rho_h^i, v_h\right)$$

$$-a^i\left(\rho_h^i, v_h\right)_{\mathscr{A}} + (a_h^i - a^i)\left(\Pi_h^i u^i, v_h\right)_{\mathscr{A}}. \tag{2.47}$$

Choosing $v_h = \tau^2\delta\theta_h^i$ *in* (2.47)*, we obtain*

$$\tau^2\left(\delta^2\theta_h^i, \delta\theta_h^i\right) + \tau^2\left(\delta\theta_h^i, \delta\theta_h^i\right) + \tau^2 a_h^i\left(\theta_h^i, \delta\theta_h^i\right)_{\mathscr{A}} = -\tau^2\left(\delta^2\rho_h^i, \delta\theta_h^i\right) - \tau^2\left(\delta\rho_h^i, \delta\theta_h^i\right)$$

$$-\tau^2 a^i\left(\rho_h^i, \delta\theta_h^i\right)_{\mathscr{A}} + \tau^2(a_h^i - a^i)\left(\Pi_h^i u^i, \delta\theta_h^i\right)_{\mathscr{A}} \tag{2.48}$$

New left-hand side of (2.48) *can be estimated as follows.*

$$\tau^2\left(\delta^2\theta_h^i, \delta\theta_h^i\right) + \tau^2\left(\delta\theta_h^i, \delta\theta_h^i\right) + \tau^2 a_h^i\left(\theta_h^i, \delta\theta_h^i\right)_{\mathscr{A}} \geq \tau^2\frac{\delta}{2}\|\delta\theta_h^i\|^2 + \tau^2\|\delta\theta_h^i\|^2 + \tau^2 m_a\frac{\delta}{2}\|\theta_h^i\|_{\mathscr{A}},$$

$$\geq \frac{\tau}{2}\|\delta\theta_h^i\|^2 - \frac{\tau}{2}\|\delta\theta_h^i\|^2 + \tau^2\|\delta\theta_h^i\|^2 + \frac{\tau}{2}m_a\|\theta_h^i\|_{\mathscr{A}}^2 - \frac{\tau}{2}m_a\|\theta_h^{i-1}\|_{\mathscr{A}}^2. \tag{2.49}$$

To estimate the right-hand side of (2.48)*, we need the following steps.*

Step1. *We estimate* $\left| -\tau^2\left(\delta^2\rho_h^i, \delta\theta_h^i\right) - \tau^2\left(\delta\rho_h^i, \delta\theta_h^i\right)\right|$.

Using Cauchy-schwarz, we get

$$\Big|-\tau^2\Big(\delta^2\rho_h^i,\delta\theta_h^i\Big)-\tau^2\Big(\delta\rho_h^i,\delta\theta_h^i\Big)\Big| \le \tau\|\delta^2\rho_h^i\|\tau\|\delta\theta_h^i\|+\tau\|\delta\rho_h^i\|\tau\|\delta\theta_h^i\|.$$

Thus,

$$\Big|-\tau^2\Big(\delta^2\rho_h^i,\delta\theta_h^i\Big)-\tau^2\Big(\delta\rho_h^i,\delta\theta_h^i\Big)\Big| \le \frac{\tau^2}{2}\|\delta^2\rho_h^i\|^2+\tau^2\|\delta\theta_h^i\|^2+\frac{\tau^2}{2}\|\delta\rho_h^i\|^2. \tag{2.50}$$

Step2. *We estimate* $\Big|-\tau^2 a^i\Big(\rho_h^i,\delta\theta_h^i\Big)_{\mathscr{A}}+\tau^2\big(a_h^i-a^i\big)\Big(\Pi_h^i u^i,\delta\theta_h^i\Big)_{\mathscr{A}}\Big|$.
Applying Cauchy-schwarz inequality and Using the inequality $ab \le \frac{\omega}{2}a^2+\frac{1}{2\omega}b^2$ *with* $\omega=\frac{m_a}{8}$, *we obtain*

$$\begin{aligned}
\Big|-\tau^2 a^i\Big(\rho_h^i,\delta\theta_h^i\Big)_{\mathscr{A}}+\tau^2\big(a_h^i-a^i\big)\Big(\Pi_h^i u^i,\delta\theta_h^i\Big)_{\mathscr{A}}\Big| &\le M_a\tau\|\rho_h^i\|_{\mathscr{A}}\|\theta_h^i-\theta_h^{i-1}\|_{\mathscr{A}}\\
&\quad +c\tau\big|a_h^i-a^i\big|\|\theta_h^i-\theta_h^{i-1}\|_{\mathscr{A}}\\
&\le \frac{m_a}{16}\tau\|\theta_h^i-\theta_h^{i-1}\|_{\mathscr{A}}^2+\frac{4M_a^2}{m_a}\tau\|\rho_h^i\|_{\mathscr{A}}^2\\
&\quad +\frac{4c^2}{m_a}\tau\big|a_h^i-a^i\big|^2+\frac{m_a}{16}\tau\|\theta_h^i-\theta_h^{i-1}\|_{\mathscr{A}}^2,\\
&\le \frac{4M_a^2}{m_a}\tau\|\rho_h^i\|_{\mathscr{A}}^2+\frac{4c^2}{m_a}\tau\big|a_h^i-a^i\big|^2\\
&\quad +\frac{m_a}{8}\tau\big(\|\theta_h^i\|_{\mathscr{A}}+\|\theta_h^{i-1}\|_{\mathscr{A}}\big)^2.
\end{aligned}$$

Using Lipschitz continuity of a, we have

$$\begin{aligned} |a_h^i - a^i| &\leq L_{M_a} \| u_h^i - u^i \| \\ &\leq L_{M_a} \| u_h^i - \Pi_h^i u^i + \Pi_h^i u^i - u^i \| \\ &\leq L_{M_a} \Big(\| \theta_h^i \| + \| \rho_h^i \| \Big). \end{aligned}$$

Thus,

$$\Big| -\tau^2 a^i \Big(\rho_h^i, \delta\theta_h^i\Big)_{\mathscr{A}} + \tau^2 (a_h^i - a^i) \Big(\Pi_h^i u^i, \delta\theta_h^i\Big)_{\mathscr{A}} \Big| \leq \frac{4M_a^2}{m_a} \tau \| \rho_h^i \|_{\mathscr{A}}^2 + \frac{4c^2}{m_a} \tau L_{M_a}^2 \Big(\| \theta_h^i \| + \| \rho_h^i \| \Big)^2 + \frac{m_a}{8} \tau \big(\| \theta_h^i \|_{\mathscr{A}} + \| \theta_h^{i-1} \|_{\mathscr{A}} \big)^2. \tag{2.51}$$

From (2.49), (2.50) *and* (2.51), *we get*

$$\frac{\tau}{2} \| \delta\theta_h^i \|^2 - \frac{\tau}{2} \| \delta\theta_h^{i-1} \|^2 + \tau^2 \| \delta\theta_h^i \|^2 + \frac{\tau}{2} m_a \| \theta_h^i \|_{\mathscr{A}}^2 - \frac{\tau}{2} m_a \| \theta_h^{i-1} \|_{\mathscr{A}}^2 \leq \frac{\tau^2}{2} \| \delta^2 \rho_h^i \|^2 + \tau^2 \| \delta\theta_h^i \|^2 + \frac{\tau^2}{2} \| \delta\rho_h^i \|^2 + \frac{4M_a^2}{m_a} \tau \| \rho_h^i \|_{\mathscr{A}}^2 + \frac{4c^2}{m_a} \tau L_{M_a}^2 \Big(\| \theta_h^i \| + \| \rho_h^i \| \Big)^2 + \frac{m_a}{4} \tau \| \theta_h^i \|_{\mathscr{A}}^2 + \frac{m_a}{4} \tau \| \theta_h^{i-1} \|_{\mathscr{A}}^2.$$

This implies,

$$\frac{\tau}{2}\|\delta\theta_h^i\|^2+\tau\frac{m_a}{2}\|\theta_h^i\|_{\mathscr{A}}^2 \leq \tau^2\|\delta^2\rho_h^i\|^2+\tau^2\|\delta\rho_h^i\|^2+\frac{8M_a^2}{m_a}\tau\|\rho_h^i\|_{\mathscr{A}}^2$$

$$+\frac{8c^2}{m_a}\tau L_{M_a}^2\Big(\|\theta_h^i\|+\|\rho_h^i\|\Big)^2+\frac{3m_a}{2}\tau\|\theta_h^{i-1}\|_{\mathscr{A}}^2+\frac{\tau}{2}\|\delta\theta_h^{i-1}\|^2.$$

Taking sum from $i=1$ to n to get

$$\frac{\tau}{2}\sum_{i=1}^{n}\|\delta\theta_h^i\|^2+\tau\frac{m_a}{2}\sum_{i=1}^{n}\|\theta_h^i\|_{\mathscr{A}}^2 \leq \tau^2\sum_{i=1}^{n}\|\delta^2\rho_h^i\|^2+\tau^2\sum_{i=1}^{n}\|\delta\rho_h^i\|^2+\frac{8M_a^2}{m_a}\tau\sum_{i=1}^{n}\|\rho_h^i\|_{\mathscr{A}}^2$$

$$+\frac{8c^2}{m_a}\tau L_{M_a}^2\sum_{i=1}^{n}\Big(\|\theta_h^i\|+\|\rho_h^i\|\Big)^2+\frac{3m_a}{2}\tau\sum_{i=1}^{n-1}\|\theta_h^i\|_{\mathscr{A}}^2+\frac{\tau}{2}\sum_{i=1}^{n-1}\|\delta\theta_h^i\|^2.$$

Now applying Gronwall's inequality, we get

$$\frac{\tau}{2}\sum_{i=1}^{n-1}\|\delta\theta_h^i\|^2+\frac{\tau}{2}\|\delta\theta_h^n\|^2+\frac{m_a}{2}\tau\sum_{i=1}^{n-1}\|\theta_h^i\|_{\mathscr{A}}^2+\frac{m_a}{2}\tau\|\theta_h^n\|_{\mathscr{A}}^2 \leq e^3\Big(\tau^2\sum_{i=1}^{n}\|\delta^2\rho_h^i\|^2+\tau^2\sum_{i=1}^{n}\|\delta\rho_h^i\|^2$$

$$+\frac{8M_a^2}{m_a}\tau\sum_{i=1}^{n}\|\rho_h^i\|_{\mathscr{A}}^2+\frac{8c^2}{m_a}\tau L_{M_a}^2\sum_{i=1}^{n}\Big(\|\theta_h^i\|+\|\rho_h^i\|\Big)^2\Big).$$

