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قسم الرياضيات

Exercise Handouts

Analysis 3

Summary of lessons and solved exercices

For students of the second year Engineering degree in Science and Technology by

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Summary

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General introduction

This document is designed for second-year engineering students in Science and Technology. It provides a rigorous and comprehensive exploration of key mathematical concepts in analysis, with a particular focus on vector calculus, series, and integral transforms. It is designed to serve as both a theoretical reference and a practical exercise guide, offering a structured approach to mastering fundamental mathematical techniques.

The content is organized into four main chapters:

Chapter 1: *Vector Analysis- Examines vector fields, gradient, divergence, and curl, as well as fundamental theorems such as Green's, Stokes', and the divergence theorem.*

Chapter 2: *Series- Explores numerical and functional series, convergence tests, and their significance in mathematical analysis.*

Chapter 3: *Fourier Series- Introduces the decomposition of periodic functions into trigonometric components, discussing convergence properties and practical applications.*

Chapter 4: *Fourier and Laplace Transforms- Investigates these powerful mathematical tools, highlighting their use in solving differential equations and analyzing signals.*

Each chapter is supplemented with a series of carefully curated exercises, reinforcing theoretical knowledge through hands-on problem-solving. By integrating theory with application, this document aims to be an invaluable resource for students, researchers, and professionals in mathematics, physics, and engineering.

Chapitre 1
Vectorial analysis.

Exercise 1: *Find the gradient vector field of f.*

1. $f(x,y) = y\sin(xy)$.
2. $f(x,y) = \sqrt{2x+3y}$.
3. $f(x,y) = \frac{1}{2}(x-y)^2$.
4. $f(x,y) = \ln(1+x^2+2y^2)$.
5. $f(x,y,z) = \sqrt{x^2+y^2+z^2}$.
6. $f(x,y,z) = x^2ye^{\frac{y}{z}}$.
7. $f(x,y,z) = 2x^2(y-z^3)$.
8. $f(x,y,z) = xyz\sin(xy)$.

Exercise 2: *Evaluate the following line integrals.*

1. $\displaystyle\int_C \frac{x}{y}dr$, *C*: $x = t^3$, $y = t^4$, $1 \leq t \leq 2$.
2. $\displaystyle\int_C xy^4 dr$, *C is the right half of the circle* $x^2+y^2 = 16$.
3. $\displaystyle\int_C (x^2y+\sin(x))dy$, *C is the arc of the parabola* $y = x^2$ *from* $(0,0)$ *to* (π,π^2).
4. $\displaystyle\int_C (x+2y)dx + x^2dy$, *C consists of the line segment from* $(0,0)$ *to* $(2,1)$ *and from* $(2,1)$ *to* $(3,0)$.
5. $\displaystyle\int_C x^2dx + y^2dy$, *C consists of the arc of the circle* $x^2+y^2 = 4$ *from* $(2,0)$ *to* $(0,2)$ *followed by the line segment from* $(0,2)$ *to* $(4,3)$.
6. $\displaystyle\int_C xe^{yz}ds$, *C is the line segment from* $(0,0,0)$ *to* $(1,2,3)$.
7. $\displaystyle\int_C z^2dx + x^2dy + y^2dz$, *C is the line segment from* $(1,0,0)$ *to* $(4,1,2)$.
8. $\displaystyle\int_C (y+z)dx + (x+z)dy + (x+y)dz$, *C is the line segment from* $(0,0,0)$ *to* $(1,0,1)$ *and from* $(1,0,1)$ *to* $(0,1,2)$.

Exercise 3: *Find a function f such that $\overrightarrow{F} = \overrightarrow{\nabla} f$, then, evaluate $\displaystyle\int_C \overrightarrow{F}.d\overrightarrow{r}$ along the given curve C.*

1. $\overrightarrow{F}(x,y) = (3+2xy^2)\overrightarrow{\mathbf{i}} + 2x^2y\overrightarrow{\mathbf{j}}$, *C is the arc of the hyperbola* $y = \frac{1}{x}$ *from* $(1,1)$ *to* $(4, \frac{1}{4})$.
2. $\overrightarrow{F}(x,y) = x^2y^3\overrightarrow{\mathbf{i}} + x^3y^2\overrightarrow{\mathbf{j}}$, *C:* $\overrightarrow{r}(t) = \langle t^3-2t, t^3+2t\rangle, 0 \le t \le 1$.
3. $\overrightarrow{F}(x,y,z) = yze^{xz}\overrightarrow{\mathbf{i}} + e^{xz}\overrightarrow{\mathbf{j}} + xye^{xz}\overrightarrow{\mathbf{k}}$, *C:* $\overrightarrow{r}(t) = (t^2+1)\overrightarrow{\mathbf{i}} + (t^2-1)\overrightarrow{\mathbf{j}} + (t^2-2t)\overrightarrow{\mathbf{k}}, 0 \le t \le 2$.
4. $\overrightarrow{F}(x,y,z) = \sin(y)\overrightarrow{\mathbf{i}} + (x\cos(y)+\cos(z))\overrightarrow{\mathbf{j}} - y\sin(z)\overrightarrow{\mathbf{k}}$, *C:* $\overrightarrow{r}(t) = \sin(t)\overrightarrow{\mathbf{i}} + t\overrightarrow{\mathbf{j}} + 2t\overrightarrow{\mathbf{k}}, 0 \le t \le \frac{\pi}{2}$.

