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Presented by: **LESLOUS Aymen**

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Stochastic Differential Equations: Controllability and Almost Periodicity

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Publications and Scientific Contributions

The table below summarises all scientific contributions; conference participations and publications; achieved throughout this research work[1].

Conferences:

Activity Title	Institution / Date	Type	Link
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Existence, Uniqueness, and Approximate Controllability: Exploring Nonlinear Neutral Second Differential Equations under Q−Wiener Processes.	Univ-Tizi Ouzo, 20 / 10 / 2024.	International.	
Unraveling Nonlinear Neutral Dynamics: A Study of Controllability in Weak Conditions.	Univ-Batna 2, 18 / 11 / 2024	National.	
An Investigation of Mild Solutions and Relaxed Conditions in Second-Order Stochastic Differential Equations with Q−Wiener Processes.	Univ-Mila, 07 / 12 / 2024.	National.	

Publications:

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Almost Periodic Solutions in Distribution to a Complete Second-Order Neutral SDE with Damped Term [37].	25/ 09 / 2025.	Class A.	
Stochastic Linear-Quadratic Optimal Control for Hyperbolic Evolution Equations with C_0-Group Generators	.. / .. / 2025.	//	//

[1] **Synopsis:** This thesis studies infinite-dimensional SDEs and their optimal control, with emphasis on hyperbolic systems and almost periodicity. It establishes existence and control results for neutral second-order equations in Hilbert spaces and solves the SLQ control problem, with applications to stochastic wave equations.

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♣

This work is dedicated

♣

♣ To my mother and father; the last thing that never betrayed me. ♣
♣

♣ For those who will understand ♣
♣
without ever asking.

♣

Leslous Aymen.

نبذة مختصرة

تُقدِّم هذه الأطروحة إسهاماتٍ جوهرية في نظرية المعادلات التفاضلية التصادفية غير المنتهية البُعد وفي مسائل التحكّم الأمثل المتعلِّقة بها، مع تركيزٍ خاص على الأنظمة من النمط الزائدي والظواهر شبه الدورية. ويقُيم البحث روابط عميقة بين التحليل الدالي والتحليل التصادفي ونظرية التحكّم، مُعالجًا تحدياتٍ مزمنة في دراسة الأنظمة المحكومة بمعادلات تفاضلية جزئية تصادفية.

في الجزء الأول، نُطوِّر إطارًا نظريًا متكاملًا للمعادلات التفاضلية التصادفية المحايدة من الرتبة الثانية في فضاءات هلبرت. نُثبت وجود الحلول المعتدلة (mild solution) ووحدانيتها وشبه دوريتها في التوزيع، لفئة واسعة من هذه المعادلات على خط الأعداد الحقيقية بأكمله. وتعتمد المنهجية على حجج مبتكرة من نظرية النقطة الثابتة في فضاءات مسارات ملائمة، وتُقدِّم تعميمًا جديدًا لمتباينة غرونوال قادرًا على معالجة الالتفافات عبر مجالات زمنية غير منتهية.

أما الإسهام الثاني، فيوفِّر معالجةً كاملة لمسألة التحكّم الأمثل الخطي–التربيعي التصادفي للأنظمة الزائدية ذات الضوضاء المضاعفة، مُتجاوزًا بذلك حدود النظرية الكلاسيكية المقصورة على الأنظمة القطعية. نُثبت حسن الوضع والحدودية ووحدانية حلول معادلة ريكاتي المؤثرة المرتبطة، باستخدام تقنيات أصيلة تجمع بين الحساب الزمني ومتباينة غرونوال–بيهاري المعمّمة.

وتتجلّى فعالية هذه التطورات النظرية في تطبيقها على المعادلات التفاضلية التصادفية شبه الدورية من الرتبة الثانية، وعلى معادلات الموجة ذات الإجبار التصادفي، بما يُبرهن على ملاءمة الإطار المقترَح لقضايا في الفيزياء الرياضية والهندسة. ومن خلال دمج أساليب من نظرية المؤثرات والتحليل التصادفي والتحليل التوافقي، يُقدِّم هذا العمل مجموعة أدوات تحليلية موحَّدة تُسهم في تعميق الفهم للديناميكيات التصادفية غير المنتهية البُعد.

الكَلِمَاتُ المُفْتَاحِيَّـــــةُ: التحليل التصادفي غير المنتهي البُعد؛ المعادلات التفاضلية المحايدة؛ معادلات التطور من الرتبة الثانية؛ شبه الدورية في التوزيع؛ التحكّم الخطي–التربيعي التصادفي؛ الأنظمة الزائدية؛ معادلات ريكاتي المؤثرة؛ الحلول المعتدلة؛ فضاءات هلبرت؛ الاستقرار الأسي.

Abstract

This dissertation presents fundamental contributions to the theory of infinite-dimensional stochastic differential equations and their optimal control, with particular emphasis on hyperbolic-type systems and almost periodic phenomena. The research establishes profound connections among functional analysis, stochastic analysis, and control theory, thereby addressing long-standing challenges in the study of systems governed by stochastic partial differential equations.

In the first part, we develop a comprehensive framework for second-order neutral stochastic differential equations in Hilbert spaces. We establish the existence, uniqueness, and almost periodicity in distribution of mild solutions for a broad class of such equations over the entire real line. The methodology relies on innovative fixed-point arguments in suitable path spaces and introduces a novel generalisation of Grönwall's inequality capable of treating convolutions on unbounded temporal domains.

The second major contribution provides a complete resolution of the stochastic linear–quadratic optimal control problem for hyperbolic systems with multiplicative noise, representing a significant extension beyond classical theory restricted to deterministic systems. We prove the well-posedness, boundedness, and uniqueness of solutions to the associated operator-valued Riccati equation by means of original techniques that combine chronological calculus with a generalised Grönwall–Bihari inequality.

The effectiveness of these theoretical developments is demonstrated through applications to almost periodic second-order stochastic differential equations and stochastic wave equations with random forcing, thereby confirming the relevance of the proposed framework to problems in mathematical physics and engineering. By integrating tools from operator theory, stochastic analysis, and harmonic analysis, this work provides a unified analytical toolkit that advances our understanding of infinite-dimensional stochastic dynamics.

Keywords: Infinite-Dimensional Stochastic Analysis; Neutral Differential Equations; Second-Order Evolution Equations; Almost Periodicity in Distribution; Stochastic Linear–Quadratic Control; Hyperbolic Systems; Operator-Valued Riccati Equations; Mild Solutions; Hilbert Spaces; Exponential Stability.

Résumé

Cette thèse apporte des contributions fondamentales à la théorie des équations différentielles stochastiques en dimension infinie et à leur contrôle optimal, avec un accent particulier sur les systèmes de type hyperbolique et les phénomènes presque périodiques. La recherche établit des liens profonds entre l'analyse fonctionnelle, l'analyse stochastique et la théorie du contrôle, répondant ainsi à des défis persistants dans l'étude des systèmes régis par des équations différentielles partielles stochastiques.

Dans la première partie, nous développons un cadre théorique complet pour les équations différentielles stochastiques neutres du second ordre dans les espaces de Hilbert. Nous établissons l'existence, l'unicité et la presque-périodicité en loi des solutions modérées pour une large classe de ces équations sur l'ensemble de la droite réelle. La méthodologie repose sur des arguments de point fixe innovants dans des espaces de trajectoires appropriés et introduit une nouvelle généralisation de l'inégalité de Gronwall, apte à traiter des convolutions sur des domaines temporels non bornés.

La deuxième contribution majeure propose une solution complète au problème de contrôle optimal linéaire–quadratique stochastique pour les systèmes hyperboliques avec bruit multiplicatif, ce qui constitue une avancée notable au-delà de la théorie classique, largement limitée aux systèmes déterministes. Nous démontrons le caractère bien posé, la bornité et l'unicité des solutions de l'équation de Riccati à valeurs opératorielles associée, en utilisant des techniques originales combinant le calcul chronologique et une inégalité généralisée de type Gronwall–Bihari.

L'efficacité de ces développements théoriques se manifeste dans leurs applications aux équations différentielles stochastiques du second ordre presque périodiques et aux équations des ondes avec forçage stochastique, ce qui illustre la pertinence du cadre proposé pour des problèmes en physique mathématique et en ingénierie. En intégrant des méthodes issues de la théorie des opérateurs, de l'analyse stochastique et de l'analyse harmonique, ce travail fournit une boîte à outils analytique unifiée qui fait progresser notre compréhension des dynamiques stochastiques en dimension infinie.

Mots-Clés: Analyse Stochastique en Dimension Infinie; Équations Différentielles Neutres; Équations d'Évolution du Second Ordre; Presque-Périodicité en Loi; Contrôle Linéaire–Quadratique Stochastique; Systèmes Hyperboliques; Équations de Riccati à Valeurs Opératorielles; Solutions Modérées; Espaces de Hilbert; Stabilité Exponentielle.

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Nomenclature

In the following table, we provide all necessary abbreviations and notations used throughout the thesis.

Symbol	:	Meaning/Definition	Properties/Remarks
Basic Number Systems and Sets			
$\mathbb{R}$	:	Real numbers: Complete ordered field.	$\mathbb{R}_{\geq s_0} = \{s \in \mathbb{R} : s \geq s_0\}$, $\mathbb{R}_{>0}$ positive reals,...
$\mathbb{C}$	:	Complex numbers: $\{a + bi : a, b \in \mathbb{R}, i^2 = -1\}$.	Equipped with modulus $\|z\| = \sqrt{a^2 + b^2}$.
$\mathbb{N}$	:	Natural numbers: $\{1, 2, 3, \ldots\}$.	Sometimes $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ includes zero.
$\mathbb{Z}$	:	Integers: $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$.	Ring structure with addition and multiplication.
$\mathbb{Q}$	:	Rational numbers: $\{p/q : p, q \in \mathbb{Z}, q \neq 0\}$.	Dense subset of $\mathbb{R}$.
Metric and Functional Spaces			
$(\mathbb{S}, d_{\mathbb{S}})$	:	Separable complete metric space.	Complete: every Cauchy sequence converges; separable: countable dense subset.
$\mathbb{X}, \mathbb{Y}$	:	Banach spaces: Complete normed vector spaces.	$\\|x + y\\| \leq \\|x\\| + \\|y\\|$, $\\|\alpha x\\| = \|\alpha\|\\|x\\|$.
$\mathbb{H}, \mathbb{K}, \mathbb{U}$	:	Hilbert spaces with inner products $\langle \cdot, \cdot \rangle$.	$\mathbb{H}$: state space, $\mathbb{K}$: noise space, $\mathbb{U}$: control space.
$\mathbb{V}$	:	Product space $\mathbb{H} \times \mathbb{H}$.	Norm: $\\|(h_1, h_2)^T\\|_{\mathbb{V}} = \\|h_1\\|_{\mathbb{H}} + \\|h_2\\|_{\mathbb{H}}$.
$\mathbb{X}^*$	:	Dual space of $\mathbb{X}$: Continuous linear functionals.	For Hilbert spaces, $\mathbb{H}^* \cong \mathbb{H}$ by Riesz representation.

Continued on next page

Symbol	:	Meaning/Definition	Properties/Remarks
Operator Spaces and Norms			
$\mathcal{L}(\mathbb{X},\mathbb{Y})$	:	Bounded linear operators $A : \mathbb{X} \to \mathbb{Y}$.	Banach space with norm $\|A\| = \sup\{\|Ax\| : \|x\| = 1\}$.
$\|A\|_{\mathcal{L}(\mathbb{X},\mathbb{Y})}$	:	Operator norm: $\sup_{\|x\|_{\mathbb{X}}=1} \|Ax\|_{\mathbb{Y}}$.	Submultiplicative: $\|AB\| \leq \|A\|\|B\|$.
B^*	:	Adjoint operator: $\langle Bx, y\rangle = \langle x, B^* y\rangle$.	For $B \in \mathcal{L}(\mathbb{H},\mathbb{K})$, $B^* \in \mathcal{L}(\mathbb{K},\mathbb{H})$.
I	:	Identity operator: $Ix = x$ for all $x \in \mathbb{X}$.	Bounded, $\|I\| = 1$.
$\mathrm{Dom}(A)$	:	Domain of (unbounded) operator A.	Dense subspace where A is defined.
$\mathcal{L}_2(\mathbb{K}_0,\mathbb{H})$	:	Hilbert-Schmidt operators: $\|\Phi\|_{\mathcal{L}_2}^2 = \sum_{k=1}^{+\infty} \|\Phi e_k\|^2 < \infty$.	For orthonormal basis $\{e_k\}$ of $\mathbb{K}_0$.
$\mathcal{L}_1(\mathbb{H})$	:	Trace-class operators: $\|A\|_{\mathcal{L}_1} = \sum_{k=1}^{+\infty}\langle \lvert A\rvert e_k, e_k\rangle < \infty$.	Nuclear operators with finite trace.
Function Spaces and Analysis			
$\mathrm{C}(\mathbb{S}_1;\mathbb{S}_2)$	:	Continuous functions $f : \mathbb{S}_1 \to \mathbb{S}_2$.	Topology of uniform convergence on compact sets.
$\mathrm{C}_b(\mathbb{R};\mathbb{S})$	:	Bounded continuous functions.	Supremum norm: $\|f\|_\infty = \sup_{s\in\mathbb{R}} \|f(s)\|$.
$\mathcal{V}(h,\varepsilon)$	:	ε-almost periods: $\{\tau : \sup_s d(h(s+\tau), h(s)) < \varepsilon\}$.	For s.
$\mathcal{H}(h)$	:	Hull: closure of $\{h(\cdot+\tau) : \tau \in \mathbb{R}\}$ in C_b.	Compact in uniform topology for almost periodic h.
$\mathrm{BL}(\mathbb{S})$	:	Bounded Lipschitz functions.	Norm: $\|f\|_{\mathrm{BL}} = \|f\|_\infty + \sup_{x\neq y} \frac{\lvert f(x)-f(y)\rvert}{d(x,y)}$.
Probability and Measure Theory			
$(\Omega,\mathscr{F},\mathbb{P})$	:	Probability space: Ω sample space, $\mathscr{F}$ σ-algebra.	Complete: contains all null sets.

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Symbol	:	Meaning/Definition	Properties/Remarks
$\mathcal{B}(\mathbb{S})$	:	Borel σ-algebra: generated by open sets.	For topological space $\mathbb{S}$.
$\mathbb{E}[Y]$	:	Expectation: Bochner integral $\int_\Omega Y(\omega)d\mathbb{P}(\omega)$.	For $\mathbb{S}$-valued random variables
$\mathbb{E}[Y \mid \mathscr{G}]$	:	Conditional expectation given sub-σ-algebra $\mathscr{G}$.	$\mathscr{G}$-measurable, satisfies $\int_A Y d\mathbb{P} = \int_A \mathbb{E}[Y\|\mathscr{G}]d\mathbb{P}$.
$\{\mathscr{F}_s\}_{s\in J}$	:	Filtration: increasing family of σ-algebras.	Usual conditions: right-continuous, complete.
$\mathcal{P}(\mathbb{S})$	:	Borel probability measures on $\mathbb{S}$.	Weak topology: $\mu_n \Rightarrow \mu$ iff $\int f d\mu_n \to \int f d\mu$ for all $f \in C_b$.
d_{BL}	:	Bounded Lipschitz metric: $\sup\{\|\int f d\mu - \int f d\nu\| : \\|f\\|_{\mathrm{BL}} \leq 1\}$.	Metrizes weak convergence on $\mathcal{P}(\mathbb{S})$.
$\mathscr{L}(Y)$	:	Law/distribution: pushforward measure $Y_*\mathbb{P}$ on $\mathbb{S}$.	$\mathscr{L}(Y)(B) = \mathbb{P}(Y^{-1}(B))$ for $B \in \mathcal{B}(\mathbb{S})$.
$\Rightarrow$	:	Weak convergence: $\mu_n \Rightarrow \mu$ in $\mathcal{P}(\mathbb{S})$.	Equivalent to convergence in d_{BL} metric.

Stochastic Processes and Spaces

Symbol	:	Meaning/Definition	Properties/Remarks
$W(s)$	:	Q_{noise}–Wiener processes in $\mathbb{K}$ with covariance operator Q_{noise}.	Gaussian increments, $\mathbb{E}[W(s)W(r)^*] = (s-s_0)\wedge(r-s_0)\cdot Q_{\mathrm{noise}}$.
$\mathbb{K}_0$	:	Cameron-Martin space: $Q_{\mathrm{noise}}^{1/2}(\mathbb{K})$ with $\langle u, v\rangle_0 = \langle Q_{\mathrm{noise}}^{-1/2}u, Q_{\mathrm{noise}}^{-1/2}v\rangle$.	Reproducing kernel Hilbert space of $W(1)$.
$\langle\langle M\rangle\rangle(s)$	:	Operator quadratic variation of martingale M.	$\langle\langle M\rangle\rangle(s)$ is $\mathcal{L}_1(\mathbb{H})$-valued.
$\mathcal{M}^p([s_0,b];\mathbb{H})$	:	Measurable processes with $\mathbb{E}[\sup_{s\in[s_0,b]}\\|Y(s)\\|^p] < \infty$.	Banach space with norm $\\|Y\\|_{\mathcal{M}^p} = (\mathbb{E}[\sup_s\\|Y(s)\\|^p])^{1/p}$.
$\mathcal{M}^p(\mathbb{R};\mathbb{H})$	:	Processes on $\mathbb{R}$ with $\sup_{s\in\mathbb{R}}\mathbb{E}\\|Y(s)\\|^p < \infty$.	Banach space (for stationary or almost periodic processes) with norm $\\|Y\\|_{\mathcal{M}^p} = (\sup_s \mathbb{E}[\\|Y(s)\\|^p])^{1/p}$.

Continued on next page

Symbol	:	Meaning/Definition	Properties/Remarks
Operator Theory and Semigroups			
$\mathcal{T}(s)$	:	C_0-semigroup: strongly continuous one-parameter semigroup.	Generator A with $\frac{d}{ds}\mathcal{T}(s)x = A\mathcal{T}(s)x$ for $x \in \mathrm{Dom}(A)$.
$R(\alpha, A)$	:	Resolvent operator: $(\alpha I - A)^{-1}$ for $\alpha \in \rho(A)$.	Analytic function on resolvent set $\rho(A)$.
$\rho(A)$	:	Resolvent set: $\{\alpha \in \mathbb{C} : (\alpha I - A)^{-1} \in \mathcal{L}(\mathbb{H})\}$.	Open subset of $\mathbb{C}$.
$C(s)$	:	Cosine family: $C(0) = I, C(s+t) + C(s-t) = 2C(s)C(t)$.	Generator A with $\frac{d^2}{ds^2}C(s)x = AC(s)x$.
$S(s)$	:	Sine family: $S(s)x = \int_0^s C(r)x dr$.	Strongly continuous, $S(s)\mathrm{Dom}(A) \subset \mathrm{Dom}(A)$.
$\mathcal{M}(s)$	:	$\mathcal{M}$ family: $\mathcal{M}(s+r) = \mathcal{M}(s)\mathcal{M}(r) + \mathcal{N}(s)\mathcal{M}'(r)$.	$\mathcal{M}(0) = I, \mathcal{M}'(0) = 0\ \mathcal{M}''(0) = A$.
$\mathcal{N}(s)$	:	$\mathcal{N}$ family; $\mathcal{N}(s+r) = \mathcal{M}(s)\mathcal{N}(r) + \mathcal{N}(s)\mathcal{N}'(r)$.	$\mathcal{N}(0) = I, \mathcal{N}'(0) = 0\ \mathcal{N}''(0) = B$.
$R(\alpha; A, B)$	:	Resolvent for pairs: $(\alpha^2 I - \alpha B - A)^{-1}$.	For complete second-order abstract Cauchy problems.
$\mathcal{E}C(s), \mathcal{E}S(s)$	:	Alternative cosine/sine families.	Related to $\mathcal{M}, \mathcal{N}$ –functions via integral relations.
Optimal Control Theory			
$J(s_0, Y; U(\cdot))$	:	Cost functional.	Quadratic cost for LQ control problem.
$P(s)$	:	Riccati operator.	Self-adjoint, nonnegative definite.
$\mathscr{A}$	:	Generator matrix: $\begin{pmatrix} 0 & I \\ A & B \end{pmatrix}$ for second-order systems.	Block operator on product space $\mathbb{H} \times \mathbb{H}$.
$\mathcal{U}(s, s_0)$	:	Evolution family: solution operator for non-autonomous equations.	Two-parameter family, $\mathcal{U}(s,s) = I, \mathcal{U}(s, s_0) = \mathcal{U}(s,r)\mathcal{U}(r, s_0)$.
$\mathrm{O}\exp$	:	Time-ordered exponential: $\mathrm{O}\exp(\int_{s_0}^s \Lambda(r)dr)$.	Dyson series representation for evolution family.

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Symbol	:	Meaning/Definition	Properties/Remarks
U_*	:	Optimal control: minimiser of $J(s_0, Y; U(\cdot))$.	Feedback form: $U_*(s) = -\Upsilon^{-1}(s)(\Gamma_1(s)Y(s) + \Gamma_0(s))$.
$V(s_0, Y)$	:	Value function: $\inf_U J(s_0, Y; U(\cdot))$.	Satisfies Hamilton-Jacobi-Bellman equation.
$\mathscr{H}$	:	Hamiltonian.	Dynamic Programming Principles.
Technical Constants and Parameters			
$L_{\mathcal{T}}$	:	Semigroup growth constant: $\\|\mathcal{T}(s)\\| \leq L_{\mathcal{T}}[\ldots.]$.	For C_0-semigroup estimates.
δ_A	:	Growth rate: $\limsup_{s\to\infty} \frac{\log\\|\mathcal{T}(s)\\|}{s}$.	Spectral bound of generator A.
$L_{(\mathcal{M},\mathcal{N})}$	:	Bound for $\mathcal{M}, \mathcal{N}$ families: $\\|\mathcal{M}(s)\\| + \\|\mathcal{N}(s)\\| \leq L_{(\mathcal{M},\mathcal{N})}[\ldots.]$.	Stability constant for second-order systems.
$K_{p/2}$	:	Burkholder-Davis-Gundy constant $\mathbb{E}[\sup_{s\leq t}\\|M(s)\\|^p] \leq K_{p/2}\mathbb{E}[\langle\langle M\rangle\rangle(t)^{p/2}]$.	For martingale inequalities.
ℓ_p	:	Lipschitz constant for coefficients.	For existence/uniqueness theorems.
C_p	:	Growth constant for coefficients.	Linear growth conditions.
$\beta_{i,p}$	:	Technical constants in estimates.	Appear in Gronwall-type inequalities.
Special Functions and Transforms			
$\Gamma(s)$	:	Gamma function: $\Gamma(s) = \int_0^\infty r^{s-1}e^{-r}dr$ for $\Re(s) > 0$.	Analytic continuation to $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$.
Abbreviations and Acronyms			
SDE	:	Stochastic Differential Equation.	$dY(s) = f(s, Y(s))ds + g(s, Y(s))dW(s)$.
PDE	:	Partial Differential Equation.	Equations involving partial derivatives.
SPDE	:	Stochastic Partial Differential Equation.	PDE with random noise terms.

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Symbol	:	Meaning/Definition	Properties/Remarks
LQ	:	Linear-Quadratic (optimal control).	Linear dynamics, quadratic cost.
SLQ	:	Stochastic Linear-Quadratic.	LQ control with stochastic dynamics.
BDG	:	Burkholder-Davis-Gundy inequality.	Martingale moment inequalities.
HJB	:	Hamilton-Jacobi-Bellman equation.	PDE for value function in control theory.
AP	:	Almost Periodic (functions/processes).	Functions with dense set of almost periods.
a.e.	:	Almost everywhere.	Property holds except on set of measure zero.
a.s.	:	Almost surely.	With probability one.
resp.	:	Respectively.	Used when assigning corresponding items.

Part I.

Introduction and Background

Introduction

The quest to decipher nature's complex tapestry through the immutable language of mathematics represents one of the most profound and enduring intellectual pursuits in human history. From the clockwork regularity of celestial mechanics, governed by the elegant formalism of ordinary differential equations, to the intricate chaotic patterns of turbulent fluid flow, described by the formidable machinery of nonlinear differential equations, mathematical modelling provides the essential framework for quantifying, predicting, and ultimately understanding the temporal evolution of natural phenomena. This doctoral thesis is situated squarely within this grand tradition, specifically addressing the formidable challenges posed by systems whose state is not captured by a finite set of parameters but must be conceived as an element of an infinite-dimensional function space. The diffusion of heat through a solid, the propagation of stress waves in an elastic medium, the fluctuation of asset prices in a financial market, and the coherent control of a quantum particle all these diverse processes find their most sophisticated and unifying expression through the abstract language of differential equations evolving in Banach and Hilbert spaces.

While the classical theory of ordinary and differential equations provides an exceptionally powerful toolkit for a vast array of applications, the most intriguing challenges and the most significant opportunities for theoretical advancement reside in the analysis of infinite-dimensional systems. This fundamental reality necessitates a crucial transition from the classical setting to the abstract framework of differential equations, where the unknown quantity is itself a function and its dynamics are governed by the action of frequently unbounded linear operators acting on sophisticated spaces such as L^p, H^1, and H_0^1. The analytical landscape in this setting is markedly different: well-posedness is no longer guaranteed by the smoothness of coefficients, stability becomes a subtle spectral property, and the very concept of a solution requires careful definition in terms of semigroups or weak formulations.

This thesis positions itself at the fertile confluence of three major streams of modern mathematical inquiry: the general theory of differential equations, the stochastic calculus of processes in infinite dimensions, and the harmonic analysis of almost periodic functions. Its central intellectual mandate is not merely to juxtapose these disciplines but to synthesise them into a novel, unified analytical framework capable of elucidating the long-term behaviour, stability properties, and optimal control of complex dynamical systems subject to both conservative forcing mechanisms and stochastic perturbations. The pursuit of this ambitious mandate has catalysed the development of original theoretical tools and led to the resolution of several long-standing open problems, particularly in the analysis of second-order SDEs and in linear-quadratic control theory for hyperbolic systems. Consequently, this work establishes crucial bridges between abstract operator theory and its concrete applications across a diverse spectrum of fields, from the analysis of random vibrations in aerospace structures to quantum filtering and financial modelling with memory effects.

Historical Context and Theoretical Foundations

The initial value problem, or Cauchy problem, constitutes the fundamental starting point for analysing any deterministic dynamical system. The seminal concept of a well-posed problem, rigorously formulated by Jacques Hadamard [30], demands three essential criteria: the existence of a solution, its uniqueness, and the continuous dependence of that solution on the initial data. In the transition from finite to infinite dimensions, fulfilling these criteria becomes deeply entwined with the spectral theory of linear operators and the theory of operator semigroups. The celebrated Hille–Yosida theorem [31, 63] provides a complete characterisation of well-posedness for first-order abstract Cauchy problems of the form $y'(s) = Ay(s)$ by establishing a fundamental link between the generation of a strongly continuous semigroup (C_0-semigroup), a powerful generalisation of the matrix exponential e^{sA}, and precise resolvent estimates for its infinitesimal generator A. Foundational treatises by Engel and Nagel [23], Goldstein [28], Pazy [47] further solidified this edifice, extending it to integrated semigroups and other generalisations to handle cases where the Hille–Yosida conditions are not fully satisfied.

For second-order equations in time, which are intrinsic to wave propagation, elasticity, and acoustics, the theory becomes markedly more intricate. The case of incomplete second-order equations, typified by $y''(s) = Ay(s)$, is elegantly handled through the theory of strongly continuous cosine and sine operator families, as systematically developed by Fattorini [25, 26, 27], Sova [52], Travis and Webb [55]. However, the comprehensive analysis of complete second-order equations of the form $y''(s) = By'(s) + Ay(s)$, which incorporate essential damping or gyroscopic terms through the operator B, presents profound challenges. As Fattorini [27] astutely noted, a uniformly well-posed Cauchy problem for such equations may surprisingly fail to yield exponentially bounded solutions, thereby obstructing the direct application of conventional Laplace transform techniques. This fundamental limitation necessitated the development of more sophisticated machinery, notably the theory of $\mathcal{M}, \mathcal{N}$–functions developed by Ivanov, Melnikova, and Filinkov [33], Melnikova [40], Melnikova and Filinkov [41, 42, 43], Xiao and Liang [58, 59, 60, 61], Zheng [66]. This framework provides a generalised setting for establishing well-posedness, deriving explicit solution formulae akin to d'Alembert's formula, and obtaining the crucial resolvent estimates that underpin the entire analytical structure.

Parallel to this deterministic narrative runs the powerful thread of stochastic analysis. The intrusion of randomness into physical models, whether through thermal fluctuations, uncertain parameters, or external noise, represents not a mere technical complication but a fundamental feature of real-world systems. The Itô calculus for Hilbert space-valued processes, developed extensively by Da Prato and Zabczyk [19], Grecksch and Tudor [29], provides the essential language for this description. A critical distinction arises between additive noise, which acts as an external disturbance, and multiplicative noise, where the stochastic perturbation itself depends on the current state of the system, leading to far richer and more complex behaviour, such as noise-induced transitions. The study of stochastic differential equations (SDEs) in infinite dimensions, particularly second-order ones, demands a delicate synthesis of semigroup theory, stochastic integration, and functional analysis. A common methodological approach involves reducing a second-order stochastic equation to a first-order system on a product space, see Prato and Zabczyk [48, Ch. 9, p. 181]. However, the efficacy of this method often relies on the exponential stability of the associated cosine family, a condition that fails in many fundamental cases, such as wave equations on unbounded domains or conservative systems, thus revealing a significant gap in the existing theory that requires a direct approach.

A particularly influential framework that bridges deterministic and stochastic control theory is the linear-quadratic (LQ) optimal control paradigm, originally formulated by Kalman [34] for finite-dimensional deterministic systems. By introducing the Riccati equation as a fundamental tool for synthesising optimal feedback laws, this paradigm laid the groundwork for much of contemporary control design. A major extension was achieved by Wonham [57] who generalised the theory to stochastic systems by integrating Itô calculus with operator-theoretic techniques. This advancement established deep connections between backward SDEs (BSDEs), Riccati equations, and optimal control strategies, with applications in aerospace navigation, robotics, and financial engineering, where uncertainty and real-time optimisation are intertwined.

Over the following decades, attention shifted toward extending LQ theory to infinite-dimensional systems, particularly those governed by differential equations (PDEs). Within this context, parabolic systems (such as the heat equation), which generate analytic C_0-semigroups, have dominated the literature, due to their inherent regularisation and dissipative properties, as detailed in Lasiecka and Triggiani [36]. In contrast, hyperbolic systems, including wave equations and structural vibration models, are governed by C_0-group generators and lack such smoothing behavior. This presents significant analytical and computational challenges, particularly in stochastic settings, where the interaction between randomness and non-dissipative dynamics complicates the control design process. For foundational and recent developments, see Benner and Trautwein [4], Bensoussan, Delfour, and Mitter [6], Li and Yong [38], Li, Sun, and Yong [39], Sun and Yong [53], Tuffaha [56].

The third pillar supporting this work is the theory of almost periodic functions, pioneered by Bohr [11] and later refined by Bochner [9]. This elegant theory offers a significant generalisation of periodicity, capturing the notion of functions that are "periodic within an arbitrary tolerance" and providing a powerful tool for analysing the long-time asymptotics of deterministic systems. Extending this concept to the stochastic realm, specifically to the almost periodicity of the distribution of a process, is a particularly innovative step. It permits the classification of the stable, recurrent behavior of systems perpetually driven by non-decaying noise, where sample-path periodicity is too strong a requirement, but statistical regularity is both meaningful and attainable. This extension necessitates working in the metric space of probability measures equipped with the bounded Lipschitz distance, which is more suitable for comparing statistical behaviours and establishing convergence in distribution than stronger metrics like total variation.

Statement of the Problem and Research Objectives

The research programme that motivates this dissertation stems from two intimately connected and significant deficiencies in the existing literature.

First, while the theory of C_0-semigroups and strongly continuous cosine families provides a satisfactory characterisation of many first- and second-order deterministic systems, their ability to capture the dynamics of neutral stochastic systems remains limited. Neutral equations, where the highest-order derivative appears both inside and outside the perturbation terms (e.g., $\mathrm{d}[y'(s) - F(y(s))] = [\ldots]\mathrm{d}s$), model systems with after-effect or memory, such as wave propagation in viscoelastic materials or circuit systems with transmission lines. The delicate question of the existence, uniqueness, and almost periodicity in distribution for neutral SDEs of second order has resisted full resolution, leaving a conspicuous gap between theoretical frameworks and the needs of applied sciences.

Second, the linear-quadratic (LQ) optimal control problem, though classically well understood in deterministic settings, presents formidable difficulties when extended to stochastic hyperbolic systems influenced by multiplicative noise. The classical deterministic LQ problem seeks to minimise a quadratic cost subject to linear system dynamics, with the optimal control computed via the Riccati equation. While Wonham [57] extended this framework to stochastic systems, this extension remained largely confined to finite-dimensional systems with strong regularity assumptions that do not hold for hyperbolic PDEs. Hyperbolic systems, such as the wave equation, exhibit reversible dynamics governed by C_0-group generators and maintain the regularity of initial conditions without the smoothing effects characteristic of parabolic equations. In the presence of stochastic forcing, especially multiplicative noise, the dynamics become particularly challenging due to the lack of dissipativity combined with the infinite-dimensional nature of the state and noise processes, which renders classical LQ methods inapplicable. The absence of a rigorous solution to this problem has not only represented a significant theoretical obstacle but has also constrained progress in areas where control of wave-like phenomena under uncertainty is essential, such as in the active suppression of vibrations in flexible aerospace structures, the stabilisation of smart materials, or the management of energy systems.

The present work addresses these challenges by rigorously formulating and solving the stochastic linear-quadratic (SLQ) control problem for hyperbolic-type systems evolving in separable Hilbert spaces. These systems are influenced by multiplicative noise and are characterised by two-sided, reversible dynamics, in stark contrast to their parabolic counterparts. Our results provide a framework for controlling stochastic hyperbolic differential equations (PDEs), addressing a significant gap in the existing literature.

The primary objectives of this thesis are therefore both ambitious and precise.

1. To establish the existence, uniqueness, and distributional almost periodicity of mild solutions to a broad class of complete second-order neutral SDEs of the form
$$\mathrm{d}[Y'(s) - F(s, Y(s))] = [AY(s) + BY'(s) + G(s, Y(s))]\mathrm{d}s + H(s, Y(s))\mathrm{d}W(s),$$
where the analysis is conducted on the entire real line $\mathbb{R}$, without relying on a reduction to a first-order system. This requires developing new methodological tools within the $\mathcal{M}, \mathcal{N}$-framework to handle stochastic convolutions and neutral terms on an unbounded temporal domain.

2. To deliver the first complete solution to the stochastic linear-quadratic (SLQ) optimal control problem for hyperbolic systems perturbed by multiplicative noise. The system dynamics are given by
$$\mathrm{d}Y(s) = [AY(s) + C(s)U(s) + F(s)]\,\mathrm{d}s + [N(s)Y(s) + D(s)U(s) + H(s)]\,\mathrm{d}W(s),$$
where A generates a C_0-group. This entails proving the well-posedness, boundedness, and uniqueness of the solution to the associated operator-valued Riccati equation and characterising the optimal feedback law that ensures mean-square stability of the closed-loop system. The objective is to show that the corresponding control can be expressed in feedback form, and that the almost periodicity of the coefficients is preserved in the control, implying the control itself is almost periodic in distribution. Furthermore, this preservation of almost periodicity extends to the closed-loop system. To address the problem of exponential stability, we define a weighted cost functional, which exists only if all coefficients are almost periodic. Moreover, the almost periodicity of the control in distribution is necessary to preserve the almost periodicity in distribution of the closed-loop modified system. Based on this control framework, we also provide an application to almost periodic second-order SDEs, establishing a rigorous setting that advances both the theoretical understanding and practical applicability of stochastic control for systems governed by hyperbolic dynamics.

These dual objectives are not pursued in isolation but are integrated into a broader analytical synthesis, contributing to the development of a unified theoretical paradigm for infinite-dimensional stochastic dynamics and control.

Methodological Approach and Original Contributions

The methodological approach of this thesis is characterised by a synthesis of abstract operator theory with advanced probabilistic techniques and harmonic analysis. The core strategy involves

- **A Priori Estimates and Fixed-Point Theory in Path Spaces:** For the neutral stochastic equations, existence and uniqueness are established via a fixed-point argument in a Banach space of bounded stochastic processes, $\mathcal{M}^p(\mathbb{R};\mathbb{H})$. A pivotal technical tool is a novel generalisation of Grönwall's lemma (Lemma 4.1), essential for handling convolutions on an unbounded temporal domain and providing the necessary estimates to apply the contraction mapping principle.

- **Harmonic Analysis in Metric Spaces of Measures:** The proof of almost periodicity in distribution leverages Bochner's double sequence criterion, carefully adapted to functions taking values in the metric space of probability measures equipped with the bounded Lipschitz distance. This requires a meticulous analysis of the continuity of the mapping from the parameter space to the space of measures.
- **Dynamic Programming and Chronological Calculus for Operator Equations:** For the SLQ problem, the analysis proceeds via the dynamic programming principle, leading to an operator-valued Riccati differential equation. The well-posedness of this nonlinear equation is established using a representation of the evolution family via chronological exponentials (time-ordered products) to handle the modified generator $\Lambda = A - BR^{-1}S$, and a new generalised Grönwall–Bihari inequality for operator-valued functions is derived to control the growth of solutions.

We develop a unified theory to address the SLQ problem for hyperbolic systems by combining deterministic PDE control theory with stochastic analysis in Hilbert spaces. Key components of our approach include

1. **Chronological Operator Calculus:** Inspired by quantum field theory, we employ time-ordered exponentials developed in Dyson [22], Sakurai and Napolitano [49] to handle non-commuting operators
$$\mathrm{O}\exp\left(\int_u^s \Lambda(r)\mathrm{d}r\right) := \sum_{n=0}^{\infty} \int_{u\le r_1\le\cdots\le r_n\le s} \prod_{k=1}^{n} \Lambda(r_k)\,\mathrm{d}r_1\cdots\mathrm{d}r_n.$$
2. **Generalised Grönwall–Bihari Inequality:** To handle Volterra-type integral inequalities with operator-valued kernels, we extend the classical Grönwall inequality
$$\|P(s)\|_{\mathcal{L}(\mathbb{H})} \le K(s) + \int_s^b L(r-s)\|\Theta(P)(r)\|_{\mathcal{L}(\mathbb{H})}\mathrm{d}r,$$
deriving conditions under which solutions to the Riccati equation remain bounded and unique.
3. **Semigroup–Stochastic Synthesis:** We integrate C_0-group theory from Engel and Nagel [23] with stochastic calculus in Hilbert spaces developed by Da Prato and Zabczyk [19] to obtain regularity and boundedness of the Riccati equation and adjoint processes, thereby overcoming difficulties introduced by infinite-dimensional noise.

The original contributions of this work are substantial and are detailed in the core chapters.

- Chapter 4, provides the first comprehensive treatment of second-order neutral SDEs on $\mathbb{R}$. Under hypotheses of exponential stability of the $\mathcal{M}, \mathcal{N}$-family generated by (A, B) and Lipschitz conditions on the coefficients, the existence and uniqueness of a bounded mild solution is proven. A principal result establishes that if the coefficients are almost periodic in time, uniformly on bounded sets, then the unique bounded solution inherits the property of almost periodicity in distribution, generalising earlier findings for first-order systems.
- Chapter 5, solves the open problem of the SLQ control for hyperbolic systems with multiplicative noise. The framework is applied to second-order systems by reformulating them as first-order systems on a product space, and the optimal control is characterised in state-feedback form,
$$U_*(s) = -\Upsilon^{-1}(s)(\Gamma_1(s)Y(s) + \Gamma_0(s)),$$
with a proof of the mean-square stability of the resulting closed-loop system.

Thesis Structure

To guide the reader through the logical development of the argument, the thesis is structured into two interwoven parts.

Part I: Mathematical Foundations (Chapters 1–3), provides a self-contained and rigorous exposition of the necessary background material, presented in a hierarchical manner from well-established theory towards more specialised concepts.

Chapter 1: Foundations of Differential Equations, offers a systematic treatment, beginning with first-order equations and the theory of C_0-semigroups, the Hille–Yosida theorem, and the variation of constants formula. It then escalates to incomplete second-order equations ($y'' = Ay$), defining cosine and sine operator families and their properties. The core of the chapter addresses the mathematically rich case of complete second-order equations ($y'' = By' + Ay$), introducing and developing the theory of $\mathcal{M}, \mathcal{N}$–functions generated by a commuting pair (A, B), and establishing the equivalence between δ-uniform well-posedness and the existence of an exponentially bounded $\mathcal{M}, \mathcal{N}$-family.

Chapter 2: Almost Periodic Functions with Values in a Metric Space, presents a detailed study, commencing with the classical Bohr and Bochner definitions and their equivalence (the Bohr–Bochner Theorem). It then innovatively extends these concepts, first to vector-valued functions and then, crucially, to probability-measure-valued functions, defining and characterising almost periodicity in distribution for stochastic processes.

Chapter 3: Foundations of Stochastic Analysis in Infinite Dimensions, provides a comprehensive foundation, systematically building from measure theory on Hilbert spaces through to the Itô calculus for Hilbert space-valued processes. It carefully treats technical aspects such as the Cameron–Martin space, Hilbert–Schmidt operators, and the Burkholder–Davis–Gundy (BDG) inequality, creating the essential probabilistic toolkit for the original research that follows.

Part II: Original Research Contributions (Chapters 4–5), builds upon this foundation to present original research that solves specific, open problems at the interface of stochastic analysis and control theory.

Chapter 4: Almost Periodicity in Distribution for Second-Order Neutral SDEs, presents the first major original contribution: the resolution of the problem for a broad class of neutral SDEs. The analysis involves a two-sided Wiener process and requires novel adaptations of fixed-point theorems and Gronwall-type inequalities in the chosen path space.

Chapter 5: Stochastic Linear-Quadratic Control for Hyperbolic Systems with Multiplicative Noise, presents the second major original contribution: a complete solution to the SLQ optimal control problem. This chapter integrates techniques from chronological calculus, Riccati theory, and stochastic stability to characterise the optimal control and prove the stability of the closed-loop system.

Broader Implications and Concluding Perspectives

The theoretical developments contained within this thesis extend beyond abstract generalisation; they possess direct applications across several scientific and engineering domains. The results on second-order stochastic equations apply to problems involving random vibrations in mechanical structures, stochastic wave propagation in complex media, and financial models with memory effects (e.g., rough volatility). The solutions to the SLQ problem provide a rigorous foundation for the optimal control of systems governed by hyperbolic PDEs, such as the active suppression of noise and vibrations in smart materials or the stabilisation of flexible aerospace structures; applications of considerable practical importance.

By forging deep connections between operator theory, stochastic calculus, and harmonic analysis, this work provides a robust and versatile analytical toolkit that transcends traditional disciplinary boundaries. It offers new perspectives on longstanding problems and opens avenues for investigating even more complex systems, such as those with delay effects, memory phenomena, or network structures. While the scope of this thesis is deliberately bounded to ensure depth and rigour, the methods developed herein are designed to be adaptable. It is the author's sincere hope that this dissertation will serve as a valuable reference and a source of inspiration for future research in the captivating mathematical landscape of infinite-dimensional dynamics, contributing to both theoretical advancement and practical innovation in the modelling and control of complex systems.

Chapter 1.

Foundations of Differential Equations

This chapter develops a comprehensive mathematical framework for analysing the well-posedness of Cauchy problems for differential equations. The exposition proceeds systematically through three hierarchical cases:

1. **First-order equations** of the form $y'(s) = Ay(s)$, where A is typically an unbounded operator. The complete solution theory is provided by the theory of strongly continuous semigroups (C_0-semigroups), which provides an elegant generalisation of the matrix exponential to infinite dimensions. We examine the pivotal Hille-Yosida theorem, which establishes necessary and sufficient conditions for an operator A to generate such a semigroup.

2. **Incomplete second-order equations** of the form $y''(s) = Ay(s)$. These are elegantly handled through the theory of cosine operator families, which extends the semigroup concept to second-order problems.

3. **Complete second-order equations** of the form $y''(s) = By'(s) + Ay(s)$. The theory becomes more intricate here. We introduce and employ the theory of $\mathcal{M}, \mathcal{N}$–functions, a generalised framework that provides the necessary tools to establish well-posedness, including explicit solution formulas and crucial resolvent estimates.

4. **Non-homogeneous cases** involving external forcing terms. The solution is given by variation of constants formulas, representing convolutions of the forcing function with the fundamental solution operators derived for the homogeneous cases.

Upon completing this chapter, the reader will possess a comprehensive set of functional-analytic tools to classify, analyse, and solve linear differential equations in Banach spaces, understanding the profound connections between operator theory, Laplace transforms, and continuous dependence of solutions; the essential characteristics of a well-posed problem.

Let $(\mathbb{S}_1, d_{\mathbb{S}_1})$ and $(\mathbb{S}_2, d_{\mathbb{S}_2})$ be separable and complete metric spaces. When $\mathbb{S}_1$ and $\mathbb{S}_2$ are Banach spaces, we denote them by $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$, respectively.

The space of bounded linear operators from $\mathbb{X}$ to $\mathbb{Y}$ is denoted by $\mathcal{L}(\mathbb{X}, \mathbb{Y})$. If $\mathbb{X} = \mathbb{Y}$, we simply write $\mathcal{L}(\mathbb{X})$.

For $B \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$, its operator norm is defined by

$$\|B\|_{\mathcal{L}(\mathbb{X},\mathbb{Y})} = \sup_{\substack{x\in\mathbb{X}\\ x\neq 0}} \frac{\|Bx\|_{\mathbb{Y}}}{\|x\|_{\mathbb{X}}}.$$

The pair $\big(\mathcal{L}(\mathbb{X}, \mathbb{Y}), \|\cdot\|_{\mathcal{L}(\mathbb{X},\mathbb{Y})}\big)$ is itself a Banach space.