Thus,

$$\frac{\tau}{2}\sum_{i=1}^{n}\|\delta\theta_h^i\|^2+\frac{m_a}{2}\tau\sum_{i=1}^{n}\|\theta_h^i\|_{\mathscr{A}}^2 \leq e^3\Big(\tau^2\sum_{i=1}^{n}\|\delta^2\rho_h^i\|^2+\tau^2\sum_{i=1}^{n}\|\delta\rho_h^i\|^2+\frac{8M_a^2}{m_a}\tau\sum_{i=1}^{n}\|\rho_h^i\|_{\mathscr{A}}^2$$

$$+\frac{16c^2}{m}L_{M_a}^2\tau\sum_{i=1}^{n}\Big(\|\theta_h^i\|^2+\|\rho_h^i\|^2\Big)\Big).$$

Again,

$$\frac{\tau}{2}\sum_{i=1}^{n}\|\delta\theta_h^i\|^2+\frac{m_a}{2}\tau\sum_{i=1}^{n}C\|\nabla\theta_h^i\|^2+\frac{m_a}{2}\min\big(b(x)\big)\tau\sum_{i=1}^{n}\|\theta_h^i\|^2\le e^3\Big(\tau^2\sum_{i=1}^{n}\|\delta^2\rho_h^i\|^2$$

$$+\tau^2\sum_{i=1}^{n}\|\delta\rho_h^i\|^2+\frac{8M_a^2}{m_a}\tau\sum_{i=1}^{n}\|\rho_h^i\|_{\mathscr{A}}^2$$

$$+\frac{16c^2}{m_a}L_{M_a}^2\tau\sum_{i=1}^{n}\|\theta_h^i\|^2+\frac{16c^2}{m_a}L_{M_a}^2\tau\sum_{i=1}^{n}\|\rho_h^i\|^2\Big),$$

So,

$$\tau\sum_{i=1}^{n}\|\delta\theta_h^i\|^2+\tau\sum_{i=1}^{n}\|\nabla\theta_h^i\|^2+\tau\sum_{i=1}^{n}\|\theta_h^i\|^2\le\tau^2\|\delta\theta_h^0\|^2+C\Big(\tau^2\sum_{i=1}^{n}\|\delta^2\rho_h^i\|^2$$

$$+\tau^2\sum_{i=1}^{n}\|\delta\rho_h^i\|^2+\tau\sum_{i=1}^{n}\|\rho_h^i\|_{\mathscr{A}}^2+\tau\sum_{i=1}^{n}\|\rho_h^i\|^2\Big).$$

This implies

$$\|\theta_h\|_{L^2(0,T,\tau,H^1(\Omega))} \leq \tau^2 \|\delta\theta_h^0\|^2 + C\Big(\tau^2 \sum_{i=1}^{n} \|\delta^2\rho_h^i\|^2 + \tau^2 \sum_{i=1}^{n} \|\delta\rho_h^i\|^2 + \tau \sum_{i=1}^{n} \|\rho_h^i\|_{\mathscr{A}}^2 + \tau \sum_{i=1}^{n} \|\rho_h^i\|^2\Big), \tag{2.52}$$

We have

$$\begin{aligned} \|\delta\theta_h^0\|^2 &\leq \|\delta(\Pi_h^0 u^0 - u_h^0)\|^2 \\ &\leq \|\delta(\Pi_h^0 u^0 - u^0)\|^2 + \|\delta(u^0 - u_h^0)\|^2. \end{aligned}$$

If we take $u_h^0 = \Pi_h^0 u^0$*, then*

$$\begin{aligned} \tau^2\|\delta\theta_h^0\|^2 &\leq Ch^2\|u^0 - u^{-1}\|_{H^2(\Omega)}^2, \\ &\leq Ch^2\|u^0 - (u^0 - \tau u^1)\|_{H^2(\Omega)}^2, \\ &\leq Ch^2\|u^1\|_{H^2(\Omega)}^2. \end{aligned} \tag{2.53}$$

So,

$$\tau^2\|\delta\theta_h^0\|^2 \leq Ch^4\|u^1\|_{H^2(\Omega)}^2.$$

Again, we note that

$$\delta^2\rho_h^i = \frac{1}{\tau}\int_{t_{i-1}}^{t_i} \partial_s\delta\rho_h(s)ds.$$

This shows

$$\|\delta^2 \rho_h^i\|^2 \leq \frac{1}{\tau^2}\|\rho_{htt}\|^2_{L^2(t_{i-1},t_i;L^2(\Omega))},$$

and

$$\begin{aligned}
\tau^2 \sum_{i=1}^{n} \|\delta^2 \rho_h^i\|^2 &\leq \sum_{i=1}^{n} \|\rho_{htt}\|^2_{L^2(t_{i-1},t_i;L^2(\Omega))}, \\
&\leq \|\rho_{htt}\|^2_{L^2(0,T;L^2(\Omega))}, \\
&\leq Ch^4 \|u_{tt}\|^2_{L^2(0,T;H^2(\Omega))}.
\end{aligned}$$

Thus,

$$\tau^2 \sum_{i=1}^{n} \|\delta^2 \rho_h^i\|^2 \leq Ch^4 \|u_{tt}\|^2_{L^2(0,T;H^2(\Omega))}. \tag{2.54}$$

Further

$$\begin{aligned}
\|\delta \rho_h^i\| &\leq \frac{1}{\tau}\|\rho_{ht}^i\|_{L^2(t_{i-1},t_i;L^2(\Omega))}, \\
\tau^2 \sum_{i=1}^{n} \|\delta \rho_h^i\|^2 &\leq \sum_{i=1}^{n} Ch^4 \|u_t^i\|^2_{L^2(t_{i-1},t_i;H^2(\Omega))}, \\
\sum_{i=1}^{n} \|\delta \rho_h^i\|^2 &\leq Ch^4 \|u_t\|^2_{L^2(0,T;H^2(\Omega))}.
\end{aligned}$$

So,

$$\tau^2 \sum_{i=1}^{n} \|\delta \rho_h^i\|^2 \leq Ch^4 \|u_t\|^2_{L^2(0,T;H^2(\Omega))}. \tag{2.55}$$

Also

$$\begin{aligned}
\|\rho_h^i\|^2 &\leq Ch^2 \|u^i\|^2_{H^2(\Omega)}, \\
\tau \sum_{i=1}^{n} \|\rho_h^i\|^2 &\leq Ch^4 \tau \sum_{i=1}^{n} \|u^i\|^2_{H^2(\Omega)}, \\
\tau \sum_{i=1}^{n} \|\rho_h^i\|^2 &\leq Ch^4 \|u\|^2_{L^2(0,T,\tau;H^2(\Omega))}.
\end{aligned}$$

Finally

$$
\begin{aligned}
\|\rho_h^i\|_{\mathscr{A}}^2 &= \left(A\nabla\rho_h^i, \nabla\rho_h^i\right) + \left(b(x)\rho_h^i, \rho_h^i\right), \\
&\leq C\left(\|\nabla\rho_h^i\|^2 + \|\rho_h^i\|^2\right), \\
\tau\sum_{i=1}^{n}\|\rho_h^i\|_{\mathscr{A}} &\leq C\tau\sum_{i=1}^{n}\|\rho_h^i\|_{H^1(\Omega)}^2 \\
&\leq C\|\rho_h\|_{L^2(0,T,\tau;H^1(\Omega))}^2, \\
&\leq Ch^2\|u\|_{L^2(0,T,\tau;H^2(\Omega))}^2.
\end{aligned}
$$

New using the estimates $(2.53-2.55)$ *in* (2.52)*, we get*

$$\tau\sum_{i=1}^{n}\|\theta_h^i\|_{H^1(\Omega)}^2 \leq C(h^2+h^4),$$

So,

$$\|\theta_h\|_{L^2(0,T,\tau,H^1(\Omega))}^2 \leq C(h^2+h^4).$$

Where C is is constant depending on $\|u\|_{L^2(0,T,\tau;H^2(\Omega))}^2$, $\|u_{tt}\|_{L^2(0,T;H^2(\Omega))}^2$, $\|u_t\|_{L^2(0,T;H^2(\Omega))}^2$ *and* $\|u^1\|_{H^2(\Omega)}^2$.

We conclude

$$\|u^i - u_h^i\|_{L^2(0,T,\tau;H^1(\Omega))} \leq C(h+h^2). \blacksquare$$

2.6 Numerical experiment.

In this section, we set up a numerical experiment to find an approximate solution of problem (2.1)*, if we use Rothe's approximation in time discretization and finite element* *scheme in the spatial discretization in wich we prescribe the computational domain* $\Omega = (0,1)$, *the time interval* $(0,1)$ *i.e.* $T = 1$ *and we take* $A(x) = b(x) = 1$.