Exercise 4: *Use Green's Theorem to evaluate the line integral along the given positively oriented curve.*

1. $\int_C (y + e^{\sqrt{x}})dx + (2x + \cos(y^2))dy$, *C is the boundary of the region enclosed by the parabolas* $y = x^2$ *and* $x = y^2$.
2. $\int_C y^4dx + 2xy^3dy$, *C is the ellipse* $x^2 + 2y^2 = 2$.
3. $\oint_C \langle xy, x^2y^3\rangle.\overrightarrow{N}ds$, *C is the triangle with vertices* $(0,0), (1,0)$ *and* $(1,2)$.
4. $\oint_C \langle y - \cos(y), y\rangle.\overrightarrow{N}ds$, *C is the circle* $(x-3)^2 + (y+4)^2 = 4$.

Exercice 5: *Find the curl and the divergence of the vector field.*

1. $\overrightarrow{F}(x,y,z) = \langle xy^2z^2, y^4z^3, x^2y^2z\rangle$
2. $\overrightarrow{F}(x,y,z) = \langle \ln(2y+3z), \ln(x+3z), \ln(x+2y)\rangle$
3. $\overrightarrow{F}(x,y,z) = e^x\sin(y), e^y\sin(z), e^z\sin(x)\rangle$
4. $\overrightarrow{F}(x,y,z) = \langle \arctan(xy), \arctan(yz), \arctan(zx)\rangle$

Exercice 6. *Evaluate the surface integral.*

1. $\iint_S (x+y+z)dS$ *S is the parallelogram with parametric equation* $x = u+v$, $y = u-v$, $z = 1+2u+v$, $0 \le u \le 2$, $0 \le v \le 1$.

2. $\displaystyle\iint_S x^2yz dS$, *S is the part of the plane* $z = 1+2x+3y$ *that lies above the rectangle* $[0,3]\times[0,2]$.

3. $\displaystyle\iint_S \langle -x,-y,z^3\rangle .dS$, *S is the part of the cone* $z = \sqrt{x^2+y^2}$ *between the planes* $z = 1$ *and* $z = 3$ *with downward orientation.*

4. $\displaystyle\iint_S \langle y,-x,2z\rangle .dS$, *S is the hemisphere* $x^2+y^2+z^2 = 4$, $z \geq 0$, *oriented downward.*

Exercice 7. *Use the Divergence Theorem to calculate the surface integral* $\displaystyle\iint_S \overrightarrow{F}.dS$*; that is, calculate the flux of* F *across* S.

1. $\overrightarrow{F}(x,y,z) = xye^z\,\overrightarrow{\mathbf{i}} + xy^2z^3\,\overrightarrow{\mathbf{j}} - ye^z\,\overrightarrow{\mathbf{k}}$. *S is the surface of the coordinate planes and the planes* $x = 3,, y = 2$ *and* $z = 1$.

2. $\overrightarrow{F}(x,y,z) = (x^3+y^3)\,\overrightarrow{\mathbf{i}} + (y^3+z^3)\,\overrightarrow{\mathbf{j}} + (z^3+x^3)\,\overrightarrow{\mathbf{k}}$. *S is the sphere with center the origin and radius 2.*

3. $\overrightarrow{F}(x,y,) = xe^y\,\overrightarrow{\mathbf{i}} + (z-e^y)\,\overrightarrow{\mathbf{j}} + xy\,\overrightarrow{\mathbf{k}}$. *S is the ellipsoid* $x^2+2y^2+3z^2 = 4$.

