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Entitled

The semigroup theory and it's application on semilinear parabolic equations

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Abstract

In this Master's thesis, we are interested by the theory of strongly continuous semigroups of bounded linear operators on Banach spaces and its applications to prove the existence of local and global to solutions of semilinear parabolic partial differential equations.

In the first chapter, we give a general introduction on the thesis.

In the second chapter, we treat the notion of strongly continuous semigroups and their properties.

In the third chapter, we treat some abstract semilinear parabolic equations which the operators generate a strongly continuous semigroups on Banach spaces.

In the last chapter, we study some concrete examples of semilinear parabolic equations on bounded and unbounded domains.

ملخص

في مذكرة الماستر هذه نحن مهتمون بنظرية أنصاف الزمر ذات المؤثرات الخطية المحدودة في فضاءات بناخ وتطبيقاتها لإثبات وجود محلي وكلي لحلول معادلات تفاضلية ذات مشتقات جزئية من النوع التكافئي نصف الخطي.

في الفصل الأول ، نعطي مقدمة عامة حول المذكرة.

في الفصل الثاني ، نتطرق إلى مفهوم أنصاف الزمر المستمرة مع دراسة أهم خواصها.

في الفصل الثالث ، ندرس بعض أشكال مجردة لمعادلات تفاضلية تكافئية نصف خطية ذات مشتقات جزئية التي تكون المؤثرات فيها مولدة لأنصاف زمر مستمرة على فضاءات بناخ.

في الفصل الأخير ، ندرس بعض الأمثلة الملموسة لمعادلات تفاضلية تكافئية نصف خطية معرفة على نطاقات محدودة وغير محدودة.

Résumé

Nous intéressons dans ce mémoire de Master par la théorie des semi-groupes fortement continu d'opérateurs linéaires bornés sur les espaces de Banach et ses applications pour prouver l'existence locale et globale de solutions d'équations aux dérivées partielles de type paraboliques semi-linéaires.

Dans le premier chapitre, nous donnons une intriduction générale sur le mémoire.

Dans le deuxième chapitre, nous traitons la notion de semi-groupes fortement continus et ses propriétés.

Dans le troisième chapitre, nous traitons quelques formes abstraites d'équations paraboliques semi-linéaires dont les opérateurs génèrent des semi-groupes fortement continus sur des espace de Banach.

Dans le dernier chapitre, nous étudions quelques exemples concrets d'équations paraboliques semi-linéaires sur des domaines bornés et non bornés.

Notations

$\mathcal{L}(X,Y)$ the space of linear, continuous mapping from X to Y

$D(A)$ the Banach space with $\|u\|_{D(A)} = \|u\| + \|Au\|$

$D(\Omega)$ the space of C_c^{∞} in Ω

$D\prime(\Omega)$ the space of distributions on Ω

$C_c(\Omega)$ the space of continuous functions with compact support in Ω

$C_c^{\infty}(\Omega)$ $D(\Omega)$

$L^p(\Omega)$ the space of measurable functions on Ω such that $|u|^p$ is integrable $1 < p < \infty$

$L^{\infty}(\Omega)$ the space of measurable functions from Ω to $\mathbb{R}$ such that there exists a constant $C \geq 0$ satisfies : $|u(x)| < C$
for almost every x$\in \Omega$

$\|u\|_{L^p}$ $\left(\int |u|^p \, dx\right)^{\frac{1}{p}}$

D^{α} $\frac{\partial^{|\alpha|}}{\partial^{a_1} X_1 ....\partial^{a_n} Xn}$

$W^{m,p}(\Omega)$ $\left\{\text{f} \in L^p \text{ , D}^{\alpha} f \in L^p(\Omega) \text{ for all a} \in N^N \text{ such that } |a| \leq m\right\}$

$W_0^{m,p}(\Omega)$ the closure of $D(\Omega)$ with respect the norm $\|.\|_{W^{\mathrm{m,p}}}$

$H^m(\Omega)$ $W^{m,1}(\Omega)$

$\|u\|_{W^{m,p}}$ $\sum\limits_{|a| \leq m} \|D^{\alpha} u\|_{L^p}$ for $u \in W^{m,p}(\Omega)$

$C_{b,u}(I,X)$ the space of uniformly continuous and bounded functions from I to X

INTRODUCTION

0.1 Semigroup of linear bounded operators

A strongly continuous semigroup of bounded linear operators on a Banach space X or C_0-semigroup, also known as a strongly continuous one-parameter semigroup, is a generalization of the exponential function. Just as exponential functions provide solutions of scalar linear constant coefficient ordinary differential equations, strongly continuous semigroups provide solutions of linear constant coefficient ordinary differential equations in Banach spaces. Such differential equations in Banach spaces arise from e.g. delay differential equations and partial differential equations.

Formally, a strongly continuous semigroup is a representation of the semigroup (R+,+) on some Banach space X that is continuous in the strong operator topology. Thus, strictly speaking, a strongly continuous semigroup is not a semigroup, but rather a continuous representation of a very particular semigroup.

0.2 Semigroup methods in partial differential equations

Semigroup theory can be used to study some problems in the field of partial differential equations. Roughly speaking, the semigroup approach is to regard a time-dependent partial differential equation as an ordinary differential equation on a function space but we are going to treat the case of a partial differential equations is a semilinear problem

0.3 The heat equation

The heat equation is a partial differential equation that describes how the distribution of some quantity (such as heat) evolves over time in a solid medium, as it spontaneously flows from places where it is higher towards places where it is lower. It is a special case of the diffusion equation.

This equation was first developed and solved by Joseph Fourier in 1822 to describe heat flow. However, it is of fundamental importance in diverse scientific fields

0.3.1 Homogeneous heat equation

The homogeneous heat equation is given by :

$$\left\{ \begin{array}{ll} \partial_t u(x,t) = \partial_x^2 u(x,t) & t > 0 \\ u(x,0) = u_0(x) & t = 0 \end{array} \right\} \tag{P}$$

We are not interested in this form since we are studying the semigroup theory so lets transform P to P' and it will be our new case

$$\left\{ \begin{array}{ll} \partial_t u(x,t) = Au(x,t) & t > 0 \\ u(x,0) = u_0(x) & t = 0 \end{array} \right\} \tag{P'}$$

such that A is the generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ in an appropriete Banach space X.

0.3.2 Semilinear heat equation

We consider the initial value problem for a semilinear heat equation

$$\left\{ \begin{array}{ll} \partial_t u(x,t) = a\Delta u(x,t) + f(u) & t > 0 \\ u(x,0) = u_0(x) & t = 0 \end{array} \right\} \tag{P''}$$

where Δ is the Laplace operator on R^d $(d \in \mathbb{N}^*)$, a is a positive real constant and f is a given function.
If we denote by $Au(x,t) = a\Delta u(x,t)$ we obtain the abstract Cauchy problem :

$$\left\{ \begin{array}{ll} \frac{du(t)}{dt} = Au(t) + f(u) & t > 0 \\ u(0) = u_0(x) & \end{array} \right\}$$

Chapter 1

m-dissipative operators

1.1 Unbounded operators in Banach space

Throughout this chapter, X is a Banach space, endowed with the norm $\|.\|$.

Definition 1.1.1 (Casenave and Haraux p18): A linear unbounded operator in X is a pair (D, A), where D is a linear subspace of X and A is a linear mapping $D \to X$. We say that A is bounded if there exists $c > 0$ such that

$$\|Au\| \leq c$$

for all $u \in \{x \in D, \|x\| \leq 1\}$. Otherwise, A is not bounded.

Remark 1.1.2. Note that a linear unbounded operator can be either bounded or not bounded. This somewhat strange terminology is in general use and should not lead to misunderstanding in our applications.

Remark 1.1.3: If A is bounded, A is the restriction to D of an operator $\widetilde{A} \in \mathcal{L}(Y, X)$, where Y is a closed linear subspace of X, such that $D \subset Y$. If A is not bounded, there exists no operator $\widetilde{A} \in \mathcal{L}(Y, X)$ with Y closed in X and $D \subset Y$, such that $\widetilde{A}_{|D} = A$.

Definition 1.1.4(Casenave and Haraux p18): Let (D, A) be a linear operator in X. The graph $G(A)$ of A and the range $R(A)$ of A are defined by

$$G(A) = \{(u, f) \in X \times X; u \in D \text{ and } f = Au\}$$
$$R(A) = A(D)$$

$G(A)$ is a linear subspace of $X \times X$, and $R(A)$ is a linear subspace of X

Remark 1.1.5: In this chapter, a linear unbounded operator is just called an operator where there is no risk of confusion. As usual, we denote the pair (D, A) by A with $D(A) = D$, meaning the domain of A is D. Note, however, that when one defines an operator, it is absolutely necessary to define its domain.**Remark 1.1.5**: When $D(A) = X$, it follows from Theorem 1.1.2 that $A \in \mathcal{L}(X)$ if and only if $G(A)$ is closed in X. More generally, for not bounded operators, it is very useful to know whether or not the graph is closed pass 2 .

1.2 Definition and main properties of m-dissipative operators

Definition 1.2.1(Casenave and Haraux p19): An operator A in X is dissipative if

$$\|u-\lambda Au\| \geq \|u\|, \text{ for all } \ u \in D(A) \ \text{ and all } \ \lambda > 0$$

Definition 1.2.2(Casenave and Haraux p19): An operator A in X is m -dissipative if
(i) A is dissipative;
(ii) for all $\lambda > 0$ and all $f \in X$, there exists $u \in D(A)$ such that $u - \lambda Au = f$.

Remark 1.2.3: If A is m -dissipative in X, it is clear, from Definitions 1.2 .1 and 1.2.2, that for all $f \in X$ and all $\lambda > 0$, there exists a unique solution u of the equation $u - \lambda Au = f$. In addition, one has $\|u\| \leq \|f\|$.

