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Course support for the module

MATHEMATICAL ANALYSIS I

Intended for first year of LMD computer science
& first year of computer engineering

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Summary

General Introduction

The objective of this course is to provide a transition between the knowledge of analysis acquired in high school and the foundational concepts that will form one of the pillars of mathematical training in undergraduate and engineering studies.

This document presents the Analysis 1 course that I teach during the first semester of the first year in the Computer Science Engineering program.

It contains the main mathematical analysis tools that students must understand and master. The document can be used as a reference text for first-year computer science students who will face mathematical problems and wish to learn techniques to solve them.

The course is divided into six chapters covering the fundamental topics of Analysis 1 :

First chapter covers Field of real numbers, absolute value, the greatest integer function, upper and lower bounds, the completeness axioms, the Archimedean property, the density of rational numbers, extend real line, and includes related exercises.

Chapter Two presents different forms of complex numbers, Euler's identity, the n^{th} roots of complex numbers, and includes exercises.

Chapter Three covers bounded and monotonic sequences, lower and upper limits, subsequences, limits and their properties, convergence and divergence, adjacent, recurrence and Cauchy sequences, and the Bolzano-Weierstrass theorem. .

Chapter Four introduces special classes of functions, explores limits, continuity and discontinuity, covers fundamental theorems of continuous functions, the reciprocal function, the order of a variable, and includes exercises..

Chapter Five presents the definitions and properties of differentiable functions, discusses Theorems on differentiable functions, Taylor's formula, convexity and asymptotes of a curve, and includes exercises.

The last chapter covers elementary functions, including logarithmic and exponential functions, hyperbolic functions and their inverses, and includes exercises."

Chapter 1

Real numbers

1.1 Number Sets

In mathematics, we often study sets whose elements are real numbers. Some special sets of numbers that are frequently encountered are defined as follows :

- **N** is the set of Natural numbers : $N = \{0, 1, 2, 3, ...\}$

- **Z** is the set of Integers : The set of integers $\mathbb{Z}$ includes all positive and negative whole numbers, as well as zero, like this

$$Z = \{....., -3, -2, -1, 0, 1, 2, 3, ...\}$$

- **D** is the set of Decimal numbers : $D = \{\frac{p}{10^n} \,|\, p \in \mathbb{Z}, n \in \mathbb{N}\}$.

  Example 1.1. $1.234 = \frac{1234}{10^3}$ is a decimal number.

- **Q** is the set of Rational numbers : $\mathbb{Q} = \{\frac{p}{q} \,|\, p \in \mathbb{Z}, q \in \mathbb{Z}^*\}$.
  Rational numbers are numbers that can be expressed as the quotient of two integers, i.e., a fraction, with a non-zero denominator. Note that all terminating or repeating decimals (also known as periodic decimal expansions) are rational numbers.

  Example 1.2. $\frac{3}{5} = 0.6$ (terminating decimals).

$$\frac{1}{3} = 0.33333...and \quad 1.179\mathbf{325325325}... \quad \text{(repeating decimals)}$$

- **R** is the set of Real numbers, that can be represented by any decimal expansion, limited or not.

  Example 1.3. 123.10100010000100001.........

- **R****Q** is the set of Irrational numbers, which are real numbers that cannot be expressed as the quotient of two integers. In other words, an Irrational number is a real number that cannot be written in the form $\frac{p}{q}$, where p and q are integers, and $q \neq 0$.

  Example 1.4. $-\sqrt{2}$, π, and e.
  These numbers have decimal expansions that are non-terminating and non-repeating.

- **C** is the set of Complex numbers is defined as :

$$\mathbf{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

  Where i is the imaginary unit satisfying $i^2 = -1$.
  Recall that a complex number is formed by adding a real number a to a real b multiple of i, where $i^2 = -1$. The real number a is called the real part, and the real number b is called the imaginary part of the complex number.

Remark 1.1. We have

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{D} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

Lemma 1.1. A number is rational if and only if it admits a periodic or finite decimal writing.

Proof. The direct implication ($\Rightarrow$) is based on Euclid's Division. For the converse ($\Leftarrow$) let's examine an example to illustrate how it works :

Let us show that $x = 12,34\underline{202120212021}$... is rational number.
Here, the repeating decimal starts two digits after the decimal point ; therefore, we multiply by 100 :

$$100.x = 1234,\underline{202120212021}..... \quad (1)$$

Now, we will shift everything to the left by the length of one period. Therefore, we multiply by 10000 to shift the decimal point four digits to the left.

$$10000.100.x = 12342021,\underline{20212021}..... \quad (2)$$

The fractional parts after the decimal point in lines (1) and (2) are identical. Therefore, when we subtract equation (1) from equation (2), we obtain

$$10000.100.x - 100.x = 12342021 - 1234$$

so

$$999900.x = 12340787$$

therefore

$$x = \frac{12340787}{999900}.$$

So of course $x \in \mathbb{Q}$.

Example 1.5. Prove that $\sqrt{2}$ is not rational number.

Proof by Contradiction. Assume that $\sqrt{2}$ is a rational number. By definition, a rational number can be expressed as

$$\sqrt{2} = \frac{p}{q}, \quad p \in \mathbb{Z},\ q \in \mathbb{Z}^*,$$

where p and q have no common factors other than 1.
Squaring both sides of the equation

$$(\sqrt{2})^2 = \frac{p^2}{q^2} \Longrightarrow p^2 = 2q^2.$$

p^2 is even, then p is even. Thus, we can express p as :

$$p = 2p'$$

where p' is an integer. Substituting $p = 2p'$ into the equation $p^2 = 2q^2$:

$$4p'^2 = 2q^2 \Longrightarrow q^2 = 2p'^2.$$

The equation $q^2 = 2p'^2$ implies that q^2 is even, By the same reasoning as before, q must also be even. Thus, we can express q as :

$$q = 2m, \quad m \in \mathbb{Z}^*$$

We have both p and q are even, and have 2 as a common factor. However, this contradicts our initial assumption that p and q have no common factors other than 1. We conclude that $\sqrt{2}$ is not a rational number. Therefore, $\sqrt{2}$ is irrational.

1.2 Operations with Real numbers

There are several fundamental properties concerning the operations of addition and multiplication on real numbers that are essential in algebra.
The set of real numbers, denoted $\mathbb{R}$, is equipped with two internal operations : addition $(+)$ and multiplication $(\times)$, satisfying the following axioms :

- **Commutative property of addition** : $\forall (x,y) \in \mathbb{R}^2, x+y=y+x$
- **Associative property of addition** : $\forall (x,y,z) \in \mathbb{R}^3$,

$$(x+y)+z=x+(y+z)$$

- **Identity property of addition** : $\forall x \in \mathbb{R}, x+0=0+x=x$
- **Additive inverse property** : $\forall x \in \mathbb{R}, x+(-x)=(-x)+x=0$
- **Commutative property of multiplication** : $\forall (x,y) \in \mathbb{R}^2$,

$$x \times y = y \times x$$

- **Associative property of multiplication** :$\forall (x,y,z) \in \mathbb{R}^3$,

$$(x \times y) \times z = x \times (y \times z)$$

- **Identity property of multiplication** : $\forall x \in \mathbb{R}, x \times 1 = 1 \times x = x$
- **Multiplicative inverse property** : $\forall x \in \mathbb{R}^*, x \times \frac{1}{x} = \frac{1}{x} \times x = 1$
- **Distributive property** : $\forall (x,y,z) \in \mathbb{R}^3, x \times (y+z) = x \times y + x \times z.$

1.3 The field of real numbers

1.3.1 Commutative Field

The set of real numbers $\mathbb{R}$, equipped with the usual addition and multiplication operations, forms a **commutative field**.

Remark.1.2. Generally, any set, such as $\mathbb{R}$, whose elements satisfy the above properties is known as a field.

For example, the set of integers $\mathbb{Z}$ is not a field because it does not satisfy the property of multiplicative inverses. However, in $\mathbb{Z}$, only 1 and -1 have multiplicative inverses. But, there is no integer n such that $2 \times n = 1$, because $\frac{1}{2}$ is not an integer.

1.3.2 Totally Ordered Field

Proposition 1.1. $(\mathbb{R}, +, \times, \leq)$ is a totally order field.

Proposition 1.1. means that $\leq$ is a total order relation in $\mathbb{R}$; that satisfies the following properties for all elements x, y, and z in $\mathbb{R}$:

1. **Reflexivity** : $\forall x \in \mathbb{R},\ x \leq x$.
2. **Antisymmetry** : If $(x \leq y \text{ and } y \leq x) \Longrightarrow x = y$.
3. **Transitivity** : If $(x \leq y \text{ and } y \leq z) \Longrightarrow x \leq z$.
4. $\leq$ is **total relation** : $\forall x, y \in \mathbb{R}, (x \leq y)$ or $(y \leq x)$.

Remark 1.3. A relation R on a set A is called a total order if, for every pair of elements $x, y \in A$, is comparable.
This property is known as comparability or the trichotomy property, ensuring that every pair of elements in A is comparable under R.

1.3.3 Commutative Archimedean Field

A field F is said to be a **commutative Archimedean field** if :
- F is a commutative field,
- F satisfies the Archimedean property :

$$\forall x \in F,\ \exists n \in \mathbb{N} \text{ such that } x < n.$$

Proposition 1.2. $(\mathbb{R}, +, \cdot)$ is commutative Archimedean field.

From these axioms, many properties of $\mathbb{R}$ can be derived. Some examples are given in the next :
- $\forall x, y, z \in \mathbb{R},\ x \leq y \Longrightarrow x + z \leq y + z$
- $\forall x, y \in \mathbb{R},\ (x \leq y \text{ and } z \geq 0) \Longrightarrow x \times z \leq y \times z$

- $\forall x, y, z, t \in \mathbb{R},\ (x \leq y \text{ and } z \leq t) \Longrightarrow x + z \leq y + t$
- $x \leq y \Longrightarrow -x \geq -y$
- $x > 0 \Longrightarrow \frac{1}{x} > 0$
- $(x \leq 0) \wedge (y \geq 0) \Longrightarrow x.y \leq 0$
- $0 < x < y \Longrightarrow 0 < \frac{1}{y} < \frac{1}{x}$
- $\forall m \in \mathbb{N}^*, 0 < x < y \Longrightarrow 0 < x^m < y^m$.

1.4 Principle of Mathematical Induction

Mathematical induction is a method of proof used to show that a property holds for all natural numbers from a certain starting point. It involves three main steps :

1. For $n = 0$ or $n = 1$, prove that the property is true.
2. Assume that the property is true for n.
3. Use the induction hypothesis to prove that the property is true for $n + 1$.

Example 1.6. Prove that for all integers $n \geq 1$:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

For $n = 1$:

$$1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1 \quad \text{is true}$$

Assume the formula holds for n, i.e.,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

We must show that :

$$1 + 2 + \cdots + n + (n+1) = \frac{(n+1)(n+2)}{2}$$

Using the induction hypothesis :

$$
\begin{aligned}
1+2+\cdots+n+(n+1) &= \left(\frac{n(n+1)}{2}\right)+(n+1) \\
&= \frac{n(n+1)+2(n+1)}{2} \\
&= \frac{(n+1)(n+2)}{2}
\end{aligned}
$$

So the formula also holds for $n+1$.
By the principle of mathematical induction, the formula

$$1+2+\cdots+n = \frac{n(n+1)}{2}$$

is true for all integers $n \geq 1$.

1.5 Absolute value

Definition 1.1. The absolute value of a real number x, denoted as $|x|$, represents the distance of x from zero on the real number line, it is always positive.
Mathematical definition : Absolute value for a real number x is defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Example 1.7.

$$|-9| = 9, \quad |9| = 9, \quad |-\frac{2}{3}| = \frac{2}{3}, \quad \text{and} \quad |0| = 0.$$

1.5.1 R-Valued Field

The set of real numbers $\mathbb{R}$ is a **valued field**, i.e., a field equipped with an absolute value function :

$$|\cdot| : \mathbb{R} \to \mathbb{R}^+$$

Satisfying the following properties for all $x, y \in \mathbb{R}$:
- (Positivity) : $|x| \geq 0$ and $|x| = 0 \iff x = 0$

- (Multiplicatives) : $|xy| = |x||y|$

– (Triangle inequality) : $|x+y| \leq |x|+|y|$.

Properties 1.1. For all x, y, and $r \in \mathbb{R}$, where $r>0$. The following properties are satisfied

1. $|x| \geq 0, \quad |-x|=|x|, \quad x \leq |x|$, and $\quad |x|=0 \Longleftrightarrow x=0$

2. $\sqrt{x^2}=|x|$, and $\quad |x|^2=x^2$

3. $|x.y|=|x|.|y|$, and $\forall\ x \neq 0, \ |\frac{1}{x}|=\frac{1}{|x|}$

4. $|x| \leq r \Longleftrightarrow -r \leq x \leq r$ and $|x| \geq r \Longleftrightarrow x \geq r$ or $x \leq -r$

5. Triangle inequality : $\forall x, y \in \mathbb{R}, \quad |x+y| \leq |x|+|y|$.

   Indeed. $\forall x, y \in \mathbb{R}$, we have
   $$|x+y|^2=(x+y)^2=x^2+y^2+2xy$$
   and
   $$(|x|+|y|)^2=|x|^2+|y|^2+2|x||y|=x^2+y^2+2|x||y|.$$
   Apply the inequality $xy \leq |xy|$. Since $|xy|=|x||y|$, we have :
   $$2x.y \leq 2|x|.|y|.$$
   So
   $$|x+y|^2 \leq (|x|+|y|)^2.$$
   Take the square root of both sides, therefore
   $$|x+y| \leq |x|+|y|.$$

   Remark 1.4. Inequality becomes equality if x and y have the same sign.

6. Second triangle inequality : $||x|-|y|| \leq |x-y|$.

   Indeed. $\forall x, y \in \mathbb{R}$, we have
   $$|x|=|x-y+y| \leq |x-y|+|y|$$

$$\Longrightarrow |x| - |y| \leq |x - y| \qquad (1)$$

we have also

$$|y| = |y - x + x| \leq |y - x| + |x|$$

then

$$|y| - |x| \leq |y - x| = |x - y|$$

$$\Longrightarrow -|x - y| \leq |x| - |y| \qquad (2)$$

From (1), (2), and properties 4, we obtain

$$||x| - |y|| \leq |x - y|.$$

Training exercise 1.1.

1. Let the function $f(x) = |x-3|+|x+3|$, by writing f without the absolute value :

$$|x - 3| = \begin{cases} x - 3, & \text{if} \quad x \geq 3 \\ 3 - x, & \text{if} \quad x < 3 \end{cases}$$

furthermore

$$|x + 3| = \begin{cases} x + 3, & \text{if} \quad x \geq -3 \\ -3 - x, & \text{if} \quad x < -3 \end{cases}$$

So

$$f(x) = \begin{cases} -2x, & \text{if} \quad x < -3 \\ 6, & \text{if} \quad x \in [-3, 3[\\ 2x, & \text{if} \quad x \geq 3. \end{cases}$$

2. Solve the following equation

$$|4x + 8| - |x - 3| = 3.$$

Using definition of absolute value, then

$$|4x + 8| = \begin{cases} 4x + 8, & \text{if} \quad x \geq -2 \\ -4x - 8, & \text{if} \quad x < -2 \end{cases}$$

$$|x - 3| = \begin{cases} x - 3, & \text{if} \quad x \geq 3 \\ -x + 3, & \text{if} \quad x < 3 \end{cases}$$

Solve in each interval :

- $x < -2$. In this interval, both $4x+8$ and $x-3$ are negative. Substitute these into the original equation :

$$-4x - 8 + x - 3 = 3 \Longrightarrow x = \frac{-14}{3}.$$

Since $\frac{-14}{3} \simeq -4.67$, which is less than -2, this solution is valid in this interval.

- $-2 \leq x < 3$. In this interval $4x+8$ is positive and $x-3$ is negative. substitute these into the original equation, then

$$4x + 8 + x - 3 = 3 \Longrightarrow x = \frac{-2}{5}.$$

Since $\frac{-2}{5} \simeq -0.4$, which is between -2 and 3, this solution is valid in this interval.

- $x \geq 3$. In this interval, both $4x+8$ and $x-3$ are positive, substitute these into the original equation

$$4x + 8 - x + 3 = 3 \Longrightarrow x = \frac{-8}{3}.$$

Since $\frac{-8}{3} \simeq -2.67$, which is less than 3, this solution is not valid in this interval.

Therefore the solutions to the equation are : $x = \frac{-14}{3}$, and $x = \frac{-2}{5}$.

3. Solve the following inequality :

$$|x+2| > |3x+5|$$

Using definition of absolute value, then

$$|x+2| = \begin{cases} x+2, & \text{if} \quad x \geq -2 \\ -x-2, & \text{if} \quad x < -2 \end{cases}$$

and

$$|3x+5| = \begin{cases} 3x+5, & \text{if} \quad x \geq \frac{-5}{3} \\ -3x-5, & \text{if} \quad x < \frac{-5}{3} \end{cases}$$

We will analyze the inequality in each of these intervals.

- $x < -2$. In this interval, both $x+2$ and $3x+5$ are negative. Substitute these into the inequality, then

$$-x-2 > -3x-5 \Longrightarrow x > \frac{-3}{2}$$

Since $x < -2$ in this interval, there is no solution because $\frac{-3}{2}$ is greater than -2.

- $-2 \leq x < \frac{-5}{3}$. In this interval, $x+2 \geq 0$ and $3x+5 < 0$. Substitute these into the inequality, then

$$x+2 > -3x-5 \Longrightarrow x > \frac{-7}{4}$$

Since $\frac{-7}{4} \simeq -1.75$, which is greater than -2, the solution in this interval is

$$\frac{-7}{4} < x < \frac{-5}{3}.$$

- $x \geq \frac{-5}{3}$. In this interval, both $x+2$ and $3x+5$ are positive. Substitute these into the inequality, then

$$x+2 > 3x+5 \Longrightarrow x < \frac{-3}{2}$$

Since $x \geq \frac{-5}{3} \simeq -1.67$, which is less than $\frac{-3}{2}$, the solution in this interval is

$$\frac{-5}{3} \leq x < \frac{-3}{2}.$$

Combining the solutions from all intervals, the solution to the inequality is : $\frac{-7}{4} < x < \frac{-3}{2}$.

1.6 The greatest integer function

Definition 1.2. The greatest integer function of a real number x, denoted by $[x]$ is the largest integer value less than or equal to x. This is written as :

$$\begin{aligned} f: \quad \mathbb{R} &\longrightarrow \mathbb{Z} \\ x &\longrightarrow [x]. \end{aligned}$$

Example 1.8. $[1.65] = 1, \quad [0.016] = 0, \quad [-3.14] = -4, \quad [-1.96] = -2.$

Properties 1.2. $\forall x, y \in \mathbb{R}$, the following properties are satisfied :

1. $\forall x \in \mathbb{R}, \ \ [x] \in \mathbb{Z}$
2. $[x] \leq x < [x] + 1$, and $x - 1 < [x] \leq x$
3. $[x] = k \Longrightarrow x \in [k, k+1[$
4. $\forall x \in \mathbb{R}$, and $m \in \mathbb{Z}$, $[x+m] = [x] + m$
5. $\forall x, y \in \mathbb{R}$, if $x \leq y \Longrightarrow [x] \leq [y]$.

Proof.
4- For any $x \in \mathbb{R}$, we have

$$[x] \leq x \leq [x] + 1$$

which implies

$$[x] + m \leq x + m \leq [x] + m + 1,$$

for all $m \in \mathbb{Z}$. On the other hand

$$[x+m] \leq x + m \leq [x+m] + 1.$$

Since $[x+m]$ is the largest integer less than or equal to $x+m$, then

$$[x] + m \leq [x+m] \qquad (1)$$

Similarly, $[x+m]+1$ is the smallest integer greater than or equal to $x+m$, so

$$[x+m] + 1 \leq [x] + m + 1.$$

After simplifying, we obtain

$$[x+m] \leq [x] + m \qquad (2)$$

From (1) and (2), we conclude $[x+m] = [x] + m$.

5- Let $x, y \in \mathbb{R}$, and assume $x \leq y$. Let $m = [x]$ and $n = [y]$, then

$$m \leq x < m+1, \quad \text{and} \quad n \leq y < n+1,$$

so

$$m \leq x \leq y < n+1.$$

Thus, we have

$$m < n+1 \Longrightarrow m \leq n$$

because m and n are integers numbers. Therefore $[x] \leq [y]$.

Training exercise 1.2. Solve the following equation with greatest integer function :

a) $[x]^2 + [x+1] - 3 = 0$

b) $[\frac{x-1}{2}] = -2$

Solution.

a)

$$[x]^2 + [x+1] - 3 = 0$$

Using property 4, we obtain

$$[x]^2 + [x] + 1 - 3 = 0$$

Noted $[x] = y$, then the equation becomes

$$y^2 + y - 2 = 0$$

this equation admits two solutions $y = 1$ or $y = -2$.
Which implies $[x] = 1$ or $[x] = -2$. Therefore the solutions to the equation are : $x \in [-2, -1[\cup[1, 2[$.

b) According to property 3, we have

$$[\frac{x-1}{2}] = -2 \implies -2 \leq \frac{x-1}{2} < -1$$

which implies

$$-4 \leq x - 1 < -2 \Longrightarrow -3 \leq x < -1.$$

Therefore the solutions to the equation are : $x \in [-3, -1[$.

1.7 Intervals

Definition 1.3. A subset I of $\mathbb{R}$ is called an interval if :

$$\forall a \in I,\ \forall b \in I,\ \forall x \in \mathbb{R},\ (a \leq x \leq b) \Rightarrow x \in I.$$

Let a and b be two real numbers such that $a \leq b$.

The table below summarizes the types of bounded or unbounded intervals

Notation	**Type**	**Included Endpoints**	**Set Definition**
$]a, b[$	Open interval	None	$\{x \in \mathbb{R} \mid a < x < b\}$
$[a, b]$	Closed interval	a and b	$\{x \in \mathbb{R} \mid a \leq x \leq b\}$
$[a, b[$	Half-open (right)	a only	$\{x \in \mathbb{R} \mid a \leq x < b\}$
$]a, b]$	Half-open (left)	b only	$\{x \in \mathbb{R} \mid a < x \leq b\}$
$]a, +\infty[$	Infinite (open)	None	$\{x \in \mathbb{R} \mid x > a\}$
$[a, +\infty[$	Infinite (closed left)	a only	$\{x \in \mathbb{R} \mid x \geq a\}$
$]-\infty, b[$	Infinite (open)	None	$\{x \in \mathbb{R} \mid x < b\}$
$]-\infty, b]$	Infinite (closed right)	b only	$\{x \in \mathbb{R} \mid x \leq b\}$

1.8 Upper and Lower Bounds. Completeness Axioms

Definition 1.4. Let A be a subset of $\mathbb{R}$, a real number M is called an **upper bound** of A if

$$\text{for all } x \in A \quad \text{we have} \quad x \leq M, \quad .$$

If A has an upper bound, then we say that A is **bounded above**.

Example 1.9.

a) Let $A = [-1, 3[,\ \forall x \in A,\ x < 3$, then $M = 3$ is an upper bound of A.
Any real number $M' \geq 3$ is also an upper bound of A. So A is bounded above.

b) Let $A = \{x^2, -2 < x < 1\}$, $M = 4$ is an upper bound of A. Any real number $M' \geq 4$ is also an upper bound of A. So A is bounded above.

c) Let $A = \mathbb{N} = \{1, 2, 3, ...\}$. A has no upper bound. Therefore A is not bounded above.

Definition 1.5. Let A be a subset of $\mathbb{R}$, a real number m is called **lower bound** of A if

$$\text{for all } \ x \in A, \quad \text{we have} \quad x \geq m.$$

If A has a lower bound, then we say that A is **bounded below**.

Example 1.10.

a) Let $A = [-1, 3[$, for all $x \in A$, $x \geq -1$, then $m = -1$ is a lower bound of A. Any real number $m' \leq -1$ is also a lower bound of A. Therefore A is bounded below.

b) Let $A = \{x^2, -2 < x < 1\}$, for all $x \in A$, $x \geq 0$, then $m = 0$ is a lower bound of A. Any real number $m' \leq 0$ is a lower bound of A. Therefore A is bounded below.