Exercice 8. *Verify that Stokes' Theorem is true for the given vector field* $\overrightarrow{F}$ *and surface S.*

1. $\overrightarrow{F}(x,y,z) = -y\,\overrightarrow{\mathbf{i}} + x\,\overrightarrow{\mathbf{j}} - 2\,\overrightarrow{\mathbf{k}}$. *S is the cone* $z^2 = x^2+y^2$, $0 \leq z \leq 4$, *oriented downward.*

2. $\overrightarrow{F}(x,y,z) = -2yz\,\overrightarrow{\mathbf{i}} + y\,\overrightarrow{\mathbf{j}} + 3x\,\overrightarrow{\mathbf{k}}$. *S is the part of the paraboloid* $z = 5-x^2-y^2$ *that lies above the plane* $z = 1$, *oriented upward.*

3. $\overrightarrow{F}(x,y,z) = y\,\overrightarrow{\mathbf{i}} + z\,\overrightarrow{\mathbf{j}} + x\,\overrightarrow{\mathbf{k}}$. *S is the hemisphere* $x^2+y^2+z^2 = 1$, $y \geq 0$, *oriented in the direction of the positive y-axis.*

Exercice 9. *(Examen)*

1. *Find the work done when a force* $\overrightarrow{F} = \langle x^2-y^2+x, -2xy-y\rangle$ *moves a particle in the* $xy-$*plane from* $(0,0)$ *to* $(1,1)$ *along the parabola* $y^2 = x$. *Is the work done different when the path is the straight line* $y = x$?

2. *The electrostatic potential at* $(0,0,-a)$ *of a charge of constant density* σ *on hemisphere* $S: x^2+y^2+z^2=a^2, z\geq 0$ *is*

$$U=\iint \frac{\sigma}{\sqrt{x^2+y^2+(z+a)}}dS$$

Evaluate U.

Exercice 10. *(Exam)*

1. *Let* $\int (3x-5ydx+(x-6y)dy$ *on* C, *where* C *is the ellipse* $\frac{x^2}{4}+y^2=1$ *in the counterclockwise. Evaluate the integral by Green's theorem and directly.*

2. *Let the vector field* $\overrightarrow{F}=\langle 2x-4y-5z,-4x+2y,-5x+6z\rangle$. *Verify that* $\overrightarrow{F}$ *is conservative and find its potential* f.

Exercise 11 *(Exam)*

1. *Given* $\overrightarrow{F}=\langle 12x^2+3y^2+5y, 6xy-3y^2+5x\rangle$, *find* f *such that* $\overrightarrow{\nabla}f=\overrightarrow{F}$ *and evaluate* $\int_C \overrightarrow{F}.d\overrightarrow{r}$, C *is the arc of the hyperola* $y=\frac{1}{x}$ *from* $(1,1)$ *to* $(4,\frac{1}{4})$.

2. *Use Stokes theorem to evaluate* $\int_C \overrightarrow{F}.d\overrightarrow{r}$, *where* $\overrightarrow{F}=\langle x^3-x^5, 7e^x, z^5+z^9\rangle$ *and where* C *is the boundary of the part of the plane* $6x+3y+z=12$ *in the first octant, oriented counterclockwise.*

3. *For* $\overrightarrow{F}=\langle xy^2, yz^2, x^2z\rangle$, *use the divergence theorem to evaluate* $\iint_S \overrightarrow{F}.d\overrightarrow{S}$ *where* S *is the sphere of radius* 3 *centred at origin. Orient the surface with the outward pointing normal vector.*

Solution of exercise 1.

Reminder.

•*If f is a scalar function of two variables, its gradient $\overrightarrow{\nabla} f$ is defined by*

$$\overrightarrow{\nabla} f(x,y) = f_x(x,y)\,\overrightarrow{\mathbf{i}} + f_y(x,y)\,\overrightarrow{\mathbf{j}}.$$

and is called a **gradient vector field** *on* $\mathbb{R}^2$.

• *If f is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$ given by*

$$\overrightarrow{\nabla} f(x,y,z) = f_x(x,y,z)\,\overrightarrow{\mathbf{i}} + f_y(x,y,z)\,\overrightarrow{\mathbf{j}} + f_z(x,y,z)\,\overrightarrow{\mathbf{k}},$$

and is called a **gradient vector field** *on* $\mathbb{R}^3$.

Solution.

1. $\overrightarrow{\nabla} f(x,y) = \langle y^2\cos(xy), \sin(xy) + yx\cos(xy)\rangle$.
2. $\overrightarrow{\nabla} f(x,y) = \left\langle \frac{2}{2\sqrt{2x+3y}}, \frac{3}{2\sqrt{2x+3y}} \right\rangle = \left\langle \frac{1}{\sqrt{2x+3y}}, \frac{3}{2\sqrt{2x+3y}} \right\rangle$.
3. $\overrightarrow{\nabla} f(x,y) = \langle (x-y), -(x-y)\rangle = \langle x-y, y-x\rangle$.
4. $\overrightarrow{\nabla} f(x,y) = \left\langle \frac{2x}{1+x^2+2y^2}, \frac{4y}{1+x^2+2y^2} \right\rangle$.
5. $\overrightarrow{\nabla} f(x,y,z) = \left\langle \frac{2x}{2\sqrt{x^2+y^2+z^2}}, \frac{2y}{2\sqrt{x^2+y^2+z^2}}, \frac{2z}{2\sqrt{x^2+y^2+z^2}} \right\rangle = \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle$.
6. $\overrightarrow{\nabla} f(x,y,z) = \left\langle 2xye^{\frac{y}{z}}, x^2\left(e^{\frac{y}{z}} + \frac{y}{z}e^{\frac{y}{z}}\right), x^2y\frac{-y}{z^2}e^{\frac{y}{z}} \right\rangle = \left\langle 2xye^{\frac{y}{z}}, x^2e^{\frac{y}{z}}\left(1+\frac{y}{z}\right), -\frac{x^2y^2}{z^2}e^{\frac{y}{z}} \right\rangle$.
7. $\overrightarrow{\nabla} f(x,y,z) = \langle 4x(y-z^3), 2x^2, -6x^2z^2\rangle$.
8. $\overrightarrow{\nabla} f(x,y,z) = \langle yz\left(\sin(xy) + xy\cos(xy)\right), xz\left(\sin(xy) + yx\cos(xy)\right), xy\sin(xy)\rangle$.