In order using Newton's we take initial guess u^0 *and* u^1 *as follows*

$u^0 = 0,$

and,

$$u^1 = \begin{cases} 1, & \text{at interior node} \\ \\ 0, & \text{at boundary node} \end{cases}$$

The tolerance for stopping iteration is defined to be 10^{-15}, *we have considered the step length* $h = \frac{1}{10}, \frac{1}{20}, \frac{1}{30}, \frac{1}{40}$ *and* $\tau = 0.001$. *We plot the error in loglog-plot.*

2.6.1 Exemple

We choose $f(x,t,u)$ *according to test solution* $u(x,t) = x(1-x)te^{-t^2}$ *and* $a(l(u)) = 1 + \cos(l(u))$. *The table below gives the error and the order of convergence of the solution and Fig.1 shows the results of error in loglog-plot*

h	$\Vert u^i - u_h^i \Vert_{H^1(\Omega)}$	*Order*
$\frac{1}{10}$	$9.8689e-003$	$--$
$\frac{1}{20}$	$5.6454e-003$	0.8058
$\frac{1}{30}$	$3.9796e-003$	0.8624
$\frac{1}{40}$	$3.0748e-003$	0.8966

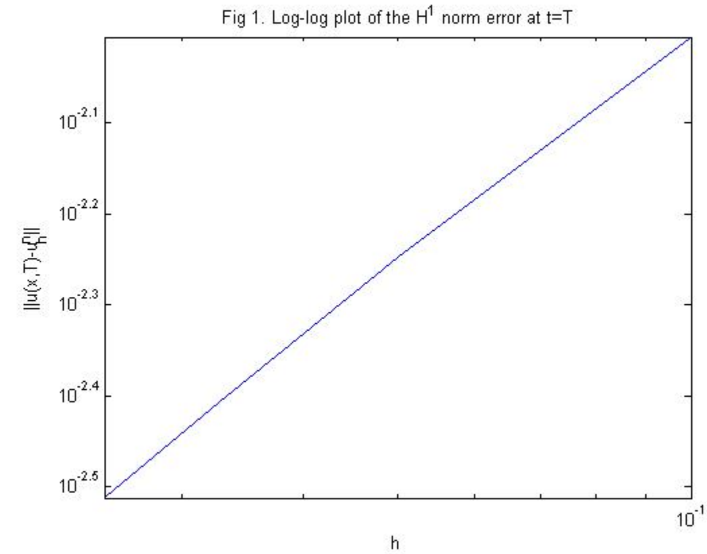

Figure 4 : The results of error in log log-plot.

3

Rothe-H^1-Galerkin Mixed finite element approximation for Nonlocal evolution equation

3.1 Introduction

In this chapter,we combine H^1-Galerkin mixed finite element method for the spatial discretization and Rothe's method to time discretization of nonlocal hyperbolic equa-

tion.

The principle of the use H^1-Galerkin mixed finite element is to introduce the auxiliary variable p that depends on u, which leads to split our equation into a system of two equations, thus eliminating the difficulties explained in [21] caused by the presence of the non-local term in the Jacobian matrix of the Newton Raphson method. In other words, this system achieves the sparsity of the Jacobian matrix and thus the calculations are easier.

The optimal a priori error estimates for both semi discrete and fully discrete schemes to functions u and p are proved in H^1 and L^2, respectively. Finally the convergence of the obtained scheme is verified by a numerical experiment.

3.2 Position of the problem

Let Ω is a simply connected bounded domain of $\mathbb{R}^k$, $(k = 1,2,3)$ with Lipschitz continuous boundary $\partial\Omega$. Consider the following non local hyperbolic problem.

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \dfrac{\partial u}{\partial t} + a(l(u))(-div(A(x)\nabla u) + b(x)u) = f(x,t) & in\ Q = \Omega \times [0,T] \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & in\ \Omega \\ u = 0 & on\ \Sigma \times [0,T]. \end{cases} \tag{3.1}$$

Where $T \geq 0$ and a is a function depends of $l(u)$ with

$$l(u) = \int_\Omega u(x,t)\,dx, \tag{3.2}$$

and satisfies

$$0 < m_0 \leq a(s) \leq M_0 < \infty, \quad \forall s \in \mathbb{R}. \tag{3.3}$$

Here $A(x)$ *is a invertible and symmetrical matrix with uniformly delimited and measurable components,* $b(x)$ *is bounded function.*

3.3 The variational formulation

Let us make the following choice of auxiliary variable

$$p = A(x)\nabla u, \tag{3.4}$$

If we set

$$\sigma(x) = A^{-1}(x).$$

Then, Problem (1.1) *can be written as the mixed system*

$$\begin{cases} \nabla u = \sigma(x) p, \\ \\ \dfrac{\partial^2 u}{\partial t^2} + \dfrac{\partial u}{\partial t} + a(l(u))(-div(p) + b(x)u) = f(x,t). \end{cases} \tag{3.5}$$

Using the fact that $u_t(0,x) = u_{tt}(1,x) = 0$*, Green formula implies*

$$(\frac{\partial^2 u}{\partial t^2}, \nabla w) = -(\sigma(x)\frac{\partial^2 p}{\partial t^2}, w), \; (\frac{\partial u}{\partial t}, \nabla w) = -(\sigma(x)\frac{\partial p}{\partial t}, w).$$

This allows us to write the weak formulation of (3.5) *as : Find a pair* $(u,p) \in H_0^1 \times H^1$ *such that*

$$\begin{cases} (\nabla u, \nabla v) - (\sigma(x) p, \nabla v) = 0, \\ \\ (\sigma(x)\dfrac{\partial^2 p}{\partial t^2}, w) + (\sigma(x)\dfrac{\partial p}{\partial t}, w) + a(l(u))[(\nabla p, \nabla w) - (b(x)u, \nabla w)] = -(f, \nabla w). \end{cases} \tag{3.6}$$

3.4 Semi-Discretization

In this section we present a semi-discrete scheme by using the H^1-Galerkin mixed finite element method and extract a priori error estimates.

Let $V_h \subset H_0^1(\Omega)$, $W_h \subset H^1(\Omega)$ be the discrete finite spaces which comprises of piecewise linear polynomials of order k and r, respectively, satisfying approximation properties (see [17])

$$\inf_{\phi_h \in V_h} \{\|\phi - \phi_h\|_{0,p} + h\|\phi - \phi_h\|_{1,p}\} \leq Ch^{k+1}\|\phi\|_{k+1,p}, \ \phi \in H_0^1(\Omega) \cap W^{k+1,p}(\Omega),$$

and

$$\inf_{\omega_h \in W_h} \{\|\omega - \omega_h\|_{0,p} + h\|\omega - \omega_h\|_{1,p}\} \leq Ch^{r+1}\|\omega\|_{r+1,p}, \ \omega \in W^{r+1,p}(\Omega).$$

The semi-discrete H^1-Galerkin mixed finite element approximation for (3.6) *reads as : find $(u_h, p_h) \in V_h \times W_h$ such that*

$$\begin{cases} (\nabla u_h, \nabla v_h) - (\sigma p_h, \nabla v_h) = 0, \\ \\ (\sigma \dfrac{\partial^2 p_h}{\partial t^2}, w_h) + (\sigma \dfrac{\partial p_h}{\partial t}, w_h) + a(l(u_h))[(\nabla p_h, \nabla w_h) - (bu_h, \nabla w_h)] = -(f, \nabla w_h). \end{cases} \tag{3.7}$$

Integrating (3.7) *over the interval* $[0, T]$, *we get*

$$\begin{cases} \int_{[0,T]} (\nabla u_h, \nabla v_h) - \int_{[0,T]} (\sigma p_h, \nabla v_h) = 0, \\ \\ -\int_{[0,T]} (\sigma \dfrac{\partial p_h}{\partial t}, \dfrac{\partial w_h}{\partial t}) - (p_1, w(.,0)) + \int_{[0,T]} (\sigma \dfrac{\partial p_h}{\partial t}, w_h) \\ \\ \qquad + \int_{[0,T]} a(l(u_h))[(\nabla p_h, \nabla w_h) - (bu_h, \nabla w_h)] = -\int_{[0,T]} (f, \nabla w_h), \end{cases} \tag{3.8}$$

where $p_1 = A\nabla u_1$.

3.4.1 A priori error estimates

In order to get an optimal convergence, we will decompose the error in the following way

$$u - u_h = u - \tilde{u}_h + -\tilde{u}_h - u_h = \theta_u + \zeta_u, \tag{3.9}$$

$$p - p_h = p - \tilde{p}_h + -\tilde{p}_h - p_h = \theta_p + \zeta_p, \tag{3.10}$$

where the elliptic projections $\tilde{u}_h \in V_h$ *and* $\tilde{p}_h \in W_h$ *satisfy*

$$(\nabla p - \nabla \tilde{p}_h, \nabla w_h) + \lambda(p - \tilde{p}_h, w_h) = 0, \ \ \forall w_h \in W_h, \tag{3.11}$$

$$(\nabla p - \nabla \tilde{p}_h, \nabla w_h) = 0, \ \ \forall v_h \in V_h, \tag{3.12}$$

and λ *is chosen so that the first equation is* H^1*-Coercive.*

Hence, we may proceed as in [56] *and obtain*

$$\sum_{i=0}^{2} \| \frac{\partial^i \theta_p}{\partial t^i} \|_j \leq C h^{r+1-j} \sum_{i=0}^{2} \| \frac{\partial^i p}{\partial t^i} \|_{r+1}, \ \ j = 0, 1, \tag{3.13}$$

$$\sum_{i=0}^{2} \| \frac{\partial^i \theta_u}{\partial t^i} \|_j \leq C h^{k+1-j} \sum_{i=0}^{2} \| \frac{\partial^i u}{\partial t^i} \|_{k+1}, \ \ j = 0, 1. \tag{3.14}$$

Theorem 3.4.1. *Assume that* $p_h(0) = \tilde{p}_h(0)$ *and* $p_{ht}(0) = \tilde{p}_{ht}(0)$. *There exists positive constant* C *such that*

$$\left\| (u - u_h)(t) \right\|_1 \leq C h^{min(k+1, r+1)}.$$

$$\left\| (p - p_h)(t) \right\| \leq C h^{min(k+1, r+1)}.$$

Proof. *From equations* (3.6), (3.7) *and by employing* (3.11), (3.12), *we get*

$$\left(\nabla\zeta_u,\nabla v_h\right)=\left(\sigma\theta_p,\nabla v_h\right)+\left(\sigma\zeta_p,\nabla v_h\right), \tag{3.15}$$

$$\begin{aligned}\left(\sigma\zeta_{p_{tt}},w_h\right)+\left(\sigma\zeta_{p_t},w_h\right)+a(l(u_h))\left(\nabla\zeta_p,\nabla w_h\right)&=-\left(\sigma\theta_{p_{tt}},w_h\right)-\left(\sigma\theta_{p_t},w_h\right)\\ &+a(l(u))\left(\lambda\theta_p,w_h\right)-[a(l(u))-a(l(u_h))]\left(\nabla\tilde{p}_h,\nabla w_h\right)+a(l(u))\left(b\theta_u,\nabla w_h\right)\\ &+[a(l(u))-a(l(u_h))]\left(b\tilde{u}_h,\nabla w_h\right)+a(l(u_h))\left(b\zeta_u,\nabla w_h\right).\end{aligned} \tag{3.16}$$