Definition 1.6. A subset A of $\mathbb{R}$ is said to be bounded, if it is both bounded above and bounded below.

Example 1.11.

a) $A = [-1, 3[$, it has un upper bound and a lower bound, then is bounded.

b) $A = \{x^2, -2 < x < 1\}$, it has un upper bound and a lower bound, then is bounded.

c) $A = \mathbb{N} = \{1, 2, 3, ...\}$, it has no upper bound, then A is not bounded.

Definition 1.7. If M is an upper bound of A and $M \in A$, then M is called the maximum of A, denoted by $\max A$.

Definition 1.8. If m is a lower bound of A and $m \in A$, then m is called the minimum of A, denoted by $\min A$.

Example 1.12.

a) Let $A = [-1, 3[$. The maximum of A does not exist, and $\min A = -1$.

b) Let $A = \{x^2, -2 \leq x \leq 1\}$, then $\max A = 4, \min A = 0$.

c) Let $A = \mathbb{N} = \{1, 2, 3, ...\}$, then $\min A = 0$, the maximum of A does not exist.

Definition 1.9. Let A be a nonempty subset of $\mathbb{R}$ that is bounded above, we say that α is the **supremum** of A if α is the smallest upper bound of A, and we denote it by $\sup A$.

Definition 1.10. Let A be a nonempty subset of $\mathbb{R}$ that is bounded above. Then $\alpha = \sup A$ if and only if :

i) $x \leq \alpha$ for all $x \in A$

ii) If M is an upper bound of A, then $\alpha \leq M$.

Remark 1.5. That means α is the smallest of all upper bounds of A.

Proposition 1.3. Let A be a nonempty subset of $\mathbb{R}$ that is bounded above. Then $\alpha = \sup A$, if and only if the following conditions are satisfied :

i) $x \leq \alpha$ for all $x \in A$

ii) For every $\varepsilon > 0$, there exists $a \in A$ such that $\alpha - \varepsilon < a$.

Remark 1.6.

- If A is nonempty and bounded above, then exists $\alpha = \sup A \in \mathbb{R}$
- If A is nonempty and not bounded above, then $\sup A = \infty$
- If $A = \emptyset$, then $\sup A = -\infty$. Any real number is an upper bound of $\emptyset$.

Example 1.13

a) Let $A =]-\infty, 2[$, then $\sup A = 2$

b) Let $A = \{x^2, -2 < x < 1\}$, then $\sup A = 4$.

c) Let $A = \mathbb{N}$, then $\sup A$ does not exist.

Definition 1.11. Let A be a nonempty subset of $\mathbb{R}$ that is bounded below. We say that β is the infimum of A if β is the largest lower bound of A, and we denote it by $\inf A$.

Definition 1.12. Let A be a nonempty subset of $\mathbb{R}$ that is bounded below. Then $\beta = \inf A$ if and only if :

a) $\beta \leq x$ for all $x \in A$

b) If m is a lower bound of A, then $m \leq \beta$.

Proposition 1.4. Let A be a nonempty subset of $\mathbb{R}$ that is bounded below. Then $\beta = \inf A$ if and only if the following conditions satisfied :

i) $\beta \leq x$ for all $x \in A$

ii) For every $\varepsilon > 0$, there exists $b \in A$ such that $b < \beta + \varepsilon$.

Remark 1.7.

- When the supremum (respectively, the infimum) exists, it is unique.
- The supremum of A (respectively, the infimum) does not necessarily belong to the set A.
- If the maximum of A (respectively, the minimum of A) exists, then $\sup A = \max A$ (respectively $\inf A = \min A$).
- If the supremum of A (respectively , the infimum of A) belongs to A, then $\max A = \sup A$ (respectively, $\min A = \inf A$).
- If the supremum of A (respectively, the infimum of A) does not belong to A, then $\max A$ (respectively, $\min A$) does not exist.

Properties 1.3.

1. Let A and B be two nonempty bounded subsets of $\mathbb{R}$ such that $A \subset B$. Then :

$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

Indeed :

We know that for all $x \in A$, we have :

$$\inf A \leq x \leq \sup A \Rightarrow \inf A \leq \sup A.$$

Also, since every element $x \in A$ also belongs to B, it follows that $\inf B \leq x$ for all $x \in A$. This means that $\inf B$ is a lower bound of A. But $\inf A$ is the greatest lower bound of A, so :

$$\inf B \leq \inf A.$$

Similarly, since every $x \in A$ is in B, we have $x \leq \sup B$ for all $x \in A$. Thus, $\sup B$ is an upper bound of A. But $\sup A$ is the least upper bound of A, so :

$$\sup A \leq \sup B.$$

2. Let A and B be two nonempty bounded subsets of $\mathbb{R}$, so

$$\sup(A \cup B) = \max(\sup A, \sup B)$$

and

$$\inf(A \cup B) = \min(\inf A, \inf B).$$

Example 1.14.
Let $A = [0, 1], B = [-1, 2]$, it is clear that $A \subset B$.

$$\sup A = 1 \quad \sup B = 2, \quad \inf A = 0, \quad \inf B = -1.$$

Note that

$$\sup A \leq \sup B, \quad \inf B \leq \inf A.$$

Example 1.15. Find sup, inf, max, and min of the following subsets

1. $A =]-1, 3[\cup]4, 8[$

$$\sup A = \max(\sup(]-1, 3[), \sup(]4, 8[)) = \max(3, 8) = 8,$$
$$\inf A = \min(\inf(]-1, 3[), \inf(]4, 8[)) = \min(-1, 4) = -1,$$
$$\max A \ \text{ and } \ \min A \ \text{ does not exists.}$$

2. $B = \{\frac{n+1}{n};\ n \in \mathbb{N}^*\} = \{1 + \frac{1}{n},\ n \in \mathbb{N}^*\}$
it is clear that

$$\forall\ n > 0,\ 1 < 1 + \frac{1}{n} \leq 2.$$

Therefore

$$\sup B = \max B = 2$$
$$\inf B = 1, \ \text{ and } \ \min B \ \text{ does not exists.}$$

3. $C = \{(\sin x + \cos x)^2, \ 0 \leq x \leq \pi\}$
Assume that

$$y = (\sin x + \cos x)^2 = \sin^2 x + \cos^2 x + 2 \sin x . \cos x$$

so

$$y = 1 + \sin(2x)$$

we have $0 \leq x \leq \pi$, then

$$-1 \leq \sin(2x) \leq +1$$

$$0 \leq 1 + \sin(2x) \leq 2.$$

Therefore

$$\sup C = \max C = 2$$

and

$$\inf C = \min C = 0.$$

Training exercise 1.3. Suppose that A and B are subset of $\mathbb{R}$ nonempty and bounded from above. Define

$$A + B = \{a + b, \ \ a \in A \ \text{ and } \ b \in B.\}$$

Prove that $A + B$ is bounded from above and $\sup(A + B) = \sup A + \sup B$.
Answer :
Let $\alpha = \sup A$ and $\beta = \sup B$.
Take any $x \in A + B$, then there exist $a \in A$ and $b \in B$ such that $x = a + b$.
Since $\alpha = \sup A$, $\beta = \sup B$ then

$$a \leq \alpha \ \text{ and } \ b \leq \beta$$

It follows that $\quad x = a + b \leq \alpha + \beta$, then $\alpha + \beta$ is an upper bound of $A + B$, and so $A + B$ is bounded from above.
$\sup(A + B) = \alpha + \beta$ if

- $x \leq \alpha + \beta$ for all $x \in A + B$

- For any $\varepsilon > 0$, then exist $u \in A + B$ such that $\alpha + \beta - \varepsilon < u$.

Take any ε, since $\alpha = \sup A$, there exist $a \in A$ such that $\alpha - \frac{\varepsilon}{2} < a$.
Similarly, since $\beta = \sup B$, there exist $b \in B$ such that $\beta - \frac{\varepsilon}{2} < b$.
Then

$$(\alpha + \beta) - \varepsilon < a + b,$$

Let $u = a + b \in A + B$ and $(\alpha + \beta) - \varepsilon < u$

$$\sup(A + B) = \alpha + \beta = \sup A + \sup B.$$

1.9 Archimedean property

The completeness axiom implies the Archimedean property, which states that every real number is strictly less than some natural number.

Theorem 1.1. (Archimedean property for $\mathbb{R}$). For each $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $x < n$.

Proof. Assume, for contradiction, that there exists a real number $x \in \mathbb{R}$ such that,

$$x \geq n \text{ for all } n \in \mathbb{N}.$$

Thus, $\mathbb{N} \subset \mathbb{R}$ is bounded above. Hence, by the completeness axiom, $\sup(\mathbb{N}) = \alpha$ exists. Now because $\alpha - 1 < \alpha$ there is an $m \in \mathbb{N}$ such that $\alpha - 1 < m$. Therefore, $\alpha < m + 1 = n \in \mathbb{N}$; contradicting the fact that α is an upper bound for $\mathbb{N}$. This contradiction completes the proof.

In the next theorem, we show that the Archimedean property implies two useful results.

Theorem 1.2. Each of the following statements holds :

a) For all $x \in \mathbb{R}$ and $y \in \mathbb{R}$, if $x > 0$, then there exists $n \in \mathbb{N}$ such that

$$y < nx$$

b) For all $x \in \mathbb{R}$, if $x > 0$, then there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < x$.

Proof. We begin by proving statement (a).
Let $x, y \in \mathbb{R}$ where $x > 0$. Consider the real number $\frac{y}{x}$. By Theorem 1.1, there is an $n \in \mathbb{N}$ such that

$$\frac{y}{x} < n$$

Multiplying both sides of this inequality by x, we obtain

$$y < nx.$$

Thus, we have shown that for $x > 0$, there exists an $n \in N$ such that $y < nx$. The proof of (a) is complete.

Now, to prove statement (b), let $x > 0$. From (a) where we take $y = 1$, we conclude that there exists $n \in \mathbb{N}$ such that

$$0 < 1 < nx.$$

Thus, we have

$$0 < \frac{1}{n} < x.$$

This completes the proof of (b).

1.10 The Density of the Rational Numbers

Definition 1.13. Let $A \subset \mathbb{R}$. We say that A is dense in $\mathbb{R}$ if, for all $x, y \in \mathbb{R}$ with $x < y$, there exists $a \in A$ such that $x < a < y$.

Theorem 1.3. (Density of $\mathbb{Q}$ in $\mathbb{R}$).
For all $x, y \in \mathbb{R}$, if $x < y$, then there exists a $q \in \mathbb{Q}$ such that $x < q < y$.

Proof. Let $x, y \in \mathbb{R}$ with $x < y$. Then $y - x > 0$.
By the Archimedean Property (Theorem 1.2(a)), there exists $n \in \mathbb{N}$ such that

$$\frac{1}{n} < y - x.$$

Multiplying both sides by n, we get :

$$1 < n(y - x) \quad \Rightarrow \quad nx < ny - 1.$$

Since $nx < ny - 1$, and the set of integers $\mathbb{Z}$ is dense in $\mathbb{R}$, there exists an integer $m \in \mathbb{Z}$ such that :

$$nx < m < ny.$$

Dividing the inequality by n, we obtain :

$$x < \frac{m}{n} < y.$$

Let $q = \frac{m}{n} \in \mathbb{Q}$. Then $x < q < y$, as desired.

1.11 Extend Real Line

Definition 1.14. The **extended real line**, denoted by $\overline{\mathbb{R}}$, is the set :

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$$

Properties 1.4.

1. $\forall x \in \overline{\mathbb{R}}, -\infty \leq x \leq +\infty.$

2. $\forall x \in \mathbb{R}, x + (+\infty) = (+\infty) + x = +\infty$, and $x + (-\infty) = (-\infty) + x = -\infty$
   $(+\infty) + (+\infty) = (+\infty)$, $(-\infty) + (-\infty) = (-\infty)$

3. $\forall x > 0, x(+\infty) = +\infty$, and $x(-\infty) = -\infty$

4. $\forall x < 0, x(+\infty) = -\infty$, and $x(-\infty) = +\infty$

5. $(+\infty) \cdot (+\infty) = +\infty$, $(-\infty) \cdot (-\infty) = +\infty$
   $(+\infty) \cdot (-\infty) = (-\infty) \cdot (+\infty) = -\infty$

6. $\forall x \in \mathbb{R}, \frac{x}{+\infty} = \frac{x}{-\infty} = 0.$

Corollary 1.1. Every nonempty subset of the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ has a supremum and an infimum in $\overline{\mathbb{R}}$.

Example 1.16. Let $A = \{x \in \mathbb{R} : x > 100\} \subset \overline{\mathbb{R}}$.
Then :

$$\inf A = 100, \quad \sup A = +\infty$$

Thus, A has both a supremum and an infimum in $\overline{\mathbb{R}}$, even though it is not bounded above in $\mathbb{R}$.

Properties 1.5.

- $\overline{\mathbb{R}}$ is not a field because arithmetic operations like $\infty - \infty$ or ∞/∞ are undefined.
- The total order on $\mathbb{R}$ can be extended to $\overline{\mathbb{R}}$. This set can be equipped with a total order defined by :

  $$-\infty \leq x \leq +\infty \quad \text{for all } x \in \overline{\mathbb{R}}.$$

  Thus, $\overline{\mathbb{R}}$ is a totally ordered set, but not a field.

– $\overline{\mathbb{R}}$ is compact. Every sequence in $\mathbb{R}$ that diverges has a limit in $\overline{\mathbb{R}}$ (possibly $\pm\infty$).

1.12 Topological Properties of Real numbers

Definition 1.15. Neighborhood in $\mathbb{R}$
Let $V \subseteq \mathbb{R}$ be a nonempty set and let $a \in \mathbb{R}$. We say that V is a **neighborhood** of a if there exists an open interval I such that :

$$a \in I \quad \text{and} \quad I \subseteq V.$$

In other words, a set V is a neighborhood of a point a if it contains an open interval around a.

Example 1.17. The interval $[-1, 3]$, and $]-2, 1]$ are two neighborhoods of 0, because

$$0 \in]-1, 3[\subset [-1, 3]$$

and

$$0 \in]-2, 1[\subset]-2, 1]$$

but $[0, 1]$ is not a neighborhood of a point 0.

Definition 1.15. Open Subsets of $\mathbb{R}$
A subset $A \subseteq \mathbb{R}$ is said to be open if it is a neighborhood of each of its points. That is, for every $a \in A$, there exists an open interval $I \subseteq \mathbb{R}$ such that :

$$a \in I \subseteq A.$$

Example 1.18. Open interval $]a, b[$ is open set of $\mathbb{R}$. Indeed

$$\forall x \in]a, b[: \ x \in]a, b[\subset]a, b[.$$

Remark 1.8. An arbitrary union of open subsets of $\mathbb{R}$ is an open subset of $\mathbb{R}$.

Definition 1.16. Closed Subsets of $\mathbb{R}$
A subset $A \subseteq \mathbb{R}$ is said to be closed if its complement $\mathbb{R} \setminus A$ is open.

Equivalently, A is closed if it contains all its limit points. That is, if $(x_n) \subseteq A$ and $\lim_{n\to\infty} x_n = \ell \in \mathbb{R}$, then $\ell \in A$.

Example 1.19. Closed interval $]a,b[$ is closed set of $\mathbb{R}$. Indeed

$$[a,b]^c = [-\infty, a[\cup]b, +\infty[.$$

Definition 1.17. Compact subset $A \subset \mathbb{R}$ is a set that is
– closed
– bounded.

This result follows from the **Heine-Borel Theorem** :

Theorem 1.4. A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.

Example 1.20.
– The closed interval $[a,b]$ with $a \leq b$ is compact.
– A finite set such as $\{1, \sqrt{2}, \pi\}$ is compact.
– The set $\left\{\frac{1}{n} \mid n \in \mathbb{N}^*\right\} \cup \{0\}$ is compact.
– The open interval $]a,b[$: not closed, then not compact
– The set $[a,+\infty)$: not bounded, then not compact
– $\mathbb{R}$: not bounded, then not compact.

1.13 Exercises

Exercise 1.1. Prove that

1. $\sqrt[3]{45+29\sqrt{2}}+\sqrt[3]{45-29\sqrt{2}}$ is integers number.
2. If $r \in \mathbb{Q}$ and $x \notin \mathbb{Q}$ then $r+x \notin \mathbb{Q}$, and if $r \neq 0$ then $rx \notin \mathbb{Q}$.
3. $\sqrt{2}$ is irrational number, and deduce that $\sqrt{7}+\sqrt{2} \notin \mathbb{Q}$.
4. $\frac{\ln 3}{\ln 2}$ is not rational number.
5. Deduce that $\sqrt{18} \notin \mathbb{Q}$.

Exercise 1.2.

1. Let $(a,b) \in \mathbb{Q}^+ \times \mathbb{Q}^+$ such that $\sqrt{ab} \notin \mathbb{Q}$. Prove that $\sqrt{a}+3\sqrt{b} \notin \mathbb{Q}$.
2. Knowing that if m is prime then $\sqrt{m}$ is irrational, show that $\sqrt{5}+\sqrt[3]{2}$ is irrational.

Exercise 1.3.

1. Prove the following relations
   a) $|x+y| = |x|+|y| \Longleftrightarrow x.y \geq 0$.
   b) $\sqrt{x^2+y^2} \leq |x|+|y|$
   c) $\sqrt{x+y} \leq \sqrt{x}+\sqrt{y}, \quad \forall x,y \in \mathbb{R}^+$.
   d) $|\sqrt{x}-\sqrt{y}| \leq \sqrt{|x-y|}$

2. Solve the following equations in $\mathbb{R}$
   a) $||x+2|-|x-4|| = 2$

   b) $|\frac{2x-1}{x+1}| = 1$

Exercise 1.4.

1. Let $[x]$ be the integer part of x ; to show that $\forall x \in \mathbb{R}$

   a) $x \leq y \Longrightarrow [x] \leq [y]$.

b) $[x]+[y]\leq[x+y]\leq[x]+[y]+1$

2. Solve the following equations :

a) $x^2-4x=[x]$ in the interval $[0,2]$

b) $[x]+|x-1|=x$

3. Calculate $\lim\limits_{n\longrightarrow+\infty}\frac{1}{n^2}([x]+[2x]+[3x]+...+[nx])$.

Exercise 1.5.

Determine the sup, inf, max, and min of the following parts of $\mathbb{R}$.

$$A=]-5,5],\quad B=[-1,1]\cup]2,4[,\quad C=\{\frac{5}{n};n\in\mathbb{N}^*\}$$

$$D=\{-\frac{n+1}{n};\ n\in\mathbb{N}^*\},\quad E=\{(-1)^n+\frac{1}{n},\ n\in\mathbb{N}^*\}\},\quad F=\{x\in\mathbb{R}/3x^2+8x-3<0\},$$

$$G=\{\ \frac{2x-1}{x+4};\ |x-5|<2\}.$$

Exercise 1.6. :

1. Let A be a nonempty subset of $\mathbb{R}$. define $-A=\{-x,\ x\in\mathbb{R}\}$.
   a) Prove that if A is bounded below then $-A$ is bounded above.

   b) Prove that if A is bounded below then $\inf A=-\sup(-A)$.

2. Prove that if $B=\{\varepsilon x,x\in A\}$ then $\sup B=\varepsilon\sup A$.

3. Prove that $\sup(A\cup B)=\max(\sup A,\sup B)$.

Exercise 1.7.

Determine the sup, inf, max, and min of the following parts of $\mathbb{R}$.

$$A=[-1,\sqrt{2}]\cap\mathbb{Q},\qquad B=\{(-1)^n+\frac{3}{n^2},\ n\in\mathbb{N}^*\},\qquad C=\{\frac{m+n}{m.n},\ m,n\in\mathbb{N}^*\}.$$

Chapter 2

The Field of Complex Numbers

This section presents the fundamental concepts of complex numbers, such as operations on complex numbers, their representation in various forms, and an introduction to solving equations of the form $z^n = c$.

2.1 Algebraic expression of a Complex Number

A complex number is represented by $z = a + ib$, where $a \in \mathbb{R}$ is the real part, $b \in \mathbb{R}$ is the imaginary part, and i is the imaginary unit defined by $i^2 = -1$.
We can write

$$z = \mathrm{Re}(z) + i\,\mathrm{Im}(z)$$

This form is called the algebraic (or Cartesian) expression of the complex number z. The set of all complex numbers is denoted by $\mathbb{C}$.

Notes :

- If $b = 0$, then z is a real number.

- If $a = 0$, then z is a purely imaginary number.

Properties 2.1. Let z be a complex number. We have the following properties :

1. If $z = 0$, then $a = b = 0$
2. If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ then $z_1 = z_2 \Longleftrightarrow a_1 = a_2$ and $b_1 = b_2$.

2.2 Operations on Complex numbers

In this section, we present the basic operations on complex numbers, including addition, subtraction, multiplication, and division.

1. The complex conjugate of a complex number $z = a + ib$ is given by $\overline{z} = a - ib$.

   Example 2.1.
$$z_1 = -9 + i3, \text{ then } \overline{z}_1 = -9 - i3,$$
$$z_2 = \frac{1}{5} - i\frac{2}{3}, \text{ then } \overline{z}_2 = \frac{1}{5} + i\frac{2}{3}.$$

2. Addition, subtraction, and multiplication follow the same rules as for polynomials, except that after multiplication, one must simplify by using $i^2 = -1$.

   Example 2.2.
$$(6 + i3) + (-2 - i) = (6 - 2) + i(3 - 1) = 4 + i2.$$
$$(-2 + 5i) - (3 - i) = (-2 - 3) + i(5 + 1) = -5 + i6$$
$$(2 + 5i)(1 - 3i) = 2 - i - 15i^2 = 17 - i.$$

   To divide z by w, multiply $\frac{z}{w}$ by $\frac{\overline{w}}{\overline{w}}$ so that the denominator becomes a real number.

   Example 2.3.
$$\frac{2 + 3i}{1 - 5i} = \frac{2 + 3i}{1 - 5i} \times \frac{1 + 5i}{1 + 5i} = \frac{-13 + 13i}{26} = -\frac{1}{2} + i\frac{1}{2}.$$

Remark 2.1.

- The arithmetic operations on complex numbers satisfy the same properties as those on real numbers, such as commutativity and associativity. For example :
$$z \cdot w = w \cdot z, \quad z + w = w + z, \quad (z + w) + u = z + (w + u), \quad \text{etc.}$$
- For any complex number z and any integer n, the power z^n is defined as follows :
  - If $n > 0$, then z^n is the product of z multiplied by itself n times :
$$z^n = \underbrace{z \cdot z \cdot \ldots \cdot z}_{n \text{ factors}}.$$
  - If $n = 0$, then $z^0 = 1$, provided $z \neq 0$.
  - If $n < 0$, then z^n is defined as the reciprocal of z^{-n}, provided $z \neq 0$:
$$z^n = \frac{1}{z^{-n}}.$$

2.2.1 Modulus of complex number

The modulus $|z|$ of a complex number $z = a + ib$ is defined as

$$|z| = \sqrt{a^2 + b^2}.$$

Example 2.4.
Calculate the modulus of complex number $z = 4 - 2i$.

$$|z| = \sqrt{(4)^2 + (-2)^2} = \sqrt{20} = 2\sqrt{5}.$$

Properties 2.2. For all $z, z_1, z_2 \in \mathbb{C}$, the following properties hold :

1. $|z| = 0 \Longleftrightarrow z = 0$
2. $z = \overline{\overline{z}}$
3. $|z| = |\overline{z}| = |-z| = |-\overline{z}|$
4. $|z_1 + z_2| \leq |z_1| + |z_2|$ (Triangle inequality)
5. $|z_1 z_2| = |z_1| \cdot |z_2|$
6. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad \text{for } z_2 \neq 0$
7. $z \cdot \overline{z} = |z|^2$
8. z is purely real $\Longleftrightarrow z = \overline{z}$
9. z is purely imaginary $\Longleftrightarrow z = -\overline{z}$
10. $\mathrm{Re}(z) = \frac{z+\overline{z}}{2}, \quad \mathrm{Im}(z) = \frac{z-\overline{z}}{2i}$

2.3 Geometric representation of Complex numbers

The complex plane is the plane formed by all complex numbers, equipped with a Cartesian coordinate system. The horizontal axis, called the **real axis**, represents the real part of the complex number, while the vertical axis, called the **imaginary axis**, represents the imaginary part.
A complex number $z = a+ib$ can be represented in the complex plane as the ordered pair (a, b).
Alternatively, z can be viewed as a vector $\vec{OP}$, where the initial point O is the origin and the terminal point P is the point (a, b) in the plane.