Solution of exercise 2.

Reminder.

- **The scalar line integral.**

  Let f be a continuous function with a domain that includes curve C with parameterization $\overrightarrow{r}(t) = \langle x(t), y(t), z(t)\rangle, a \leq t \leq b$. Then,

$$\int_C f(x,y,z)ds = \int_a^b f(\overrightarrow{r}(t))||\overrightarrow{r}'(t)||dr = \int_a^b f(\overrightarrow{r}(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt.$$

  Similarly,

$$\int_C f(x,y)ds = \int_a^b f(\overrightarrow{r}(t))||\overrightarrow{r}'(t)||dr = \int_a^b f(\overrightarrow{r}(t))\sqrt{(x'(t))^2 + (y'(t))^2}dt,$$

  if C is a planar curve with parameterization $\overrightarrow{r}(t) = \langle x(t), y(t)\rangle, a \leq t \leq b$ and f is a function of two variables.

- **The vector line integral.**

  The vector line integral of a vector field $\overrightarrow{F}$ along oriented smooth curve C with parameterization $\overrightarrow{r}(t), a \leq t \leq b$ is

$$\int_C \overrightarrow{F}.d\overrightarrow{r} = \int_C \overrightarrow{F}.\overrightarrow{T}ds = \int_a^b \overrightarrow{F}(\overrightarrow{r}(t)).\overrightarrow{r}'(t)dt.$$

  where T is the unit tangent vector.
  If $\overrightarrow{F} = \langle F_1, F_2, F_3\rangle$, the vector line integral of a vector field $\overrightarrow{F}$ along oriented smooth curve C with parameterization $\overrightarrow{r}(t), a \leq t \leq b$ is

$$\begin{aligned}\int_C \overrightarrow{F}.d\overrightarrow{r} &= \int_C F_1dx + F_2dy + F_3dz \\ &= \int_a^b \left(F_1(\overrightarrow{r}(t))\frac{dx}{dt} + F_2(\overrightarrow{r}(t))\frac{dy}{dt} + F_3(\overrightarrow{r}(t))\frac{dz}{dt}\right)dt.\end{aligned}$$

  Similary, if $\overrightarrow{F} = \langle F_1, F_2\rangle$, we have

$$\begin{aligned}\int_C \overrightarrow{F}.d\overrightarrow{r} &= \int_C F_1dx + F_2dy \\ &= \int_a^b \left(F_1(\overrightarrow{r}(t))\frac{dx}{dt} + F_2(\overrightarrow{r}(t))\frac{dy}{dt}\right)dt.\end{aligned}$$

- **Properties of vector line integral** *Let $\overrightarrow{F}$ and $\overrightarrow{G}$ be continuous vector fields with domains that include the oriented smooth curve C. Then*

  1. $\displaystyle\int_C \left(\overrightarrow{F} + \overrightarrow{G}\right).d\overrightarrow{r} = \int_C \overrightarrow{F}.d\overrightarrow{r} + \int_C \overrightarrow{G}.d\overrightarrow{r}$.
  2. $\displaystyle\int_C \kappa\overrightarrow{F}.d\overrightarrow{r} = \kappa\int_C \overrightarrow{F}.d\overrightarrow{r}$, *where κ is a constant .*
  3. $\displaystyle\int_C \overrightarrow{F}.d\overrightarrow{r} = -\int_{-C} \overrightarrow{F}.d\overrightarrow{r}$.
  4. *Suppose instead that C is a piecewise smooth curve in the domains of $\overrightarrow{F}$ and $\overrightarrow{G}$, where $C = C_1 + C_2 + \cdots + C_n$ and $C_1, C_2, \cdots, C_n$ are smooth curves such that the endpoint of C_i is the starting point of C_{i+1} . Then*

$$\int_C \overrightarrow{F}.d\overrightarrow{r} = \int_{C_1} \overrightarrow{F}.d\overrightarrow{r} + \int_{C_2} \overrightarrow{F}.d\overrightarrow{r} + \cdots + \int_{C_n} \overrightarrow{F}.d\overrightarrow{r}.$$

Solution.