Applying the differential operator $\frac{\partial}{\partial t}$ *on* (3.15), *we find*

$$\left(\nabla\zeta_{ut},\nabla v_h\right)=\left(\sigma\theta_{pt},\nabla v_h\right)+\left(\sigma\zeta_{pt},\nabla v_h\right), \tag{3.17}$$

choosing $v_h=\zeta_u$ *in* (3.15) *and* $w_h=\zeta_{pt}$ *in* (3.16), *we arrive at*

$$\begin{aligned}\|\nabla\zeta_u\| &\le \|\sigma^{\frac{1}{2}}\theta_p\|+\|\sigma^{\frac{1}{2}}\zeta_p\|\\ &\le C\left(\|\theta_p\|+\|\zeta_p\|\right).\end{aligned} \tag{3.18}$$

$$\frac{d}{dt}\left\|\sigma^{\frac{1}{2}}\zeta_{p_t}\right\|^2+\left\|\sigma^{\frac{1}{2}}\zeta_{p_t}\right\|^2+a(l(u_h))\frac{d}{dt}\left\|\nabla\zeta_p\right\| = -\left(\sigma\theta_{p_{tt}},\zeta_{p_t}\right)-\left(\sigma\theta_{p_t},\zeta_{p_t}\right)$$

$$+a(l(u))\left(\lambda\theta_p,\zeta_{p_t}\right)-[a(l(u))-a(l(u_h))]\left(\nabla\tilde{p}_h,\nabla\zeta_{p_t}\right)+a(l(u))\left(b\theta_u,\nabla\zeta_{p_t}\right)$$

$$+[a(l(u))-a(l(u_h))]\left(b\tilde{u}_h,\nabla\zeta_{p_t}\right)+a(l(u_h))\left(b\zeta_u,\nabla\zeta_{p_t}\right), \tag{3.19}$$

Now, by setting $v_h=\zeta_{ut}$ *in* (3.17)*, we have*

$$\begin{aligned}\left\|\nabla\zeta_{ut}\right\| &\leq \left\|\sigma^{\frac{1}{2}}\theta_{pt}\right\|+\left\|\sigma^{\frac{1}{2}}\zeta_{pt}\right\|,\\ &\leq C\left(\left\|\theta_{pt}\right\|+\left\|\zeta_{pt}\right\|\right).\end{aligned} \tag{3.20}$$

Integration in time in inequality (3.19) *yields*

$$\|\sigma^{\frac{1}{2}}\zeta_{p_t}\|^2+\int_0^t \|\sigma^{\frac{1}{2}}\zeta_{p_t}\|^2 ds+\int_0^t a(l(u_h))\frac{d}{dt}\|\nabla\zeta_p\| ds=-\int_0^t \big(\sigma\theta_{p_{tt}},\zeta_{p_t}\big)ds-\int_0^t \big(\sigma\theta_{p_t},\zeta_{p_t}\big)ds$$
$$+\int_0^t a(l(u))\big(\lambda\theta_p,\zeta_{p_t}\big)ds-\int_0^t [a(l(u))-a(l(u_h))]\big(\nabla\tilde{p}_h,\nabla\zeta_{p_t}\big)ds$$
$$+\int_0^t a(l(u))\big(b\theta_u,\nabla\zeta_{p_t}\big)ds+\int_0^t [a(l(u))-a(l(u_h))]\big(b\tilde{u}_h,\nabla\zeta_{p_t}\big)ds$$
$$+\int_0^t a(l(u_h))\big(b\zeta_u,\nabla\zeta_{p_t}\big)ds, \tag{3.21}$$

thus,

$$\|\sigma^{\frac{1}{2}}\zeta_{p_t}\|^2+\int_0^t \|\sigma^{\frac{1}{2}}\zeta_{p_t}\|^2 ds+\int_0^t a(l(u_h))\frac{d}{dt}\|\nabla\zeta_p\| ds\leq C\Big(\int_0^t \|\theta_{p_{tt}}\|^2 ds+\int_0^t \|\zeta_{p_t}\|^2 ds$$
$$+\int_0^t \|\theta_{p_t}\|^2 ds+\int_0^t \|\theta_p\|^2 ds\Big)-\int_0^t [a(l(u))-a(l(u_h))]\big(\nabla\tilde{p}_h,\nabla\zeta_{p_t}\big)ds$$
$$+\int_0^t a(l(u))\big(b\theta_u,\nabla\zeta_{p_t}\big)ds+\int_0^t [a(l(u))-a(l(u_h))]\big(b\tilde{u}_h,\nabla\zeta_{p_t}\big)$$
$$+\int_0^t a(l(u_h))\big(b\zeta_u,\nabla\zeta_{p_t}\big)ds, \tag{3.22}$$

on the other hand, we see

$$
\begin{aligned}
\int_0^t [a(l(u)) - a(l(u_h))]\big(\nabla \tilde{p}_h, \nabla \zeta_{p_t}\big) ds &= \int_0^t \big([a(l(u)) - a(l(u_h))]\nabla \tilde{p}_h, \nabla \zeta_{p_t}\big) ds \\
&= \int_0^t \frac{d}{dt}\Big([a(l(u)) - a(l(u_h))]\nabla \tilde{p}_h, \nabla \zeta_p\Big) ds - \int_0^t \Big(\frac{d}{dt}\big([a(l(u)) - a(l(u_h))]\nabla \tilde{p}_h\big), \nabla \zeta_p\Big) ds \\
&= \Big([a(l(u)) - a(l(u_h))]\nabla \tilde{p}_h, \nabla \zeta_p\Big) - \int_0^t \Big(\frac{d}{dt}\big([a(l(u)) - a(l(u_h))]\nabla \tilde{p}_h\big).\nabla \zeta_p\Big) ds,
\end{aligned}
\tag{3.23}
$$

Thanks to Cauchy-Schwarz and Poincarè inequalities we conclude

$$
\begin{aligned}
\Big([a(l(u)) - a(l(u_h))]\nabla \tilde{p}_h, \nabla \zeta_p\Big) &\le C\|u - u_h\|\,\|\nabla \zeta_p\| \\
&\le \epsilon\|\nabla \zeta_p\|^2 + C\|\theta_u + \zeta_u\|^2 \\
&\le \epsilon\|\nabla \zeta_p\|^2 + C\big(\|\theta_u\|^2 + \|\nabla \zeta_u\|^2\big) \\
&\le \epsilon\|\nabla \zeta_p\|^2 + C\big(\|\theta_u\|^2 + \|\zeta_p\|^2 + \|\theta_p\|^2\big) \\
&\le \epsilon\|\nabla \zeta_p\|^2 + C\big(\|\theta_u\|^2 + \int_0^t \|\zeta_{p_t}\|^2 ds + \|\theta_p\|^2\big).
\end{aligned}
\tag{3.24}
$$

Now making use of the estimates (3.18), (3.20) *we have*

$$\int_0^t \Big(\frac{d}{dt}\big([a(l(u)) - a(l(u_h))]\nabla \tilde{p}_h\big), \nabla\zeta_p\Big)ds = \int_0^t \Big(\big([a(l(u)) - a(l(u_h))]\nabla \tilde{p}_{h_t}\big), \nabla\zeta_p\Big)ds$$

$$+\int_0^t \Big(\frac{d}{dt}\big([a(l(u)) - a(l(u_h))]\big)\nabla \tilde{p}_h, \nabla\zeta_p\Big)ds$$

$$\leq \int_0^t \|u - u_h\| \|\nabla \tilde{p}_{h_t}\| \|\nabla\zeta_p\| ds + \int_0^t \|u_t - u_{ht}\| \|\nabla \tilde{p}_h\| \|\nabla\zeta_p\| ds$$

$$\leq C\Big(\int_0^t (\|\theta_u\|^2 + \|\zeta_u\|^2 + \|\nabla\zeta_p\|^2)ds + \int_0^t (\|\theta_{u_t}\|^2 + \|\zeta_{u_t}\|^2 + \|\nabla\zeta_p\|^2)ds\Big)$$

$$\leq C\int_0^t \Big(\|\theta_u\|^2 + \|\zeta_u\|^2 + \|\theta_{u_t}\|^2 + \|\zeta_{u_t}\|^2 + \|\nabla\zeta_p\|^2\Big)ds$$

$$\leq C\int_0^t \Big(\|\theta_u\|^2 + \|\nabla\zeta_u\|^2 + \|\theta_{u_t}\|^2 + \|\nabla\zeta_{u_t}\|^2 + \|\nabla\zeta_p\|^2\Big)ds$$

$$\leq C\int_0^t \Big(\|\theta_u\|^2 + \|\zeta_p\|^2 + \|\theta_p\|^2 + \|\theta_{u_t}\|^2 + \|\zeta_{p_t}\|^2 + \|\theta_{p_t}\|^2 + \|\nabla\zeta_p\|^2\Big)ds$$

$$\leq C\int_0^t \Big(\|\theta_u\|^2 + \|\theta_p\|^2 + \|\theta_{u_t}\|^2 + \|\zeta_{p_t}\|^2 + \|\theta_{p_t}\|^2 + \|\nabla\zeta_p\|^2\Big)ds. \tag{3.25}$$

Substituting (3.24) *and* (3.25) *into* (3.23) *we see that*

$$\int_0^t [a(l(u)) - a(l(u_h))]\big(\nabla \tilde{p}_h, \nabla \zeta_{p_t}\big) ds \le \epsilon \big\|\nabla \zeta_p\big\|^2 + C\Big(\big\|\theta_u\big\|^2 + \big\|\theta_p\big\|^2 + \int_0^t \big\|\theta_u\big\|^2 ds$$

$$+ \int_0^t \big\|\theta_p\big\|^2 ds + \int_0^t \big\|\theta_{u_t}\big\|^2 ds + \int_0^t \big\|\zeta_{p_t}\big\|^2 ds + \int_0^t \big\|\theta_{p_t}\big\|^2 ds + \int_0^t \big\|\nabla \zeta_p\big\|^2 ds\Big). \tag{3.26}$$