– The modulus of z is the length of the vector $\vec{OP}$:

$$|z| = OP = \sqrt{a^2 + b^2}.$$

Example 2.5. If $z = 3 + 2i$, then

$$|z| = \sqrt{3^2 + 2^2} = \sqrt{13}.$$

– The argument of z, denoted $\arg(z)$, is the angle (measured in radians or degrees) between the positive real axis and the vector $\vec{OP}$, taken in the counterclockwise direction.

2.4 Trigonometric Form of a Complex Number

Given a nonzero complex number $z = a + ib$, we can express the point (a, b) in polar coordinates using r and θ, where $r = |z| = OP$ is the modulus of z, and $\theta = \arg(z)$ is the argument (angle) :

$$a = r\cos\theta, \quad b = r\sin\theta.$$

Substituting into the expression for z, we obtain :

$$z = a + ib = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta),$$

which is called the **polar form** or **trigonometric form** of the complex number.

Example 2.6. We want to express z in polar form : $z = r(\cos\theta + i\sin\theta)$.

1. Let $z = \sqrt{6} + i\sqrt{2}$.
   – First, compute the modulus :

$$r = |z| = \sqrt{(\sqrt{6})^2 + (\sqrt{2})^2} = \sqrt{6+2} = \sqrt{8} = 2\sqrt{2}.$$

   – Next, compute the argument :

$$\begin{cases} \cos(\theta) = \frac{\sqrt{6}}{2\sqrt{2}} = \frac{\sqrt{3}}{2} \\ \sin(\theta) = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2} \end{cases} \implies \theta = \frac{\pi}{6}.$$

   – Therefore, the polar form of z is :

$$z = 2\sqrt{2}\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right).$$

2. Let $z = -3 + 3i$.
   – First, compute the modulus :

$$r = |z| = \sqrt{(-3)^2 + 3^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}.$$

   – Next, compute the argument θ :

$$\begin{cases} \cos(\theta) = -\frac{3}{3\sqrt{2}} = -\frac{\sqrt{2}}{2} \\ \sin(\theta) = \frac{3}{3\sqrt{2}} = \frac{\sqrt{2}}{2} \end{cases} \implies \theta = \frac{3\pi}{4}.$$

   – Therefore, the polar form of z is :

$$z = 3\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right).$$

2.5 Exponential of Complex numbers

Definition 2.1. The complex exponential function is defined as e^z, where $z \in \mathbb{C}$. The number e is the base of the natural logarithm and is defined as the value of e^z at $z = 1$.

Properties 2.3. The complex exponential function e^z, where $z \in \mathbb{C}$, has the following properties :

1. For any complex numbers $z_1, z_2 \in \mathbb{C}$,

$$e^{z_1+z_2} = e^{z_1}e^{z_2}.$$

   This property is similar to the exponential law for real numbers.

2. For any complex number $z \in \mathbb{C}$ and any integer n,

$$e^{nz} = (e^z)^n.$$

3. The complex exponential function is periodic with a period of $2\pi i$. Specifically, for any $z \in \mathbb{C}$,

$$e^{z+2\pi i} = e^z.$$

   This property reflects the fact that the complex exponential function repeats itself after every multiple of $2\pi i$.

4. The derivative of e^z with respect to z is the same as the function itself :

$$\frac{d}{dz}e^z = e^z.$$

   This is a fundamental property of the complex exponential function, analogous to the derivative of the exponential function in the real case.

5. For any non-zero complex number z,

$$e^{\ln z} = z \quad \text{and} \quad \ln(e^z) = z.$$

   Note that the complex logarithm is multivalued, meaning that $\ln z$ can have multiple values, differing by integer multiples of $2\pi i$.

2.5.1 Euler's Formula

For any real number θ, Euler's formula states that

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This formula connects the complex exponential function with trigonometric functions, and it is fundamental in the study of complex numbers and their geometric interpretation.

More generally

$$e^{x+iy} = e^x.e^{iy} = e^x(\cos y + i\sin y) \quad \text{for all real numbers } x \text{ and } y.$$

Example 2.7. Let's compute an example to illustrate the concept. If $z = 1 + i\pi$, then :

$$e^{1+i\pi} = e^1(\cos\pi + i\sin\pi)$$
$$e^{1+i\pi} = e(\cos\pi + i\sin\pi)$$
$$e^{1+i\pi} = e(-1 + 0i)$$
$$e^{1+i\pi} = -e$$

Thus, $e^{1+i\pi} = -e$.

2.6 Exponential form of Complex numbers

The exponential form of a complex number is a way of representing the complex number using Euler's formula. Given a nonzero complex number $z = a + ib$, we can express z in polar form :

$$z = (r\cos\theta) + i(r\sin\theta)$$
$$z = r(\cos\theta + i\sin\theta)$$

using Euler's formula

$$z = re^{i\theta}.$$

Where $r = |z|$ is the modulus of the complex number, and $\theta = \arg(z)$ is the argument.

Example 2.8.

$$z = -3 + 3i \quad \text{Algebraic form}$$
$$z = 3\sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}) \quad \text{Polar form}$$
$$z = 3\sqrt{2}e^{i\frac{3\pi}{4}} \quad \text{Exponential form.}$$

Properties 2.4. Here are the main properties of complex numbers in exponential form :

- Two complex numbers $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ are equal if and only if :

$$r_1 = r_2 \text{ and } \theta_1 = \theta_2 + 2k\pi, \quad \text{for some integer k.}$$

- For $z = ei\theta$, then

$$\overline{z} = re^{-i\theta}$$

- Based on formula of Euler and that

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin(\theta).$$

  Since

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

- $|e^{i\theta}| = 1$, for every real number θ.

- If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then :

$$z_1 \cdot z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

  and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} \quad \text{for } z_2 \neq 0$$

– **Powers (De Moivre's Theorem)**

$$z^n = (re^{i\theta})^n = r^n e^{in\theta}$$

2.7 Nth roots of Complex numbers

To solve the equation $z^n = c$ for complex numbers, you can follow a structured approach using polar (or exponential) form of complex numbers.

– Express c in polar form, as follows $c = re^{i\theta}$.

– We assume z has polar form

$$z = r'e^{i\phi}$$

Raising both sides to the power n

$$z^n = (r'e^{i\phi})^n = r'^n e^{in\phi}$$

Setting this equal to c

$$r'^n e^{in\phi} = re^{i\theta}$$

which implies that

$$r' = \sqrt[n]{r}, \quad \text{and} \quad \phi = \frac{\theta + 2k\pi}{n}, \quad k \in \mathbb{Z}.$$

So

$$z = \sqrt[n]{r} e^{i(\frac{\theta+2k\pi}{n})}$$

Since complex arguments repeat every 2π, only n values of k (usually $0 \leq k < n$) yield distinct solutions.
Thus, the general form of the n^{th} roots of c is :

$$z_k = \sqrt[n]{r} \cdot e^{i\left(\frac{\theta+2\pi k}{n}\right)}, \quad \text{for } k = 0, 1, \ldots, n-1.$$

Example 2.9. Consider the equation $z^2 = 4i$.
In other words, we are trying to find the "square root of $4i$".
The number $4i$ can be written in polar form

$$4e^{i\frac{\pi}{2}}.$$

Now, we need to find the square roots of $4e^{i\frac{\pi}{2}}$. The formula for the square roots of a complex number $re^{i\theta}$ is

$$z_k = \sqrt{r} \cdot e^{i\left(\frac{\theta+2\pi k}{2}\right)}, \quad \text{for } k = 0, 1.$$

So

$$z_k = 2 \cdot e^{i\left(\frac{\frac{\pi}{2}+2\pi k}{2}\right)}, \quad \text{for } k = 0, 1.$$

Thus, the two possible values for z are :

$$z_1 = 2e^{i\frac{\pi}{4}}, \quad z_2 = 2e^{i\frac{5\pi}{4}}.$$

We can now convert these polar forms back into rectangular form

$$z_1 = 2(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})), \quad z_2 = 2(\cos(\frac{5\pi}{4}) + i\sin(\frac{5\pi}{4})).$$

2.8 Exercises.

Exercise 2.1.

1. Compute real and imaginary part of $z = \frac{i-4}{2i-3}$
2. Compute the absolute value and the conjugate of
$$z = (1+i)^6, \quad w = i^{17}.$$
3. Write in the "algebraic" form the following complex numbers
$$z = i^5 + i + 1, \quad w = (3+3i)^8$$
4. Write the given complex number in the algebraic form
$$2e^{i\frac{i\pi}{4}}, \quad \sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$$
5. Write in the "trigonometric" form the following complex numbers
$$z = 6i, \quad w = (\cos\frac{\pi}{3} - i\sin\frac{\pi}{3})^7$$
6. Compute the cube roots of $z = -8$.

Exercise 2.2.

1. Prove that $(1+i)^6 = -8i$
2. Deduce solution of equation (E) $:z^2 = -8i$
3. Write the two solutions of (E) in algebraic form, and in exponential form.
4. Find all $z \in \mathbb{C}$ such that $(\frac{z-1}{z+1})^2 = 2i$

Exercise 2.3.
Establish the following equalities :

1. $(\cos(\frac{\pi}{7}) + i\sin(\frac{\pi}{7}))(\frac{1-i\sqrt{3}}{2})(1+i) = \sqrt{2}(\cos(\frac{5\pi}{84}) + i\sin(\frac{5\pi}{84}))$
2. $(1-i)(\cos(\frac{\pi}{5}) + i\sin(\frac{\pi}{5}))(\sqrt{3}-i) = 2\sqrt{2}(\cos(\frac{13\pi}{60}) - i\sin(\frac{13\pi}{60}))$
3. $\frac{\sqrt{2}(\cos(\frac{\pi}{12})+i\sin(\frac{\pi}{12}))}{1+i} = \frac{\sqrt{3}}{2} - \frac{1}{2}i.$

Chapter 3

Sequences of Real Numbers

Suppose for each positive integer n, we are given a real number a_n. Then, the list of numbers $a_1, a_2, \dots, a_n, \dots$ is called a **sequence**, and this ordered list is usually written as (a_n) or $\{a_n\}$. We define a sequence as follows :

Definition 3.1. A sequence of real numbers is a function defined on the set $\mathbb{N}$, of natural numbers whose range is contained in the set $\mathbb{R}$ of real numbers.

$$u : \quad \mathbb{N} \longrightarrow \mathbb{R}$$

$$n \longrightarrow u_n.$$

- The value u_n is called the **general term** of the sequence $(u_n)_{n\in\mathbb{N}}$.

- The value u_0 is called the **first term** of the sequence.

- We also consider sequences $(u_n)_{n\in\mathbb{N}}$ that are defined only from a certain index n_0.

For example, the sequence with general term $u_n = \sqrt{n-2}$ is only defined for $n \geq 2$.

Remark 3.1. A sequence can be defined explicitly by a formula or implicitly by a recurrence relation :

- By an explicit formula, for the general term of the sequence (u_n) ; that is, expressing u_n directly in terms of n.

- By a recurrence relation, which defines each term of the sequence in terms of one or more of the preceding terms.

Example 3.1.

1. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence defined by :

$$u_n = 2 - \frac{1}{n+1}, \forall n \in \mathbb{N}$$

then $u_0 = 1$, $u_1 = \frac{3}{2}$, $u_2 = \frac{5}{3}, ...$

2. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence defined by :

$$u_{n+1} = \frac{1}{\sqrt{u_n + 1}}, \quad \text{and } u_0 = 1, \forall n \in \mathbb{N}^*$$

then $u_1 = \frac{1}{\sqrt{2}}$, $u_2 = \frac{1}{\sqrt{\frac{1}{\sqrt{2}}+1}}$,...

3.1 Bounded sequence

- A sequence $(u_n)_{n\in\mathbb{N}}$ is bounded above if and only if :

$$\exists\, M \in \mathbb{R}, \forall\, n \in \mathbb{N};\ u_n \leq M.$$

- A sequence $(u_n)_{n\in\mathbb{N}}$ is bounded below if and only if :

$$\exists\, m \in \mathbb{R}, \forall\, n \in \mathbb{N};\ u_n \geq m.$$

- A sequence is said to be bounded if it is both bounded above and bounded below, or if there exists $P \in \mathbb{R}^+$ such that

$$|u_n| \leq P, \quad \forall n \in \mathbb{N}.$$

Example 3.2.

1. For all $n \in \mathbb{N}$, $u_n = \sin(n)$. Then the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded. Indeed, $|u_n| \leq 1$ for all $n \in \mathbb{N}$.

2. The sequence $(u_n)_{n\in\mathbb{N}}$; where $u_n = n^3$ is bounded below by 0 but it is not bounded above. Therefore it is not bounded.

3. The sequence $(u_n)_{n\in\mathbb{N}}$; where $u_n = \frac{(-1)^n\times 4n}{\sqrt{n^2+1}}$ is bounded. Indeed,

$$|u_n| = |\frac{(-1)^n \times 4n}{\sqrt{n^2+1}}|.$$

   Since $|(-1)^n| = 1$ for all n, this simplifies to :

$$|u_n| = \frac{4n}{\sqrt{n^2+1}}.$$

   We now bound $|u_n|$ from above

$$|u_n| \leq \frac{4n}{\sqrt{n^2}} = 4.$$

   Therefore u_n is bounded.

3.2 Monotony of a Real Sequence

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.

- We say that u_n is increasing if $\forall n \in \mathbb{N}, u_{n+1} \geq u_n$,(i.e : $u_{n+1} - u_n \geq 0$)
- We say that u_n is decreasing if $\forall n \in \mathbb{N}, u_{n+1} \leq u_n$, (i.e : $u_{n+1} - u_n \leq 0$)
- We say that u_n is monotone if it is either increasing or decreasing.

Remark 3.2.

- If $(u_n)_{n\in\mathbb{N}}$ is a sequence with strictly positive terms, then it is increasing (respectively, decreasing) if and only if $\forall n \in \mathbb{N}$, $\frac{u_{n+1}}{u_n} \geq 1$ (respectively, $\forall n \in \mathbb{N}$, $\frac{u_{n+1}}{u_n} \leq 1$).
- We say that the sequence is strictly increasing, strictly decreasing, or strictly monotonic if the corresponding inequalities are strict.

Example 3.3.

1. For $u_n = \frac{n-1}{2n+1} + 3$, $\forall n \in \mathbb{N}$, the sequence $(u_n)_{n\in\mathbb{N}}$ is increasing. Indeed,

$$u_{n+1} - u_n = \frac{n}{2n+3} - \frac{n-1}{2n+1} = \frac{3}{(2n+3)(2n+1)} \geq 0,\ \forall n \in \mathbb{N}.$$

2. For $v_n = \frac{2^n}{(n)!}$, $\forall n \in \mathbb{N}^*$, the sequence $(v_n)_{n\in\mathbb{N}^*}$ is decreasing. Indeed, since $v_n > 0$, $\forall n \in N^*$, we compute

$$\frac{v_{n+1}}{v_n} = \frac{2^{n+1}}{(n+1)!}.\frac{(n)!}{2^n} = \frac{2}{n+1} \leq 1$$

for all $n \in N^*$, which shows that the sequence is increasing.

3. Let the sequence (u_n) be defined by :

$$\begin{cases} u_1 = \frac{1}{2}, \\ u_{n+1} = u_n^2 + \frac{3}{16} \end{cases}$$

- Prove that $\forall n \geq 1$, $\frac{1}{4} < u_n < \frac{3}{4}$.
- Determine whether the sequence u_n is decreasing .

Proof.

- For $n = 1$, we have

$$\frac{1}{4} < u_1 = \frac{1}{2} < \frac{3}{4}, \quad \text{is true}$$

Assume that $\frac{1}{4} < u_n < \frac{3}{4}$, for same $n \geq 1$. We will show that this implies

$$\frac{1}{16} < u_n^2 < \frac{9}{16} \Longleftrightarrow \frac{1}{16} + \frac{3}{16} < u_n^2 + \frac{3}{16} < \frac{9}{16} + \frac{3}{16}$$
$$\Longleftrightarrow \frac{1}{4} < u_{n+1} < \frac{3}{4}.$$

By mathematical induction

$$\forall n \geq 1, \ \frac{1}{4} < u_n < \frac{3}{4}.$$

- Monotonicity

$$u_{n+1} - u_n = u_n^2 + \frac{3}{16} - u_n = u_n^2 - u_n + \frac{3}{16}.$$

Define the function $f(x) = x^2 - x + \frac{3}{16}$, and compute the discriminant

$$\Delta = (-1)^2 - 4(1)(\frac{3}{16}) = \frac{1}{4}.$$

The roots are

$$x_1 = \frac{1}{4}, \quad x_2 = \frac{3}{4}.$$

Since $f(x) < 0$ between its roots, so

$$u_{n+1} < u_n \implies (u_n) \text{ is strictly decreasing.}$$

3.3 Lower limit and upper limit of a sequence

Definition 3.2. For each sequence (u_n), we define its upper limit $\overline{\lim} u_n$ and lower limit $\underline{\lim} u_n$ as follows. We set

$$q_n = \sup_{k \geq n} u_k, \quad \text{and} \quad p_n = \inf_{k \geq n} u_k,$$

We define

$$\overline{\lim} u_n = \inf_n q_n, \quad \text{and} \quad \underline{\lim} u_n = \sup_n p_n.$$

Theorem 3.1. Let $(u_n)_{n \in \mathbb{N}}$ be a real sequence

$$\lim_{n \longrightarrow +\infty} u_n = \ell, \quad \text{if and only if} \quad \overline{\lim} u_n = \underline{\lim} u_n = \ell.$$

Example 3.4. Let $u_n = \frac{1}{n}$. Calculate upper limit and lower limit of u_n

$$q_1 = \sup\{1, \frac{1}{2}, ..., \frac{1}{n}, ...\} = 1, \quad q_2 = \frac{1}{2}, ..., q_n = \frac{1}{n}.$$

Hence

$$\overline{\lim} u_n = \inf_n q_n = \inf\{1, \frac{1}{2}, ..., \frac{1}{n}, ...\} = 0$$

and

$$p_1 = \inf_{k \geq 1} \frac{1}{k} = \inf\{1, \frac{1}{2}, ..., \frac{1}{n}, ...\} = 0, \quad p_2 = \inf_{k \geq 2} \frac{1}{k} = 0, ..., p_n = 0.$$

Hence

$$\underline{\lim} u_n = \sup_n p_n = 0.$$

we have

$$\overline{\lim} u_n = \underline{\lim} u_n = 0$$

Therefore

$$\lim_{n \longrightarrow +\infty} u_n = 0.$$

3.4 Extracted Sequence (Subsequence)

Definition 3.3. We say that a sequence (v_n) is an extracted sequence or a subsequence of a sequence (u_n) if there is an application $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$ strictly increasing such that

$$\forall n \in \mathbb{N}, \quad v_n = u_{\varphi(n)}$$

Example 3.5. Let $(u_n)_{n\in N*}$ be a real sequence such that

$$u_n = \frac{(-1)^n}{n},$$

we can extract two subsequences $(u_{2n})_{n\in N*}$ and $(u_{2n+1})_{n\in N}$, we have

$$u_{2n} = \frac{1}{2n}, \ \ and \ \ u_{2n+1} = \frac{-1}{2n+1}$$

Remark 3.3. If φ is a strictly increasing application of $\mathbb{N}$ to $\mathbb{N}$, we have

$$\forall n \in \mathbb{N}, \ \ \varphi(n) \geq n.$$

Theorem 3.2. (Monotone Subsequence Theorem)
Every sequence of real numbers has a monotonic subsequence.

3.5 Convergence and Divergence of Sequence

Definition 3.4. A sequence of real numbers $(u_n)_{n\in\mathbb{N}}$ is said to converge to a real number ℓ if

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall n \geq N_\varepsilon, \ |u_n - \ell| < \varepsilon.$$

We denote this by :

$$\lim_{n\to+\infty} u_n = \ell.$$

We also say that ℓ is the limit of the sequence (u_n).

Example 3.6. We consider the sequence $(u_n)_{n\in\mathbb{N}}$ defined by :

$$u_n = \frac{2^n + (-1)^n}{2^n}.$$

Prove that the sequence (u_n) converges to 1, i.e.,

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall n \geq N_\varepsilon \Rightarrow |u_n - 1| < \varepsilon.$$

We compute :

$$|u_n - 1| = \left| \frac{2^n + (-1)^n}{2^n} - 1 \right|.$$

Let's simplify this

$$|u_n - 1| = \left|\frac{(-1)^n)}{2n}\right| = \frac{1}{2^n}.$$

So

$$|u_n - 1| < \varepsilon \Longleftrightarrow \frac{1}{2^n} < \varepsilon.$$

then

$$2^n > \frac{1}{\varepsilon} \Longleftrightarrow n > \frac{\ln(\frac{1}{\varepsilon})}{\ln(2)}$$

Thus, choosing $N_\varepsilon = [\frac{\ln(\frac{1}{\varepsilon})}{\ln(2)}] + 1$ ensures convergence.

Theorem 3.3. If a sequence $(u_n)_{n\in\mathbb{N}}$ converges, then its limit is unique.

Proof. Let's assume by contradiction that $(u_n)_{n\in\mathbb{N}}$ converges to two different limits ℓ_1 and ℓ_2 such that $\ell_1 \neq \ell_2$. Then we have

$$\lim u_n = \ell_1 \Longleftrightarrow \forall\varepsilon > 0, \exists N_{\varepsilon 1} \in \mathbb{N}, \forall n \geq N_{\varepsilon 1}, |u_n - \ell_1| < \frac{\varepsilon}{2}$$

and

$$\lim u_n = \ell_2 \Longleftrightarrow \forall\varepsilon > 0, \exists N_{\varepsilon 2} \in \mathbb{N}, \forall n \geq N_{\varepsilon 2}, |u_n - \ell_2| < \frac{\varepsilon}{2}$$

Let's note $N_\varepsilon = \max(N_{\varepsilon 1}, N_{\varepsilon 2})$, then for all $n > N_\varepsilon$, we have

$$|\ell_1 - \ell_2| = |(\ell_1 - u_n) + (u_n - \ell_2| \leq |(u_n - \ell_1)| + |(u_n - \ell_2| \leq \varepsilon$$

This leads to $|\ell_1 - \ell_2| < \varepsilon$. must hold. This implies that $|\ell_1 - \ell_2| = 0$, which contradicts the assumption that $\ell_1 \neq \ell_2$. Therefore, we conclude that $\ell_1 = \ell_2$, and the limit is unique.

Definition 3.5. A sequence is said to be divergent if it does not converge to a finite real number.

Example 3.7. There are two types of divergence :

1. **Divergence of infinite type :** In this case, the sequence tends to $+\infty$ or $-\infty$.
   For example, the sequence with general term $u_n = 2n + 4$ diverges to $+\infty$.
2. **Divergence of undefined-limit type :** In this case, the sequence has no finite or infinite limit.
   That is, the limit does not exist in the extended real line. For example, the sequence with general term $u_n = (-1)^n$.

Example 3.8. Study the convergence of the following sequence $u_n = (-1)^n$.
This sequence alternates between two values :

$$u_0 = 1, \quad u_1 = -1, \quad u_2 = 1, \quad u_3 = -1, \quad \dots$$

So we observe that

$$u_{2n} = 1, \text{ and } u_{2n+1} = -1.$$

Hence, the sequence does not approach a single value as $n \to \infty$. More precisely, the sequence does **not converge**.

Definition 3.6. Let a real sequence $(u_n)_{n\in\mathbb{N}}$.