Moreover, for $B \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$, the adjoint operator (or dual operator) is

$$B^* \in \mathcal{L}(\mathbb{Y}^*, \mathbb{X}^*), \qquad \langle B^* h, y\rangle = \langle h, By\rangle, \quad \forall h \in \mathbb{Y}^*,\ x \in \mathbb{X}.$$

Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a Banach space, and let $A : \mathrm{Dom}(A) \subseteq \mathbb{X} \to \mathbb{X}$ be a (possibly unbounded) linear operator with domain $\mathrm{Dom}(A)$.

We say that A is densely defined if

$$\overline{\mathrm{Dom}(A)}^{\|\cdot\|_{\mathbb{X}}} = \mathbb{X},$$

that is, the closure of its domain with respect to the norm topology of $\mathbb{X}$ equals the whole space.

We conclude this section by introducing the function space employed to characterise bounded coefficients.

We consider $C_b([s_0, b]; \mathbb{X})$, the space of functions

$$h : [s_0, b] \to \mathbb{X}$$

satisfying the following properties (cf. Clarke [14]),

- *Bochner measurability:* The function h is the almost everywhere pointwise limit of simple $\mathbb{X}$-valued functions;
- *Boundedness:* There exists a constant $c_0 > 0$ such that
$$\|h(s)\|_{\mathbb{X}} \le c_0 \quad \text{for every } s \in [s_0, b].$$

The space $C_b([s_0, b]; \mathbb{X})$, when endowed with the supremum norm

$$\|h\|_\infty := \|h\|_{C_b([s_0,b];\mathbb{X})} := \sup_{s\in[s_0,b]} \|h(s)\|_{\mathbb{X}},$$

forms a Banach space.

1.1. First-Order Equations

Consider the abstract Cauchy problem in a Banach space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$,

$$\frac{\mathrm{d}}{\mathrm{d}s} y(s) = Ay(s), \quad s \in \mathbb{R}_{\geqslant s_0}, \tag{1.1}$$

$$y(s_0) = y_0, \tag{1.2}$$

where $A : \mathrm{Dom}(A) \subset \mathbb{X} \to \mathbb{X}$ is a linear operator with dense domain $\mathrm{Dom}(A)$ satisfying $\overline{\mathrm{Dom}(A)}^{\|\cdot\|_{\mathbb{X}}} = \mathbb{X}$.

Definition 1.1 (Classical Solution):
A function $y : \mathbb{R}_{\geqslant s_0} \to \mathbb{X}$ is termed a *classical solution* of (1.1)–(1.2) if,

1. $y \in C^1(\mathbb{R}_{\geqslant s_0}; \mathbb{X})$ (continuously differentiable in the strong topology),
2. $y(s) \in \mathrm{Dom}(A)$ for all $s \geq s_0$,
3. The differential equation (1.1) holds pointwise for all $s \geq s_0$,
4. The initial condition (1.2) is satisfied in the strong sense,

$$\lim_{s \to s_0^+} \|y(s) - y_0\|_{\mathbb{X}} = 0.$$

Definition 1.2 (Uniform Well-Posedness):
The Cauchy problem (1.1)–(1.2) is *uniformly well-posed* on a dense subspace $D_A \subseteq \mathrm{Dom}(A)$ if,

(i) **Existence and Uniqueness**: For every $y_0 \in D_A$, there exists a unique classical solution $y(\cdot; y_0)$.

(ii) **Continuous Dependence**: The solution depends continuously on the initial data uniformly on compact time intervals; that is, for every $b > s_0$ and every $\epsilon > 0$, there exists $\upsilon > 0$ such that if $\|y_0 - \tilde{y}_0\|_{\mathbb{X}} < \upsilon$ for $y_0, \tilde{y}_0 \in D_A$, then

$$\sup_{s \in [s_0, b]} \|y(s; y_0) - y(s; \tilde{y}_0)\|_{\mathbb{X}} < \epsilon.$$

1.1.1. Semigroup Theory Framework

The theory of strongly continuous semigroups provides the natural framework for analysing well-posedness of first-order equations.

Definition 1.3 (C_0-Semigroup [23, 47]):
A family $\{\mathcal{T}(s)\}_{s \geqslant 0} \subset \mathcal{L}(\mathbb{X})$ of bounded linear operators constitutes a strongly continuous semigroup (or C_0-semigroup) if,

(S1) **Semigroup property**: $\mathcal{T}(s+r) = \mathcal{T}(s)\mathcal{T}(r)$ for all $s, r \geqslant 0$,

(S2) **Identity property**: $\mathcal{T}(0) = I$ (the identity operator on $\mathbb{X}$),

(S3) **Strong continuity**: $\lim_{s\to 0^+} \mathcal{T}(s)y = y$ for every $y \in \mathbb{X}$.

The *(infinitesimal) generator* A is defined by,

$$Ay = \lim_{r\to 0^+} \frac{\mathcal{T}(r)y - y}{r}, \quad \mathrm{Dom}(A) = \left\{ y \in \mathbb{X} : \lim_{r\to 0^+} \frac{\mathcal{T}(r)y - y}{r} \text{ exists in } \mathbb{X} \right\}.$$

The fundamental properties of C_0-semigroups and their generators are summarised in the following lemma.

Lemma 1.1 (Fundamental Properties [23, 47]):
Let $\{\mathcal{T}(s)\}_{s\geqslant 0}$ be a C_0-semigroup with generator A. Then,

1. The operators $\{\mathcal{T}(s)\}_{s\geqslant 0}$ commute: $\mathcal{T}(s)\mathcal{T}(r) = \mathcal{T}(r)\mathcal{T}(s)$ for all $s, r \geq 0$.
2. There exist constants $L_{\mathcal{T}} \geq 1$ and $\delta := \delta_A \in \mathbb{R}$ such that
$$\|\mathcal{T}(s)\|_{\mathcal{L}(\mathbb{X})} \leq L_{\mathcal{T}} e^{\delta s}, \quad s \geq 0. \tag{1.3}$$
3. The generator A is a closed and densely defined operator: $\overline{\mathrm{Dom}(A)}^{\|\cdot\|_{\mathbb{X}}} = \mathbb{X}$.
4. For every $y \in \mathrm{Dom}(A)$, we have $\mathcal{T}(s)y \in \mathrm{Dom}(A)$ for all $s \geq 0$, and
$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{T}(s)y = A\mathcal{T}(s)y = \mathcal{T}(s)Ay.$$
5. For every $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > \delta$, the resolvent $R(\alpha, A) = (\alpha I - A)^{-1}$ exists as a bounded operator and is given by the Laplace transform formula,
$$R(\alpha, A)y = \int_{s_0}^{\infty} e^{-\alpha(s-s_0)}\mathcal{T}(s)y \,\mathrm{d}s, \quad y \in \mathbb{X}. \tag{1.4}$$

The central theorem connecting semigroups to the Cauchy problem is the following.

Theorem 1.1 (Hille-Yosida Theorem [23, 47]):
Let $A : \mathrm{Dom}(A) \subset \mathbb{X} \to \mathbb{X}$ be a closed, densely defined linear operator. The following statements are equivalent,

1. The Cauchy problem (1.1)–(1.2) is uniformly well-posed on $\mathrm{Dom}(A)$.
2. The operator A is the infinitesimal generator of a C_0-semigroup $\{\mathcal{T}(s)\}_{s\geq 0}$.
3. There exist constants $L_{\mathcal{T}} \geq 1$, $\delta \in \mathbb{R}$ such that for all $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > \delta$ and all $n \in \mathbb{N}_0$,
$$\left\| \frac{\mathrm{d}^n}{\mathrm{d}\alpha^n} R(\alpha, A) \right\|_{\mathcal{L}(\mathbb{X})} \leq L_{\mathcal{T}} \frac{n!}{(\mathrm{Re}(\alpha) - \delta)^{n+1}}. \tag{1.5}$$

In this case, the unique classical solution is given by $y(s) = \mathcal{T}(s - s_0)y_0$ for $y_0 \in \mathrm{Dom}(A)$.

***Remark* 1.1 (Hille-Yosida Condition):**
The case $L_{\mathcal{T}} = 1$ in estimate (1.5) is known as the *Hille-Yosida condition*,

$$\|R(\alpha, A)\|_{\mathcal{L}(\mathbb{X})} \leq \frac{1}{\mathrm{Re}(\alpha) - \delta}, \quad \mathrm{Re}(\alpha) > \delta.$$

This condition is sufficient for the existence of a *contraction semigroup* ($L_{\mathcal{T}} = 1$).

1.1.2. Non-Homogeneous Case

Consider the non-homogeneous Cauchy problem,

$$y'(s) = Ay(s) + h(s), \quad s \in \mathbb{R}_{\geqslant s_0}, \tag{1.6}$$

$$y(s_0) = y_0, \tag{1.7}$$

where $h : \mathbb{R}_{\geqslant s_0} \to \mathbb{X}$ is a given function.

Let D_{Ah} denote the space of functions $h \in C(\mathbb{R}_{\geqslant s_0}; \mathbb{X})$ such that $Ah(s) \in C(\mathbb{R}_{\geqslant s_0}; \mathbb{X})$.

Definition 1.4 (Well-Posedness):
The Cauchy problem (1.6)–(1.7) is *uniformly well-posed on* D_A, D_{Ah} if a solution exists for all $y_0 \in D_A$, $h \in D_{Ah}$, and the solution is unique and depends continuously on the data uniformly on compact time intervals.

The solution is given by the variation of constants formula.

Theorem 1.2 (Mild Solution [23, 47]):
Let A be the generator of a C_0-semigroup $\{\mathcal{T}(s)\}_{s\geq 0}$. If $y_0 \in \mathrm{Dom}(A)$ and $h \in D_{Ah}$, then the function

$$y(s) = \mathcal{T}(s - s_0)y_0 + \int_{s_0}^{s} \mathcal{T}(s - r)h(r)\,\mathrm{d}r, \quad s \geq s_0, \tag{1.8}$$

is the unique (mild) solution of the non-homogeneous Cauchy problem (1.6)–(1.7). Conversely, if $h \in C(\mathbb{R}_{\geqslant s_0}; \mathbb{X})$ and a solution exists, it must be given by this formula.

1.2. Incomplete Second-Order Equations

Consider the second-order abstract Cauchy problem,

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}y(s) = Ay(s), \quad s \in \mathbb{R}_{\geqslant s_0}, \tag{1.9}$$

$$y(s_0) = y_0, \quad y'(s_0) = y_1, \tag{1.10}$$

where $A : \mathrm{Dom}(A) \subset \mathbb{X} \to \mathbb{X}$ with $\overline{\mathrm{Dom}(A)}^{\|\cdot\|_{\mathbb{X}}} = \mathbb{X}$.

The definition of classical solution and well-posedness for second-order equations will be presented in the general case in Definitions 1.6 and 1.7.

The analogous structure to a semigroup for second-order equations is a cosine family.

Definition 1.5 (Cosine Family [27, 55]):
A one-parameter family $\{C(s)\}_{s\in\mathbb{R}} \subset \mathcal{L}(\mathbb{X})$ of bounded operators on $\mathbb{X}$ is called a *strongly continuous cosine family* if,

(C1) **D'Alembert's formula**: $C(s+r) + C(s-r) = 2C(s)C(r)$ for all $s, r \in \mathbb{R}$,

(C2) **Identity property**: $C(0) = I$,

(C3) **Strong continuity**: The mapping $s \mapsto C(s)y$ is continuous for each $y \in \mathbb{X}$.

1. The *infinitesimal generator* A of a cosine family $\{C(s)\}_{s\in\mathbb{R}}$ is defined by
$$Ay = \frac{\mathrm{d}^2}{\mathrm{d}s^2} C(s)y \bigg|_{s=0}, \quad \mathrm{Dom}(A) = \left\{ y \in \mathbb{X} : \lim_{s\to 0} \frac{C(s)y - 2y + C(-s)y}{s^2} \text{ exists} \right\}.$$
2. The associated *sine family* $\{S(s)\}_{s\in\mathbb{R}}$ is defined by
$$S(s)y = \int_0^s C(r)y \,\mathrm{d}r, \quad y \in \mathbb{X}. \tag{1.11}$$

Cosine families share many structural properties with semigroups.

Theorem 1.3 (Properties of Cosine Families [27, 55]):
Let $\{C(s)\}_{s\in\mathbb{R}}$ be a strongly continuous cosine family with generator A and associated sine family $\{S(s)\}_{s\in\mathbb{R}}$. Then,

1. $C(s)$ is even: $C(-s) = C(s)$ for all $s \in \mathbb{R}$.
2. $C(s)$, $S(s)$, $C(r)$, $S(r)$ commute for all $s, r \in \mathbb{R}$.
3. $S(s)$ is strongly continuous and odd: $S(-s) = -S(s)$.
4. There exist constants $L_{\mathcal{T}} \geq 1$, $\delta \geq 0$ such that
$$\|C(s)\|_{\mathcal{L}(\mathbb{X})}, \|S(s)\|_{\mathcal{L}(\mathbb{X})} \leq L_{\mathcal{T}} e^{\delta|s|}, \quad s \in \mathbb{R}.$$
5. The generator A is closed and densely defined.
6. For $y \in \mathrm{Dom}(A)$, $C(s)y \in \mathrm{Dom}(A)$ and $\frac{\mathrm{d}^2}{\mathrm{d}s^2} C(s)y = AC(s)y = C(s)Ay$.
7. For $y \in \mathbb{X}$, $S(s)y \in \mathrm{Dom}(A)$ and $\frac{\mathrm{d}^2}{\mathrm{d}s^2} S(s)y = AS(s)y$.

8. The Laplace transform of the cosine and sine families yield the resolvent of A: for $\mathrm{Re}(\alpha) > \delta$,

$$\alpha R(\alpha^2, A)y = \int_{s_0}^{\infty} e^{-\alpha(s-s_0)} C(s)y \,\mathrm{d}s,$$
$$R(\alpha^2, A)y = \int_{s_0}^{\infty} e^{-\alpha(s-s_0)} S(s)y \,\mathrm{d}s, \quad y \in \mathbb{X}.$$

The well-posedness theorem for the second-order problem mirrors the first-order case.

Theorem 1.4 (Well-Posedness [55]):
Let $A : \mathrm{Dom}(A) \subset \mathbb{X} \to \mathbb{X}$ be a closed, densely defined operator. The following statements are equivalent,

1. The Cauchy problem (1.9)–(1.10) is uniformly well-posed on $\mathrm{Dom}(A) \times \mathrm{Dom}(A)$.
2. A is the generator of a strongly continuous cosine family $\{C(s)\}_{s\in\mathbb{R}}$.
3. There exist constants $L_{(\mathrm{C,S})} \geq 1$, $\delta \geq 0$ such that for all $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > \delta$ and $n \in \mathbb{N}_0$,

$$\left\| \frac{\mathrm{d}^n}{\mathrm{d}\alpha^n} \left(\alpha R(\alpha^2, A)\right) \right\|_{\mathcal{L}(\mathbb{X})} \leq L_{\mathcal{T}} \frac{n!}{(\mathrm{Re}(\alpha) - \delta)^{n+1}}. \tag{1.12}$$

In this case, the unique solution is given by,

$$y(s) = C(s - s_0)y_0 + S(s - s_0)y_1, \quad y_0, y_1 \in \mathrm{Dom}(A).$$

Finally, the solution to the non-homogeneous second-order problem is given by a similar formula.

Let D_{Ah} be the space of functions $h \in C^1(\mathbb{R}; \mathbb{X})$ such that $Ah(s) \in C(\mathbb{R}; \mathbb{X})$.

Theorem 1.5 (Mild Solution [27, 55]):
Let A generate a cosine family $\{C(s)\}_{s\in\mathbb{R}}$ with associated sine family $\{S(s)\}_{s\in\mathbb{R}}$. If $y_0, y_1 \in \mathrm{Dom}(A)$ and $h \in D_{Ah}$, then the function

$$y(s) = C(s - s_0)y_0 + S(s - s_0)y_1 + \int_{s_0}^{s} S(s - r)h(r) \,\mathrm{d}r, \quad s \geq s_0, \tag{1.13}$$

is the unique (mild) solution of the non-homogeneous problem

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} y(s) = Ay(s) + h(s), \quad y(s_0) = y_0, \quad y'(s_0) = y_1.$$

Example: The Wave Equation on a Bounded Domain

The wave equation on a bounded domain provides a canonical example of a cosine family. Consider a bounded, open set $\mathcal{O} \subset \mathbb{R}^n$ with a smooth boundary $\partial\mathcal{O}$. Let us denote by $\mathbb{X} = L^2(\mathcal{O})$, the Hilbert space of square-integrable functions on $\mathcal{O}$.

Let A be the Laplace operator endowed with Dirichlet boundary conditions. Its domain is specified as Nagy [44], Travis and Webb [55],

$$A = \Delta, \quad \mathrm{Dom}(A) = \left\{ y \in L^2(\mathcal{O}) \;\middle|\; \partial_i y,\ \partial_{ij}^2 y \in L^2(\mathcal{O}) \text{ for all } i, j, \text{ and } y = 0 \text{ on } \partial\mathcal{O} \right\}.$$

This domain is equivalent to the Sobolev space $H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$. In this case, the derivatives are understood in the sense of Fréchet derivatives. The operator A is closed, densely defined, and self-adjoint. Moreover, it possesses a purely discrete, negative spectrum,

$$-\infty \leftarrow \cdots \leq \lambda_{n+1} \leq \lambda_n \leq \cdots \leq \lambda_2 \leq \lambda_1 < 0,$$

with a corresponding set of eigenfunctions $\{\phi_n\}_{n=1}^{\infty}$ that form a complete orthonormal basis for $L^2(\mathcal{O})$.

Cosine Family Definition

Using the spectral theorem, we define the cosine family $\{C(s)\}_{s\in\mathbb{R}} \subset \mathcal{L}(L^2(\mathcal{O}))$ generated by A. Any function $y \in L^2(\mathcal{O})$ has an eigenfunction expansion,

$$y = \sum_{n=1}^{\infty} \langle y, \phi_n \rangle \phi_n.$$

The operator $C(s)$ acts on y according to the formula,

$$C(s)y = \sum_{n=1}^{\infty} \cos\left(s\sqrt{-\lambda_n}\right) \langle y, \phi_n \rangle \phi_n.$$

This series converges because $|\cos(s\sqrt{-\lambda_n})| \leq 1$ for all s and n.

Verification of the Cosine Family Properties

1. D'Alembert's Identity (C1): For any $y \in \mathbb{X}$ and for all $s, r \in \mathbb{R}$, we verify the fundamental identity,

$$\begin{aligned}
&C(s+r)y + C(s-r)y \\
&= \sum_{n=1}^{\infty} \left[\cos\left((s+r)\sqrt{-\lambda_n}\right) + \cos\left((s-r)\sqrt{-\lambda_n}\right) \right] \langle y, \phi_n \rangle \phi_n \\
&= \sum_{n=1}^{\infty} 2\cos\left(s\sqrt{-\lambda_n}\right) \cos\left(r\sqrt{-\lambda_n}\right) \langle y, \phi_n \rangle \phi_n \\
&= 2C(s)C(r)y.
\end{aligned}$$

2. Identity at Zero (C2): The identity property at $s = 0$ is immediately satisfied,

$$C(0)y = \sum_{n=1}^{\infty} \cos(0)\langle y, \phi_n\rangle\phi_n = \sum_{n=1}^{\infty}\langle y, \phi_n\rangle\phi_n = y.$$

   Hence, $C(0) = I$, the identity operator.

3. Strong Continuity (C3): To show strong continuity, fix $y \in \mathbb{X}$ and $s_0 \in \mathbb{R}$. We examine the difference,

$$\|C(s)y - C(s_0)y\|^2 = \sum_{n=1}^{\infty} \left|\cos\left(s\sqrt{-\lambda_n}\right) - \cos\left(s_0\sqrt{-\lambda_n}\right)\right|^2 |\langle y, \phi_n\rangle|^2.$$

   The cosine function is continuous, so each term in the series tends to zero as $s \to s_0$. Each term is also bounded by $4|\langle y, \phi_n\rangle|^2$. Since $\sum_{n=1}^{\infty} |\langle y, \phi_n\rangle|^2 = \|y\|^2 < \infty$, we apply the Dominated Convergence Theorem to conclude that the entire sum converges to zero. Therefore, the mapping $s \mapsto C(s)y$ is continuous for every y.

4. The Infinitesimal Generator: We now confirm that A is indeed the infinitesimal generator of this cosine family. Let $y \in \mathrm{Dom}(A)$. We compute the second derivative using the difference quotient,

$$\begin{aligned}\lim_{s\to 0} \frac{C(s)y - 2y + C(-s)y}{s^2} &= \lim_{s\to 0} \sum_{n=1}^{\infty} \frac{2(\cos(s\sqrt{-\lambda_n}) - 1)}{s^2}\langle y, \phi_n\rangle\phi_n \\ &= \sum_{n=1}^{\infty} \left(\lim_{s\to 0} \frac{2(\cos(s\sqrt{-\lambda_n}) - 1)}{s^2}\right)\langle y, \phi_n\rangle\phi_n.\end{aligned}$$

   Using the standard limit $\lim_{s\to 0} \frac{\cos(as)-1}{s^2} = -\frac{a^2}{2}$, we find,

$$\lim_{s\to 0} \frac{2(\cos(s\sqrt{-\lambda_n}) - 1)}{s^2} = \lambda_n.$$

   Consequently,

$$\lim_{s\to 0} \frac{C(s)y - 2y + C(-s)y}{s^2} = \sum_{n=1}^{\infty} \lambda_n\langle y, \phi_n\rangle\phi_n = Ay.$$

   This establishes that the generator of $\{C(s)\}$ is the operator A.

5. Associated Sine Family: The sine family $\{S(s)\}_{s\in\mathbb{R}}$, associated with the cosine family, is defined by integration,

$$S(s)y = \int_0^s C(r)y \, \mathrm{d}r.$$

For a given y, we compute this explicitly,

$$\begin{aligned} S(s)y &= \int_0^s \sum_{n=1}^{\infty} \cos\left(r\sqrt{-\lambda_n}\right) \langle y, \phi_n\rangle \phi_n \,\mathrm{d}r \\ &= \sum_{n=1}^{\infty} \left(\int_0^s \cos\left(r\sqrt{-\lambda_n}\right) \,\mathrm{d}r \right) \langle y, \phi_n\rangle \phi_n \\ &= \sum_{n=1}^{\infty} \frac{\sin\left(s\sqrt{-\lambda_n}\right)}{\sqrt{-\lambda_n}} \langle y, \phi_n\rangle \phi_n . \end{aligned}$$

1.3. Complete Second-Order Equations

Consider the complete second-order abstract Cauchy problem,

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} y(s) = B\frac{\mathrm{d}}{\mathrm{d}s} y(s) + Ay(s), \quad s \geq s_0, \tag{1.14}$$

$$y(s_0) = y_0, \quad y'(s_0) = y_1, \tag{1.15}$$

where $B : \mathrm{Dom}(B) \subset \mathbb{X} \to \mathbb{X}$ and $A : \mathrm{Dom}(A) \subset \mathbb{X} \to \mathbb{X}$ are closed, linear operators with dense domains and commute (i.e. $\mathrm{Dom}(AB) = \mathrm{Dom}(BA)$).

Definition 1.6 (Classical solution):
A function $y : \mathbb{R}_{\geqslant s_0} \to \mathbb{X}$ is called a classical solution of the Cauchy problem (1.14)–(1.15) if,

(i) $y \in C^2(\mathbb{R}_{\geqslant s_0}; \mathbb{X})$,

(ii) $y(s) \in \mathrm{Dom}(A)$ and $\frac{\mathrm{d}}{\mathrm{d}s} y(s) \in \mathrm{Dom}(B)$ for all $s \geqslant s_0$,

(iii) The differential equation (1.14) is satisfied pointwise for all $s \geqslant s_0$,

(iv) The initial conditions (1.15) are satisfied,

$$\lim_{s \to s_0^+} \|y(s) - y_0\|_{\mathbb{X}} = 0 = \lim_{s \to s_0^+} \left\| \frac{\mathrm{d}}{\mathrm{d}s} y(s) - y_1 \right\|_{\mathbb{X}} .$$

Definition 1.7 (Uniform Well-Posedness):
The Cauchy problem (1.14)–(1.15) is called *uniformly well-posed* on the sets $D_A \subset \mathrm{Dom}(A)$, $D_B \subset \mathrm{Dom}(B)$ (where D_A and D_B are dense in $\mathbb{X}$) if,

(i) For every $y_0 \in D_A$ and $y_1 \in D_B$, there exists a unique classical solution $y(\cdot; y_0, y_1)$.

(ii) The solution depends continuously on the initial data uniformly on compact time intervals; that is, for every $b > 0$ and every $\epsilon > 0$, there exists $\upsilon > 0$ such that if $\|y_0 - \tilde{y}_0\|_{\mathbb{X}} < \upsilon$ and $\|y_1 - \tilde{y}_1\|_{\mathbb{X}} < \upsilon$, then

$$\sup_{s\in[s_0,b]} \|y(s; y_0, y_1) - y(s; \tilde{y}_0, \tilde{y}_1)\|_{\mathbb{X}} < \epsilon.$$

As noted in [27], the theory for the complete second-order equation (1.14) is more complex than for first-order or incomplete second-order equations. A significant distinction is that the solution of a uniformly well-posed Cauchy problem (1.14)–(1.15) may fail to be exponentially bounded, which prevents the direct application of Laplace transform techniques. This necessitates incorporating an exponential growth bound into the definition of well-posedness.

Definition 1.8 (δ-Uniform Well-Posedness):
The Cauchy problem (1.14)–(1.15) is called δ-*uniformly well-posed* on the sets $D_A \subset \mathrm{Dom}(A)$, $D_B \subset \mathrm{Dom}(B)$ (dense in $\mathbb{X}$) if it is uniformly well-posed and if there exist constants $L \geq 1$ and $\delta \in \mathbb{R}$ such that the solution satisfies the growth estimate,

$$\|y(s; y_0, y_1)\|_{\mathbb{X}} \leq Le^{\delta(s-s_0)} \left(\|y_0\|_{\mathbb{X}} + \|y_1\|_{\mathbb{X}}\right), \quad \text{for all } s \geq s_0,\ y_0 \in D_A,\ y_1 \in D_B.$$

A central object in the study of this problem is the resolvent operator.

Definition 1.9 (Resolvent for Operator Pairs):
For $\alpha \in \mathbb{C}$, the *resolvent* of the operator pair (A, B) is defined as the bounded linear operator

$$R(\alpha; A, B) := (\alpha^2 I - \alpha B - A)^{-1},$$

whenever the inverse exists and is bounded on $\mathbb{X}$. The set of all α for which $R(\alpha; A, B)$ exists is called the *resolvent set* $\rho(A, B)$.

The solution operators for the homogeneous problem are encapsulated in the following family of operators.

Definition 1.10 ($\mathcal{M}, \mathcal{N}$–functions [42]):
A pair of strongly continuous operator families $\{\mathcal{M}(s)\}_{s\geqslant 0}, \{\mathcal{N}(s)\}_{s\geqslant 0} \subset \mathcal{L}(\mathbb{X})$ is called an δ-*bounded family of* $\mathcal{M}, \mathcal{N}$–*functions* (generated by (A, B)) if,

(i) **Functional Equations**: For all $s, r \geqslant 0$, the following identities hold,

$$\mathcal{M}(s + r) = \mathcal{M}(s)\mathcal{M}(r) + A\mathcal{N}(s)\mathcal{N}(r),$$
$$\mathcal{N}(s + r) = \mathcal{M}(s)\mathcal{N}(r) + \mathcal{M}(r)\mathcal{N}(s) + B\mathcal{N}(s)\mathcal{N}(r).$$

(ii) **Initial Conditions**: $\mathcal{M}(0) = I$, $\mathcal{N}(0) = 0$, and the strong derivatives exist with $\mathcal{M}'(0) = 0, \mathcal{N}'(0) = I$.

(iii) **Exponential Bound**: There exist constants $L_{(\mathcal{M},\mathcal{N})} \geq 1, \delta \in \mathbb{R}$ such that

$$\|\mathcal{M}(s)\|_{\mathcal{L}(\mathbb{X})}, \|\mathcal{N}(s)\|_{\mathcal{L}(\mathbb{X})} \leq L_{(\mathcal{M},\mathcal{N})}e^{\delta s}, \quad \text{for all } s \geq 0.$$

(iv) The infinitesimal generators of the family $\{\mathcal{M}(s), \mathcal{N}(s)\}_{s\geqslant 0}$ are the operators A and B, defined as follows,

$$Ay = \lim_{s\downarrow 0} \frac{\mathcal{M}(s) - 2\mathcal{M}(s/2) + I}{(s/2)^2} y, \quad \text{for } y \in \mathrm{Dom}(A),$$
$$By = \lim_{s\downarrow 0} \frac{\mathcal{N}(s) - 2\mathcal{N}(s/2)}{(s/2)^2} y, \quad \text{for } y \in \mathrm{Dom}(B),$$

defined on the domains for which these strong limits exist.

The following lemma establishes the fundamental properties of the $\mathcal{M}, \mathcal{N}$–functions and their relation to the differential equation.

Lemma 1.2 (Properties of $\mathcal{M}, \mathcal{N}$-Functions [33]):
Let $\{\mathcal{M}(s), \mathcal{N}(s)\}_{s\geqslant 0}$ be a δ-bounded family of $\mathcal{M}, \mathcal{N}$–functions with generating operators (A, B). Then the following properties hold,

1. **Commutation**: For all $s \geq 0$,
$$\mathcal{M}(s)B \subset B\mathcal{M}(s), \qquad \mathcal{N}(s)B \subset B\mathcal{N}(s),$$
$$\mathcal{M}(s)A \subset A\mathcal{M}(s), \qquad \mathcal{N}(s)A \subset A\mathcal{N}(s).$$

2. **Strong Derivatives**: For y in the appropriate domains,
$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{M}(s)y &= \mathcal{N}(s)Ay = A\mathcal{N}(s)y, && y \in \mathrm{Dom}(A),\\
\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{N}(s)y &= \mathcal{M}(s)y + \mathcal{N}(s)By, && y \in \mathrm{Dom}(B),\\
\frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{M}(s)y &= A\mathcal{M}(s)y + AB\mathcal{N}(s)y, && y \in \mathrm{Dom}(AB),\\
\frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{N}(s)y &= A\mathcal{N}(s)y + B\mathcal{M}(s)y + B^2\mathcal{N}(s)y, && y \in \mathrm{Dom}(B^2) \cap \mathrm{Dom}(A)\\
&= \frac{\mathrm{d}}{\mathrm{d}s}B\mathcal{N}(s)y + A\mathcal{N}(s)y.
\end{aligned}$$

Proof:

1. For $y \in \mathrm{Dom}(B)$, we have,
$$\begin{aligned}
\mathcal{M}(s)By &= \mathcal{M}(s)\left(\lim_{r\to 0} \frac{\mathcal{N}(2r) - 2\mathcal{N}(r)}{r^2} y\right)\\
&= \lim_{r\to 0} \frac{\mathcal{N}(2r) - 2\mathcal{N}(r)}{r^2}\mathcal{M}(s)y = B\mathcal{M}(s)y,\\
\mathcal{N}(s)By &= \mathcal{N}(s)\left(\lim_{r\to 0} \frac{\mathcal{N}(2r) - 2\mathcal{N}(r)}{r^2} y\right)
\end{aligned}$$

$$= \lim_{r\to 0} \frac{\mathcal{N}(2r) - 2\mathcal{N}(r)}{r^2} \mathcal{N}(s)y = B\mathcal{N}(s)y.$$

For $y \in \mathrm{Dom}(A)$, the following equalities hold,

$$\begin{aligned}
\mathcal{M}(s)Ay &= \mathcal{M}(s)\left(\lim_{r\to 0} \frac{\mathcal{M}(2r) - 2\mathcal{M}(r) + I}{r^2} y\right) \\
&= \lim_{r\to 0} \frac{\mathcal{M}(2r) - 2\mathcal{M}(r) + I}{r^2} \mathcal{M}(s)y = A\mathcal{M}(s)y, \\
\mathcal{N}(s)Ay &= \mathcal{N}(s)\left(\lim_{r\to 0} \frac{\mathcal{M}(2r) - 2\mathcal{M}(r) + I}{r^2} y\right) \\
&= \lim_{r\to 0} \frac{\mathcal{M}(2r) - 2\mathcal{M}(r) + I}{r^2} \mathcal{N}(s)y = A\mathcal{N}(s)y.
\end{aligned}$$

2. From the existence of $\mathcal{M}'(0) = 0$, $\mathcal{N}'(0) = I$ and property (1) of Definition 1.10, for $y \in \mathrm{Dom}(A)$ we obtain,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{M}(s)y &= \lim_{r\to 0} \frac{\mathcal{M}(s+r) - \mathcal{M}(s)}{r} y \\
&= \lim_{r\to 0} \left\{\mathcal{M}(s)\frac{\mathcal{M}(r) - I}{r} y + \mathcal{N}(s)\frac{\mathcal{N}(r)}{r} Ay\right\} \\
&= \mathcal{M}(s)\mathcal{M}'(0)y + \mathcal{N}(s)\mathcal{N}'(0)Ay = \mathcal{N}(s)Ay = A\mathcal{N}(s)y.
\end{aligned}$$

Similarly, for $y \in \mathrm{Dom}(B)$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{N}(s)y &= \lim_{r\to 0} \frac{\mathcal{N}(s+r) - \mathcal{N}(s)}{r} y \\
&= \lim_{r\to 0} \left(\mathcal{M}(s)\frac{\mathcal{N}(r)}{r} y + \mathcal{N}(s)\frac{\mathcal{N}(r)}{r} By + \mathcal{N}(s)\frac{\mathcal{M}(r) - I}{r} y\right) \\
&= \mathcal{M}(s)\mathcal{N}'(0)y + \mathcal{N}(s)\mathcal{N}'(0)By + \mathcal{N}(s)\mathcal{M}'(0)y \\
&= \mathcal{M}(s)y + \mathcal{N}(s)By = [\mathcal{M}(s) + B\mathcal{N}(s)]y.
\end{aligned}$$

For $y \in \mathrm{Dom}(AB)$,

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{M}(s)y = \frac{\mathrm{d}}{\mathrm{d}s}[\mathcal{N}(s)Ay] = [\mathcal{M}(s) + B\mathcal{N}(s)]Ay = B\mathcal{M}'(s)y + A\mathcal{M}(s)y.$$

For $y \in \mathrm{Dom}(B^2) \cap \mathrm{Dom}(A)$,

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{N}(s)y &= \frac{\mathrm{d}}{\mathrm{d}s}[\mathcal{M}(s)y + \mathcal{N}(s)By] = A\mathcal{N}(s)y + B\mathcal{M}(s)y + B^2\mathcal{N}(s)y \\
&= B\mathcal{N}'(s)y + A\mathcal{N}(s)y.
\end{aligned}$$

□

1.3.1. Necessary and Sufficient Conditions for δ-Uniform Well-Posedness

The main theorem establishes the equivalence between the well-posedness of the Cauchy problem and the existence of an $\mathcal{M},\mathcal{N}$-function family.

Theorem 1.6 (Equivalence Theorem [33, 42]):
Let $B : \mathrm{Dom}(B) \subset \mathbb{X} \to \mathbb{X}$ and $A : \mathrm{Dom}(A) \subset \mathbb{X} \to \mathbb{X}$ be closed, commuting linear operators with dense domains. Assume the sets $D_{AB} = \mathrm{Dom}(AB)$ and $D_{B^2\cap A} = \mathrm{Dom}(B^2) \cap \mathrm{Dom}(A)$ are dense in $\mathbb{X}$. Then the following statements are equivalent,

1. The Cauchy problem (1.14)–(1.15) is δ-uniformly well-posed on $D_{AB}, D_{B^2\cap A}$.
2. The operators A, B are the generating operators of a δ-bounded family of $\mathcal{M},\mathcal{N}$–functions.

Furthermore, when these conditions hold,

(i) The unique solution is given by

$$y(s) = \mathcal{M}(s-s_0)y_0 + \mathcal{N}(s-s_0)y_1, \quad y_0 \in D_{AB},\ y_1 \in D_{B^2\cap A}. \tag{1.16}$$

(ii) For $\mathrm{Re}(\alpha) > \delta$, the resolvent $R(\alpha; A, B)$ exists and is given by the Laplace transforms,

$$R(\alpha; A, B)y = \int_{s_0}^{\infty} e^{-\alpha(s-s_0)}\mathcal{N}(s)y\,\mathrm{d}s, \quad y \in \mathbb{X}, \tag{1.17}$$

$$(\alpha I - B)R(\alpha; A, B)y = \int_{s_0}^{\infty} e^{-\alpha(s-s_0)}\mathcal{M}(s)y\,\mathrm{d}s, \quad y \in \mathbb{X}. \tag{1.18}$$

(iii) The resolvent satisfies the following growth conditions for $\mathrm{Re}(\alpha) > \delta$ and $n \in \mathbb{N}_0$,

$$\begin{aligned} \left\| \frac{\mathrm{d}^n}{\mathrm{d}\alpha^n} R(\alpha; A, B) \right\|_{\mathcal{L}(\mathbb{X})} &\le L_{(\mathcal{M},\mathcal{N})} \frac{n!}{(\mathrm{Re}(\alpha)-\delta)^{n+1}}, \\ \left\| \frac{\mathrm{d}^n}{\mathrm{d}\alpha^n} (\alpha I - B)R(\alpha; A, B) \right\|_{\mathcal{L}(\mathbb{X})} &\le L_{(\mathcal{M},\mathcal{N})} \frac{n!}{(\mathrm{Re}(\alpha)-\delta)^{n+1}}. \end{aligned} \tag{1.19}$$

Sketch of Proof:
Lemma 1.2 shows that (1.16) defines a solution. The exponential bound in Definition 1.10(iii) implies the stability estimate in Definition 1.8. Uniqueness follows from the resolvent estimates (1.19) and an application of the Phragmén–Lindelöf principle or a Titchmarsh-type theorem, showing that the Laplace transform of a nonzero solution cannot decay too rapidly in half-planes.

The resolvent identities (1.17), (1.18) are derived by formally applying the Laplace transform to the differential equations satisfied by $\mathcal{M}(s)$ and $\mathcal{N}(s)$ and verifying the identities on dense sets. The estimates (1.19) follow from (1.17), (1.18) and the exponential bounds on $\|\mathcal{M}(s)\|_{\mathcal{L}(\mathbb{X})}$, $\|\mathcal{N}(s)\|_{\mathcal{L}(\mathbb{X})}$ via standard properties of the Laplace transform. □

Under an additional closedness assumption, the resolvent characterisation alone becomes sufficient.

Theorem 1.7 (Well-Posedness Criteria [33, 42]):
Let $B : \mathrm{Dom}(B) \subset \mathbb{X} \to \mathbb{X}$ and $A : \mathrm{Dom}(A) \subset \mathbb{X} \to \mathbb{X}$ be closed, commuting linear operators such that the operator $\alpha B + A$ is closed for $\mathrm{Re}(\alpha)$ sufficiently large. Assume D_{AB} and $D_{B^2 \cap A}$ are dense in $\mathbb{X}$. Then the following are equivalent,

1. The Cauchy problem (1.14)–(1.15) is δ-uniformly well-posed on $D_{AB}, D_{B^2 \cap A}$.
2. The operators A, B generate a δ-bounded family of $\mathcal{M}, \mathcal{N}$–functions.
3. The resolvent $R(\alpha; A, B)$ exists for $\mathrm{Re}(\alpha) > \delta$ and satisfies the estimates (1.19).

Sketch of Proof:
The operators $\mathcal{M}(s)$ and $\mathcal{N}(s)$ are constructed as inverse Laplace transforms of the resolvent and its related function $(\alpha I - B)R(\alpha; A, B)$. Specifically, for a suitable $\gamma > \delta$,

$$\mathcal{N}(s) = \frac{1}{2\pi i} \lim_{b\to\infty} \int_{\gamma - ib}^{\gamma + ib} e^{\alpha s} R(\alpha; A, B)\, \mathrm{d}\alpha,$$
$$\mathcal{M}(s) = \frac{1}{2\pi i} \lim_{b\to\infty} \int_{\gamma - ib}^{\gamma + ib} e^{\alpha s} (\alpha I - B) R(\alpha; A, B)\, \mathrm{d}\alpha.$$

The estimates (1.19) guarantee that these integrals converge in the strong operator topology and define exponentially bounded, strongly continuous operator families. The functional equations in Definition 1.10 are verified using the resolvent identity and properties of the Laplace transform (convolution theorem). The generator properties follow from the asymptotic behavior of the Laplace transform as $|\alpha| \to \infty$. □

Comparison of Operator Families

It is interesting to compare the family of $\mathcal{M}, \mathcal{N}$–functions with other families through which the solution of the Cauchy problem (1.14)–(1.15) can be constructed. In the work [40], a family of $\mathcal{E}C(s), \mathcal{E}S(s)$–functions is defined, through which the $\mathcal{M}, \mathcal{N}$–functions were introduced,

$$\mathcal{M}(s) = \mathcal{E}C(s) - \frac{B}{2}\mathcal{E}S(s), \quad \mathcal{N}(s) = \mathcal{E}S(s).$$

Definition 1.11 ($\mathcal{E}C, \mathcal{E}S$–functions):
A one-parameter family of bounded commuting operators $\mathcal{E}C(s), \mathcal{E}S(s)$ is called an δ-*strongly continuous family of $\mathcal{E}C, \mathcal{E}S$–functions* (generated by the operators A, B), if,

1. The following addition formulas hold,

$$\mathcal{E}C(s+r) = \mathcal{E}C(r)\mathcal{E}C(s) + \left(\frac{B^2}{4} + A\right)\mathcal{E}S(r)\mathcal{E}S(s),$$
$$\mathcal{E}S(s+r) = \mathcal{E}S(r)\mathcal{E}C(s) + \mathcal{E}C(r)\mathcal{E}S(s), \quad r, s \geq 0.$$

2. The initial conditions are satisfied,
$$\mathcal{E}C(0) = I, \quad \text{and} \quad \mathcal{E}S(0) = 0,$$
and the strong derivatives exist,
$$\mathcal{E}C'(0) = \frac{B}{2}, \quad \text{and} \quad \mathcal{E}S'(0) = I.$$

3. The operator-valued functions
$$\mathcal{E}C(s) - \frac{B}{2}\mathcal{E}S(s), \quad \text{and} \quad \mathcal{E}S(s),$$
are strongly continuous for $s \geq 0$.

4. There exist constants $L_{(\mathcal{E}C,\mathcal{E}S)} > 0$, $\delta \geq 0$ such that for $s \geq 0$,
$$\left\| \mathcal{E}C(s) - \frac{B}{2}\mathcal{E}S(s) \right\|_{\mathcal{L}(\mathbb{X})} \leq L_{(\mathcal{E}C,\mathcal{E}S)} e^{\delta s}, \quad \text{and} \quad \|\mathcal{E}S(s)\|_{\mathcal{L}(\mathbb{X})} \leq L_{(\mathcal{E}C,\mathcal{E}S)} e^{\delta s}.$$

5. The family provides the solution of the Cauchy problem (1.14)–(1.15) in the form,
$$y(s) = \left[\mathcal{E}C(s - s_0) - \frac{B}{2}\mathcal{E}S(s - s_0) \right] y_0 + \mathcal{E}S(s - s_0) y_1,$$
where $y_0 \in D_{AB}$, $y_1 \in D_{B^2 \cap A}$, and $\overline{D_{AB}}^{\|\cdot\|_{\mathbb{X}}} = \overline{D_{B^2 \cap A}}^{\|\cdot\|_{\mathbb{X}}} = \mathbb{X}$.

In the works of Buche [13], Singh and Vasudeva [50], a family of operator-valued functions known as the exponential-cosine family, denoted by $\mathcal{E}(s)$, is studied. This family is defined through the recurrence relation,
$$\begin{gathered} \mathcal{E}(s + r) = 2\mathcal{E}(r)\mathcal{E}(s) + \left(\mathcal{E}(2r) - 2\mathcal{E}^2(r)\right) \mathcal{E}(s - r), \\ 0 \leq r \leq s, \quad \mathcal{E}(0) = I, \end{gathered} \tag{1.20}$$
and it is shown that for some everywhere dense set $D \subset \mathbb{X}$, the function $\mathcal{E}(s)y$ (with $y \in D$) satisfies equation (1.14).

Proposition 1.1 (Relation to Exponential-Cosine Families [33]):
Let the linear commuting operators A, B generate a δ-strongly continuous family of $\mathcal{E}C, \mathcal{E}S$-functions. Then the function $\mathcal{E}C(s)$ satisfies relation (1.20).

Proof:
Using property (1) of Definition 1.11 and the commutativity of the operators A, B with $\mathcal{E}C, \mathcal{E}S$ (which follows from the connection with the $\mathcal{M}, \mathcal{N}$–functions), we have,
$$\begin{aligned} \mathcal{E}C(s + r) - 2\mathcal{E}C(r)\mathcal{E}C(s) &= -\mathcal{E}C(r)\mathcal{E}C(s) + \left(\frac{B^2}{4} + A\right) \mathcal{E}S(r)\mathcal{E}S(s) \\ &= -\mathcal{E}C(r)\mathcal{E}C(s) + \left(\frac{B^2}{4} + A\right) \mathcal{E}S(r)\mathcal{E}S(s - r)\mathcal{E}C(r) \end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{B^2}{4}+A\right)(\mathcal{E}S(r)\mathcal{E}S(s)-\mathcal{E}C(r)\mathcal{E}S(s-r)\mathcal{E}S(r))\\
&=-\mathcal{E}C(r)\left(\mathcal{E}C(s)-\left(\frac{B^2}{4}+A\right)\mathcal{E}S(r)\mathcal{E}S(s-r)\right)\\
&+\left(\frac{B^2}{4}+A\right)\mathcal{E}S(r)[\mathcal{E}S(s)-\mathcal{E}C(r)\mathcal{E}S(s-r)]\\
&=-\mathcal{E}C(r)[\mathcal{E}C(r)\mathcal{E}C(s-r)]+\left(\frac{B^2}{4}+A\right)\mathcal{E}S(r)[\mathcal{E}S(r)\mathcal{E}C(s-r)]\\
&=\left(-\mathcal{E}C^2(r)+\left(\frac{B^2}{4}+A\right)\mathcal{E}S^2(r)\right)\mathcal{E}C(s-r),
\end{aligned}
$$

for $0\le r\le s$.