Definition 1.2.4(Casenave and Haraux p19): Let A be an m -dissipative operator in X and $\lambda > 0$. For all $f \in X$, we denote by $J_\lambda j$ or by $(I-\lambda A)^{-1} f$ the solution u of the equation

$$u - \lambda Au = f$$

Remark 1.2.5: By Remark 1.2.3, one has $J_\lambda \in \mathcal{L}(X)$ and $\|J_\lambda\|_{\mathcal{L}(X)} \leq 1$.

Proposition 1.2.6(Casenave and Haraux p19): Let A be a dissipative operator in X. The following properties are equivalent.
(i) A is m -dissipative in X
(ii) there exists $\lambda_0 > 0$ such that for all $f \in X$, there exists a solution $u \in D(A)$ of

$$u - \lambda_0 Au = f$$

Proof. It is clear that (i) $\Rightarrow$ (ii). Let us show that (ii) $\Rightarrow$ (i), Let $\lambda > 0$: Note that the equation $u - \lambda Au = f$ is equivalent to

$$u - \lambda_0 Au = \frac{\lambda_0}{\lambda} f + \left(1 - \frac{\lambda_0}{\lambda}\right) u$$

since A is dissipative and $R(I - \lambda_0 A) = X$, the operator $J_{\lambda_0} = (I - \lambda_0 A)^{-1}$ can be defined as in Definition 1.2.4.
This operator is a contraction on X.
Next, note that the preceding equation is also equivalent to

$$u = J_{\lambda_0}\left(\frac{\lambda_0}{\lambda} f + \left(1 - \frac{\lambda_0}{\lambda}\right) u\right)$$

If $2\lambda > \lambda_0$, this last equation is $u = F(u)$, where F is Lipschitz continuous $x \rightarrow X$, with a Lipschitz constant $k = |(\lambda - \lambda_0)/\lambda| < 1$. Applying Theorem 1.1.1, there exists a solution u of $u - \lambda Au = f$, for all $\lambda \in (\lambda_0/2, \infty)$.
Iterating this argument n times, there exists a solution for all $\lambda \in (2^{-n}\lambda_0, \infty)$ $n \geq 1$. since n is arbitrary there exists a solution for all $\lambda > 0$.

Proposition 1.2.7(Casenave and Haraux p20): If A is m -dissipative, then $G(A)$ is closed in X.

Proof. since $J_1 \in \mathcal{L}(X), G(J_1)$ is closed. It follows that $G(I - A)$ is closed, and so $G(A)$ is closed.

Corollary 1.2.8 : Let A be an m -dissipative operator. For every $u \in D(A)$ let $\|u\|_{D(A)} = \|u\| + \|Au\|$. Then $\left(D(A), \|\cdot\|_{D(A)}\right)$ is a Banach space, and

$$A \in \mathcal{L}(D(A), X)$$

Remark 1.2.9: In what follows, and in particular in Chapters 2 and 3, $D(A)$ means the Banach space $\left(D(A), \|\cdot\|_{D(A)}\right)$.

Proposition 1.2.10(Casenave and Haraux p20): If A is m -dissipative, then $\lim_{\lambda \downarrow 0} \|J_\lambda u - u\| = 0$ for all

$$u \in \overline{D(A)}$$

Proof: We have $\|J_\lambda - I\| \leq 2$, and by density we need only consider the case $u \in D(A)$. We have

$$J_\lambda u - u = J_\lambda(u - (I - \lambda A)u)$$

and so $\|J_\lambda u - u\| \leq \|u - (I - \lambda A)u\| = \lambda\|Au\| \rightarrow 0$, as $\lambda \downarrow 0$.

Definition 1.2.11(Casenave and Haraux p20): Let A be an m -dissipative operator. For $\lambda > 0$, we denote by A_λ the operator defined by

$$A_\lambda = AJ_\lambda = \frac{J_\lambda - I}{\lambda}$$

We have $A_\lambda \in \mathcal{L}(X)$ and $\|A_\lambda\|_{\mathcal{L}(X)} \leq 2/\lambda$.

Proposition 1.2.12(Casenave and Haraux p20): If A is m -dissipative and if $\overline{D(A)} = X$, then $A_\lambda u \longrightarrow Au$ as $\lambda \downarrow 0$ for all $u \in D(A)$.

Proof. Let $u \in D(A)$. By Proposition 1.2.10, one has

$$J_\lambda Au - Au \longrightarrow 0 \quad \text{as } \lambda \downarrow 0$$

On the other hand, it follows easily from Definition 1.2 .11 that

$$A_\lambda u = J_\lambda A u$$

Thus,

$$\|A_\lambda u - Au\| = \|J_\lambda Au - Au\| \to 0 \quad \text{as } \lambda \downarrow 0$$

hence the result.

1.3 m-dissipative operator in X with dense domain

Proposition 1.3.1: Let A be an m -dissipative operator in X with dense domain. There exists a Banach space $\bar{X}$, and an m -dissipative operator $\bar{A}$ in $\bar{X}$, such that
(i) $X \hookrightarrow \bar{X}$, with dense embedding;
(ii) for all $u \in X$, the norm of u in $\bar{X}$ is equal to $\|J_1 u\|$
(iii) $D(\bar{A}) = X$, with equivalent norms;
(iv) $\bar{A}u = Au$, for all $u \in D(A)$
In addition, $\bar{X}$ and $\bar{A}$ satisfying (i)-(iv) are unique, up to isomorphism.

Proof: For $u \in X$, we define $\|u\| = \|J_1 u\|$. It is clear that $\|\cdot\|$ is a norm on X. Let $\bar{X}$ be the completion of X for the norm $\|.\|$. $\bar{X}$ is unique, up to an isomorphism, and $X \hookrightarrow \bar{X}$, with dense embedding. On the other hand, observe that

$$J_1 Au = J_1 u - u, \quad \forall u \in D(A)$$

Thus,

$$\|Au\| \leq \|u\| + \|u\| \leq 2\|u\|, \quad \forall u \in D(A)$$

Hence, A can be extended to an operator $\tilde{A} \in \mathcal{L}(X, Y)$. We define the linear operator $\bar{A}$ on $\bar{X}$ by

$$\|Au\| \leq \|u\| + \|u\| \leq 2\|u\|, \quad \forall u \in D(A)$$

It is clear that $\bar{A}$ satisfies (iii) and (iv). Now, let us show that $\bar{A}$ is dissipative. Take $\lambda > 0$. Let $u \in D(A)$ and let $v = J_1 u$. One has

$$v - \lambda Av = J_1(u - \lambda Au)$$

since A is dissipative, it follows that By continuity of $\widetilde{A}$ m-dissipative operators and so $\bar{A}$ is dissipative. Finally, let $f \in \bar{X}$,
and $(f_n)_{n\geq 0} \subset X$, with $f_n \longrightarrow f$ in $\bar{X}$ as $n \to \infty$. Set $u_n = J_1 f_n$. since $(f_n)_{n\geq 0}$ is a Cauchy sequence in $\bar{X}, (u_n)_{n\geq 0}$ is also a Cauchy sequence in X; and so there exists $u \in X$, such that $u_n \longrightarrow u$ in X as $n \to \infty$. We have

$$f_n = u_n - Au_n = u_n - \tilde{A}u_n$$

since $\widetilde{A} \in \mathcal{L}(X, Y)$; it follows that $f = u - \tilde{A}u = u - \bar{A}u$. Hence $\bar{A}$ is m dissipative. The uniqueness of $\bar{A}$ follows from the uniqueness of $\widetilde{A}$.

Corollary 1.3.2 : If $x \in X$ is such that $\bar{A}x \in X$, then $x \in D(A)$ and $Ax = \bar{A}x$

Proof: Let $f = x - \bar{A}x \in X$. since A is m -dissipative, there exists $y \in D(A)$ such that $y - Ay = f$.
By Proposition 1.3.1 (iii), we have $(x-y) - \bar{A}(x-y) = 0$ and since $\bar{A}$ is dissipative, we obtain $x = y$

1.4 Unbounded operators in Hilbert spaces

Throughout this section, we assume that X is a Hilbert space, and we denote by $\langle \cdot, \cdot \rangle$ its scalar product. If A is a linear operator in X with dense domain, then

$$G\left(A^{\star}\right) = \{(v, \varphi) \in X \times X ; \langle \varphi, u \rangle = \langle v, f \rangle \text{ for all } (u, f) \in G(A)\}$$

defines a linear operator A^* (the adjoint of A). The domain of A^* is

$$D\left(A^*\right) = \{v \in X, \exists C < \infty, |\langle Au, v \rangle| \leq C\|u\|, \forall u \in D(A)\}$$

and A^* satisfies

$$\langle A^* v, u \rangle = \langle v, Au \rangle, \forall u \in D(A)$$

Indeed, the linear mapping $u \mapsto \langle v, Au \rangle$, defined on $D(A)$ for all $v \in D\left(A^*\right)$, can be extended to a unique linear mapping $\varphi \in X' \approx X$, denoted by $\varphi = A^* v$. It is clear that $G\left(A^*\right)$ is systematically closed. Finally, it follows easily that if $B \in \mathcal{L}(X)$, then $(A+B)^* = A^* + B^*$.

Proposition 1.4.1(Casenave and Haraux p22): $(\overline{R(A)})^{\perp} = \{v \in D\left(A^*\right) ; A^* v = 0\}$.

Proof: One has $v \in (\overline{R(A)})^{\perp} \Leftrightarrow \langle v, Au \rangle = 0, \forall u \in D(A) \Leftrightarrow (0, v) \in G\left(A^*\right)$ This last property is equivalent to $v \in D\left(A^*\right)$ and $A^* v = 0$; hence the result.