– The sequence $(u_n)_{n\in\mathbb{N}}$ tends to $+\infty$ if

$$\forall A > 0, \exists N \in \mathbb{N}, \forall n \geq N \Longrightarrow u_n \geq A.$$

– The sequence $(u_n)_{n\in\mathbb{N}}$ tends to $-\infty$ if

$$\forall A > 0, \exists N \in \mathbb{N}, \forall n \geq N \Longrightarrow u_n \leq -A.$$

Theorem 3.4. If a sequence $(u_n)_{n\in N}$ converges to ℓ ; then the sequence $(|u_n|)_{n\in N}$ converges to $|\ell|$.

Proof. Let the sequence $(u_n)_{n\in\mathbb{N}}$ tends to ℓ. For all $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that :

$$\forall n \in \mathbb{N}, n \geq N, \ |u_n - \ell| \leq \varepsilon,$$

but we have, $||u_n| - |\ell|| \leq |u_n - \ell|$, we deduce that

$$\forall n \in \mathbb{N}, n \geq N, \ ||u_n| - |\ell|| \leq \varepsilon.$$

Therefore

$$\lim_{n\to+\infty} |u_n| = |\ell|.$$

Remark 3.4. The converse is not true. Indeed, consider the following example :

$$u_n = (-1)^n,$$

we have $|u_n| = |(-1)^n| = 1, \quad \forall n \in \mathbb{N}$, then $|u_n|$ converges to 1, but u_n is not convergent.

Theorem 3.5. Every convergent sequence is bounded.

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence that converges to a limit $\ell \in \mathbb{R}$. Then by virtue of the Theorem 3.4. $(|u_n|)_{n\in N}$ converges to $|l|$, in other words

$$\forall\varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq N,\ ||u_n| - |\ell|| \leq \varepsilon.$$

that's to say $|\ell| - \varepsilon \leq |u_n| \leq \varepsilon + |\ell|$. Let's choose $\varepsilon = 1$, then

$$\text{for } n \geq N,\ \ |u_n| \leq 1 + |\ell|.$$

By then posing

$$M = \max\{|u_1|, |u_2|, ..., |u_N|, 1 + |\ell|\}$$

we obtain

$$|u_n| \leq M,\ \forall n \in \mathbb{N}.$$

Therefore $(u_n)_{n\in\mathbb{N}}$ is bounded.

Remark 3.5. The converse is clearly false, as illustrated by the sequence $u_n = (-1)^n, \forall n \in \mathbb{N}$, which is bounded since $|(-1)^n| = 1,\ \forall n \in \mathbb{N}$, yet it does not converge.

Property 3.1. Let (a_n) be a bounded sequence and (b_n) a sequence such that

$$\lim_{n\to\infty} b_n = 0.$$

Then,

$$\lim_{n\to\infty} (a_n b_n) = 0.$$

Proof.
Since (a_n) is bounded, there exists a constant $M > 0$ such that

$$|a_n| \leq M \quad \text{for all } n \in \mathbb{N}.$$

Since $\lim_{n\to\infty} b_n = 0$, for any $\varepsilon > 0$, there exists a number $N \in \mathbb{N}$ such that for all $n > N$,

$$|b_n| < \frac{\varepsilon}{M}.$$

Thus, for all $n > N$,

$$|a_n b_n| = |a_n| \cdot |b_n| < M \cdot \frac{\varepsilon}{M} = \varepsilon.$$

Therefore, $\lim_{n\to\infty} a_n b_n = 0$.

Example 3.9. We consider the sequence $u_n = (-1)^n$, Recall that this sequence does not converge, but it is bounded. We also consider the sequence $v_n = \frac{1}{n}$, which converges to 0. According to the previous property 3.1, the product of these two sequences converges to 0. That is :

$$\lim_{n\longrightarrow+\infty} \frac{(-1)^n}{n} = 0.$$

3.6 Limits and inequalities

Theorem 3.6. (Gendarmes' Theorem)
Let $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$, and $(w_n)_{n\in\mathbb{N}}$ be real sequences. If $(u_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ both converge to the same limit ℓ, and if there exists n_0 such that $\forall n \geq n_0$,

$$u_n \leq v_n \leq w_n,$$

then the sequence v_n also converges to ℓ.

Proof.
Let $\varepsilon > 0$. Since $\lim\limits_{n\to+\infty} u_n = \ell$, $\exists n_1 \in \mathbb{N}$ such that $\forall n \geq n_1$,

$$|u_n - \ell| < \varepsilon \Longrightarrow \ell - \varepsilon < u_n < \ell + \varepsilon.$$

Similarly, since $\lim\limits_{n\to+\infty} w_n = \ell$, $\exists n_2 \in \mathbb{N}$ such that $\forall n \geq n_2$,

$$|w_n - \ell| < \varepsilon \Longrightarrow \ell - \varepsilon < w_n < \ell + \varepsilon.$$

Let's choose $\forall n \geq N = \max(n_1, n_2)$, then for all $n \geq N$, we have

$$\ell - \varepsilon < u_n \leq v_n \leq w_n < \ell + \varepsilon$$

Hence, for all $n \geq N$,

$$|v_n - \ell| < \varepsilon.$$

This shows that (v_n) converges to ℓ.

Example 3.10. Calculate the following limits using Gendarmes' Theorem :

$$1)\ u_n = \frac{n \sin n}{(n+1)^2}, \quad 2)\ v_n = \sum_{k=1}^{n} \frac{n}{n^2+k}, \quad 3)\ w_n = \frac{1}{n^2}\sum_{k=1}^{n}[kx], x \in \mathbb{R}.$$

Solution :

1. We know that
$$\frac{-n}{(n+1)^2} \leq \frac{n\sin n}{(n+1)^2} \leq \frac{n}{(n+1)^2},$$
both sequences have the same limit
$$\lim_{n\longrightarrow+\infty} \frac{-n}{(n+1)^2} = \lim_{n\longrightarrow+\infty} \frac{n}{(n+1)^2} = 0,$$
we conclude $\lim\limits_{n\to+\infty} u_n = 0$.

2. For every integer k of $\{1, 2, 3, ..., n\}$, we have
$$\frac{n}{n^2+n} \leq \frac{n}{n^2+k} \leq \frac{n}{n^2+1}.$$
Now apply this to the entire sum
$$\frac{n^2}{n^2+n} \leq v_n \leq \frac{n^2}{n^2+1},$$
both sequences have the same limit
$$\lim_{n\to+\infty} \frac{n^2}{n^2+n} = \lim_{n\to+\infty} \frac{n^2}{n^2+1} = 1.$$
we conclude by the Gendarmes' Theorem
$$\lim_{n\to+\infty} v_n = 1.$$

3. For all $x \in \mathbb{R}$, we have $kx - 1 < [kx] \leq kx$.
Summing over $k = 1$ to n, we obtain :
$$\sum_{k=1}^{n}(kx-1) < \sum_{k=1}^{n}[kx] \leq \sum_{k=1}^{n} kx.$$
This simplifies to
$$x\sum_{k=1}^{n} k - n < \sum_{k=1}^{n}[kx] \leq x\sum_{k=1}^{n} k.$$
Using the identity $\sum\limits_{k=1}^{n} k = \dfrac{n(n+1)}{2}$, we get :
$$x \cdot \frac{n(n+1)}{2} - n < \sum_{k=1}^{n}[kx] \leq x \cdot \frac{n(n+1)}{2}.$$

Now divide the entire inequality by n^2 to get bounds on w_n :

$$\frac{1}{n^2}\left(x \cdot \frac{n(n+1)}{2} - n\right) < w_n \leq \frac{1}{n^2} \cdot x \cdot \frac{n(n+1)}{2}.$$

Simplify each bound :

$$\frac{x(n+1)}{2n} - \frac{1}{n} < w_n \leq \frac{x(n+1)}{2n}.$$

As $n \to \infty$, both bounds converge to $\frac{x}{2}$, so by the Gendarmes' Theorem :

$$\lim_{n\to\infty} w_n = \frac{x}{2}.$$

Corollary 3.1. If $(u_n)_{n\in\mathbb{N}}$ converges to ℓ, and $u_n \geq 0$ for all $n \in \mathbb{N}$, then $\ell \geq 0$. (Similarly, if $u_n \leq 0$, then $\ell \leq 0$).

Proof. We will prove by contradiction. Suppose that $u_n \geq 0$ and $\ell \leq 0$. Let

$$\varepsilon = \frac{|\ell|}{2} \Longrightarrow \ell + \varepsilon = \ell + \frac{|\ell|}{2} < 0.$$

Since

$$u_n \longrightarrow \ell \Longrightarrow \exists n_\varepsilon \in \mathbb{N}, \forall n > n_\varepsilon \Longrightarrow \ell - \varepsilon < u_n < \ell + \varepsilon < 0.$$

contradicting the assumption that $u_n \geq 0 \ \forall n \in \mathbb{N}$.
Therefore, our assumption that $\ell < 0$ must be false, so $\ell \geq 0$.

Corollary 3.2.. If $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are two convergent sequences such that $u_n \geq v_n$ for all $n \in N$, then

$$\lim_{n\longrightarrow+\infty} u_n \geq \lim_{n\longrightarrow+\infty} v_n.$$

Proof. Consider the sequence $(w_n)_{n\in\mathbb{N}}$ defined by

$$w_n = u_n - v_n \geq 0, \quad \text{because } u_n \geq v_n.$$

We have

$$\lim_{n\longrightarrow+\infty} w_n = (\lim_{n\longrightarrow+\infty} u_n) - (\lim_{n\longrightarrow+\infty} v_n)$$

which according to the previous corollary 3.1, implies that

$$\lim_{n\longrightarrow+\infty} w_n \geq 0.$$

Therefore

$$\lim_{n \longrightarrow +\infty} u_n \geq \lim_{n \longrightarrow +\infty} v_n.$$

Theorem 3.7. If the sequence $(u_n)_{n \in \mathbf{N}}$ converges to ℓ, then every subsequence extracted from $(u_n)_{n \in N}$ also converges to ℓ.

Proof. Let $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$ be a strictly increasing function that defines a subsequence v_n of u_n that is $v_n = u_{\varphi(n)}$.
Since $u_n \longrightarrow \ell$, then $\forall \varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\forall n \geq N, \; |u_n - \ell| \leq \varepsilon \qquad (1).$$

Now, since $\varphi(n) \geq n$, it follows that $\varphi(n) \geq N$ whenever $n \geq N$. Therefore, inequality (1) remains valid if we replace n by $\varphi(n)$, and we get

$$|u_{\varphi(n)} - \ell| \leq \varepsilon.$$

Consequently $u_{\varphi(n)} = v_n$ converges to ℓ.

3.7 Convergence of monotone sequences

We now state the fundamental theorem of convergence of monotone sequences.

Theorem 3.8.

a) If a sequence of real numbers is increasing and bounded from above, then it converges.

b) If a sequence of real numbers is decreasing and bounded from below, then it converges.

Training exercise 3.1. Study the nature of the sequence $(u_n)_{n \in \mathbb{N}^*}$ defined by

$$u_1 = 1, \quad u_{n+1} = \sqrt{2 + u_n}, \quad n \geq 1.$$

- Prove that $u_n < 2$ for all $n \in \mathbb{N}^*$
- Prove that u_n is increasing sequence
- Deduce that u_n is convergent, and compute its limit.

Solution.

- For $n = 1$,
$$u_1 = 1 < 2.$$
Assume $u_n < 2$ for some $n \geq 1$. Then
$$u_n + 2 < 4 \Longrightarrow \sqrt{u_n + 2} < 2 \Longrightarrow u_{n+1} < 2.$$
Therefore $u_n < 2, \forall n \in \mathbb{N}$.

- For $n = 1$, we have $u_1 = 1 \leq u_2 = \sqrt{3}$.
Suppose that $u_n - u_{n-1} \geq 0$ then :
$$u_{n+1} - u_n = \sqrt{u_n + 2} - \sqrt{u_{n-1} + 2} = \frac{u_n - u_{n-1}}{\sqrt{u_n + 2} + \sqrt{u_{n-1} + 2}} \geq 0$$
Therefore u_n is increasing.

- $(u_n)_{n \in \mathbb{N}^*}$ is increasing, and bounded above then it is convergent. Also
$$\lim_{n \longrightarrow \infty} u_n = \lim_{n \longrightarrow \infty} u_{n+1} = \ell$$
then
$$\lim_{n \longrightarrow \infty} u_{n+1} = \lim_{n \longrightarrow \infty} \sqrt{u_n + 2} \Longleftrightarrow \sqrt{\ell + 2} = \ell$$
we obtain the equation $\ell^2 - \ell - 2 = 0$, so $\ell = 2$ or -1.
But since $u_n \geq 1$ and increasing, the only possible limit is $\ell = 2$.

3.8 Limits and properties

It is natural to wonder how the limits of sequences behave with respect to operations. In this sense, the limit behaves as simply as possible when the sequences are convergent.

Proposition 3.1. Let (u_n) and (v_n) be two sequences converging to the limit ℓ_1 and ℓ_2 respectively. Then
$$\lim_{n \to \infty} (u_n \pm v_n) = \ell_1 \pm \ell_2$$
$$\lim_{n \to \infty} (u_n \times v_n) = \ell_1 \times \ell_2$$
$$\lim_{n \to \infty} \lambda \times u_n = \lambda \ell_1, \ \lambda \in \mathbb{R}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{\ell_1}{\ell_2}, \text{ if } \ell_2 \neq 0$$
$$\lim_{n\to+\infty} |u_n| = |\ell_1|$$

These properties allow us to calculate the limits using already well-known limits.

Remark 3.6. It is possible for the sum of two divergent sequences to be convergent.

Example 3.11. Let two sequences be defined by $u_n = 2n$ and $v_n = -2n+e^{-n}$. Both $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are divergent. However, their sum $(u_n + v_n)_{n\in\mathbb{N}}$ is convergent because

$$u_n + v_n = e^{-n},$$

and since $\lim\limits_{n\to+\infty} e^{-n} = 0$, we have

$$\lim_{n\to+\infty} (u_n + v_n) = 0.$$

3.9 Adjacent Sequences

Definition 3.7. Two real sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are said to be adjacent if :

$$\begin{cases} (u_n)_{n\in\mathbb{N}} \text{ is increasing,} \\ (v_n)_{n\in\mathbb{N}} \text{ is decreasing,} \\ \text{and } \lim\limits_{n\to\infty} (u_n - v_n) = 0. \end{cases}$$

Example 3.12. Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two sequences defined by

$$u_n = \sum_{k=0}^{n} \frac{1}{k!}, \qquad v_n = u_n + \frac{1}{n.n!}.$$

Show that (u_n) and (v_n) are adjacent.

Solution. For all $n \in \mathbb{N}$, we have

$$u_{n+1} - u_n = \sum_{k=0}^{n+1} \frac{1}{k!} - \sum_{k=0}^{n} \frac{1}{k!} = \frac{1}{(n+1)!} > 0.$$

Therefore (u_n) is increasing.

$$v_{n+1} - v_n = u_{n+1} + \frac{1}{(n+1).(n+1)!} - u_n - \frac{1}{n.n!}$$

$$= \frac{1}{(n+1)!} - \frac{(n+1)^2 - n}{n(n+1)(n+1)!} = \frac{-1}{n(n+1)(n+1)!} < 0.$$

Therefore (v_n) is decreasing.
Moreover, for all $n \in \mathbb{N}$,

$$v_n - u_n = \frac{1}{n.n!} \Longrightarrow \lim_{n \longrightarrow +\infty} \frac{1}{n.n!} = 0.$$

All three criteria are satisfied. Therefore $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are adjacent.

Theorem 3.9. If the sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are adjacent then they converge to the same limit.

3.10 Recurrence Sequences

Definition 3.8. A Recurrence sequence is a sequence defined by their first term(s) and a recurrence relation which can be of the form $u_{n+1} = f(u_n)$ where f is a function.

Example 3.13. Let's consider

$$u_0 = 0, \quad u_{n+1} = \frac{2}{u_n^2 + 1}.$$

This generates

$$u_1 = 2, \ \ u_2 = \frac{2}{5}, \ \ u_3 = \frac{50}{29}, ...$$

3.10.1 Monotonicity of a Recurrence Sequence

Let f be a function defined on an interval I, and suppose the sequence (u_n) defined

$$u_{n+1} = f(u_n).$$

– If f is decreasing on I, then the sequence (u_n) is not guaranteed to be monotonic.

- If f is increasing, then the sequence (u_n) is monotonic, and its direction can be determined by comparing $f(u_0)$ to u_0
  - If $f(u_0) - u_0 > 0$, then the sequence $(u_n)_{n\in\mathbb{N}}$ is increasing.
  - If $f(u_0) - u_0 < 0$, then the sequence $(u_n)_{n\in\mathbb{N}}$ is decreasing.

Example 3.14. Define the recurrence sequence

$$u_0 = 0, \quad u_{n+1} = \frac{1}{2}u_n + 1$$

Let's define

$$f(x) = \frac{1}{2}x + 1 \quad (\textit{increasing function})$$

Compute $f(u_0) - u_0$

$$f(u_0) - u_0 = f(0) - 0 = 1 > 0$$

So the sequence is increasing.

Theorem 3.10. Let $f : I \longrightarrow I$ be a continuous function defined on an interval $I \subset \mathbb{R}$, and let (u_n) be a sequence defined by

$$u_{n+1} = f(u_n), \quad \text{with } u_0 \in I.$$

If the sequence (u_n) converges to a limit $\ell \in I$, then $f(\ell) = \ell$.

Example 3.15. Calculate the limit of the recurrence sequence given in the example 3.14.
The sequence u_n is increasing and bounded above then u_n converges to ℓ, such that

$$\ell = f(\ell) \Longleftrightarrow \ell = \frac{1}{2}\ell + 1 \Longrightarrow \ell = 2.$$

Therefore $\lim u_n = 2$.

3.11 Cauchy's Convergence Criterion

Definition 3.9. A sequence (u_n) is called a Cauchy sequence, if

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \ \forall p, q \geq n_0, \ |u_p - u_q| < \varepsilon.$$

Theorem 3.11. A real sequence (u_n) converges if and only if it is a Cauchy sequence.

Proof. Suppose $\lim u_n = \ell$. Then by definition for convergence, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n > n_0 \Longrightarrow |u_n - \ell| < \frac{\varepsilon}{2}$$

then

$$|u_p - u_q| = |(u_p - \ell) - (u_q - \ell)| \leq |(u_p - \ell)| + |(u_q - \ell)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

for all $p, q \geq n_0$.

Example 3.17. Using Cauchy criterion of convergence, examine the convergence of sequence (u_n) where

$$u_n = \sum_{k=1}^{n} \frac{1}{k!}.$$

A sequence (u_n) is convergent if and only if it is a Cauchy sequence, i.e,
For every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$,

$$|u_n - u_m| < \varepsilon.$$

$$\begin{aligned} |u_n - u_m| &= |\frac{1}{(m+1)!} + \frac{1}{(m+2)!} + ... + \frac{1}{n!}| \\ &\leq \frac{1}{2^m} + \frac{1}{2^{m+1}} + ... + \frac{1}{2^{n-1}} \\ &= \frac{1}{2^m}(\frac{1 - \frac{1}{2^{n-m-1}}}{1 - \frac{1}{2}}) \\ &= \frac{1}{2^{m-1}}(1 - \frac{1}{2^{n-m-1}}) < \frac{1}{2^{m-1}} < \varepsilon. \end{aligned}$$

$\forall \varepsilon > 0,\ \forall m > [1 - \frac{\ln(\varepsilon}{\ln(2)}] + 1 \Longrightarrow |u_n - u_m| < \varepsilon$.
Therefore (u_n) is Cauchy sequence, then the sequence (u_n) converges.

3.12 Bolzano-Weierstrass Property

Theorem 3.12. Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence of real numbers. That is, there exists a constant $M > 0$ such that for all $n \in \mathbb{N}$, we have

$$|u_n| \leq M.$$

By the **Monotone Subsequence Theorem**, every sequence of real numbers has a monotonic subsequence. Therefore, there exists a subsequence $(u_{\varphi(n)})_{n\in\mathbb{N}}$ of (u_n) that is either increasing or decreasing.

Since (u_n) is bounded, every subsequence of (u_n) is also bounded. In particular, the monotonic subsequence $(u_{\varphi(n)})$ is bounded.

A bounded monotonic sequence of real numbers converges by the Theorem 3.8. Hence, the subsequence $(u_{\varphi(n)})$ converges.

3.13 Exercise.

Exercise 3.1.
Calculate the limit of the following sequences with the general term

$$u_n = \frac{n}{n^2+1} + \frac{n}{n^2+2} + \frac{n}{n^2+3} + \cdots + \frac{n}{n^2+n}, \quad v_n = \frac{1}{n!}(1! + 2! + \cdots + n!),$$
$$w_n = \sqrt{n}(\sqrt{n-1} - \sqrt{n}).$$

Exercise 3.2.
Let the sequence (u_n) defined by the general term :

$$u_n = \frac{2^n + (-1)^n}{2^n}, n \in \mathbb{N}$$

Show that $\lim u_n = 1$. For what values of n, $|u_n - 1|$ less then ε and less then 10^{-4}.

Exercise 3.3.
Determine which of the following sequences are bounded

$$u_n = n^{(-1)^n}, \quad v_n = \Sigma_{k=1}^{n} \frac{1}{k+n}.$$

Study the monotony of the following sequences and deduce possibly their nature :

$$u_n = \Sigma_{k=1}^{n} \frac{k^2}{n^2}, \quad v_n = \Sigma_{k=1}^{n} \frac{1}{k+n}, \quad w_n = \frac{1 \times 3 \times 5 \times \cdots (2n-1)}{2 \times 4 \times 6 \times \cdots \times (2n)}.$$

Exercise 3.4.
Let the sequence (u_n) be defined by :

$$u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n}, \ n \geq 1.$$

1. Show that the sequences u_{2k} and u_{2k+1} are adjacent.
2. Deduce the nature of (u_n).

Exercise 3.5.
We consider the sequence $(u_n)_{n\in\mathbb{N}}$ of real numbers whose general term is defined by recurrence

$$u_0 = 2, \quad u_{n+1} = \sqrt{2u_n - 1}.$$

1. Show that, for all $n \in \mathbb{N}, u_n \geq 1$
2. Show that the sequence $(u_n)_{n\in\mathbb{N}}$ is decreasing
3. Deduce that the sequence $(u_n)_{n\in\mathbb{N}}$ is convergent and determine its limit.

We consider the sequences (v_n) and (w_n) of real numbers defined for all $n \geq 1$ by

$$v_n = \sum_{k=0}^{n} \frac{1}{k!}, \qquad w_n = v_n + \frac{1}{n!}.$$

Show that these two sequences are convergent and have the same limit.

Exercise 3.6.
Let the sequence (u_n) be defined by :

$$u_n = \begin{cases} u_1 = \frac{1}{2}, \\ u_{n+1} = u_n^2 + \frac{3}{16} \end{cases}.$$

1. Prove that $\forall n \geq 1,\ \frac{1}{4} < u_n < \frac{3}{4}$.
2. Study the nature of the sequence u_n and calculate its limit if it is convergent.
3. Let $E = \{u_n, n \geq 1\}$. Determine $\sup E$ and $\inf E$.

Exercise 3.7. Find $\inf u_n$, $\sup u_n$, $\liminf u_n$ and $\limsup u_n$ if :

$$u_n = \frac{(-1)^n}{n} + \frac{1+(-1)^n}{2}; \ \ u_n = 1 + \frac{n}{n+1}\cos\frac{n\pi}{2}.$$

Chapter 4

Real Functions of a Real Variable

This chapter is devoted to the study of functions of a real variable, which are commonly used to model various problems in mathematics, mechanics, and other fields.

4.1 Preliminaries

Definition 4.1. Let $E \subset \mathbb{R}$. A function f of a real variable x defined on the set E is any mapping from E into $\mathbb{R}$; that is, each element of E is associated with a unique element of $\mathbb{R}$. This is denoted as :

$$f : E \to \mathbb{R}, \quad x \mapsto f(x).$$

The domain of definition of f is the set defined by

$$D(f) := \{x \in E \mid f(x) \text{ exists}\}.$$

A set $f(E) = \{y = f(x) \mid x \in D(f)\}$ is called the range of f and is denoted by $\mathrm{Im}(f)$.