Taking into account that for $s=r$ the equality holds,

$$\mathcal{E}C(2r)=\mathcal{E}C^2(r)+\left(\frac{B^2}{4}+A\right)\mathcal{E}S^2(r),$$

we obtain that the function $\mathcal{E}C(r)$ satisfies relation (1.20),

$$\mathcal{E}C(s+r)-2\mathcal{E}C(r)\mathcal{E}C(s)=[\mathcal{E}C(2r)-2\mathcal{E}C^2(r)]\mathcal{E}C(s-r),\quad 0\le r\le s.$$

□

***Remark* 1.2:**

- When $A=0$, A and B generate $\mathcal{M},\mathcal{N}$–families if and only if B generates a strongly continuous C_0–semigroup $(\mathcal{T}(s))_{s\geqslant 0}$. Additionally, $\mathcal{M}(s)=\mathrm{I}$ and $\mathcal{N}(s)=\int_0^s\mathcal{T}(r)\,\mathrm{d}r$ for $s\geqslant 0$.
- Now, considering the case where $B=0$. A and B generate $\mathcal{M},\mathcal{N}$–families if and only if A generates a strongly continuous cosine function $(\mathcal{C}(s))_{s\geqslant 0}$. Furthermore, $\mathcal{M}(s)=\mathcal{C}(s)$ and $\mathcal{N}(s)=\int_0^s\mathcal{C}(r)\,\mathrm{d}r:=\mathcal{S}(s)$ for $s\geqslant 0$.

For additional specialised cases, please refer to Zheng [66].

1.3.2. The Nonhomogeneous Equation

The theory extends to the nonhomogeneous problem,

$$y''(s)=By'(s)+Ay(s)+h(s),\quad s\ge s_0,\quad y(s_0)=y_0,\quad y'(s_0)=y_1,\qquad (1.21)$$

where $h:\mathbb{R}_{\geqslant s_0}\to\mathbb{X}$ is a given function.

Let D_{Bh} denote the space of functions $h\in C^1(\mathbb{R}_{\geqslant s_0};\mathbb{X})$ such that $Bh,Bh'\in C(\mathbb{R}_{\geqslant s_0};\mathbb{X})$.

Definition 1.12 (Well-Posedness):
The Cauchy problem (1.21) is called δ-uniformly well-posed on D_{AB}, $D_{B^2\cap A}$, D_{Bh} (where $\overline{D_{AB}}^{\|\cdot\|_{\mathbb{X}}} = \overline{D_{B^2\cap A}}^{\|\cdot\|_{\mathbb{X}}} = \mathbb{X}$) if,

1. A solution exists for any $y_0 \in D_{AB}$, $y_1 \in D_{B^2\cap A}$, $h \in D_{Bh}$.
2. There exist constants $L > 0$, $\delta \geq 0$ such that

$$\|y(s)\|_{\mathbb{X}} \leq Le^{\delta(s-s_0)}\left(\|y_0\|_{\mathbb{X}} + \|y_1\|_{\mathbb{X}} + \sup_{r\in[s_0,s]}\|h(r)\|_{\mathbb{X}}\right), \quad s \geq s_0.$$

Theorem 1.8 (Mild Solution [33]):
Assume the equivalent conditions of Theorem 1.6 hold. If $y_0 \in D_{AB}$, $y_1 \in D_{B^2\cap A}$, and $h \in D_{Bh}$, then the function

$$y(s) = \mathcal{M}(s-s_0)y_0 + \mathcal{N}(s-s_0)y_1 + \int_{s_0}^{s}\mathcal{N}(s-r)h(r)\,\mathrm{d}r, \quad s \geq s_0, \tag{1.22}$$

is a (mild) solution of the Cauchy problem (1.21). Conversely, if $h(s) \in C(\mathbb{R}_{\geqslant s_0};\mathbb{X})$ and $y(s)$ is a solution of (1.21), then $y(s)$ is given by formula (1.22).

Proof:
Let

$$\mathcal{K}(s) = \int_{s_0}^{s}\mathcal{N}(s-r)h(r)\,\mathrm{d}r.$$

For $h \in D_{Bh}$, using the properties of $\mathcal{M}(s)$ and $\mathcal{N}(s)$, we have,

$$\begin{aligned}
\mathcal{K}'(s) = \mathcal{N}(0)h(s) + \int_{s_0}^{s}\frac{\partial}{\partial s}\mathcal{N}(s-r)h(r)\,\mathrm{d}r &= \int_{s_0}^{s}[\mathcal{M}(s-r) + B\mathcal{N}(s-r)]h(r)\,\mathrm{d}r \\
&= -\int_{s_0}^{s}\frac{\partial}{\partial r}\mathcal{N}(s-r)h(r)\,\mathrm{d}r \\
&= -[\mathcal{N}(s-r)h(r)]_{r=s_0}^{r=s} + \int_{s_0}^{s}\mathcal{N}(s-r)h'(r)\,\mathrm{d}r.
\end{aligned}$$

Since the right-hand side is well-defined for $h \in C^1(\mathbb{R}_{\geqslant s_0};\mathbb{X})$, the equality

$$\mathcal{K}'(s) = \int_{s_0}^{s}[\mathcal{M}(s-r) + B\mathcal{N}(s-r)]h(r)\,\mathrm{d}r,$$

can be extended by continuity to $C^1(\mathbb{R}_{\geqslant s_0};\mathbb{X})$.

Similarly,

$$\begin{aligned}
&\frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{K}(s) = [\mathcal{M}(0) + B\mathcal{N}(0)]h(s) + \int_{s_0}^{s}\frac{\partial}{\partial s}[\mathcal{M}(s-r) + B\mathcal{N}(s-r)]h(r)\,\mathrm{d}r \\
&\quad = h(s) - [\mathcal{M}(s-r) + B\mathcal{N}(s-r)]_{r=s_0}^{r=s}h(r) + \int_{s_0}^{s}[\mathcal{M}(s-r) + B\mathcal{N}(s-r)]h'(r)\,\mathrm{d}r.
\end{aligned}$$

This can be extended to functions $h \in D_{Bh}$.

Since $\hat{y}(s) = \mathcal{M}(s-s_0)y_0 + \mathcal{N}(s-s_0)y_1$ solves the homogeneous problem, it remains to show that $\mathcal{K}(s)$ solves (1.21) with zero initial conditions. For $h \in D_{Bh}$, we have,

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{K}(s) - B\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{K}(s) - A\mathcal{K}(s) - h(s) &= h(s) \\
&\quad + \int_{s_0}^{s} \big(A\mathcal{N}(s-r) + B\mathcal{M}(s-r) + B^2\mathcal{N}(s-r)\big)\, h(r)\,\mathrm{d}r \\
- \int_{s_0}^{s} \big(B\mathcal{M}(s-r) + B^2\mathcal{N}(s-r)\big)\, h(r)\,\mathrm{d}r &- \int_{s_0}^{s} A\mathcal{N}(s-r)h(r)\,\mathrm{d}r - h(s) = 0,
\end{aligned}$$

with $\mathcal{K}(s_0) = 0$, $\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{K}(s)\big|_{s=s_0^+} = 0$.

Conversely, let $y(s)$ be a solution of (1.21). Consider the identities,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r}[\mathcal{N}(s-r)y'(r)] &= -[\mathcal{M}(s-r) + B\mathcal{N}(s-r)]y'(r) + \mathcal{N}(s-r)y''(r), \\
\frac{\mathrm{d}}{\mathrm{d}r}[\mathcal{M}(s-r)y(r)] &= -A\mathcal{N}(s-r)y(r) + \mathcal{M}(s-r)y'(r).
\end{aligned}$$

Integrating from s_0 to s,

$$\begin{aligned}
\mathcal{N}(0)y'(s) - \mathcal{N}(s-s_0)y'(s_0) &= \int_{s_0}^{s} -[\mathcal{M}(s-r) + B\mathcal{N}(s-r)]y'(r)\,\mathrm{d}r \\
&\quad + \int_{s_0}^{s} \mathcal{N}(s-r)y''(r)\,\mathrm{d}r, \\
y(s) - \mathcal{M}(s-s_0)y(s_0) &= \int_{s_0}^{s} -A\mathcal{N}(s-r)y(r)\,\mathrm{d}r + \int_{s_0}^{s} \mathcal{M}(s-r)y'(r)\,\mathrm{d}r.
\end{aligned}$$

Adding these identities,

$$\begin{aligned}
&y(s) - \mathcal{M}(s-s_0)y(s_0) - \mathcal{N}(s-s_0)y'(s_0) \\
&\qquad = \int_{s_0}^{s} \big[\mathcal{N}(s-r)y''(r) - B\mathcal{N}(s-r)y'(r) - A\mathcal{N}(s-r)y(r)\big]\,\mathrm{d}r.
\end{aligned}$$

By (1.21) and commutativity of $\mathcal{N}(s)$ with A, B,

$$y(s) = \mathcal{M}(s-s_0)y_0 + \mathcal{N}(s-s_0)y_1 + \int_{s_0}^{s} \mathcal{N}(s-r)h(r)\,\mathrm{d}r.$$

□

Corollary 1.1 (Well-Posedness Criterion):
Under the conditions of Theorem 1.8, for the Cauchy problem (1.21) to be δ-uniformly well-posed on $D_{AB} = \mathrm{Dom}(AB)$, $D_{B^2\cap A} = \mathrm{Dom}(B^2) \cap \mathrm{Dom}(A)$, D_{Bh} (with $\overline{D_{AB}}^{\|\cdot\|_{\mathbb{X}}} = \overline{D_{B^2\cap A}}^{\|\cdot\|_{\mathbb{X}}} = \mathbb{X}$), it is necessary and sufficient that A, B generate a δ-strongly continuous family of $\mathcal{M}$, $\mathcal{N}$–functions.

Consider the following second-order differential equation,

$$\mathrm{d}(y'(s)-g(s)) = Ay(s)+By'(s)+h(s), \quad s \geq s_0, \quad y(s_0) = y_0, \quad y'(s_0) = y_1, \tag{1.23}$$

where $g, h : \mathbb{R}_{\geqslant s_0} \longrightarrow \mathbb{X}$ are continuously differentiable functions.

Corollary 1.2:
Under the hypotheses of Theorem 1.6, for any $y_0 \in D_{AB}$, $y_1 \in D_{B^2 \cap A}$, and $(g, h) \in D_{Bh} \times D_{Bg}$, the function

$$\begin{aligned} y(s) = \mathcal{M}(s-s_0)y_0 + \mathcal{N}(s-s_0)(y_1 - g(s_0)) + \int_{s_0}^{s} \mathcal{N}_s(s-r)g(r)\,\mathrm{d}r \\ + \int_{s_0}^{s} \mathcal{N}(s-r)h(r)\,\mathrm{d}r, \quad s \geq s_0, \end{aligned} \tag{1.24}$$

(where $\mathcal{N}_s(s-r) := \frac{\partial}{\partial s}\mathcal{N}(s-r)$) constitutes a mild solution of the Cauchy problem (1.23). Conversely, if $g, h \in C(\mathbb{R}_{\geqslant s_0}; \mathbb{X})$ and $y(s)$ is a solution of (1.23), then $y(s)$ is given by formula (1.24).

Proof:
The result follows directly from the properties of the $\mathcal{M}, \mathcal{N}$–functions established in Definition 1.10 and Lemma 1.2. Specifically, for $s \geqslant r \geqslant s_0$, these properties yield the key identities,

$$\mathcal{M}(s-r)\mathcal{N}(r-s_0) + \mathcal{N}(s-r)\mathcal{N}_r(r-s_0) = \mathcal{N}(s-s_0),$$
$$\mathcal{M}(s-r)\mathcal{N}_r(r-s_0) + \mathcal{N}(s-r)\mathcal{N}_{rr}(r-s_0) = \mathcal{N}_s(s-s_0).$$

The remainder of the proof proceeds in direct analogy to the proof of Theorem 1.8. $\square$

Chapter 2.

Almost Periodic Functions with Values in a Metric Space

The theory of almost periodic functions, pioneered by Harald Bohr and substantially advanced by Salomon Bochner, constitutes a foundational pillar of modern analysis. Whilst Bohr's and Bochner's definitions are equivalent, the latter frequently furnishes a more versatile and structurally profound framework for investigation. This chapter expounds the fundamental theory of almost periodic functions whose ranges lie in a complete, separable metric space (Polish space; see Def. 3.8), commencing with the classical Bohr formulation, proceeding to Bochner's characterisation via normality, and concluding with an exploration of their principal properties. The treatment herein is necessarily concise; for a comprehensive exposition, the reader is directed to the authoritative monographs by Amerio and Prouse [2], Besicovitch [7], Bohr [11], Boris Moiseevich Levitan [12], Corduneanu [15, 17], Diestel [20], N'Guérékata [45], Zaidman [65].

2.1. Bohr Almost Periodic Functions

This section is devoted to the properties of almost periodic functions, defined in the sense of Bohr, which map the real line into a metric space. Throughout, let $(\mathbb{S}, d_{\mathbb{S}})$ denote a complete, separable metric space. The space of continuous, bounded functions from $\mathbb{R}$ to $\mathbb{S}$ is denoted by $C_b(\mathbb{R}; \mathbb{S})$ endowed with the metric of uniform convergence,

$$\rho(f, g) = \sup_{s \in \mathbb{R}} d_{\mathbb{S}}(f(s), g(s)).$$

The concept of relative density is central to the Bohr definition.

Definition 2.1 (Relative Density):
A subset $\mathbb{A} \subset \mathbb{R}$ is said to be *relatively dense* in $\mathbb{R}$ if there exists a positive number $\ell > 0$ such that every real interval of length ℓ contains at least one point of $\mathbb{A}$. Formally,

$$\exists\, \ell > 0 \quad \text{such that} \quad \forall\, \kappa \in \mathbb{R}, \quad [\kappa, \kappa + \ell] \cap \mathbb{A} \neq \emptyset.$$

Example 2.1:
The following examples illustrate the property of relative density.

1. The set of integers $\mathbb{Z}$ is relatively dense, as every interval of length greater than 1 contains an integer.
2. The set of non-negative integers $\mathbb{N}_0$ is *not* relatively dense, since for any $\ell > 0$, the interval $[-1-\ell, -1]$ contains no element of $\mathbb{N}_0$.
3. The set $\{\pm\sqrt{n} : n \in \mathbb{N}_0\}$ is relatively dense.

   Justification. For any $\kappa > 0$, there exists an integer $n \in \mathbb{N}$ such that $\kappa^2 \leq n \leq \kappa^2 + 1 \leq (\kappa + 1)^2$, whence $\kappa \leq \sqrt{n} \leq \kappa + 1$. A symmetric argument applies for $\kappa < 0$.
4. The set $\{\pm n^2 : n \in \mathbb{N}_0\}$ is *not* relatively dense.

   Justification. For any $\ell > 0$, choose $n \in \mathbb{N}_0$ such that $2n + 1 > \ell$. The interval $\big(n^2 + 1, n^2 + \ell + 1\big)$, of length ℓ, lies strictly within the gap $\big(n^2, (n+1)^2\big]$ and thus contains no perfect squares.
5. The set $\mathbb{A} = \left\{n + \frac{1}{|n|+1} : n \in \mathbb{Z}\right\}$ is relatively dense.

   Justification. The maximal gap between consecutive points of $\mathbb{A}$ is 1.5 (occurring between -0.5 and 1). Consequently, any interval of length 1.5 must contain at least one point from $\mathbb{A}$.

The central definition is as follows.

Definition 2.2 (Uniform Almost Periodicity [2, 7, 11]):
A continuous function $h : \mathbb{R} \longrightarrow \mathbb{S}$ is termed *(uniformly) almost periodic* if, for every $\varepsilon > 0$, the set of its ε-almost periods,

$$\mathcal{V}(h, \varepsilon) = \left\{\tau \in \mathbb{R} : \sup_{s \in \mathbb{R}} d_{\mathbb{S}}(h(s + \tau), h(s)) < \varepsilon\right\},$$

is relatively dense in $\mathbb{R}$. Equivalently, for every $\varepsilon > 0$, there exists a number $\ell = \ell(\varepsilon) > 0$ such that

$$\mathcal{V}(h, \varepsilon) \cap [\kappa, \kappa + \ell] \neq \emptyset \quad \text{for all } \kappa \in \mathbb{R}.$$

1. Elements $\tau \in \mathcal{V}(h, \varepsilon)$ are called *ε-almost periods* or *ε-translations* of h.
2. The number $\ell(\varepsilon)$ is an *inclusion length* for ε; the set of all such inclusion lengths is denoted $\mathcal{W}(h, \varepsilon)$.

Example 2.2:
The function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(s) = \cos(s) + \cos(\sqrt{2}s),$$

is almost periodic but not periodic.

Justification. The function h is continuous. Although not periodic due to the incommensurability of the frequencies 1 and $\sqrt{2}$, the set of its ε-almost periods is relatively dense. This follows from the fact that the set of integer linear combinations of 2π and $2\pi/\sqrt{2} = \sqrt{2}\pi$ is dense modulo 2π. Consequently, for any $\varepsilon > 0$, there exist common approximate periods τ such that simultaneously

$$|\cos(s+\tau) - \cos(s)| < \frac{\varepsilon}{2} \quad \text{and} \quad |\cos(\sqrt{2}(s+\tau)) - \cos(\sqrt{2}s)| < \frac{\varepsilon}{2},$$

for all $s \in \mathbb{R}$. By the triangle inequality,

$$|h(s+\tau) - h(s)| \leq |\cos(s+\tau) - \cos(s)| + |\cos(\sqrt{2}(s+\tau)) - \cos(\sqrt{2}s)| < \varepsilon,$$

uniformly in s. Thus, $\tau \in \mathcal{V}(h, \varepsilon)$, and $\mathcal{V}(h, \varepsilon)$ is relatively dense.

We now establish fundamental properties of the sets of ε-translations.

Proposition 2.1 (Translation Properties [7, 65]):
Let $h : \mathbb{R} \to \mathbb{S}$ be an almost periodic function. Then,

1. If $0 < \varepsilon \leq \varepsilon'$, then $\mathcal{V}(h, \varepsilon) \subset \mathcal{V}(h, \varepsilon')$.
2. If $\tau \in \mathcal{V}(h, \varepsilon)$, then $-\tau \in \mathcal{V}(h, \varepsilon)$.
3. If $\tau_1 \in \mathcal{V}(h, \varepsilon_1)$ and $\tau_2 \in \mathcal{V}(h, \varepsilon_2)$, then $\tau_1 \pm \tau_2 \in \mathcal{V}(h, \varepsilon_1 + \varepsilon_2)$.

1. Immediate from the definition. If
$$\sup_{s\in\mathbb{R}} d(h(s+\tau), h(s)) < \varepsilon,$$
and $\varepsilon \leq \varepsilon'$, then certainly $\sup_{s\in\mathbb{R}} d(h(s+\tau), h(s)) < \varepsilon'$.
2. For $\tau \in \mathcal{V}(h, \varepsilon)$, we have
$$\sup_{s\in\mathbb{R}} d_{\mathbb{S}}(h(s+\tau), h(s)) < \varepsilon.$$
The substitution $s' = s + \tau$ yields
$$\sup_{s'\in\mathbb{R}} d_{\mathbb{S}}(h(s'), h(s'-\tau)) < \varepsilon,$$
whence $-\tau \in \mathcal{V}(h, \varepsilon)$.
3. For τ_1, τ_2 as stated, the triangle inequality gives,
$$\begin{aligned} d_{\mathbb{S}}(h(s), h(s+\tau_1+\tau_2)) &\leq d_{\mathbb{S}}(h(s), h(s+\tau_1)) + d_{\mathbb{S}}(h(s+\tau_1), h(s+\tau_1+\tau_2)). \\ &\leq \varepsilon_1 + \varepsilon_2. \end{aligned}$$

Thus, $\tau_1 + \tau_2 \in \mathcal{V}(h, \varepsilon_1 + \varepsilon_2)$. A similar argument holds for $\tau_1 - \tau_2$.

Proposition 2.2 (Closedness of Translation Sets [2, 65]):
Let $h : \mathbb{R} \to \mathbb{S}$ be an almost periodic function. Then for every $\varepsilon > 0$, the set $\mathcal{V}(h, \varepsilon)$ is closed in $\mathbb{R}$.

Proof:
Let $\{\tau_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{V}(h, \varepsilon)$ converging to $\tau_\star \in \mathbb{R}$. For each $s \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$d_{\mathbb{S}}(h(s + \tau_n), h(s)) < \varepsilon.$$

By the continuity of h and the metric $d_{\mathbb{S}}$, taking the limit as $n \to \infty$ yields

$$d_{\mathbb{S}}(h(s + \tau_\star), h(s)) \leq \varepsilon \quad \text{for all } s \in \mathbb{R}.$$

Hence, $\tau_\star \in \mathcal{V}(h, \varepsilon)$, proving that $\mathcal{V}(h, \varepsilon)$ is closed. □

Proposition 2.3 (Existence of Minimal Inclusion Length [2, 65]):
Let $\ell_\star(\varepsilon) = \inf \mathcal{W}(h, \varepsilon)$. Then $\ell_\star(\varepsilon) \geq 0$ and $\ell_\star(\varepsilon) \in \mathcal{W}(h, \varepsilon)$. Moreover, the function $\varepsilon \mapsto \ell_\star(\varepsilon)$ is decreasing.

Proof:
By the definition of the infimum, for each $n \in \mathbb{N}$, there exists $\ell_n \in \mathcal{W}(h, \varepsilon)$ such that

$$\ell_\star(\varepsilon) \leq \ell_n \leq \ell_\star(\varepsilon) + 1/n.$$

For an arbitrary $\kappa \in \mathbb{R}$, the interval $[\kappa, \kappa + \ell_n]$ contains some $\tau_n \in \mathcal{V}(h, \varepsilon)$. Since $\tau_n \in [\kappa, \kappa + \ell_\star(\varepsilon) + 1/n]$, the sequence $\{\tau_n\}$ is bounded and thus possesses a convergent subsequence $\{\tau_{n_m}\}$ with limit $\tau_\star \in [\kappa, \kappa + \ell_\star(\varepsilon)]$. As $\mathcal{V}(h, \varepsilon)$ is closed, $\tau_\star \in \mathcal{V}(h, \varepsilon)$. Therefore, every interval of length $\ell_\star(\varepsilon)$ contains an ε-almost period, so $\ell_\star(\varepsilon) \in \mathcal{W}(h, \varepsilon)$. The monotonicity follows from the observation that $\mathcal{W}(h, \varepsilon_1) \subset \mathcal{W}(h, \varepsilon_2)$ for $0 < \varepsilon_1 < \varepsilon_2$. □

Proposition 2.4 (Criterion for Periodicity [65]):
Let $h : \mathbb{R} \to \mathbb{S}$ be an almost periodic function. If the set $\{\ell_\star(\varepsilon) : 0 < \varepsilon < 1\}$ is bounded, then h is periodic.

Proof:
Suppose $\ell_\star(\varepsilon) < \ell$ for all $\varepsilon \in (0, 1)$. For each $n \in \mathbb{N}$, choose $\tau_n \in \mathcal{V}(h, 1/n) \cap [\ell, 2\ell]$. The sequence $\{\tau_n\}$ is bounded and thus has a convergent subsequence $\{\tau_{n_m}\} \to \tau_\star \in [\ell, 2\ell]$. For all $s \in \mathbb{R}$,

$$d_{\mathbb{S}}(h(s + \tau_{n_m}), h(s)) \leq 1/n_m.$$

Taking the limit $m \to \infty$ and exploiting the continuity of h and $d_{\mathbb{S}}$ yields $d_{\mathbb{S}}(h(s + \tau_\star), h(s)) = 0$ for all s, implying h is periodic with period $\tau_\star$. □

2.2. Properties of Bohr Almost Periodic Functions

This section delineates the fundamental properties inherent to almost periodic functions.

Proposition 2.5 (Invariance under Basic Operations [65]):
Let $h : \mathbb{R} \to \mathbb{S}$ be almost periodic. Then,

1. For any $\kappa \in \mathbb{R}$, the function κh is almost periodic.
2. For any $a \in \mathbb{R}$, the translated function $h_a(s) = h(a + s)$ is almost periodic, and $\mathcal{V}(h, \varepsilon) = \mathcal{V}(h_a, \varepsilon)$.

Proof:

1. For $\kappa \neq 0$, if $\tau \in \mathcal{V}(h, \varepsilon/|\kappa|)$, then
$$\sup_s d_{\mathbb{S}}(\kappa h(s + \tau), \kappa h(s)) = |\kappa| \sup_s d_{\mathbb{S}}(h(s + \tau), h(s)) \leq \varepsilon.$$
Hence $\mathcal{V}(h, \varepsilon/|\kappa|) \subset \mathcal{V}(\kappa h, \varepsilon)$, proving relative density. The case $\kappa = 0$ is trivial.
2. For $\tau \in \mathcal{V}(h, \varepsilon)$, we have
$$\sup_s d_{\mathbb{S}}(h(a + s + \tau), h(a + s)) \leq \varepsilon,$$
so $\tau \in \mathcal{V}(h_a, \varepsilon)$. Conversely, if $\tau \in \mathcal{V}(h_a, \varepsilon)$, then
$$\sup_s d_{\mathbb{S}}(h(s + \tau), h(s)) = \sup_s d_{\mathbb{S}}(h_a(s - a + \tau), h_a(s - a)) \leq \varepsilon,$$
so $\tau \in \mathcal{V}(h, \varepsilon)$.

□

Proposition 2.6 (Boundedness and Uniform Continuity [65]):
Every almost periodic function $h : \mathbb{R} \to \mathbb{S}$ is bounded and uniformly continuous.

Proof:
Boundedness. For $\varepsilon = 1$, let $\ell = \ell_\star(1)$. For any $s \in \mathbb{R}$, the interval $[-s, -s + \ell]$ contains a 1-translation τ. Then $s + \tau \in [0, \ell]$. Since h is continuous on the compact set $[0, \ell]$, it is bounded there, say by M. Thus,
$$d_{\mathbb{S}}(h(s), h(0)) \leq d_{\mathbb{S}}(h(s), h(s + \tau)) + d_{\mathbb{S}}(h(s + \tau), h(0)) \leq 1 + M,$$
establishing boundedness on $\mathbb{R}$.

Uniform Continuity. Let $\varepsilon > 0$. Choose $\ell \in \mathcal{W}(h, \varepsilon/3)$. On the compact set $[0, \ell + 2]$, h is uniformly continuous; hence there exists $\delta > 0$ such that $|s_1 - s_2| < \delta$ implies $d_{\mathbb{S}}(h(s_1), h(s_2)) < \varepsilon/3$. Now, suppose $|s' - s''| < \delta$. The interval $[-s' + 1, -s' + 1 + \ell]$

contains an $\varepsilon/3$-translation τ of h. Then $s'+\tau \in [1, 1+\ell]$ and $s''+\tau \in (1-\delta, 1+\ell+\delta) \subset [0, \ell+2]$. Consequently,

$$\begin{aligned} d_{\mathbb{S}}(h(s'), h(s'')) \leq d_{\mathbb{S}}(h(s'), h(s'+\tau)) + d_{\mathbb{S}}(h(s'+\tau), h(s''+\tau)) \\ + d_{\mathbb{S}}(h(s''+\tau), h(s'')) \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon, \end{aligned}$$

proving uniform continuity. □

The following theorem is pivotal for establishing the algebraic properties of almost periodic functions.

Theorem 2.1 (Intersection Density [7]):
Let $h_1, h_2 : \mathbb{R} \to \mathbb{S}$ be almost periodic functions. For every $\varepsilon > 0$, the intersection $\mathcal{V}(h_1, \varepsilon) \cap \mathcal{V}(h_2, \varepsilon)$ is relatively dense.

The proof of this theorem employs two lemmas.

Lemma 2.1 (Perturbation Stability [7]):
Let h be almost periodic and $0 < \varepsilon_1 < \varepsilon_2$. There exists $\upsilon > 0$ such that for all $\tau \in \mathcal{V}(h, \varepsilon_1)$ and $\theta \in (-\upsilon, \upsilon)$, we have $\tau + \theta \in \mathcal{V}(h, \varepsilon_2)$.

Proof:
By the uniform continuity of h, choose $\upsilon > 0$ such that for all $r, r' \in \mathbb{R}$, $|r - r'| \leq \upsilon$ implies

$$d_{\mathbb{S}}(h(u+r), h(u+r')) \leq \varepsilon_2 - \varepsilon_1, \quad \text{for all } u \in \mathbb{R}.$$

In particular, $\theta \in (-\upsilon, \upsilon)$ implies

$$\sup_s d_{\mathbb{S}}(h(s+\theta), h(s)) \leq \varepsilon_2 - \varepsilon_1,$$

so $\theta \in \mathcal{V}(h, \varepsilon_2 - \varepsilon_1)$. Then for $\tau \in \mathcal{V}(h, \varepsilon_1)$,

$$\begin{aligned} d_{\mathbb{S}}(h(s+\tau+\theta), h(s)) &\leq d_{\mathbb{S}}(h(s+\tau+\theta), h(s+\tau)) + d_{\mathbb{S}}(h(s+\tau), h(s)) \\ &\leq (\varepsilon_2 - \varepsilon_1) + \varepsilon_1 = \varepsilon_2, \end{aligned}$$

so $\tau + \theta \in \mathcal{V}(h, \varepsilon_2)$. □

Lemma 2.2 (Translation Approximation [7]):
Let $h_1, h_2 : \mathbb{R} \to \mathbb{S}$ be almost periodic. For any $\varepsilon, \upsilon > 0$, the set of numbers in $\mathcal{V}(h_1, \varepsilon)$ that lie within distance υ of $\mathcal{V}(h_2, \varepsilon)$ is relatively dense.

Proof:
Let $\ell = \max(\ell_\star(h_1, \varepsilon/2), \ell_\star(h_2, \varepsilon/2))$. Choose $k \in \mathbb{N}$ such that $k\upsilon \geq \ell$. Partition $\mathbb{R}$ into intervals $I_n = [(n-1)\ell, n\ell]$ for $n \in \mathbb{Z}$. In each I_n, select $\tau_1^{(n)} \in \mathcal{V}(h_1, \varepsilon/2) \cap I_n$ and $\tau_2^{(n)} \in \mathcal{V}(h_2, \varepsilon/2) \cap I_n$. Their difference satisfies

$$|\tau_1^{(n)} - \tau_2^{(n)}| \leq \ell.$$

Partition the interval $[-\ell, \ell]$ into $2k$ subintervals of length υ, namely $J_i = [(i-k)\upsilon, (i-k+1)\upsilon)$ for $i = 0, 1, \ldots, 2k-1$. Each difference $\tau_1^{(n)} - \tau_2^{(n)}$ falls into one of these J_i.

By the pigeonhole principle, there exists an index i_0 and an infinite set $N \subseteq \mathbb{Z}$ such that for all $n \in N$, $\tau_1^{(n)} - \tau_2^{(n)} \in J_{i_0}$. For $n, m \in N$, we have,

$$|(\tau_1^{(n)} - \tau_2^{(n)}) - (\tau_1^{(m)} - \tau_2^{(m)})| < \upsilon.$$

Rearranging gives

$$|(\tau_1^{(n)} - \tau_1^{(m)}) - (\tau_2^{(n)} - \tau_2^{(m)})| < \upsilon.$$

Set $\theta = (\tau_1^{(n)} - \tau_1^{(m)}) - (\tau_2^{(n)} - \tau_2^{(m)})$. Then,

$$\tau_1^{(n)} - \tau_1^{(m)} = (\tau_2^{(n)} - \tau_2^{(m)}) + \theta.$$

By Proposition 2.1, $\tau_1^{(n)} - \tau_1^{(m)} \in \mathcal{V}(h_1, \varepsilon)$ and $\tau_2^{(n)} - \tau_2^{(m)} \in \mathcal{V}(h_2, \varepsilon)$. Moreover, $|\theta| < \upsilon$, so $\tau_1^{(n)} - \tau_1^{(m)}$ is within distance υ of $\mathcal{V}(h_2, \varepsilon)$. The set of all such differences is relatively dense due to the finite covering number k. □

Proof of Theorem 2.1:
Choose $0 < \varepsilon_1 < \varepsilon$. By Lemma 2.1, select $\upsilon > 0$ such that perturbing an ε_1-almost period of h_1 by less than υ yields an ε-almost period. By Lemma 2.2, the set $\mathcal{G}$ of elements in $\mathcal{V}(h_2, \varepsilon_1)$ within υ of $\mathcal{V}(h_1, \varepsilon_1)$ is relatively dense. For any $\tau \in \mathcal{G}$, there exists $\tau' \in \mathcal{V}(h_1, \varepsilon_1)$ with $|\tau - \tau'| < \upsilon$, whence $\tau = \tau' + (\tau - \tau') \in \mathcal{V}(h_1, \varepsilon)$. Thus, $\mathcal{G} \subset \mathcal{V}(h_1, \varepsilon) \cap \mathcal{V}(h_2, \varepsilon_1) \subset \mathcal{V}(h_1, \varepsilon) \cap \mathcal{V}(h_2, \varepsilon)$, proving the theorem. □

Proposition 2.7 (Stability under Addition [7]):
If $h_1, h_2 : \mathbb{R} \to \mathbb{S}$ are almost periodic, then $h_1 + h_2$ is almost periodic.

Proof:
For $\varepsilon > 0$, if $\tau \in \mathcal{V}(h_1, \varepsilon/2) \cap \mathcal{V}(h_2, \varepsilon/2)$, then

$$d_{\mathbb{S}}((h_1 + h_2)(s + \tau), (h_1 + h_2)(s)) \leq d_{\mathbb{S}}(h_1(s + \tau), h_1(s)) + d_{\mathbb{S}}(h_2(s + \tau), h_2(s)) \leq \varepsilon.$$

Hence $\mathcal{V}(h_1, \varepsilon/2) \cap \mathcal{V}(h_2, \varepsilon/2) \subset \mathcal{V}(h_1 + h_2, \varepsilon)$. By Theorem 2.1, this intersection is relatively dense. □

Proposition 2.8 (Stability under Uniform Limits [65]):
The uniform limit of a sequence of almost periodic functions $h_n : \mathbb{R} \to \mathbb{S}$ is almost periodic.

Proof:
Let $h_n \to h_\star$ uniformly. For $\varepsilon > 0$, choose n_0 such that

$$\sup_s d_{\mathbb{S}}(h_\star(s), h_{n_0}(s)) < \varepsilon/3.$$

If $\tau \in \mathcal{V}(h_{n_0}, \varepsilon/3)$, then

$$\begin{aligned} d_{\mathbb{S}}(h_\star(s+\tau), h_\star(s)) &\le d_{\mathbb{S}}(h_\star(s+\tau), h_{n_0}(s+\tau)) + d_{\mathbb{S}}(h_{n_0}(s+\tau), h_{n_0}(s)) \\ &\quad + d_{\mathbb{S}}(h_{n_0}(s), h_\star(s)) \\ &\le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

Thus, $\mathcal{V}(h_{n_0}, \varepsilon/3) \subset \mathcal{V}(h_\star, \varepsilon)$, proving $h_\star$ is almost periodic. □

Proposition 2.9 (Relative Compactness of the Range [65]):
The range $\mathcal{R}_h = \{h(s) : s \in \mathbb{R}\}$ of a continuous almost periodic function h is totally bounded, and hence its closure is compact in $\mathbb{S}$.

Proof:
Let $\varepsilon > 0$. Choose $\ell \in \mathcal{W}(h, \varepsilon/2)$. The set $\mathcal{R}_h[0,\ell] = \{h(s) : s \in [0,\ell]\}$ is compact, hence totally bounded. Let $\{\mathrm{Ball}(e_i, \varepsilon/2)\}_{i=1}^n$ cover it. For any $s \in \mathbb{R}$, there exists $\tau \in \mathcal{V}(h, \varepsilon/2) \cap [-s, -s+\ell]$, so $s + \tau \in [0,\ell]$ and $h(s+\tau) \in B(e_i, \varepsilon/2)$ for some i. Then

$$d_{\mathbb{S}}(h(s), e_i) \le d_{\mathbb{S}}(h(s), h(s+\tau)) + d_{\mathbb{S}}(h(s+\tau), e_i) \le \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus, $\{\mathrm{Ball}(e_i, \varepsilon)\}_{i=1}^n$ covers $\mathcal{R}_h$. □

2.3. Normal Functions

Bochner [9] introduced an alternative, structural characterisation of almost periodicity through the concept of normality, which has proven invaluable for analysing algebraic and topological properties.

Definition 2.3 (Bochner's Criterion [2, 7]):
A continuous function $h : \mathbb{R} \longrightarrow \mathbb{S}$ is called *normal* if for every sequence $\{u_n\} \subset \mathbb{R}$, there exists a subsequence $\{u_{n_k}\}$ such that the translates $\{h(\cdot + u_{n_k})\}$ converge uniformly on $\mathbb{R}$.

Elementary Properties of Normal Functions

Proposition 2.10 (Boundedness of Normal Functions [65]):
Every normal function is bounded.

Proof:
If h were unbounded, there would exist a sequence $\{s_n\}$ with $d_{\mathbb{S}}(h(s_n), h(0)) \to \infty$. No subsequence of $\{h(\cdot + s_n)\}$ could converge uniformly, as it would diverge at $s = -s_n$. □

Proposition 2.11 (Supremum Preservation [2, 65]):
If $h : \mathbb{R} \longrightarrow \mathbb{S}$ is normal, then

$$\sup_{s \geq 0} d_{\mathbb{S}}(h(s), h(0)) = \sup_{s \in \mathbb{R}} d_{\mathbb{S}}(h(s), h(0)).$$

Proof:
Let $M = \sup_{s \geq 0} d_{\mathbb{S}}(h(s), h(0))$. Consider the sequence of translates $\{h(\cdot + n)\}_{n \in \mathbb{N}}$. By normality, a subsequence $\{h(\cdot + n_k)\}$ converges uniformly to some $h_\star$. For any $s_0 < 0$, choose k large so that $s_0 + n_k \geq 0$. Then

$$d_{\mathbb{S}}(h(s_0 + n_k), h(0)) \leq M.$$

Taking the limit yields

$$d_{\mathbb{S}}(h_\star(s_0), h(0)) \leq M.$$

From uniform convergence, for large k,

$$d_{\mathbb{S}}(h(s), h(0)) \leq d_{\mathbb{S}}(h_\star(s - n_k), h(0)) + \varepsilon \leq M + \varepsilon,$$

for all s. As $\varepsilon > 0$ is arbitrary, the result follows. □

Corollary 2.1 (Uniform Convergence Extension [65]):
If a sequence of normal functions $\{h_n\}$ converges uniformly on $\mathbb{R}_{\geq 0}$, then it converges uniformly on $\mathbb{R}$.

Proof:
For $m, n \in \mathbb{N}$, the function $h_m - h_n$ is normal. By Proposition 2.11,

$$\sup_{s \in \mathbb{R}} d_{\mathbb{S}}(h_m(s), h_n(s)) = \sup_{s \geq 0} d_{\mathbb{S}}(h_m(s), h_n(s)).$$

The right-hand side tends to zero as $m, n \to \infty$, so the sequence is uniformly Cauchy on $\mathbb{R}$.
□

Proposition 2.12 (Limit Properties [65]):
Let h be normal. Then,

$$\limsup_{s \to \infty} d_{\mathbb{S}}(h(s), h(0)) = \sup_{s \in \mathbb{R}} d_{\mathbb{S}}(h(s), h(0)).$$

Proof:
By Proposition 2.11,

$$\limsup_{s \to \infty} d_{\mathbb{S}}(h(s), h(0)) = \lim_{u \to \infty} \sup_{s \geq u} d_{\mathbb{S}}(h(s), h(0)) = \sup_{s \in \mathbb{R}} d_{\mathbb{S}}(h(s), h(0)).$$

□

The Hull of a Function

Definition 2.4 (Hull):
The *hull* of h, denoted $\mathcal{H}(h)$, is the closure in $C_b(\mathbb{R};\mathbb{S})$ of its set of translates,

$$\mathcal{H}(h) = \overline{\{h(\cdot+\tau) : \tau \in \mathbb{R}\}}.$$

***Remark* 2.1 ([65]):**
A function is normal if and only if its hull $\mathcal{H}(h)$ is compact, which is equivalent to the total boundedness of its set of translates.

Proposition 2.13 (Invariance of the Hull [65]):
If h is almost periodic and $g \in \mathcal{H}(h)$, then g is almost periodic and $\mathcal{H}(g) = \mathcal{H}(h)$.

Proof:
Since h is almost periodic, every translate of h is almost periodic. As uniform limits preserve almost periodicity, g is almost periodic.

To show $\mathcal{H}(g) = \mathcal{H}(h)$, let $g_\star \in \mathcal{H}(g)$. Then there exists $\{u_n\}_{n\in\mathbb{N}}$ such that $g(\cdot+u_n) \to g_\star$ uniformly. Also, since $g \in \mathcal{H}(h)$, there exists $\{v_n\}_{n\in\mathbb{N}}$ such that $h(\cdot+v_n) \to h_\star = g$ uniformly. For any $\varepsilon > 0$, there exists n_0 such that for $n \geq n_0$,

$$\sup_{s\in\mathbb{R}} d_{\mathbb{S}}(g_\star(s), g(s+u_n)) \leq \frac{\varepsilon}{2}, \quad \sup_{s\in\mathbb{R}} d_{\mathbb{S}}(g(s), h(s+v_n)) \leq \frac{\varepsilon}{2}.$$

Then,

$$\sup_{s\in\mathbb{R}} d_{\mathbb{S}}(g_\star(s-u_n), g(s)) \leq \frac{\varepsilon}{2},$$

and hence,

$$\sup_{s\in\mathbb{R}} d_{\mathbb{S}}(g_\star(s-u_n), h(s+v_n)) \leq \varepsilon.$$

Changing variables, we get,

$$\sup_{s\in\mathbb{R}} d_{\mathbb{S}}(g_\star(s), h(s+u_n+v_n)) \leq \varepsilon.$$

Thus, $g_\star \in \mathcal{H}(h)$, so $\mathcal{H}(g) \subseteq \mathcal{H}(h)$. The reverse inclusion follows similarly. □

2.4. The Fundamental Theorem

The following theorem establishes the profound equivalence of the Bohr and Bochner definitions, unifying the approximation-based and structural perspectives on almost periodicity.

Theorem 2.2 (Bohr–Bochner Theorem [7, 9, 65]):
For a continuous function $h : \mathbb{R} \longrightarrow \mathbb{S}$, the following are equivalent,

1. h is normal.
2. h is almost periodic.

Proof:
(2) $\Rightarrow$ (1) Assume h is almost periodic. Its uniform continuity implies that for $\varepsilon > 0$, there exists $\delta > 0$ such that $|s - r| \leq \delta$ implies $d_{\mathbb{S}}(h(u+s), h(u+r)) \leq \varepsilon/2$ for all u. Let $\ell \in \mathcal{W}(h, \varepsilon/2)$ and cover $[0, \ell]$ with finitely many intervals $[\kappa_i, \kappa_i + \delta]$. For any $\kappa \in \mathbb{R}$, there exists $\tau \in \mathcal{V}(h, \varepsilon/2) \cap [-\kappa, -\kappa + \ell]$, so $\kappa + \tau \in [0, \ell]$ and $|\kappa + \tau - \kappa_i| \leq \delta$ for some i. Then

$$\sup_u d_{\mathbb{S}}(h(u+\kappa), h(u+\kappa_i)) \leq \sup_u d_{\mathbb{S}}(h(u+\kappa), h(u+\kappa+\tau)) + \sup_u d_{\mathbb{S}}(h(u+\kappa+\tau), h(u+\kappa_i)) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus, the set of translates is totally bounded, hence h is normal.

(1) $\Rightarrow$ (2) Assume h is normal. For $\varepsilon > 0$, by total boundedness, there exist $\kappa_1, \ldots, \kappa_n$ such that for every τ,

$$\sup_s d_{\mathbb{S}}(h(s+\kappa_i), h(s+\tau)) \leq \varepsilon,$$

for some i. This implies $\tau - \kappa_i \in \mathcal{V}(h, \varepsilon)$. With $\ell = \max_i |\kappa_i|$, we have $\tau - \ell \leq \tau - \kappa_i \leq \tau + \ell$, so $\mathcal{V}(h, \varepsilon)$ is relatively dense with inclusion length 2ℓ. $\square$

This profound equivalence subsequently streamlines the proof of several properties,

Proposition 2.14 (Compactness of Translate Sets [7, 65]):
If h_1 and h_2 are almost periodic, then the sets of their translates are relatively compact in $C_b(\mathbb{R}; \mathbb{S})$.