Proposition 1.4.2(Casenave and Haraux p22): A is dissipative in X if and only if :

$$\langle Au, u \rangle \leq 0 \text{ for all } u \in D(A)$$

Proof: If A is dissipative, one has

$$-2\lambda \langle Au, u \rangle + \lambda^2 \|Au\|^2 = \|u - \lambda Au\|^2 - \|u\|^2 \geq 0, \quad \forall \lambda > 0, \forall u \in D(A)$$

Dividing by λ and letting $\lambda \downarrow 0$, we obtain

$$\langle Au, u \rangle \leq 0, \text{ for all } u \in D(A)$$

Conversely, if the last property is satisfied, then for all $\lambda > 0$ and $u \in D(A)$ we have

$$\|u - \lambda Au\|^2 = \|u\|^2 - 2\lambda \langle Au, u \rangle + \lambda^2 \|Au\|^2 \geq \|u\|^2$$

and then A is dissipative.

Corollary 1.4.3 : If A is m -dissipative in X, then $D(A)$ is dense in X

Proof: Let $z \in (D(A))^{\perp}$, and let $u = J_1 z \in D(A)$. We have

$$0 = \langle z, u\rangle = \langle u - Au, u\rangle$$

Hence,

$$\|u\|^2 = \langle Au, u\rangle \leq 0$$

It follows that $u = z = 0$; and so $D(A)$ is dense in X.

Corollary 1.4.4: If A is m -dissipative in X, then

$$J_\lambda u \longrightarrow u \quad \text{as } \lambda \downarrow 0, \text{ for all } u \in X$$

and

$$A_\lambda u \to Au \quad \text{as } \lambda \downarrow 0, \text{ for all } u \in D(A)$$

Proof: We apply Corollary 1.2.3 and Propositions 1.2 .10 and 1.2.12 .

Theorem 1.4.5(Casenave and Haraux p23): Let A be a linear dissipative operator in X with dense domain. Then A is m -dissipative if and only if A^* is dissipative and $G(A)$ is closed.

Proof: If A is m -dissipative, then $G(A)$ is closed, by Proposition 1.2 .7 . Let us show that A^* is dissipative.
Let $v \in D(A^*)$. We have since $\langle A^*v, J_\lambda v\rangle \longrightarrow \langle A^*v, v\rangle$ as $\lambda \downarrow 0$, it follows that A^* is dissipative. Conversely, since A is dissipative and $G(A)$ is closed, it is clear that $R(I - A)$ is closed in X. On the other hand, by Proposition 1.4 .1 , one has

$$(\overline{R(I-A)})^{\perp} = \{v \in D(A^*); v - A^*v = 0\} = \{0\}$$

since A^* is dissipative. Therefore $R(I - A) = X$, and A is m -dissipative, by Proposition 1.2 .6.

Definition 1.4.6:(Casenave and Haraux p24) Let A be a linear operator in X with dense domain. We say that A is self-adjoint (respectively skew-adjoint) if $A^* = A$ (respectively

$$A^* = -A)$$

Remark 1.4.7: The equality $A^* = \pm A$ has to be taken in the sense of operators. It means that $D(A) = D(A^*)$and $A^*u = \pm Au$, for all $u \in D(A)$.

Corollary 1.4.8 : If A is a self-adjoint operator in X, and if $A \leq 0$ (i.e. $\langle Au, u\rangle \leq 0$, for all $u \in D(A)$),then A is m -dissipative.

Proof: By Proposition 1.4.2: A is dissipative. since $A^* = A, A^*$ is dissipative. Finally, $G(A^*)$ is closed, so that $G(A)$ is closed. We finish th e proof by applying Theorem 1.4 .5.

Corollary 1.4.9: If A is a skew-adjoint operator in X, then A and $-A$ are m -dissipative.

Proof. Let $u \in D(A)$. One has $\langle Au, u\rangle = \langle u, A^*u\rangle = -\langle u, Au\rangle$. Hence $\langle Au, u\rangle = 0$. It follows from Proposition 1.4.2, that A and $-A$ are dissipative. We conclude as in Corollary 1.4 .8.

Corollary 1.4.10. Let A be a linear operator in X with dense domain, such that $G(A) \subset G(A^*)$ and $A \leq 0$. Then A is m -dissipative if and only if A is self-adjoint.

Proof: Applying Corollary 1.4.8, we need only show that if A is m -dissipative then A is self-adjoint. Let $(u, f) \in G(A^*)$, and let $g = u - A^*u = u - f$. since A is m -dissipative, there exists $v \in D(A)$ such that $g = v - Av$, and since $G(A) \subset G(A^*)$, we have $v \in D(A^*)$ and $g = v - A^*v$. Therefore $(v-u) - A^*(v-u) = 0$ and since A^* is dissipative (Theorem 1.4 .5), we obtain $u = v$. Thus, $(u, f) \in G(A)$ and so $A = A^*$.

Corollary 1.4.11 : Let A be a linear operator in X with dense domain. Then A and $-A$ are m -dissipative if and only if A is skew-adjoint.

Proof: Applying Corollary 1.4.9, it suffices to show that if A and $-A$ are m dissipative, then A is skew-adjoint. Applying Proposition 1.4.2 to A and $-A$ we obtain

$$\langle Au, u\rangle = 0, \quad \text{for all } u \in D(A)$$

For all $u, v \in D(A)$, we obtain

$$\langle Au, v\rangle + \langle Av, u\rangle = \langle A(u+v), u+v\rangle - \langle Au, u\rangle - \langle Av, v\rangle = 0$$

Therefore $G(-A) \subset G(A^*)$. It remains to show that $G(A^*) \subset G(-A)$. Consider $(u, f) \in G(A^*)$ and let $g = u - A^*u = u - f$. since $-A$ is m -dissipative, there exists $v \in D(A)$ such that $g = v + Av$, and since $G(-A) \subset G(A^*)$, we have $v \in D(A^*)$ and $f = v - A^*v$. Hence $(v-u) - A^*(v-u) = 0$ and since $-A^*$ is dissipative (Theorem 1.4.5), we obtain $u = v$. Therefore, $(u, f) \in G(A^*)$; and so

$$A = -A^*$$

1.5 The Laplactian in an open subset of $\mathbb{R}^N$: L^2 theory

Let Ω be any open subset of $\mathbb{R}^N$, and let $Y = L^2(\Omega)$. We can consider either realvalued functions or complex-valued functions, but in both cases, Y is considered as a real Hilbert space
We define the linear operator B in Y by

$$\begin{cases} D(B) = \{u \in H_0^1(\Omega); \Delta u \in L^2(\Omega)\} \\ Bu = \Delta u, \quad \forall u \in D(B) \end{cases}$$

Proposition 1.5.1(Casenave and Haraux p26): B is m -dissipative with dense domain. More precisely, B is self-adjoint and $B \leq 0$.

We need the following lemma.

Lemma 1.5.2. We have :

$$\int_{\Omega} v\Delta u dx = -\int_{\Omega} \nabla u \cdot \nabla v dx \ , \quad \text{for all} \ \ u \in D(B) \ \ \text{and all} \ \ v \in H_0^1(\Omega). \tag{1.1}$$

Proof: (1.1) is satisfied by $v \in \mathcal{D}(\Omega)$. The lemma follows by density, since both terms of (1.1) are continuous in v on $H_0^1(\Omega)$.

Proof of Proposition 1.5.1: First, $\mathcal{D}(\Omega) \subset D(B)$, and so $D(B)$ is dense in Y. Let $u \in D(B)$. Applying (1.1) with $v = u$, we obtain $(Bu, u) \leq 0$, so that B is dissipative (Proposition 1.4 .2). The bilinear continuous mapping

$$b(u, u) = \int (uv + \nabla u \cdot \nabla v) dx \tag{1.2}$$

is coercive In H_0 (Ω). It followe from :for all $\rho \in L^2(\alpha)$, there exists $u \in H_0^1(\Omega)$ such that

$$\int (uv + \nabla u \cdot \nabla v) dx = \int f v dx, \quad \forall v \in H_0(\Omega) \tag{1.3}$$

We obtain

$$u - \Delta u = \rho_1$$

In the sense of distributions.
Since, In addition $u \in H_0^1(\Omega)$, we obtaln $u \in D(B)$ and $u - Bu = f$.
Therefore B in m -dianipative. Finally, for all $u, v \in D(B)$, we have, by (1.1)

$$(Bu, v) \text{ w } \langle u, Bv \rangle$$

Therefore $G(B) \subset G(B^*)$, and by Corollary 1.4.10, it follows that B is offer adjoint.

Remark 1.5.3: If Ω has a bounded boundary of class C^2, then $D(B) = H^2(\Omega) \cap H_0^1(\Omega)$, with equivalent norms

1.5.1 The Laplacian In an open subset of $\mathbb{R}^N$: C_0 theory

Let Ω be a bounded open subset of $\mathbb{R}^N$, and let $Z= L^\infty(\Omega)$. We define the linear operator C in Z by

$$\begin{cases} D(C) = \{u \in H_0^1(\Omega) \cap Z, \Delta u \in Z\} \\ Cu = \Delta u, \quad Vu \in D(C) \end{cases}$$

Proposition 1.5.4(Casenave and Haraux p27) : C is $m-$dissipative in Z.

Proof. First, let us show that C is dissipative. Let $\lambda > 0, f \in Z$. and let $M = \|f\|_{L^\infty}$.