Definition 4.2. The graph of the function f is the set of ordered pairs of real numbers $(x, f(x))$, where $x \in D(f)$. We write :

$$\Gamma(f) = \{(x, f(x)) \mid x \in D(f)\}.$$

Example 4.1. Give the largest possible domain of the following functions :

1. $f(x) = \frac{x^2}{x-3}$
2. $g(x) = \ln(x+1)$

3. $h(x) = 2 - \sqrt{9 - x^2}$

Solution.

1. $f(x) = \frac{x^2}{x-3}$
   For this function, we must ensure that the denominator is not equal to zero. Therefore, the domain is all real numbers except $x = 3$.
   $$D(f) = \{x \in \mathbb{R} \mid x \neq 3\}.$$

2. $g(x) = \ln(x+1)$
   The natural logarithm function is defined only for positive arguments. Therefore, we need :
   $$x + 1 > 0 \quad \Rightarrow \quad x > -1.$$
   Thus, the largest domain of $g(x)$ is :
   $$D(g) = \{x \in \mathbb{R} \mid x > -1\}.$$

3. $h(x) = 2 - \sqrt{9 - x^2}$
   For this function, we must ensure that the expression inside the square root is non-negative, we need :
   $$9 - x^2 \geq 0 \quad \Rightarrow \quad -3 \leq x \leq 3.$$
   Thus, the largest domain of $h(x)$ is :
   $$D(h) = \{x \in \mathbb{R} \mid -3 \leq x \leq 3\}.$$

Definition 4.3.

– A function f is called injective if each element in the codomain has at most one preimage.

– It is called surjective if each element in the codomain has at least one preimage.

– It is called bijective if it is both injective and surjective, i.e., if each element in the codomain has exactly one preimage.

Remark 4.1. Also recall that a function has an inverse if and only if it is bijective.

4.1.1 Even, odd and periodic functions

Definition 4.4. A function f, defined on a symmetric interval I (that is to say, $\forall x \in I, -x \in I$), is said to be :

- **Even** if and only if $\forall x \in I, f(-x) = f(x)$,
- **Odd** if and only if $\forall x \in I, f(-x) = -f(x)$.

Geometrically :

- If f is even, then its graph is symmetrical with respect to the y-axis.
- If f is odd, then its graph is symmetrical with respect to the origin.

Example 4.2.

- Let $f(x) = \dfrac{e^{x^2}}{x^4+1}$.
  Compute $f(-x)$:
  $$f(-x) = \frac{e^{(-x)^2}}{(-x)^4+1} = \frac{e^{x^2}}{x^4+1} = f(x)$$
  Thus, f is an even function on $\mathbb{R}$. Its graph is symmetric with respect to the y-axis.
- We want to determine whether the function
  $$g(x) = \frac{x^2 \cos x}{\sin^2(x)+1}$$
  is even. Compute $g(-x)$:
  $$g(-x) = \frac{(-x)^2 \cos(-x)}{\sin^2(-x)+1} = \frac{x^2 \cos x}{\sin^2 x+1} = g(x)$$
  g is an even function on $\mathbb{R}$.
- Consider the function $h(x) = x^3$. This function is odd because :
  $$h(-x) = (-x)^3 = -x^3 = -h(x), \quad \forall x \in \mathbb{R}.$$
  Therefore, $h(x) = x^3$ is an odd function. Its graph is symmetric with respect to the origin.

Definition 4.5. A function f is called periodic if there exists $T > 0$ such that

$$f(x+T) = f(x) \quad \text{for all } x \in \mathbb{R}.$$

The smallest such positive number T is called the **period** of the function f.

Remark 4.2. If T is a period of a function f, then for any integer $k \neq 0$, kT is also a period of f, because :

$$f(x+kT) = f((x+(k-1)T)+T) = f(x+(k-1)T) = \cdots = f(x).$$

Example 4.3.

1. The function $f(x) = x - [x]$, where $[x]$ denotes the integer part of x, is periodic with period 1. Indeed

$$\forall x \in \mathbb{R},\ f(x+1) = x+1-[x+1] = x+1-[x]-1 = x+[x] = f(x).$$

2. The function $f(x) = \sin(x)$ is periodic with period 2π, since

$$\sin(x+2\pi) = \sin(x), \quad \forall x \in \mathbb{R}.$$

4.1.2 Bounded and monotonic functions

Definition 4.6. Let $f : E \longrightarrow \mathbb{R}$ be a real-valued function, we say that :

– f is bounded above on E if

$$\exists M \in \mathbb{R} \text{ such that } \forall x \in E,\ f(x) \leq M.$$

– f is bounded below on E if

$$\exists m \in \mathbb{R} \text{ such that } \forall x \in E,\ m \leq f(x).$$

– f is bounded on E if it is both bounded above and bounded below, that is :

$$\exists m, M \in \mathbb{R} \text{ such that } \forall x \in E,\ m \leq f(x) \leq M$$

or equivalently,

$$\exists C > 0 \text{ such that } \forall x \in E,\ |f(x)| \leq C.$$

Example 4.4.

1. $f(x) = \cos x$ is bounded on $\mathbb{R}$. Indeed
$$-1 \leq \cos(x) \leq 1.$$
The infimum and supremum of f on $\mathbb{R}$ are :
$$\inf_{x\in\mathbb{R}} f(x) = -1, \quad \sup_{x\in\mathbb{R}} f(x) = 1.$$

2. $f(x) = e^{-\frac{1}{x}}$ is bounded on $]0, +\infty[$. The image of f over this interval is :
$$f(]0, +\infty[) =]0, 1[.$$
Therefore,
$$\sup_{x\in]0,+\infty[} f(x) = 1, \quad \inf_{x\in]0,+\infty[} f(x) = 0.$$

3. The function $x \mapsto \ln x$ is not bounded, because as $x \to 0^+$, we have :
$$\ln x \to -\infty,$$
and as $x \to +\infty$, we have :
$$\ln x \to +\infty.$$
This implies that the logarithmic function is neither bounded above nor bounded below.

Definition 4.7. A function f defined on $E \subset \mathbb{R}$ is said to be :

– Increasing on E if
$$\forall x_1, x_2 \in E,\ x_1 \leq x_2 \Longrightarrow f(x_1) \leq f(x_2).$$

– Decreasing on E if
$$\forall x_1, x_2 \in E,\ x_1 \leq x_2 \Longrightarrow f(x_1) \geq f(x_2).$$

– If f is either increasing or decreasing on E, we say that f is monotone on E.

Remark 4.3.
If the inequalities in the definitions above are strict, we obtain the notions of **strictly increasing** and **strictly decreasing** functions.

Example 4.5.

1. Let $f(x) = 2x + 3$. We show that f is an increasing function on $\mathbb{R}$.

$$\text{Let } x_1, x_2 \in \mathbb{R} \text{ such that } x_1 \leq x_2.$$
$$f(x_1) = 2x_1 + 3, \quad f(x_2) = 2x_2 + 3.$$

Since $2x_1 + 3 \leq 2x_2 + 3$, we get :

$$f(x_1) \leq f(x_2).$$

Therefore, f is increasing on $\mathbb{R}$.
Alternatively, we can observe that the derivative of $f(x) = 2x + 3$ is :

$$f'(x) = 2 > 0 \quad \text{for all } x \in \mathbb{R},$$

which confirms that f is strictly increasing on $\mathbb{R}$.

2. Let $f(x) = \dfrac{1}{x}$. We analyze the monotonicity of f on the interval $]0, +\infty[$.

$$\text{Let } x_1, x_2 \in]0, +\infty[\text{ with } x_1 < x_2.$$

Then,
$$f(x_1) = \frac{1}{x_1} > \frac{1}{x_2} = f(x_2).$$

Thus,
$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2),$$

which shows that f is strictly decreasing on $]0, +\infty[$.

Proposition 4.1. The sum of two increasing (respectively, decreasing) functions is an increasing (respectively, decreasing) function.

Corollary 4.1. Let f be strictly monotone on set $E \subset \mathbb{R}$. Then f is injective on E.

4.1.3 Maximum and Minimum local of function

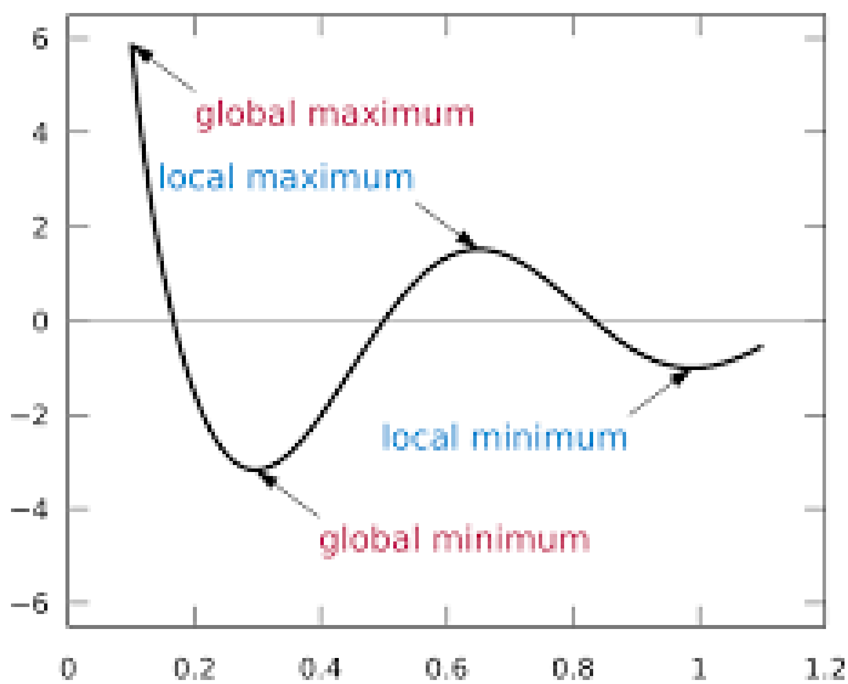
Definition 4.8.

- A function f is said to have a local maximum at a point x_0 if there exists an interval I around x_0 such that

$$f(x_0) \geq f(x), \quad \forall x \in I$$

- A function f is said to have a local minimum at a point x_0 if there exists an interval I around x_0 such that

$$f(x_0) \leq f(x), \quad \forall x \in I$$



4.2 Limit of function

Definition 4.9. (**Neighborhood of a Point**)
Let $x_0 \in \mathbb{R}$. A neighborhood of x_0 is any open interval of the form

$$]x_0 - \delta, x_0 + \delta[, \quad \text{with } \delta > 0.$$

Definition 4.10. Let f be a function defined on a neighborhood of a point x_0. We say that f has a limit $\ell \in \mathbb{R}$ at the point x_0 if :

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \ |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \varepsilon.$$

Example 4.6. Let $f(x) = \dfrac{x^2-1}{x-1}$ for $x \neq 1$. We want to find $\lim_{x\to 1} f(x)$.

We simplify the expression :

$$f(x) = \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1 \quad \text{for } x \neq 1.$$

So,

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}(x+1) = 2.$$

Even though f is not defined at $x = 1$, it has a limit at that point.

Example 4.7. Let $f(x) = 5x - 3$. Show that

$$\lim_{x\to 1} f(x) = 2,$$

using the epsilon-delta definition.

We want to show that :

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } 0 < |x-1| < \delta \Rightarrow |f(x) - 2| < \varepsilon.$$

Now compute :

$$|f(x) - 2| = |5x - 3 - 2| = |5x - 5| = 5|x - 1|.$$

So to ensure that $|f(x) - 2| < \varepsilon$, we need :

$$5|x-1| < \varepsilon \Rightarrow |x-1| < \frac{\varepsilon}{5}.$$

Thus, we can choose $\delta = \frac{\varepsilon}{5}$. Then :

$$0 < |x-1| < \delta \Rightarrow |f(x) - 2| < \varepsilon.$$

Therefore

$$\lim_{x\to 1} f(x) = 2.$$

Theorem 4.1. If f has a limit at point x_0, then this limit is unique.

Proof. (by contradiction). Let

$$\lim_{x \to x_0} f(x) = \ell_1 \quad \text{and} \quad \lim_{x \to x_0} f(x) = \ell_2,$$

Assume, that $\ell_1 \neq \ell_2$.
Let $\varepsilon = \frac{|\ell_1 - \ell_2|}{3}$, which is strictly positive. Since $\lim_{x \to x_0} f(x) = \ell_1$, there exists $\delta_1 > 0$ such that for all x satisfying $0 < |x - x_0| < \delta_1$, we have

$$|f(x) - \ell_1| < \varepsilon.$$

Similarly, since $\lim_{x \to x_0} f(x) = \ell_2$, there exists $\delta_2 > 0$ such that for all x satisfying $0 < |x - x_0| < \delta_2$, we have

$$|f(x) - \ell_2| < \varepsilon.$$

Let $\delta = \min(\delta_1, \delta_2)$. For any x such that $0 < |x - x_0| < \delta$, both of the following inequalities hold :

$$|f(x) - \ell_1| < \varepsilon \quad \text{and} \quad |f(x) - \ell_2| < \varepsilon.$$

Now consider the difference $|\ell_1 - \ell_2|$. Using the triangle inequality, we get

$$|\ell_1 - \ell_2| \leq |f(x) - \ell_1| + |f(x) - \ell_2| < \varepsilon + \varepsilon = \frac{2|\ell_1 - \ell_2|}{3}.$$

This is a contradiction, because we assumed that $\ell_1 \neq \ell_2$, so the inequality $|\ell_1 - \ell_2| < \frac{2|\ell_1 - \ell_2|}{3}$ cannot hold. Therefore, $\ell_1 = \ell_2$, proving the uniqueness of the limit.

Definition 4.11.

– The right limit of a function f at a point x_0 is the limit of $f(x)$ as x approaches x_0 from values greater than x_0. It is denoted as :

$$\lim_{x \to x_0^+} f(x) = \ell,$$

which means :

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that for all } x, \text{ if } 0 < x - x_0 < \delta, \text{ then } |f(x) - \ell| < \varepsilon.$$

– The left limit of a function f at a point x_0 is the limit of $f(x)$ as x approaches x_0 from values less than x_0. It is denoted as :

$$\lim_{x\to x_0^-} f(x) = \ell,$$

which means :

$$\forall\varepsilon > 0, \exists\delta > 0 \text{ such that for all } x, \text{ if } 0 < x_0 - x < \delta, \text{ then } |f(x) - \ell| < \varepsilon.$$

Remark 4.4.

– If the limit of f exists at the point x_0, then both the right and left limits also exist, and we have :

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x).$$

– If

$$\lim_{x\to x_0^-} f(x) \neq \lim_{x\to x_0^+} f(x),$$

then f does not have a limit at the point x_0.

Example 4.8. Evaluate the limit :

$$\lim_{x\to 0} \frac{x^2 + 2|x|}{x}$$

$D_f = \mathbb{R}^*$. We have

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

so

$$\frac{x^2 + 2|x|}{x} = \begin{cases} \frac{x^2+2x}{x} = x + 2, & \text{if } x \geq 0 \\ \frac{x^2-2x}{x} = x - 2, & \text{if } x < 0 \end{cases}$$

We analyze this limit by considering the left and right limits separately. Thus :

$$\lim_{x\to 0^+} \frac{x^2 + 2|x|}{x} = \lim_{x\to 0^+} (x + 2) = 2$$

and

$$\lim_{x\to 0^-} \frac{x^2 + 2|x|}{x} = \lim_{x\to 0^-} (x - 2) = -2$$

Since

$$\lim_{x\to 0^-} \frac{x^2+2|x|}{x} \neq \lim_{x\to 0^+} \frac{x^2+2|x|}{x},$$

Therefore $\lim_{x\to 0} \frac{x^2+2|x|}{x}$ does not exist.

4.2.1 Finite limit at infinity

We say that a function $f(x)$ has a finite limit at infinity if :

$$\lim_{x\to+\infty} f(x) = \ell \quad \text{or} \quad \lim_{x\to-\infty} f(x) = \ell$$

where $\ell \in \mathbb{R}$, meaning that the function approaches a real (finite) number as x tends to positive or negative infinity.

Mathematically

$$\lim_{x\to+\infty} f(x) = \ell \Longleftrightarrow \forall\varepsilon > 0,\ \exists A > 0, \forall x \in \mathbb{R}, x > A \Rightarrow |f(x) - \ell| < \varepsilon.$$

$$\lim_{x\to-\infty} f(x) = \ell \Longleftrightarrow \forall\varepsilon > 0,\ \exists A > 0, \forall x \in \mathbb{R}, x < -A \Rightarrow |f(x) - \ell| < \varepsilon.$$

Example 4.9.

1.
$$\lim_{x\to+\infty} \frac{3x+1}{x+2} = 3,$$
and
$$\lim_{x\to-\infty} \frac{1}{x} = 0.$$

2. We want to prove that
$$\lim_{x\to\infty} \frac{1}{x^2+1} = 0.$$
For every $\varepsilon > 0$, there exists $B(\varepsilon) > 0$ such that
$$\forall x > B(\varepsilon) \Longrightarrow \left|\frac{1}{x^2+1} - 0\right| < \varepsilon$$
Given that
$$\left|\frac{1}{x^2+1} - 0\right| < \varepsilon \Longleftrightarrow x^2+1 > \frac{1}{\varepsilon}$$

$$\iff x^2 > \frac{1}{\varepsilon} - 1$$

$$\iff x > \sqrt{\frac{1}{\varepsilon} - 1}$$

Therefore, we choose $B(\varepsilon) = \sqrt{\frac{1}{\varepsilon} - 1}$. By the definition of a limit at infinity, we conclude :

$$\lim_{x\to\infty} \frac{1}{x^2+1} = 0$$

4.3 Infinite limit

4.3.1 Infinite limit at point

We say that a function $f(x)$ has an **infinite limit at a point** $x_0 \in \mathbb{R}$ if :

$$\lim_{x\to x_0} f(x) = +\infty \quad \text{or} \quad \lim_{x\to x_0} f(x) = -\infty$$

This means that as x approaches x_0, the values of $f(x)$ increase or decrease without bound.

Mathematically

$$\lim_{x\to x_0} f(x) = +\infty \iff \forall M > 0,\ \exists \delta > 0, \forall x \in \mathbb{R},\ |x - x_0| < \delta \implies f(x) > M.$$

$$\lim_{x\to x_0} f(x) = -\infty \iff \forall M > 0,\ \exists \delta > 0 \forall x \in \mathbb{R},\ |x - x_0| < \delta \implies f(x) < -M.$$

Example 4.10.

$$\lim_{x\to 0} \frac{1}{x^2} = +\infty$$

and

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty \quad \text{and} \quad \lim_{x\to 0^+} \frac{1}{x} = +\infty.$$

4.3.2 Infinite limit at infinity

We say that a function $f(x)$ has an infinite limit at infinity if :

$$\lim_{x\to +\infty} f(x) = +\infty \quad \text{or} \quad \lim_{x\to +\infty} f(x) = -\infty$$

$$\lim_{x\to-\infty} f(x) = +\infty \quad \text{or} \quad \lim_{x\to-\infty} f(x) = -\infty$$

This means that as x increases or decreases without bound, the values of $f(x)$ also grow without bound in the positive or negative direction.

Mathematically

$$\lim_{x\to+\infty} f(x) = +\infty \iff \forall A > 0, \exists B > 0, \forall x \in \mathbb{R}, x > B \Longrightarrow f(x) > A.$$

$$\lim_{x\to-\infty} f(x) = +\infty \iff \forall A > 0, \exists B > 0, \forall x \in \mathbb{R}, x < -B \Longrightarrow f(x) > A.$$

$$\lim_{x\to+\infty} f(x) = -\infty \iff \forall A > 0, \exists B > 0, \forall x \in \mathbb{R}, x > B \Longrightarrow f(x) < -A.$$

$$\lim_{x\to-\infty} f(x) = -\infty \iff \forall A > 0, \exists B > 0, \forall x \in \mathbb{R}, x < -B \Longrightarrow f(x) < -A.$$

Example 4.11.

1. $$\lim_{x\to+\infty} \frac{x^2}{x+1} = +\infty$$

2. $$\lim_{x\to-\infty} e^{-x^3} = +\infty$$

3. We want to show that :

$$\lim_{x\to+\infty} \ln(x^3) = +\infty$$

We say that $\lim_{x\to+\infty} f(x) = +\infty$ if :

$$\forall A > 0,\ \exists B > 0 \text{ such that } x > B \Rightarrow f(x) > A$$

note that

$$\ln(x^3) = 3\ln(x)$$

We want

$$3\ln(x) > A \Rightarrow \ln(x) > \frac{A}{3} \Rightarrow x > e^{A/3}$$

Choose $B = e^{A/3}$. Then for all $x > B$, we have :

$$\ln(x^3) = 3\ln(x) > A$$

By definition, we conclude :

$$\lim_{x\to+\infty} \ln(x^3) = +\infty$$

4.3.3 Indeterminate Forms

The following are the standard indeterminate forms encountered in limits :

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\cdot\infty,\ \infty-\infty,\ 0^0,\ \infty^0,\ 1^\infty.$$

Example 4.12. Let's calculate the following limits :

1.

$$\lim_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x}}}$$

The direct evaluation of this limit at $+\infty$ results in an indeterminate form $\frac{\infty}{\infty}$. Then

$$\lim_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x}}}=\lim_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{x(1+\frac{1}{\sqrt{x}})}}$$

$$=\lim_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{x}(\sqrt{(1+\frac{1}{\sqrt{x}})}}=\lim_{x\to+\infty}\frac{1}{\sqrt{1+\frac{1}{\sqrt{x}}}}=1.$$

2.

$$\lim_{x\to0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$$

The direct evaluation of this limit at $x=0$ results in an indeterminate form $\frac{0}{0}$. To resolve this, we multiply the numerator and denominator by the conjugate of the numerator :

$$\frac{\sqrt{1+x}-\sqrt{1-x}}{x}\cdot\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}=\frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})}$$

Canceling the x gives :

$$\frac{2}{\sqrt{1+x}+\sqrt{1-x}}$$

Taking the limit as $x\to0$:

$$\lim_{x\to0}\frac{2}{\sqrt{1+x}+\sqrt{1-x}}=\frac{2}{1+1}=1$$

Thus, the limit is 1.

4.4 Theorems on Limits

4.4.1 Operations on Limits

Let

$$\lim_{x\to x_0} f(x) = \ell_1 \quad \text{and} \quad \lim_{x\to x_0} g(x) = \ell_2$$

Then the following operations hold :

1. $\lim_{x\to x_0} (f(x) \pm g(x)) = \ell_1 \pm \ell_2$
2. $\lim_{x\to x_0} f(x) \cdot g(x) = \ell_1 \cdot \ell_2$
3. (if $\ell_2 \neq 0$) $\lim_{x\to x_0} \dfrac{f(x)}{g(x)} = \dfrac{\ell_1}{\ell_2}$
4. $\lim_{x\to x_0} \lambda \cdot f(x) = \lambda \cdot \ell_1$
5. $\lim_{x\to x_0} [f(x)]^n = \ell_1^n, \qquad \lim_{x\to x_0} \sqrt[n]{f(x)} = \sqrt[n]{\ell_1}$ (if defined)
6. $f(x) \leq g(x) \Longrightarrow \ell_1 \leq \ell_2$.

Theorem 4.2.(Gendarme's Theorem)
Suppose that

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = \ell$$

If

$$f(x) \leq g(x) \leq h(x)$$

in a neighborhood of x_0, then

$$\lim_{x\to x_0} g(x) = \ell.$$

Example 4.13. Let us evaluate the following limits :

1. We want to evaluate the limit :

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$$

We use the fact that for all real numbers $x \neq 0$, we have :

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

Multiplying all sides of the inequality by $x^2 \geq 0$, we obtain :

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

Now, take the limit of the left and right sides as $x \to 0$:

$$\lim_{x\to 0}(-x^2) = 0 \quad \text{and} \quad \lim_{x\to 0}(x^2) = 0.$$

By the Gendarme's Theorem, it follows that

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$

2.

$$\lim_{x\to +\infty} \frac{[x]}{x}$$

We know that for any real number x, the following inequality holds :

$$x - 1 < [x] \leq x$$

Dividing all parts of the inequality by $x > 0$:

$$\frac{x-1}{x} < \frac{[x]}{x} \leq 1$$

Simplifying :

$$1 - \frac{1}{x} < \frac{[x]}{x} \leq 1$$

Now, taking the limit as $x \to +\infty$:

$$\lim_{x\to +\infty}\left(1 - \frac{1}{x}\right) = 1 \quad \text{and} \quad \lim_{x\to +\infty} 1 = 1$$

Therefore, by the **Gendarme's Theorem** :

$$\lim_{x\to +\infty} \frac{[x]}{x} = 1.$$

4.5 Continuous function

Definition 4.12.(Continuity at a Point)
Let $f : I \to \mathbb{R}$ be a function, where $I \subset \mathbb{R}$ is an interval, and let $x_0 \in I$.