1. *We have $\overrightarrow{r}(t) = \langle x(t), y(t)\rangle = \langle t^3, t^4\rangle$, $1 \leq t \leq 2$. Then,*

$$||\overrightarrow{r'}(t)|| = \sqrt{(3t^2)^2 + (4t^3)^2} = \sqrt{9t^4 + 16t^6} = \sqrt{t^4(9 + 16t^2)} = t^2\sqrt{9 + 16t^2}.$$

Therefore,

$$\int_C \frac{x}{y}dr = \int_1^2 \frac{t^3}{t^4}t^2\sqrt{9 + 16t^2}dt = \int_1^2 t\sqrt{9 + 16t^2}dt = \left[\frac{(9 + 16t^2)^{\frac{3}{2}}}{32\frac{3}{2}}\right]_1^2 = \frac{1}{48}[73^{\frac{3}{2}} - 25^{\frac{3}{2}}].$$

2. *The parameterization of C is,*

$$\overrightarrow{r}(t) := \begin{cases} x(t) = 4\cos(t) \\ y(t) = 4\sin(t), \end{cases} \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}.$$

Then,

$$\overrightarrow{r'}(t) = \langle -4\sin(t), 4\cos(t)\rangle \quad and \quad ||\overrightarrow{r'}(t)|| = \sqrt{16\sin^2(t + 16\cos^2(t)} = 4.$$

Therefore,

$$\begin{aligned}\int_C xy^4 dr &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\cos(t)(4\sin(t))^4 4dt = 4096\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t)\sin^4(t)dt \\ &= 4096\left[\frac{\sin^5(t)}{5}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 4096\frac{2}{5} = \frac{8192}{5}.\end{aligned}$$

3. *The parameterization of C is,*

$$\overrightarrow{r}(t) = \langle t, t^2\rangle, 0 \le t \le \pi,$$

with,

$$\overrightarrow{r'}(t) = \langle 1, 2t\rangle.$$

Then,

$$\begin{aligned}\int_C (x^2y+\sin(x))dy &= \int_0^{\pi} ((t)^2t^2+\sin(t))2tdt = \int_0^{\pi}(t^4+\sin(t))2tdt = 2\int_0^{\pi}(t^5+t\sin(t))dt \\ &= 2\left[\frac{t^6}{6}\right]_0^{\pi} + 2\left[-t\cos(t)\right]_0^{\pi} + 2\int_0^{\pi}\cos(t)dt \\ &= \frac{\pi^6}{3} + 2\pi + 2\left[\sin(t)\right]_0^{\pi} \\ &= \frac{\pi^6+6\pi}{3}.\end{aligned}$$

4. *Here, we write C as the union of two curves, that is, $C = C_1 \cup C_2$, where C_1 is the line segment from $(0,0)$ to $(2,1)$, and, C_2 is the line segment from $(2,1)$ to $(3,1)$. Then, the parameterization of C is the parameterization of C_1 and C_2, and we have,*

$$\begin{aligned}&\overrightarrow{r_1}(t) = (2,1)t + (0,0)(1-t) = \langle 2t, t\rangle, &&0 \le t \le 1, \\ &\overrightarrow{r_2}(t) = (3,0)t + (2,1)(1-t) = \langle t+2, 1-t\rangle, &&0 \le t \le 1,\end{aligned}$$

where $\overrightarrow{r}_1(t), \overrightarrow{r}_2(t)$ are the parameterization of C_1 and C_2 respectively. Which gives,

$$\overrightarrow{r'_1}(t) = \langle 2, 1\rangle, \overrightarrow{r'_2}(t) = \langle 1, -1\rangle.$$

Then,

$$\begin{aligned}
\int_C (x+2y)dx + x^2dy &= \int_{C_1\cup C_2} (x+2y)dx + x^2dy \\
&= \int_{C_1} (x+2y)dx + x^2dy + \int_{C_2} (x+2y)dx + x^2dy \\
&= \int_0^1 (2t+2t)2dt + (2t)^2 1dt \\
&\quad + \int_0^1 ((t+2)+2(1-t))1dt + (t+2)^2(-1)dt \\
&= \int_0^1 (8t+4t^2)dt + \int_0^1 ((4-t)-(t+2)^2)dt \\
&= 8\left[\frac{t^2}{2}\right]_0^1 + 4\left[\frac{t^3}{3}\right]_0^1 + \left[-\frac{(4-t)^2}{2}\right]_0^1 - \left[\frac{(t+2)^3}{3}\right]_0^1 \\
&= 4 + \frac{4}{3} + \left(-\frac{9}{2}+8\right) - \left(9-\frac{8}{3}\right) \\
&= \frac{16}{3} + \frac{7}{2} - \frac{19}{3} \\
&= \frac{-2+7}{2} \\
&= \frac{5}{2}.
\end{aligned}$$