Let $u \in H_0^1(\Omega)$ be a solution of

$$u - \lambda\Delta u = f_1, \text{ in } \mathcal{D}'(\Omega)$$

In particular, this equation is satisfied in $L^2(\Omega)$, and we have

$$(u - M) - \lambda\Delta(u - M) = f - M_1, \text{ in } L^2(\Omega)$$

On the other hand, $v = (u - M)' \in H_d^1(\Omega)$, with $\nabla v = 1_{(|\omega|>M)}\nabla u$. Applying Lemma 1.5.2 , we obtain

$$\int v^2 dx + \varphi \int_{(|u|>M\}} |\nabla u|^2 dx = \int (f - M)v dx \leq 0$$

Therefore $\int v^2 dx \leq 0$, and so $v = 0$. We conclude that $u \leq M$ a.e. on $\mathbb{R}$.
Smilarly, we show that $u \geq -M$ a.e. on Ω.
Hence $u \in L^\infty(\Omega)$, and $\|u\|_{L^\infty} \leq \|f\|_{L^\infty}$. It follows that C is dissipative.

Now let $f \in L^\infty(\Omega) \subset L^2(\Omega)$. By *proposition* 1.5.1, there exists $u \in H_0(\Omega)$, with $\Delta u \in L^2(\Omega)$, a solutlon of $u - \Delta u = f_1$ in $L^2(\Omega)$.
We already know that $u \in L^\infty(\Omega)$, so that $u \in D(C)$, and $u - Cu = f$, therefore C is m-dissipative.

Lemma 1.5.5: If Ω has a Lipschitz continuous boundary, then $D(C) \subset C_0(\Omega) = \{u \in C(\bar{\Omega}); u_{|\theta\cap} = 0\}$.

Remark 1.5.6. It follows from Lemme 1.5.5 that in general the domain of C is not dense in Z.
The fact that the domain is dense will turn out to be very important (see Chapter 2). This is the reason why we are led to consider another example. We now set $X = C_0(\Omega)$, and we define the operator A as follows:

$$\begin{cases} D(A) = \{u \in X \cap H_0^1(\Omega), \Delta u \in X\} \\ Au = \Delta u, \quad \forall u \in D(A) \end{cases}$$

Proposition 1.5.7(**Casenave and Haraux p28**) : Assume that Ω has a Lipschitz continuous boundary. Then A is m -dissipative, with dense domain.

Proof. $\mathcal{D}(\Omega)$ is dense in X, and $\mathcal{D}(\Omega) \subset D(A)$; and so $D(A)$ is dense in X. On the other hand, X is equipped with the norm of $L^\infty(\Omega)$, and so $X \hookrightarrow Z$ and $G(A) \subset G(C)$. Since C is dissipntive, A is also dissipative.
Now let $f \in X \hookrightarrow L^\infty(\Omega)$. since C is m -dissipative, there exists $u \in D(C)$, such that $u - \Delta u = f$.
By Lemma 1.5.5, we have $u \in X$, and so $\triangle u \in X$. Therefore, $u \in D(A)$ and $u - Au = f$. Hence A is m -dissipative.

Remark 1.5.8: In the three examples of $1.5.1 and $1.5.2, note that the same formula (the Laplacian) corresponds to several operators that enjoy different properties (since they are defined in different domains).
In particular, the expression the operator $\triangle$ has a meaning only if we specify the space in which this operator applies and its domain.

Chapter 2

The semigroup generated by an m-dissipative operator

2.1 Strongly Continuous Semigroups

Let us fix some notations. From now on, we take X to be a complex Banach space with norm $\|.\|$. We denote by $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on X endowed with the operator norm, which again is denoted by $\|.\|$.
The identity operator on X is denoted by $Id \in \mathcal{L}(X)$, and $\mathbb{R}_+$ denotes the interval $[0, +\infty)$.

Definition 2.1.1(A.Batkai and S.Piazzera p3): A family $(T(t))t \geq 0$ of bounded linear operators on a Banach space X is called a strongly continuous semigroup (or C_0 -semigroup) if the following properties hold:
(i) $T(0) = \text{Id}$
(ii) $T(t+s) = T(t)T(s)$ for all t, $s \geq 0$.
(iii) The orbit maps $t \mapsto T(t)x$ are continuous from $\mathbb{R}_+$ into X for every $x \in X$.

Sometimes C_0 -semigroups are also called linear semidynamical systems.

We will see that the orbit maps of semigroups occur as solutions of differential equations in Banach spaces.

The key definition for this fact is the following.

Definition 2.1.2(A.Batkai and S.Piazzera p4): Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X and let $D(A)$ be the subspace of X defined as

$$D(A) := \left\{ x \in X : \lim_{h\to 0} \frac{1}{h}(T(h)x - x) \text{ exists} \right\} \tag{2.1}$$

For every $x \in D(A)$, we define

$$Ax := \lim_{h\to 0} \frac{1}{h}(T(h)x - x) \tag{2.2}$$

The operator $A : D(A) \subseteq X \to X$ is called the generator of the semigroup $(T(t))_{t\geq 0}$.

In the following, we will denote the operator A with domain $D(A)$ by the pair $(A, D(A))$.

Lemma 2.1.3: For the generator $(A, D(A))$ of a strongly continuous semigroup $(T(t))_{t\geq 0}$. The following properties hold :
(i) $A : D(A) \subseteq X \to X$ is a linear operator.
(ii) If $x \in D(A)$, then $T(t)x \in D(A)$ and

$$\frac{d}{dt}T(t)x = T(t)Ax = AT(t)x \quad \text{for all } t \geq 0 \tag{2.3}$$

As a consequence the orbit map is continuously differentiable on $D(A)$.
(iii) For every $t \geq 0$ and $x \in X$, one has

$$\int_0^t T(s)xds \in D(A) \tag{2.4}$$

(iv) For every $t \geq 0$ we have the identities

$$\begin{aligned} T(t)x - x &= A\int_0^t T(s)xds, \text{ if } x \in X \\ &= \int_0^t T(s)Axds \text{ if } x \in D(A) \end{aligned} \tag{2.5}$$

With the help of this lemma, one can show the following theorem.

Theorem 2.1.4(A.Batkai and S.Piazzera p5): Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X with generator $(A, D(A))$. Then $(A, D(A))$ is a closed operator and the domain $D(A)$ is dense in X. Moreover, if $(S(t))_{t\geq 0}$ is another strongly continuous semigroup with the same generator $(A, D(A))$, then

$$S(t) = T(t) \text{ for all } t \geq 0$$

As a consequence of the above theorem, there is a one-to-one correspondence between strongly continuous semigroups and their generators. Therefore, we will say that an operator $(A, D(A))$ generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X if $(A, D(A))$ is the generator of the semigroup $(T(t))_{t\geq 0}$.
Since $(A, D(A))$ is a closed operator, $D(A)$ is a Banach space with the graph norm $\|.\|_A$. We will denote this Banach space by X_1, i.e., $X_1 := (D(A), \|.\|_A)$.

Example 2.1.5: (Uniformly Continuous Semigroups.) Let X be a Banach space and $A \in \mathcal{L}(X)$ a linear bounded operator. Define

$$T(t) := e^{tA} := \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} \quad \text{for } t \geq 0 \tag{2.6}$$

Then $(T(t))_{t>0}$ is a strongly continuous semigroup with generator (A, X).

Actually, the semigroup is uniformly continuous, i.e., the map $t \mapsto T(t)$ is continuous from $\mathbb{R}_+$ to $\mathcal{L}(X)$. Moreover, one can prove that a semigroup is uniformly continuous if and only if its generator is a bounded linear operator.

Example 2.1.6: (Multiplication Semigroups.) Let Ω be a locally compact metric space, $q : \Omega \to \mathbb{C}$ a continuous function with real part bounded above, that is, $\sup_{\omega\in\Omega} \Re q(\omega) < \infty$. On the Banach space $X := C_0(\Omega)$ of continuous functions that vanish at infinity, define the multiplication operators

$$T(t)f := e^{tq}f, \quad f \in X \text{ and } t \geq 0 \tag{2.7}$$

Then the family $(T(t))t \geq 0$ is a strongly continuous semigroup on X, called the multiplication semigroup, and its generator is given by the multiplication operator

$$Af = q.f$$

with domain $D(A) = \{f \in X : qf \in X\}$.

Example 2.1.7: (Shift Semigroups.) Let X be one of the following Banach spaces:
$C_{ub}(\mathbb{R})$ of all bounded, uniformly continuous functions on $\mathbb{R}$ endowed with the supremum norm $\|.\|_\infty$.
$C_0(\mathbb{R})$ of all coutinuous functions on $\mathbb{R}$ vanishing at infinity endowed with the supremum norm $\|.\|_\infty$.
$L^p(\mathbb{R}), 1 \leq p < \infty$, of all p -integrable functions on $\mathbb{R}$ endowed with the corresponding p -norm $\|.\|_p$.
For $f \in X$ and $t \geq 0$, we call

$$(T_l(t)f)(s) := f(s+t), \quad s \in \mathbb{R}$$

the left shift or translation (of f by t), while

$$(T_r(t)f)(s) := f(s-t), \quad s \in \mathbb{R}$$

is the right shift or translation (of f by t). The families $(T_i(t))_{t\geq 0}$ and $(T_r(t))_{t\geq 0}$ are strongly continuous semigroups on X with generators

$$A_l f = f' \quad \text{and} \quad A_r f = -f',$$

respectively, and domains $D(A_l) = D(A_r) = \{f \in X : f \text{ is differentiable and} f' \in X\}$ if $X = C_{ub}(\mathbb{R})$ or $C_0(\mathbb{R})$, and

$$D(A_l) = D(A_r) = W^{1.p}(\mathbb{R})$$

if $X = L^p(\mathbb{R})$.

In the following we will show some more properties of strongly continuous semigroups. One of the important features is the following proposition.