1. We say that f is **continuous at** x_0 if this following three conditions must be satisfied :
   (a) $f(x_0)$ is defined,
   (b) $\lim\limits_{x \to x_0} f(x)$ exists,
   (c) $\lim\limits_{x \to x_0} f(x) = f(x_0)$.
2. We say that f is **left-continuous at** x_0 if
$$\lim_{x \to x_0^-} f(x) = f(x_0)$$
3. We say that f is **right-continuous at** x_0 if
$$\lim_{x \to x_0^+} f(x) = f(x_0)$$

Example 4.14. Consider the function

$$h(x) = \begin{cases} \frac{1}{2}x + 1 & \text{if } x > 2 \\ -x + 1 & \text{if } x \leq 2 \end{cases}$$

We study the continuity of h at the point $x = 2$.

$$h(2) = -2 + 1 = -1$$

Left limit as $x \to 2^-$, we find :

$$\lim_{x \to 2^-} h(x) = -2 + 1 = -1 = h(2).$$

Then the function h is left continuous at 2.
Right limit as $x \to 2^+$, we find :

$$\lim_{x \to 2^+} h(x) = \frac{1}{2} \cdot 2 + 1 = 1 + 1 = 2 \neq h(2).$$

Since

$$\lim_{x \to 2^-} h(x) \neq \lim_{x \to 2^+} h(x).$$

Therefore, the limit $\lim_{x \to 2} h(x)$ does not exist, and the function is not continuous at $x = 2$.

Definition 4.13.(Epsilon-Delta Definition of Continuity)
Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and let $x_0 \in \mathbb{R}$.

1. We say that f is continuous at x_0 if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall x \in \mathbb{R}, |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

2. We say that f is **left-continuous at** x_0 if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall x \in \mathbb{R}, 0 < x_0 - x < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

3. We say that f is **right-continuous at** x_0 if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall x \in \mathbb{R}, 0 < x - x_0 < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

Example 4.15.

1. Prove that $f(x) = x^2$ is continuous at $x_0 = 2$
   We want to find $\delta > 0$ such that :

$$|x - 2| < \delta \Rightarrow |x^2 - 4| < \varepsilon$$

   Note that :

$$|x^2 - 4| = |x - 2||x + 2|$$

   we have,

$$1 < x < 3 \Rightarrow 3 < x + 2 < 5 \Rightarrow |x + 2| < 5$$

   Then :

$$|x^2 - 4| = |x - 2||x + 2| < 5 \cdot |x - 2|$$

   We want

$$5 \cdot |x - 2| < \varepsilon \Rightarrow |x - 2| < \frac{\varepsilon}{5}$$

   So, we choose

$$\delta = \frac{\varepsilon}{5}$$

For all $\varepsilon > 0$, choosing $\delta = \frac{\varepsilon}{5}$, such that

$$|x-2| < \delta \Rightarrow |x^2-4| < \varepsilon$$

Hence, $f(x) = x^2$ is continuous at $x = 2$ according to the epsilon-delta definition.

2. Let the function f be defined by :

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x^2}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We want to study the continuity of f at the point $x_0 = 0$.

We have :

$$|f(x) - f(0)| = |x \sin\left(\frac{1}{x^2}\right)| \leq |x|,$$

since $|\sin\left(\frac{1}{x^2}\right)| \leq 1$ for all $x \neq 0$.

Let $\varepsilon > 0$. Choose $\delta = \varepsilon$. Then, if $|x| \leq \delta$, we get :

$$|f(x) - f(0)| \leq |x| \leq \delta = \varepsilon.$$

Therefore, f is continuous at the point $x_0 = 0$.

Definition 4.14.(Continuity on an Interval)
A real-valued function f is said to be continuous on a given interval I if it is continuous at every point of that interval.

Remark 4.5. All of the following functions are continuous on their domains of definition :

- Polynomial functions : $f(x) = a_n x^n + \cdots + a_1 x + a_0$
- Rational functions : $f(x) = \frac{P(x)}{Q(x)}$, where $Q(x) \neq 0$
- Exponential functions : $f(x) = a^x$, $a > 0$
- Logarithmic functions : $f(x) = \ln(x)$, $x > 0$

– Trigonometric functions : $\sin(x), \cos(x), \tan(x), \ldots$ on their domains.

Theorem 4.3. Let f and g be functions continuous on an interval $I \subseteq \mathbb{R}$, and let $c \in \mathbb{R}$. Then, the following functions are also continuous on I :

$$f + g,\ f - g,\ c \cdot f,\ f \cdot g,\ \frac{f}{g}, \text{provided } g(x) \neq 0\ \forall x \in I.$$

Example 4.16. Let $g(x) = \sqrt{x}$ and $h(x) = \ln(x)$ defined on the interval $I =]0, +\infty[$. Both functions are continuous on I.

Then, the following functions are also continuous on I

– $f_1(x) = g(x) + h(x) = \sqrt{x} + \ln(x)$

– $f_2(x) = g(x) \cdot h(x) = \sqrt{x} \cdot \ln(x)$

– $f_3(x) = \frac{g(x)}{h(x)} = \frac{\sqrt{x}}{\ln(x)}$, continuous on $]0, 1[\cup]1, +\infty)$.

Theorem 4.4. (Continuity of the Composition of Functions)
If f is continuous at a and g is continuous at $f(a)$, then the composition $g \circ f$ is necessarily continuous at a. Moreover, we have :

$$\lim_{x \to a} g(f(x)) = g\left(\lim_{x \to a} f(x)\right) = g(f(a)).$$

Example 4.17. We consider the function :

$$f(x) = \ln(2 + \sin x)$$

Note that :

$$-1 \leq \sin x \leq 1, \quad \text{for all } x \in \mathbb{R}$$

So

$$1 \leq 2 + \sin(x) \leq 3.$$

Since

$$2 + \sin x > 0 \quad \text{for all } x \in \mathbb{R}$$

Therefore, the function $f(x)$ is defined for all real numbers x and continuous on $\mathbb{R}$.

4.6 Discontinuity of a Function

1. If the function f is not defined at x_0, then f is discontinuous at x_0.
2. If f is defined in a neighborhood of x_0, then f is said to be discontinuous at x_0 if there exists $\varepsilon > 0$ such that for every $\delta > 0$,

$$\exists x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\} \quad \text{such that} \quad |f(x) - f(x_0)| \geq \varepsilon.$$

3. If the one-sided limits exist but are not equal :

$$\lim_{x \to x_0^-} f(x) \neq \lim_{x \to x_0^+} f(x),$$

then f is discontinuous at x_0, and x_0 is a point of discontinuity of the first kind.

4. If at least one of the one-sided limits does not exist or is not finite :

$$\lim_{x \to x_0^-} f(x) \quad \text{or} \quad \lim_{x \to x_0^+} f(x) \quad \text{does not exist or is infinite,}$$

then f is discontinuous at x_0, and x_0 is a point of discontinuity of the second kind.

5. If the limit exists but is not equal to the function value :

$$\lim_{x \to x_0} f(x) \quad \text{exists and is finite, but} \quad \lim_{x \to x_0} f(x) \neq f(x_0),$$

then f is discontinuous at x_0. This is called a removable discontinuity.

Example 4.18.

1. Undefined at x_0

$$f(x) = \frac{1}{x}, \quad \text{undefined at } x_0 = 0.$$

Since f is not defined at 0, it is discontinuous at 0.

2. Discontinuity (First Kind)

$$f(x) = \begin{cases} 1 & \text{if } x < 0, \\ 2 & \text{if } x \geq 0. \end{cases}$$

$$\lim_{x \to 0^-} f(x) = 1 \neq \lim_{x \to 0^+} f(x) = 2.$$

So f is discontinuous at $x = 0$, and this is a discontinuity of the first kind.

3. Discontinuity of the Second Kind

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

$$\text{As } x \to 0,\ \frac{1}{x} \to \infty \Rightarrow f(x) \text{ oscillates without limit.}$$

The limit does not exist at 0, so this is a discontinuity of the second kind.

4. Removable Discontinuity

$$f(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{if } x \neq 1, \\ 3 & \text{if } x = 1. \end{cases}$$

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1 \quad \text{for } x \neq 1.$$

So,

$$\lim_{x \to 1} f(x) = 2 \neq f(1) = 3.$$

This is a removable discontinuity at $x = 1$.

4.7 Uniform Continuity on an Interval

Definition 4.15. Let $f : I \to \mathbb{R}$ be a function defined on an interval $I \subseteq \mathbb{R}$. We say that f is **uniformly continuous** on I if :

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \forall x, y \in I,\ |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Remark 4.6. In ordinary continuity, δ may depend on both ε and the point x. In uniform continuity, δ depends only on ε, not on the point.

Theorem 4.5. (Heine-Cantor)
If f is continuous on a closed and bounded interval $[a, b]$, then f is uniformly continuous on $[a, b]$.

Example 4.19.

1. Let $f(x) = x^2$ on the closed interval $[0, 1]$. Since f is continuous on a compact interval, it is uniformly continuous.

2. Let $f(x) = \frac{1}{x}$ on the interval $]0, 1[$. Then f is continuous on $]0, 1[$, but not uniformly continuous. Indeed, as $x \to 0^+$, $f(x) \to \infty$, and we cannot find a single δ that works for all $x, y \in]0, 1[$ for a given ε.

4.8 Extension by Continuity

Definition 4.16. Let f be a function defined on a set $I \setminus \{a\}$. If the limit $\lim_{x\to a} f(x)$ exists and is finite, we can define :

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \neq a \\ \lim_{x\to a} f(x) & \text{if } x = a \end{cases}$$

Then, $\tilde{f}$ is called the **continuous extension** of f at the point a. In this case, $\tilde{f}$ is continuous at $x = a$.

Example 4.20.

1. Consider the function
$$f(x) = \frac{\sin x}{x}$$
which is defined for all $x \neq 0$.
We know that :
$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$
We define the extended function $\tilde{f}$ by :
$$\tilde{f}(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$
Thus, $\tilde{f}$ is continuous at $x = 0$, and hence continuous on $\mathbb{R}$.

2. Consider the function
$$f(x) = e^{1/x}, \quad \text{defined for } x \neq 0$$
We know that
$$\lim_{x\to 0^+} e^{1/x} = +\infty$$
Since we cannot define $f(0)$ in such a way that the function becomes continuous at $x = 0$.

   The function $f(x) = e^{1/x}$ does not admit a continuous extension at $x = 0$.

4.9 Fundamental Theorem's

Theorem 4.6.(Intermediate Value Theorem)
Let f be a function continuous on the closed interval $[a, b]$, and let N be a real number between $f(a)$ and $f(b)$, i.e.,

$$f(a) < N < f(b) \quad \text{or} \quad f(b) < N < f(a)$$

Then there exists a number $c \in]a, b[$ such that :

$$f(c) = N$$

Interpretation : A continuous function on an interval takes every intermediate value between its endpoints.

Theorem 4.7.
Let $f : [a, b] \to \mathbb{R}$ be a continuous function. If :

$$f(a) \cdot f(b) < 0$$

then there exists at least one point $c \in]a, b[$ such that :

$$f(c) = 0.$$

This is particularly useful for proving that an equation $f(x) = 0$ has a solution in $]a, b[$ if $f(a)f(b) < 0$.

Example 4.21.

1. Let us consider the function :

$$f(x) = x^3 - x - 1$$

   Prove that $f(x) = 0$ admits a solution in the interval $]1, 2[$. We evaluate the function at the endpoints of the interval $[1, 2]$:

$$f(1) = 1^3 - 1 - 1 = -1$$

$$f(2) = 2^3 - 2 - 1 = 8 - 2 - 1 = 5$$

   we have

$$f(1) \cdot f(2) < 0.$$

Because f is a polynomial (and thus continuous on $[1,2]$), the hypotheses of the Intermediate Value Theorem are satisfied.
There exists $c \in]1,2[$ such that $f(c) = 0$
So, the equation $x^3 - x - 1 = 0$ has at least one real root in the interval $(1,2)$.

2. Let us consider the function :

$$g(x) = x \sin x + \cos x - x^2$$

With the Intermediate Value Theorem applied to prove that the equation $g(x) = 0$ has at least one positive solution and one negative solution.

This function is continuous on $\mathbb{R}$ because it is composed of continuous elementary functions (product, sine, cosine, square, etc.).
moreover we have

$$g(-\pi) = (-\pi)\sin(-\pi) + \cos(-\pi) - (-\pi)^2$$
$$= 1 - \pi^2 < 0$$

and

$$g(0) = 0 \cdot \sin(0) + \cos(0) - 0^2 = 1 > 0.$$
$$g(\pi) = (\pi)\sin(\pi) + \cos(\pi) - (\pi)^2$$
$$= -1 - \pi^2 < 0$$

We see that :

$$g(-\pi) \cdot g(0) < 0 \quad \text{and} \quad g(\pi) \cdot g(0) < 0$$

Therefore, by the Intermediate Value Theorem, there exists $c_1 \in]-\pi, 0[$ such that : $g(c_1) = 0$, and $c_2 \in]0, \pi[$ such that $g(c_2) = 0$. So, the equation has at least one positive solution and one negative solution.

Theorem 4.8. Let $f : I \to \mathbb{R}$ be a continuous and strictly monotonic function on an interval $I \subset \mathbb{R}$. If there exist $a, b \in I$ such that :

$$f(a) \cdot f(b) < 0,$$

then the equation $f(x) = 0$ has a **unique solution** in the interval $]a, b[$.

Theorem 4.9. The image of a closed interval under a continuous function is a closed interval. In other words, if $f : [a, b] \to \mathbb{R}$ is continuous, then $f([a, b])$ is a closed interval in $\mathbb{R}$.

Theorem 4.10. If f is continuous on $[a, b]$, then :

- f is bounded on $[a, b]$,
- f attains its bounds : there exist $x_{\min}, x_{\max} \in [a, b]$ such that

$$f(x_{\min}) = \min_{x\in[a,b]} f(x), \quad f(x_{\max}) = \max_{x\in[a,b]} f(x).$$

Theorem 4.11. Let $f : I \to \mathbb{R}$ be a continuous and strictly monotonic function defined on an interval $I \subset \mathbb{R}$. Then

1. f is bijective from I onto its image $f(I)$,
2. The inverse function $f^{-1} : f(I) \to I$ exists,
3. f^{-1} is continuous and strictly monotonic on $f(I)$, and its monotonicity is the same as that of f.

Example 4.22. Let $f(x) = \sqrt{x} \quad$ on $[0, +\infty[$.
Thus is continuous function, strictly increasing, then its inverse function

$$f^{-1}(x) = x^2 \quad \text{on } [0, +\infty[$$

is also continuous.

4.10 Order of a Variable - Landau Notation (Asymptotic Equivalence)

Definition 4.17.

1. We say that a function f is **negligible compared to** a function g as $x \to x_0$ if :

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0$$

In this case, we write :

$$f(x) = o(g(x)) \quad \text{as } x \to x_0$$

Interpretation : $f(x)$ becomes insignificant in comparison with $g(x)$ near x_0.

Example 4.23.
Let $f(x) = x^2$ and $g(x) = x$. Then :

$$\frac{f(x)}{g(x)} = \frac{x^2}{x} = x \to 0 \quad \text{as } x \to 0$$

So we conclude :

$$x^2 = o(x) \quad \text{as } x \to 0$$

This expresses that x^2 is negligible compared to x near 0.

2. We say that f is **dominated by** g as $x \to x_0$, and we write :

$$f(x) = O(g(x)) \quad \text{as } x \to x_0$$

if and only if there exists a constant $C > 0$ and a neighborhood of x_0 such that :

$$|f(x)| \leq C \cdot |g(x)| \quad \text{for all } x \text{ sufficiently close to } x_0.$$

Interpretation : This means that $f(x)$ is **a**t most of the same order of magnitude as $g(x)$ near x_0. In other words, f does not grow faster than g, up to a constant factor.

Example 4.22. Let $f(x) = 3x^2 + 5x$ and $g(x) = x^2$.
Then :

$$\frac{f(x)}{g(x)} = \frac{3x^2 + 5x}{x^2} = 3 + \frac{5}{x}$$

- As $x \to 0$: the expression is unbounded, so $f(x) \neq O(x^2)$ near 0. - As $x \to \infty$: $\frac{f(x)}{x^2} = 3 + \frac{5}{x} \to 3$, so we conclude :

$$f(x) = O(x^2) \quad \text{as } x \to \infty$$

Remark 4.7. If $g \neq 0$ in a neighborhood of x_0, then :

- $f = o(g)$ if and only if $\lim_{x \to x_0} \dfrac{f(x)}{g(x)} = 0$,
- $f = O(g)$ if and only if $f(x) \cdot g(x)$is bounded in a neighborhood of x_0.

Definition 4.18. Let f and g be two functions defined on $]0, +\infty[$. We define that :

- $f = o(g)$ as $x \to +\infty$, if and only if

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = 0.$$

- $f = O(g)$ as $x \to +\infty$, if and only if there exists a constant $C > 0$ and a real number $x_1 > 0$ such that :

$$|f(x)| \leq C \cdot |g(x)| \quad \text{for all } x > x_1.$$

Similarly, we define the relations $f = o(g)$ and $f = O(g)$ as $x \to -\infty$.

4.10.1 Equivalence functions

Definition 4.19. Let f and g be two functions defined in a neighborhood of x_0, except possibly at x_0. We say that f is equivalent to g as $x \to x_0$, written $f(x) \sim g(x)$, if

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1.$$

Example 4.23. Let us consider the functions

$$f(x) = \sin x \quad \text{and} \quad g(x) = x,$$

We have :

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Therefore, we can write :

$$\sin x \sim x \quad \text{as } x \to 0.$$

This means that $\sin(x)$ and x are asymptotically equivalent near 0.

Theorem 4.12. Let f and g two functions defined in the neighborhood of x_0 except perhaps at x_0. We suppose that $f \sim g$ at x_0 then, if $\lim_{x \to x_0} f(x)$ exists then $\lim_{x \to x_0} g(x)$ exists also, and we have

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x).$$

4.11 Exercises

Exercise 4.1.
Calculate the limit of the following functions

1. $\lim\limits_{x\longrightarrow+\infty} \dfrac{x+\sqrt{x}}{x+1}$

2. $\lim\limits_{x\longrightarrow+\infty} \sqrt{x-3}-\sqrt{x+1}$

3. $\lim\limits_{x\longrightarrow0} \dfrac{x^2+|x|}{x^2-|x|}$

4. $\lim\limits_{x\longrightarrow5} \dfrac{\sqrt{(x-5)^2}}{x-5}$

5. $\lim\limits_{x\longrightarrow0} \dfrac{cosx-1}{x}$

6. $\lim\limits_{x\longrightarrow\frac{\pi}{4}} \dfrac{\tan x-1}{x-\frac{\pi}{4}}$

7. $\lim\limits_{x\longrightarrow0} \dfrac{\tan 5x}{\sin 3x}$.

Exercise 4.2.
Using the definition of the limit of a function, show that

$$1)\ \lim_{x\longrightarrow2} \frac{x^2-1}{x^2+1} = \frac{3}{5},\quad 2)\ \lim_{x\longrightarrow1} \frac{2}{(x-1)^2} = +\infty,\quad 3)\ \lim_{x\longrightarrow\infty} \frac{x^2-1}{x^2+1} = 1.$$

Exercise 4.3.
Study the continuity of the following functions

$$f(x) = \begin{cases} x\sin(\frac{3}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad g(x) = \begin{cases} \frac{\sqrt{x+1}-1}{\tan x} & \text{if } \ x \neq 0 \\ \frac{1}{2} & \text{if } x = 0 \end{cases} \ ;\ h(x) = \begin{cases} \frac{\sin(x-2)}{x^2-2x} & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$$

Determine a and b so that the function f is continuous at $x_0 = 2$

$$f(x) = \begin{cases} \frac{x^2+x-a}{x-2}, & \text{if } x > 2 \\ \frac{2x+b}{3}, & \text{if } x \leq 2. \end{cases}$$

Exercise 4.4.

1. Let f be a real function defined by

$$f(x) = \begin{cases} e^x - a & \text{if} \quad x < 0 \\ b\ln(1+x), & \text{if} \quad x \geq 0 \end{cases}$$

   Determine a and b so that f is continuous and differentiable on $\mathbb{R}$.

2. We define the real function g as follows

$$g(x) = \frac{|x+1|}{(x+1)(x^2 - x + 1)}$$

   Can we extend g by continuity at -1.

Exercise 4.5.
Let f be a function defined on $\mathbb{R}$ by

$$f(x) = x^5 - x^3 + x - 2$$

1. Show that $f(x) = 0$ admits a solution α with $1 < \alpha < 2$.
2. Determine the sign of the function $f(x)$, $\forall x \in \mathbb{R}$.

Chapter 5

Differentiable functions

5.1 Differentiable functions

Definition 5.1. Let $I \subset \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}$ be a real-valued function. Let $x_0 \in I$. We say that f **is differentiable at** x_0 if the following limit exists and is finite :

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \ell \in \mathbb{R}.$$

This limit, if it exists, is called the **derivative of** f **at** x_0 and is denoted by

$$f'(x_0) \quad \text{or} \quad \frac{df}{dx}(x_0).$$

5.1.1 Right and Left Derivative

Definition 5.2.

– We say that a function f is left-differentiable at a point $x_0 \in \mathbb{R}$ if the following limit from the left exists and is finite

$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} = f'_-(x_0) \quad \text{(left derivative)}$$

– We say that a function f is right-differentiable at a point $x_0 \in \mathbb{R}$ if the following limit from the right exists and is finite

$$\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = f'_+(x_0) \quad \text{(right derivative)}$$

Remark 5.1. For f to be differentiable at x_0, it is necessary and sufficient that f is differentiable from the left and from the right at x_0, and that the two limits are equal. i.e.,

$$\lim_{x\to x_0^-}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{x\to x_0^+}\frac{f(x)-f(x_0)}{x-x_0}=f'(x_0).$$

Definition 5.3. A function defined on an open interval I of $\mathbb{R}$ to $\mathbb{R}$ is said to be differentiable on I if it is differentiable at every point in I.

Example 5.1.

1. Consider :

$$f(x)=\begin{cases}x^2 & \text{if } x\le 1\\ 2x-1 & \text{if } x>1\end{cases}$$

Left derivative at $x=1$

$$f'_-(1)=\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^-}\frac{x^2-1}{x-1}=\lim_{x\to 1^-}\frac{(x-1)(x+1)}{x-1}=\lim_{x\to 1^-}x+1=2.$$

Right derivative at $x=1$

$$f'_+(1)=\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^+}\frac{2x-1-1}{x-1}=\lim_{x\to 1^+}\frac{2x-2}{x-1}=\lim_{x\to 1^+}2=2.$$

Thus, f is differentiable at $x=1$ and $f'(1)=2$.

2. The function $f(x)=|x|$ is differentiable for $x\neq 0$, but it is not differentiable at $x=0$. Indeed

The function $f(x)=|x|$ is piecewise defined as :

$$f(x)=\begin{cases}x & \text{if } x\ge 0,\\ -x & \text{if } x<0.\end{cases}$$

To study the differentiability at $x=0$, we need to check the left and right derivatives at $x=0$.