By the same arguments we get

$$\int_0^t [a(l(u)) - a(l(u_h))]\big(b \tilde{u}_h, \nabla \zeta_{p_t}\big) ds \le \epsilon \big\|\nabla \zeta_p\big\|^2 + C\Big(\big\|\theta_u\big\|^2 + \big\|\theta_p\big\|^2 + \int_0^t \big\|\theta_u\big\|^2 ds$$

$$+ \int_0^t \big\|\theta_p\big\|^2 ds + \int_0^t \big\|\theta_{u_t}\big\|^2 ds + \int_0^t \big\|\zeta_{p_t}\big\|^2 ds + \int_0^t \big\|\theta_{p_t}\big\|^2 ds + \int_0^t \big\|\nabla \zeta_p\big\|^2 ds\Big). \tag{3.27}$$

Our next target is to estimate $\int_0^t (a(l(u))\big(b\theta_u, \nabla \zeta_{pt}\big) ds$. *For this purpose, we have*

$$\int_0^t (a(l(u))\big(b\theta_u, \nabla \zeta_{p_t}\big) ds = \int_0^t \big(b\theta_u a(l(u)), \nabla \zeta_{p_t}\big) ds$$

$$= \frac{1}{2}\int_0^t \frac{d}{dt}\big(b\theta_u a(l(u)), \nabla \zeta_p\big) ds - \frac{1}{2}\int_0^t \big(\frac{d}{dt}[b\theta_u] a(l(u)), \nabla \zeta_p\big) ds$$

$$= \frac{1}{2} a(l(u))\big(b\theta_u, \nabla \zeta_p\big) - \frac{1}{2}\int_0^t a(l(u))\big(b\theta_{u_t}, \nabla \zeta_p\big) ds, \tag{3.28}$$

It follows from

$$(a(l(u))\big(b\theta_u, \nabla \zeta_p\big) \le C\big\|\theta_u\big\|^2 + \epsilon \big\|\nabla \zeta_p\big\|^2, \tag{3.29}$$

that

$$\int_0^t a(l(u))\big(b\theta_{ut}, \nabla\zeta_p\big)ds \le C\Big(\int_0^t \|\theta_{ut}\|^2 ds + \int_0^t \|\nabla\zeta_p\|^2 ds\Big), \tag{3.30}$$

This allows us to write

$$\int_0^t (a(l(u))\big(b\theta_u, \nabla\zeta_{pt}\big)ds \le \epsilon\|\nabla\zeta_p\|^2 + C\Big(\|\theta_u\|^2 + \int_0^t \|\theta_{ut}\|^2 ds + \int_0^t \|\nabla\zeta_p\|^2 ds\Big). \tag{3.31}$$

Analogously we estimate

$$\begin{aligned}\int_0^t (a(l(u_h))\big(b\zeta_u, \nabla\zeta_{p_t}\big)ds &\le \epsilon\|\nabla\zeta_p\|^2 + C\Big(\|\zeta_u\|^2 + \int_0^t \|\zeta_{u_t}\|^2 ds + \int_0^t \|\nabla\zeta_p\|^2 ds\Big)\\ &\le \epsilon\|\nabla\zeta_p\|^2 + C\Big(\|\nabla\zeta_u\|^2 + \int_0^t \|\nabla\zeta_{u_t}\|^2 ds + \int_0^t \|\nabla\zeta_p\|^2 ds\Big)\\ &\le \epsilon\|\nabla\zeta_p\|^2 + C\Big(\|\theta_p\|^2 + \|\zeta_p\|^2 + \int_0^t \|\theta_{p_t}\|^2 ds + \int_0^t \|\zeta_{p_t}\|^2 ds + \int_0^t \|\nabla\zeta_p\|^2 ds\Big).\end{aligned} \tag{3.32}$$

Combining (3.26) − (3.27), (3.31) − (3.32) *and* (3.22), *we come to the inequality*

$$\|\zeta_{p_t}\|^2+\int_0^t\|\zeta_{p_t}\|^2ds+m\|\nabla\zeta_p\|^2\leq\epsilon\int_0^t\|\nabla\zeta_p\|^2ds+C\Big(\|\theta_p\|^2+\|\theta_u\|^2+\int_0^t\|\theta_p\|^2ds$$
$$+\int_0^t\|\theta_u\|^2ds+\int_0^t\|\zeta_{p_t}\|^2ds+\int_0^t\|\theta_{p_t}\|^2ds$$
$$+\int_0^t\|\theta_{u_t}\|^2ds+\int_0^t\|\nabla\zeta_p\|^2ds+\int_0^t\|\theta_{p_{tt}}\|^2ds\Big),\tag{3.33}$$

According to Gronwall's lemma with a suitable ϵ, we obtain

$$\|\zeta_{p_t}\|^2+\int_0^t\|\zeta_{p_t}\|^2ds+m\|\nabla\zeta_p\|^2\leq C\Big(\|\theta_p\|^2+\|\theta_u\|^2+\int_0^t\|\theta_p\|^2ds+\int_0^t\|\theta_u\|^2ds$$
$$+\int_0^t\|\theta_{p_t}\|^2ds+\int_0^t\|\theta_{u_t}\|^2ds+\int_0^t\|\theta_{p_{tt}}\|^2ds\Big),\tag{3.34}$$

Thanks to (3.13) *and* (3.14)*, we have*

$$\|\zeta_{p_t}\|^2+\int_0^t\|\zeta_{p_t}\|^2ds+m\|\nabla\zeta_p\|^2\leq C\Big(h^{r+1}+h^{k+1}\Big).\tag{3.35}$$

Now, it follows from

$$\|\zeta_p\|\leq\int_0^t\|\zeta_{p_t}\|ds,$$

that

$$\begin{aligned}\|\zeta_u\|^2 &\leq \|\zeta_u\|_1\\ &\leq C\|\nabla\zeta_u\|^2\\ &\leq C\Big(\|\theta_p\|^2+\|\zeta_p\|^2\Big)\\ &\leq C\Big(\|\theta_p\|^2+\int_0^t\|\zeta_{pt}\|ds\Big)\\ &\leq C\Big(h^{k+1}+h^{r+1}\Big),\end{aligned} \tag{3.36}$$

where C *is constant depending on* $\|p\|_{L^\infty(0,T;H^{r+1})}$, $\|u\|_{L^\infty(0,T;H^{k+1})}$, $\|p\|_{L^2(0,T;H^{r+1})}$, $\|p_t\|_{L^2(0,T;H^{r+1})}$, $\|p_{tt}\|_{L^2(0,T;H^{r+1})}$, $\|u\|_{L^2(0,T;H^{k+1})}$ *and* $\|u_t\|_{L^2(0,T;H^{k+1})}$. *This achieves the proof.*■

3.5 Full Discretization

In this section, we focus on time discretization by using Rothe's method and prove the existence and uniqueness of a fully discrete scheme solution with a priori estimates and error estimates.

We subdivide the interval $[0,T]$ *into* n *subintervals of length* $\tau=\frac{T}{n}$ *and denote by* p_h^i, u_h^i *the values of* p_h, u_h *respectively at* $t=i\tau$, *for* $i=0,...,n$. *Let* p_h^{-1} *be defined as* $p_h^{-1}(x)=p_h^0(x)-\tau p_h^1(x)$, *the recurrent approximation scheme for* $i=1,...,n$ *becomes*

$$\begin{cases} \textit{Find } u_h^i \cong u(.,t_i) \in V_h \ \textit{ and } p_h^i \cong p(.,t_i) \in V_h \ i = 1,2,...,n \textit{ such that,} \\ \\ (\nabla u_h^i, \nabla v_h) - (\sigma p_h^i, \nabla v_h) = 0, \ \forall (v_h, w_h) \in V_h \times W_h, \\ \\ \left(\sigma\delta^2 p_h^i, w_h\right) + \left(\sigma\delta p_h^i, w_h\right) + a(l(u_h^i))[(\nabla p_h^i, \nabla w_h) - (bu_h^i, \nabla w_h)] \\ \\ \qquad\qquad\qquad = -\left(f^i, \nabla w_h\right), \ \forall (v_h, w_h) \in V_h \times W_h, \end{cases} \tag{3.37}$$

here $\delta p_h^i = \dfrac{p_h^i - p_h^{i-1}}{\tau}$ *and* $\delta^2 p_h^i = \dfrac{\delta p_h^i - \delta p_h^{i-1}}{\tau}$.

Let us introduce the Rothe's functions by a piecewise linear interpolation with respect to the time t*, defined as follows*

$$p_h^n = p_h^{i-1} + (t - t_{i-1})\delta p_h^i \quad \forall t \in [t_{i-1}, t_i], \ 1 \le i \le n \tag{3.38}$$

$$\delta p_h^n = \delta p_h^{i-1} + (t - t_{i-1})\delta^2 p_h^i \quad \forall t \in [t_{i-1}, t_i], \ 1 \le i \le n, \tag{3.39}$$

together with the auxiliary functions

$$\bar{u}_h^n = \begin{cases} u_h^i & t \in [t_{i-1}, t_i], \\ \\ u_h^0 & t \in [-\tau, 0], \end{cases} \tag{3.40}$$

and

$$\bar{p}_h^n = \begin{cases} p_h^i & t \in [t_{i-1}, t_i], \\ \\ p_h^0 & t \in [-\tau, 0]. \end{cases} \tag{3.41}$$

We denote by $\bar{f}^n$ *the function*

$$\bar{f}^n = \begin{cases} f^i & t \in [t_{i-1}, t_i], \quad . \\ \\ 0 & t = 0. \end{cases} \tag{3.42}$$

Then, the problem (3.37) *can be rewritten as follows :*

$$\begin{cases} \forall (v_h, w_h) \in V_h \times W_h, \ \textit{with } v(x,T) = 0 \textit{ and } w(x,T) = 0. \\ \\ (\nabla \bar{u}_h^n, \nabla v_h) - (\sigma \bar{p}_h^n, \nabla v_h) = 0, \ \forall (v_h, w_h) \in V_h \times W_h, \\ \\ \big(\sigma \partial_t \delta p_h^n, w_h\big) + \big(\sigma \partial_t p_h^n, w_h\big) + a(l(\bar{u}_h^n))[(\nabla \bar{p}_h^n, \nabla w_h) - (b\bar{u}_h^n, \nabla w_h)] \\ \\ \qquad\qquad = -\big(\bar{f}^n, \nabla w_h\big), \ \forall (v_h, w_h) \in V_h \times W_h, \end{cases} \tag{3.43}$$

we integrate the above equations over $[0, T]$*, we get*

$$\begin{cases} \forall (v_h, w_h) \in V_h \times W_h, \ \textit{with } v(x,T) = 0 \textit{ and } w(x,T) = 0. \\ \\ \int_{[0,T]} (\nabla \bar{u}_h^n, \nabla v_h) - \int_{[0,T]} (\sigma \bar{p}_h^n, \nabla v_h) = 0, \\ \\ -\int_{[0,T]} \big(\sigma \delta p_h^n, \partial_t w_h\big) - \big(\delta p_h^n(0), w_h(.,0)\big) + \int_{[0,T]} \big(\sigma \partial_t p_h^n, w_h\big) \\ \\ \qquad + \int_{[0,T]} a(l(\bar{u}_h^n))[(\nabla \bar{p}_h^n, \nabla w_h) - (b\bar{u}_h^n, \nabla w_h)] = -\int_{[0,T]} \big(\bar{f}^n, \nabla w_h\big), \end{cases} \tag{3.44}$$

3.5.1 Existence and uniqueness

Proposition 3.5.1. *The problem* (3.37) *admits a unique solution* (p_h^i, u_h^i), *for* $1 \le i \le n$.