Therefore,

$$\int_C (x+2y)dx + x^2dy = \frac{5}{2}.$$

5. *C is the union of two curves, that is, $C = C_1 \cup C_2$, where C_1 is the arc of the circle $x^2+y^2=4$ from $(2,0)$ to $(0,2)$, and, C_2 is the line segment from $(0,2)$ to $(4,3)$. The parameterization of C is the parameterization of C_1 and C_2, and we have,*

$$\begin{aligned}
&\overrightarrow{r}_1(t) = \langle 2\cos(t), 2\sin(t)\rangle, && 0 \le t \le \tfrac{\pi}{2}, \\
&\overrightarrow{r}_2(t) = (4,3)t + (0,2)(1-t) = \langle 4t, t+2\rangle, && 0 \le t \le 1,
\end{aligned}$$

where $\overrightarrow{r}_1(t), \overrightarrow{r}_2(t)$ are the parameterization of C_1 and C_2 respectively. Which gives,

$$\overrightarrow{r'}_1(t) = \langle -2\sin(t), 2\cos(t)\rangle, \overrightarrow{r'}_2(t) = \langle 4, 1\rangle.$$

Then,

$$\begin{aligned}
\int_C x^2dx + y^2dy &= \int_{C_1\cup C_2} x^2dx + y^2dy \\
&= \int_{C_1} x^2dx + y^2dy + \int_{C_2} x^2dx + y^2dy \\
&= \int_0^{\frac{\pi}{2}} (2\cos(t))^2(-2\sin(t))dt + (2\sin(t))^2(2\cos(t))dt \\
&\quad + \int_0^1 (4t)^2 4dt + (t+2)^2 1dt \\
&= -8\int_0^{\frac{\pi}{2}} \cos^2(t)\sin(t)dt - \sin^2(t)\cos(t)dt + \int_0^1 64t^2dt + (t+2)^2dt \\
&= 8\left[\frac{\cos^3(x)}{3}\right]_0^{\frac{\pi}{2}} + 8\left[\frac{\sin^3(x)}{3}\right]_0^{\frac{\pi}{2}} + 64\left[\frac{t^3}{3}\right]_0^1 + \left[\frac{(t+2)^3}{3}\right]_0^1 \\
&= -\frac{8}{3} + \frac{8}{3} + \frac{64}{3} + 9 - \frac{8}{3} \\
&= \frac{64+27-8}{3} \\
&= \frac{83}{3}.
\end{aligned}$$

Therefore,

$$\int_C x^2dx + y^2dy = \frac{83}{3}.$$

6. *The parameterization of C is,*

$$\overrightarrow{r}(t) = (1,2,3)t + (0,0,0)(1-t) = \langle t, 2t, 3t\rangle, 0 \le t \le 1.$$

Then,

$$\overrightarrow{r'}(t) = \langle 1,2,3\rangle \quad \text{and} \quad ||\overrightarrow{r'}(t)|| = \sqrt{1^2+2^2+3^2} = \sqrt{14}.$$

Therefore,

$$\int_C xe^{yz}ds = \int_0^1 te^{2t3t}\sqrt{14}dt = \sqrt{14}\int_0^1 te^{6t^2}dt = \sqrt{14}\left[\frac{e^{6t^2}}{12}\right]_0^1 = \frac{\sqrt{14}}{12}(e^6-1).$$

7. *The parameterization of* C *is,*

$$\overrightarrow{r}(t) = (4,1,2)t + (1-t)(1,0,0) = \langle 3t+1, t, 2t\rangle, 0 \leq t \leq 1,$$

with,

$$\overrightarrow{r'}(t) = \langle 3,1,2\rangle.$$

Then,

$$\begin{aligned}
\int_C z^2dx + x^2dy + y^2dz &= \int_0^1 (2t)^2 3dt + (3t+1)^2 1dt + (t)^2 2dt \\
&= \int_0^1 (12t^2 + (3t+1)^2 + 2t^2)dt \\
&= \int_0^1 (14t^2 + (3t+1)^2)dt \\
&= 14\left[\frac{t^3}{3}\right]_0^1 + \left[\frac{(3t+1)^3}{9}\right]_0^1 \\
&= \frac{14}{3} + \frac{64}{9} - \frac{1}{9} \\
&= \frac{105}{9} \\
&= \frac{35}{3}.
\end{aligned}$$