Proposition 2.1.8(A.Batkai and S.Piazzera p7): For every strongly continuous semigroup $(T(t))_{t\geq 0}$, there exist constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that

$$\|T(t)\| \leq Me^{\omega t}, \text{ for all } t \geq 0 \tag{2.8}$$

By means of Proposition 2.1.8 we can define a very important constant.

Definition 2.1.9(A.Batkai and S.Piazzera p7): Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$. We call :

$$\omega_0(A) := \inf\left\{\omega \in \mathbb{R} : \exists M > 0 \text{ such that } \|T(t)\| \leq Me^{\omega t}, \quad \forall t \geq 0\right\} \tag{2.9}$$

the semigroup's growth bound.

Example 2.1.10: For the uniformly continuous semigroup of Example 2.1.5, we have $\omega_0(A) \leq \|A\|$.
For the multiplication semigroups of Example 2.1.6, we have :
$\omega_0(A) = \sup_{\omega\in\Omega} \Re q(\omega)$ if $X = C_0(\Omega)$,
$\omega_0(A) = \text{ess}\cdot\sup_{\omega\in\Omega} \Re(\omega)$ if $X = L^p(\Omega, \mu)$ $1 \leq p < \infty$.
For the shift semigroup of Example 2.1.7, we have $\omega_0(A) = 0$.

We have seen in Theorem 2.1.4 that the generator $(A, D(A))$ of a strongly continuous semigroup is always a closed operator, therefore by the closed graph theorem, if $(A, D(A))$ is bijective, its inverse becomes a bounded operator on X. This motivates the following definition.

Definition 2.1.11(A.Batkai and S.Piazzera p7):. Let $(A, D(A))$ be a closed operator on a Banach space X. We call the sets

$$\rho(A) := \{\lambda \in \mathbb{C} : \lambda - A \text{ is bijective }\}$$

the resolvent set of A, and

$$\sigma(A) := \mathbb{C}\backslash\rho(A)$$

the spectrum of A, respectively.
For $\lambda \in \rho(A)$, we call $R(\lambda, A) := (\lambda - A)^{-1}$ the resolvent of A at λ.

The following result states that the resolvent of the generator is given by the Laplace transform of the semigroup (at least in a right half plane).

Theorem 2.1.1(A.Batkai and S.Piazzera p8): Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))$ on a Banach space X. Then the following properties hold :
(i) For every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega_0(A)$, we have $\lambda \in \rho(A)$ and

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda x}T(s)xds, \text{ for all } x \in X \tag{2.10}$$

(ii) If $\lambda \in \mathbb{C}$ such that $R(\lambda)x := \int_0^\infty e^{-\lambda s}T(s)xds$ exists for all $x \in X$

$$\text{then } \lambda \in \rho(A) \text{ and } R(\lambda, A) = R(\lambda)\ . \tag{2.11}$$

Now we can state the basic theorem in semigroup theory, which characterizes generators of strongly continuous semigroups by means of their resolvents only

2.2 Abstract Cauchy Problems

The aim of this section is to show how to solve abstract (i.e., Banach spacevalued) initial value problems using operator semigroups.

Definition 2.2.1(A.Batkai and S.Piazzera p9): Let X be a Banach space, $A : D(A) \subseteq X \to X$ a linear operator and $x \in X$
(i) The initial value problem

$$\begin{cases} u'(t) = Au(t) & \text{for } t > 0 \\ u(0) = x \end{cases} \tag{2.12}$$

is called the abstract Cauchy problem associated to $(A, D(A))$ with initial value x.
(ii) A function $u : \mathbb{R}_+ \to X$ is called a (classical) solution of (2.12) if u is continuously differentiable, $u(t) \in D(A)$ for all $t \geq 0$, and (2.12) holds.

The following proposition follows from Lemma 2.1.3.

Proposition 2.2.2(A.Batkai and S.Piazzera p10): Let $(A, D(A))$ be the generator of a strongly continues semigroup $(T(t))_{e\blacksquare 0.}$ Then, for every $x \in D(A)$, the function

$$u : t \mapsto u(t) := T(t)x \tag{2.13}$$

is the unique classical solution of (ACP) with initial value x.

Actually, there is no hope to have a classical solution of **(2.12)** if the initial value x is not in $D(A)$. This suggests that more general concepts of "solutions" might be useful.

Definition 2.2.3(A.Batkai and S.Piazzera p10): A continuous function $u : \mathbb{R}_+ \to X$ is called a mild solution of **(2.12)** i f $\int_0^t u(s)ds \in D(A)$ for all $t \geq 0$ and

$$u(t) = x + A \int_0^t u(s)ds \quad \text{for all } t \geq 0 \tag{2.14}$$

We can now generalize Proposition 2.2.2 to mild solutions.

Proposition 2.2.4(A.Batkai and S.Piazzera p10): Let $(A, D(A))$ be the generator of a strongly continumes semigroup $(T(t))_{t\geq 0}$. Then, for every $x \in X$, the function

$$u : t \mapsto u(t) := T(t)x$$

is the unique mild solution of **(2.12)** with initial value x. So far,
if $(A, D(A))$ generates a semigroup, we have existence and uniqueness of solutions of **(2.12)**. In addition, we can characterize generators of semigroups by means ofthe associated abstract Cauchy problem.

Definition 2.2.5(A.Batkai and S.Piazzera p10) : The abstract Cauchy problem

$$\begin{cases} u'(t) = Au(t) & \text{for } t > 0 \\ u(0) = x \end{cases}$$

associated to an operator $A : D(A) \subseteq X \to X$ is called well-posed if the domain $D(A)$ is dense in X, for every $x \in D(A)$ there exists a unique classical solution u_x of (2,12), and for every sequence $(x_n)_{n\in\mathbb{N}}$ in $D(A)$ satisfying $\lim_{n\to\infty} x_n = 0$ one has $\lim_{n\to\infty} u_{x_n}(t) = 0$ uniformly for all t in compact intervals $[0, T]$.

We are now ready to state the main result of this section.

Theorem 2.2.6(A.Batkai and S.Piazzera p11) : For a closed operator $A : D(A) \subseteq X \to X$, the associated abstract Cauchy problem **(2.12)** is well-posed if and only if $(A, D(A))$ generates a strongly continuous semigroup on X. Therefore, to solve an abstract Cauchy problem means to show that the operator $(A, D(A))$ generates a strongly continuous semigroup

2.2.1 Contraction semigroups and their generators

Definition 2.3.1(Casenave and Haraux p39): A one-parameter family $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ is a contraction semigroup on X provided that
(i) $\|T(t)\| \leq 1$ for all $t \geq 0$
(ii) $T(0) = I$
(iii) $T(t+s) = T(t)T(s)$ for all $s, t \geq 0$
(iv) for all $x \in X$, the function $t \mapsto T(t)x$ belongs to $C([0,\infty), X)$

Definition 2.3.2(Casenave and Haraux p39): The generator of $(T(t))_{t\geq 0}$ is the linear operator L defined by

$$D(L) = \left\{ x \in X; \frac{T(t)x - x}{h} \text{ has a limit in } X \text{ as } h \downarrow 0 \right\}$$

and

$$Lx = \lim_{h,0} \frac{T(t)x - x}{h}, \text{ for all } x \in D(L)$$

The following proposition justifies the introduction of m-dissipative operators In Chapter 1

Proposition 2.3.3(Casenave and Haraux p39): Let $(T(t))_{t\geq 0}$ be a contraction semigroup in X and let L be its generator. Then L is m -dissipative and $D(L)$ is dense in X.

Proof: We proceed in three steps.

Step 1. L is dissipative. For all $x \in D(L), \lambda > 0$, and $h > 0$, we have

$$\left\| x - \lambda \frac{T(h)x - x}{h} \right\| \geq \left\| \left(1 + \frac{\lambda}{h}\right) x \right\| - \frac{\lambda}{h} \|T(h)x\| \geq \|x\|$$

hence the result, letting $h \perp 0$.

Step 2. L is m -dissipative. We define the operator J by

$$Jx = \int_0^\infty e^{-t} T(t) x dt$$

for all $x \in X$. It is clear that $J \in \mathcal{L}(X)$, with $\|J\| \leq 1$. For $x \in X$ and $h > 0$ we have

$$\begin{aligned}
\frac{T(h)-I}{h} Jx &= \frac{1}{h}\int_0^\infty e^{-t}(T(t+h)x - T(t)x)dt \\
&= \frac{1}{h}\int_h^\infty e^{-(t-h)}T(t)xdt - \frac{1}{h}\int_0^\infty e^{-t}T(t)xdt \\
&= \frac{e^h-1}{h}\int_0^\infty e^{-t}T(t)xdt - \frac{e^h}{h}\int_0^h e^{-t}T(t)xdt
\end{aligned}$$

Letting $h \searrow 0$, we obtain

$$\lim_{h\searrow 0} \frac{T(h)-I}{h} Jx = Jx - x$$

and so $Jx \in D(L)$, with $LJx = Jx - x$, i.e. $Jx - LJx = x$.

Step 3. For all $x \in X$ and $t > 0$, we set

$$x_t = \frac{1}{t}\int_0^t T(s)xds$$

It is clear that $x_t \longrightarrow x$ as $t \downarrow 0$.

To show that $D(L)$ is dense, it suffices to prove that $x_t \in D(L)$, for all $t > 0$. Now we have, for all $h > 0$

$$\begin{aligned}
t\frac{T(h)-I}{h} x_t &= \frac{1}{h}\int_h^{t+h} T(s)xds - \frac{1}{h}\int_0^t T(s)xds \\
&= \frac{1}{h}\int_t^{t+h} T(s)xds - \frac{1}{h}\int_0^h T(s)xds
\end{aligned}$$

As $h \searrow 0$, the cerm on the right-lund side converges to $T(t)x - x$, and so $x_t \in D(L)$ with $tLx_l = T(t)x - x$.