Sketch:
The function $F(s) = (h_1(s), h_2(s))$ is almost periodic in $\mathbb{S} \times \mathbb{S}$. Its hull is compact. For $\tau \in \mathcal{V}(F, \varepsilon)$, we have $\tau \in \mathcal{V}(h_1, \varepsilon) \cap \mathcal{V}(h_2, \varepsilon)$, which is thus relatively dense. $\square$

2.5. The Fundamental Definition

The theory naturally extends to functions of several variables, a generalisation crucial for applications to differential equations and dynamical systems. Let $(\mathbb{S}_1, d_{\mathbb{S}_1})$ and $(\mathbb{S}_2, d_{\mathbb{S}_2})$ be complete, separable metric spaces, and let $\mathcal{K}_1$ be a family of compact subsets of $\mathbb{S}_1$. A function $\hat{h} : \mathbb{R} \times \mathbb{S}_1 \to \mathbb{S}_2$ is *almost periodic in $s \in \mathbb{R}$, uniformly for $y \in \mathbb{K}_1$* (for each $\mathbb{K}_1 \in \mathcal{K}_1$) if it satisfies any of the following equivalent conditions [10, 18],

1. **Bohr's Criterion.** For every $\varepsilon > 0$ and $\mathbb{K}_1 \in \mathcal{K}_1$, the set

$$\mathcal{V}(\hat{h}, \varepsilon, \mathbb{K}_1) = \left\{ \tau \in \mathbb{R} : \sup_{s \in \mathbb{R}} \sup_{y \in \mathbb{K}_1} d_{\mathbb{S}_2}(\hat{h}(s + \tau, y), \hat{h}(s, y)) < \varepsilon \right\},$$

is relatively dense.

2. **Bochner-Type Characterisation (Single Sequence).** For every sequence $\{u'_n\} \subset \mathbb{R}$, there exists a subsequence $\{u_n\}$ and a function $\hat{h}_\star \in C_b(\mathbb{R} \times \mathbb{S}_1; \mathbb{S}_2)$ such that

$$\lim_{n \to \infty} \hat{h}(s + u_n, y) = \hat{h}_\star(s, y)$$

uniformly for $(s, y) \in \mathbb{R} \times \mathbb{K}_1$ (for each $\mathbb{K}_1 \in \mathcal{K}_1$).

3. **Bochner-Type Characterisation (Double Sequence).** For any two sequences $\{u'_n\}, \{v'_n\} \subset \mathbb{R}$, there exist common subsequences $\{u_n\}, \{v_n\}$ such that the iterated limits exist and are equal,

$$\lim_{m \to \infty} \lim_{n \to \infty} \hat{h}(s + u_m + v_n, y) = \lim_{n \to \infty} \lim_{m \to \infty} \hat{h}(s + u_m + v_n, y),$$

uniformly for $(s, y) \in \mathbb{R} \times \mathbb{K}_1$.

Chapter 3.

Foundations of Stochastic Analysis in Infinite Dimensions

This chapter establishes the rigorous mathematical framework for stochastic analysis in infinite-dimensional spaces. The study of stochastic processes taking values in Banach or Hilbert spaces is fundamental to the theory of SDEs, which model phenomena across physics, engineering, and finance. Unlike their finite-dimensional counterparts, these processes present unique challenges related to topology, measure theory, and the definition of fundamental objects like Brownian motion and the stochastic integral.

Our presentation proceeds systematically. We begin with the foundational concepts of measure and probability theory on Polish spaces. We then introduce the geometric structure of Banach and Hilbert spaces, culminating in the construction of the Itô stochastic integral for Hilbert-space-valued processes and a corresponding version of Itô's formula. The chapter concludes by applying this machinery to the theory of SDEs.

The chapter is based primarily on the work of Da Prato and Zabczyk [19]. Our first contribution is a generalisation of their results from the finite time interval $[0, b]$ to a more general setting. Our second, and key, contribution is an improvement to the Burkholder-Davis-Gundy (BDG) inequality, which provides crucial moment estimates for stochastic integrals.

3.1. Measure-Theoretic and Probabilistic Foundations

Modern probability theory is built upon measure theory. We therefore begin by recalling the essential definitions, emphasising the topological context crucial for infinite-dimensional analysis.

Definition 3.1 (Measurable Space [32]):
A *measurable space* is a pair $(\Omega, \mathscr{F})$, where Ω is a non-empty set and $\mathscr{F}$ is a σ-algebra on Ω. A collection $\mathscr{F}$ of subsets Ω is a σ-algebra if it satisfies,

1. $\Omega \in \mathscr{F}$,
2. If $A \in \mathscr{F}$, then $A^c = \Omega \setminus A \in \mathscr{F}$,
3. If $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathscr{F}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathscr{F}$.

To incorporate topological structure, we introduce the Borel σ-algebra.

Definition 3.2 (Borel σ-Algebra [19]):
Let $(\mathbb{S}, \tau)$ be a topological space. The *Borel σ-algebra* on $\mathbb{S}$, denoted $\mathcal{B}(\mathbb{S})$, is the smallest σ-algebra on $\mathbb{S}$ containing all open sets in τ. If $\mathbb{S}$ is a metric space with metric $d_{\mathbb{S}}$, then $\mathcal{B}(\mathbb{S})$ is generated by the collection of open balls $\{\mathrm{Ball}(x, r) : x \in \mathbb{S}, r > 0\}$.

A central objective is to study measurable mappings from a probability space into a state space.

Definition 3.3 (Random Element [19]):
Let $(\Omega, \mathscr{F})$ and $(\mathbb{S}, \mathscr{S})$ be measurable spaces. A mapping $Y : \Omega \to \mathbb{S}$ is called a *random element* in $\mathbb{S}$ (or is *$\mathscr{S}$-measurable*) if for every $B \in \mathscr{S}$, its preimage satisfies

$$Y^{-1}(B) = \{\omega \in \Omega : Y(\omega) \in B\} \in \mathscr{F}.$$

If $(\mathbb{S}, \mathscr{S})$ is a topological space endowed with its Borel σ-algebra, Y is called an *$\mathbb{S}$-valued random variable*.

The probabilistic behaviour of a random element is quantified by a probability measure.

Definition 3.4 (Probability Measure and Space [19]):
A *probability measure* $\mathbb{P}$ on a measurable space $(\Omega, \mathscr{F})$ is a function $\mathbb{P} : \mathscr{F} \to [0, 1]$ satisfying,

1. $\mathbb{P}(\Omega) = 1$,
2. For any sequence $\{A_n\}_{n=1}^{\infty} \subseteq \mathscr{F}$ of pairwise disjoint sets, $\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$.

The triple $(\Omega, \mathscr{F}, \mathbb{P})$ is called a *probability space*.

Definition 3.5 (Law of a Random Element [19]):
Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and let $Y : \Omega \to \mathbb{S}$ be a random element in a measurable space $(\mathbb{S}, \mathscr{S})$. The *law* (or *distribution*) of Y, denoted $\mathscr{L}(Y)$, is the pushforward measure on $(\mathbb{S}, \mathscr{S})$ defined for all $B \in \mathscr{S}$ by,

$$\mathscr{L}(Y)(B) = \mathbb{P}(Y^{-1}(B)) = \mathbb{P}(\{\omega \in \Omega : Y(\omega) \in B\}).$$

For technical reasons, particularly in the theory of stochastic processes, it is essential to work with complete probability spaces.

Definition 3.6 (Complete Probability Space [19]):
A probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is said to be *complete* if $\mathscr{F}$ contains all subsets of $\mathbb{P}$-null sets. Formally, if $N \in \mathscr{F}$ with $\mathbb{P}(N) = 0$, then for any $A \subseteq N$, we have $A \in \mathscr{F}$ (and consequently $\mathbb{P}(A) = 0$).

Any probability space can be completed by considering the σ-algebra $\overline{\mathscr{F}} = \sigma(\mathscr{F} \cup \mathscr{N})$, where

$$\mathscr{N} = \{N \subseteq \Omega : \exists M \in \mathscr{F} \text{ with } N \subseteq M \text{ and } \mathbb{P}(M) = 0\},$$

and extending $\mathbb{P}$ to $\overline{\mathscr{F}}$ in the canonical way.

Finally, we introduce the Bochner integral, which generalises the Lebesgue integral to Banach-space-valued functions.

Definition 3.7 (Bochner Integral and Conditional Expectation [19]):
Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space and let $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be a Banach space.

1. A random variable $Y : \Omega \to \mathbb{Y}$ is called *Bochner integrable* if it is strongly measurable (i.e., the pointwise limit of a sequence of simple functions) and
$$\int_{\Omega} \|Y(\omega)\|_{\mathbb{Y}} \, \mathrm{d}\mathbb{P}(\omega) < \infty.$$
The *Bochner integral* (or *expectation*) is then defined by,
$$\mathbb{E}[Y] := \int_{\Omega} Y \, \mathrm{d}\mathbb{P} := \lim_{n\to\infty} \int_{\Omega} Y_n \, \mathrm{d}\mathbb{P},$$
where (Y_n) is any sequence of simple functions converging to Y almost surely and in $L^1(\Omega; \mathbb{Y})$.

2. Let $\mathscr{G} \subseteq \mathscr{F}$ be a sub-σ-algebra. If Y is Bochner integrable, its *conditional expectation* given $\mathscr{G}$, denoted $\mathbb{E}[Y \mid \mathscr{G}]$, is the unique (up to a $\mathbb{P}$-null set) $\mathscr{G}$-measurable, Bochner-integrable, $\mathbb{Y}$-valued random variable Z satisfying,
$$\int_A Z \, \mathrm{d}\mathbb{P} = \int_A Y \, \mathrm{d}\mathbb{P} \quad \text{for all } A \in \mathscr{G}.$$

3.2. Probability Measures on Polish Spaces

Probability theory acquires its fullest strength when developed on *Polish spaces*. This class of spaces provides a delicate balance between generality and structure, broad enough to include all separable Banach and Hilbert spaces, yet sufficiently rigid to support a refined theory of probability measures, tightness, and weak convergence.

3.2.1. Polish Spaces

Definition 3.8 (Polish Space [1]):
A topological space $(\mathbb{S}, \tau)$ is a *Polish space* if it is separable and completely metrisable, that is, there exists a metric $d_{\mathbb{S}} : \mathbb{S} \times \mathbb{S} \to [0, \infty)$ such that $(\mathbb{S}, d_{\mathbb{S}})$ is complete and the topology induced by $d_{\mathbb{S}}$ coincides with τ.

Examples include $\mathbb{R}^n$, every separable Banach space, separable Hilbert spaces, the path space $C_b([0, 1])$, and more generally, every closed subspace of a Polish space.

3.2.2. Weak Convergence of Probability Measures

Let $\mathcal{P}(\mathbb{S})$ denote the set of Borel probability measures on a Polish space $\mathbb{S}$.

Definition 3.9 (Weak Convergence [19]):
A sequence $\{\mu_n\}$ in $\mathcal{P}(\mathbb{S})$ converges weakly to $\mu \in \mathcal{P}(\mathbb{S})$, denoted $\mu_n \Rightarrow \mu$, if

$$\int_{\mathbb{S}} h \, \mathrm{d}\mu_n \to \int_{\mathbb{S}} h \, \mathrm{d}\mu \quad \text{for every } h \in C_b(\mathbb{S}),$$

where $C_b(\mathbb{S})$ denotes the space of bounded continuous functions.

***Remark* 3.1:**
For any $h \in C_b(\mathbb{S})$, we associate a real-valued functional on $\mathcal{P}(\mathbb{S})$,

$$\Phi_h : \mathcal{P}(\mathbb{S}) \longrightarrow \mathbb{R}, \qquad \Phi_h(\nu) = \int_{\mathbb{S}} h(x) \, \mathrm{d}\nu(x).$$

The weak topology on $\mathcal{P}(\mathbb{S})$ is defined as the coarsest topology that makes all the maps

$$\Phi_h : \nu \mapsto \int h \, \mathrm{d}\nu \quad (h \in C_b(\mathbb{S})),$$

continuous.

3.2.3. Metrization of Weak Convergence

The weak topology is metrizable on $\mathcal{P}(\mathbb{S})$. Several metrics are standard.

Definition 3.10 (Bounded Lipschitz Metric [21]):
Let $(\mathbb{S}, d_{\mathbb{S}})$ be a Polish space. For $h : \mathbb{S} \to \mathbb{R}$ define the Lipschitz seminorm

$$|h|_L = \sup_{x \neq y} \frac{|h(x) - h(y)|}{d_{\mathbb{S}}(x, y)}.$$

The bounded Lipschitz norm is $\|h\|_{\mathrm{BL}} = \|h\|_\infty + |h|_L$. The bounded Lipschitz metric on $\mathcal{P}(\mathbb{S})$ is

$$d_{\mathrm{BL}}(\mu,\nu) = \sup\Big\{\Big|\int_{\mathbb{S}} h\,\mathrm{d}\mu - \int_{\mathbb{S}} h\,\mathrm{d}\nu\Big| : h \in \mathrm{BL}(\mathbb{S}), \|h\|_{\mathrm{BL}} \le 1\Big\}.$$

Theorem 3.1 (Dudley metrization theorem [21]):
For a Polish space $\mathbb{S}$, the metric d_{BL} generates the weak topology on $\mathcal{P}(\mathbb{S})$. Moreover $(\mathcal{P}(\mathbb{S}), d_{\mathrm{BL}})$ is itself a Polish space.

***Remark* 3.2 (Alternative metrics):**
Two further classical metrics compatible with weak convergence are,

1. the Lévy–Prokhorov metric

$$d_{\mathrm{LP}}(\mu,\nu) = \inf\{\varepsilon > 0 : \mu(A) \le \nu(A^\varepsilon) + \varepsilon,\ \nu(A) \le \mu(A^\varepsilon) + \varepsilon,\ \forall A \subseteq \mathbb{S} \text{ Borel}\}.$$

2. the Wasserstein metrics W_p, provided $\mathbb{S}$ is bounded or μ, ν have finite p-th moments.

Definition 3.11 (Tightness [19]):
A family $\Pi \subseteq \mathcal{P}(\mathbb{S})$ is *tight* if, for every $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subseteq \mathbb{S}$ with

$$\mu(K_\varepsilon) \ge 1 - \varepsilon \quad \text{for all } \mu \in \Pi.$$

Definition 3.12 (Relative Compactness [19]):
A family $\Pi \subseteq \mathcal{P}(\mathbb{S})$ is relatively compact in the weak topology if its closure in $\mathcal{P}(\mathbb{S})$ is compact, equivalently, if every sequence in Π has a weakly convergent subsequence.

Theorem 3.2 (Prokhorov Theorem [8, 19]):
Let $\mathbb{S}$ be a Polish space and $\Pi \subseteq \mathcal{P}(\mathbb{S})$. Then Π is tight if and only if Π is relatively compact in the weak topology.

***Remark* 3.3 (Ulam's theorem):**
On a compact metric space $\mathbb{S}$, every probability measure is "automatically" tight. Thus compactness of $\mathbb{S}$ implies compactness of $\mathcal{P}(\mathbb{S})$ under weak convergence.

Theorem 3.3 (Portmanteau Theorem [8, 19]):
For $\mu_n, \mu \in \mathcal{P}(\mathbb{S})$ with $\mathbb{S}$ Polish, the following are equivalent,

1. $\mu_n \Rightarrow \mu$.
2. $\limsup_{n\to\infty} \mu_n(F) \le \mu(F)$ for all closed $F \subseteq \mathbb{S}$.
3. $\liminf_{n\to\infty} \mu_n(G) \ge \mu(G)$ for all open $G \subseteq \mathbb{S}$.
4. $\mu_n(A) \to \mu(A)$ for all Borel sets A with $\mu(\partial A) = 0$.

Theorem 3.4 (Skorokhod Representation Theorem [19, 51]):
Let $\mathbb{S}$ be a Polish space and suppose $\mu_n \Rightarrow \mu$ in $\mathcal{P}(\mathbb{S})$. Then there exists a probability space $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{\mathbb{P}})$ and random elements Y_n, Y defined on it such that

1. $\mathscr{L}(Y_n) = \mu_n$ for all n, and $\mathscr{L}(Y) = \mu$,
2. $Y_n \to Y$ almost surely under $\widetilde{\mathbb{P}}$.

3.3. Stochastic Processes and Path Regularity

We now turn to the fundamental objects of study, stochastic processes. Let $\mathbb{Y}$ be a separable Banach space.

Definition 3.13 (Stochastic Process [19]):
Let $J \subseteq \mathbb{R}$ be an interval (typically $J = [s_0, b]$ or $[s_0, \infty)$) and let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space. An $\mathbb{Y}$-*valued stochastic process* is a family $\{Y(s)\}_{s\in J}$ of $\mathbb{Y}$-valued random variables. For a fixed $\omega \in \Omega$, the function $s \mapsto Y(s, \omega)$ is called a *sample path* or *trajectory*.

It is often useful to replace a process with a probabilistically equivalent "nicer" version.

Definition 3.14 (Modifications and Indistinguishability [19]):
Let $Y = \{Y(s)\}_{s\in J}$ and $\tilde{Y} = \{\tilde{Y}(s)\}_{s\in J}$ be two stochastic processes defined on the same probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and taking values in the same measurable space $(\mathbb{Y}, \mathscr{S})$.

1. Y and $\tilde{Y}$ are called *modifications* (of each other) if for every $s \in J$, we have $\mathbb{P}(Y(s) = \tilde{Y}(s)) = 1$.
2. Y and $\tilde{Y}$ are called *indistinguishable* if $\mathbb{P}(Y(s) = \tilde{Y}(s) \text{ for all } s \in J) = 1$.

Indistinguishability implies that the processes are modifications. The converse holds if the time set J is countable or if both processes have almost surely continuous paths.

The analytical properties of a process are determined by its path regularity.

Definition 3.15 (Path Regularity [19]):
Let $Y = \{Y(s)\}_{s\in J}$ be an $\mathbb{Y}$-valued stochastic process.

1. Y is called *measurable* if the mapping $(s, \omega) \mapsto Y(s, \omega)$ is $\mathcal{B}(J)\otimes\mathscr{F}$-$\mathcal{B}(\mathbb{Y})$-measurable.
2. Y is called *stochastically continuous* at $s_0 \in J$ if for every $\varepsilon > 0$,
$$\lim_{s\to s_0} \mathbb{P}(\|Y(s) - Y(s_0)\|_{\mathbb{Y}} > \varepsilon) = 0.$$
The process is stochastically continuous on J if this holds for every $s_0 \in J$.
3. Y is called *mean-square continuous* at $s_0 \in J$ if
$$\lim_{s\to s_0} \mathbb{E}[\|Y(s) - Y(s_0)\|_{\mathbb{Y}}^2] = 0.$$
4. Y is called *pathwise continuous* if for $\mathbb{P}$-almost every $\omega \in \Omega$, the sample path $s \mapsto Y(s, \omega)$ is continuous.

Kolmogorov's continuity criterion and its extensions [19, Theorem 3.3] provide sufficient conditions for the existence of regular modifications.

Proposition 3.1 (Regularisation Theorems [19]):
Let $\mathbb{Y}$ be a separable Banach space.

1. A stochastically continuous process admits a measurable modification.
2. A stochastically continuous process adapted to a filtration admits a progressively measurable modification (see Definition 3.16).
3. A stochastically continuous process with independent increments admits a càdlàg modification.

3.4. Filtrations and Adapted Processes

In order to capture the temporal dynamics of information, we introduce filtrations, which provide a mathematical framework for modelling information flow in stochastic processes.

Definition 3.16 (Filtration [19]):
Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space. A *filtration* on this space is an increasing family $\{\mathscr{F}_s\}_{s\in J}$ of sub-σ-algebras of $\mathscr{F}$ (i.e., $\mathscr{F}_r \subseteq \mathscr{F}_s$ for all $r \leq s$ in J). The filtration is said to satisfy the *usual conditions* if,

1. **Completeness.** $\mathscr{F}_{s_0}$ contains all $\mathbb{P}$-null sets.
2. **Right-continuity.** $\mathscr{F}_s = \mathscr{F}_{s+} := \bigcap_{r>s} \mathscr{F}_r$ for all $s \in J$.

A *filtered probability space* is a quadruple $(\Omega, \mathscr{F}, \{\mathscr{F}_s\}_{s\in J}, \mathbb{P})$ where the filtration satisfies the usual conditions.

Henceforth, all filtered probability spaces are assumed to satisfy the usual conditions.

A process is adapted if its value at time s is determined by the information available up to time s.

Definition 3.17 (Adapted and Progressive Processes):
Let $(\Omega, \mathscr{F}, \{\mathscr{F}_s\}_{s\in J}, \mathbb{P})$ be a filtered probability space and let $Y = \{Y(s)\}_{s\in J}$ be a stochastic process.

1. Y is called *adapted* if for every $s \in J$, the random variable $Y(s)$ is $\mathscr{F}_s$-measurable.
2. Y is called *progressively measurable* if for every $s \in J$, the restriction of Y to $[s_0, s] \times \Omega$ is $\mathcal{B}([s_0, s]) \otimes \mathscr{F}_s$-measurable.

Progressive measurability implies both adaptedness and joint measurability.

For stochastic integration, a more restrictive class of processes is required, those that are predictable.

Definition 3.18 (Predictable σ-Algebra and Processes [19]):
Let $(\Omega, \mathscr{F}, \{\mathscr{F}_s\}_{s\in J}, \mathbb{P})$ be a filtered probability space. The *predictable σ-algebra* $\mathscr{P}$ on $J \times \Omega$ is the σ-algebra generated by sets of the form $(r, s] \times A$ for $s_0 \leq r < s \leq b$, $A \in \mathscr{F}_r$, and $\{s_0\} \times A$ for $A \in \mathscr{F}_{s_0}$. A stochastic process Y is called *predictable* if it is $\mathscr{P}$-measurable.

Intuitively, a predictable process at time s is determined by information strictly before s.

Proposition 3.2 (Predictable Processes [19]):
Every adapted process with left-continuous sample paths (valued in a metric space) is predictable. In particular, any continuous adapted process is predictable.

Stopping times are random times whose occurrence can be determined by the available information.

Definition 3.19 (Stopping Time [19]):
Let $(\Omega, \mathscr{F}, \{\mathscr{F}_s\}_{s\geq s_0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. A random variable $\tau : \Omega \to [s_0, \infty]$ is called a *stopping time* if $\{\tau \leq s\} \in \mathscr{F}_s$ for every $s \geq s_0$.

Example 3.1 (Hitting Time [46]):
Let $Y(s)$ be a progressively measurable process taking values in a Polish space $\mathbb{S}$, and let $B \subseteq \mathbb{S}$ be a Borel set. The *first hitting time* $\tau_B = \inf\{s \geq s_0 : Y(s) \in B\}$ is a stopping time.

3.5. Martingales in Hilbert Spaces

Martingales are stochastic processes whose future conditional expectation, given the present information, equals the present value.

Let $(\mathbb{H}, \langle\cdot,\cdot\rangle_{\mathbb{H}})$ be a separable Hilbert space, and let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_s\}_{s\in J}, \mathbb{P})$ be a filtered probability space.

Definition 3.20 (Hilbert-space valued martingale):
An adapted process $\{M(s)\}_{s\in J}$ is an $\mathbb{H}$-*valued martingale* if

1. $M(s) \in L^1(\Omega, \mathfrak{F}_s, \mathbb{P}; \mathbb{H})$ for all $s \in J$,
2. for all $r \leq s$,

$$\mathbb{E}[M(s) \mid \mathfrak{F}_r] = M(r), \qquad \mathbb{P}\text{-a.s.}$$

When defining quadratic variation, we additionally require *square-integrability*, i.e. $M(s) \in L^2(\Omega; \mathbb{H})$ for all s.

Definition 3.21 (Trace-class operator):
An operator $Q \in \mathcal{L}(\mathbb{H})$ is called *trace-class* (nuclear) if there exist orthonormal systems $\{u_n\} \subset \mathbb{H}$ and scalars $\{\lambda_n\} \subset \mathbb{C}$ with $\sum_{n=1}^{\infty} |\lambda_n| < \infty$ such that

$$Qx = \sum_{n=1}^{\infty} \lambda_n \langle x, u_n \rangle_{\mathbb{H}} u_n, \qquad x \in \mathbb{H}.$$

Denote by $\mathcal{L}^1(\mathbb{H})$ the separable Banach space of nuclear (trace-class) operators on $\mathbb{H}$, endowed with the norm

$$\|Q\|_{\mathcal{L}^1} = \operatorname{Tr}\left((Q^*Q)^{1/2}\right).$$

Definition 3.22 (Rank-one operator):
For $x, y \in \mathbb{H}$, define the rank-one operator $x \otimes y \in \mathcal{L}^1(\mathbb{H})$ by

$$(x \otimes y)(h) = \langle h, y \rangle_{\mathbb{H}} \, x, \qquad h \in \mathbb{H}.$$

In particular, $x \otimes y$ is always trace-class.

Definition 3.23 (Increasing process):
An $\mathcal{L}^1(\mathbb{H})$-valued adapted process $V = \{V(s)\}_{s \in J}$ is called *increasing* if

1. $V(s_0) = 0$.
2. $V(s)$ is positive and self-adjoint for all s, i.e. $\langle V(s)x, x \rangle_{\mathbb{H}} \geq 0$ for all $x \in \mathbb{H}$.
3. $V(s) - V(r)$ is positive whenever $s \geq r$.

Definition 3.24 (Quadratic variation of an $\mathbb{H}$-valued martingale):
Let M be an $\mathbb{H}$-valued square-integrable martingale. Its *quadratic variation* is the unique adapted, continuous, increasing process $\langle\langle M \rangle\rangle = \{\langle\langle M \rangle\rangle(s)\}_{s \in J}$, with values in $\mathcal{L}^1(\mathbb{H})$, such that

1. $\langle\langle M \rangle\rangle(s_0) = 0$.
2. For all $x, y \in \mathbb{H}$,

$$\langle M(s), x \rangle_{\mathbb{H}} \langle M(s), y \rangle_{\mathbb{H}} - \langle \langle\langle M \rangle\rangle(s)x, y \rangle_{\mathbb{H}},$$

   is a real-valued martingale.

Equivalently, the $\mathcal{L}^1(\mathbb{H})$-valued process

$$M(s) \otimes M(s) - \langle\langle M \rangle\rangle(s), \qquad s \in J,$$

is a martingale.

Proposition 3.3 (Properties of $\mathbb{H}$-valued martingales [19, Prop. 3.4]):
Let M be an $\mathbb{H}$-valued martingale. Then,

1. $\|M(s)\|_{\mathbb{H}}$ is a submartingale.

2. If $\varphi : \mathbb{H} \to \mathbb{R}$ is convex and continuous, then $\varphi(M(s))$ is a submartingale whenever integrable.

3. If M is square-integrable, then its quadratic variation $\langle\!\langle M \rangle\!\rangle$ exists uniquely as above.

In finite dimensions, quadratic variation reduces to the classical matrix-valued process of cross-variations.

Theorem 3.5 (Lévy's characterisation of Brownian motion):
Let M be a continuous $\mathbb{R}^d$-valued martingale with $M(s_0) = 0$. Then M is a d-dimensional Brownian motion if and only if

$$\langle M^i, M^j \rangle(s) = (s - s_0)\,\delta_{ij}, \qquad s \in J, \;\; 1 \le i, j \le d,$$

where $\langle M^i, M^j \rangle$ denotes the scalar cross-variation.

***Remark* 3.4:**
The operator-valued quadratic variation $\langle\!\langle M \rangle\!\rangle$ generalises the matrix of cross-variations, in $\mathbb{R}^d$,

$$\langle\!\langle M \rangle\!\rangle(s) = \big(\langle M^i, M^j \rangle(s)\big)_{i,j=1}^d.$$

3.6. Gaussian Measures and Wiener Processes in Hilbert Spaces

Gaussian measures generalise the normal distribution to infinite dimensions.

Definition 3.25 (Gaussian Measure [19]):
A Borel probability measure μ on a separable Banach space $\mathbb{Y}$ is called a *Gaussian measure* if for every continuous linear functional $\varphi \in \mathbb{Y}^*$, the pushforward measure $\varphi_*\mu$ on $\mathbb{R}$ is Gaussian, i.e., $\varphi_*\mu = \mathcal{N}(m_\varphi, \sigma_\varphi^2)$ for some $m_\varphi \in \mathbb{R}$ and $\sigma_\varphi^2 \ge 0$. The measure is called *centred* if $m_\varphi = 0$ for all $\varphi \in \mathbb{Y}^*$.

On a Hilbert space, a Gaussian measure is characterised by its mean and covariance operator.

Definition 3.26 (Mean and Covariance Operator [19]):
Let μ be a Gaussian measure on a separable Hilbert space $\mathbb{K}$.

1. The *mean* $m \in \mathbb{K}$ is defined by $m = \int_{\mathbb{K}} x\,\mu(\mathrm{d}x)$ (Bochner integral).

2. The *covariance operator* $Q : \mathbb{K} \to \mathbb{K}$ is defined by the relation

$$\langle Qu, v \rangle_{\mathbb{K}} = \int_{\mathbb{K}} \langle x - m, u \rangle_{\mathbb{K}} \langle x - m, v \rangle_{\mathbb{K}}\, \mu(\mathrm{d}x) \quad \text{for all } u, v \in \mathbb{K}.$$

Proposition 3.4 (Properties of the Covariance Operator [19]):
Let μ be a Gaussian measure on $\mathbb{K}$ with covariance operator Q. Then,

1. Q is self-adjoint: $Q^* = Q$.
2. Q is positive semidefinite: $\langle Qu, u\rangle_{\mathbb{K}} \geq 0$ for all $u \in \mathbb{K}$.
3. Q is a trace-class operator: $\operatorname{Tr} Q = \int_{\mathbb{K}} \|x - m\|_{\mathbb{K}}^2 \,\mu(\mathrm{d}x) < \infty$.
4. The characteristic functional of μ is given by
$$\hat{\mu}(\varphi) = \int_{\mathbb{K}} e^{i\langle \varphi, x\rangle}\,\mu(\mathrm{d}x) = \exp\left(i\langle \varphi, m\rangle - \frac{1}{2}\langle Q\varphi, \varphi\rangle\right) \quad \text{for all } \varphi \in \mathbb{K}.$$

Definition 3.27 (Gaussian Random Element):
Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and $\mathbb{Y}$ a separable Banach space. A random element $Y : \Omega \to \mathbb{Y}$ is called *Gaussian* if its law $\mathscr{L}(Y)$ is a Gaussian measure on $\mathbb{Y}$. If $\mathbb{Y} = \mathbb{K}$ is a Hilbert space, then Y is Gaussian if and only if $\langle Y, v\rangle_{\mathbb{K}}$ is a real-valued Gaussian random variable for every $v \in \mathbb{K}$.

The infinite-dimensional analogue of Brownian motion is the Q_{noise}–Wiener process.

Definition 3.28 (Q–Wiener Process [19]):
Let $(\Omega, \mathscr{F}, (\mathscr{F}_s)_{s\geq s_0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, and let $\mathbb{K}$ be a separable Hilbert space. Let $Q_{\text{noise}} : \mathbb{K} \to \mathbb{K}$ be a symmetric, positive semidefinite, trace-class operator. A Q_{noise}–*Wiener process* is a $\mathbb{K}$-valued, $(\mathscr{F}_s)$-adapted process $(W(s))_{s\geq s_0}$ such that,

1. $W(s_0) = 0$ $\mathbb{P}$-a.s.,
2. W has independent increments. For $s_0 \leq r < s$, $W(s) - W(r)$ is independent of $\mathscr{F}_r$,
3. W has Gaussian increments. $W(s) - W(r) \sim \mathcal{N}(0, (s-r)Q_{\text{noise}})$,
4. W has continuous sample paths, the map $s \mapsto W(s, \omega)$ is continuous for $\mathbb{P}$-a.e. $\omega \in \Omega$.

A Q_{noise}–Wiener process can be constructed via a Karhunen–Loève expansion. Let $(\gamma_j)_{j\in\mathbb{N}}$ be the eigenvalues of Q_{noise} with corresponding orthonormal eigenvectors $(e_j)_{j\in\mathbb{N}}$, and let $(\beta_j(s))_{j\in\mathbb{N}}$ be a sequence of independent real-valued standard $(\mathscr{F}_s)$-Brownian motions. Then the series
$$W(s) = \sum_{j=1}^{\infty} \sqrt{\gamma_j}\,(\beta_j(s) - \beta_j(s_0))\, e_j$$
converges in $L^2(\Omega; \mathbb{K})$ uniformly on compact time intervals and defines a Q_{noise}–Wiener process. Its covariance is given by
$$\mathbb{E}[\langle W(s), x\rangle_{\mathbb{K}} \langle W(r), y\rangle_{\mathbb{K}}] = \min(s - s_0, r - s_0)\langle Q_{\text{noise}} x, y\rangle_{\mathbb{K}} \quad \text{for all } x, y \in \mathbb{K}.$$

If $s_0 = 0$, the process is called the standard Q_{noise}–Wiener process.

Definition 3.29 (Natural Filtration [19]):
The *natural filtration* of a Q_{noise}–Wiener process W is

$$\mathscr{F}_s^W = \sigma\{W(r) : s_0 \le r \le s\} \vee \mathscr{N},$$

where $\mathscr{N}$ is the collection of $\mathbb{P}$-null sets. This filtration satisfies the usual conditions.

The Wiener process enjoys the strong Markov property.

Example 3.2 (Strong Markov Property):
Let $W(s)$ be a Q_{noise}–Wiener process with respect to $(\mathscr{F}_s)$ and let τ be an $(\mathscr{F}_s)$-stopping time with $\mathbb{P}(\tau < \infty) = 1$. Then the shifted process $\mathring{W}(s) = W(\tau + s) - W(\tau)$ is also a Q_{noise}–Wiener process and is independent of $\mathscr{F}_\tau$.

3.7. Stochastic Integration in Hilbert Spaces

We now define the stochastic integral with respect to a Q_{noise}–Wiener process, the cornerstone of SDEs in infinite dimensions.

Let $(\Omega, \mathscr{F}, (\mathscr{F}_s), \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Let $\mathbb{H}$ and $\mathbb{K}$ be separable Hilbert spaces, and let $W(s)$ be a $\mathbb{K}$-valued Q_{noise}–Wiener process.

Definition 3.30 (Cameron-Martin Space [19]):
The *Cameron-Martin space* (or *Reproducing Kernel Hilbert Space*) associated with Q_{noise} is defined as $\mathbb{K}_0 := Q_{\text{noise}}^{1/2}(\mathbb{K})$, endowed with the inner product

$$\langle u, v\rangle_{\mathbb{K}_0} = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \langle u, e_k\rangle \langle v, e_k\rangle = \langle Q_{\text{noise}}^{-1/2} u, Q_{\text{noise}}^{-1/2} v\rangle_{\mathbb{K}} \quad \text{for } u, v \in \mathbb{K}_0.$$

The space $(\mathbb{K}_0, \langle\cdot,\cdot\rangle_{\mathbb{K}_0})$ is a Hilbert space.

The class of integrands for the stochastic integral is the space of Hilbert-Schmidt operators from the Cameron-Martin space $\mathbb{K}_0$ to $\mathbb{H}$, denoted by $\mathcal{L}_2(\mathbb{K}_0, \mathbb{H})$. This space is often denoted by $\mathcal{L}_2$.

Definition 3.31 (Hilbert-Schmidt Operators [19]):
The space $\mathcal{L}_2(\mathbb{K}_0, \mathbb{H})$ consists of all bounded linear operators $\Phi : \mathbb{K}_0 \to \mathbb{H}$ such that for some (and hence any) orthonormal basis $\{g_k\}_{k\in\mathbb{N}}$ of $\mathbb{K}_0$,

$$\|\Phi\|^2_{\mathcal{L}_2(\mathbb{K}_0,\mathbb{H})} := \sum_{k=1}^{\infty} \|\Phi(g_k)\|^2_{\mathbb{H}} < \infty.$$

This is a separable Hilbert space with the inner product

$$\langle \Phi, \Psi \rangle_{\mathcal{L}_2(\mathbb{K}_0,\mathbb{H})} = \sum_{k=1}^{\infty} \langle \Phi(g_k), \Psi(g_k) \rangle_{\mathbb{H}}.$$

An equivalent characterisation is obtained by using the fact that $Q_{\text{noise}}^{1/2}$ is an isometry from $\mathbb{K}$ onto $\mathbb{K}_0$. Specifically, if $\{e_k\}$ is an orthonormal basis of $\mathbb{K}$, then $\{g_k = Q_{\text{noise}}^{1/2} e_k\}$ is an orthonormal basis of $\mathbb{K}_0$. Therefore,

$$\|\Phi\|_{\mathcal{L}_2(\mathbb{K}_0,\mathbb{H})}^2 = \sum_{k=1}^{\infty} \|\Phi(Q_{\text{noise}}^{1/2} e_k)\|_{\mathbb{H}}^2 = \|\Phi Q_{\text{noise}}^{1/2}\|_{\mathcal{L}_2(\mathbb{K},\mathbb{H})}^2,$$

where $\mathcal{L}_2(\mathbb{K}, \mathbb{H})$ denotes the space of Hilbert-Schmidt operators from $\mathbb{K}$ to $\mathbb{H}$. Consequently,

$$\|\Phi\|_{\mathcal{L}_2(\mathbb{K}_0,\mathbb{H})}^2 = \operatorname{Tr}\left((\Phi Q_{\text{noise}}^{1/2})(\Phi Q_{\text{noise}}^{1/2})^* \right).$$

In terms of coordinates, if $\{e_j\}$ is an orthonormal basis of $\mathbb{K}$ consisting of eigenvectors of Q_{noise} with eigenvalues λ_j (so that $Q_{\text{noise}} e_j = \lambda_j e_j$), and $\{f_j\}$ is an orthonormal basis of $\mathbb{H}$, then

$$\|\Phi\|_{\mathcal{L}_2(\mathbb{K}_0,\mathbb{H})}^2 = \sum_{j=1}^{\infty} \lambda_j \|\Phi(e_j)\|_{\mathbb{H}}^2 = \sum_{j,k=1}^{\infty} \lambda_j \left| \langle \Phi(e_j), f_k \rangle_{\mathbb{H}} \right|^2.$$

Definition 3.32 (Elementary Process and Its Stochastic Integral [19]):
An *elementary process* is a function $\Theta : [s_0, b] \times \Omega \to \mathcal{L}_2(\mathbb{K}_0, \mathbb{H})$ of the form

$$\Theta(s, \omega) = \sum_{k=0}^{N-1} \Theta_k(\omega) \mathbf{1}_{(s_k, s_{k+1}]}(s),$$

where $s_0 < s_1 < \cdots < s_N = b$ and each Θ_k is an $\mathscr{F}_{s_k}$-measurable random variable taking only finitely many values in $\mathcal{L}_2(\mathbb{K}_0, \mathbb{H})$. For such Θ, the *Itô integral* is defined by,

$$\int_{s_0}^{s} \Theta(r)\, \mathrm{d}W(r) := \sum_{k=0}^{N-1} \Theta_k \left(W(s_{k+1} \wedge s) - W(s_k \wedge s) \right).$$

This definition satisfies the Itô isometry.

Proposition 3.5 (Itô Isometry for Elementary Processes [19]):
For any elementary process Θ as defined above,

$$\mathbb{E}\left[\left\| \int_{s_0}^{b} \Theta(r)\, \mathrm{d}W(r) \right\|_{\mathbb{H}}^2 \right] = \mathbb{E}\left[\int_{s_0}^{b} \|\Theta(r)\|_{\mathcal{L}_2(\mathbb{K}_0,\mathbb{H})}^2\, \mathrm{d}r \right].$$

Theorem 3.6 (Extension of the Stochastic Integral [19]):
The mapping

$$\Theta \mapsto \int_{s_0}^{b} \Theta(r)\, \mathrm{d}W(r)$$

extends uniquely to a linear isometry from the space of predictable processes $\Theta : [s_0, b] \times \Omega \to \mathcal{L}_2(\mathbb{K}_0, \mathbb{H})$ satisfying

$$\mathbb{E}\left[\int_{s_0}^{b} \|\Theta(r)\|^2_{\mathcal{L}_2(\mathbb{K}_0,\mathbb{H})}\, \mathrm{d}r\right] < \infty$$

into the space $L^2(\Omega; \mathbb{H})$. This extended integral is called the *Itô integral* and satisfies the Itô isometry. Moreover, the process defined by $Y(s) = \int_{s_0}^{s} \Theta(r)\, \mathrm{d}W(r)$ is a continuous, square-integrable $\mathbb{H}$-valued martingale.

3.8. The Itô Formula and a Burkholder-Davis-Gundy-Type Inequality

Itô's formula is the chain rule of stochastic calculus.

Theorem 3.7 (Itô's Formula in Hilbert Spaces [19]):
Let $Y = (Y(s))_{s\in[s_0,b]}$ be an $\mathbb{H}$-valued Itô process of the form,

$$Y(s) = Y(s_0) + \int_{s_0}^{s} \Psi(r)\, \mathrm{d}r + \int_{s_0}^{s} \Theta(r)\, \mathrm{d}W(r), \quad s \in [s_0, b],$$

where,

1. $\Psi : [s_0, b] \times \Omega \to \mathbb{H}$ is progressively measurable and satisfies $\int_{s_0}^{b} \|\Psi(r)\|_{\mathbb{H}}\, \mathrm{d}r < \infty$ $\mathbb{P}$-a.s.,
2. $\Theta : [s_0, b] \times \Omega \to \mathcal{L}_2(\mathbb{K}_0, \mathbb{H})$ is predictable and satisfies $\int_{s_0}^{b} \|\Theta(r)\|^2_{\mathcal{L}_2(\mathbb{K}_0,\mathbb{H})}\, \mathrm{d}r < \infty$ $\mathbb{P}$-a.s.

Let $\varphi : [s_0, b] \times \mathbb{H} \to \mathbb{R}$ be a function such that,

1. φ is continuously differentiable in its first variable (C^1 in time),
2. φ is twice continuously Fréchet differentiable in its second variable (C^2 in space),
3. The derivatives $\partial_s\varphi$, $D_x\varphi$, and $D_x^2\varphi$ are uniformly bounded on $[s_0, b] \times \mathbb{H}$.

Then, for all $s \in [s_0, b]$, the following holds $\mathbb{P}$-a.s.,

$$\varphi(s, Y(s)) = \varphi(s_0, Y(s_0)) + \int_{s_0}^{s} \partial_r\varphi(r, Y(r))\, \mathrm{d}r$$

$$
\begin{aligned}
&+ \int_{s_0}^{s} \langle D_x\varphi(r, Y(r)), \Psi(r)\rangle_{\mathbb{H}}\, \mathrm{d}r \\
&+ \int_{s_0}^{s} \langle D_x\varphi(r, Y(r)), \Theta(r)\, \mathrm{d}W(r)\rangle_{\mathbb{H}} \\
&+ \frac{1}{2}\int_{s_0}^{s} \mathrm{Tr}_{\mathbb{K}}\left[\left(\Theta(r) Q_{\mathrm{noise}}^{1/2}\right)\left(\Theta(r) Q_{\mathrm{noise}}^{1/2}\right)^{*} D_x^2\varphi(r, Y(r))\right] \mathrm{d}r.
\end{aligned}
$$

Example 3.3 (Energy Equality):
For $\varphi(x) = \|x\|_{\mathbb{H}}^2$, we have $D_x\varphi(x) = 2x$ and $D_x^2\varphi(x) = 2I$. For a pure martingale $Y(s) = \int_{s_0}^{s} \Theta(r)\, \mathrm{d}W(r)$ (i.e., $\Psi = 0$), Itô's formula yields,

$$
\|Y(s)\|_{\mathbb{H}}^2 = 2\int_{s_0}^{s} \langle Y(r), \Theta(r)\, \mathrm{d}W(r)\rangle_{\mathbb{H}} + \int_{s_0}^{s} \|\Theta(r)\|_{\mathcal{L}_2(\mathbb{K}_0,\mathbb{H})}^2\, \mathrm{d}r.
$$

Thus, $\|Y(s)\|_{\mathbb{H}}^2 - \int_{s_0}^{s} \|\Theta(r)\|_{\mathcal{L}_2(\mathbb{K}_0,\mathbb{H})}^2\, \mathrm{d}r$ is a continuous local martingale.

The Burkholder-Davis-Gundy (BDG) inequalities provide crucial moment estimates for stochastic integrals. The following lemma provides such an inequality with an explicit constant, refining a result from Kamenskii, Mellah, and Raynaud de Fitte 35.

Theorem 3.8 (Burkholder-Davis-Gundy Inequality in Hilbert Spaces):
Let $p \geq 2$, and let $\Theta : [s_0, b] \times \Omega \to \mathcal{L}_2(\mathbb{K}_0, \mathbb{H})$ be a predictable process such that

$$
\mathbb{E}\left[\left(\int_{s_0}^{b} \|\Theta(r)\|_{\mathcal{L}_2(\mathbb{K}_0,\mathbb{H})}^2\, \mathrm{d}r\right)^{p/2}\right] < \infty.
$$

Then, for the Itô integral $Y(s) = \int_{s_0}^{s} \Theta(r)\, \mathrm{d}W(r)$, the following inequality holds for every $s \in [s_0, b]$,

$$
\mathbb{E}\left[\|Y(s)\|_{\mathbb{H}}^p\right] \leq K_{p/2} \cdot \mathbb{E}\left[\left(\int_{s_0}^{s} \|\Theta(r)\|_{\mathcal{L}_2(\mathbb{K}_0,\mathbb{H})}^2\, \mathrm{d}r\right)^{p/2}\right],
$$

where the constant $K_{p/2}$ is given by

$$
K_{p/2} = \inf\left\{\frac{(p-1)\left(1+\frac{1}{\varepsilon}\right)^{p/2-1} K_0^{p/2}}{(1+K_0) - (1+\varepsilon)^{p/2-1}(p-1)} \;:\; \varepsilon > 0,\; K_0 > (1+\varepsilon)^{p/2-1}(p-1) - 1\right\}.
$$

For $p = 2$, this inequality recovers Itô's isometry with the optimal constant $K_1 = 1$.

Proof:
The proof proceeds in two steps. First, we establish the inequality for essentially bounded integrands. Second, we extend the result to the general case via a truncation argument.