8. *we see that* C *is the union of two curves, that is,* $C = C_1 \cup C_2$*, where* C_1 *is the line segment from* $(0,0,0)$ *to* $(1,0,1)$*, and,* C_2 *is the line segment from* $(1,0,1)$ *to* $(0,1,2)$*. Then, the parameterization of* C *is the parameterization of* C_1 *and* C_2*, and we have,*

$$\begin{aligned}
\overrightarrow{r_1}(t) &= (1,0,1)t + (0,0,0)(1-t) = \langle t, 0, t\rangle, && 0 \leq t \leq 1, \\
\overrightarrow{r_2}(t) &= (0,1,2)t + (1,0,1)(1-t) = \langle 1-t, t, t+1\rangle, && 0 \leq t \leq 1,
\end{aligned}$$

where $\overrightarrow{r_1'}(t), \overrightarrow{r_2'}(t)$ *are the parameterization of* C_1 *and* C_2 *respectively. which gives,*

$$\overrightarrow{r_1'}(t) = \langle 1,0,1\rangle \quad and \quad \overrightarrow{r_2'}(t) = \langle -1,1,1\rangle.$$

Then,

$$\int_C (y+z)dx + (x+z)dy + (x+y)dz = \int_{C_1\cup C_2} (y+z)dx + (x+z)dy + (x+y)dz$$

$$\begin{aligned}
&= \int_{C_1} (y+z)dx + (x+z)dy + (x+y)dz + \int_{C_2} (y+z)dx + (x+z)dy + (x+y)dz \\
&= \int_0^1 (0+t)1dt + (t+t)0dt + (t+0)1dt + \int_0^1 (t+(t+1))(-1)dt \\
&\quad +((1-t)+(t+1))1dt + ((1-t)+t)1dt \\
&= \int_0^1 (t+0+t)dt + \int_0^1 ((-2t-1)+2+1)dt \\
&= \int_0^1 2tdt + \int_0^1 2(-t+1)dt \\
&= [t^2]_0^1 + \left[-(-t+1)^2\right]_0^1 \\
&= 1+1 \\
&= 2.
\end{aligned}$$

Therefore,

$$\int_C (y+z)dx + (x+z)dy + (x+y)dz = 2.$$

Solution of exercise 3.

Reminder.

The fundamental theorem of line integral.

Let C be a piecewise smooth curve with parameterization $\overrightarrow{r}(t), a \leq t \leq b$. Let f be a function of two or three variables with first-order partial derivatives that exist and are continuous on C. Then,

$$\int_C \overrightarrow{\nabla} f.d\overrightarrow{r} = f(\overrightarrow{r}(b)) - f(\overrightarrow{r}(a)).$$

Solution.

1. • *Finding the potential function f.*
   Since $\overrightarrow{F} = \overrightarrow{\nabla} f$, we have

$$\overrightarrow{F} = \langle 3+2xy^2, 2x^2y\rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \overrightarrow{\nabla} f.$$

Then, using the integration with respect to x, we obtain,

$$\int \frac{\partial f(x,y)}{\partial x}dx = \int 3 + 2xy^2 dx \Rightarrow f(x,y) = 3x + x^2y^2 + K(y), K : \mathbb{R} \to \mathbb{R} \quad \text{is a function.}$$

Now, we differentiate $f(x,y)$ with respect to y to have,

$$\frac{\partial f(x,y)}{\partial y} = 2x^2y + K'(y).$$

While f is the potential function of $\overrightarrow{F}$, we have $2x^2y + K'(y) = 2x^2y$, then,

$$K'(y) = 0,$$

which gives by integrating with respect to y,

$$K(y) = C, \quad C \text{ is a constant.}$$

Therefore,

$$f(x,y) = 3x + x^2y^2 + C, \quad C \text{ is a constant.}$$

If we take $C = 0$,

$$f(x,y) = 3x + x^2y^2.$$

- *Computing* $\int_C \overrightarrow{F}.d\overrightarrow{r}$.

We have $\overrightarrow{r}(t) = \langle t, \frac{1}{t} \rangle, 1 \leq t \leq 4$. Then, using the Fundamental theorem of line integral, we obtain,

$$\begin{aligned}
\int_C \overrightarrow{F}.d\overrightarrow{r} &= \int_C \overrightarrow{\nabla} f.d\overrightarrow{r} \\
&= f(\overrightarrow{r}(4)) - f(\overrightarrow{r}(1)) \\
&= f(4, 1/4) - f(1,1) \\
&= 3.4 + 4^2.\frac{1}{4^2} - 3.1 + 1^2.1^2 \\
&= 13 - 4 \\
&= 9.
\end{aligned}$$

2. • *Finding the potential function f.*
Since $\overrightarrow{F} = \overrightarrow{\nabla} f$, *we have*

$$\overrightarrow{F} = \langle x^2y^3, x^3y^2\rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \overrightarrow{\nabla} f.$$