2.3 The Hille-Yosida-Phillips Theorem

Theorem 2.4.1(Casenave and Haraux p33): For all $x \in X$, the sequence $u_\lambda(t) = T_\lambda(t)x$ converges uniformly on bounded intervals of $[0, T]$ to a function $u \in C([0,\infty), X)$, as $\lambda \downarrow 0$.

We set $T(t)x = u(t)$, for all $x \in X$ and $t \geq 0$. Then

$$\begin{aligned}
&T(t) \in \mathcal{L}(X) \text{ and } \|T(t)\| \leq 1, \quad \forall t \geq 0; \\
&T(0) = I_i \\
&T(t+s) = T(t)T(s), \ \forall s,t \geq 0
\end{aligned}$$

In addition, for all $x \in D(A), u(t) = T(t)x$ is the unique solution of the problem

$$\begin{cases} u \in C([0,\infty), D(A)) \cap C^1([0,\infty), X) : \\ u'(t) = Au(t), \quad \downarrow t \geq 0 \\ u(0) = x \end{cases}$$

Finally,

$$T(t)Ax = AT(t)x$$

for all $x \in D(A)$ and $t \geq 0$

Theorem 2.4.2(Casenave and Haraux p37) : Assume that A is a skew-adjoint operator. Then $(T(t))_{t\geq 0}$ can be extended to a one-parameter group $T(t) : \mathbb{R} \to \mathcal{L}(X)$ such that

$$\begin{aligned} & T(t)x \in C(R, X), \quad \forall x \in X \\ & \|T(t)x\| = \|x\|_1 \quad \forall x \in X, t \in \mathbb{R} \\ & T(0) = I \\ & T(s+t) = T(s)T(t), \quad \forall s, t \in R \end{aligned}$$

In addition, for all $x \in D(A), u(t) = T(t)x$ satisfies $u \in C(\mathbb{R}, D(A)) \cap C^1(R, X)$ and

$$u'(t) = Au(t)$$

for all $t \in \mathbb{R}$.

Remark 3.4.2: The conclusions of Theorem 2.4.2 may be satisficd without assuming that A is skew-adjoint. Indeed, it suffices that A and $-A$ are m -dissipative.

Theorem 2.4.3(Hille-Yosida-Phillips Theorem): A linear operator A is the generator of a contraction semigroup in X if and only if A is m -dissipative With dense domain.

Proof: If A is the generator of a contraction semigroup in X, Proposition 2.3 .3 shows that A is m -dissipative with dense domain. Conversely, assume that A is m -dissipative with dense domain, and let $(T(t))_{t\geq 0}$ be the semigroup corresponding to A given by Theorem 2.4.1. Then, $(T(t))_{t\geq 0}$ is clearly a contraction semigroup. Denote its generator by L and let us show that $L = A$.
For all $x \in D(A)$ and $h > 0$, we have (Theorem 2.4 .1)

$$T(h)x = x + \int_0^t T(s)Axds_1$$

and so $x \in D(L)$ with $Lu = Au$, Consequently, $G(A) \subset G(L)$.
Finally, let $y \in D(L)$: Since A is m -dissipative, there exists $x \in D(A)$ such that $x - Ax = y - Ly$; and since $G(A) \subset G(L)$, we have $(x-y) - L(x-y) = 0$ L being dissipative, we have $x = y$, and so $G(L) \subset G(A)$, It follows that $A = L_1$ which completes the proof.

The following result show the uniqueness of the semigroup generated by an m-disalpatlve operator with dense domain.

Proposition 2.4.2 : Let A be an m -dissipative operator with dense domain. Assume that A is the generator of a contraction semigroup $(S(t))_{t\geq 0}$. Then $(S(t))_i \geq 0$ is the semigroup correspouding to A given by Theorem 2.4.1.

Proof. Let $(T(t))_{t\geq 0}$ be the semigroup corresponding to A given by Theorem 2.4.1. Let $x \in D(A)$, and $u(t) = S(t)x$. For all $t \geq 0$ and $h > 0$, we have

$$\frac{u(t+h) - u(t)}{h} = \frac{S(h) - I}{h} u(t) = S(t) \frac{S(h)x - x}{h} \to S(t)Ax \quad \text{as} \quad h \downarrow 0$$

We deduce that $S(t)x \in D(A)$, for 'all $t \geq 0$, and that

$$AS(t)x = S(t)Ax = \frac{d^+u}{dt}(t)$$

for all $t \geq 0$. Thus $u \in C([0,\infty), D(A)) \cap C^1([0,\infty), X)$ and $u'(t) = Au(t)$, for $t \geq 0$. Therefore, by Theorem 2.1.1, we have $S(t)x = T(t)x$; hence the result, by density.

The following definition is related to Theorem 2.4.2.

Definition 2.4.3 : A one-parameter family $(T(t))$ of linear operators is said to be an isometry group in X provided that
(i) $\|T(t)x\| = \|x\|$ for all $x \in X$ and all $t \in \mathbb{R}_i$
(ii) $T(0) = I$
(iii) $T(t+s) = T(t)T(s)$ for all $s, t \in \mathbb{R}$
(iv) for all $x \in X$ the function $t \mapsto T(t)x$ belongs to $C(\mathbb{R}, X)$.

Further to Theorem 2.4.2 and Remark 2.4.3 we have the following result.

Proposition 2.4.4 : Let A be an m -dissipative operator with dense domain. A is the restriction to $\mathbb{R}_+$ of an isometry group if and only if $-A$ is m -dicsipstive.

Proof. It is clear by Theorem 2.4.2 and Remark 2.4.3 that the condition $-A$ is m-dissipative is sufficient. Assume that $(T(t))_{\mathrm{t}\geq 0} \geq 0$ is the restriction to $\mathbb{R}_+$ of an isometry group $(T(t))_{t\in\mathcal{R}}$, and set $U(t) = T(-t)$, for $t \geq 0$. Then $(U(t))_{t\geq 0}$ is a contraction semigroup. Let B be its generator. For all $h > 0$ and $x \in X$, we have

$$\frac{U(h) - I}{h} x = \frac{T(-h) - I}{h} x = -U(h) \frac{T(h) - I}{h} x$$

We deduce immediately that $B = -A$; hence the result.

Chapter 3

Existence and Uniqueness of Solutions to a Semilinear Evolution equations

The aim of this chapter is to give some results about the existence and uniqueness of mild, strong, and classical solutions of a nonlocal Cauchy problem for a semilinear evolution equation.
The method of semigroups and the Banach theorem aboul the fixed point are used to prove the existence and uniqueness of solutions of the problem considered.

The semilinear evolution equation considered here is of the form :

$$\frac{du(t)}{dt} = Au(t) + f(t, u(t)), \quad t \in]0, T] \tag{3.1}$$

$$u(0) = u_0 \tag{3.2}$$

where $T \in \mathbb{R}_+^*$, is a constant.

We assume that :

(H1) A is the generator of a C_0 semigroup $S = \{S(t)\}_{t \geqslant 0}$, on a Banach space $(X, \|\cdot\|)$.
(H2) $f : [0, T] \times X \longrightarrow X$ is a continuous map in t and uniformly lipschitzian over $[0, T] \times X$; namely, there exists a constant $L \in \mathbb{R}_+^*$ such that :

$$\|f(t, u) - f(t, v)\| \leq L \|u - v\|, \text{ for all } u, v \in X.$$

(H3) $u_0 \in X$.

Then, we have the following result :

Therorem 3.1. Under the assumptions (H1)-(H2)-(H3), The problem (3.1)-(3.2) has a unique mild solution $u \in \mathcal{C}([0, T]; X)$.

Proof. A mild solution u is a continuous map $u : [0, T] \longrightarrow X$ satisfies the integral equation :

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s, u(s))ds \tag{3.3}$$

Consider the map $F : \mathcal{C}([0,T];X) \longrightarrow \mathcal{C}([0,T];X)$ defined by :

$$F(u)(t) = S(t)u_0 + \int_0^t S(t-s)f(s,u(s))ds \tag{3.4}$$

where $\mathcal{C}([0,T];X)$ is the Banach space of continuous maps $u : [0,T] \longrightarrow X$ endowed with the norm :

$$\|u\|_\infty = \sup_{t\in[0,T]} \|u(t)\| \tag{3.5}$$

In the sequel we denote by $F(u) = Fu$ and $Fu(t) = (F(u))(t)$.