Proof. *We can write the problem* (3.37) *as*

$$\begin{cases} \text{Find } u_h^i \cong u(.,t_i) \in V_h \quad \text{and } p_h^i \cong p(.,t_i) \in W_h \; i = 1,2,...,n \text{ such that,} \\ (\nabla u_h^i, \nabla v_h) - (\sigma p_h^i, \nabla v_h) = 0, \;\; \forall (v_h, w_h) \in V_h \times W_h, \\ (\frac{1}{2} + \frac{1}{\tau})\big(\sigma p_h^i, w_h\big) + a(l(u_h^i))[(\nabla p_h^i, \nabla w_h) - (b u_h^i, \nabla w_h)] = G(w_h), \\ \qquad\qquad\qquad \forall (v_h, w_h) \in V_h \times W_h, \end{cases} \tag{3.45}$$

where

$$G(w_h) = -\big(f^i, \nabla w_h\big) + (\frac{2}{\tau^2} + \frac{1}{\tau})\big(\sigma p_h^{i-1}, w_h\big) - \frac{1}{\tau^2}\big(\sigma p_h^{i-2}, w_h\big) \tag{3.46}$$

Let us introduce $\Psi : V_h \times W_h \to V_h' \times W_h'$ *such that*

$$\Psi(u_h, p_h) := \Big(w_h \to (\frac{1}{\tau^2} + \frac{1}{\tau})\big(\sigma p_h, w_h\big) + a(l(u_h))[(\nabla p_h, \nabla w_h) - (b u_h, \nabla w_h)], \\ v_h \to (\nabla u_h, \nabla v_h) - (\sigma p_h, \nabla v_h). \Big)$$

Then, for $(u_h, p_h) \in V_h \times W_h$, *we have*

$$(\frac{1}{\tau^2} + \frac{1}{\tau})\big(\sigma p_h, w_h\big) + a(l(u_h))[(\nabla p_h, \nabla w_h) - (b u_h, \nabla w_h)] = 0, \tag{3.47}$$

$$(\nabla u_h, \nabla v_h) - (\sigma p_h, \nabla v_h) = 0. \tag{3.48}$$

It's easy to verify that for $w_h = p_h$ in (3.47) *and $v_h = u_h$ in* (3.48) *we get*

$$(\frac{1}{\tau^2} + \frac{1}{\tau})\|\sigma^{\frac{1}{2}} p_h\|^2 + m_0\|\nabla p_h\|^2 \leq C\|u_h\|\|\nabla p_h\|, \tag{3.49}$$

$$\|\nabla u_h\| \leq C\|p_h\|. \tag{3.50}$$

Now, Poincarré inequality shows that

$$(\frac{1}{\tau^2} + \frac{1}{\tau})\|p_h\|^2 + m_0\|\nabla p_h\|^2 \leq (C\|\nabla p_h\|)\|p_h\|. \tag{3.51}$$

Hence, the inequality $ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$ for $\epsilon = \frac{\tau^2}{2}$, allows us to say that

$$\Big(\frac{1}{\tau^2} + \frac{1}{\tau}\Big)\|p_h\|^2 + m\|\nabla p_h\|^2 \leq C\tau^2\|\nabla p_h\|^2 + \frac{1}{\tau^2}\|p_h\|^2. \tag{3.52}$$

This leads us to the estimate

$$0 \leq \frac{1}{\tau}\|p_h\|^2 + \|\nabla p_h\|^2 \leq 0. \tag{3.53}$$

Therefore $u_h = p_h = 0$. Consequently, Ψ_h is injective, which means it is bijective.

This allows us to solve (3.37) *in sequence for $i = 1, ..., n$.* ■

3.5.2 A priori estimates

Lemma 3.5.2. *For $i = 1, \dots, s$, $1 \le s \le n$, the following estimates hold*

$$\|\nabla u_h^i\| \le C. \tag{3.54}$$

$$\|\delta p_h^s\|^2 + \sum_{i=1}^{s} \|\delta p_h^i - \delta p_h^{i-1}\|^2 + \sum_{i=1}^{s} \tau \|\delta p_h^i\|^2 + \|\nabla p_h^s\|^2 + \sum_{i=1}^{s} \|\nabla p_h^i - \nabla p_h^{i-1}\|^2 \le C. \tag{3.55}$$

Proof. *Set $v_h = u_h^i$ and $w_h = \tau \delta p_h^i$ in* (3.37)*. Then*

$$\begin{cases} \|\nabla u_h^i\|^2 = (\sigma p_h^i, \nabla u_h^i), \\ \big(\sigma(\delta p_h^i - \delta p_h^{i-1}), \delta p_h^i\big) + \tau \|\sigma^{\frac{1}{2}} \delta p_h^i\|^2 + a(l(u_h^i))(\nabla p_h^i, \nabla p_h^i - \nabla p_h^{i-1}) = \\ \qquad a(l(u_h^i))(b u_h^i, \nabla p_h^i - \nabla p_h^{i-1}) - \big(f^i, \nabla p_h^i - \nabla p_h^{i-1}\big), \end{cases} \tag{3.56}$$

by using the Abel's summing formula for $(3.56)_b$*, we get*

$$\begin{cases} \|\nabla u_h^i\| \le C \|p_h^i\|, \\ \|\sigma^{\frac{1}{2}} \delta p_h^i\|^2 - \|\sigma^{\frac{1}{2}} \delta p_h^{i-1}\|^2 + \|\sigma^{\frac{1}{2}} (\delta p_h^i - \delta p_h^{i-1})\|^2 + \tau \|\sigma^{\frac{1}{2}} \delta p_h^i\|^2 \\ \qquad + m_0 [\|\nabla p_h^i\|^2 - \|\nabla p_h^{i-1}\|^2 + \|\nabla p_h^i - \nabla p_h^{i-1}\|^2] \le \\ \qquad M_0 \|b u_h^i\| \|\nabla p_h^i - \nabla p_h^{i-1}\| + \|f^i\| \|\nabla p_h^i - \nabla p_h^{i-1}\|, \end{cases} \tag{3.57}$$

summing from $i=1$ to s for $(3.57)_b$, we obtain

$$\|\sigma^{\frac{1}{2}}\delta p_h^s\|^2+\sum_{i=1}^{s}\|\sigma^{\frac{1}{2}}(\delta p_h^i-\delta p_h^{i-1})\|^2+\sum_{i=1}^{s}\tau\|\sigma^{\frac{1}{2}}\delta p_h^i\|^2+m_0\sum_{i=1}^{s}\|\nabla p_h^i\|^2+m_0\sum_{i=1}^{s}\|\nabla p_h^i-\nabla p_h^{i-1}\|\le$$

$$+\|\sigma^{\frac{1}{2}}\delta p_h^0\|^2-\|\nabla p_h^0\|^2+M_0\sum_{i=1}^{s}\|bu_h^i\|\|\nabla p_h^i-\nabla p_h^{i-1}\|+\sum_{i=1}^{s}\|f^i\|\|\nabla p_h^i-\nabla p_h^{i-1}\|.$$

Now making use of Poincarè inequality and $\epsilon-$Young inequality with particular choice $\epsilon=\frac{m_0}{2}$, we have

$$\|\sigma^{\frac{1}{2}}\delta p_h^s\|^2+\sum_{i=1}^{s}\|\sigma^{\frac{1}{2}}(\delta p_h^i-\delta p_h^{i-1})\|^2+\sum_{i=1}^{s}\tau\|\sigma^{\frac{1}{2}}\delta p_h^i\|^2+m_0\sum_{i=1}^{s}\|\nabla p_h^i\|^2$$

$$+\frac{m_0}{2}\sum_{i=1}^{s}\|\nabla p_h^i-\nabla p_h^{i-1}\|\le C.$$

Therefore

$$\|\delta p_h^s\|^2+\sum_{i=1}^{s}\|(\delta p_h^i-\delta p_h^{i-1})\|^2+\sum_{i=1}^{s}\tau\|\delta p_h^i\|^2+m_0\sum_{i=1}^{s}\|\nabla p_h^i\|^2+\frac{m_0}{2}\sum_{i=1}^{s}\|\nabla p_h^i-\nabla p_h^{i-1}\|\le C.$$

By means of the Poincarè inequality for $(3.57)_a$ once more, we deduce that

$\|\nabla u_h^i\|^2\le C$. *This achieves the proof.* ■

Corollary 3.5.3. *Suppose that $\bar{p}_{h\tau}^n=\bar{p}_h^n(.,t-\tau)$. There exists a positive constant C such*

that

$$\|\bar{u}_h^n\|_{L^2([0,T],H_0^1(\Omega))} \leq C,\ \|\bar{p}_h^n\|_{L^2([0,T],H_0^1(\Omega))} \leq C,\ \|\partial_t p_h^n\|_{L^2([0,T],L^2(\Omega))} \leq C, \tag{3.58}$$

$$\|p_h^n - \bar{p}_h^n\|_{L^2([0,T];H_0^1(\Omega))} \leq C\tau,\ \|p_h^n - \bar{p}_h^n\|_{L^2([0,T];L^2(\Omega))} \leq C\tau^2, \tag{3.59}$$

$$\|p_h^n - \bar{p}_{h\tau}^n\|_{L^2([0,T];L^2(\Omega))} \leq C\tau^2,\ \|\delta p_h^n - \partial_t p_h^n\|_{L^2([0,T];L^2(\Omega))} \leq C\tau. \tag{3.60}$$

3.5.3 A priori error estimates

We denote by $e_p = p_h - p_h^n$, $e_u = u_h - u_h^n$ *and* $e_f = f - \bar{f}^n$.