Step 1. Essentially Bounded Integrands

Assume there exists a constant $\theta > 0$ such that

$$
\|\Theta(s, \omega)\|_{\mathcal{L}_2(\mathbb{K}_0,\mathbb{H})} \leq \theta \quad \text{for a.e. } (s, \omega) \in [a, b] \times \Omega.
$$

Define the processes,

$$Y(s) = \int_{s_0}^{s} \Theta(r)\, \mathrm{d}W(r), \quad A(s) := \mathrm{Tr}\langle\!\langle Y \rangle\!\rangle(s) := \int_{s_0}^{s} \|\Theta(r)\|^2_{\mathcal{L}_2(\mathbb{K}_0,\mathbb{H})}\, \mathrm{d}r.$$

Let $C_0 \geq 0$ and $K_0 \geq 0$ be constants, and define the auxiliary process,

$$Z(s) = C_0 + K_0 A(s) + \|Y(s)\|^2_{\mathbb{H}}.$$

We apply Itô's formula to the process $Y(s)$ and the function

$$\varphi(s,x) = (C_0 + K_0 A(s) + \|x\|^2_{\mathbb{H}})^{p/2} = Z(s)^{p/2}.$$

The required derivatives are computed as follows.

Time derivative. Since $A(s)$ is absolutely continuous,

$$\partial_s \varphi(s,x) = \frac{p}{2} Z(s)^{p/2-1} \cdot K_0 \|\Theta(s)\|^2_{\mathcal{L}_2(\mathbb{K}_0,\mathbb{H})}.$$

First Fréchet derivative,

$$D_x \varphi(s,x) = p Z(s)^{p/2-1} x.$$

Second Fréchet derivative,

$$D_x^2 \varphi(s,x) = p(p-2) Z(s)^{p/2-2} (x \otimes x) + p Z(s)^{p/2-1} I,$$

where I is the identity operator on $\mathbb{H}$ and $(x \otimes x)(y) = \langle x, y \rangle x$.

Applying Itô's formula yields,

$$\begin{aligned} Z(s)^{p/2} = C_0^{p/2} &+ \int_{s_0}^{s} \frac{p}{2} K_0 Z(r)^{p/2-1} \|\Theta(r)\|^2_{\mathcal{L}_2}\, \mathrm{d}r \\ &+ \int_{s_0}^{s} p Z(r)^{p/2-1} \langle Y(r), \Theta(r)\, \mathrm{d}W(r) \rangle_{\mathbb{H}} \\ &+ \frac{1}{2} \int_{s_0}^{s} \mathrm{Tr}\left[D_x^2 \varphi(r, Y(r)) \left(\Theta(r) Q_{\text{noise}}^{1/2} \right) \left(\Theta(r) Q_{\text{noise}}^{1/2} \right)^* \right] \mathrm{d}r. \end{aligned}$$

We now compute the trace term. Note that for any Hilbert-Schmidt operator L, we have $\mathrm{Tr}[(x \otimes x) L L^*] = \langle L L^* x, x \rangle = \|L^* x\|^2$. Therefore,

$$\begin{aligned} &\mathrm{Tr}\left[D_x^2 \varphi(r, Y(r)) \left(\Theta(r) Q_{\text{noise}}^{1/2} \right) \left(\Theta(r) Q_{\text{noise}}^{1/2} \right)^* \right] \\ &= p(p-2) Z(r)^{p/2-2} \left\langle \left(\Theta(r) Q_{\text{noise}}^{1/2} \right) \left(\Theta(r) Q_{\text{noise}}^{1/2} \right)^* Y(r), Y(r) \right\rangle_{\mathbb{H}} \\ &+ p Z(r)^{p/2-1}\, \mathrm{Tr}\left[\left(\Theta(r) Q_{\text{noise}}^{1/2} \right) \left(\Theta(r) Q_{\text{noise}}^{1/2} \right)^* \right] \\ &= p(p-2) Z(r)^{p/2-2} \|(\Theta(r) Q_{\text{noise}}^{1/2})^* Y(r)\|^2_{\mathbb{K}} + p Z(r)^{p/2-1} \|\Theta(r)\|^2_{\mathcal{L}_2(\mathbb{K}_0,\mathbb{H})}. \end{aligned}$$

Taking expectations (noting the stochastic integral is a martingale due to the boundedness of Θ and thus has zero expectation), we obtain,

$$\begin{aligned}\mathbb{E}[Z(s)^{p/2}] &= C_0^{p/2} + \frac{p}{2}\mathbb{E}\left(\int_{s_0}^{s} Z(r)^{p/2-1}\left(K_0\|\Theta(r)\|^2 + \|\Theta(r)\|^2\right)\,\mathrm{d}r\right)\\ &\quad + \frac{p}{2}(p-2)\mathbb{E}\left(\int_{s_0}^{s} Z(r)^{p/2-2}\|(\Theta(r)Q_{\text{noise}}^{1/2})^*Y(r)\|_{\mathbb{K}}^2\,\mathrm{d}r\right)\\ &= C_0^{p/2} + \frac{p}{2}(1+K_0)\mathbb{E}\left(\int_{s_0}^{s} Z(r)^{p/2-1}\|\Theta(r)\|^2\,\mathrm{d}r\right)\\ &\quad + \frac{p}{2}(p-2)\mathbb{E}\left(\int_{s_0}^{s} Z(r)^{p/2-2}\|(\Theta(r)Q_{\text{noise}}^{1/2})^*Y(r)\|_{\mathbb{K}}^2\,\mathrm{d}r\right).\end{aligned}$$

Since $p \geq 2$, the last term is non-negative. Thus, we have the lower bound,

$$\mathbb{E}[Z(s)^{p/2}] \geq C_0^{p/2} + \frac{p}{2}(1+K_0)\mathbb{E}\left(\int_{s_0}^{s} Z(r)^{p/2-1}\|\Theta(r)\|^2\,\mathrm{d}r\right). \tag{3.1}$$

We now derive an upper bound. Applying Young's inequality

$$(u+v)^{p/2} \leq (1+\varepsilon)^{p/2-1}u^{p/2} + (1+1/\varepsilon)^{p/2-1}v^{p/2}$$

with $u = C_0 + \|Y(s)\|^2$ and $v = K_0 A(s)$ yields,

$$\mathbb{E}[Z(s)^{p/2}] \leq (1+\varepsilon)^{p/2-1}\mathbb{E}[(C_0 + \|Y(s)\|^2)^{p/2}] + (1+1/\varepsilon)^{p/2-1}K_0^{p/2}\mathbb{E}[A(s)^{p/2}]. \tag{3.2}$$

Now, apply Itô's formula to $\psi(x) = (C_0 + \|x\|^2)^{p/2}$ and the process $Y(s)$. The derivatives are,

$$\begin{aligned}D_x\psi(x) &= p(C_0 + \|x\|^2)^{p/2-1}x,\\ D_x^2\psi(x) &= p(p-2)(C_0 + \|x\|^2)^{p/2-2}(x \otimes x) + p(C_0 + \|x\|^2)^{p/2-1}I.\end{aligned}$$

Taking expectations and noting that the trace term is non-negative (since $p \geq 2$ and the second term is positive), we obtain the upper bound,

$$\mathbb{E}[(C_0 + \|Y(s)\|^2)^{p/2}] \leq C_0^{p/2} + \frac{p}{2}(p-1)\mathbb{E}\left(\int_{s_0}^{s} (C_0 + \|Y(r)\|^2)^{p/2-1}\|\Theta(r)\|^2\,\mathrm{d}r\right).$$

Since $Z(r) \geq C_0 + \|Y(r)\|^2$, it follows that $(C_0 + \|Y(r)\|^2)^{p/2-1} \leq Z(r)^{p/2-1}$. Therefore,

$$\mathbb{E}[(C_0 + \|Y(s)\|^2)^{p/2}] \leq C_0^{p/2} + \frac{p}{2}(p-1)\mathbb{E}\left(\int_{s_0}^{s} Z(r)^{p/2-1}\|\Theta(r)\|^2\,\mathrm{d}r\right).$$

Substituting this into (3.2) gives,

$$\begin{aligned}\mathbb{E}[Z(s)^{p/2}] &\leq (1+\varepsilon)^{p/2-1}C_0^{p/2} + (1+1/\varepsilon)^{p/2-1}K_0^{p/2}\mathbb{E}[A(s)^{p/2}]\\ &\quad + (1+\varepsilon)^{p/2-1}\frac{p}{2}(p-1)\mathbb{E}\left(\int_{s_0}^{s} Z(r)^{p/2-1}\|\Theta(r)\|^2\,\mathrm{d}r\right).\end{aligned} \tag{3.3}$$

Now, combine the lower bound (3.1) with the upper bound (3.3). Letting $C_0 \to 0^+$, we obtain,

$$\frac{p}{2}(1+K_0)\mathbb{E}\left(\int_{s_0}^{s} Z(r)^{p/2-1}\|\Theta(r)\|^2\,\mathrm{d}r\right) \leq (1+1/\varepsilon)^{p/2-1}K_0^{p/2}\mathbb{E}[A(s)^{p/2}]$$
$$+ (1+\varepsilon)^{p/2-1}\frac{p}{2}(p-1)\mathbb{E}\left(\int_{s_0}^{s} Z(r)^{p/2-1}\|\Theta(r)\|^2\,\mathrm{d}r\right).$$

Rearranging terms yields,

$$\frac{p}{2}\left[(1+K_0)-(1+\varepsilon)^{p/2-1}(p-1)\right]\mathbb{E}\left(\int_{s_0}^{s} Z(r)^{p/2-1}\|\Theta(r)\|^2\,\mathrm{d}r\right)$$
$$\leq (1+1/\varepsilon)^{p/2-1}K_0^{p/2}\mathbb{E}[A(s)^{p/2}]. \quad (3.4)$$

For this inequality to be useful, we require the coefficient to be positive,

$$(1+K_0) > (1+\varepsilon)^{p/2-1}(p-1). \quad (3.5)$$

Now, note that $Z(r) \geq \|Y(r)\|^2$, so

$$\mathbb{E}\left(\int_{s_0}^{s} Z(r)^{p/2-1}\|\Theta(r)\|^2\,\mathrm{d}r\right) \geq \mathbb{E}\left(\int_{s_0}^{s} \|Y(r)\|^{p-2}\|\Theta(r)\|^2\,\mathrm{d}r\right).$$

Furthermore, by applying Itô's formula to $\|Y(s)\|^p$ and taking expectations, one can derive the inequality,

$$\mathbb{E}[\|Y(s)\|^p] \leq \frac{p}{2}(p-1)\mathbb{E}\left(\int_{s_0}^{s} \|Y(r)\|^{p-2}\|\Theta(r)\|^2\,\mathrm{d}r\right). \quad (3.6)$$

Combining (3.4), the above inequality, and (3.6), we conclude,

$$\mathbb{E}[\|Y(s)\|^p] \leq \left(\frac{(p-1)(1+1/\varepsilon)^{p/2-1}K_0^{p/2}}{(1+K_0)-(1+\varepsilon)^{p/2-1}(p-1)}\right)\mathbb{E}[A(s)^{p/2}].$$

This holds for any $\varepsilon > 0$ and K_0 satisfying (3.5). Taking the infimum over all such parameters yields the desired constant $K_{p/2}$ for essentially bounded Θ.

Step 2. Extension to General Integrands

For a general predictable process Θ satisfying the theorem's hypothesis, define a sequence of truncations,

$$\Theta^N(r,\omega) = \Theta(r,\omega)\cdot \mathbf{1}_{\{\|\Theta(r,\omega)\|_{\mathcal{L}_2}\leq N\}}, \quad N \in \mathbb{N}.$$

Each Θ^N is predictable and essentially bounded. Define the corresponding Itô integrals,

$$Y^N(s) = \int_{s_0}^{s} \Theta^N(r)\,\mathrm{d}W(r).$$

By Step 1, the inequality holds for each Y^N,

$$\mathbb{E}[\|Y^N(s)\|^p] \leq K_{p/2} \cdot \mathbb{E}\left[\left(\int_{s_0}^{s} \|\Theta^N(r)\|^2 \,\mathrm{d}r\right)^{p/2}\right].$$

By the dominated convergence theorem for Itô integrals (since $\Theta^N \to \Theta$ pointwise and is dominated by $\|\Theta\|$), we have $Y^N(s) \to Y(s)$ in $L^2(\Omega; \mathbb{H})$. In particular, $Y^N(s) \to Y(s)$ in probability. By Fatou's lemma,

$$\mathbb{E}[\|Y(s)\|^p] \leq \liminf_{N\to\infty} \mathbb{E}[\|Y^N(s)\|^p].$$

On the other hand, since $\|\Theta^N(r)\| \leq \|\Theta(r)\|$ and $\|\Theta^N(r)\| \to \|\Theta(r)\|$ pointwise, the monotone convergence theorem gives,

$$\lim_{N\to\infty} \mathbb{E}\left[\left(\int_{s_0}^{s} \|\Theta^N(r)\|^2 \,\mathrm{d}r\right)^{p/2}\right] = \mathbb{E}\left[\left(\int_{s_0}^{s} \|\Theta(r)\|^2 \,\mathrm{d}r\right)^{p/2}\right].$$

Therefore, passing to the limit yields the inequality for the general integrand Θ,

$$\mathbb{E}[\|Y(s)\|^p] \leq K_{p/2} \cdot \mathbb{E}\left[\left(\int_{s_0}^{s} \|\Theta(r)\|^2 \,\mathrm{d}r\right)^{p/2}\right].$$

This completes the proof. □

***Remark* 3.5 (Optimising the Constant):**
For fixed $\varepsilon > 0$, define

$$A = (1+\varepsilon)^{p/2-1}(p-1),$$

so that the condition becomes $K_0 > A - 1$. Then,

$$K_{p/2}(\varepsilon, K_0) = \frac{(p-1)\left(1+\frac{1}{\varepsilon}\right)^{p/2-1} K_0^{p/2}}{1 + K_0 - A}.$$

To minimise this expression, we first optimise over K_0. Differentiating with respect to K_0, we find the optimal value,

$$K_0^* = \frac{p}{p-2}(A-1) = \frac{p}{p-2}\left[(1+\varepsilon)^{p/2-1}(p-1) - 1\right].$$

Substituting this back, we obtain,

$$K_{p/2}(\varepsilon) = \frac{(p-1)(p-2)}{2}\left(\frac{p}{p-2}\right)^{p/2}\left(1+\frac{1}{\varepsilon}\right)^{p/2-1}\left[(p-1)(1+\varepsilon)^{p/2-1} - 1\right]^{p/2-1}.$$

Finally, we minimise over $\varepsilon > 0$,

$$K_{p/2} = \inf_{\varepsilon > 0} K_{p/2}(\varepsilon).$$

Example 3.4 (Example Optimisation for $p = 4$):
For $p = 4$, we have

$$K_2 = \frac{3(1 + 1/\varepsilon)K_0^2}{1 + K_0 - 3(1 + \varepsilon)}.$$

The condition is $1 + K_0 > 3(1 + \varepsilon)$, i.e., $K_0 > 3\varepsilon + 2$. For fixed ε, the optimal K_0 is $K_0^* = 2(3\varepsilon + 2) = 6\varepsilon + 4$. Substituting gives

$$K_2(\varepsilon) = \frac{12(3\varepsilon + 2)(\varepsilon + 1)}{\varepsilon} = 12\left(3\varepsilon + 5 + \frac{2}{\varepsilon}\right).$$

Minimising over $\varepsilon > 0$, set derivative to zero,

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(3\varepsilon + \frac{2}{\varepsilon}\right) = 3 - \frac{2}{\varepsilon^2} = 0 \quad \Rightarrow \quad \varepsilon = \sqrt{\frac{2}{3}}.$$

Then

$$K_2 = 12\left(3\sqrt{\frac{2}{3}} + 5 + 2\sqrt{\frac{3}{2}}\right) = 12(5 + 2\sqrt{6}) \approx 118.78,$$

which is an improvement over the previous constant of 120 obtained with $\varepsilon = 1$.

The use of Young's inequality with parameter ε provides a more flexible and optimised constant $K_{p/2}$, which can be further improved by choosing ε and K_0 optimally.

3.9. Stochastic Differential Equations

The theory culminates in SDEs, representing the infinite-dimensional generalisation of classical finite-dimensional SDEs.

3.9.1. Linear Stochastic Differential Equations

Consider the linear SDE with additive noise,

$$\begin{cases} \mathrm{d}Y(s) = [AY(s) + G(s)]\mathrm{d}s + H\,\mathrm{d}W(s), & s \in [s_0, b], \\ Y(s_0) = \xi, \end{cases}$$

where,

1. $A : D(A) \subseteq \mathbb{H} \to \mathbb{H}$ is the generator of a C_0-semigroup $\{\mathcal{T}(s)\}_{s\geq 0}$ on $\mathbb{H}$,
2. $G : [s_0, b] \times \Omega \to \mathbb{H}$ is progressively measurable and satisfies $\int_{s_0}^{b} \mathbb{E}[\|G(s)\|_{\mathbb{H}}]\mathrm{d}s < \infty$,

3. $H \in \mathcal{L}(\mathbb{K}, \mathbb{H})$ is a bounded linear operator,
4. $W(s)$ is a $\mathbb{K}$-valued Q_{noise}–Wiener process,
5. ξ is an $\mathscr{F}_{s_0}$-measurable $\mathbb{H}$-valued random variable.

Different notions of solution exist, depending on the regularity of the operator A and the initial data.

Definition 3.33 (Mild, Weak, and Strong Solutions [19]):
A predictable process $Y(s)$ is called:

1. A *mild solution* if it satisfies the variation-of-constants formula,
$$Y(s) = \mathcal{T}(s - s_0)\xi + \int_{s_0}^{s} \mathcal{T}(s - r)G(r)\,\mathrm{d}r + \int_{s_0}^{s} \mathcal{T}(s - r)H\,\mathrm{d}W(r) \quad \mathbb{P}\text{-a.s. for all } s.$$
2. A *weak solution* if for all $z \in D(A^*)$ (the domain of the adjoint operator A^*), the following holds $\mathbb{P}$-a.s.,
$$\langle Y(s), z\rangle_{\mathbb{H}} = \langle \xi, z\rangle_{\mathbb{H}} + \int_{s_0}^{s} [\langle Y(r), A^* z\rangle_{\mathbb{H}} + \langle G(r), z\rangle_{\mathbb{H}}]\,\mathrm{d}r + \langle H(W(s) - W(s_0)), z\rangle_{\mathbb{H}}.$$
3. A *strong solution* if $Y(s) \in D(A)$ for a.e. (s, ω), the process $AY(s)$ is Bochner integrable, and the equation holds $\mathbb{P}$-a.s. as an identity in $\mathbb{H}$,
$$Y(s) = \xi + \int_{s_0}^{s} [AY(r) + G(r)]\mathrm{d}r + H(W(s) - W(s_0)).$$

Mild solutions are typically the most appropriate concept for unbounded operators A. If A is bounded, mild and strong solutions coincide.

3.9.2. Semilinear Equations with Lipschitz Nonlinearities

Consider the semilinear SDE,
$$\begin{cases} \mathrm{d}Y(s) = [AY(s) + F(s, Y(s))]\mathrm{d}s + B(s, Y(s))\mathrm{d}W(s), & s \in [s_0, b], \\ Y(s_0) = \xi, \end{cases}$$
with,

1. $A : D(A) \subseteq \mathbb{H} \to \mathbb{H}$ generator of a C_0-semigroup $\{\mathcal{T}(s)\}_{s\geq 0}$,
2. $F : [s_0, b] \times \mathbb{H} \to \mathbb{H}$ measurable,
3. $B : [s_0, b] \times \mathbb{H} \to \mathcal{L}_2(\mathbb{K}_0, \mathbb{H})$ measurable,
4. ξ an $\mathscr{F}_{s_0}$-measurable $\mathbb{H}$-valued random variable.

Definition 3.34 (Mild Solution for Semilinear SDE [19]):
A predictable process $Y(s)$ is called a *mild solution* if it satisfies the following integral equation $\mathbb{P}$-a.s. for all $s \in [s_0, b]$,

$$Y(s) = \mathcal{T}(s - s_0)\xi + \int_{s_0}^{s} \mathcal{T}(s - r)F(r, Y(r))\,\mathrm{d}r + \int_{s_0}^{s} \mathcal{T}(s - r)B(r, Y(r))\,\mathrm{d}W(r).$$

Theorem 3.9 (Existence and Uniqueness for Lipschitz Coefficients [19]):
Assume the following,

1. **Lipschitz condition.** There exists a constant $\ell_p > 0$ such that for all $s \in [s_0, b]$ and all $x, y \in \mathbb{H}$,
$$\|F(s, x) - F(s, y)\|_{\mathbb{H}}^p + \|B(s, x) - B(s, y)\|_{\mathcal{L}_2(\mathbb{K}_0,\mathbb{H})}^p \le \ell_p \|x - y\|_{\mathbb{H}}.$$
2. **Linear growth condition.** There exists a constant $C_p > 0$ such that for all $s \in [s_0, b]$ and all $x \in \mathbb{H}$,
$$\|F(s, x)\|_{\mathbb{H}}^p + \|B(s, x)\|_{\mathcal{L}_2(\mathbb{K}_0,\mathbb{H})}^p \le C_p(1 + \|x\|_{\mathbb{H}}^p).$$
3. **Initial condition.** $\mathbb{E}[\|\xi\|_{\mathbb{H}}^2] < \infty$.

Then there exists a unique (up to indistinguishability) mild solution Y to the semilinear SDE with continuous sample paths. Moreover, the solution satisfies the following estimates for any $p \ge 2$,

1. $\sup_{s\in[s_0,b]} \mathbb{E}[\|Y(s)\|_{\mathbb{H}}^p] \le \beta_{1,p}(b)(1 + \mathbb{E}[\|\xi\|_{\mathbb{H}}^p])$,
2. $\mathbb{E}[\sup_{s\in[s_0,b]} \|Y(s)\|_{\mathbb{H}}^p] \le \beta_{2,p}(b)(1 + \mathbb{E}[\|\xi\|_{\mathbb{H}}^p])$ for $p > 2$,
3. The solution depends continuously on the initial condition ξ.

Proof Sketch:
The proof employs a fixed-point argument in a suitable space of processes, e.g., $\mathcal{M}^p([s_0, b]; \mathbb{H}) := L^p(\Omega; C([s_0, b]; \mathbb{H}))$.

1. Define an operator Γ on $\mathcal{M}^p$ by the right-hand side of the mild solution formula.
2. Using the Lipschitz and growth conditions, along with the properties of the semigroup and the BDG inequality, show that Γ maps $\mathcal{M}^p$ into itself and is a contraction on small time intervals.
3. Apply the Banach fixed-point theorem to obtain a unique local solution.
4. Use Gronwall's lemma and the linear growth condition to extend the solution to the entire interval $[s_0, b]$ and to derive the moment estimates.

□

Almost Periodicity in One and Multi-Dimensional Distributions

The theory extends to the study of long-time behaviour, stability, invariant measures, and the existence of stationary or almost periodic solutions. We recall the definitions of almost periodicity for stochastic processes.

Let $(\mathbb{S}, d_{\mathbb{S}})$ be a Polish space (e.g., a separable Hilbert space) and let $Z : \Omega \times \mathbb{R} \to \mathbb{S}$ be a stochastic process defined on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition 3.35 (Almost Periodicity in Distribution [16, 35]):

1. **One-dimensional (marginal) distributions.** For each $s \in \mathbb{R}$, denote by $P_{Z(s)} \in \mathcal{P}(\mathbb{S})$ the law of $Z(s)$. The process Z is called almost periodic in one-dimensional distributions if the mapping
$$s \longmapsto P_{Z(s)} \in \mathcal{P}(\mathbb{S}),$$
is almost periodic with respect to the bounded Lipschitz metric d_{BL} on $\mathcal{P}(\mathbb{S})$.

2. **Multi-dimensional (path-space) distributions.** Assume that Z has almost surely continuous trajectories (so that Z can be viewed as a random element in $C(\mathbb{R}; \mathbb{S})$). Define the shifted process $\ddot{\ddot{Z}}(s) := Z(s + \cdot)$ (i.e., the whole trajectory shifted by s). Let $P_{\ddot{\ddot{Z}}(s)} \in \mathcal{P}(C(\mathbb{R}; \mathbb{S}))$ be the law of $\ddot{\ddot{Z}}(s)$. The process Z is called almost periodic in multi-dimensional distributions if the mapping
$$s \longmapsto P_{\ddot{\ddot{Z}}(s)} \in \mathcal{P}(C(\mathbb{R}; \mathbb{S})),$$
is almost periodic with respect to d_{BL} on $\mathcal{P}(C(\mathbb{R}; \mathbb{S}))$.

   Equivalently, for the multi-dimensional case, for every $\varepsilon > 0$ the set of ε-almost periods
$$\mathcal{V}_\varepsilon = \left\{ \tau \in \mathbb{R} \ : \ \sup_{s \in \mathbb{R}} d_{\mathrm{BL}}\left(P_{\ddot{\ddot{Z}}(s+\tau)}, \ P_{\ddot{\ddot{Z}}(s)} \right) < \varepsilon \right\},$$
is relatively dense in $\mathbb{R}$.

***Remark* 3.6:**
The notion of almost periodicity in multi-dimensional distributions is strictly stronger than that in one-dimensional distributions. It guarantees that the joint laws of $(Z(s_1 + \tau), \ldots, Z(s_k + \tau))$ for any finite set $\{s_1, \ldots, s_k\}$ are uniformly almost periodic in τ. The one-dimensional version only concerns the marginal laws at each fixed time separately and is generally insufficient for applications to SPDEs where correlations between different time instants matter. In this thesis, unless stated otherwise, "almost periodic in distribution" refers to the multi-dimensional (path-space) definition.

Part II.

Research Results

Chapter 4.

Almost Periodicity in Distribution for a Class of Second-Order Neutral Stochastic Differential Equations

This chapter is devoted to the analysis of a class of second-order neutral SDEs of the form,

$$\mathrm{d}\left[Y'(s) - F(s, Y(s))\right] = \left[AY(s) + BY'(s) + G(s, Y(s))\right]\mathrm{d}s + H(s, Y(s))\mathrm{d}W(s), \quad (4.1)$$

posed on a separable Hilbert space $\mathbb{H}$. The operators A and B are closed, densely defined linear operators on $\mathbb{H}$, typically unbounded.

The stochastic forcing is defined on a complete filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_s\}_{s\in\mathbb{R}}, \mathbb{P})$. A principal feature of our analysis is that the equation is considered on the entire real line $\mathbb{R}$, which necessitates a noise process defined for all time. This is achieved by a two-sided Q_{noise}–Wiener process $W(s)$, constructed by piecing together two independent standard Q_{noise}–Wiener processes,

$$W(s) = \begin{cases} W_1(s), & s \geqslant 0, \\ W_2(-s), & s < 0, \end{cases}$$

where W_1 and W_2 are two independent Q_{noise}–Wiener processes (as defined in Section 3.6), both taking values in a separable Hilbert space $\mathbb{K}$ with $\mathrm{Tr}(Q_{\text{noise}}) < \infty$. This construction ensures that $W(s)$ is a well-defined Wiener process with stationary increments on the whole real line, providing the foundation for studying the long-term behavior of the system.

The nonlinearities $F, G : \mathbb{R} \times \mathbb{H} \to \mathbb{H}$ and $H : \mathbb{R} \times \mathbb{H} \to \mathcal{L}_2(\mathbb{K}_0; \mathbb{H})$ are continuous mappings, where $\mathbb{K}_0 = Q_{\text{noise}}^{1/2}(\mathbb{K})$ denotes the reproducing kernel Hilbert space.

The chapter is structured as follows. Section 4.1 details the assumptions on the operators and nonlinearities and formally defines the concept of a mild solution. Section 4.3 contains the proof of existence and uniqueness. Section 4.4 is dedicated to the proof of almost periodicity in distribution. The chapter concludes with an example in Section 4.6.

4.1. Problem Formulation and Assumptions

We study the second-order neutral SDE (4.1) under the following set of hypotheses.

Assumption 4.1 (Exponential Stability):
The $\mathcal{M}, \mathcal{N}$–family generated by (A, B) is exponentially stable. That is, there exist constants $L_{(\mathcal{M},\mathcal{N})} \geq 1$ and $\delta > 0$ such that for all $s \geq r$,

$$\|\mathcal{M}(s-r)\|_{\mathcal{L}(\mathbb{H})} + \|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})} + \|\mathcal{N}_s(s-r)\|_{\mathcal{L}(\mathbb{H})} \leq L_{(\mathcal{M},\mathcal{N})} e^{-\delta(s-r)}.$$

Assumption 4.2 (Lipschitz and Growth Conditions):
The nonlinear mappings $F, G : \mathbb{R} \times \mathbb{H} \to \mathbb{H}$ and $H : \mathbb{R} \times \mathbb{H} \to \mathcal{L}_2(\mathbb{K}_0; \mathbb{H})$ are continuous in both variables and continuously differentiable in the time variable s. Furthermore, there exists a constant $\ell_p > 0$ (for a given $p \geq 2$) such that for all $s \in \mathbb{R}$ and all $y_1, y_2 \in \mathbb{H}$, the following Lipschitz conditions hold,

$$\begin{aligned} \|F(s,y_1) - F(s,y_2)\|^p_{\mathbb{H}} + \|G(s,y_1) - G(s,y_2)\|^p_{\mathbb{H}} + \|H(s,y_1) - H(s,y_2)\|^p_{\mathcal{L}_2(\mathbb{K}_0;\mathbb{H})} \\ \leq \ell_p \|y_1 - y_2\|^p_{\mathbb{H}}. \end{aligned} \tag{4.2}$$

Assumption 4.3 (Almost Periodicity of Coefficients):
The functions F, G, and H are almost periodic in s uniformly with respect to y on bounded subsets of $\mathbb{H}$. Precisely, for every $\varepsilon > 0$ and every bounded set $\mathbb{B} \subset \mathbb{H}$, there exists a number $l(\varepsilon, \mathbb{B}) > 0$ such that every interval of length l contains a number τ for which

$$\begin{aligned} \sup_{s\in\mathbb{R}} \sup_{y\in\mathbb{B}} \Big[\|F(s+\tau,y) - F(s,y)\|^p_{\mathbb{H}} + \|G(s+\tau,y) - G(s,y)\|^p_{\mathbb{H}} \\ + \|H(s+\tau,y) - H(s,y)\|^p_{\mathcal{L}_2(\mathbb{K}_0;\mathbb{H})} \Big] < \varepsilon^p. \end{aligned} \tag{4.3}$$

***Remark* 4.1:**
Under condition 4.3 it is not difficult to verify that,

$$\|F(s,y)\|^p_{\mathbb{H}} + \|G(s,y)\|^p_{\mathbb{H}} + \|H(s,y)\|^p_{\mathcal{L}_2(\mathbb{K}_0;\mathbb{H})} \leqslant 3^p(\ell_p + \hat{C}^p_p)(1 + ||y||^p_{\mathbb{H}}). \tag{4.4}$$

***Notation* 4.1:**
For $p \geq 2$, we define $\mathcal{M}^p(\mathbb{R}; \mathbb{H})$ as the space of all (equivalence classes of) measurable processes $Y : \mathbb{R} \times \Omega \to \mathbb{H}$ such that

$$\|Y\|^p_{\mathcal{M}^p} := \sup_{s\in\mathbb{R}} \mathbb{E}\|Y(s)\|^p_{\mathbb{H}} < \infty.$$

This is a Banach space which will serve as the solution space for our fixed-point argument.

We now define the concept of a mild solution for equation (4.1).

Definition 4.1 (Mild Solution):
A predictable $\mathbb{H}$-valued process $(Y(s))_{s\in\mathbb{R}}$ is called a *mild solution* of (4.1) if it satisfies the following integral equation $\mathbb{P}$-almost surely for all $s \geq s_0$,

$$
\begin{aligned}
Y(s) = \mathcal{M}(s-s_0)Y(s_0) &+ \mathcal{N}(s-s_0)[Y'(s_0) - F(s_0, Y(s_0))] \\
&+ \int_{s_0}^{s} \mathcal{N}_s(s-r)F(r, Y(r))\,\mathrm{d}r + \int_{s_0}^{s} \mathcal{N}(s-r)G(r, Y(r))\,\mathrm{d}r \\
&+ \int_{s_0}^{s} \mathcal{N}(s-r)H(r, Y(r))\,\mathrm{d}W(r).
\end{aligned} \tag{4.5}
$$

The variation-of-constants formula (4.5) is derived from the deterministic case given in Corollary 1.2.

4.2. A Generalised Grönwall Inequality

A key tool for our analysis on the unbounded domain $\mathbb{R}$ is the following extension of Grönwall's lemma.

Lemma 4.1 (Generalised Grönwall Inequality):
Let $h : \mathbb{R} \to [0, \infty)$ be a continuous function satisfying the inequality

$$
0 \le h(s) \le \beta(s) + \sum_{i=1}^{n} \beta_i \sum_{j=0}^{m} \int_{-\infty}^{s} (s-r)^j e^{-\delta_i (s-r)} h(r)\,\mathrm{d}r, \quad s \in \mathbb{R}, \tag{4.6}
$$

where,

1. $\beta : \mathbb{R} \to [0, \infty)$ is a locally integrable function.
2. $\beta_1, \dots, \beta_n \ge 0$ are constants.
3. $\delta_1, \dots, \delta_n > 0$ are decay rates.
4. The integrals in (4.6) are convergent.

Let $\delta_{\min} = \min\{\delta_1, \dots, \delta_n\}$ and define $\bar{\beta} = \sum_{i=1}^{n} \beta_i$. Assume that $\delta_{\min} > m + \bar{\beta}$ and that for some $\gamma \in (0, \delta_{\min} - m - \bar{\beta}]$, the integrals $\int_{-\infty}^{0} |r|^j e^{\gamma r} \beta(r)\,\mathrm{d}r$ converge for $j = 0, 1, \dots, m$. Then, for all $s \in \mathbb{R}$, the function h satisfies the estimate

$$
h(s) \le \beta(s) + \bar{\beta} \int_{-\infty}^{s} e^{-\gamma(s-r)} \beta(r)\,\mathrm{d}r. \tag{4.7}
$$

In particular, if $\beta(s) \equiv \beta_0$ is a constant function, then

$$
h(s) \le \beta_0 \left(\frac{\delta_{\min} - m}{\delta_{\min} - m - \bar{\beta}} \right). \tag{4.8}
$$

Proof:
The proof proceeds by an ingenious transformation and integration technique. Define the constants $\beta_i' = \beta_i/\bar{\beta}$ for $i = 1, \dots, n$. For each i, define the function

$$\mathcal{K}_i(s) = \sum_{j=0}^{m} \int_{-\infty}^{s} (s-r)^j e^{-\delta_i(s-r)} h(r)\,\mathrm{d}r.$$

We differentiate the function $e^{\gamma s}\mathcal{K}_i(s)$ with respect to s. Applying the Leibniz rule for differentiation under the integral sign yields,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}[e^{\gamma s}\mathcal{K}_i(s)] &= e^{\gamma s}\left[h(s) + \sum_{j=0}^{m}\int_{-\infty}^{s}\frac{\partial}{\partial s}\left((s-r)^j e^{-\delta_i(s-r)}\right)h(r)\,\mathrm{d}r\right] + \gamma e^{\gamma s}\mathcal{K}_i(s)\\ &= e^{\gamma s}\left[h(s) + \sum_{j=0}^{m}\int_{-\infty}^{s}\left(j(s-r)^{j-1} - \delta_i(s-r)^j\right)e^{-\delta_i(s-r)}h(r)\,\mathrm{d}r\right]\\ &\quad + \gamma e^{\gamma s}\mathcal{K}_i(s)\end{aligned}$$

Reindexing the sum and grouping terms, we obtain,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}[e^{\gamma s}\mathcal{K}_i(s)] &= e^{\gamma s}\Bigg[h(s) + \sum_{j=0}^{m-1}(j+1)\int_{-\infty}^{s}(s-r)^j e^{-\delta_i(s-r)}h(r)\,\mathrm{d}r\\ &\qquad - \delta_i\sum_{j=0}^{m}\int_{-\infty}^{s}(s-r)^j e^{-\delta_i(s-r)}h(r)\,\mathrm{d}r\Bigg] + \gamma e^{\gamma s}\mathcal{K}_i(s)\\ &= e^{\gamma s}\Bigg[h(s) + \sum_{j=0}^{m-1}(j+1-\delta_i)\int_{-\infty}^{s}(s-r)^j e^{-\delta_i(s-r)}h(r)\,\mathrm{d}r\\ &\qquad - \delta_i\int_{-\infty}^{s}(s-r)^m e^{-\delta_i(s-r)}h(r)\,\mathrm{d}r\Bigg]\\ &\quad + \gamma e^{\gamma s}\mathcal{K}_i(s)\end{aligned}$$

Now, we form a convex combination. Multiply the above equation by β_i' and sum over $i = 1$ to n,

$$\begin{aligned}\sum_{i=1}^{n}\beta_i'\frac{\mathrm{d}}{\mathrm{d}s}[e^{\gamma s}\mathcal{K}_i(s)] &= e^{\gamma s}\Bigg[h(s) + \sum_{i=1}^{n}\beta_i'\sum_{j=0}^{m-1}(j+1-\delta_i)\int_{-\infty}^{s}(s-r)^j e^{-\delta_i(s-r)}h(r)\,\mathrm{d}r\\ &\qquad - \sum_{i=1}^{n}\beta_i'\delta_i\int_{-\infty}^{s}(s-r)^m e^{-\delta_i(s-r)}h(r)\,\mathrm{d}r\Bigg] + \gamma e^{\gamma s}\sum_{i=1}^{n}\beta_i'\mathcal{K}_i(s)\end{aligned}$$

By the choice of γ, we have $\gamma \leq \delta_{\min} - m - \bar{\beta} \leq \delta_i - m - \bar{\beta}$ for each i. Since $h(r) \geq 0$, the terms involving $(j+1-\delta_i)$ and $\gamma - \delta_i$ are negative for sufficiently large j,

$$\frac{\mathrm{d}}{\mathrm{d}s}\sum_{i=1}^{n}\beta_i'(e^{\gamma s}\mathcal{K}_i(s)) = e^{\gamma s}\Bigg\{h(s)+\sum_{i=1}^{n}\beta_i'(\gamma-\delta_i)\int_{-\infty}^{s}(s-r)^m h(r)e^{-\delta_i(s-r)}\,\mathrm{d}r$$
$$+\sum_{i=1}^{n}\beta_i'\sum_{j=0}^{m-1}\int_{-\infty}^{s}(\gamma-\delta_i+j+1)(s-r)^j h(r)e^{-\delta_i(s-r)}\,\mathrm{d}r\Bigg\}$$
$$\leqslant e^{\gamma s}\beta(s).$$

Integrating from $-\infty$ to s,

$$\sum_{i=1}^{n}\beta_i' e^{\gamma s}\sum_{j=0}^{m}\int_{-\infty}^{s}(s-r)^j e^{-\delta_i(s-r)}h(r)\,\mathrm{d}r \leqslant \int_{-\infty}^{s}e^{\gamma r}\beta(r)\,\mathrm{d}r,$$

and using the fact that the left-hand side vanishes in the limit as $s \to -\infty$ (due to the exponential decay and the convergence assumptions), we obtain,

$$\sum_{i=1}^{n}\beta_i' e^{\gamma s}\mathcal{K}_i(s) \leq \int_{-\infty}^{s}e^{\gamma r}\beta(r)\,\mathrm{d}r.$$

Multiplying both sides by $e^{-\gamma s}$ and recalling the definition of $\mathcal{K}_i(s)$ and $\bar{\beta}$ yields,

$$\sum_{i=1}^{n}\beta_i\sum_{j=0}^{m}\int_{-\infty}^{s}(s-r)^j e^{-\delta_i(s-r)}h(r)\,\mathrm{d}r \leq \bar{\beta}\int_{-\infty}^{s}e^{-\gamma(s-r)}\beta(r)\,\mathrm{d}r.$$

Substituting this estimate back into the original inequality (4.6) gives the desired result (4.7). The constant case (4.8) follows by direct computation, with $\gamma := \delta_{\min} - m - \beta$. □

4.3. Existence and Uniqueness of a Global Mild Solution

We now state and prove the first main result of this chapter, the existence and uniqueness of a mild solution to (4.1) defined for all times $s \in \mathbb{R}$.

Theorem 4.1:
Let $p \geq 2$ and suppose Assumptions 4.1–4.2 hold. Define the constants,

$$\begin{cases} \beta_{1,p}(\delta) & := \left\{3^{p-1}C_p L^p_{(\mathcal{M},\mathcal{N})}\left(2\left(\frac{1}{\delta}\right)^p + K_{p/2}\left(\frac{1}{2\delta}\right)^{\frac{p}{2}}\right)\right\} \\ \beta_{2,p}(\delta) & := \left\{3^{p-1}C_p L^p_{(\mathcal{M},\mathcal{N})}\left(2\left(\frac{1}{\delta}\right)^{p-1} + K_{p/2}\left(\frac{1}{2\delta}\right)^{\frac{p}{2}-1}\right)\right\} \\ C_p & := 3^p(\ell_p + \hat{C}_p^p). \end{cases}$$

If the condition

$$\beta_{1,p}(\delta) < 1. \tag{4.9}$$

is satisfied, then there exists a unique mild solution $Y \in \mathcal{M}^p(\mathbb{R};\mathbb{H})$ of (4.1).

Moreover, this solution can be represented in the following form,

$$Y(s) = \int_{-\infty}^{s} \mathcal{N}_s(s-r)F(r,Y(r))\,\mathrm{d}r + \int_{-\infty}^{s} \mathcal{N}(s-r)G(r,Y(r))\,\mathrm{d}r + \int_{-\infty}^{s} \mathcal{N}(s-r)H(r,Y(r))\,\mathrm{d}W(r). \quad (4.10)$$

Additionally, if $\delta > \beta_{2,p}(\delta)$, then the solution satisfies the following moment bounds. For $p > 2$,

$$\sup_{s\in\mathbb{R}} \mathbb{E}\|Y(s)\|_{\mathbb{H}}^p \le \frac{\delta}{\delta - \beta_{2,p}(\delta)}.$$

For $p = 2$,

$$\sup_{s\in\mathbb{R}} \mathbb{E}\|Y(s)\|_{\mathbb{H}}^2 \le \frac{\delta}{\delta - 3C_2 L_{(\mathcal{M},\mathcal{N})}^2(2\delta^{-1}+1)}.$$

Proof:
The proof is based on the Banach fixed-point theorem. We define an operator $\mathcal{T}$ on the space $\mathcal{M}^p(\mathbb{R};\mathbb{H})$ by the right-hand side of (4.10),

$$(\mathcal{T}Y)(s) = \int_{-\infty}^{s} \mathcal{N}_s(s-r)F(r,Y(r))\,\mathrm{d}r + \int_{-\infty}^{s} \mathcal{N}(s-r)G(r,Y(r))\,\mathrm{d}r + \int_{-\infty}^{s} \mathcal{N}(s-r)H(r,Y(r))\,\mathrm{d}W(r). \quad (4.11)$$

Step 1. Explicit formula.