We have for any two elements $u, v \in \mathcal{C}([0,T];X)$:

$$\begin{aligned} \|Fu(t) - Fv(t)\| &\leq \int_0^t \|S(t-s)\| \, \|f(s,u(s)) - f(s,v(s))\| \, ds \\ &\leq \sup_{t\in[0,T]} \|S(t)\| \int_0^t L \, \|u(s) - v(s)\| \, ds \\ &\leq MLt \sup_{t\in[0,T]} \|u(s) - v(s)\| \end{aligned}$$

Hence

$$\|F(u)(t) - F(v)(t)\| \leq ML \, \|u - v\|_\infty \, t, \text{ for all } t \in [0,T] \tag{3.6}$$

where :

$$M = \sup_{t\in[0,T]} \|S(t)\| \tag{3.7}$$

and

$$\|u - v\|_\infty = \sup_{t\in[0,T]} \|u(s) - v(s)\| \tag{3.8}$$

Let $F^2 = F \circ F$ (F round F) and $F^n = F \circ F \circ \cdots \circ F$ (n times), then by (H2), (3.7) and (3.8) we have :

$$\begin{aligned} \left\|F^2u(t) - F^2v(t)\right\| &= \|F(Fu)(t) - F(Fv)(t)\| \\ &= \left\| \int_0^t S(T-s) \left[f(s,Fu(s)) - f(s,Fv(s))\right] ds \right\| \\ &\leq ML \int_0^t \|Fu(s) - Fv(s)\| \, ds \end{aligned} \tag{3.9}$$

From (3.6) and (3.9) we get :

$$\begin{aligned} \left\|F^2u(t) - F^2v(t)\right\| &\leq ML \int_0^t ML \, \|u - v\|_\infty \, sds \\ &\leq \frac{(ML)^2}{2} \|u - v\|_\infty \, t^2 \\ &\leq \frac{(MLt)^2}{2} \|u - v\|_\infty \end{aligned} \tag{3.10}$$

By induction on n it follows that

$$\|F^n u(t) - F^n v(t)\| \leq \frac{(MLt)^n}{n!} \|u - v\|_\infty \,, \text{ for all } t \in [0,T]\,. \tag{3.11}$$

and hence

$$\|F^n u - F^n v\|_\infty \leq \frac{(MLT)^n}{n!} \|u - v\|_\infty \,. \tag{3.12}$$

As $\lim\limits_{n\to\infty} \frac{(MLT)^n}{n!} = 0$, then for n large enough we will have $\frac{(MLT)^n}{n!} = K < 1$.
By the well known fixed point principle, the map F^n has a unique fixed point $u \in ([0,T]\,;X)$, that's to say :

$$F^n u = u \tag{3.13}$$

Let $G = F^n$. As $G \circ F = F \circ G = F^{n+1}$, then from 3.13 we get :

$$G(Fu) = F(Gu) = Fu \tag{3.14}$$

Hence Fu is a fixed point of G, however G has a unique point u, then :

$$Fu = u \tag{3.15}$$

that's to say u is a fixed point of F.

Let's prove that u is the unique fixed point of F : Assume that $v \in \mathcal{C}([0,T]\,;X)$ is an other fixed point of F, hence :

$$F(v) = v \tag{3.16}$$

We have from 3.16 and 3.15 :

$$\begin{aligned} G(v) &= F^n v = F^{n-1}(Fv) = F^{n-1} v \\ &= F^{n-2}(Fv) = F^{n-2} v = \cdots \\ &= Fv = v \end{aligned} \tag{3.17}$$

that's to say v is a fixed point of G, but G has a unique fixed point u, then $v = u$.

Now, if we have solely the following slightly less weak condition on the function f :

(H2)' $f : [0,\infty[\times X \longrightarrow X$ is continuous and and uniformly lipschitzian in u on a bounded interval $[0,T]$, $T \in \mathbb{R}_+^*$, that's, there exists a consant $L = L(T) \in \mathbb{R}_+$ such that :

$$\|f(t,u) - f(t,v)\| \leq L(T)\,\|u - v\|\,, \text{ for all } t \in [0,T] \text{ and all } u, v \in X$$

Then we the following result of local existence.

Theorem 3.2. (Goldstein p 87) Under the assumptions (H1)-(H2)'-(H3)**,** The problem (3.1)-(3.2) has a unique mild local solution $u \in \mathcal{C}([0,T']\,;X)$, where $0 < T' \leq T$.

Proof. Define $F : \mathcal{C}([0,T];X) \longrightarrow \mathcal{C}([0,T];X)$ by :

$$F(u)(t) = S(t)u_0 + \int_0^t S(t-s)f(s,u(s))ds,\ t \in [0,T] \tag{3.4}$$

and

$$Q = \{v \in \mathcal{C}([0,T];X) : v(0) = u_0\} \tag{3.18}$$

It's easy to verify that M is a complete metric space.
As A is the generator of a C_0 semigroup S then there exist two real constants $M \geq 1$ and $\omega \geq 0$ such that :

$$\|S(t)\| \leq Me^{\omega t},\ \text{for all}\ t \geq 0. \tag{3.19}$$

Consequently :

$$\begin{aligned} \|Fu(t) - Fv(t)\| &\leq \int_0^t \|S(t-s)\| \, \|f(s,u(s)) - f(s,v(s))\| \, ds \\ & \quad Me^{\omega t}\int_0^t L(T)\, \|u(s) - v(s)\| \, ds \\ &\leq Me^{\omega t} tL(T)\, \|u - v\|_\infty,\ \text{for all}\ t \in [0,\tau] \end{aligned} \tag{3.20}$$

where $\|u\|_\infty = \sup_{s\in[0,T]} \|u(s)\|$. Hence :

$$\|Fu - Fv\|_\infty \leq Me^{\omega\tau} TL(\tau)\, \|u - v\|_\infty \tag{3.21}$$

As $\lim_{T\to 0^+} Me^{\omega\tau}TL(T) = 0$ and $F(Q) \subset Q$ for T sufficiently small, then for $T = T'$ sufficiently small, F is a contaction mapping from Q into Q, and by the fixed point principle, the mapping F has a unique fixed point which is the unique mild solution.

If in the condition (H2)' is satified for every $T \in \mathbb{R}_+^*$:

(H2)" $f : [0,\infty[\times X \longrightarrow X$ is continuous and and uniformly lipschitzian in u on bounded intervals, that's, ***for every*** $T \in \mathbb{R}_+$ there exists a consant $L = L(T) \in \mathbb{R}_+$ such that :

$$\|f(t,u) - f(t,v)\| \leq L(T)\, \|u - v\|,\ \text{for all}\ t \in [0,T]\ \text{ and all}\ u,v \in X$$

Then we the following result of gocal existence.

Theorem 3.3. (Global existence) Under the assumptions (H1)-(H2)"-(H3)**,** The problem (3.1)-(3.2) has a unique mild global solution $u \in \mathcal{C}([0,\infty[;X)$.

Proof. The local existence is provided by the theorem 3.2.
For an arbitrary $T \in \mathbb{R}_+$ we have from theorem 3.1 : The problem (3.1)-(3.2) has a unique mild solution $u \in \mathcal{C}([0,T];X)$. As T is arbitrary, then the solution exists and it is defined over $[0,\infty[$.

Chapter 4

Applications

In this section we treat some concrete equations.

4.1 Semilinear equations on unbounded domains

Consider the following semilinear equation on unbounded domains

$$Au \equiv u_t - au_{xx} - bu_x = pu + \varphi u \equiv f(u), \quad x \in \mathbb{R}, \ t > 0 \tag{4.1}$$

$$u(x,0) = u_0(x), \ \ x \in \mathbb{R} \tag{4.2}$$

Here u is the unknown function in this equation.
(H1) The coefficient of diffusion a is in $\mathbb{R}_+^*$.
(H2) The coefficient b and p are in $\mathbb{R}^*$, they are called the coefficients of evolution.
(H3) $\varphi : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ $((x,t) \longmapsto \varphi(x,t))$ is a given mapping such that, for every fixed $t \in \mathbb{R}^+$, $\varphi_t : \mathbb{R} \longrightarrow \mathbb{R}$ $(x \longmapsto \varphi_t(x) = \varphi(x,t))$ is a uniformly continuous mapping on $\mathbb{R}$, and $t \longmapsto \|\varphi(t)\| = \sup_{x\in\mathbb{R}} |\varphi(x,t)|$ is bounded on every bounded interval of time $[0,T]$.
(H4) $u_0 : \mathbb{R} \longrightarrow \mathbb{R}$ is a given bounded and uniformly continuous mapping on $\mathbb{R}$.

We seek for solutions in the space $X = BUC(\mathbb{R})$ of bounded and uniformly continuous functions on $\mathbb{R}$ equipped with the norm :

$$\|u\| = \sup_{x\in\mathbb{R}} |u(x)| \tag{4.3}$$

Theorem 4.1. Under the assumptions (H1)-(H2)-(H3)-(H4), the problem (4.1)-(4.2) has a unique global mild solution.

Proof. We can prove by using Fourier transformation that the operator $Au = au_{xx} + bu_x$ defined on the domain

$$D(A) = \{u \in X : u', \ u'' \in X\} \tag{4.4}$$

generates a C_0-semigroup of contractions $S = \{S(t)\}_{t\geq 0}$ on the Banach space X given explicitly by the expression :

$$[S(t)u](x) = \frac{1}{\sqrt{4\pi at}}\int_{-\infty}^{+\infty}\left[\exp\left(-\frac{|x+bt-\xi|^2}{4at}\right)\right]u(\xi)d\xi \tag{4.5}$$

$$S(0) = I \tag{4.6}$$

where I is the identity operator.

Note that :

$$\varphi,\ u \in X \Longrightarrow \varphi u \in X \tag{4.7}$$

According to theorem 3.3, it suffices to chek the validity of the assumption (H2)'.

$$\begin{aligned} \|f(t,u) - f(t,v)\| &\leq \|pu + \varphi u - (pv + \varphi v)\| \\ &\leq |p|\,\|u-v\| + \|\varphi(t)\|\,\|u-v\| \\ &\leq L(T)\,\|u-v\| \end{aligned} \tag{4.8}$$

where $L(T) = |p| + \sup\limits_{t\in[0,T]} \|\varphi(t)\|$.

Hence, the problem (4.1)-(4.2) has a unique local solution $u \in \mathcal{C}([0,\infty[\,;X)$.

4.2 Semilinear equations on bounded domains

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz continuous boundary and $T \in \mathbb{R}_+^*$ given positive real number. Consider the following semilnear parabolic equation :

$$u_t(x,t) = \Delta u(x,t) + f(t,u(x,t)),\ x\in\Omega,\ t\in\,]0,T] \tag{4.9}$$

$$u(x,t) = 0,\ x\in\partial\Omega,\ t\in\,]0,T] \tag{4.10}$$

$$u(x,0) = u_0(x) \tag{4.11}$$

We look for solutions u defined on $[0,T]$ which values are in $L^2(\Omega)$.