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{f(x)-0}{x}=\lim_{x\to 0^-}\frac{-x}{x}=-1.$$

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{f(x)-0}{x}=\lim_{x\to 0^+}\frac{x}{x}=1.$$

The left and right derivatives at $x = 0$ are not equal :

$$f'_-(0) = -1, \quad f'_+(0) = 1.$$

Thus, the function $f(x) = |x|$ is not differentiable at $x = 0$.

5.1.2 Differential

If f is differentiable at a, the **differential** of f at a, denoted df_a, is the linear map defined by :

$$df_a(h) = f'(a) \cdot h$$

It provides the best linear approximation to f near the point a :

$$f(a+h) \approx f(a) + f'(a)h.$$

Example 5.2. Let $f(x) = x^2$. We will compute the derivative, the differential, and the linear approximation at the point $a = 3$.
Derivative :

$$f'(x) = 2x \Longrightarrow f'(3) = 6$$

Differential at $a = 3$ The differential is :

$$df_3(h) = f'(3) \cdot h = 6h.$$

Linear approximation

$$f(3+h) \approx 6h + 9.$$

5.2 Geometric Interpretation

If the function f is differentiable at x_0, then the graph (Γ) has a tangent line at x_0. The equation of this tangent line (T) is given by the following formula :

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Remark 5.2. If the function f has a left derivative ℓ_- and a right derivative ℓ_+ at x_0, such that $\ell_- \neq \ell_+$, then the graph (Γ_f) of f has two half-tangents at $M_0(x_0, f(x_0))$, and we say that M_0 is a **corner point** of (Γ_f).

For example $f(x) = |x|$, has a corner point at $x = 0$.

Proposition 5.1. If f is differentiable at $x = a$, then f is continuous at $x = a$.

Proof. We have

$$\lim_{x\to a}[f(x) - f(a)] = \lim_{x\to a}[\frac{f(x) - f(a)}{x - a}][x - a]$$

And since f is differentiable at $x = a$, then

$$\lim_{x\to a}[f(x) - f(a)] = \lim_{x\to a}[\frac{f(x) - f(a)}{x - a}] \lim_{x\to a}[x - a] = f'(a) \cdot 0 = 0$$

So

$$\lim_{x\to a}[f(x) - f(a)] = 0 \Longrightarrow \lim_{x\to a} f(x) = f(a).$$

Therefore f is continuous at a.

Remark 5.3.

- For f to be differentiable at $x = a$, it must also be continuous at $x = a$.
- If f were not continuous at $x = a$, the derivative could not exist because there would be a discontinuity at that point.
- Thus, differentiability at a point $x = a$ guarantees continuity at that point.

5.3 Operations on differentiable functions

Theorem 5.1. Let f and g be two functions differentiable at x_0, and let $\alpha, \beta \in \mathbb{R}$. Then the functions $f + g$, fg, $\alpha f + \beta g$, and $\frac{f}{g}$ (if $g(x_0) \neq 0$) are also differentiable at x_0, and we have

$$\begin{aligned}
(f + g)'(x_0) &= f'(x_0) + g'(x_0) \\
(fg)'(x_0) &= f'(x_0)g(x_0) + f(x_0)g'(x_0) \\
(\alpha f + \beta g)'(x_0) &= \alpha f'(x_0) + \beta g'(x_0) \\
\left(\frac{f}{g}\right)'(x_0) &= \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}, \quad \text{if } g(x_0) \neq 0
\end{aligned}$$

Proposition 5.2. Let f be differentiable at x_0, and let g be differentiable at $f(x_0)$. Then the composition $g \circ f$ is differentiable at x_0, and we have

$$(g \circ f)'(x_0) = g'\big(f(x_0)\big) \cdot f'(x_0).$$

Proof. We want to compute the limit

$$\lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0}$$

We have

$$\frac{g(f(x)) - g(f(x_0))}{x - x_0} = \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \frac{f(x) - f(x_0)}{x - x_0} =$$

Since f is differentiable at x_0, and g is differentiable at $f(x_0)$, we can write

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

and

$$\lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} = g'(f(x_0)$$

Therefore

$$\lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} = g'\big(f(x_0)\big) \cdot f'(x_0).$$

Example 5.3.
Let

$$f(x) = x^2 + 1 \quad \text{and} \quad g(u) = \sin(u)$$

We define the composition :

$$(g \circ f)(x) = g(f(x)) = \sin(x^2 + 1)$$

To differentiate $g(f(x))$

$$(g \circ f)'(x) = g'\big(f(x)\big) \cdot f'(x)$$

We compute each part :

$$f'(x) = 2x \quad \text{and} \quad g'(u) = \cos(u) \Rightarrow g'\big(f(x)\big) = \cos(x^2 + 1)$$

So,

$$(g \circ f)'(x) = \cos(x^2 + 1) \cdot 2x.$$

Proposition 5.3. (Derivative of an Inverse Function)
Let f be a bijective function that is differentiable at a point x_0, and suppose that $f'(x_0) \neq 0$. Then the inverse function f^{-1} is differentiable at $y_0 = f(x_0)$, and its derivative is given by :

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'\left(f^{-1}(y_0)\right)}.$$

Example 5.4. Let $f(x) = e^x$. Then the inverse function is $f^{-1}(y) = \ln(y)$.
The derivative of f is $f'(x) = e^x$. Therefore

$$(\ln(y))' = \left(f^{-1}\right)'(y) = \frac{1}{e^{\ln(y)}} = \frac{1}{y}.$$

5.4 Higher-Order Derivatives

Definition 5.4. Let f be a real function, differentiable on an interval $I \subset \mathbb{R}$. We say that f is **n-times differentiable** on I if all its successive derivatives $f', f'', f^{(3)}, \ldots, f^{(n)}$ exist on I.
$f^{(n)}$ is called the **n-th derivative** of f, and we have by recurrence :

$$f^{(0)} = f(x),\ f^{(n)} = (f^{(n-1)})'.$$

Example 5.5. Let $f(x) = x^n$. We will calculate the first derivatives using the recurrence relation :

$$f'(x) = nx^{n-1}$$
$$f''(x) = n(n-1)x^{n-2}$$
$$f^{(3)}(x) = n(n-1)(n-2)x^{n-3}$$

$$f^{(n)}(x) = n! \quad \text{(where } n! \text{ is the factorial of } n\text{)}.$$

5.4.1 Leibniz Formula

Let f and g be functions that are n-times differentiable. Then the nth derivative of their product is given by :

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} C_n^k f^{(k)}(x) \cdot g^{(n-k)}(x)$$

Where $C_n^k = \frac{n!}{k!(n-k)!}$. This is known as the **Leibniz formula**.

Example 5.6. Let $f(x) = x^2$, $g(x) = e^x$. Compute $(fg)^{(3)}(x)$.
We know :

$$f(x) = x^2, \quad f'(x) = 2x, \quad f''(x) = 2, \quad f^{(n)}(x) = 0,\ n \geq 3$$

$$g(x) = e^x, \quad g^{(n)}(x) = e^x \quad \text{for all } n$$

Using Leibniz's formula :

$$(fg)^{(3)}(x) = C_3^0 f(x)g^{(3)}(x) + C_3^1 f'(x)g^{(2)}(x) + C_3^2 f''(x)g^{(1)}(x) + C_3^3 f^{(3)}(x)g(x)$$
$$= 1 \cdot x^2 e^x + 3 \cdot 2xe^x + 3 \cdot 2e^x + 0 = (x^2 + 6x + 6)e^x.$$

5.5 Taylor's Formula

Let f be a function that is n-times continuously differentiable on an interval around a point $a \in \mathbb{R}$. Then, for all x near a, we have :

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where $R_n(x)$ is the remainder term of the Taylor approximation.

Example 5.7.
Let $f(x) = e^x$. Since all derivatives of e^x are equal to e^x, we have :

$$f^{(n)}(x) = e^x \quad \text{and} \quad f^{(n)}(0) = 1$$

The Taylor polynomial of order n for $f(x)$ at $a = 0$ is :

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + R_n(x).$$

where $R_n(x)$ is the remainder term.

5.5.1 Taylor's Formula with Lagrange Remainder

Let f be a function that is $(n+1)$-times continuously differentiable on an open interval containing a and x. Then there exists a point ξ between a and x such that :

$$f(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

The final term is known as the **Lagrange remainder**.

Example 5.8. Taylor Expansion of $\sin(x)$ with Lagrange Remainder.
Let $f(x) = \sin(x)$. We expand f around $a = 0$ up to order 3. We have

$$\begin{aligned} f(x) &= \sin(x) & & \\ f'(x) &= \cos(x), & f'(0) &= 1 \\ f''(x) &= -\sin(x), & f''(0) &= 0 \\ f^{(3)}(x) &= -\cos(x), & f^{(3)}(0) &= -1 \\ f^{(4)}(x) &= \sin(x), & f^{(4)}(\xi) &= \sin(\xi) \end{aligned}$$

Taylor Expansion with Lagrange Remainder

$$\sin(x) = x - \frac{x^3}{3!} + \frac{\sin(\xi)}{4!}x^4 \quad \text{for some } \xi \in (0, x).$$

5.5.2 Taylor Maclaurin Formula

If $a = 0$, We obtain what is known as the Taylor Maclaurin formula with the Lagrange remainder

$$f(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\frac{f^{(n+1)}(\theta x)}{(n+1)!}(x-a)^{n+1},$$

where $0 < \theta < 1$.

5.6 Theorems on Differentiable Functions

5.6.1 Global and Local extremum - Fermat's Theorem

Definition 5.5. A function f is said to have a **local extremum** at a point x_0 if there exists an interval I around x_0 such that :

- $f(x_0) \geq f(x)$ for all $x \in I$ (then x_0 is a local maximum), or
- $f(x_0) \leq f(x)$ for all $x \in I$ (then x_0 is a local minimum).

Definition 5.6.

- We say that a function f has a global maximum (respectively, a global minimum) at the point x_0 if :

$$f(x_0) \geq f(x) \quad \text{(respectively, } f(x_0) \leq f(x)) \text{ for all } x \in \text{Domain}(f).$$

- We say that f has a **global extremum** at the point x_0 if f has either a global maximum or a global minimum at x_0.

Theorem 5.2. (Fermat's Theorem)
If a function f has a local extremum at a point c, and f is differentiable at c, then :

$$f'(c) = 0.$$

5.6.2 Rolle's Theorem

Theorem 5.3. (**Rolle's theorem**)
If a real-valued function f is continuous on a closed interval $[a, b]$, differentiable on the open interval $]a, b[$, and $f(a) = f(b)$, then there exists at least one c in the open interval $]a, b[$ such that

$$f'(c) = 0.$$

Example 5.9.

1. Let the function f be defined on $[0; 1]$ by

$$f(x) = x^2 - x$$

   f is continuous on $[0; 1]$, and differentiable on an interval $]0; 1[$, and

$$f(0) = f(1) = 0.$$

   Therefore according to Rolle's Theorem there exists a $c \in]0, 1[$ such that $f'(c) = 0 \Longleftrightarrow 2c - 1 = 0 \Longrightarrow c = \frac{1}{2}$.

2. To show that the equation

$$4x^3 - 18x^2 + 22x - 6 = 0$$

has at least one solution in the open interval $]1, 3[$, we define the function :

$$f(x) = x^4 - 6x^3 + 11x^2 - 6x$$

Observe that f is a polynomial function, and therefore continuous and differentiable on $\mathbb{R}$.
Compute the values at the endpoints of the interval :

$$f(1) = 1^4 - 6(1)^3 + 11(1)^2 - 6(1) = 0, \quad f(3) = 3^4 - 6(3)^3 + 11(3)^2 - 6(3) = 0$$

then by **Rolle's Theorem**, there exists $c \in]1, 3[$ such that

$$f'(c) = 0$$

thus

$$f'(x) = 4x^3 - 18x^2 + 22x - 6$$

Therefore, the equation $4x^3 - 18x^2 + 22x - 6 = 0$ has at least one solution in $]1, 3[$, as a consequence of Rolle's Theorem.

3. Let the function f be defined on $[-1; 1]$ by

$$f(x) = |x|.$$

but This function is continuous on $[-1; 1]$ and satisfies $f(-1) = f(1) = 1$ but it is not differentiable at 0 ($f'_R = 1$ and $f'_L = -1$). Therefore there does not exist a $c \in] - 1, 1[$ such that $f'(c) = 0$.

5.6.3 Theorem of finite increments

Theorem 5.4. Mean Value Theorem (Lagrange)
Let $f : [a; b] \longrightarrow \mathbb{R}$ be a continuous function on $[a; b]$, and differentiable on $]a; b[$. Then there exists a point $c \in]a; b[$, such that

$$f(b) - f(a) = (b - a).f'(c).$$

Corollary 5.1. If f is differentiable on an interval $I \subset R$, then, for all distinct $x_1; x_2 \in I$, there exists a point c between x_1 and x_2 such that :

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(c).$$

Example 5.10.
Prove $\forall x \in [0,+\infty[,\ e^x > x+1$ using the Mean Value Theorem. Let us consider the function :

$$f(x) = e^x$$

The function f is continuous and differentiable on $[0,x]$ for any $x > 0$.

Apply the Mean Value Theorem, for $x > 0$, there exists a $c \in]0,x[$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{e^x - 1}{x}$$

Since $f'(x) = e^x$, we have :

$$\frac{e^x - 1}{x} = e^c \quad \text{for some } c \in]0,x[$$

Multiply both sides by x :

$$e^x - 1 = xe^c \quad \Rightarrow \quad e^x = 1 + xe^c$$

Now, since $c > 0$, we know that $e^c > 1$. So

$$e^x = 1 + xe^c > 1 + x \cdot 1 = x + 1$$

Therefore, for all $x > 0$, $e^x > x + 1$.

Theorem 5.5.(L'Hopital's Rule)
Let f and g be functions that are differentiable on an open interval I containing a. Suppose that :

- $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0 \quad \text{or} \quad \pm\infty$,
- $g'(x) \neq 0$ for all x near a (except possibly at a),
- The limit $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$ exists or is $\pm\infty$.

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Remarks 5.4.

- This rule can also be applied as $x \to \infty$ or $x \to -\infty$.
- The rule applies to indeterminate forms of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

5.7 Convexity of a Curve

Definition 5.7. A function $f : I \to \mathbb{R}$, defined on an interval $I \subseteq \mathbb{R}$, is called **convex** if for all $x, y \in I$ and $t \in [0, 1]$, we have :

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

This inequality means that the graph of f lies *below the chord* connecting any two points on the graph.

Corollary 5.2. The graph of f is said to be

– **Convex** on I if :

$$f''(x) \geq 0 \quad \text{for all } x \in I.$$

– **Concave** on I if :

$$f''(x) \leq 0 \quad \text{for all } x \in I.$$

Geometric Interpretation : If f is convex, then the graph of the function lies *above* any of its tangents. If f is concave, the graph lies *below* its tangents.

Example 5.11. The function $f(x) = |x|$ is convex on $\mathbb{R}$. Indeed

We verify the definition of convexity : for all $x, y \in \mathbb{R}$ and for all $t \in [0, 1]$, using triangular inequality we obtain

$$|tx + (1-t)y| \leq t|x| + (1-t)|y|.$$

5.7.1 Point of Inflexion

Definition 5.8. Let f be a differentiable function on an interval $I \subseteq \mathbb{R}$, and let $x_0 \in I$. Let Γ_f denote the graph of f.

We say that x_0 is a **point of inflection** of f if the graph Γ_f changes concavity at the point $M_0 = (x_0, f(x_0))$, that is, the curve crosses its tangent at M_0.

Theorem 5.6. (**Point of Inflexion**)
A point $x_0 \in I$ is called a point of inflection if the concavity of f changes at x_0, that is :

$$f''(x_0) = 0 \quad \text{and } f'' \text{ changes sign at } x_0.$$

5.8 Asymptotes of a Curve

Let $f : D \to \mathbb{R}$ be a real-valued function defined on a domain $D \subseteq \mathbb{R}$. An **asymptote** is a line that the graph of a function approaches as the variable tends toward a finite value or toward infinity.

– **Vertical Asymptote :** The line $x = a$ is a vertical asymptote of f if :

$$\lim_{x \to a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = \pm\infty$$

– **Horizontal Asymptote :** The line $y = L$ is a horizontal asymptote if :

$$\lim_{x \to \pm\infty} f(x) = L$$

– **Oblique (Slant) Asymptote :** If there exists a line $y = ax + b$ such that :

$$\lim_{x \to \pm\infty} [f(x) - (ax + b)] = 0$$

then $y = ax + b$ is an oblique asymptote of the graph of f.

Example 5.12.

1. Consider the function

$$f(x) = \frac{1}{x - 2}.$$

This function is undefined at $x = 2$. We compute the limits :

$$\lim_{x \to 2^-} \frac{1}{x - 2} = -\infty, \quad \lim_{x \to 2^+} \frac{1}{x - 2} = +\infty.$$

Therefore, the line $x = 2$ is a vertical asymptote.

2. Consider the function

$$f(x) = \frac{1}{x + 1}.$$

We compute the limits at infinity

$$\lim_{x \to \infty} \frac{1}{x + 1} = 0, \quad \lim_{x \to -\infty} \frac{1}{x + 1} = 0.$$

Both limits equal 0, so the line $y = 0$ is a horizontal asymptote of the graph of $f(x)$.

3. Consider the function
$$f(x) = \frac{x^2+1}{x}.$$
We can rewrite it as
$$f(x) = x + \frac{1}{x}.$$
As $x \to \infty$ or $x \to -\infty$, the term $\frac{1}{x} \to 0$. We verify
$$\lim_{x\to\infty} (f(x) - x) = \lim_{x\to\infty} \frac{1}{x} = 0.$$
The same limit holds as $x \to -\infty$. The line $y = x$ is an oblique (slant) asymptote.

5.9 Construct the Graph of a Function

Let a real function $f : \mathbb{R} \to \mathbb{R}$, the graph of f can be studied and drawled by following these steps :

1. Determine the domain of definition of the function.

2. **Symmetry :**
   – If $f(-x) = f(x)$, then the function is even (symmetric about the y-axis).
   – If $f(-x) = -f(x)$, then the function is odd (symmetric about the origin).

3. To study the asymptotic behavior of a function. We distinguish three types of asymptotes : Vertical asymptotes, Horizontal asymptotes, Oblique asymptotes.

4. **Limits and Continuity :** Analyze the limits and discontinuities.

5. **First Derivative** $f'(x)$
   – Study the sign of $f'(x)$ to determine intervals of increase or decrease.
   – Critical points occur where $f'(x) = 0$ or is undefined.
   – Use the First Derivative Test to identify local maxima and minima.
6. **Second Derivative ($f''(x)$) :**
   – Study the sign of $f''(x)$ to determine intervals of concavity.

- Points where $f''(x) = 0$ and concavity changes are inflection points.

7. **Graph Sketching :**
   - Plot the points : extremum, inflection points.
   - Draw asymptotes lines.
   - Draw the curve using the information above.

Example 5.13.

Analysis of the Function $f(x) = \ln\left(\frac{e^{2x}+5}{e^x-2}\right)$

- The function is defined when the denominator is strictly positive and the whole expression inside the logarithm is positive :

$$e^x - 2 > 0 \Rightarrow x > \ln(2)$$

  So the domain is :

$$D_f =]\ln(2),\ +\infty[$$

- Limit at $\ln(2)^+$

$$\lim_{x\to\ln(2)^+} f(x) = \ln\left(\frac{e^{2\ln(2)}+5}{e^{\ln(2)}-2}\right) = \ln\left(\frac{4+5}{0^+}\right) = +\infty$$

  There is a vertical asymptote at $x = \ln(2)$.

- Behavior at Infinity

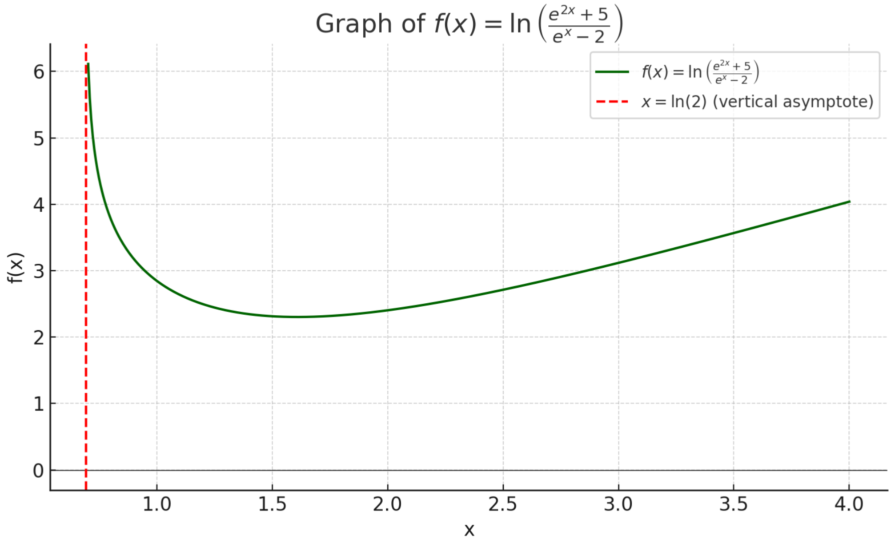
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \ln\left(\frac{e^{2x}+5}{e^x-2}\right) = \lim_{x\to+\infty} \ln(\frac{e^x+5e^{-x}}{1-2e^{-x}}) = +\infty.$$

- First derivative is used to study the increasing or decreasing of the function

$$f'(x) = \frac{(2(e^x-2)e^x - e^{2x} - 5)e^x}{(e^x-2)(e^{2x}+5)}$$

- Second Derivative is used to analyze concavity and inflection points.

$$f''(x) = \frac{2e^x(e^{4x}+10e^{3x}-30e^{2x}+40e^x+25)}{e^{6x}-4e^{5x}+14e^{4x}-40e^{3x}+65e^{2x}-100e^x+100}$$



Vertical asymptote at $x = \ln(2)$ (dashed red line).
The function increases for $x > \ln(5)$, and tends to $+\infty$.

5.10 Exercises

Exercise 5.1.
Study the differentiability at x_0 of functions :

$$f(x) = (x-1)|x-1|;\ x_0 = 1, \quad g(x) = |x-1| + |x+1|, \quad x_0 = -1$$

$$h(x) = x + (x-1)\arcsin\sqrt{\frac{x}{x+1}};\ \ x_0 = 1.$$

Exercise 5.2.
Let $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$.

1. Is f differentiable at $x = 0$?
2. Is f' continuous at $x = 0$?

Under what condition does the function

$$g(x) = \begin{cases} x^n \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \\ & \end{cases}$$

admits a continuous derivative at the point $x_0 = 0$?

Exercise 5.3.
Determine the values of α and β for which the function

$$f(x) = \begin{cases} \alpha + \beta x^2, & \text{if } |x| < 1 \\ \frac{1}{|x|}, & \text{if } |x| \geq 1, \end{cases}$$

is continuous and differentiable on $\mathbb{R}$.

Exercise 5.4.
Calculate the derivatives of the following functions

$$f(x) = \ln(1+\ln(x)), \quad g(x) = e^{\ln(x)+2}, \quad h(x) = \begin{cases} x(1+e^{\frac{1}{x}})^{-1}, & \text{if; } x < 0 \\ \ln(3+\sqrt[3]{x^5}), & \text{if } \ x \geq 0. \end{cases} \quad L(x) = \tan^2(1-2x)$$

Exercise 5.5.
Let the function $y(x) = \frac{2-x^2}{x^2}$.

1. Show that $y'(x) \neq 0$ on $[-1, 1]$
2. Is there a contradiction with the theorem of Rolle ?

Exercise 5.6.
Using the Mean Value Theorem, establish the following inequalities :

1. For all $x \in [0, +\infty[$,
$$\frac{x}{1+x} \leq \ln(x+1) \leq x.$$
2. For all $x, y \in \mathbb{R}$,
$$|\sin(x) - \sin(y)| \leq |x - y|.$$

Exercise 5.7.

1. Write the Taylor-Lagrange formula of order 5 at 0 for the function $\sin(x)$.
2. What is the value of the limit :
$$\lim_{x \to 0} \frac{\sin x - x}{x^2}?$$
3. Show that for all $x \geq 0$,
$$x - \frac{x^3}{6} \leq \sin x \leq x - \frac{x^3}{3} + \frac{x^5}{120}.$$
What happens when x is negative ?