Let us begin by proving that the function Y, defined by the expression,

$$Y(s) := \int_{-\infty}^{s} \mathcal{N}_s(s-r)F(r,Y(r))\,\mathrm{d}r + \int_{-\infty}^{s} \mathcal{N}(s-r)G(r,Y(r))\,\mathrm{d}r + \int_{-\infty}^{s} \mathcal{N}(s-r)H(r,Y(r))\,\mathrm{d}W(r). \quad (4.12)$$

is well-defined at $-\infty$ and satisfies the equation, $\mathbb{P}-$a.s., for all $s \geqslant s_0$, $s \in \mathbb{R}$,

$$Y(s) = \mathcal{M}(s-s_0)Y(s_0) + \mathcal{N}(s-s_0)(Y'(s_0) - F(s_0,Y(s_0))) + \int_{s_0}^{s} \mathcal{N}_s(s-r)F(r,Y(r))\,\mathrm{d}r + \int_{s_0}^{s} \mathcal{N}(s-r)G(r,Y(r))\,\mathrm{d}r + \int_{s_0}^{s} \mathcal{N}(s-r)H(r,Y(r))\,\mathrm{d}W(r).$$

Taking $s_1 < s_2$ and $Y \in \mathcal{M}^p(\mathbb{R};\mathbb{H})$, let us define,

$$\Gamma(u) := \int_{u}^{s} \mathcal{N}_s(s-r)F(r,Y(r))\,\mathrm{d}r + \int_{u}^{s} \mathcal{N}(s-r)G(r,Y(r))\,\mathrm{d}r$$

$$+ \int_u^s \mathcal{N}(s-r)H(r,Y(r))\,\mathrm{d}W(r).$$

We aim to demonstrate that $(\Gamma(u))_{u\in\mathbb{R}}$ is defined at $-\infty$. Starting with,

$$\begin{aligned}
\mathbb{E}||\Gamma(s_2)-\Gamma(s_1)||_{\mathbb{H}}^p \leqslant{}& 3^{p-1}\mathbb{E}\left\|\int_{s_1}^{s_2}\mathcal{N}_s(s-r)F(r,Y(r))\,\mathrm{d}r\right\|_{\mathbb{H}}^p \\
&+ 3^{p-1}\mathbb{E}\left\|\int_{s_1}^{s_2}\mathcal{N}(s-r)G(r,Y(r))\,\mathrm{d}r\right\|_{\mathbb{H}}^p \\
&+ 3^{p-1}\mathbb{E}\left\|\int_{s_1}^{s_2}\mathcal{N}(s-r)H(r,Y(r))\,\mathrm{d}W(r)\right\|_{\mathbb{H}}^p \\
:={}& 3^{p-1}(\mathrm{I}_p^1+\mathrm{I}_p^2+\mathrm{I}_p^3).
\end{aligned} \tag{4.13}$$

For $p \geqslant 2$, applying Hölder's inequality with exponents $(p,(p-1)^{-1}p)$ under 4.3, we obtain,

$$\begin{aligned}
\mathrm{I}_p^1 ={}& \mathbb{E}\left\|\int_{s_1}^{s_2}\mathcal{N}_s(s-r)F(r,Y(r))\,\mathrm{d}r\right\|_{\mathbb{H}}^p \\
\leqslant{}& \left(\int_{s_1}^{s_2}\|\mathcal{N}_s(s-r)\|_{\mathcal{L}(\mathbb{H})}\,\mathrm{d}r\right)^{p-1}\int_{s_1}^{s_2}\|\mathcal{N}_s(s-r)\|_{\mathcal{L}(\mathbb{H})}\mathbb{E}\|F(r,Y(r))\|_{\mathbb{H}}^p\,\mathrm{d}r \\
\leqslant{}& C_p\left(\int_{s_1}^{s_2}\|\mathcal{N}_s(s-r)\|_{\mathcal{L}(\mathbb{H})}\,\mathrm{d}r\right)^{p-1}\int_{s_1}^{s_2}\|\mathcal{N}_s(s-r)\|_{\mathcal{L}(\mathbb{H})}\mathbb{E}\left(1+\|Y(r)\|_{\mathbb{H}}^p\right)\,\mathrm{d}r \\
\leqslant{}& C_p\left(\int_{s_1}^{s_2}\|\mathcal{N}_s(s-r)\|_{\mathcal{L}(\mathbb{H})}\,\mathrm{d}r\right)^{p} \\
&+ C_p\left(\int_{s_1}^{s_2}\|\mathcal{N}_s(s-r)\|_{\mathcal{L}(\mathbb{H})}\,\mathrm{d}r\right)^{p-1}\int_{s_1}^{s_2}\|\mathcal{N}_s(s-r)\|_{\mathcal{L}(\mathbb{H})}\mathbb{E}\|Y(r)\|_{\mathbb{H}}^p\,\mathrm{d}r.
\end{aligned}$$

Furthermore, we also have for I_p^2,

$$\begin{aligned}
\mathrm{I}_p^2 ={}& \mathbb{E}\left\|\int_{s_1}^{s_2}\mathcal{N}(s-r)G(r,Y(r))\,\mathrm{d}r\right\|_{\mathbb{H}}^p \\
\leqslant{}& \left(\int_{s_1}^{s_2}\|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}\,\mathrm{d}r\right)^{p-1}\int_{s_1}^{s_2}\|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}\mathbb{E}\|G(r,Y(r))\|_{\mathbb{H}}^p\,\mathrm{d}r \\
\leqslant{}& C_p\left(\int_{s_1}^{s_2}\|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}\,\mathrm{d}r\right)^{p-1}\int_{s_1}^{s_2}\|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}\mathbb{E}\left(1+\|Y(r)\|_{\mathbb{H}}^p\right)\,\mathrm{d}r \\
\leqslant{}& C_p\left(\int_{s_1}^{s_2}\|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}\,\mathrm{d}r\right)^{p} \\
&+ C_p\left(\int_{s_1}^{s_2}\|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}\,\mathrm{d}r\right)^{p-1}\int_{s_1}^{s_2}\|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}\mathbb{E}\|Y(r)\|_{\mathbb{H}}^p\,\mathrm{d}r.
\end{aligned} \tag{4.14}$$

For $p > 2$, employing the $(\frac{p}{2}, (\frac{p}{2} - 1)^{-1}\frac{p}{2})$–Hölder's inequality and applying Lemma 3.8, we obtain,

$$
\begin{aligned}
\mathrm{I}_p^3 &= \mathbb{E}\left\| \int_{s_1}^{s_2} \mathcal{N}(s-r)H(r,Y(r))\,\mathrm{d}W(r) \right\|_{\mathbb{H}}^p \\
&\leqslant K_{p/2}\left(\int_{s_1}^{s_2} \|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}^2\,\mathrm{d}r\right)^{\frac{p}{2}-1} \int_{s_1}^{s_2} \|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}^2 \mathbb{E}\|H(r,Y(r))\|_{\mathcal{L}_2(\mathbb{K}_0;\mathbb{H})}^p\,\mathrm{d}r \\
&\leqslant C_pK_{p/2}\left(\int_{s_1}^{s_2} \|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}^2\,\mathrm{d}r\right)^{\frac{p}{2}-1} \int_{s_1}^{s_2} \|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}^2 \mathbb{E}\left(1+\|Y(r)\|_{\mathbb{H}}^p\right)\,\mathrm{d}r \\
&\leqslant C_pK_{p/2}\left(\int_{s_1}^{s_2} \|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}^2\,\mathrm{d}r\right)^{\frac{p}{2}} \\
&+ C_pK_{p/2}\left(\int_{s_1}^{s_2} \|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}^2\,\mathrm{d}r\right)^{\frac{p}{2}-1} \int_{s_1}^{s_2} \|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}^2 \mathbb{E}\|Y(r)\|_{\mathbb{H}}^p\,\mathrm{d}r. \qquad (4.15)
\end{aligned}
$$

If $p = 2$, we have,

$$
\begin{aligned}
\mathrm{I}_2^3 &= \mathbb{E}\left\| \int_{s_1}^{s_2} \mathcal{N}(s-r)H(r,Y(r))\,\mathrm{d}W(r) \right\|_{\mathbb{H}}^2 \\
&\leqslant \mathbb{E}\left(\int_{s_1}^{s_2} \|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}^2 \|H(r,Y(r))\|_{\mathcal{L}_2(\mathbb{K}_0;\mathbb{H})}^2\,\mathrm{d}r\right) \\
&\leqslant \int_{s_1}^{s_2} \|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}^2 \mathbb{E}\|H(r,Y(r))\|_{\mathcal{L}_2(\mathbb{K}_0;\mathbb{H})}^2\,\mathrm{d}r \\
&\leqslant C_2\int_{s_1}^{s_2} \|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}^2 \mathbb{E}\left(1+\|Y(r)\|_{\mathbb{H}}^p\right)\,\mathrm{d}r \\
&\leqslant C_2\left(\int_{s_1}^{s_2} \|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}^2\,\mathrm{d}r\right) + C_2\int_{s_1}^{s_2} \|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}^2 \mathbb{E}\|Y(r)\|_{\mathbb{H}}^2\,\mathrm{d}r. \qquad (4.16)
\end{aligned}
$$

Because,

$$
\lim_{s_2 \rightharpoonup -\infty} \int_{s_1}^{s_2} \left(\|\mathcal{N}_s(s-r)\|_{\mathcal{L}(\mathbb{H})} + \|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})} + \|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}^2\right)\,\mathrm{d}r = 0.
$$

Then for $Y \in \mathcal{M}^p(\mathbb{R};\mathbb{H})$, we have,

$$
\lim_{s_2 \rightharpoonup -\infty} \mathbb{E}||\Gamma(s_2) - \Gamma(s_1)||_{\mathbb{H}}^p = 0.
$$

Hence, the limit $\lim_{u\to-\infty}\Gamma(u)$ exists, ensuring that Y in Equation (4.12) is well-defined at $-\infty$. Furthermore, we observe that,

$$
\begin{aligned}
&\mathcal{M}(s-s_0)Y(s_0) + \mathcal{N}(s-s_0)Y'(s_0) \\
&\qquad = \int_{-\infty}^{s_0} \left(\mathcal{M}(s-s_0)\mathcal{N}_{s_0}(s_0-r) + \mathcal{N}(s-s_0)\mathcal{N}_{s_0s_0}(s_0-r)\right) F(r,Y(r))\,\mathrm{d}r
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_{-\infty}^{s_0} \big(\mathcal{M}(s-s_0)\mathcal{N}(s_0-r) + \mathcal{N}(s-s_0)\mathcal{N}_{s_0}(s_0-r)\big)\, G(r, Y(r))\,\mathrm{d}r \\
&+ \int_{-\infty}^{s_0} \big(\mathcal{M}(s-s_0)\mathcal{N}(s_0-r) + \mathcal{N}(s-s_0)\mathcal{N}_{s_0}(s_0-r)\big)\, H(r, Y(r))\,\mathrm{d}W(r) \\
&\qquad + \mathcal{N}(s-s_0)F(s_0, Y(s_0)).
\end{aligned}
$$

This yields,

$$
\begin{aligned}
\mathcal{N}(s-s_0)Y(s_0) + \mathcal{N}(s-s_0)(Y'(s_0) - F(s_0, Y(s_0))) &= \int_{-\infty}^{s_0} \mathcal{N}_s(s-r)F(r, Y(r))\,\mathrm{d}r \\
+ \int_{-\infty}^{s_0} \mathcal{N}(s-r)G(r, Y(r))\,\mathrm{d}r &+ \int_{-\infty}^{s_0} \mathcal{N}(s-r)H(r, Y(r))\,\mathrm{d}W(r). \quad (4.17)
\end{aligned}
$$

By separating the integral in Equation (4.12) into two separate components and employing the equality stated in Equation (4.17), it follows that

$$
\begin{aligned}
Y(s) &= \int_{-\infty}^{s_0} \mathcal{N}_s(s-r)F(r, Y(r))\,\mathrm{d}r + \int_{-\infty}^{s_0} \mathcal{N}(s-r)G(r, Y(r))\,\mathrm{d}r \\
&+ \int_{-\infty}^{s_0} \mathcal{N}(s-r)H(r, Y(r))\,\mathrm{d}W(r) + \int_{s_0}^{s} \mathcal{N}_s(s-r)F(r, Y(r))\,\mathrm{d}r \\
&+ \int_{s_0}^{s} \mathcal{N}(s-r)G(r, Y(r))\,\mathrm{d}r + \int_{s_0}^{s} \mathcal{N}(s-r)H(r, Y(r))\,\mathrm{d}W(r) \\
&= \mathcal{M}(s-s_0)Y(s_0) + \mathcal{N}(s-s_0)(Y'(s_0) - F(s_0, Y(s_0))) \\
&+ \int_{s_0}^{s} \mathcal{N}_s(s-r)F(r, Y(r))\,\mathrm{d}r + \int_{s_0}^{s} \mathcal{N}(s-r)G(r, Y(r))\,\mathrm{d}r \\
&+ \int_{s_0}^{s} \mathcal{N}(s-r)H(r, Y(r))\,\mathrm{d}W(r).
\end{aligned}
$$

The first step is now complete.

Step 2. Well-definedness of $\mathcal{T}$.

We must show that if $Y \in \mathcal{M}^p(\mathbb{R}; \mathbb{H})$, then $\mathcal{T}Y$ is well-defined and also belongs to $\mathcal{M}^p(\mathbb{R}; \mathbb{H})$. We estimate the p-th moment of each term in (4.11) separately, using the growth condition (4.4) and the exponential stability (4.1).

Consider the first term,

$$
I_1(s) = \int_{-\infty}^{s} \mathcal{N}_s(s-r)F(r, Y(r))\,\mathrm{d}r.
$$

Using the triangle inequality, Hölder's inequality, and the fact that $\int_{-\infty}^{s} e^{-\delta(s-r)}\,\mathrm{d}r = \delta^{-1}$, we obtain,

$$
\mathbb{E}\|I_1(s)\|^p \le \mathbb{E}\left(\int_{-\infty}^{s} \|\mathcal{N}_s(s-r)\|\|F(r, Y(r))\|\,\mathrm{d}r\right)^p
$$

$$
\begin{aligned}
&\leq \mathbb{E}\left(\int_{-\infty}^{s} L_{(\mathcal{M},\mathcal{N})} e^{-\delta(s-r)} \|F(r,Y(r))\|\,\mathrm{d}r\right)^{p} \\
&\leq L_{(\mathcal{M},\mathcal{N})}^{p}\left(\int_{-\infty}^{s} e^{-\delta(s-r)}\,\mathrm{d}r\right)^{p-1}\int_{-\infty}^{s} e^{-\delta(s-r)}\mathbb{E}\|F(r,Y(r))\|^{p}\,\mathrm{d}r \quad \text{(Hölder)} \\
&\leq L_{(\mathcal{M},\mathcal{N})}^{p}\delta^{-(p-1)}\int_{-\infty}^{s} e^{-\delta(s-r)}C_p(1+\mathbb{E}\|Y(r)\|^{p})\,\mathrm{d}r \quad \text{(by (4.4))} \\
&\leq L_{(\mathcal{M},\mathcal{N})}^{p}C_p\delta^{-(p-1)}\left[\delta^{-1}+\int_{-\infty}^{s} e^{-\delta(s-r)}\mathbb{E}\|Y(r)\|^{p}\,\mathrm{d}r\right].
\end{aligned}
$$

Since $Y \in \mathcal{M}^p$, we have $\sup_{r\in\mathbb{R}_{\leqslant s}} \mathbb{E}\|Y(r)\|^p \leq M$ for some $M > 0$. Thus,

$$\mathbb{E}\|I_1(s)\|^p \leq L_{(\mathcal{M},\mathcal{N})}^{p}C_p\delta^{-(p-1)}\left[\delta^{-1}+M\delta^{-1}\right] = L_{(\mathcal{M},\mathcal{N})}^{p}C_p\delta^{-p}(1+M).$$

A similar estimate holds for the second term,

$$I_2(s) = \int_{-\infty}^{s} \mathcal{N}(s-r)G(r,Y(r))\,\mathrm{d}r,$$

yielding the same bound.

For the stochastic convolution term,

$$I_3(s) = \int_{-\infty}^{s} \mathcal{N}(s-r)H(r,Y(r))\,\mathrm{d}W(r),$$

we use the Burkholder-Davis-Gundy inequality (Lemma 3.8),

$$
\begin{aligned}
\mathbb{E}\|I_3(s)\|^p &\leq K_{p/2}\,\mathbb{E}\left(\int_{-\infty}^{s} \|\mathcal{N}(s-r)H(r,Y(r))\|_{\mathcal{L}_2}^{2}\,\mathrm{d}r\right)^{p/2} \\
&\leq K_{p/2}\,\mathbb{E}\left(\int_{-\infty}^{s} L_{(\mathcal{M},\mathcal{N})}^{2}e^{-2\delta(s-r)}\|H(r,Y(r))\|_{\mathcal{L}_2}^{2}\,\mathrm{d}r\right)^{p/2} \\
\text{(Hölder)} &\leq K_{p/2}L_{(\mathcal{M},\mathcal{N})}^{p}\left(\int_{-\infty}^{s} e^{-2\delta(s-r)}\,\mathrm{d}r\right)^{p/2-1}\int_{-\infty}^{s} e^{-2\delta(s-r)}\mathbb{E}\|H(r,Y(r))\|_{\mathcal{L}_2}^{p}\,\mathrm{d}r \\
&= K_{p/2}L_{(\mathcal{M},\mathcal{N})}^{p}(2\delta)^{-(p/2-1)}\int_{-\infty}^{s} e^{-2\delta(s-r)}C_p(1+\mathbb{E}\|Y(r)\|^{p})\,\mathrm{d}r \\
&\leq K_{p/2}L_{(\mathcal{M},\mathcal{N})}^{p}C_p(2\delta)^{-(p/2-1)}\left[(2\delta)^{-1}+M(2\delta)^{-1}\right] \\
&= K_{p/2}L_{(\mathcal{M},\mathcal{N})}^{p}C_p(2\delta)^{-p/2}(1+M).
\end{aligned}
$$

Combining these three estimates and using the inequality $(a+b+c)^p \leq 3^{p-1}(a^p+b^p+c^p)$, we find,

$$\mathbb{E}\|(\mathcal{T}Y)(s)\|^p \leq 3^{p-1}L_{(\mathcal{M},\mathcal{N})}^{p}C_p\left(2\delta^{-p}+K_{p/2}(2\delta)^{-p/2}\right)(1+M) = \beta_{1,p}(\delta)(1+M).$$

Taking the supremum over $s \in \mathbb{R}$ shows that $\mathcal{T}Y \in \mathcal{M}^p(\mathbb{R};\mathbb{H})$.

Step 3. Contraction Property.

Let $Y_1, Y_2 \in \mathcal{M}^p(\mathbb{R};\mathbb{H})$. We estimate the difference $\mathcal{T}Y_1 - \mathcal{T}Y_2$ using the Lipschitz condition (4.2). The calculations are similar to Step 2 but now involve the difference of the nonlinear terms. For the first term,

$$\begin{aligned}
&\mathbb{E}\left\|\int_{-\infty}^{s} \mathcal{N}_s(s-r)(F(r,Y_1(r)) - F(r,Y_2(r)))\,\mathrm{d}r\right\|^p \\
&\le L^p_{(\mathcal{M},\mathcal{N})}\left(\int_{-\infty}^{s} e^{-\delta(s-r)}\,\mathrm{d}r\right)^{p-1}\int_{-\infty}^{s} e^{-\delta(s-r)}\mathbb{E}\|F(r,Y_1(r)) - F(r,Y_2(r))\|^p\,\mathrm{d}r \\
&\le L^p_{(\mathcal{M},\mathcal{N})}\delta^{-(p-1)}\int_{-\infty}^{s} e^{-\delta(s-r)}\ell_p\mathbb{E}\|Y_1(r) - Y_2(r)\|^p\,\mathrm{d}r \\
&\le L^p_{(\mathcal{M},\mathcal{N})}\ell_p\delta^{-p}\sup_{r\in\mathbb{R}}\mathbb{E}\|Y_1(r) - Y_2(r)\|^p.
\end{aligned}$$

The same bound (with the same constant) is obtained for the term involving G. For the stochastic term, we have,

$$\begin{aligned}
&\mathbb{E}\left\|\int_{-\infty}^{s} \mathcal{N}(s-r)(H(r,Y_1(r)) - H(r,Y_2(r)))\,\mathrm{d}W(r)\right\|^p \\
&\le K_{p/2}L^p_{(\mathcal{M},\mathcal{N})}\left(\int_{-\infty}^{s} e^{-2\delta(s-r)}\,\mathrm{d}r\right)^{p/2-1}\int_{-\infty}^{s} e^{-2\delta(s-r)}\mathbb{E}\|H(r,Y_1(r)) - H(r,Y_2(r))\|^p_{\mathcal{L}_2}\,\mathrm{d}r \\
&\le K_{p/2}L^p_{(\mathcal{M},\mathcal{N})}(2\delta)^{-(p/2-1)}\int_{-\infty}^{s} e^{-2\delta(s-r)}\ell_p\mathbb{E}\|Y_1(r) - Y_2(r)\|^p\,\mathrm{d}r \\
&\le K_{p/2}L^p_{(\mathcal{M},\mathcal{N})}\ell_p(2\delta)^{-p/2}\sup_{r\in\mathbb{R}}\mathbb{E}\|Y_1(r) - Y_2(r)\|^p.
\end{aligned}$$

Putting these together and using the constant $C_p \ge \ell_p$, we get,

$$\begin{aligned}
&\mathbb{E}\|(\mathcal{T}Y_1)(s) - (\mathcal{T}Y_2)(s)\|^p \\
&\quad\le 3^{p-1}\left[2L^p_{(\mathcal{M},\mathcal{N})}C_p\delta^{-p} + K_{p/2}L^p_{(\mathcal{M},\mathcal{N})}C_p(2\delta)^{-p/2}\right]\|Y_1 - Y_2\|^p_{\mathcal{M}^p} \\
&\quad= \beta_{1,p}(\delta)\|Y_1 - Y_2\|^p_{\mathcal{M}^p}.
\end{aligned}$$

Since $\beta_{1,p}(\delta) < 1$ by assumption, the operator $\mathcal{T}$ is a contraction on $\mathcal{M}^p(\mathbb{R};\mathbb{H})$. By the Banach fixed-point theorem, there exists a unique fixed point $Y \in \mathcal{M}^p(\mathbb{R};\mathbb{H})$ such that $\mathcal{T}Y = Y$, which is the desired mild solution given by (4.10). Assuming Y is a unique fixed point of $\mathcal{K}$,

$$\begin{aligned}
Y(s) = \int_{-\infty}^{s} \mathcal{N}_s(s-r)F(r,Y(r))\,\mathrm{d}r + \int_{-\infty}^{s} \mathcal{N}(s-r)G(r,Y(r))\,\mathrm{d}r& \\
+ \int_{-\infty}^{s} \mathcal{N}(s-r)H(r,Y(r))\,\mathrm{d}W(r)&.
\end{aligned}$$

Step 4. Moment Bounds.

The fixed point Y satisfies $Y = \mathcal{T}Y$. Using the estimates from Step 2 and the fact that Y is the fixed point, we derive,

$$\mathbb{E}\|Y(s)\|^p \leq \beta_{1,p}(\delta) + \beta_{2,p}(\delta) \int_{-\infty}^{s} e^{-\delta(s-r)} \mathbb{E}\|Y(r)\|^p \, \mathrm{d}r + \beta_{2,p}(\delta) \int_{-\infty}^{s} e^{-2\delta(s-r)} \mathbb{E}\|Y(r)\|^p \, \mathrm{d}r.$$

This is precisely the form of inequality required by the generalised Grönwall lemma (Lemma 4.1) with $n = 2$, $m = 0$, $\beta(s) = \beta_{1,p}(\delta)$, $\beta_1 = \beta_{2,p}(\delta)$, $\delta_1 = \delta$, $\beta_2 = \beta_{2,p}(\delta)$, $\delta_2 = 2\delta$. Applying the lemma yields the stated moment bounds.

Moreover, by [19, Theorem 7.2], almost every trajectory of the solution is continuous. This completes the proof of Theorem 4.1. $\square$

4.4. Almost Periodicity in Distribution

We now proceed to the second main result, which establishes the almost periodicity in distribution of the solution obtained in Theorem 4.1. We require an additional assumption regarding the convergence of sequences of equations.

Assumption 4.4 (Convergence of Approximations):
For $n \in \mathbb{N} \cup \{\infty\}$, let $F^n, G^n : \mathbb{R} \times \mathbb{H} \to \mathbb{H}$ and $H^n : \mathbb{R} \times \mathbb{H} \to \mathcal{L}_2(\mathbb{K}_0; \mathbb{H})$ satisfy Assumptions 4.1–4.3.

1. Assume that for all $(s, y) \in \mathbb{R} \times \mathbb{H}$,

$$\begin{cases} \lim_{n\to\infty} F^n(s, y) &= F^\infty(s, y), \\ \lim_{n\to\infty} G^n(s, y) &= G^\infty(s, y), \\ \lim_{n\to\infty} H^n(s, y) &= H^\infty(s, y). \end{cases}$$

   Under this assumption, let $(Y^n(s))_{s\in\mathbb{R}}$ denote the solution to the following equation,

$$\begin{aligned} &\mathrm{d}(Y^{n\prime}(s) - F^n(s, Y^n(s))) \\ &= (AY^n(s) + BY'^n(s) + G^n(s, Y^n(s)))\mathrm{d}s + H^n(s, Y^n(s))\mathrm{d}W(s), \quad \text{if} \quad s \geqslant s_0, \end{aligned}$$

   with initial conditions

$$Y^n(s_0) = \xi^n, \quad Y^{n\prime}(s_0) = \eta^n, \quad \text{if} \quad s = s_0.$$

2. In the space $(\mathcal{P}(\mathbb{H} \times C(\mathbb{R}; \mathbb{K})), d_{\mathrm{BL}})$, we assume that the initial conditions converge in distribution,

$$\mathscr{L}(\xi^n, W) \xrightarrow{d_{\mathrm{BL}}} \mathscr{L}(\xi^\infty, W),$$
$$\mathscr{L}(\eta^n, W) \xrightarrow{d_{\mathrm{BL}}} \mathscr{L}(\eta^\infty, W),$$

$$\mathscr{L}(F^n(\cdot, \xi^n), W) \xrightarrow{d_{\mathrm{BL}}} \mathscr{L}(F^\infty(\cdot, \xi^\infty), W).$$

The following proposition, which we state without proof, establishes the continuous dependence of the solution on the coefficients and initial data. Its proof follows standard techniques from the theory of SDEs (cf. Da Prato and Tudor [18]).

Proposition 4.1:
Under assumptions 4.1–4.4, we have in $(\mathcal{P}(\mathrm{C}([s_0, b]; \mathbb{H})), d_{\mathrm{BL}})$, for any $b > s_0$,

$$\lim_{n\to\infty} d_{\mathrm{BL}}\left(\mathscr{L}(Y^n|_{[s_0,b]}), \mathscr{L}(Y^\infty|_{[s_0,b]})\right) = 0.$$

Theorem 4.2:
Let $p = 2$. Suppose Assumptions 4.1–4.4 hold. Furthermore, assume the constants satisfy,

$$\beta_{1,2}(\delta) := \left\{3C_2 L^2_{(\mathcal{M},\mathcal{N})}\left(2\left(\frac{1}{\delta}\right)^2 + \left(\frac{1}{2\delta}\right)\right)\right\} < 1,$$

$$\text{and} \qquad 2\beta_{2,2}(\delta) := \left\{6C_2 L^2_{(\mathcal{M},\mathcal{N})}\left(2\left(\frac{1}{\delta}\right) + 1\right)\right\} < \delta,$$

Then, the unique mild solution $Y \in \mathcal{M}^2(\mathbb{R}; \mathbb{H})$ of (4.1) is almost periodic in distribution.

Proof:
The solution's existence and uniqueness were established via Theorem 4.1. Let us show that Y is almost periodic in distribution. We will utilise the definition provided in Section 2.5 for this purpose. Let $\{u'_n\} \subset \mathbb{R}$ and $\{v'_n\} \subset \mathbb{R}$. Our objective is to show the existence of subsequences $\{u_n\} \subset \{u'_n\}$ and $\{v_n\} \subset \{v'_n\}$ with the same indices, such that for every $s \in \mathbb{R}$, the limits

$$\lim_{n\to\infty} \lim_{m\to\infty} \mu(s + u_n + v_m), \quad \text{and} \quad \lim_{n\to\infty} \mu(s + u_n + v_n),$$

exist and are equal, where $\mu(s) := \mathscr{L}(Y(s))$. Using Assumption 4.3, we have

$$\begin{cases} \lim_{n\to\infty} \lim_{m\to\infty} F(s + u_n + v_m, y) = \lim_{n\to\infty} F(s + u_n + v_n, y) := F^0(s, y), \\ \lim_{n\to\infty} \lim_{m\to\infty} G(s + u_n + v_m, y) = \lim_{n\to\infty} G(s + u_n + v_n, y) := G^0(s, y), \\ \lim_{n\to\infty} \lim_{m\to\infty} H(s + u_n + v_m, y) = \lim_{n\to\infty} H(s + u_n + v_n, y) := H^0(s, y). \end{cases} \tag{4.18}$$

These limits exist pointwise with respect to $s \in \mathbb{R}$ and $y \in \mathbb{B}_2$. Consider now the sequences defined by $\{x_n := u_n + v_n\}$. For each integer n, we consider a semilinear SDE,

$$\begin{aligned} &\mathrm{d}(Y'_n(s) - F(s + x_n, Y_n(s))) \\ &\qquad = (AY_n(s) + BY'_n(s) + G(s + x_n, Y_n(s)))\mathrm{d}s + H(s + x_n, Y_n(s))\mathrm{d}W(s). \end{aligned}$$

The mild solution for $(Y_n(s))_{s\in\mathbb{R}}$ is given by,

$$Y_n(s) = \int_{-\infty}^{s} \mathcal{N}_s(s-r)F(r+x_n, Y_n(r))\,\mathrm{d}r + \int_{-\infty}^{s} \mathcal{N}(s-r)G(r+x_n, Y_n(r))\,\mathrm{d}r \\ + \int_{-\infty}^{s} \mathcal{N}(s-r)H(r+x_n, Y_n(r))\,\mathrm{d}W(r).$$

Additionally, we have another SDE,

$$\mathrm{d}(Y^{0\prime}(s) - F^0(s, Y^0(s))) \\ = (AY^0(s) + BY'^0(s) + G^0(s, Y^0(s)))\mathrm{d}s + H^0(s, Y^0(s))\mathrm{d}W(s).$$

With its mild solution,

$$Y^0(s) = \int_{-\infty}^{s} \mathcal{N}_s(s-r)F^0(r, Y^0(r))\,\mathrm{d}r + \int_{-\infty}^{s} \mathcal{N}(s-r)G^0(r, Y^0(r))\,\mathrm{d}r \\ + \int_{-\infty}^{s} \mathcal{N}(s-r)H^0(r, Y^0(r))\,\mathrm{d}W(r).$$

Now, consider the process,

$$Y(s+x_n) = \int_{-\infty}^{s+x_n} \mathcal{N}_s(s-(r-x_n))F(r, Y(r))\,\mathrm{d}r + \int_{-\infty}^{s+x_n} \mathcal{N}(s-(r-x_n))G(r, Y(r))\,\mathrm{d}r \\ + \int_{-\infty}^{s+x_n} \mathcal{N}(s-(r-x_n))H(r, Y(r))\,\mathrm{d}W(r).$$

By making the change of variable $z := r - x_n$, we get,

$$Y(s+x_n) = \int_{-\infty}^{s} \mathcal{N}_s(s-r)F(r+x_n, Y(r+x_n))\,\mathrm{d}r + \int_{-\infty}^{s} \mathcal{N}(s-r)G(r+x_n, Y(r+x_n))\,\mathrm{d}r \\ + \int_{-\infty}^{s} \mathcal{N}(s-r)H(r+x_n, Y(r+x_n))\,\mathrm{d}W_n(r),$$

where,

$$W_n(s) := W(s+x_n) - W(x_n) \underset{\mathscr{L}(W(s))}{=} W(s).$$

Due to the independence of the increments of $W(s)$, we conclude that the process,

$$Y(s+x_n) \underset{\mathscr{L}(Y(s))}{=} Y_n(s).$$

Now, we aim to demonstrate that $Y_n(s)$ converges in $(p=2)-$mean to $Y^0(s)$ for each fixed $s \in \mathbb{R}$, i.e.

$$\lim_{n\to\infty} \mathbb{E}\|Y_n(s) - Y^0(s)\|_{\mathbb{H}}^2 = 0.$$

We start the analysis from,

$$\mathbb{E}\|Y_n(s) - Y^0(s)\|_{\mathbb{H}}^2 \\ \leqslant 3\mathbb{E}\left\| \int_{-\infty}^{s} \mathcal{N}_s(s-r)(F(r+x_n, Y_n(r)) - F^0(r, Y^0(r)))\,\mathrm{d}r \right\|_{\mathbb{H}}^2$$

$$
\begin{aligned}
&+ 3\mathbb{E}\left\|\int_{-\infty}^{s}\mathcal{N}(s-r)(G(r+x_n,Y_n(r))-G^0(r,Y^0(r)))\,\mathrm{d}r\right\|_{\mathbb{H}}^2\\
&+ 3\mathbb{E}\left\|\int_{-\infty}^{s}\mathcal{N}(s-r)(H(r+x_n,Y_n(r))-H^0(r+x_n,Y^0(r)))\,\mathrm{d}W(r)\right\|_{\mathbb{H}}^2\\
&:= 3(\mathrm{I}^1+\mathrm{I}^2+\mathrm{I}^3).
\end{aligned}\tag{4.19}
$$

For I^1, we get,

$$
\begin{aligned}
\mathrm{I}^1 &= \mathbb{E}\left\|\int_{-\infty}^{s}\mathcal{N}_s(s-r)(F(r+x_n,Y_n(r))-F^0(r,Y^0(r)))\,\mathrm{d}r\right\|_{\mathbb{H}}^2\\
&\leqslant 2\mathbb{E}\left\|\int_{-\infty}^{s}\mathcal{N}_s(s-r)(F(r+x_n,Y_n(r))-F(r+x_n,Y^0(r)))\,\mathrm{d}r\right\|_{\mathbb{H}}^2\\
&\quad+ 2\mathbb{E}\left\|\int_{-\infty}^{s}\mathcal{N}_s(s-r)(F(r+x_n,Y^0(r))-F^0(r,Y^0(r)))\,\mathrm{d}r\right\|_{\mathbb{H}}^2\\
&:= 2(\mathrm{J}^1+\mathrm{J}^2).
\end{aligned}\tag{4.20}
$$

Now, using $(2,2)-$Hölder's inequality, we obtain,

$$
\begin{aligned}
\mathrm{J}^1 &= \mathbb{E}\left\|\int_{-\infty}^{s}\mathcal{N}_s(s-r)(F(r+x_n,Y_n(r))-F(r+x_n,Y^0(r)))\,\mathrm{d}r\right\|_{\mathbb{H}}^2\\
&\leqslant \left(\int_{-\infty}^{s}\|\mathcal{N}_s(s-r)\|_{\mathcal{L}(\mathbb{H})}\,\mathrm{d}r\right)\\
&\cdot\int_{-\infty}^{s}\|\mathcal{N}_s(s-r)\|_{\mathcal{L}(\mathbb{H})}\mathbb{E}\|F(r+x_n,Y_n(r))-F(r+x_n,Y^0(r))\|_{\mathbb{H}}^2\,\mathrm{d}r\\
&\leqslant C_2\left(\int_{-\infty}^{s}\|\mathcal{N}_s(s-r)\|_{\mathcal{L}(\mathbb{H})}\,\mathrm{d}r\right)\int_{-\infty}^{s}\|\mathcal{N}_s(s-r)\|_{\mathcal{L}(\mathbb{H})}\mathbb{E}\|Y_n(r)-Y^0(r)\|_{\mathbb{H}}^2\,\mathrm{d}r.
\end{aligned}\tag{4.21}
$$

And for J^2, we have,

$$
\begin{aligned}
\mathrm{J}^2 &= \mathbb{E}\left\|\int_{-\infty}^{s}\mathcal{N}_s(s-r)(F(r+x_n,Y^0(r))-F^0(r,Y^0(r)))\,\mathrm{d}r\right\|_{\mathbb{H}}^2\\
&\leqslant \left(\int_{-\infty}^{s}\|\mathcal{N}_s(s-r)\|_{\mathcal{L}(\mathbb{H})}\,\mathrm{d}r\right)\\
&\cdot\int_{-\infty}^{s}\|\mathcal{N}_s(s-r)\|_{\mathcal{L}(\mathbb{H})}\mathbb{E}\|F(r+x_n,Y^0(r))-F^0(r,Y^0(r))\|_{\mathbb{H}}^2\,\mathrm{d}r\\
&\leqslant \left(\int_{-\infty}^{s}\|\mathcal{N}_s(s-r)\|_{\mathcal{L}(\mathbb{H})}\,\mathrm{d}r\right)^2\\
&\cdot\sup_{r\in]-\infty,s]}\mathbb{E}\|F(r+x_n,Y^0(r))-F^0(r,Y^0(r))\|_{\mathbb{H}}^2.
\end{aligned}\tag{4.22}
$$

For I^2, we have,

$$
\mathrm{I}^2 = \mathbb{E}\left\|\int_{-\infty}^{s}\mathcal{N}(s-r)(G(r+x_n,Y_n(r))-G^0(r,Y^0(r)))\,\mathrm{d}r\right\|_{\mathbb{H}}^2
$$

$$\begin{aligned}
&\leqslant 2\mathbb{E}\left\|\int_{-\infty}^{s}\mathcal{N}(s-r)(G(r+x_n,Y_n(r))-G(r+x_n,Y^0(r)))\,\mathrm{d}r\right\|_{\mathbb{H}}^{2}\\
&+2\mathbb{E}\left\|\int_{-\infty}^{s}\mathcal{N}(s-r)(G(r+x_n,Y^0(r))-G^0(r,Y^0(r)))\,\mathrm{d}r\right\|_{\mathbb{H}}^{2}\\
&:=2(\mathrm{J}^3+\mathrm{J}^4).
\end{aligned}\tag{4.23}$$

For J^3, we obtain that

$$\begin{aligned}
\mathrm{J}^3&=\mathbb{E}\left\|\int_{-\infty}^{s}\mathcal{N}(s-r)(G(r+x_n,Y_n(r))-G(r+x_n,Y^0(r)))\,\mathrm{d}r\right\|_{\mathbb{H}}^{2}\\
&\leqslant\left(\int_{-\infty}^{s}\|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}\,\mathrm{d}r\right)\\
&\cdot\int_{-\infty}^{s}\|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}\mathbb{E}\|G(r+x_n,Y_n(r))-G(r+x_n,Y^0(r))\|_{\mathbb{H}}^{2}\,\mathrm{d}r\\
&\leqslant C_2\left(\int_{-\infty}^{s}\|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}\,\mathrm{d}r\right)\int_{-\infty}^{s}\|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}\mathbb{E}\|Y_n(r)-Y^0(r)\|_{\mathbb{H}}^{2}\,\mathrm{d}r.
\end{aligned}\tag{4.24}$$

Furthermore,

$$\begin{aligned}
\mathrm{J}^4&=\mathbb{E}\left\|\int_{-\infty}^{s}\mathcal{N}(s-r)(G(r+x_n,Y^0(r))-G^0(r,Y^0(r)))\,\mathrm{d}r\right\|_{\mathbb{H}}^{2}\\
&\leqslant\left(\int_{-\infty}^{s}\|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}\,\mathrm{d}r\right)\\
&\cdot\int_{-\infty}^{s}\|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}\mathbb{E}\|G(r+x_n,Y^0(r))-G^0(r,Y^0(r))\|_{\mathbb{H}}^{2}\,\mathrm{d}r\\
&\leqslant\left(\int_{-\infty}^{s}\|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}\,\mathrm{d}r\right)^2\\
&\cdot\sup_{r\in]-\infty,s]}\mathbb{E}\|G(r+x_n,Y^0(r))-G^0(r,Y^0(r))\|_{\mathbb{H}}^{2}.
\end{aligned}\tag{4.25}$$

And for I^3 we have,

$$\begin{aligned}
\mathrm{I}^3&=\mathbb{E}\left\|\int_{-\infty}^{s}\mathcal{N}(s-r)(H(r+x_n,Y_n(r))-H^0(r+x_n,Y^0(r)))\,\mathrm{d}W(r)\right\|_{\mathbb{H}}^{2}\\
&\leqslant 2\mathbb{E}\left\|\int_{-\infty}^{s}\mathcal{N}(s-r)(H(r+x_n,Y_n(r))-H(r+x_n,Y^0(r)))\,\mathrm{d}W(r)\right\|_{\mathbb{H}}^{2}\\
&+2\mathbb{E}\left\|\int_{-\infty}^{s}\mathcal{N}(s-r)(H(r+x_n,Y^0(r))-H^0(r,Y^0(r)))\,\mathrm{d}W(r)\right\|_{\mathbb{H}}^{2}\\
&:=2(\mathrm{J}^5+\mathrm{J}^6).
\end{aligned}\tag{4.26}$$

By applying Lemma 3.8, we derive,

$$\mathrm{J}^5=\mathbb{E}\left\|\int_{-\infty}^{s}\mathcal{N}(s-r)(H(r+x_n,Y_n(r))-H(r+x_n,Y^0(r)))\,\mathrm{d}W(r)\right\|_{\mathbb{H}}^{2}$$

$$\begin{aligned}
&\leqslant \mathbb{E}\left(\int_{-\infty}^{s} \|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}^{2} \|H(r+x_n, Y_n(r)) - H(r+x_n, Y^0(r))\|_{\mathcal{L}_2^0}^{2}\, \mathrm{d}r\right)\\
&\leqslant \int_{-\infty}^{s} \|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}^{2} \mathbb{E}\|H(r+x_n, Y_n(r)) - H(r+x_n, Y^0(r))\|_{\mathcal{L}_2^0}^{2}\, \mathrm{d}r\\
&\leqslant C_2 \int_{-\infty}^{s} \|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}^{2} \mathbb{E}\|Y_n(r) - Y^0(r)\|_{\mathbb{H}}^{2}\, \mathrm{d}r. \qquad (4.27)
\end{aligned}$$

In addition,

$$\begin{aligned}
\mathrm{J}^6 &= \mathbb{E}\left\|\int_{-\infty}^{s} \mathcal{N}(s-r)(H(r+x_n, Y_n(r)) - H^0(r, Y^0(r)))\, \mathrm{d}W(r)\right\|_{\mathbb{H}}^{2}\\
&\leqslant \mathbb{E}\left(\int_{-\infty}^{s} \|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}^{2} \|H(r+x_n, Y_n(r)) - H^0(r, Y^0(r))\|_{\mathcal{L}_2^0}^{2}\, \mathrm{d}r\right)\\
&\leqslant \int_{-\infty}^{s} \|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}^{2} \mathbb{E}\|H(r+x_n, Y_n(r)) - H^0(r, Y^0(r))\|_{\mathcal{L}_2^0}^{2}\, \mathrm{d}r\\
&\leqslant \left(\int_{-\infty}^{s} \|\mathcal{N}(s-r)\|_{\mathcal{L}(\mathbb{H})}^{2}\, \mathrm{d}r\right)\\
&\quad \cdot \sup_{r\in]-\infty,s]} \mathbb{E}\|H(r+x_n, Y_n(r)) - H^0(r, Y^0(r))\|_{\mathcal{L}_2^0}^{2}. \qquad (4.28)
\end{aligned}$$

Includes the inequalities from (4.19)–(4.28),

$$\begin{aligned}
\mathbb{E}\|Y_n(s) - Y^0(s)\|_{\mathbb{H}}^{2} &\leqslant \left\{6C_2 L_{(\mathcal{M},\mathcal{N})}^{2}\left(2\left(\frac{1}{\delta}\right)^2 + \left(\frac{1}{2\delta}\right)\right)\right\}\alpha_n(s)\\
&\quad + 6C_2 L_{(\mathcal{M},\mathcal{N})}^{2}\left(2\left(\frac{1}{\delta}\right)\right)\int_{-\infty}^{s} e^{-\delta(s-r)}\mathbb{E}\|Y_n(r) - Y^0(r)\|_{\mathbb{H}}^{2}\, \mathrm{d}r\\
&\quad + 6C_2 L_{(\mathcal{M},\mathcal{N})}^{2}\int_{-\infty}^{s} e^{-2\delta(s-r)}\mathbb{E}\|Y_n(r) - Y^0(r)\|_{\mathbb{H}}^{2}\, \mathrm{d}r. \qquad (4.29)
\end{aligned}$$

For a sequence,

$$\begin{aligned}
\alpha_n(s) := \sup_{r\in]-\infty,s]} \mathbb{E}\Big\{&\|F(r+x_n, Y_n(r)) - F^0(r, Y^0(r))\|_{\mathbb{H}}^{2}\\
&+ \|G(r+x_n, Y_n(r)) - G^0(r, Y^0(r))\|_{\mathbb{H}}^{2}\\
&+ \|H(r+x_n, Y_n(r)) - H^0(r, Y^0(r))\|_{\mathcal{L}_2^0}^{2}\Big\},
\end{aligned}$$

which converges to 0 as $n \to \infty$ because $\sup_{s\in\mathbb{R}} \mathbb{E}\|Y^0(s)\|^2 < \infty$ which implies that $(Y^0(s))_s$ is tight relatively to bounded sets. Since $\delta > \left\{6C_2 L_{(\mathcal{M},\mathcal{N})}^{2}\left(2\left(\frac{1}{\delta}\right)+1\right)\right\}$, we deduce from Lemma 4.1 that,

$$\lim_{n\to\infty} \mathbb{E}\|Y_n(s) - Y^0(s)\|_{\mathbb{H}}^{2} = 0.$$

Hence $Y_n(s)$ converges in distribution to $Y^0(s)$. But, since the distribution of $Y_n(s)$ is the same as that of $Y(s+x_n)$, we deduce that $Y(s+x_n)$ converges in distribution to $Y^0(s)$, i.e.

$$\lim_{n\to\infty} \mu(s+u_n+v_n) := \mathscr{L}(Y^0(s)).$$

By analogy and using (4.18) we can easily deduce that,

$$\lim_{n\to\infty}\lim_{m\to\infty}\mu(s+u_n+v_n):=\mathscr{L}(Y^0(s)).$$

We have demonstrated the almost periodicity of Y in one-dimensional-distributions. To establish the almost periodicity in multi-dimensional distributions for Y, we employ Proposition 4.1. For a fixed $s_0 \in \mathbb{R}$, consider

$$\xi^n := Y(s_0+x_n), \qquad \eta^n := Y'(s_0+x_n),$$

and

$$F^n(s,y) := F(s+x_n,y), \qquad G^n(s,y) := G(s+x_n,y), \qquad H^n(s,y) := H(s+x_n,y).$$

From the previous discussions, we know that ξ^n (resp. η^n) converges in distribution to some variable $Y(s_0)$ (resp. $Y'(s_0)$). Consequently, ξ^n (resp. η^n) is tight, and thus (ξ^n, W) (resp. (η^n, W)) is also tight. Therefore, we can select $Y(s_0)$ (resp. $Y'(s_0)$) such that (ξ^n, W) (resp. (η^n, W)) converges in distribution to $(Y(s_0), W)$ (resp. $(Y'(s_0), W)$). Then, according to Proposition 4.1, for every $b > s_0$, $Y(\cdot + u_n)$ converges in distribution on $\mathcal{P}(C([s_0,b];\mathbb{H}))$ to the (unique in distribution) solution to,

$$\begin{aligned} Y(s) := \mathcal{M}(s-s_0)Y(s_0)+\mathcal{N}(s-s_0)\big(Y'(s_0)-F(s_0,Y(s_0))\big)+\int_{s_0}^{s}\mathcal{N}_s(s-r)F(r,Y(r))\,\mathrm{d}r \\ +\int_{s_0}^{s}\mathcal{N}(s-r)G(r,Y(r))\,\mathrm{d}r+\int_{s_0}^{s}\mathcal{N}(s-r)H(r,Y(r))\,\mathrm{d}W(r). \end{aligned}$$

Observe that Y remains independent of the selected interval $[s_0, b]$, implying that the convergence occurs in $\mathcal{P}(C(\mathbb{R};\mathbb{H}))$. Similarly, $Y^n := Y(\cdot + u_n)$ converges in distribution on $\mathcal{P}(C(\mathbb{R};\mathbb{H}))$ to a continuous process X. For $s \geqslant s_0$, the expression for $X(s)$ is given by,

$$\begin{aligned} X(s) := \mathcal{M}(s-s_0)X(s_0)+\mathcal{N}(s-s_0)\big(Y'(s_0)-F(s_0,Y(s_0))\big)+\int_{s_0}^{s}\mathcal{N}_s(s-r)F(r,X(r))\,\mathrm{d}r \\ +\int_{s_0}^{s}\mathcal{N}(s-r)G(r,X(r))\,\mathrm{d}r+\int_{s_0}^{s}\mathcal{N}(s-r)H(r,X(r))\,\mathrm{d}W(r). \end{aligned}$$

However, based on (4.18), $Y(\cdot + x_n)$ converges in distribution to the identical process X. Consequently, we can assert that Y exhibits almost periodicity in multi-dimensional distributions. □

4.5. A Specific Instantiation

Within the theoretical framework established in Theorems 4.1 to 4.2, a general condition (Condition 4.1) was formulated to streamline the analytical approach. In this section, we employ a specialised methodology based on fixed-point theorems to analyse a particular instance of this framework. We shall make crucial use of a Gronwall-type inequality presented in Lemma 4.1 of Kamenskii et al. [35], specifically considering the case where the parameter m is set to zero. Our approach demonstrates that a certain relaxation of the original condition is permissible, thereby highlighting the fundamental importance of Lemma 4.1. We therefore replace Condition 4.1 with the following alternative set of hypotheses.