We assume :

(H4) $f : L^2(\Omega) \longrightarrow L^2(\Omega)$ is a Lipschitz continuous function on bounded subsets of $L^2(\Omega)$, namely, for all $M \subset L^2(\Omega)$ bounded subset, there exists a constant $\alpha = \alpha(M) \in \mathbb{R}_+^*$ such that :

$$\|f(t,u) - f(t,v)\|_2 \leq \alpha\,\|u-v\|_2\,,\text{ for all } u,v\in M$$

(H5) $u_0 \in .L^2(\Omega)$.

Define the linear operator :

$$A : L^2(\Omega) \longrightarrow L^2(\Omega) \tag{4.12}$$

by :

$$D(A) = \left\{u \in H_0^1(\Omega) : Au \in L^2(\Omega)\right\} \tag{4.13}$$

$$Au = \Delta u \tag{4.14}$$

where the derivation in $\Delta u = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ is taken in the distributions sens.

Theorem 4.2. The operator A is the generator of a contraction C_0-semigroup on the space $L^2(\Omega)$.

Proof. Applying the Hill-Yosida theorem.
(i) We notice that $C_0^\infty(\Omega) \subset D(A)$ and as $\overline{C_0^\infty(\Omega)} = L^2(\Omega)$, then $\overline{D(A)} = L^2(\Omega)$.

(ii) Proving that $]0,\infty[\subset \rho(A)$ and $\|R_\lambda(A)\| = \left\|(\lambda I - A)^{-1}\right\| \leq \frac{1}{\lambda}$, for all $\lambda > 0$:
Let $g \in L^2(\Omega)$, we seek to a function $u \in D(A)$ such that : $\lambda u - Au = g$. In this caase we write $u = R_\lambda(A)g$.
The varational formulation of the problem :

$$\lambda u - Au = g,\ u \in D(A) \tag{4.15}$$

is :

$$b(u,v) = l(v),\ \text{for all}\ \ v \in H_0^1(\Omega) \tag{4.16}$$

where :

$$b(u,v) = \lambda \langle u,v\rangle_{H_0^1(\Omega)} + \int_\Omega \nabla u.\nabla v dx,\ l(v) = \int_\Omega g v dx \tag{4.17}$$

We can easly verify that :
- $l(.)$ is a continuous linear form over $H_0^1(\Omega)$.
- $b(.,.)$ is a bilinear continuous form over $H_0^1(\Omega)$.

Also, $b(.,.)$ is coercive ($H_0^1(\Omega)$-elliptic) form over $H_0^1(\Omega)$ due to the follwing estimate :

$$b(v,v) \geq \min\{1,\lambda\} \|v\|^2_{H_0^1(\Omega)},\ \text{for all}\ \ \lambda > 0 \tag{4.18}$$

Then, by Lax-Milgram theorem, the equation (4.16) has a unique solution u.
Take $v = u$ (u is the solution) in (4.16) and since $\int_\Omega \nabla u.\nabla v dx = \sum_{j=1}^{n} \int_\Omega \left(\frac{\partial^2 u}{\partial x_j^2}\right)^2 dx \geq 0$
we obtain :

$$\lambda \|u\|_2^2 \leq \|g\|_2 . \|u\|_2$$

therefore :

$$\|u\|_2 \leq \frac{1}{\lambda} \|g\|_2 \tag{4.19}$$

that's to say R_λ exists and $\|R_\lambda\| \leq \frac{1}{\lambda}$.
It is clear that A is closed.
Hence, A generates a contraction C_0-semigroup S on $L^2(\Omega)$.

Gronwall's Inequality. (C. Corduneanu p.14)
Let the inequality

$$x(t) \leq h(t) + \int_{t_0}^{t} k(s)x(s)ds,\ \text{for all}\ \ t \in [t_0, T[\tag{4.20}$$

where $t_0 \in \mathbb{R}$ and $T \in \left]t_0, \infty\right]$. If $x,\ k \in \mathcal{C}\left([t_0, T[\,, \mathbb{R}\right)$ and $k(.)$ is a nonnegative function on $[t_0, T[$. Then we have :

$$x(t) \leq h(t) + \int_{t_0}^{t} h(s)k(s)\exp\left\{\int_{s}^{t} k(\tau)d\tau\right\} ds, \text{ for all } t \in [t_0, T[\tag{4.21}$$

If, in addition, $h(.)$ is nondecreasing, then

$$x(t) \leq h(t)\exp\left\{\int_{t_0}^{t} k(s)ds\right\}, \text{ for all } t \in [t_0, T[\tag{4.22}$$

The mild solution u of the equation is written as follows :

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(u(s))ds \tag{4.23}$$

Result 1. The problem (4.9)-(4.10)-(4.11) under the assumptions (H4)-(H5) admits a uniqe mild solution on u on $[0, T]$.

Proof. Let $R \in \mathbb{R}_+^*$ and :

$$B(u_0, R) = \left\{u \in L^2(\Omega) : \|u - u_0\|_2 \leq R\right\} \tag{4.24}$$

$$\widehat{B}(u_0, R) = \left\{u \in \mathcal{C}\left([0, T], L^2(\Omega)\right) : \|u - u_0\|_\infty \leq R\right\} \tag{4.25}$$

We define the operator $L : \mathcal{C}\left([0, T], L^2(\Omega)\right) \longrightarrow \mathcal{C}\left([0, T], L^2(\Omega)\right)$ as follows :

$$Lu(t) = S(t)u_0 + \int_0^t S(t-s)f(u(s))ds \tag{4.26}$$

Soit $u,\ v \in \widehat{B}(u_0, R)$, then we have :

$$\begin{aligned} \|Lu(t) - Lv(t)\|_2 &\leq \int_0^t \|f(u(s)) - f(v(s))\|_2\, ds \\ &\leq \alpha\int_0^t \|(u(s) - v(s)\|_2\, ds \end{aligned}$$

Hence

$$\|Lu - Lv\|_\infty \leq \alpha T\, \|u - v\|_\infty \tag{4.27}$$

Also :

$$\begin{aligned} \|Lu(t) - u_0\|_2 &\leq \|S(t)u_0 - u_0\|_2 + \int_0^t \|f(u(s))\|_2\, ds \\ &\leq \|S(t)u_0 - u_0\|_2 + \int_0^t \|f(u(s)) - f(u_0)\|_2\, ds + \int_0^t \|f(u_0)\|_2\, ds \\ &\leq \|S(t)u_0 - u_0\|_2 + \alpha\int_0^t \|u(s) - u_0\|_2\, ds + t\,\|f(u_0)\|_2 \\ &\leq \|S(t)u_0 - u_0\|_2 + T\,\|f(u_0)\|_2 + \alpha t\,\|u - u_0\|_\infty \end{aligned}$$

Hence

$$\|Lu - u_0\|_\infty \leq \|S(.)u_0 - u_0\|_\infty + T\,\|f(u_0)\|_2 + \alpha T\,\|u - u_0\|_\infty \tag{4.28}$$

From (4.27) and (4.28) we get that there exists $T^* \in \mathbb{R}_+^*$ small enough such that $\alpha T = \beta < 1$ and $\|S(.)u_0 - u_0\|_\infty + T\,\|f(u_0)\|_2 + \alpha T\,\|u - u_0\|_\infty \leq R$.
Consequetly :

$$LB(u_0, R) \subset B(u_0, R) \tag{4.29}$$

and :

$$L \text{ is a contractante function from } B(u_0, R) \text{ into itself} \tag{3.30}$$

Then, by the fixed point principle, there exists a unique point $u \in \mathcal{C}\left([0, T^*], L^2(\Omega)\right)$ such that $Lu = u$, i.e :

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(u(s))ds, \text{ for all } t \in [0, T^*]$$

Result 2. (Casenave) We can prove that there exists a function $T : L^2(\Omega) \longrightarrow]0, \infty]$ such that :
For all $u_0 \in L^2(\Omega)$, there exists a unique solution $u \in \mathcal{C}\left([0, T(u_0)[, L^2(\Omega)\right)$ to (4.9)-(4.10)-(4.11) under the hypothesis (H4)-(H5), and we have the following alternatives :
(i) $T(u_0) = \infty$.
(ii) If $T(u_0) < \infty$, then $\lim\limits_{t \to T(u_0)} \|u(t)\|_2 = \infty$.

Result 3. If moreover we have :
(H6) $\|f(t,u)\|_2 \leq k(t)\,\|u\|_2$, for all $u \in L^2(\Omega)$ and all $t \geq 0$, where $k(.)$ is a nonnegative continuous function.
Then, the solution u is global.

Proof. We have from (H6) :

$$\|u(t)\|_2 \leq \|u_0\|_2 + \int_0^t k(s)\,\|u(s)\|_2\,ds \tag{3.31}$$

Then, by Gronwall's inequality we obtain :

$$\|u(t)\|_2 \leq \|u_0\|_2 \exp\left\{\int_0^t k(s)\right\} \equiv \varphi(t) \tag{3.32}$$

As the function φ is continuous, we conclude from the result 2, that the maximal time of existence is ∞.

Bibliography

[1] A.Batkai and S.Piazzera Semigroups for delay equations

[2] T. Casenave and A. Haraux, An introduction to semilinear evolution equations, Clarendon Press. Oxford, 1998.

[3] C. Corduneanu, Principles of Differential and Integral Equations, Chelsea Publishing Company, 2nd Edition, 1977.

[4] J.A. Goldstein-Semigroups of Operators -Theory and Applications, Oxford University Press, 1985.

[5] A. Pazy, Semigroups of linear operators and applications to Partial differential equations, Springer-Verlag, 1983.