Chapter 6

Elementary functions

6.1 Logarithm and Exponential

6.1.1 Natural logarithm

Definition 6.1. The natural logarithm, denoted $\ln(x)$, is the logarithm to the base e, where $e \approx 2.71828$.

It is defined for all $x > 0$ and is the inverse of the exponential function e^x. That is :

$$\ln(x) = y \quad \text{if and only if} \quad e^y = x.$$

Properties 6.1.

- $\ln(x)$ is defined on $]0, +\infty[$
- $\ln(x)$ is continuous and differentiable on $]0, +\infty[$, and

$$\frac{d}{dx}\ln(x) = \frac{1}{x}, \quad x > 0$$

- For all $x > 0$, $\ln(x)$ is increasing
- $\lim\limits_{x \to 0} \dfrac{\ln(x+1)}{x} = 1$
- $\ln(ab) = \ln(a) + \ln(b)$,
- $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$,
- $\ln(a^r) = r\ln(a)$, for all $a, b > 0$ and $r \in \mathbb{R}$

6.1.2 Exponential function

Definition 6.2. The exponential function, denoted $\exp(x)$ or e^x, is defined for all real numbers x.

It is the inverse of the natural logarithm function $\ln(x)$. That is :

$$\exp(x) = e^x, \quad \text{and} \quad \ln(e^x) = x.$$

Properties 6.2. The exponential function satisfies the following properties :

1. For all $x > 0$, $\exp(\ln(x)) = x$ and for all $x \in \mathbb{R}$, $\ln(\exp(x)) = x$
2. $\exp(a + b) = \exp(a) \times \exp(b)$
3. $\exp(nx) = (\exp(x))^n$, for all $n \in \mathbb{N}$
4. The exponential function is continuous and strictly increasing, and satisfies :
$$\lim_{x\to-\infty} \exp(x) = 0 \quad \text{and} \quad \lim_{x\to+\infty} \exp(x) = +\infty$$
5. The exponential function is differentiable and :
$$\frac{d}{dx}\exp(x) = \exp(x), \quad \text{for all } x \in \mathbb{R}$$
It is convex, and satisfies the inequality :
$$\exp(x) \geq 1 + x$$

6.1.3 Logarithm with an arbitrary base

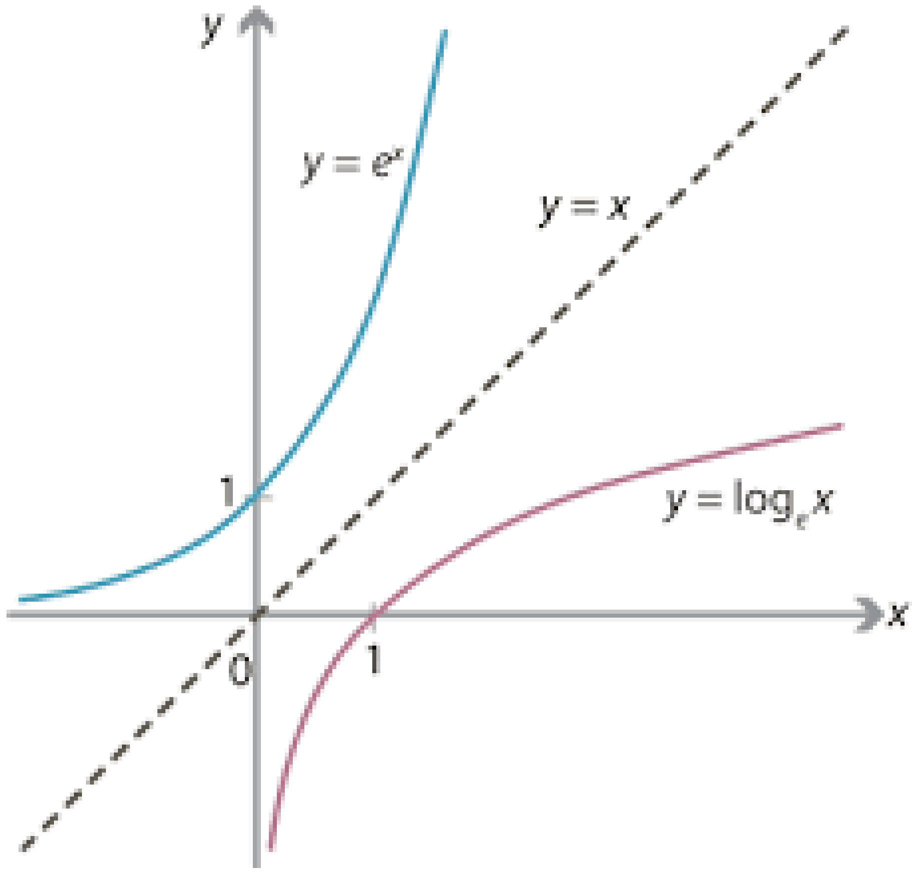
Definition 6.3. The logarithm with an arbitrary base $a > 0$, $a \neq 1$, is denoted by $\log_a(x)$. It is defined for all $x > 0$, and it is the inverse of the exponential function x^a. That is :

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}.$$

Properties 6.3. Let a be a strictly positive real number such that $a \neq 1$. For all $x, y \in\ ]0, +\infty[$ and $n \in \mathbb{Z}$, the following properties hold :

1. If $a > 1$ and $x \in]1, +\infty[$, then the logarithmic function $\log_a(x)$ is strictly increasing and concave.
2. If $x \in]0, 1[$, then the function $\log_a(x)$ is strictly decreasing and convex.
3. $\log_a(x \times y) = \log_a(x) + \log_a(y)$
4. $\log_a\left(\frac{1}{x}\right) = -\log_a(x)$
5. $\log_a(x^n) = n\log_a(x)$
6. $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$.

graphical representation of the log and exp



6.1.4 Power Function

By definition, for $a > 0$ and $b \in \mathbb{R}$

$$a^b = \exp(b \ln(a))$$

Remark 6.1.

1. $\sqrt{a} = a^{1/2} = \exp\left(\frac{1}{2}\ln(a)\right)$
2. The n-th root of a can be written as :

$$\sqrt[n]{a} = a^{1/n} = \exp\left(\frac{1}{n}\ln(a)\right)$$

Proposition 6.1. Let $x, y > 0$ and $a, b \in \mathbb{R}$. The following properties hold :

- $x^{a+b} = x^a \cdot x^b$
- $x^{-a} = \frac{1}{x^a}$
- $(xy)^a = x^a \cdot y^a$
- $(x^a)^b = x^{ab}$
- $\ln(x^a) = a\ln(x)$

6.2 Hyperbolic functions

Definition 6.4.

- The hyperbolic sine (sinh) and hyperbolic cosine (cosh) functions are defined on $\mathbb{R}$ with values in $\mathbb{R}$ as follows :

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

- The hyperbolic tangent and hyperbolic cotangent are defined by :

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \quad x \neq 0$$

Properties 6.4.

1. Fundamental Identity
$$\cosh^2(x) - \sinh^2(x) = 1$$
2. The hyperbolic sine (sinh) and hyperbolic cosine (cosh) functions are differentiable on $\mathbb{R}$, and their derivatives are :
$$\sinh'(x) = \cosh(x), \qquad \cosh'(x) = \sinh(x)$$
3. The function sinh is **odd**, strictly increasing on $\mathbb{R}$, and
$$\lim_{x\to-\infty} \sinh(x) = -\infty, \qquad \lim_{x\to+\infty} \sinh(x) = +\infty$$
4. The function cosh is **even**, strictly decreasing on $]-\infty, 0]$, and strictly increasing on $[0, +\infty)$ and
$$\lim_{x\to-\infty} \cosh(x) = \lim_{x\to+\infty} \cosh(x) = +\infty$$

5. The hyperbolic tangent function tanh is differentiable on $\mathbb{R}$, and for every $x \in \mathbb{R}$, its derivative is given by :

$$\tanh'(x) = 1 - \tanh^2(x)$$

6. The hyperbolic cotangent function coth is differentiable on $\mathbb{R} \setminus \{0\}$, and for every $x \in \mathbb{R} \setminus \{0\}$, its derivative is given by :

$$\coth'(x) = -\frac{1}{\sinh(x)}.$$

7. The function (tanh) is **odd** and **strictly increasing** on $\mathbb{R}$.

$$\lim_{x\to\infty} \tanh(x) = 1, \qquad \lim_{x\to-\infty} \tanh(x) = -1.$$

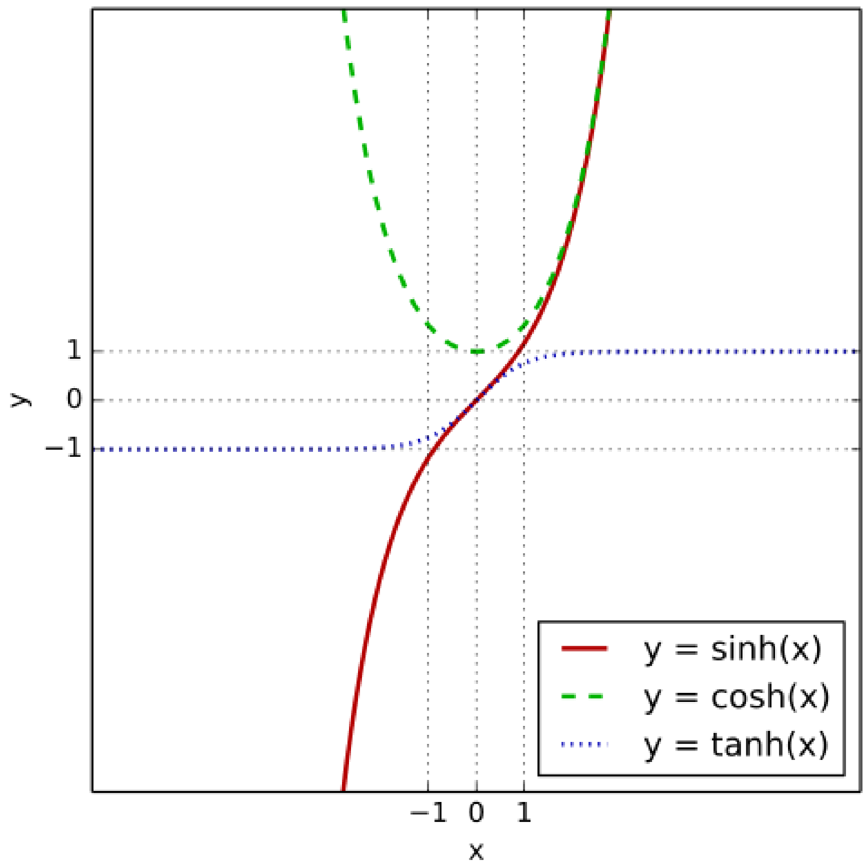
8. The function coth is **odd** and **strictly decreasing** on $\mathbb{R} \setminus \{0\}$

$$\lim_{x\to\infty} \coth(x) = 1, \qquad \lim_{x\to-\infty} \coth(x) = -1.$$

9. Parity

$$\begin{aligned} \sinh(-x) &= -\sinh(x) \quad \text{(odd function)} \\ \cosh(-x) &= \cosh(x) \quad \text{(even function)} \\ \tanh(-x) &= -\tanh(x) \quad \text{(odd function)} \end{aligned}$$

graphical representation of the hyperbolic functions sinh, cosh, tanh



6.3 Reciprocal Hyperbolic Functions

Definition 6.5. The hyperbolic sine function is a bijection from $\mathbb{R}$ to $\mathbb{R}$. Its inverse is called the inverse hyperbolic sine and is denoted by $\operatorname{argsh}(x)$

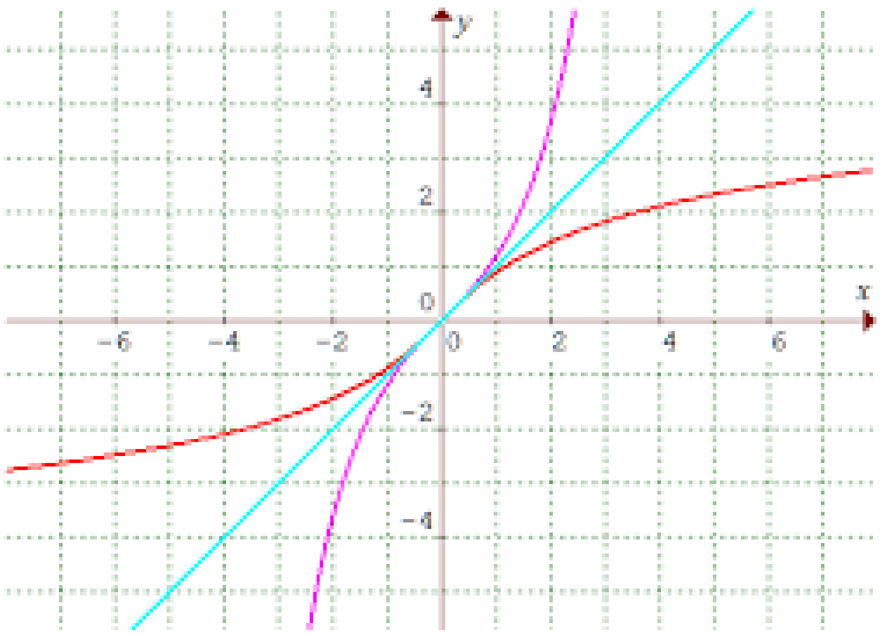
$$\operatorname{argsh} : \begin{array}{c} \mathbb{R} \longrightarrow \mathbb{R} \\ x \longmapsto \operatorname{argsh}(x), \end{array}$$

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R} : y = \operatorname{argsh}(x) \Longrightarrow x = \sinh(y).$$

Properties 6.5. The function argsh(x) (also written as $\sinh^{-1}(x)$) has the following properties :

- $\forall x \in \mathbb{R} : \text{argsh}(\sinh x) = x$, and $\forall x \in [0, \pi] : \sinh(\text{argsh}(x)) = x$.
- it is continuous on $\mathbb{R}$,
- strictly increasing,
- odd : $\text{argsh}(-x) = -\text{argsh}(x)$,
- differentiable on $\mathbb{R}$, in particular on the interval $[-1, 1]$. Its derivative is given by :

$$\frac{d}{dx}\text{argsh}(x) = \frac{1}{\sqrt{x^2+1}}$$



graphical representation of the Argsh and Sinh.

Definition 6.6. The hyperbolic cosine function is a bijection from $[0, +\infty[$ to $[1, +\infty[$. Its inverse is called the inverse hyperbolic cosine and is denoted by argch(x)

$$\text{argch} : \begin{array}{l} [1, +\infty[\longrightarrow [0, +\infty[\\ x \longmapsto \text{argch}(x), \end{array}$$

$$\forall x \in [1, +\infty[, \forall y \in [0, +\infty[: y = \text{argch}(x) \Longleftrightarrow x = \cosh(y).$$

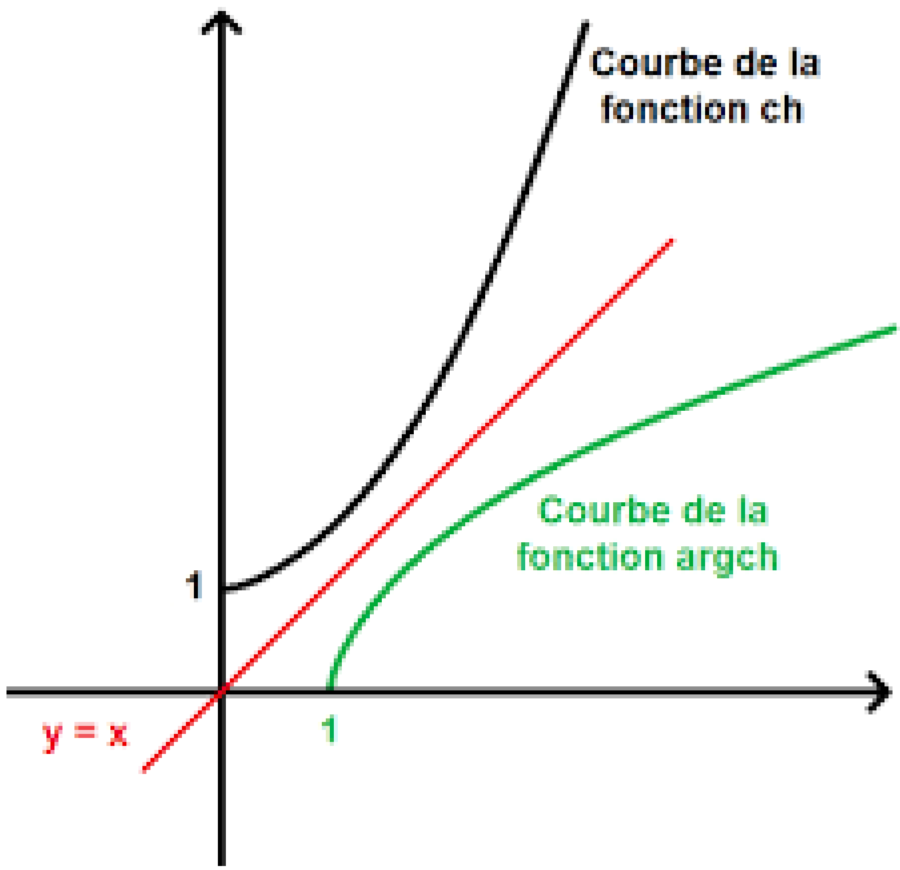
Proposition 6.6. The function argch(x) (also written as $\cosh^{-1}(x)$) has the following properties :

- it is continuous on the interval $[1, +\infty[$,
- strictly increasing,
- not an odd or even function,
- differentiable on $]1, +\infty[$.

Its derivative is given by :

$$\frac{d}{dx}\operatorname{argch}(x) = \frac{1}{\sqrt{x^2 - 1}}, \quad x > 1$$

graphical representation of the Argch and Cosh



Definition 6.7. The hyperbolic tangent function defines a bijection from $\mathbb{R}$ onto its image $]-1, 1[$. The inverse function is called the inverse hyperbolic tangent function

and is denoted by argth, that is :

$$\text{argth}: \begin{array}{l}]-1,1[\longrightarrow \mathbb{R} \\ x \longmapsto \text{argth}(x), \end{array}$$

$$\forall x \in]-1,1[, \forall y \in \mathbb{R} : y = \text{argth}(x) \Longleftrightarrow x = \tanh(y).$$

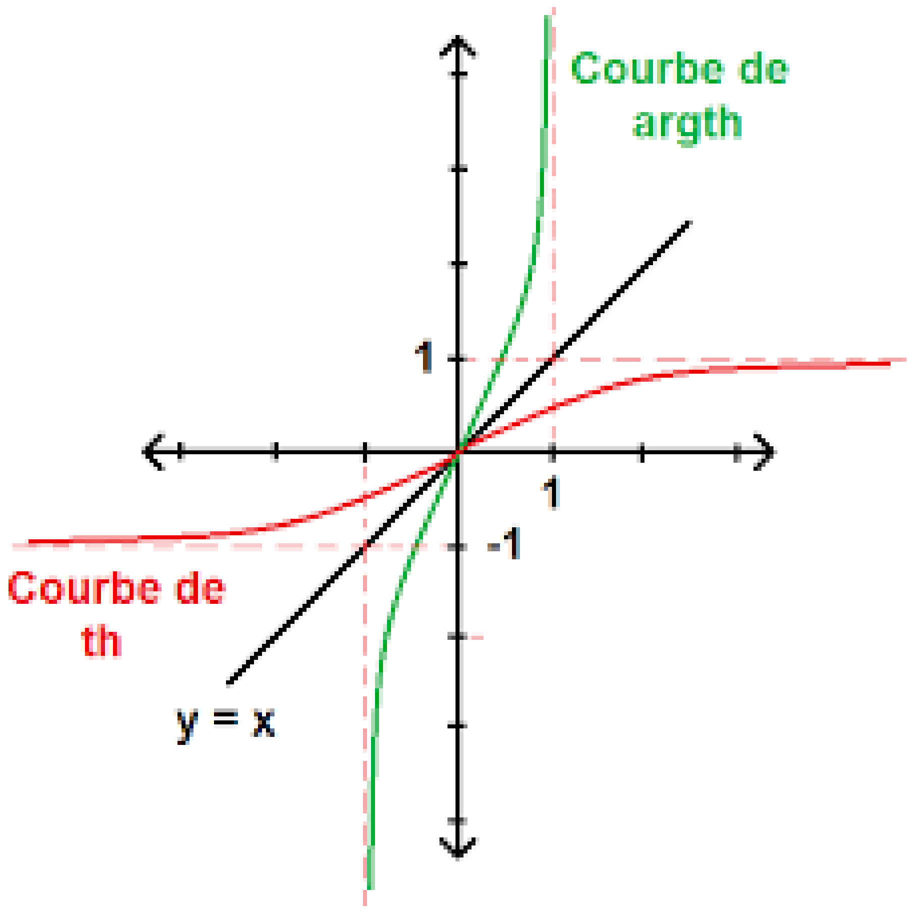
Properties 6.7. The function $\text{argth}(x)$ (also written as $\tanh^{-1}(x)$) has the following properties :

- it is continuous on the open interval $]-1,1[$,
- strictly increasing,
- odd : $\text{argth}(-x) = -\,\text{argth}(x)$,
- differentiable on $]-1,1[$.

Its derivative is given by :

$$\frac{d}{dx}\,\text{argth}(x) = \frac{1}{1-x^2}, \quad |x| < 1.$$

graphical representation of the Argth and Tanh



6.4 Exercises

Exercise 6.1.
Show that for all $x \in [-1,1]$, **we have**

$$\sin(\arccos x) = \sqrt{1-x^2} = \cos(\arcsin x).$$

Exercise 6.2.

1. Calculate :

$$\arcsin\left(\sin\frac{\pi}{3}\right), \quad \arccos\left(\cos\frac{\pi}{3}\right), \quad \arccos\left(\sin\frac{\pi}{3}\right).$$

2. Calculate :

$$\arccos\left(\cos\frac{4\pi}{3}\right), \quad \arccos\left(\cos\frac{7\pi}{3}\right), \quad \arcsin\left(\sin\frac{2\pi}{3}\right), \quad \arcsin\left(\sin\frac{7\pi}{3}\right).$$

Exercise 6.3.
Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by :

$$f(x) = \operatorname{argcosh}\left(\sqrt{1+x^2}\right).$$

1. Determine the domain of definition of the function f.
2. Compute $\operatorname{argcosh}(\cosh(t))$ for all $t \in \mathbb{R}$.
3. Show that for all $x \in \mathbb{R}$, we have :

$$f(x) = \operatorname{argsinh}(|x|).$$

4. Compute $f'(x)$ for all $x \in \mathbb{R}^*$.
5. Is the function f differentiable at $x = 0$?

Exercise 6.4.

1. Calculate the exact values of :

$$\arcsin\left(\frac{\sqrt{3}}{2}\right), \quad \arccos\left(-\frac{1}{2}\right), \quad \arctan(1).$$

2. Simplify the expressions :

$$\arcsin(\sin 5\pi/6), \quad \arccos(\cos 7\pi/4), \quad \arctan(\tan 3\pi).$$

3. Prove that for all $x \in [-1, 1]$,

$$\sin(\arccos x) = \sqrt{1 - x^2}.$$

4. Show that :

$$\arcsin(x) + \arccos(x) = \frac{\pi}{2}, \quad \forall x \in [-1, 1].$$

5. Calculate derivatives :

$$\frac{d}{dx}\arcsin(x), \quad \frac{d}{dx}\arccos(x), \quad \frac{d}{dx}\arctan(x).$$

6. Solve for x :

$$\arcsin(x) = \frac{\pi}{6}, \quad \arccos(x) = \frac{\pi}{3}, \quad \arctan(x) = 1.$$

7. Evaluate :

$$\arccos(\cos 5\pi/3), \quad \arcsin(\sin 11\pi/6).$$

8. Express $\arctan\left(\frac{1}{x}\right)$ in terms of $\arctan(x)$, for $x > 0$.
9. Show that :

$$\tan(\arcsin x) = \frac{x}{\sqrt{1 - x^2}}, \quad x \in (-1, 1).$$

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