Theorem 3.5.4. *There exists a constant C such that*

$$\|e_p\|^2 + m_0 \int_{[0,T]} \|\nabla e_p\|^2 + \int_{[0,T]} \|\nabla e_u\|^2 \leq C\big(\tau + \tau^2\big). \tag{3.61}$$

Proof. *Subtracting* (3.8) *from* (3.44), *setting* $v_h = e_p$ *and* $w_h = e_u$ *and applying* ϵ-*Young inequality with* $\epsilon = \frac{m_0}{4}$, *we get*

$$\begin{cases} \int_{[0,T]} \|\nabla e_u\|^2 \leq C \int_{[0,T]} \|e_p\|^2, \\ \\ \frac{1}{2}\|e_p\|^2 + \frac{m_0}{2}\int_{[0,T]} \|\nabla e_p\|^2 \leq C\big(\int_{[0,T]} \|p_h^n - \bar{p}_h^n\|^2 + \int_{[0,T]} \|e_u\|^2 \\ \\ \qquad\qquad + \int_{[0,T]} \|u_h^n - \bar{u}_h^n\|^2 + \int_{[0,T]} \|e_f\|^2 + \tau\big), \end{cases} \tag{3.62}$$

Finally, using Gronwall's lemma, Poincarè inequality and corollary 3.1*, we come at*

$$\begin{cases} \int_{[0,T]} \|\nabla e_u\|^2 \leq C \int_{[0,T]} \|e_p\|^2, \\ \\ \|e_p\|^2 + m_0 \int_{[0,T]} \|\nabla e_p\|^2 \leq C\big(\tau + \tau^2\big). \ \blacksquare \end{cases} \tag{3.63}$$

3.6 Numerical experiment.

In this section we present a numerical experiment that demonstrates the accuracy and efficiency of our theoretical results. we choose $r = k = 1$ *means that the functions u and p are approximated by piecewise linear polynomials.*

The nonlinear system of equations obtained are solved using Newton–Raphson method. To do this we give initial guess p^0*,* p^1*,* u^0 *and* u^1*. For the test example we take the computation domain* $\Omega = (0,1)$ *and the time interval* $(0,1)$ *i.e.* $T = 1$ *and also* $A(x) = b(x) = 1$.

The step length $h \in \{\frac{1}{10}, \frac{1}{20}, \frac{1}{30}, \frac{1}{40}, \frac{1}{50}\}$ *and* $\tau = 2^{-5}$*. We plot the error in loglog-plot.*

We choose $f(x,t,u)$ *according to test solution* $u(x,t) = \frac{1}{\pi} sin(\pi x) t e^{-t^2}$ *and* $a(l(u)) = 1 + \cos(l(u))$. *The table* 1 *and the table* 2 *below gives the numerical errors for u and p, respectively.*

$$\|u - u_h^i\|_{H^1(\Omega)}$$

$h \diagdown i$	2	2^2	2^3	2^4	2^5
$\frac{1}{10}$	$4.5446e-004$	$1.6166e-003$	$5.6110e-003$	$1.5106e-002$	$1.3645e-002$
$\frac{1}{20}$	$3.3278e-004$	$1.0986e-003$	$3.6608e-003$	$9.6777e-003$	$8.8336e-003$
$\frac{1}{30}$	$2.9523e-004$	$9.1652e-004$	$2.9471e-003$	$7.6638e-003$	$7.0620e-003$
$\frac{1}{40}$	$2.8246e-004$	$8.3389e-004$	$2.6005e-003$	$6.6648e-003$	$6.1934e-003$
$\frac{1}{50}$	$2.8032e-004$	$7.9394e-004$	$2.4116e-003$	$6.1023e-003$	$5.7128e-003$

Table 2

$$\|p - p_h^i\|_{L^2(\Omega)}$$

$h \diagdown i$	2	2^2	2^3	2^4	2^5
$\frac{1}{10}$	$4.1725e-004$	$1.3581e-003$	$4.7115e-003$	$1.2813e-002$	$1.1451e-002$
$\frac{1}{20}$	$1.6810e-004$	$6.7223e-004$	$2.4776e-003$	$6.8469e-003$	$6.0929e-003$
$\frac{1}{30}$	$1.1418e-004$	$4.5866e-004$	$1.6942e-003$	$4.6862e-003$	$4.1679e-003$
$\frac{1}{40}$	$8.6608e-005$	$3.4809e-004$	$1.2861e-003$	$3.5577e-003$	$3.1640e-003$
$\frac{1}{50}$	$6.9765e-005$	$2.8034e-004$	$1.0357e-003$	$2.8648e-003$	$2.5479e-003$

Table 3

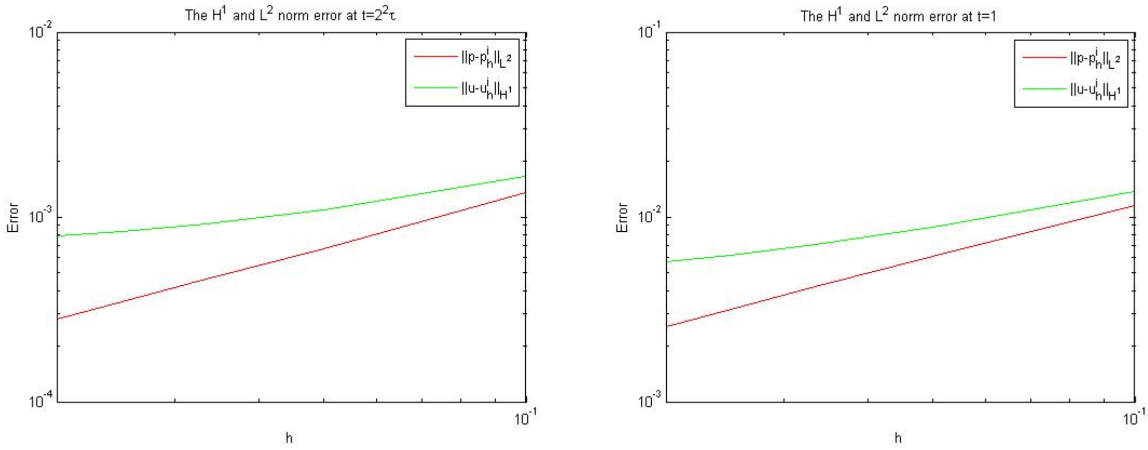

Figure 5 *: The results of error for u and p in log log-plot.*

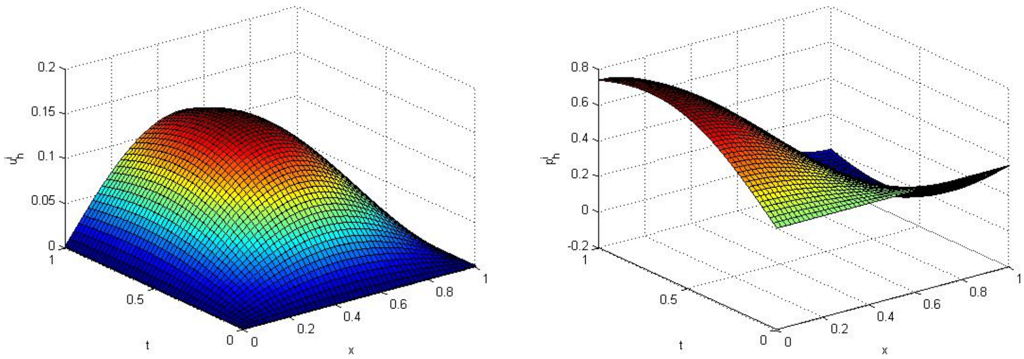

Figure 6 *: The surface for* u_h^i *and* p_h^i *on* $[0,1] \times [0,1]$.

Conclusion and perspectives

In this thesis, we proposed two works to study a problem that includes a differential equation of a hyperbolic type accompanied by a nonlocal term in addition to initial conditions and boundary conditions.

In the first work we combined the Rothe method with the finite element method to analyze and study this problem. Here, we get a fully discretization scheme that leads us to a system of non-linear equations, but we face difficulties in searching for a solution to this system caused by the presence of non-local term in the equation, so it is necessary to develop a numerical plan to get rid of these difficulties.

For the second work, we suggested the H^1-Galerkin mixed finite element method for the spatial discretization and Rothe's method to time discretization to solve the same problem. The H^1-Galerkin mixed finite element method gives us a system of two equations, thus eliminating the difficulties described in the first work. More clearly, the Jacobian matrix of the Newton Raphson method for this system is not full, therefor ensure the sparsity of the Jacobian matrix and facilitate its calculation.

In both works, a priori estimates and error estimates are given for both semi discrete and fully discrete schemes. We also complete the two works with a numerical experiment that proves our theoretical results.

Among the points of view that we may address is an interest in a posteriori analysis of the error in the study of partial differential equations. It is also possible to study and analyze PDEs of high degrees.

Bibliography

[1] G. Allaire, Analyse numérique et optimisation : Une introduction à la modélisation mathématique et simulation numérique. Editions Ecole Polytechnique, 2005.

[2] D. Bahuguna, A.K. Pani, and V. Raghavendra, Rothe´s method to semilinear hyperbolic integrodifferential equations, Appl. Math. Stochastic Anal. 3 :4, 245-252, (1990),

[3] D. Bahuguna and V. Raghavendra, Application of Rothe´s method to nonlinear Schrödinger type equations, Appl. Anal. 31 :1 (1988), 149-160.

[4] É. Bécache, P. Ciarlet, C. Hazard and É. Lunéville, La méthode des éléments finis : De la théorie à la pratique II. Compléments. Paris (2010).

[5] S. Boulaaras, M. Haiour, L^{∞}-asymptotic behavior for a finite element approximation in parabolic quasi-variational inequalities related to impulse control problem. Applied Mathematics and Computation. 217, 6443-6450, (2011).

[6] A. Bouziani, and N. Merazga, Rothe time discretisation methode applied to a quasilinear wave equation subject to integral conditions, Adv. Difference Equ. 3 (2004) 211-235.