Assumption 4.5:

1. Assume there exists a linear operator $\mathscr{A} := A + \frac{B^2}{4}$ that generates a strongly continuous cosine family $\{C(s) : s \in \mathbb{R}_+\}$ on the Hilbert space $\mathbb{H}$.
2. Assume the operator $B/2$ generates a strongly continuous semigroup $\{\mathcal{T}(s) : s \in \mathbb{R}_+\}$ on $\mathbb{H}$.
3. There exist constants $L_{(C,S)}, L_{BS}, L_{\mathcal{T}} \geq 1$ and real numbers $\delta_{B/2} > \bar{\delta}_{\mathscr{A}} > \delta_{\mathscr{A}} > 0$ such that for all $s \geq r$ with $s, r \in \mathbb{R}$, the following estimates hold,

$$\begin{cases} \|C(s-r)\|_{\mathcal{L}(\mathbb{H})} & \leq L_{(C,S)} e^{\delta_{\mathscr{A}}(s-r)}, \\ \|S(s-r)\|_{\mathcal{L}(\mathbb{H})} & \leq L_{(C,S)}(s-r) e^{\delta_{\mathscr{A}}(s-r)}, \\ \|BS(s-r)\|_{\mathcal{L}(\mathbb{H})} & \leq L_{BS}(s-r) e^{\bar{\delta}_{\mathscr{A}}(s-r)}, \\ \|\mathcal{T}(s-r)\|_{\mathcal{L}(\mathbb{H})} & \leq L_{\mathcal{T}} e^{-\delta_{B/2}(s-r)}. \end{cases}$$

Following the formalism introduced by Melnikova and Filinkov [41], we define the following operator compositions,

$$\mathcal{N}(s) = \mathcal{T}(s)S(s) =: \mathcal{E}S(s), \quad \mathcal{E}C(s) = \mathcal{T}(s)C(s),$$

and,

$$\mathcal{M}(s) := \mathcal{E}C(s) - (B/2)\mathcal{E}S(s) := \mathcal{T}(s)C(s) - (B/2)\mathcal{T}(s)S(s).$$

We now introduce several quantitative parameters that will be essential for our analysis.

For $p \geq 2$, define the following quantities,

$$\begin{cases} \beta_{3,p}(\delta) & := 4^{p-1} C_p L_{\max}^p \left[\left(\frac{1}{\delta_{\min}}\right)^p + 2\left(\frac{1}{\delta_{\min}^2}\right)^p + K_{p/2}\left(\frac{1}{4\delta_{\min}^3}\right)^{p/2} \right], \\ \beta_{4,p}(\delta) & := 4^{p-1} C_p L_{\max}^p \left[\left(\frac{1}{\delta_{\min}}\right)^{p-1} + 2\left(\frac{1}{\delta_{\min}^2}\right)^p + K_{p/2}\left(\frac{1}{4\delta_{\min}^3}\right)^{p/2-1} \right], \\ L_{\max} & := L_{\mathcal{T}} \max\left(L_{(C,S)}, L_{BS}/2\right), \\ \delta_{\min} & := \min\left(\delta_{B/2} - \delta_{\mathscr{A}}, \delta_{B/2} - \bar{\delta}_{\mathscr{A}}, 2(\delta_{B/2} - \delta_{\mathscr{A}}), 2(\delta_{B/2} - \bar{\delta}_{\mathscr{A}})\right) = \delta_{B/2} - \delta_{\mathscr{A}}. \end{cases}$$

Theorem 4.3 (Existence and Uniqueness of Mild Solutions):
Assume Conditions 4.5, 4.2, and 4.3 are satisfied. If $\beta_{3,p}(\delta) < 1$ for some $p \geq 2$, then there exists a unique mild solution $Y \in \mathcal{M}^p(\mathbb{R}, \mathbb{H}_2)$ to equation (4.1) on $\mathbb{R}$. Moreover, if $\delta_{\min} > 2 + \beta_{4,p}(\delta)$ for $p \geq 2$, then the following moment estimates hold.

For $p > 2$,

$$\mathbb{E}\|Y(s)\|^p \le \frac{\delta_{\min} - 2}{\delta_{\min} - \beta_{4,p}(\delta) - 2}.$$

For $p = 2$,

$$\mathbb{E}\|Y(s)\|^2 \le \frac{\delta_{\min} - 2}{\delta_{\min} - \beta_{4,2}(\delta) - 2}.$$

The solution admits the representation,

$$\begin{aligned} Y(s) = &\int_{-\infty}^{s} \mathcal{T}(s-r)\left[C(s-r) + \frac{B}{2}S(s-r)\right] F(r, Y(r))\,\mathrm{d}r \\ &+ \int_{-\infty}^{s} \mathcal{T}(s-r)S(s-r)G(r, Y(r))\,\mathrm{d}r \\ &+ \int_{-\infty}^{s} \mathcal{T}(s-r)S(s-r)H(r, Y(r))\,\mathrm{d}W(r). \end{aligned}$$

Proof:
We employ the contraction mapping principle in the Banach space $\mathcal{M}^p(\mathbb{R}, \mathbb{H}_2)$. Define the mapping $\Phi : \mathcal{M}^p(\mathbb{R}, \mathbb{H}) \to \mathcal{M}^p(\mathbb{R}, \mathbb{H})$ by,

$$\begin{aligned} (\Phi Y)(s) = &\int_{-\infty}^{s} \mathcal{T}(s-r)\left[C(s-r) + \frac{B}{2}S(s-r)\right] F(r, Y(r))\,\mathrm{d}r \\ &+ \int_{-\infty}^{s} \mathcal{T}(s-r)S(s-r)G(r, Y(r))\,\mathrm{d}r \\ &+ \int_{-\infty}^{s} \mathcal{T}(s-r)S(s-r)H(r, Y(r))\,\mathrm{d}W(r). \end{aligned}$$

Step 1. Well-definedness.

We verify that Φ maps into the correct space. Using the growth conditions from Assumptions 4.2-4.3 and the operator norm estimates from Assumption 4.5, we obtain for $Y \in \mathcal{M}^p(\mathbb{R}, \mathbb{H}_2)$,

$$\mathbb{E}\|(\Phi Y)(s)\|^p \le \beta_{3,p}(\delta) + \beta_{4,p}(\delta)\int_{-\infty}^{s}(1 + 2(s-r) + (s-r)^2)e^{-\delta_{\min}(s-r)}\mathbb{E}\|Y(r)\|^p\,\mathrm{d}r.$$

The condition $\delta_{\min} > 0$ ensures these integrals converge.

Step 2. Contraction property.

For $Y_1, Y_2 \in \mathcal{M}^p(\mathbb{R}, \mathbb{H}_2)$, we estimate,

$$\mathbb{E}\|(\Phi Y_1)(s) - (\Phi Y_2)(s)\|^p \le \beta_{3,p}(\delta) \sup_{s\in\mathbb{R}} \mathbb{E}\|Y_1(s) - Y_2(s)\|^p.$$

The condition $\beta_{3,p}(\delta) < 1$ implies Φ is a contraction.

Step 3. Moment estimates. Applying the Gronwall inequality from Lemma 4.1 to the fixed point equation $Y = \Phi Y$ yields the stated moment bounds when $\delta_{\min} > 2 + \beta_{4,p}(\delta)$.

The representation follows directly from the definition of mild solutions for SDEs. □

Theorem 4.4 (Almost Periodic Solutions):
Under Assumptions 4.5, 4.2-4.4, with $\beta_{3,2}(\delta) < 1$ and $2\beta_{4,2}(\delta) < \delta_{\min}$, there exists a unique mild solution $Y \in \mathcal{M}^2(\mathbb{R}, \mathbb{H})$ to equation (4.1) that is almost periodic in distribution.

Proof:
The existence and uniqueness follow from Theorem 4.3. For almost periodicity, we employ the following methodology.

Step 1. Bochner's criterion. A stochastic process is almost periodic in distribution if for every sequence $\{\tau_n\} \subset \mathbb{R}$, there exists a subsequence $\{\tau_{n_k}\}$ such that the finite-dimensional distributions of $Y(\cdot + \tau_{n_k})$ converge.

Step 2. Tightness. Using the moment estimates from Theorem 4.3 and the conditions on the coefficients, we show that the family $\{Y(\cdot + \tau) : \tau \in \mathbb{R}\}$ is tight in an appropriate function space.

Step 3. Continuity. The stochastic convolution terms are mean-square continuous due to the regularity conditions on H and the $\mathcal{M}, \mathcal{N}$ properties.

Step 4. Conclusion. By Prokhorov's theorem and the continuity of the finite-dimensional distributions, we conclude almost periodicity in distribution.

The technical details follow the approach in Theorem 4.2, augmented with the specific estimates from Theorem 4.3. □

4.6. Application to Stochastic Wave Equations

Consider the SDE with boundary conditions,

$$\begin{cases} \frac{\partial^2 y}{\partial s^2}(x,s) - \frac{\partial}{\partial s} f(s, y(x,s)) + b(x)\frac{\partial}{\partial s} y(x,s) \\ = \frac{\partial^2 y}{\partial x^2}(x,s)\mathrm{d}s + g(s, y(x,s))\mathrm{d}s + h(s, y(x,s))\mathrm{d}W(s), \\ s \in \mathbb{R}, \quad 0 \le x \le \pi, \\ y(0,s) = y(\pi,s) = 0, \quad s \in \mathbb{R}. \end{cases} \tag{4.30}$$

Let $\mathbb{H} = L^2[0,\pi]$ with inner product $\langle \cdot, \cdot \rangle$. Define the operator $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ by $Ay = y''$ with domain,

$$\mathrm{Dom}(A) = \{y \in \mathbb{H} \mid y, y', \text{are absolutely continuous, and } y(0) = y'(\pi) = 0,\ y'' \in \mathbb{H}\}$$

The operator A has discrete spectrum $\lambda_n = -n^2$ ($n \in \mathbb{N}$) with eigenfunctions $e_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$. In the spectral representation,

$$Ay = -\sum_{n=1}^{\infty} n^2 \langle y, e_n\rangle e_n.$$

Define the multiplication operator $B : \mathbb{H} \to \mathbb{H}$ by $(By)(x) = b(x)y(x)$, where $b : [0,\pi] \to \mathbb{R}_{\geqslant 0}$ is continuous. Then,

$$\mathscr{A} = A + \frac{B^2}{4} = -\sum_{n=1}^{\infty}\left[n^2 - \frac{b^2(x)}{4}\right]\langle y, e_n\rangle e_n.$$

The cosine family generated by $\mathscr{A}$ has the representation,

$$\begin{aligned}
[C(s)y](x) &= \sum_{n=1}^{\infty}\cos\left(s\sqrt{n^2 - \tfrac{b^2(x)}{4}}\right)\langle y, e_n\rangle e_n(x)\mathbf{1}_{\{n^2 > b^2(x)/4\}} \\
&\quad + \sum_{n=1}^{\infty}\cosh\left(s\sqrt{\tfrac{b^2(x)}{4} - n^2}\right)\langle y, e_n\rangle e_n(x)\mathbf{1}_{\{n^2 < b^2(x)/4\}} \\
&\quad + \sum_{n=1}^{\infty}\langle y, e_n\rangle e_n(x)\mathbf{1}_{\{n^2 = b^2(x)/4\}}.
\end{aligned}$$

The associated sine family is,

$$\begin{aligned}
[S(s)y](x) &= \sum_{n=1}^{\infty}\frac{\sin\left(s\sqrt{n^2 - \tfrac{b^2(x)}{4}}\right)}{\sqrt{n^2 - \tfrac{b^2(x)}{4}}}\langle y, e_n\rangle e_n(x)\mathbf{1}_{\{n^2 > b^2(x)/4\}} \\
&\quad + \sum_{n=1}^{\infty}\frac{\sinh\left(s\sqrt{\tfrac{b^2(x)}{4} - n^2}\right)}{\sqrt{\tfrac{b^2(x)}{4} - n^2}}\langle y, e_n\rangle e_n(x)\mathbf{1}_{\{n^2 < b^2(x)/4\}} \\
&\quad + \sum_{n=1}^{\infty} s\langle y, e_n\rangle e_n(x)\mathbf{1}_{\{n^2 = b^2(x)/4\}}.
\end{aligned}$$

The semigroup generated by $-B/2$ is,

$$[\mathcal{T}(s)y](x) = e^{-\frac{1}{2}sb(x)}y(x).$$

Define the nonlinear operators,

$$\begin{aligned}
F(s,y)(x) &= f(s, y(x)), \\
G(s,y)(x) &= g(s, y(x)), \\
H(s,y)(x) &= h(s, y(x)).
\end{aligned}$$

Then equation (4.30) can be written in the abstract form,

$$\mathrm{d}(Y'(s) - F(s, Y(s))) = AY(s)\mathrm{d}s - BY'(s)\mathrm{d}s + G(s, Y(s))\mathrm{d}s + H(s, Y(s))\mathrm{d}W(s).$$

Under the conditions of Theorem 4.4, with $\delta_{\min} = \delta_{B/2} - \delta_{\mathscr{A}} > 0$, there exists a unique almost periodic solution to this stochastic boundary value problem.

Chapter 5.

Stochastic LQ Control of Hyperbolic Systems and Applications to Almost Periodic Second-Order Equations

Following the comprehensive theoretical foundation of linear-quadratic optimal control established in the introductory chapters of this dissertation, the present chapter undertakes a specialised investigation into the stochastic linear-quadratic (SLQ) control problem for hyperbolic systems evolving within Hilbert spaces. While parabolic differential equations have received considerable attention in the literature, primarily owing to their inherent regularising and dissipative characteristics, hyperbolic systems, exemplified by wave equations and models of structural vibrations, are governed by C_0-groups and consequently lack such smoothing properties. This fundamental distinction introduces significant analytical complexities, particularly when stochastic effects are incorporated through multiplicative noise terms.

The principal aim of this chapter is to provide a rigorous mathematical formulation and resolution of the SLQ problem for this important class of systems. In pursuing this objective, we address a notable lacuna in the existing research by establishing precise conditions for well-posedness, solvability of the associated operator Riccati equations, and stability analysis of the resulting closed-loop system. The theoretical framework developed herein extends classical linear-quadratic control theory to scenarios where stochastic perturbations interact with non-dissipative infinite-dimensional dynamical evolution.

Beyond the theoretical developments, this chapter also demonstrates the applicability of our results to almost periodic second-order SDEs, which model oscillatory and recurrent phenomena subject to random disturbances. This connection not only underscores the practical relevance of the theory but also broadens its potential application to various engineering and physical systems, including stochastic wave propagation, quantum control problems, and thermoelastic structural dynamics.

5.1. Mathematical Preliminaries and Functional Analytic Framework

Let $\mathbb{H}$, $\mathbb{U}$, and $\mathbb{K}$ denote separable Hilbert spaces, representing respectively the state space, control space, and the space in which the noise process takes values. To model the stochastic forcing term in our system, we introduce a two-sided Q_{noise}–Wiener process $\{W(s)\}_{s\in\mathbb{R}}$ with values in $\mathbb{K}$, where Q_{noise} is a positive, self-adjoint, trace-class operator on $\mathbb{K}$.

The probabilistic foundation consists of a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, equipped with a filtration $\{\mathscr{F}_s\}_{s\in\mathbb{R}}$ that satisfies the usual conditions of right-continuity and completeness. This filtration is generated by the Wiener process $W(s)$ and defined formally as

$$\mathscr{F}_s := \sigma\,(W(r) : r \leq s) \vee \mathcal{N},$$

where $\mathcal{N}$ denotes the collection of all $\mathbb{P}$-null sets. The process $W(s)$ is adapted to this filtration.

For notational clarity, we introduce the following conventions for operators acting between these Hilbert spaces.

5.1.1. Solution Space Characterisation

Consider a fixed temporal interval $[s_0, b] \subset \mathbb{R}$. We define $\mathcal{M}^2([s_0, b]; \mathbb{H})$ as the space of $\mathbb{H}$-valued stochastic processes $Y : [s_0, b] \times \Omega \to \mathbb{H}$ satisfying the following properties.

1. *Progressive measurability.* For each $s \in [s_0, b]$, the mapping $(s, \omega) \mapsto Y(s, \omega)$ is measurable with respect to the σ-algebra $\mathcal{B}([s_0, s]) \otimes \mathscr{F}_s$;
2. *Path regularity.* The sample paths $s \mapsto Y(s, \omega)$ are almost surely continuous;
3. *Finite energy condition.* The process exhibits finite second moments in the supremum norm, that is,

$$\mathbb{E}\left[\sup_{s\in[s_0,b]} \|Y(s)\|_{\mathbb{H}}^2\right] < \infty.$$

The space $\mathcal{M}^2([s_0, b]; \mathbb{H})$ constitutes a Banach space when equipped with the norm

$$\|Y\|_{\mathcal{M}^2} := \left(\mathbb{E}\left[\sup_{s\in[s_0,b]} \|Y(s)\|_{\mathbb{H}}^2\right]\right)^{1/2}.$$

***Remark* 5.1:**
The two-sided Q_{noise}–Wiener process $W(s)$ can be rigorously defined over any compact interval $[s_0, s] \subset [a, b] \subset \mathbb{R}$. For comprehensive foundations of stochastic calculus in infinite

dimensions, we refer to Da Prato and Tudor [18], Da Prato and Zabczyk [19], Grecksch and Tudor [29].

5.2. Formulation of the Stochastic Differential Equation

We consider the stochastic differential equation governing the state process $Y(s)$ in the Hilbert space $\mathbb{H}$, which is described by the following differential equation for $s_0 \leq s \leq b$,

$$\mathrm{d}Y(s) = [AY(s) + C(s)U(s) + F(s)]\,\mathrm{d}s + [N(s)Y(s) + D(s)U(s) + H(s)]\,\mathrm{d}W(s), \tag{5.1}$$

with initial condition $Y(s_0) = Y \in \mathbb{H}$. Here, A is a bounded linear operator on $\mathbb{H}$ that generates a strongly continuous C_0-group $\{\mathcal{T}(s)\}_{s\in\mathbb{R}}$ on $\mathbb{H}$. The coefficient functions $C(s)$, $N(s)$, $D(s)$, $F(s)$, and $H(s)$ are time-dependent operators, and $W(s)$ represents the two-sided Q_{noise}–Wiener process on $\mathbb{K}$.

We impose the following assumptions on the system coefficients.

Assumption 5.1:

1. The operator $A : \mathrm{Dom}(A) \subset \mathbb{H} \to \mathbb{H}$ generates a C_0-group $\{\mathcal{T}(s)\}_{s\in\mathbb{R}}$ on $\mathbb{H}$ satisfying the growth condition

$$\|\mathcal{T}(s)\|_{\mathcal{L}(\mathbb{H})} \leq L_{\mathcal{T}} e^{\delta_A |s|}, \quad \forall s \in \mathbb{R},$$

   for constants $L_{\mathcal{T}} \geq 1, \delta_A \in \mathbb{R}$.

2. The coefficients satisfy the following regularity conditions,

$$\begin{cases} C \in C_b([s_0, b]; \mathcal{L}(\mathbb{U}, \mathbb{H})), \\ F \in C_b([s_0, b]; \mathbb{H}), \\ N \in C_b([s_0, b]; \mathcal{L}(\mathbb{H}, \mathcal{L}_2(Q_{\text{noise}}^{1/2}\mathbb{K}, \mathbb{H}))), \\ D \in C_b([s_0, b]; \mathcal{L}(\mathbb{U}, \mathcal{L}_2(Q_{\text{noise}}^{1/2}\mathbb{K}, \mathbb{H}))), \\ H \in C_b([s_0, b]; \mathcal{L}_2(Q_{\text{noise}}^{1/2}\mathbb{K}, \mathbb{H})). \end{cases}$$

To evaluate and compare different admissible control strategies (as defined in Definition 5.1), we associate to each control $U(\cdot)$ a quadratic cost functional, inspired by classical linear-quadratic control theory (see, e.g., Bensoussan [5], Yong and Zhou [62]).

The quadratic cost functional is defined as,

$$J(s_0, Y; U(\cdot)) := \frac{1}{2}\mathbb{E}\left[\int_{s_0}^{b} \begin{pmatrix} Y(r) \\ U(r) \end{pmatrix}^* \begin{pmatrix} Q(r) & S^*(r) \\ S(r) & R(r) \end{pmatrix} \begin{pmatrix} Y(r) \\ U(r) \end{pmatrix} \mathrm{d}r + \langle GY(b), Y(b)\rangle_{\mathbb{H}}\right].$$

Equivalently,

$$J(s_0, Y; U(\cdot)) := \frac{1}{2}\mathbb{E}\Bigg[\int_{s_0}^{b}\Big(\langle Q(r)Y(r), Y(r)\rangle_{\mathbb{H}} \\ + 2\langle S(r)Y(r), U(r)\rangle_{\mathbb{U}} + \langle R(r)U(r), U(r)\rangle_{\mathbb{U}}\Big)\mathrm{d}r \\ + \langle GY(b), Y(b)\rangle_{\mathbb{H}}\Bigg],$$

where Q, R, and G are self-adjoint, uniformly positive definite operators, and S is a uniformly bounded operator (see assumptions below).

This quadratic cost functional incorporates the following components.

1. *State penalty term* $\frac{1}{2}\langle QY, Y\rangle_{\mathbb{H}}$, which penalises deviations of the state from desired trajectories;
2. *Control penalty term* $\frac{1}{2}\langle RU, U\rangle_{\mathbb{U}}$, which penalises the magnitude of control inputs;
3. *Cross term* $\langle SY, U\rangle_{\mathbb{U}}$, which accounts for interactions between state and control variables;
4. *Terminal cost* $\frac{1}{2}\langle GY(b), Y(b)\rangle_{\mathbb{H}}$, which imposes a penalty on the final state at time b.

For well-posedness (i.e., boundedness from below and convexity), the following conditions are required (cf. Sun and Yong [54, Proposition 3.15]).

Assumption 5.2:

1. *Uniform positive definiteness.*
$$Q \in C_b([s_0, b]; \mathcal{L}(\mathbb{H})), \quad R \in C_b([s_0, b]; \mathcal{L}(\mathbb{U})), \quad S \in C_b([s_0, b]; \mathcal{L}(\mathbb{H}, \mathbb{U})),$$
with $G \in \mathcal{L}(\mathbb{H})$, and
$$\langle R(r)v, v\rangle_{\mathbb{U}} \geq \alpha\|v\|_{\mathbb{U}}^2, \quad \alpha > 0, \quad \text{a.e. } r \in [s_0, b].$$

2. *Schur complement condition.*
$$\langle (Q(r) - S^*(r)R^{-1}(r)S(r))y, y\rangle \geq \gamma\|y\|_{\mathbb{H}}^2, \quad \gamma > 0, \quad \text{a.e. } r \in [s_0, b].$$

***Remark* 5.2:**
The first condition ensures that R is invertible, while the second condition guarantees that the block matrix
$$\begin{pmatrix} Q(r) & S^*(r) \\ S(r) & R(r) \end{pmatrix}$$
is uniformly positive definite. Consequently, the Riccati equation in Lemmas 5.1–5.2 is positive definite.

Next, we define the class of control processes that are admissible within our problem formulation.

Definition 5.1 (Admissible Controls):
A control process $U : [s_0, b] \times \Omega \to \mathbb{U}$ is *admissible* if.

1. The probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_s\}, \mathbb{P})$ is complete.
2. $W(s)$ is a Q_{noise}–Wiener process with $\mathrm{Tr}(Q_{\text{noise}}) < \infty$.
3. $U(s)$ is progressively measurable and satisfies
$$\mathbb{E}\left[\int_{s_0}^{b} \|U(r)\|_{\mathbb{U}}^2 \mathrm{d}r\right] < \infty.$$
4. For any $Y \in \mathbb{H}$, the SDE (5.1) admits a unique mild solution $Y \in \mathcal{M}^2([s_0, b]; \mathbb{H})$.

The set of all admissible controls is denoted by $\mathcal{U}_{\text{ad}}[s_0, b]$.

We are now in a position to formulate the central optimisation problem.

Definition 5.2 (Stochastic Linear-Quadratic Problem):
Given an initial pair $(s_0, Y) \in [a, b) \times \mathbb{H}$, the stochastic linear-quadratic problem consists of finding an optimal control $U_* \in \mathcal{U}_{\text{ad}}[s_0, b]$ that minimises the cost functional
$$V(s_0, Y) := \inf_{U(\cdot)\in\mathcal{U}_{\text{ad}}[s_0,b]} J(s_0, Y; U(\cdot)) =: J(s_0, Y; U_*(\cdot)).$$

1. The function $V(s_0, Y)$ is referred to as the *value function* of the stochastic linear-quadratic problem. Furthermore, the value function satisfies the terminal condition
$$V(b, Y) = \frac{1}{2}\langle GY, Y\rangle_{\mathbb{H}}, \quad \forall Y \in \mathbb{H}.$$
2. An optimal control $U_*(\cdot)$, its corresponding trajectory $Y_*(\cdot)$, and the pair $(Y_*(\cdot), U_*(\cdot))$ are collectively referred to as an *optimal pair*.

5.3. Derivation of the Riccati Equation via Dynamic Programming Principles

In this section, we present the Riccati system associated with the stochastic linear-quadratic problem. The rigorous analysis of this system, including existence, uniqueness, and regularity results, will be carried out in the subsequent section. Our approach builds upon the framework developed in Barbu and Da Prato [3], Fabbri, Gozzi, and Swiech [24], Yong and Zhou [62].

The value function satisfies the Hamilton-Jacobi-Bellman equation at time s, i.e.,

$$-\frac{\partial V}{\partial s}(s,Y) = \inf_{U\in\mathbb{U}} \mathscr{H}(s,Y,U,D_Y V(s,Y), D_Y^2 V(s,Y)), \tag{5.2}$$

where the Hamiltonian $\mathscr{H}$ is defined as,

$$\begin{aligned}\mathscr{H}(s,Y,U,p,P) = \langle p, AY + C(s)U + F(s)\rangle_{\mathbb{H}} + \langle S(s)Y, U\rangle_{\mathbb{U}} \\ + \frac{1}{2}\langle Q(s)Y,Y\rangle_{\mathbb{H}} + \frac{1}{2}\langle R(s)U,U\rangle_{\mathbb{U}} \\ + \frac{1}{2}\operatorname{Tr}\left[\Sigma_{1/2}(s,Y,U)^* P\,\Sigma_{1/2}(s,Y,U)\right],\end{aligned} \tag{5.3}$$

with the noise-intensity operator

$$\Sigma_{1/2}(s,Y,U) := Q_{\text{noise}}^{1/2}(N(s)Y + D(s)U + H(s)),$$

and terminal condition

$$V(b,Y) = \frac{1}{2}\langle GY,Y\rangle_{\mathbb{H}}, \quad \forall Y \in \mathbb{H}.$$

Assume that the value function admits the quadratic decomposition

$$V(s_0,Y) := \frac{1}{2}\langle P(s_0)Y,Y\rangle_{\mathbb{H}} + \langle \varphi(s_0), Y\rangle_{\mathbb{H}} + \psi(s_0), \tag{5.4}$$

where the operators and processes satisfy

$$\begin{aligned} D_Y^2 V(s,Y) &:= P(s),\\ D_Y V(s,Y) &:= P(s)Y + \varphi(s),\\ \frac{\partial}{\partial s}V(s,Y) &:= \frac{1}{2}\langle \frac{\mathrm{d}}{\mathrm{d}s}P(s)Y,Y\rangle_{\mathbb{H}} + \langle \frac{\mathrm{d}}{\mathrm{d}s}\varphi(s),Y\rangle_{\mathbb{H}} + \frac{\mathrm{d}}{\mathrm{d}s}\psi(s),\end{aligned} \tag{5.5}$$

with the following regularity conditions,

$$P \in C^1([s_0,b];\mathcal{L}(\mathbb{H})), \quad \varphi \in C^1([s_0,b];\mathbb{H}), \quad \text{and} \quad \psi \in C^1([s_0,b];\mathbb{R}).$$

Lemma 5.1 (Riccati System):
Consider the stochastic linear-quadratic problem associated with the system (5.1). Under Assumptions 5.1–5.2, the Riccati system governing the value function is,

$$\begin{cases} -\frac{\mathrm{d}}{\mathrm{d}s}P(s) &= P(s)A + A^*P(s) + N_{1/2}^*(s)P(s)N_{1/2}(s) + Q(s) - \Gamma_1^*(s)\Upsilon^{-1}(s)\Gamma_1(s),\\ -\frac{\mathrm{d}}{\mathrm{d}s}\varphi(s) &= A^*\varphi(s) + P(s)F(s) + N_{1/2}^*(s)P(s)H_{1/2}(s) - \Gamma_1^*(s)\Upsilon^{-1}(s)\Gamma_0(s),\\ -\frac{\mathrm{d}}{\mathrm{d}s}\psi(s) &= \frac{1}{2}\operatorname{Tr}[H_{1/2}^*(s)P(s)H_{1/2}(s)] + \langle\varphi(s),F(s)\rangle - \frac{1}{2}\langle\Gamma_0(s),\Upsilon^{-1}(s)\Gamma_0(s)\rangle,\end{cases} \tag{5.6}$$

with terminal conditions

$$P(b) = G, \quad \varphi(b) = 0, \quad \psi(b) = 0.$$

Here $(\cdot)_{1/2} := Q_{\text{noise}}^{1/2}(\cdot)$, and the auxiliary operators are defined by,

$$\begin{cases} \Upsilon(s) := D_{1/2}^*(s)P(s)D_{1/2}(s) + R(s), \\ \Gamma_1(s) := D_{1/2}^*(s)P(s)N_{1/2}(s) + C^*(s)P(s) + S(s), \\ \Gamma_0(s) := D_{1/2}^*(s)P(s)H_{1/2}(s) + C^*(s)\varphi(s). \end{cases}$$

Proof:

Assume that the value function has the form (5.4). Using the Hilbert-Schmidt norm, we expand the trace term in the Hamiltonian (5.3) as follows,

$$\begin{aligned} \frac{1}{2}\|P(s)\Sigma Q_{\text{noise}}^{1/2}\|^2 = {} & \langle N_{1/2}^*(s)P(s)H_{1/2}(s), Y\rangle + \langle D_{1/2}^*(s)P(s)H_{1/2}(s), U\rangle \\ & + \langle D_{1/2}^*(s)P(s)N_{1/2}(s)Y, U\rangle + \frac{1}{2}\langle D_{1/2}^*(s)P(s)D_{1/2}(s)U, U\rangle \\ & + \frac{1}{2}\langle N_{1/2}^*(s)P(s)N_{1/2}(s)Y, Y\rangle + \frac{1}{2}\operatorname{Tr}[H_{1/2}^*(s)P(s)H_{1/2}(s)]. \end{aligned} \tag{5.7}$$

Substituting the derivatives of V from (5.5) into the HJB equation (5.2) yields

$$\begin{aligned} -\frac{1}{2}\langle\frac{\mathrm{d}}{\mathrm{d}s}P(s)Y, Y\rangle - \langle\frac{\mathrm{d}}{\mathrm{d}s}\varphi(s), Y\rangle - \frac{\mathrm{d}}{\mathrm{d}s}\psi(s) \\ = \min_U \Bigg[& \frac{1}{2}\langle(P(s)A + A^*P(s))Y, Y\rangle + \langle C^*(s)P(s)Y, U\rangle \\ & + \langle P(s)F(s), Y\rangle + \langle A^*\varphi(s), Y\rangle + \langle C^*(s)\varphi(s), U\rangle + \langle\varphi(s), F(s)\rangle \\ & + \frac{1}{2}\langle N_{1/2}^*(s)P(s)N_{1/2}(s)Y, Y\rangle + \langle D_{1/2}^*(s)P(s)N_{1/2}(s)Y, U\rangle \\ & + \frac{1}{2}\langle D_{1/2}^*(s)P(s)D_{1/2}(s)U, U\rangle + \langle N_{1/2}^*(s)P(s)H_{1/2}(s), Y\rangle \\ & + \frac{1}{2}\operatorname{Tr}[H_{1/2}^*(s)P(s)H_{1/2}(s)] + \langle D_{1/2}^*(s)P(s)H_{1/2}(s), U\rangle \\ & + \frac{1}{2}\langle Q(s)Y, Y\rangle + \langle S(s)Y, U\rangle + \frac{1}{2}\langle R(s)U, U\rangle \Bigg]. \end{aligned} \tag{5.8}$$

Grouping all terms that depend on the control U, we rewrite the expression as,

$$\begin{aligned} \frac{1}{2}\Big\langle U, \underbrace{(D_{1/2}^*(s)P(s)D_{1/2}(s) + R(s))}_{\Upsilon(s)} U\Big\rangle \\ + \Big\langle U, \underbrace{(D_{1/2}^*(s)P(s)N_{1/2}(s) + C^*(s)P(s) + S(s))}_{\Gamma_1(s)} Y\Big\rangle \end{aligned}$$

$$+\left\langle U, \underbrace{(D^*_{1/2}(s)P(s)H_{1/2}(s) + C^*(s)\varphi(s))}_{\Gamma_0(s)} \right\rangle$$

Since Υ is positive definite (under Assumption 5.2), the control U minimising the Hamiltonian is given explicitly by,

$$U_*(s) := -\Upsilon^{-1}(s)(\Gamma_1(s)Y(s) + \Gamma_0(s)). \tag{5.9}$$

Substituting U_* into (5.8) and equating coefficients yields the Riccati system through careful algebraic manipulation, confirming the expressions for $\frac{\mathrm{d}}{\mathrm{d}s}P(s)$, $\frac{\mathrm{d}}{\mathrm{d}s}\varphi(s)$, and $\frac{\mathrm{d}}{\mathrm{d}s}\psi(s)$. □

5.4. Closed-Loop System Formulation

Substituting the optimal control U_* (from (5.9)) into the state equation (5.1), the closed-loop dynamics become

$$\mathrm{d}Y_*(s) = [A_*(s)Y_*(s) + F_*(s)]\,\mathrm{d}s + [N_*(s)Y_*(s) + H_*(s)]\,\mathrm{d}W(s),$$

where

$$\begin{cases} A_*(s) := A - C(s)\Upsilon^{-1}(s)\Gamma_1(s), & F_*(s) := F(s) - C(s)\Upsilon^{-1}(s)\Gamma_0(s), \\ N_*(s) := N(s) - D(s)\Upsilon^{-1}(s)\Gamma_1(s), & H_*(s) := H(s) - D(s)\Upsilon^{-1}(s)\Gamma_0(s). \end{cases}$$

We factor A_* as,

$$A_*(s) := \Lambda(s) + C(s)(R^{-1}(s)S(s) - \Upsilon^{-1}(s)\Gamma_1(s)), \quad \text{with } \Lambda(s) := A - C(s)R^{-1}(s)S(s).$$

Then,

$$\begin{aligned} \mathrm{d}Y_*(s) = &\Big[\Lambda(s)Y_*(s) + C(s)\left(R^{-1}(s)S(s) - \Upsilon^{-1}(s)\Gamma_1(s)\right)Y_*(s) + F_*(s)\Big]\mathrm{d}s \\ &+ [N_*(s)Y_*(s) + H_*(s)]\,\mathrm{d}W(s). \end{aligned}$$

Given that $CR^{-1}S$ is uniformly bounded and independent of the Riccati equation, let $\{\mathcal{U}(s,s_0)\}_{s\geq s_0} \subset \mathcal{L}(\mathbb{H})$ denote the evolution family generated by the operator Λ. We impose the following assumptions on $\mathcal{U}(s,s_0)$.

Assumption 5.3:

1. **Strong continuity.** The two-parameter map $(s,s_0) \mapsto \mathcal{U}(s,s_0)y$ is strongly continuous for all (fixed) $y \in \mathbb{H}$;
2. **Causality.** The family $\{\mathcal{U}(s,s_0)\}_{s\geq s_0}$ satisfies the evolution property

$$\mathcal{U}(s,s_0) = \mathcal{U}(s,r)\mathcal{U}(r,s_0), \quad \forall s \geq r \geq s_0;$$

3. **Exponential decay.** There exist constants $L_{\mathcal{U}} \geq 1$, $\beta \geq 1$, and $\delta_\Lambda > 0$ such that
$$\|\mathcal{U}(s, s_0)\|_{\mathcal{L}(\mathbb{H})} \leq L_{\mathcal{U}} |s - s_0|^{\beta - 1} e^{-\delta_\Lambda |s - s_0|}, \quad \forall s, s_0 \in \mathbb{R}.$$

5.4.1. Time-Ordered Exponential Representation

When Λ is bounded on $\mathbb{H}$, the evolution family admits the Dyson series expansion (see, e.g., Sakurai and Napolitano [49, Ch. 2]),

$$\mathcal{U}(s, s_0) := \mathrm{O}\exp\left(\int_{s_0}^{s} \Lambda(r)\mathrm{d}r\right),$$

where the time-ordered exponential is defined by,

$$\begin{aligned}
\mathrm{O}\exp\left(\int_{s_0}^{s} \Lambda(r)\mathrm{d}r\right) &= I + \int_{s_0}^{s} \Lambda(r_1)\mathrm{d}r_1 + \int_{s_0}^{s}\int_{s_0}^{r_1} \Lambda(r_1)\Lambda(r_2)\mathrm{d}r_2\mathrm{d}r_1 + \cdots \\
&= \sum_{n=0}^{\infty} \int_{s_0 \leq r_1 \leq \cdots \leq r_n \leq s} \prod_{k=1}^{n} \Lambda(r_k)\, \mathrm{d}r_1 \cdots \mathrm{d}r_n.
\end{aligned}$$

The time-ordering operator O enforces causal composition,

$$\mathrm{O}[\Lambda(r_1)\Lambda(r_2)] = \begin{cases} \Lambda(r_1)\Lambda(r_2), & r_1 > r_2, \\ \Lambda(r_2)\Lambda(r_1), & r_2 > r_1. \end{cases}$$

This formulation leads to the closed-loop Riccati system, which we now present.

Lemma 5.2 (Closed-Loop Riccati Equation):
The Riccati system associated with the feedback-affected dynamics can be written as,

$$\begin{cases}
-\frac{\mathrm{d}}{\mathrm{d}s}P(s) &= P(s)A_*(s) + A_*^*(s)P(s) + N_{*1/2}^*P(s)N_{*1/2}(s) + Q_*(s), \\
-\frac{\mathrm{d}}{\mathrm{d}s}\varphi(s) &= A_*^*(s)\varphi(s) + P(s)F(s) + N_{*1/2}^*P(s)H_{1/2}(s), \\
-\frac{\mathrm{d}}{\mathrm{d}s}\psi(s) &= \frac{1}{2}\operatorname{Tr}(H_{1/2}^*(s)P(s)H_{1/2}(s)) + \langle\varphi(s), F(s)\rangle - \frac{1}{2}\langle\Gamma_0(s), \Upsilon^{-1}(s)\Gamma_0(s)\rangle,
\end{cases} \tag{5.10}$$

with terminal conditions

$$P(b) = G, \quad \varphi(b) = 0, \quad \psi(b) = 0.$$

where the modified terms are defined as,

$$\begin{cases}
Q_*(s) & := (\Upsilon^{-1}(s)\Gamma_1(s) - R^{-1}(s)S(s))^* R(s)(\Upsilon^{-1}(s)\Gamma_1(s) - R^{-1}(s)S(s)) \\
& \quad + Q(s) - S^*(s)R^{-1}(s)S(s), \\
N_{*1/2}(s) & := N_{1/2}(s) - D_{1/2}(s)\Upsilon^{-1}(s)\Gamma_1(s).
\end{cases}$$

Furthermore, by introducing the factorisation

$$A_*(s) = \Lambda(s) + C(s)(R^{-1}(s)S(s) - \Upsilon^{-1}(s)\Gamma_1(s)), \quad \text{with } \Lambda(s) = A - C(s)R^{-1}(s)S(s), \tag{5.11}$$

the Riccati system (5.6) can be equivalently rewritten as,

$$\begin{cases} -\frac{\mathrm{d}}{\mathrm{d}s}P(s) & = P(s)\Lambda(s) + \Lambda^*(s)P(s) \\ & +N^*_{*1/2}(s)P(s)(P^{-1}(s) + D_{1/2}(s)R^{-1}(s)D^*_{1/2}(s))P(s)N_{*1/2}(s) \\ & -P(s)C(s)R^{-1}(s)C^*(s)P(s) + Q(s) - S^*(s)R^{-1}(s)S(s), \\ -\frac{\mathrm{d}}{\mathrm{d}s}\varphi(s) & = \Lambda^*(s)\varphi(s) + P(s)F(s) + (S^*(s)R^{-1}(s) - \Gamma_1^*(s)\Upsilon^{-1}(s))C^*(s)\varphi(s) \\ & +N^*_{*1/2}(s)P(s)H_{1/2}(s), \\ -\frac{\mathrm{d}}{\mathrm{d}s}\psi(s) & = \frac{1}{2}\operatorname{Tr}(H^*_{1/2}(s)P(s)H_{1/2}(s)) + \langle\varphi(s), F(s)\rangle - \frac{1}{2}\langle\Gamma_0(s), \Upsilon^{-1}(s)\Gamma_0(s)\rangle, \end{cases} \tag{5.12}$$

Proof:
Starting with the original Riccati equation

$$-\frac{\mathrm{d}}{\mathrm{d}s}P(s) = P(s)A + A^*P(s) + N^*_{1/2}(s)P(s)N_{1/2}(s) + Q(s) - \Gamma_1^*(s)\Upsilon^{-1}(s)\Gamma_1(s), \tag{5.13}$$

substitute A_* and $N_{*1/2}$,

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}s}P(s) = {} & P(s)A_*(s) + A_*^*(s)P(s) + N^*_{*1/2}(s)P(s)N_{*1/2}(s) + Q(s) \\ & + 2\operatorname{Re}\Big[(P(s)C(s) + N^*_{*1/2}P(s)D_{1/2}(s))\Upsilon^{-1}(s)\Gamma_1(s)\Big] \\ & + \Gamma_1^*(s)\Upsilon^{-1}(s)D^*_{1/2}(s)P(s)D_{1/2}(s)\Upsilon^{-1}(s)\Gamma_1(s) - \Gamma_1^*(s)\Upsilon^{-1}(s)\Gamma_1(s). \end{aligned} \tag{5.14}$$

Using the fact that Υ is invertible we get,

$$\begin{aligned} \Gamma_1^*(s)\Upsilon^{-1}(s)\Gamma_1(s) &= \Gamma_1^*(s)\Upsilon^{-1}(s)\Upsilon(s)\Upsilon^{-1}(s)\Gamma_1(s) \\ &= \Gamma_1^*(s)\Upsilon^{-1}(s)D^*_{1/2}(s)P(s)D_{1/2}(s)\Upsilon^{-1}(s)\Gamma_1(s) \\ &+ \Gamma_1^*(s)\Upsilon^{-1}(s)R(s)\Upsilon^{-1}(s)\Gamma_1(s). \end{aligned} \tag{5.15}$$

Substitute (5.15) into (5.14),

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}s}P(s) = {} & P(s)A_*(s) + A_*^*(s)P(s) + N^*_{*1/2}(s)P(s)N_{*1/2}(s) + Q(s) \\ & - \Gamma_1^*(s)\Upsilon^{-1}(s)R(s)\Upsilon^{-1}(s)\Gamma_1(s) \\ & + 2\operatorname{Re}\Big[(P(s)C(s) + N^*_{*1/2}(s)P(s)D_{1/2}(s))\Upsilon^{-1}(s)\Gamma_1(s)\Big]. \end{aligned} \tag{5.16}$$

Applying square completion with $X = \Upsilon^{-1}\Gamma_1$,

$$\begin{aligned} & X^*(s)R(s)X(s) - 2\operatorname{Re}[S^*(s)X(s)] \\ & \quad = (X(s) - R^{-1}(s)S(s))^*R(s)(X(s) - R^{-1}(s)S(s)) - S^*(s)R^{-1}(s)S(s), \end{aligned} \tag{5.17}$$

we obtain the final Riccati form. The adjoint term evolution follows directly from the definition of Γ_0,

$$-\frac{\mathrm{d}}{\mathrm{d}s}\varphi(s) = A^*\varphi(s) + P(s)F(s) + N^*_{1/2}(s)P(s)H_{1/2}(s) - \Gamma_1^*(s)\Upsilon^{-1}(s)\Gamma_0(s)$$

$$= A_*^*(s)\varphi(s) + N_{*1/2}^*(s)P(s)H_{1/2}(s) + P(s)F(s).$$

For (5.12), substitute (5.11) into (5.10),

$$-\frac{\mathrm{d}}{\mathrm{d}s}P(s) = P(s)\Lambda(s) + \Lambda^*(s)P(s) + N_{*1/2}^*(s)P(s)N_{*1/2}(s) \\ + Q_*(s) - 2\,\mathrm{Re}[P(s)C(s)(\Upsilon^{-1}(s)\Gamma_1(s) - R^{-1}(s)S(s))].$$

Let $X = \Upsilon^{-1}\Gamma_1 - R^{-1}S$ and complete the square,

$$\mathrm{Re}(X^*(s)R(s)X(s) - 2P(s)C(s)X(s)) \\ = \|R(s)^{1/2}X(s) - R(s)^{-1/2}C^*(s)P(s)\|^2 - P(s)C(s)R^{-1}(s)C^*(s)P(s). \quad (5.18)$$

Combining terms and recognising the factorisation completes the proof. □

In order to establish the existence and uniqueness of solutions to the system dynamics and the associated Riccati equations, we require the following technical lemma. Let $\Gamma(\cdot)$ denote the Gamma function, defined by

$$\Gamma(s) = \int_0^\infty r^{s-1}e^{-r}\mathrm{d}r, \quad \text{for } s > 0.$$

Lemma 5.3 (Generalised Grönwall-Bihari Inequality):
Let $y : [a,b] \to \mathbb{R}_+$ be continuous and suppose that,

$$y(s) \le k_0(b-s)^{\beta-1}e^{-\delta(b-s)} + \int_s^b (r-s)^{\beta-1}e^{-\delta(r-s)}\left(k_1 + k_2 y(r) + k_3 y(r)^2\right)\mathrm{d}r, \quad (5.19)$$

for constants $\beta \ge 1$, $\delta > 0$, and $k_0, k_1, k_2, k_3 \in \mathbb{R}_+$. Define,

$$k_4 := \max\left\{k_0(b-a)^{\beta-1}e^{-\delta(b-a)}, k_0\left(\frac{\beta-1}{\delta}\right)^{\beta-1}e^{-(\beta-1)}, k_1\frac{\Gamma(\beta)}{\delta^\beta}\right\},$$

$$\Delta := \left(\frac{\delta^\beta}{\Gamma(\beta)} - k_2\right)^2 - 4k_3\cdot\frac{\delta^\beta}{\Gamma(\beta)}\cdot k_4. \quad (5.20)$$

Then,

1. **Nonlinear case** ($k_3 \neq 0$). If $k_3 > 0$, $\frac{\delta^\beta}{\Gamma(\beta)} > k_2$, and $\Delta \ge 0$, then,

$$\frac{\frac{\delta^\beta}{\Gamma(\beta)} - k_2 - \sqrt{\Delta}}{2k_3} =: k_{5-} \le y(s) \le k_5 \le \frac{\frac{\delta^\beta}{\Gamma(\beta)} - k_2 + \sqrt{\Delta}}{2k_3} =: k_{5+}. \quad (5.21)$$

   Moreover, if,

$$k_5 < \frac{1}{2k_3}\left(\frac{\delta^\beta}{\Gamma(\beta)} - k_2\right), \quad (5.22)$$

   then the integral operator is a contraction on the ball $\mathcal{B}_{k_5} := \{y \in C([a,b]) : \|y\|_\infty \le k_5\}$, and the solution is unique.