[7] H. Brezis, Analyse fonctionnelle : théorie et applications. Masson, Paris (1983).

[8] A. C. Briozzo, M. F. Natale and D. A. Tarzia, Explicit solutions for a two-phase unidimensional Lam-Clapeyron-Stefan problem with source terms in both phases. J. Math. Anal. Appl., 329:145162, 2007.

[9] J. R. Cannon, The solution of the heat equation subject to the specification of energy. Quart. Appl. Math. 21, 155-160, (1963).

[10] M. M. Cavalcanti, V. N. D. Cavalcanti, J. Ferreira and R. Benabidallah, On global solvability and asymptotic behaviour of a mixed problem for a nonlinear degenerate Kirchhoff model in moving domains. Bull. Belg. Math. Soc. Simon Stevin, 10(2):179196, 2003.

[11] A. Chaoui, F. Ellaggoune and A. Guezane-Lakoud, Full discretization of wave equation, Boundary Value Problems (2015) 2015:133 DOI : 10.1186/s13661-015-0396-3

[12] A. Chaoui and A. Guezane-Lakoud, Solution to an integrodifferential equation with integral codition, Appl. Math. comput., 266 (2015), 903-908.

[13] A. Chaoui and N. Rezgui, Solution to fractional pseudoparabolic equation with fractional integral condition, Rendiconti del Circolo Matematico di Palermo Series 2 volume 67, pages205-213(2018)

[14] S. Chaudhary, V. Srivastava, V.V.K. Srinivas Kumar and B. Srinivasan, Finite element approximation of nonlocal parabolic problem. Wiley Online Library, (2016), doi : 10.1002/num.22123.

[15] H. Che, Y. Wang and Z. Zhou, An optimal error estimates of H^1-Galerkin expanded mixed finite element methods for nonlinear viscoelasticity-type equation. Mathematical Problems in Engineering, 2011. https://doi.org/10.1155/2011/570980.

[16] H. Che, Z. Zhou, Z. Jiang and Y. Wang, H^1-Galerkin expanded mixed finite element methods for nonlinear pseudo-parabolic integro-differential equa-

tions. Numerical Methods for Partial Differential Equations, 29(3):799817, 2013. https://doi.org/10.1002/num.21731.

[17] F. Chen and Z. Zhou, An H^1-Galerkin Mixed Finite Element Approximation of a Nonlocal Hyperbolic Equation, Mathematical Modelling and Analysis, 22:5, 643-653, (2017).DOI: 10.3846/13926292.2017.1346524.

[18] M. Chipot, Elements of Nonlinear Analysis. Birkhauser Advanced Texts, Berlin, 2000.

[19] M. Chipot and B. Lavot, Some remarks on nonlocal elliptic and parabolic problems. Nonlinear Anal., 30, pp. 46194627, (1997).

[20] P. Ciarlet, É. Lunéville, La méthode des éléments finis : De la théorie à la pratique I. Concepts généraux. Paris (2009).

[21] M. Djaghout, A. Chaoui and K. Zennir, Full Discretization to an Hyperbolic Equation with Nonlocal Coefficient. Bol. Soc. Paran. Mat, (2019), doi: 10.5269/bspm.46032.

[22] M.S. El-Azab, M. El-Gamel, A numerical algorithm for solution of telegraph equations, Applied Mathematics and Computation, 190, 757-764, (2007).

[23] H. El Sossa, "Quelques méthodes d´éléments finis mixtes raffinées basées sur lutilisation des champs de Raviart-Thomas", Thesis from the University of Valenciennes and Hainaut Cambrs, (June 2001).

[24] M. Farhloul, "Méthodes d´Élément Finis Mixtes et Volumes finis", Thesis of the University of Laval, Quebec,March (1991).

[25] W. Govaerts, J. D. Pryce, Block elimination with one refinement solves bordered linear system accurately. BIT, 30, 490-507, (1990).

[26] T. Gudi, Finite element method for a nonlocal problem of Kirchhoff type. SIAM J Numer Anal 50, 657-668, (2012).

[27] A. Guezane-Lakoud and D. Belakroum, Rothe´s method for a telegraph equation with integral conditions, Nonlinear Anal. 70(2009) 3842-3853.

[28] A. Guezane-Lakoud and A. Chaoui, A functional method applied to operator equations. Banach J. Math. Anal. 3, (1) 52-60, (2009).

[29] A. Guezane-Lakoud, M. S. Jasmati and A. Chaoui, Rothe´s method for an integrodifferential equation with integral conditions, Nonlinear Analysis. 72 (2010) 1522-1530. (2010).

[30] C. Johnson and V. Thomée, "Error Estimates for some Mixed Finite Element Methods for Parabolic Type Problems", R.A.I.R.O. Analyse numérique / Numerical Analysis, vol.15, n1, 1981, p. 41-78.

[31] S.G. Krein, Linear differential equations in Banach space. A.M.S. Vol 29. (1971).

[32] N. Kikuchi and J. Kacur, Convergence of Rothe´s method in Holder spaces, Appl. Math. 48 :5(2003), 353-365.

[33] O. A. Ladyzenskaja, On solution of nonstationary operator equations. Math. Sb. 39(4), (1956).

[34] O. A. Ladyzenskaja, Solution of the third boundary value problem for quasilinear parabolic equations. Trudy Mosk. Mat. Obs. 7, (1957)(in Russian).

[35] M. G. Larson and F. Bengzon, The Finite Element Method : Theory, Implementation, and Applications. Texts in Computational Science and Engineering. Department of Mathematics Umeå University Umeå Sweden.

[36] D. Lauerova, The Rothe method and time periodic solutions to the Navier-Stokes equations and equations of magneto hydro dynamics, Appl.Math. 35 :2(1990), 89-98.

[37] H.Manouzi and T.G. Theting, "Mixed finite element approximation for the stochastic pressure equation of Wick type", IMA Journal of Numerical Analysis, 24 (2004), no.4, 605-634.

[38] N. Merazga and A. Bouziani, Rothe method for a mixed problem with an integral condition for the two dimentional diffusion equation, Abstract and Applied Analysis 2003 : 16(2003) 899-922.

[39] N.Merazga and A. Bouziani, Rothe time-discretization method for a nonlocal problem arising in thermoelasticity, J. Appl.Math. Stoch. Anal. 1 (2005) 13-28.

[40] A.C. Metaxas and R.J. Meredith, Industrial Microwave, Heating, Peter Peregrinus, London, 1993.

[41] J. Necas, application of Rothe´s method to abstract parabolic equations, Czech. Math. J. 24(1974),496-500.

[42] A. K. Pani, An H^1-Galerkin mixed finite element method for parabolic difference equations,' SIAM Journal on Numerical Analysis, vol. 35, no. 2, pp. 712-727, 1998.

[43] A. K. Pani and G. Fairweather, H^1-Galerkin mixed finite element methods for parabolic partial integro-differential equations. IMA Journal of Numerical Analysis, 22(2):231252, 2002. https://doi.org/10.1093/imanum/22.2.231.

[44] A. K. Pani, R.K. Sinha and A.K. Otta, An H^1-Galerkin mixed method for second order hyperbolic equations. Int. J. Numer. Anal. Model, 1(2):111130, 2004.

[45] V. Pluschke, Rothe´s method for parabolic problems with nonlinear degenerating coefficient. Report No. 14, des FB Mathematik und Informatik, 1996.

[46] R. Rannacher and R. Scott, Some optimal error estimates for piecewise linear finite element approximation. Math Comp 38, 437-445, (1982).

[47] G. Raugel and C. Bernardi, "Méthodes d´éléments finis mixtes pour les équations de Stokes et de Navier Stokes dans un polygone non convexe" , Calcolo 18 (1981), 255-291.

[48] K. Rektorys, On application of direct variational methods to the solution of parabolic boundary value problems of arbitrary order in space variables. Czech. Math. J. 21, 318-339, (1971).

[49] K. Rektorys, The method of discretization in time and partial differential equations. D. Reidel Publishing Company, 1982.

[50] R. J. Robalo, R. M. P. Almeida, M. d. Carmo Coimbra and J. Ferreira, A reaction-diffusion model for a class of nonlinear parabolic equations with moving boundaries: existence, uniqueness, exponential decay and simulation, Appl. Math. Modelling (2014), doi: http://dx.doi.org/10.1016/j.apm.2014.04.045.

[51] G. Roussy and J.A. Pearcy, Foundations and Industrial Applications of Microwaves and Radio Frequency Fields, JohnWiley, New York, 1995.

[52] M. L. Santos, J. Ferreira and C. A. Raposo, Existence and uniform decay for a non-linear beam equation with nonlinearity of Kirchhoff type in domains with moving boundary. Abstr. Appl. Anal., 2005(8):901919, 2005.

[53] M. P. Sapagovas and R. Yu. Chegies, On some boundary value problems with a non-local condition. Differ. Equ. 23, (7)858- 863, (1987).

[54] V. Trinoguine, "Analyse fonctionnelle", Edition Mir. Mouscou, 1985.

[55] K. Yosida, "Functional Analysis". Springer-Verlag, Sixth Edition.

[56] Z. Zhou, An H^1-Galerkin mixed finite element method for a class of heat transport equations. Applied Mathematical Modelling, 34(9):24142425, 2010. https://doi.org/10.1016/j.apm.2009.11.007.

Research activities

INTERNATIONAL PUBLICATIONS

• M. Djaghout, A. Chaoui *and K. Zennir, Full Discretization to an Hyperbolic Equation with Nonlocal Coefficient Bol. Soc. Paran. Mat, (2019), doi: 10.5269/bspm.46032.*

INTERNATIONAL COMMUNICATIONS

• full discretization for some hyperbolic equation, *International Conference on Advances in Applied Mathematics (ICAAM), Sousse, 17-20/12/2018.*

NATIONAL COMMUNICATIONS

• Approximate solution for some hyperbolic equation, *Informatics and Applied Math-*

ematics (IAM'18), 8 Mai 1945 of Guelma, 3-4/8/2018.

• Roth-finite element method for teleghraph equation with nonlocal coefficient *, Journée Nationale sur les Mathématiques appliquées (JNMA'19). Larbi Ben M'hidi university of Oum El Bouaghi, 27/06/2019.*