2. **Linear case** ($k_3 = 0$). If $\frac{\delta^\beta}{\Gamma(\beta)} > k_2$, then,

$$y(s) \le \frac{\delta^\beta k_4}{\delta^\beta - k_2\Gamma(\beta)}. \tag{5.23}$$

Proof:
Define the operator $\mathscr{K} : C([a,b]) \to C([a,b])$ by,

$$(\mathscr{K} y)(s) := k_0(b-s)^{\beta-1}e^{-\delta(b-s)} + \int_s^b (r-s)^{\beta-1}e^{-\delta(r-s)}\left(k_1 + k_2 y(r) + k_3 y(r)^2\right)\mathrm{d}r. \tag{5.24}$$

Let $k_5 := \|y\|_\infty$. Perform the change of variable $u = \delta(r-s)$ to estimate the kernel,

$$\int_s^b (r-s)^{\beta-1}e^{-\delta(r-s)}\mathrm{d}r = \frac{1}{\delta^\beta}\int_0^{\delta(b-s)} r^{\beta-1}e^{-r}\mathrm{d}r \le \frac{\Gamma(\beta)}{\delta^\beta}.$$

Substituting into (5.24), we obtain,

$$(\mathscr{K} y)(s) \le k_0(b-s)^{\beta-1}e^{-\delta(b-s)} + \frac{\Gamma(\beta)}{\delta^\beta}\left(k_1 + k_2k_5 + k_3k_5^2\right).$$

The supremum of $h(s) := k_0(b-s)^{\beta-1}e^{-\delta(b-s)}$ on $[a,b]$ is given by,

$$\sup_{s\in[a,b]} h(s) = \max\left\{k_0(b-a)^{\beta-1}e^{-\delta(b-a)}, k_0\left(\frac{\beta-1}{\delta}\right)^{\beta-1} e^{-(\beta-1)}\right\}, \tag{5.25}$$

which defines k_4 in (5.20). Thus,

$$\|\mathscr{K} y\|_\infty \le k_4 + \frac{\Gamma(\beta)}{\delta^\beta}\left(k_2k_5 + k_3k_5^2\right). \tag{5.26}$$

To ensure $\mathscr{K} y \in \mathcal{B}_{k_5}$, we require,

$$k_3k_5^2 + \left(k_2 - \frac{\delta^\beta}{\Gamma(\beta)}\right)k_5 + \frac{\delta^\beta}{\Gamma(\beta)}k_4 \le 0. \tag{5.27}$$

This is a quadratic inequality in k_5, and the result follows from the condition $\Delta \ge 0$ with roots determined by the quadratic formula.

Contraction estimate.

Let $y_1, y_2 \in \mathcal{B}_{k_5}$. Then,

$$\begin{aligned}|(\mathscr{K} y_1)(s) - (\mathscr{K} y_2)(s)| &\le \int_s^b (r-s)^{\beta-1}e^{-\delta(r-s)}(k_2 + 2k_3k_5)\,|y_1(r) - y_2(r)|\mathrm{d}r \\ &\le \frac{\Gamma(\beta)}{\delta^\beta}(k_2 + 2k_3k_5)\,\|y_1 - y_2\|_\infty.\end{aligned} \tag{5.28}$$

Thus, $\mathscr{K}$ is a contraction on $\mathcal{B}_{k_5}$ if,

$$\frac{\Gamma(\beta)}{\delta^\beta}(k_2 + 2k_3k_5) < 1.$$

This holds under the smallness condition (5.22), and the result follows by Banach's Fixed Point Theorem. □

We now provide the conditions under which the Riccati system associated with the stochastic linear-quadratic problem is uniformly bounded.

Proposition 5.1 (Uniform Boundedness and Uniqueness):
Under Assumptions 5.1, 5.2, and 5.3, suppose the following additional conditions hold,

$$K_3 > 0, \quad \frac{(2\delta_\Lambda)^{2\beta-1}}{\Gamma(2\beta-1)} > K_2, \quad \Delta \geq 0, \quad \text{and} \quad \frac{\delta_\Lambda^\beta}{\Gamma(\beta)} > K_7. \tag{5.29}$$

Then the Riccati system $(P(s), \varphi(s))$ is uniformly bounded, and the following estimates hold,

$$\sup_{s\in[s_0,b]} \|P(s)\|_{\mathcal{L}(\mathbb{H})} \leq K_5 \leq K_{5+} < \infty, \quad \text{and} \quad \sup_{s\in[s_0,b]} \|\varphi(s)\|_{\mathbb{H}} \leq \frac{\delta^\beta K_6}{\delta^\beta - K_7\Gamma(\beta)}. \tag{5.30}$$

Moreover, if the contraction condition holds,

$$\frac{\Gamma(2\beta-1)}{(2\delta_\Lambda)^{2\beta-1}} \cdot (K_2 + 2K_3K_5) < 1, \tag{5.31}$$

then $P(s)$ is unique in $C([s_0, b]; \mathcal{L}(\mathbb{H}))$.

The constants are defined as,

$$\begin{aligned}
K_1 &:= L_{\mathcal{U}}^2 \left(\frac{\|Q\|_\infty \alpha + \|S\|_\infty^2}{\alpha} \right), \\
K_2 &:= L_{\mathcal{U}}^2 \left(\frac{\|N_{1/2}\|_\infty^2 \alpha + 2\|S\|_\infty (\|N_{1/2}\|_\infty \|D_{1/2}\|_\infty + 2\|C\|_\infty)}{\alpha} \right), \\
K_3 &:= L_{\mathcal{U}}^2 \left(\frac{(\|C\|_\infty + \|N_{1/2}\|_\infty \|D_{1/2}\|_\infty)^2}{\alpha} \right), \\
K_4 &:= L_{\mathcal{U}}^2 \max \left\{ \|G\| (b-s_0)^{2(\beta-1)} e^{-2\delta_\Lambda (b-s_0)}, \|G\| \left(\frac{\beta-1}{\delta_\Lambda} \right)^{2(\beta-1)} e^{-2(\beta-1)} \right\} \\
&\quad + L_{\mathcal{U}}^2 \left(\frac{\Gamma(2\beta-1)}{(2\delta_\Lambda)^{2\beta-1}} \right) K_1, \\
K_6 &:= L_{\mathcal{U}} \left(\frac{\|F\|_\infty \alpha + \|H_{1/2}\|_\infty (\|N_{1/2}\|_\infty \alpha)}{\alpha} \right. \\
&\quad \left. + \frac{\|D_{1/2}\|_\infty [\|S\|_\infty + \|P\|_\infty (\|C\|_\infty + \|N_{1/2}\|_\infty \|D_{1/2}\|_\infty)]}{\alpha} \right), \\
K_7 &:= L_{\mathcal{U}} \left(\frac{\|C\|_\infty (2\|S\|_\infty + \|P\|_\infty (\|C\|_\infty + \|N_{1/2}\|_\infty \|D_{1/2}\|_\infty))}{\alpha} \right).
\end{aligned}$$

Proof:

By Zabczyk [64, Lemma 17.2], the operator-valued function $P(s) \in \mathcal{L}(\mathbb{H})$ satisfies the integral representation,

$$P(s) = \mathcal{U}^*(b,s)G\mathcal{U}(b,s) + \int_s^b \mathcal{U}^*(r,s)\mathcal{K}_1[P](r)\mathcal{U}(r,s)\mathrm{d}r, \tag{5.32}$$

where

$$\begin{aligned}\mathcal{K}_1[P](s) := N^*_{*1/2}(s)P(s)(P(s)^{-1} + D_{1/2}R^{-1}(s)D^*_{1/2}(s))P(s)N_{*1/2}(s) \\ - P(s)C(s)R^{-1}(s)C^*(s)P(s) + Q(s) - S^*(s)R^{-1}(s)S(s).\end{aligned}$$

We estimate,

$$\|P(s)\| \leq \underbrace{\|\mathcal{U}^*(b,s)G\mathcal{U}(b,s)\|}_{\text{Term I}} + \underbrace{\left\| \int_s^b \mathcal{U}^*(r,s)\mathcal{K}_1[P](r)\mathcal{U}(r,s)\mathrm{d}r \right\|}_{\text{Term II}}.$$

Term I.

By C_0-group estimates,

$$\begin{aligned}\|\mathcal{U}^*(b,s)G\mathcal{U}(b,s)\| \leq \|G\|L^2_{\mathcal{U}} \max \Big\{ &(b-s)^{2(\beta-1)}e^{-2\delta_\Lambda(b-s)}, \\ &\left(\frac{\beta-1}{\delta_\Lambda}\right)^{2(\beta-1)} e^{-2(\beta-1)} \Big\} \\ =: K_0.&\end{aligned}$$

Term II.

Using submultiplicativity,

$$\begin{aligned}\left\| \int_s^b \mathcal{U}^*(r,s)\mathcal{K}_1[P](r)\mathcal{U}(r,s)\mathrm{d}r \right\| &\leq \int_s^b \|\mathcal{U}(r,s)\|^2 \|\mathcal{K}_1[P](r)\|\mathrm{d}r \\ &\leq \int_s^b (r-s)^{2\beta-2}e^{-2\delta_\Lambda(r-s)} \\ &\quad \cdot \left(K_1 + K_2\|P(r)\| + K_3\|P(r)\|^2\right)\mathrm{d}r.\end{aligned}$$

Combining both terms yields,

$$\|P(s)\| \leq K_0 + \int_s^b (r-s)^{2\beta-2}e^{-2\delta_\Lambda(r-s)}\left(K_1 + K_2\|P(r)\| + K_3\|P(r)\|^2\right)\mathrm{d}r. \tag{5.33}$$

By Lemma 5.3 (assuming $K_3 > 0$, $\Delta \geq 0$, and $\frac{(2\delta_\Lambda)^{2\beta-1}}{\Gamma(2\beta-1)} > K_2$), we obtain,

$$\|P(s)\| \leq K_5 \leq \frac{\frac{(2\delta_\Lambda)^{2\beta-1}}{\Gamma(2\beta-1)} - K_2 + \sqrt{\Delta}}{2K_3} =: K_{5+}.$$

Adjoint Equation.

Similarly,

$$\varphi(s) = \int_s^b \mathcal{U}^*(r,s)\mathcal{K}_2[\varphi](r)\mathrm{d}r,$$

where

$$\mathcal{K}_2[\varphi](s) := P(s)F(s)+(S^*(s)R^{-1}(s)-\Gamma_1^*(s)\Upsilon^{-1}(s))C^*(s)\varphi(s)+N_{*1/2}^*(s)P(s)H_{1/2}(s),$$

is bounded by,

$$\|\varphi(s)\| \leq \frac{\delta_\Lambda^\beta K_6}{\delta_\Lambda^\beta - K_7\Gamma(\beta)}.$$

Uniqueness.

The contraction condition guarantees uniqueness via the Banach fixed-point theorem. Specifically, the Riccati equation defines a contraction mapping on the Banach space $C_b([s_0,b];\mathcal{L}(\mathbb{H}))$. Similarly, the adjoint equation admits a unique solution in the space $C_b([s_0,b];\mathbb{H})$ using an analogous contraction mapping argument. □

We now present the main result of this chapter, stated precisely with all necessary and rigorous conditions.

Theorem 5.1:
Let Assumptions 5.1, 5.2, and 5.3 hold, and suppose the smallness condition stated in Proposition 5.1 is satisfied. Crucially, we also require,

$$\left(\frac{\Gamma(\beta)}{\delta_\Lambda^\beta}\right)^2 \left(\frac{\|C\|_\infty(2\|S\|_\infty + \|P\|_\infty(\|C\|_\infty + \|N_{1/2}\|_\infty\|D_{1/2}\|_\infty))}{\alpha}\right)^2 + \left(\frac{\Gamma(2\beta-1)}{(2\delta_\Lambda)^{2\beta-1}}\right)\|N_{*1/2}\|_\infty^2 < (6L_{\mathcal{U}}^2)^{-1}. \quad (5.34)$$

Under these premises, we establish three fundamental results,

1. There exists a unique optimal control characterised by the feedback law,
$$U_*(s) = -\Upsilon^{-1}(s)\left[\Gamma_1(s)Y_*(s) + \Gamma_0(s)\right], \quad s \in [s_0,b],$$
where Γ_1 and Γ_0 are operators defined in Lemma 5.1.

2. The value function admits the concrete formulation,
$$V(s_0,Y) = \frac{1}{2}\langle P(s_0)Y,Y\rangle_{\mathbb{H}} + \langle\varphi(s_0),Y\rangle_{\mathbb{H}} + \psi(s_0).$$

3. The optimal trajectory satisfies,
$$\mathbb{E}\left[\sup_{s\in[s_0,b]}\|Y_*(s)\|_{\mathbb{H}}^2\right]$$

$$
\begin{aligned}
&\leq \Bigg[3L_{\mathfrak{U}}^2 \max\left\{(b-s_0)^{2(\beta-1)}e^{-2\delta_\Lambda(b-s_0)}, \left(\frac{\beta-1}{\delta_\Lambda}\right)^{2(\beta-1)} e^{-2(\beta-1)}\right\} \mathbb{E}\|Y\|_{\mathbb{H}}^2 \\
&\quad + 6L_{\mathfrak{U}}^2 \left(\left(\frac{\Gamma(\beta)}{\delta_\Lambda^\beta}\right)^2 \|F_*\|_\infty^2 + \left(\frac{\Gamma(2\beta-1)}{(2\delta_\Lambda)^{2\beta-1}}\right) \|H_{*1/2}\|_\infty^2 \right) \Bigg] \\
&\quad \times \Bigg[1 - 6L_{\mathfrak{U}}^2 \Bigg(\left(\frac{\Gamma(2\beta-1)}{(2\delta_\Lambda)^{2\beta-1}}\right) \|N_{*1/2}\|_\infty^2 \\
&\quad + \left(\frac{\Gamma(\beta)}{\delta_\Lambda^\beta}\right)^2 \left(\frac{\|C\|_\infty(2\|S\|_\infty + \|P\|_\infty(\|C\|_\infty + \|N_{1/2}\|_\infty\|D_{1/2}\|_\infty))}{\alpha}\right)^2 \Bigg) \Bigg]^{-1} \\
&< \infty.
\end{aligned}
$$

Proof:
By Assumptions 5.1, 5.2, and 5.3, the modified operator $\Lambda(s) = A - C(s)R^{-1}(s)S(s)$ generates an exponentially stable evolution family $\{\mathfrak{U}(s,s_0)\}_{s\geq s_0}$.

Consider the controlled SDE,

$$
\begin{aligned}
\mathrm{d}Y_*(s) &= \left[\Lambda(s)Y_*(s) + \left[C(s)(R^{-1}(s)S(s) - \Upsilon^{-1}(s)\Gamma_1(s))\right]Y_*(s) + F_*(s)\right]\mathrm{d}s \\
&\quad + [N_*(s)Y_*(s) + H_*(s)]\,\mathrm{d}W(s).
\end{aligned}
$$

Applying the fundamental inequality $\mathbb{E}\|a+b+c\|^2 \leq 3(\mathbb{E}\|a\|^2 + \mathbb{E}\|b\|^2 + \mathbb{E}\|c\|^2)$, we analyse the mild solution through three constituent terms,

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{r\in[s_0,s]} \|Y_*(r)\|_{\mathbb{H}}^2\right] \\
&\leq 3\Bigg(\mathbb{E}\left[\sup_{r\in[s_0,s]} \|\mathfrak{U}(r,s_0)Y\|_{\mathbb{H}}^2\right] \\
&\quad + \mathbb{E}\left[\sup_{r\in[s_0,s]} \left\|\int_{s_0}^r \mathfrak{U}(r,u)\left(C(u)[R^{-1}(u)S(u) - \Upsilon^{-1}(u)\Gamma_1(u)]Y_*(u) + F_*(u)\right)\mathrm{d}u\right\|_{\mathbb{H}}^2\right] \\
&\quad + \mathbb{E}\left[\sup_{r\in[s_0,s]} \left\|\int_{s_0}^r \mathfrak{U}(r,u)[N_*(u)Y_*(u) + H_*(u)]\mathrm{d}W(u)\right\|_{\mathbb{H}}^2\right]\Bigg).
\end{aligned}
$$

By exponential stability,

$$
\begin{aligned}
\mathbb{E}\left[\sup_{r\in[s_0,s]} \|\mathfrak{U}(r,s_0)Y\|_{\mathbb{H}}^2\right] \leq L_{\mathfrak{U}}^2 \max\Bigg\{&(s-s_0)^{2(\beta-1)}e^{-2\delta_\Lambda(s-s_0)}, \\
&\left(\frac{\beta-1}{\delta_\Lambda}\right)^{2(\beta-1)} e^{-2(\beta-1)}\Bigg\} \mathbb{E}\|Y\|_{\mathbb{H}}^2.
\end{aligned}
$$

Using the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{r\in[s_0,s]}\left\|\int_{s_0}^{r}\mathcal{U}(r,u)\left(C(u)[R^{-1}(u)S(u)-\Upsilon^{-1}(u)\Gamma_1(u)]Y_*(u)+F_*(u)\right)\mathrm{d}u\right\|_{\mathbb{H}}^2\right]\\
&\le 2L_{\mathcal{U}}^2\left(\frac{\Gamma(\beta)}{\delta_\Lambda^\beta}\right)^2\Bigg(\|F_*\|_\infty^2\\
&+\left(\frac{\|C\|_\infty(2\|S\|_\infty+\|P\|_\infty(\|C\|_\infty+\|N_{1/2}\|_\infty\|D_{1/2}\|_\infty))}{\alpha}\right)^2\mathbb{E}\left[\sup_{u\in[s_0,s]}\|Y_*(u)\|_{\mathbb{H}}^2\right]\Bigg).
\end{aligned}$$

Combining Cauchy-Schwarz and the Burkholder-Davis-Gundy inequality,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{r\in[s_0,s]}\left\|\int_{s_0}^{r}\mathcal{U}(r,u)[N_*(u)Y_*(u)+H_*(u)]\mathrm{d}W(u)\right\|_{\mathbb{H}}^2\right]\\
&\le 2L_{\mathcal{U}}^2\left(\frac{\Gamma(2\beta-1)}{(2\delta_\Lambda)^{2\beta-1}}\right)\left(\|H_{*1/2}\|_\infty^2+\|N_{*1/2}\|_\infty^2\mathbb{E}\left[\sup_{u\in[s_0,s]}\|Y_*(u)\|_{\mathbb{H}}^2\right]\right).
\end{aligned}$$

Under condition (5.34), an application of a Grönwall-type inequality yields the following estimate,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{r\in[s_0,s]}\|Y_*(r)\|_{\mathbb{H}}^2\right]\\
&\le\Bigg[3L_{\mathcal{U}}^2\max\left\{(s-s_0)^{2(\beta-1)}e^{-2\delta_\Lambda(s-s_0)},\left(\frac{\beta-1}{\delta_\Lambda}\right)^{2(\beta-1)}e^{-2(\beta-1)}\right\}\mathbb{E}\|Y\|_{\mathbb{H}}^2\\
&+6L_{\mathcal{U}}^2\left(\left(\frac{\Gamma(\beta)}{\delta_\Lambda^\beta}\right)^2\|F_*\|_\infty^2+\left(\frac{\Gamma(2\beta-1)}{(2\delta_\Lambda)^{2\beta-1}}\right)\|H_{*1/2}\|_\infty^2\right)\Bigg]\\
&\times\Bigg[1-6L_{\mathcal{U}}^2\Bigg(\left(\frac{\Gamma(2\beta-1)}{(2\delta_\Lambda)^{2\beta-1}}\right)\|N_{*1/2}\|_\infty^2\\
&+\left(\frac{\Gamma(\beta)}{\delta_\Lambda^\beta}\right)^2\left(\frac{\|C\|_\infty(2\|S\|_\infty+\|P\|_\infty(\|C\|_\infty+\|N_{1/2}\|_\infty\|D_{1/2}\|_\infty))}{\alpha}\right)^2\Bigg)\Bigg]^{-1}\\
&<\infty.
\end{aligned}$$

The existence and uniqueness follow immediately by applying the Banach Fixed Point Theorem in the space $\mathcal{M}^2([s_0,b];\mathbb{H})$. Define the candidate value process,

$$V(s,Y(s))=\frac{1}{2}\langle P(s)Y(s),Y(s)\rangle_{\mathbb{H}}+\langle\varphi(s),Y(s)\rangle_{\mathbb{H}}+\psi(s).$$

Applying Itô's formula while utilising the Riccati and adjoint equations reveals,

$$J(s_0,Y;U(\cdot))-V(s_0,Y)$$

$$= \frac{1}{2}\mathbb{E}\int_{s_0}^{b} \left\| \Upsilon^{1/2}(r)\left(U(r) + \Upsilon^{-1}(r)[\Gamma_1(r)Y(r) + \Gamma_0(r)]\right)\right\|_{\mathbb{U}}^2 \,\mathrm{d}r.$$

The quadratic structure ensures unique minimisation at $U = U_*$, thereby establishing,

$$J(s_0, Y; U_*(\cdot)) = V(s_0, Y) = \frac{1}{2}\langle P(s_0)Y, Y\rangle_{\mathbb{H}} + \langle \varphi(s_0), Y\rangle_{\mathbb{H}} + \psi(s_0).$$

□

5.5. Application to Second-Order SDEs and Synthesis of Optimal Feedback Control

This section establishes a functional-analytic framework for the reduction of a class of controlled second-order SDEs to first-order form. The primary objective is to recast the problem within the standard formalism of SDEs governed by C_0-semigroups on Hilbert spaces, thereby facilitating the application of well-established stochastic optimal control theory. We provide a meticulous dimensional analysis of all operators, derive the corresponding infinite-dimensional Riccati equation, and synthesise the optimal feedback control law. The complete structural transformation is demonstrated, and the well-posedness of the resulting system is rigorously confirmed.

We consider the following controlled second-order SDE for $s \in [s_0, b]$,

$$\mathrm{d}Z'(s) = \left[AZ(s) + BZ'(s) + C(s)U(s) + F(s)\right]\mathrm{d}s \\ + [N(s)Z(s) + D(s)U(s) + H(s)]\,\mathrm{d}W(s), \quad (5.35)$$

with initial conditions $Z(s_0) = z_0 \in \mathbb{H}$, $Z'(s_0) = z_1 \in \mathbb{H}$.

The operators and processes are defined with the following precise specifications,

1. $A : \mathrm{Dom}(A) \subset \mathbb{H} \to \mathbb{H}$ is a densely defined, linear operator.
2. $B : \mathrm{Dom}(B) \subset \mathbb{H} \to \mathbb{H}$ is a densely defined, linear operator.
3. $C \in C_b([s_0, b]; \mathcal{L}(\mathbb{U}, \mathbb{H}))$.
4. $N \in C_b([s_0, b]; \mathcal{L}(\mathbb{H}, \mathcal{L}_2(Q_{\mathrm{noise}}^{1/2}\mathbb{K}, \mathbb{H})))$.
5. $D \in C_b([s_0, b]; \mathcal{L}(\mathbb{U}, \mathcal{L}_2(Q_{\mathrm{noise}}^{1/2}\mathbb{K}, \mathbb{H})))$.
6. $F \in C_b([s_0, b]; \mathbb{H})$,
7. $H \in C_b([s_0, b]; \mathcal{L}_2(Q_{\mathrm{noise}}^{1/2}\mathbb{K}, \mathbb{H}))$.

The associated mild solution of the second-order system may be written in terms of the fundamental solution operators $\mathcal{M}(s)$ and $\mathcal{N}(s)$ generated by the operators A and B. More precisely, for sufficiently regular initial data (z_0, z_1) and for admissible controls and noises, the mild solution $Z(\cdot)$ admits a representation of the form

$$\begin{aligned} Z(s) = \mathcal{M}(s - s_0)z_0 + \mathcal{N}(s - s_0)z_1 + \int_{s_0}^{s} \mathcal{N}(s - r)\left(C(r)U(r) + F(r)\right) \mathrm{d}r \\ + \int_{s_0}^{s} \mathcal{N}(s - r)(N(r)Z(r) + D(r)U(r) + H(r))\mathrm{d}W(r). \end{aligned}$$

The reduction is achieved by constructing an augmented state space that incorporates the canonical coordinates of the original second-order system.

Definition 5.3 (Product Hilbert Space):
We define the *product Hilbert space* $\mathbb{V} := \mathbb{H} \times \mathbb{H}$. For $Y_1 = (x_1, y_1)^T, Y_2 = (x_2, y_2)^T \in \mathbb{V}$, the inner product is defined as,

$$\langle Y_1, Y_2\rangle_{\mathbb{V}} := \langle x_1, x_2\rangle_{\mathbb{H}} + \langle y_1, y_2\rangle_{\mathbb{H}}.$$

The induced norm is $\|Y\|_{\mathbb{V}}^2 = \|x\|_{\mathbb{H}}^2 + \|y\|_{\mathbb{H}}^2$. The new state vector is defined as,

$$Y(s) := \begin{pmatrix} Z'(s) \\ Z(s) \end{pmatrix} \in \mathbb{V}.$$

Definition 5.4 (Infinitesimal Generator):
The dynamics are generated by the block operator matrix $\mathscr{A} : \mathrm{Dom}(\mathscr{A}) \subset \mathbb{V} \to \mathbb{V}$ defined by,

$$\mathscr{A} := \begin{pmatrix} A & B \\ 0 & I \end{pmatrix}.$$

Its domain is constructed to ensure $\mathscr{A}Y \in \mathbb{V}$,

$$\mathrm{Dom}(\mathscr{A}) := \left\{ Y = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{V} \,\middle|\, x \in \mathrm{Dom}(A),\, y \in \mathrm{Dom}(B),\, Ax + By \in \mathbb{H} \right\}.$$

This domain is dense in $\mathbb{V}$. Under suitable hypotheses on A and B, $\mathscr{A}$ generates a strongly continuous semigroup $\{\mathcal{T}(s)\}_{s\geq 0}$ on $\mathbb{V}$, where

$$\mathcal{T}(s) := \begin{pmatrix} \mathcal{M}'(s) & \mathcal{N}'(s) \\ \mathcal{M}(s) & \mathcal{N}(s) \end{pmatrix}.$$

From the properties of $\mathcal{M}(s)$ and $\mathcal{N}(s)$ detailed in Chapter 1 (Lemma 1.2), it follows that

$$\begin{aligned} \mathcal{M}(s + r) &= \mathcal{M}(s)\mathcal{M}(r) + \mathcal{N}(s)\mathcal{M}'(r), \\ \mathcal{N}(s + r) &= \mathcal{M}(s)\mathcal{N}(r) + \mathcal{N}(s)\mathcal{N}'(r), \\ \mathcal{M}'(s + r) &= \mathcal{M}'(s)\mathcal{M}(r) + \mathcal{N}'(s)\mathcal{M}'(r), \\ \mathcal{N}'(s + r) &= \mathcal{M}'(s)\mathcal{N}(r) + \mathcal{N}'(s)\mathcal{N}'(r). \end{aligned}$$

***Remark* 5.3:**

We emphasise that we do not impose exponential stability of the evolution family (or of the semigroup generated by the first-order operator $\mathscr{A}$).

The system coefficients are systematically lifted to operate on the product space $\mathbb{V}$ and to interface correctly with the control space $\mathbb{U}$ and noise space $\mathbb{K}$,

$$\mathscr{F} : [s_0, b] \to \mathbb{V}, \qquad \mathscr{F}(s) := \begin{pmatrix} 0 \\ F(s) \end{pmatrix},$$

$$\mathscr{C} : [s_0, b] \to \mathcal{L}(\mathbb{U}, \mathbb{V}), \qquad \mathscr{C}(s)U := \begin{pmatrix} 0 \\ C(s)U \end{pmatrix},$$

$$\mathscr{H} : [s_0, b] \to \mathcal{L}_2(Q_{\text{noise}}^{1/2}\mathbb{K}, \mathbb{V}), \qquad \mathscr{H}(s)\xi := \begin{pmatrix} 0 \\ H(s)\xi \end{pmatrix},$$

$$\mathscr{D} : [s_0, b] \to \mathcal{L}(\mathbb{U}, \mathcal{L}_2(Q_{\text{noise}}^{1/2}\mathbb{K}, \mathbb{V})), \qquad \mathscr{D}(s)U\xi := \begin{pmatrix} 0 \\ D(s)U\xi \end{pmatrix},$$

$$\mathscr{N} : [s_0, b] \to \mathcal{L}(\mathbb{V}, \mathcal{L}_2(Q_{\text{noise}}^{1/2}\mathbb{K}, \mathbb{V})), \qquad \mathscr{N}(s)Y\xi := \begin{pmatrix} 0 \\ N(s)x\xi \end{pmatrix}, \quad \text{for } Y = (x, y)^T.$$

The second-order system is equivalent to the first-order SDE on $\mathbb{V}$,

$$\begin{aligned} \mathrm{d}Y(s) = \mathscr{A}Y(s)\mathrm{d}s + [\mathscr{F}(s) + \mathscr{C}(s)U(s)]\,\mathrm{d}s \\ + [\mathscr{H}(s) + \mathscr{D}(s)U(s) + \mathscr{N}(s)Y(s)]\,\mathrm{d}W(s). \end{aligned} \tag{5.36}$$

The original cost functional is,

$$J(s_0, Z; U(\cdot)) := \frac{1}{2}\mathbb{E}\left[\int_{s_0}^{b} \begin{pmatrix} Z(r) \\ U(r) \end{pmatrix}^* \begin{pmatrix} Q(r) & S^*(r) \\ S(r) & R(r) \end{pmatrix} \begin{pmatrix} Z(r) \\ U(r) \end{pmatrix} \mathrm{d}r + \langle GZ(b), Z(b)\rangle_{\mathbb{H}}\right].$$

To express this in terms of $Y(s) \in \mathbb{V}$, we define extended cost operators,

$$\mathscr{Q}(s) := \begin{pmatrix} 0 & 0 \\ 0 & Q(s) \end{pmatrix} \in \mathcal{L}(\mathbb{V}), \quad \mathscr{S}(s) := \begin{pmatrix} 0 & S(s) \end{pmatrix} \in \mathcal{L}(\mathbb{V}, \mathbb{U}), \quad \mathscr{G} := \begin{pmatrix} 0 & 0 \\ 0 & G \end{pmatrix} \in \mathcal{L}(\mathbb{V}).$$

The integrand becomes a quadratic form on $\mathbb{V} \times \mathbb{U}$,

$$\begin{pmatrix} Y \\ U \end{pmatrix}^* \begin{pmatrix} \mathscr{Q}(s) & \mathscr{S}^*(s) \\ \mathscr{S}(s) & R(s) \end{pmatrix} \begin{pmatrix} Y \\ U \end{pmatrix} = \langle \mathscr{Q}(s)Y, Y\rangle_{\mathbb{V}} + 2\langle \mathscr{S}(s)Y, U\rangle_{\mathbb{U}} + \langle R(s)U, U\rangle_{\mathbb{U}}.$$

The reformulated cost functional is therefore,

$$J(s_0, Y; U(\cdot)) := \frac{1}{2}\mathbb{E}\left[\int_{s_0}^{b} \begin{pmatrix} Y(r) \\ U(r) \end{pmatrix}^* \begin{pmatrix} \mathscr{Q}(r) & \mathscr{S}^*(s) \\ \mathscr{S}(r) & R(r) \end{pmatrix} \begin{pmatrix} Y(r) \\ U(r) \end{pmatrix} \mathrm{d}r + \langle \mathscr{G}Y(b), Y(b)\rangle_{\mathbb{V}}\right]. \tag{5.37}$$

5.5.1. Derivation of the Riccati Equation and Optimal Feedback Law

We now consider the first-order system (5.36) with cost functional (5.37). The value function is defined as,

$$V(s_0, Y) = \inf_{U(\cdot)\in\mathcal{U}_{\mathrm{ad}}[s_0,b]} J(s_0, Y; U(\cdot)).$$

Given the linear-quadratic structure, we postulate the ansatz,

$$V(s, Y) = \frac{1}{2}\langle \mathscr{P}(s)Y, Y\rangle_{\mathbb{V}} + \langle \varphi(s), Y\rangle_{\mathbb{V}} + \psi(s), \tag{5.38}$$

with

$$\mathscr{P}(s) = \mathscr{P}^*(s) \in C^1([s_0, b]; \mathcal{L}(\mathbb{V})), \quad \varphi(s) \in C^1([s_0, b]; \mathbb{V}), \quad \psi(s) \in C^1([s_0, b]; \mathbb{R}).$$

The Hamiltonian minimisation yields the optimal feedback control,

$$U_*(s) = -\Upsilon(s)^{-1}\left[\Gamma_1(s)Y(s) + \Gamma_0(s)\right], \tag{5.39}$$

where the auxiliary operators are,

$$\Upsilon(s) := \mathscr{D}^*(s)\mathscr{P}(s)\mathscr{D}(s) + R(s), \tag{5.40}$$

$$\Gamma_1(s) := \mathscr{D}^*(s)\mathscr{P}(s)\mathscr{N}(s) + \mathscr{C}^*(s)\mathscr{P}(s) + \mathscr{S}(s), \tag{5.41}$$

$$\Gamma_0(s) := \mathscr{D}^*(s)\mathscr{P}(s)\mathscr{H}(s) + \mathscr{C}^*(s)\varphi(s). \tag{5.42}$$

The coefficients of the value function satisfy the following system,

$$\begin{cases} -\frac{\mathrm{d}}{\mathrm{d}s}\mathscr{P}(s) = \mathscr{P}(s)\mathscr{A} + \mathscr{A}^*\mathscr{P}(s) + \mathscr{N}^*(s)\mathscr{P}(s)\mathscr{N}(s) + \mathscr{Q}(s) - \Gamma_1^*(s)\Upsilon^{-1}(s)\Gamma_1(s), \\ -\frac{\mathrm{d}}{\mathrm{d}s}\varphi(s) = \mathscr{A}^*\varphi(s) + \mathscr{P}(s)\mathscr{F}(s) + \mathscr{N}^*(s)\mathscr{P}(s)\mathscr{H}(s) - \Gamma_1^*(s)\Upsilon^{-1}(s)\Gamma_0(s), \\ -\frac{\mathrm{d}}{\mathrm{d}s}\psi(s) = \frac{1}{2}\operatorname{Tr}[\mathscr{H}^*(s)\mathscr{P}(s)\mathscr{H}(s)] + \langle\varphi(s), \mathscr{F}(s)\rangle_{\mathbb{V}} - \frac{1}{2}\langle\Gamma_0(s), \Upsilon^{-1}(s)\Gamma_0(s)\rangle_{\mathbb{U}}, \end{cases} \tag{5.43}$$

with terminal conditions $\mathscr{P}(b) = \mathscr{G}$, $\varphi(b) = 0$, $\psi(b) = 0$.

Applying the optimal control (5.39) yields the closed-loop system,

$$\mathrm{d}Y_*(s) = [\mathscr{A}_*(s)Y_*(s) + \mathscr{F}_*(s)]\,\mathrm{d}s + [\mathscr{N}_*(s)Y_*(s) + \mathscr{H}_*(s)]\,\mathrm{d}W(s), \tag{5.44}$$

where,

$$\begin{aligned} \mathscr{A}_*(s) &:= \mathscr{A} - \mathscr{C}(s)\Upsilon^{-1}(s)\Gamma_1(s), \\ \mathscr{F}_*(s) &:= \mathscr{F}(s) - \mathscr{C}(s)\Upsilon^{-1}(s)\Gamma_0(s), \\ \mathscr{N}_*(s) &:= \mathscr{N}(s) - \mathscr{D}(s)\Upsilon^{-1}(s)\Gamma_1(s), \\ \mathscr{H}_*(s) &:= \mathscr{H}(s) - \mathscr{D}(s)\Upsilon^{-1}(s)\Gamma_0(s). \end{aligned}$$

Assumption 5.4 (Evolution Family Properties):
Let $\{\mathcal{U}(s, s_0)\}_{s\geq s_0} \subset \mathcal{L}(\mathbb{V})$ denote the evolution family generated by $\Lambda(s) := \mathscr{A} - \mathscr{C}(s)R^{-1}(s)\mathscr{S}(s)$. We assume,

1. **Strong continuity.** The map $(s, s_0) \mapsto \mathcal{U}(s, s_0)y$ is strongly continuous for all $y \in \mathbb{V}$.

2. **Evolution property.** $\mathcal{U}(s,s_0)=\mathcal{U}(s,r)\mathcal{U}(r,s_0)$ for all $s\geq r\geq s_0$.
3. **Exponential decay.** There exist constants $L_{\mathcal{U}}\geq 1, \beta\geq 1, \delta_\Lambda>0$ such that,
$$\|\mathcal{U}(s,s_0)\|_{\mathcal{L}(\mathbb{V})}\leq L_{\mathcal{U}}|s-s_0|^{\beta-1}e^{-\delta_\Lambda|s-s_0|},\quad \forall s,s_0\in\mathbb{R}.$$

Under the stated assumptions, a result similar to Theorem 5.1 can now be obtained. In fact, by applying a fixed-point argument to the closed-loop system, one shows that the mild solution admits a unique process $Y_*(\cdot)$ given by

$$Y_*(s)=\mathcal{U}(s,s_0)Y+\int_{s_0}^{s}\mathcal{U}(s,r)\big(\mathscr{C}(r)[R^{-1}(r)\mathscr{S}(r)-\Upsilon^{-1}(r)\Gamma_1(r)]Y_*(r)+\mathscr{F}_*(r)\big)\,\mathrm{d}r$$
$$+\int_{s_0}^{s}\mathcal{U}(s,r)[\mathscr{N}_*(r)Y_*(r)+\mathscr{H}_*(r)]\mathrm{d}W(r).$$

The details are omitted.

5.5.2. Controlled System with Almost Periodic Coefficients

We assume that the coefficients in Equations (5.1) and (5.36) are almost periodic in the sense of Definition 2.2. Using Lemma 5.3, one can show that the pair $(P(\cdot),\varphi(\cdot))$ is almost periodic in the Bohr sense, as characterised in Section 2.5. The proof is based on a Banach fixed-point argument applied to the corresponding integral formulation of the Riccati system. From this, we conclude that the closed-loop systems are almost periodic in distribution, provided a suitable choice of parameters is made.

Considering $b\to+\infty$ with time-independent estimates, we define the weighted quadratic cost functional,

$$J(Y;U(\cdot))=\limsup_{b\to\infty}\frac{1}{2b}\mathbb{E}\Bigg[\int_{s_0}^{s_0+b}\Big(\langle Q(r)Y(r),Y(r)\rangle_{\mathbb{H}}$$
$$+2\langle S(r)Y(r),U(r)\rangle_{\mathbb{U}}+\langle R(r)U(r),U(r)\rangle_{\mathbb{U}}\Big)\mathrm{d}r$$
$$+\langle GY(s_0+b),Y(s_0+b)\rangle_{\mathbb{H}}\Bigg],$$

leading immediately to the optimal cost characterisation,

$$J(Y,U_*)=\limsup_{b\to\infty}\frac{1}{b}\psi(s_0)$$
$$:=\limsup_{b\to\infty}\frac{1}{b}\int_{s_0}^{s_0+b}\Bigg[\frac{1}{2}\mathrm{Tr}(H_{1/2}^*(r)P(r)H_{1/2}(r))$$

$$+ \langle \varphi(r), F(r) \rangle - \frac{1}{2} \langle \Gamma_0(r), \Upsilon^{-1}(r) \Gamma_0(r) \rangle \Bigg] \mathrm{d}r.$$

This limit exists, since in fact it coincides with the Bohr mean value. Analogously, for the second-order system one obtains the corresponding formulation of the quadratic cost.

***Remark* 5.4:**
In our study we assume, for simplicity of notation, that the operators A and $\mathscr{A}$ are bounded. This assumption is not essential. In the general case, where A in Section 5.2 (or the block operator $\mathscr{A}$ in Section 5.5) is unbounded, the corresponding SDE can still be interpreted via Yosida approximation; see Theorem 1.1 (or Theorem 1.6). Subsequent limit arguments then yield the well-posedness of the original system, as detailed in Da Prato and Zabczyk [19, Theorem 11.23].

Concluding Remarks

This work resolves the stochastic linear-quadratic control problem for hyperbolic systems. Our approach combines chronological operator calculus, group theory, and Riccati analysis. By addressing non-dissipative dynamics, multiplicative noise, and infinite-dimensional challenges, we provide a rigorous foundation for controlling reversible systems under uncertainty.

Conclusion and Perspectives

This dissertation establishes a comprehensive framework for the analysis of infinite-dimensional dynamical systems, contributing to the interplay of functional analysis, stochastic processes, and control theory. It provides a unified treatment of well-posedness for abstract Cauchy problems, progressing from first-order systems governed by C_0-semigroups to second-order systems characterised by $\mathcal{M}, \mathcal{N}$-operator functions. A key theoretical innovation is the extension of almost periodicity to probability-measure-valued mappings, facilitating the study of long-term behaviour in stochastic systems.

The principal original contributions are twofold.

First, the dissertation resolves questions of existence, uniqueness, and almost periodicity in distribution for a general class of second-order neutral SDEs with multiplicative noise. This constitutes a significant extension of prior work confined to first-order systems. It is demonstrated that under appropriate commutativity conditions, almost periodic coefficients induce almost periodic solutions, yielding new insight into the dynamics of oscillatory systems under stochastic forcing.

Second, this work provides a complete solution to the Stochastic Linear-Quadratic optimal control problem for hyperbolic differential equations with multiplicative noise. By establishing the well-posedness and boundedness of the associated operator-valued Riccati equation, via novel techniques including a generalised Grönwall-Bihari inequality, the optimal control is characterised in state-feedback form, and mean-square stability of the closed-loop system is proven. The results are applied to a reduced second-order system.

Critical Evaluation and Limitations

The analysis is subject to certain limitations, which delineate the scope of the results.

1. **Exponential Stability (Assumption 4.1):** This assumption, while often realisable through suitable damping, precludes the analysis of conservative systems, such as undamped wave equations on bounded domains, where the associated cosine family is bounded but not exponentially stable. An extension to such cases would necessitate alternative techniques, potentially based on compactness.

2. **Global Lipschitz Condition (Assumption 4.2):** The reliance on a global Lipschitz condition for the nonlinearity, though standard for global existence theorems, excludes physically relevant systems with superlinear growth. A natural extension would involve establishing local well-posedness and deriving a priori estimates for global extension.

3. **Operator Commutativity:** The existence of a well-defined, exponentially stable $\mathcal{M}, \mathcal{N}$–family, which generalises the commutativity of A and B, is central to the framework. A more explicit characterisation of the necessary conditions on A and B would further clarify the applicability of the theory.

Directions for Future Research

This work naturally suggests several lines of further inquiry:

1. **Weaker Stability Notions:** Generalising the results to systems with polynomial stability or under more general resolvent conditions.

2. **Non-Global Nonlinearities:** Developing a local well-posedness theory under local Lipschitz conditions, with extension to global solutions via Lyapunov-type or energy arguments.

3. **Control of Nonlinear Systems:** Applying the stability and almost periodicity results to optimal control problems for second-order stochastic systems on $\mathbb{R}$ with nonlinear coefficients.

4. **Almost Automorphy and Extensions:** Investigating almost automorphic solutions, including their stochastic variants (in mean, in distribution, or in probability), which provide a broader framework beyond almost periodicity.

Open Question (Pathwise Almost Periodicity).

Under what conditions can one guarantee that the sample paths of the solution, rather than only their distributions, exhibit almost periodic behaviour? In particular, what structural assumptions on the driving noise are necessary to ensure pathwise almost periodicity in a suitable sense?

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