Then, using the integration with respect to x, *we obtain,*

$$\int \frac{\partial f(x,y)}{\partial x} dx = \int x^2y^3 dx \Rightarrow f(x,y) = \frac{x^3}{3}y^3 + K(y), K : \mathbb{R} \to \mathbb{R} \quad \textit{is a function.}$$

Now, we differentiate $f(x,y)$ *with respect to* y *to have,*

$$\frac{\partial f(x,y)}{\partial y} = x^3y^2 + K'(y).$$

While f *is the potential function of* $\overrightarrow{F}$, *we have* $x^3y^2 + K'(y) = x^3y^2$, *then,*

$$K'(y) = 0,$$

which gives by integrating with respect to y,

$$K(y) = C, \quad C \textit{ is a constant.}$$

Therefore,

$$f(x,y) = \frac{x^3y^3}{3} + C, \quad C \textit{ is a constant.}$$

If we take $C = 0$,

$$f(x,y) = \frac{x^3y^3}{3}.$$

• *Computing* $\int_C \overrightarrow{F}.d\overrightarrow{r}$.
We have $\overrightarrow{r}(t) = \langle t^3 - 2t, t^3 + 2t\rangle, 0 \le t \le 1$. *Then, using the Fundamental theorem of line integral, we obtain,*

$$\begin{aligned} \int_C \overrightarrow{F}.d\overrightarrow{r} &= \int_C \overrightarrow{\nabla} f.d\overrightarrow{r} \\ &= f(\overrightarrow{r}(1)) - f(\overrightarrow{r}(0)) \\ &= f(-1,3) - f(0,0) \\ &= \frac{(-1)^3.3^3}{3} - \frac{0^3.0^3}{3} \\ &= -9. \end{aligned}$$

3. • *Finding the potential function f.*
   Since $\overrightarrow{F} = \overrightarrow{\nabla} f$, we have
$$\overrightarrow{F} = \langle yze^{xz}, e^{xz}, xye^{xz} \rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \overrightarrow{\nabla} f.$$
   Then, using the integration with respect to x, we obtain,
$$\int \frac{\partial f(x,y,z)}{\partial x} dx = \int yze^{xz} dx \Rightarrow f(x,y,z) = ye^{xz} + K(y,z), K : \mathbb{R}^2 \to \mathbb{R} \ \text{ is a function.}$$
   Now, we differentiate $f(x,y,z)$ with respect to y to have,
$$\frac{\partial f(x,y,z)}{\partial y} = e^{xz} + \frac{\partial K(y,z)}{\partial y} = e^{xz} + K_y(y,z).$$
   While f is the potential function of $\overrightarrow{F}$, we have $e^{xz} + K_y(y,z) = e^{xz}$, then,
$$K_y(y,z) = 0,$$
   which gives by integrating with respect to y,
$$K(y,z) = C(z), C : \mathbb{R} \to \mathbb{R} \ \text{ is a function.}$$
   Therefore,
$$f(x,y,z) = ye^{xz} + C(z), C : \mathbb{R} \to \mathbb{R} \ \text{ is a function.}$$
   Now, we differentiate $f(x,y,z)$ with respect to z, we get
$$\frac{\partial f(x,y,z)}{\partial z} = xye^{xz} + C'(z).$$
   As $\overrightarrow{F} = \overrightarrow{\nabla} f$, we have $xye^{xz} + C'(z) = xye^{xz}$, then,
$$C'(z) = 0$$
   which implies,
$$C(z) = U, \ \ U \text{ is constant.}$$
   Hence,
$$f(x,y,z) = ye^{xz} + U, \ \ U \text{ is constant.}$$

If we take $U = 0$,

$$f(x, y, z) = ye^{xz}.$$

- *Computing* $\displaystyle\int_C \overrightarrow{F}.d\overrightarrow{r}$.

We have $\overrightarrow{r}(t) = \langle t^2 + 1, t^2 - 1, t^2 - 2t\rangle, 0 \leq t \leq 2$. *Then, using the Fundamental theorem of line integral, we obtain,*

$$\begin{aligned}
\int_C \overrightarrow{F}.d\overrightarrow{r} &= \int_C \overrightarrow{\nabla} f.d\overrightarrow{r} \\
&= f(\overrightarrow{r}(2)) - f(\overrightarrow{r}(0)) \\
&= f(5, 3, 0) - f(1, -1, 0) \\
&= 3e^{5.0} - (-1)e^{1.0} \\
&= 3 + 1 \\
&= 4.
\end{aligned}$$

4. - *Finding the potential function* f.

Since $\overrightarrow{F} = \overrightarrow{\nabla} f$, *we have*

$$\overrightarrow{F} = \langle \sin(y), x\cos(y) + \cos(z), -y\sin(z)\rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \overrightarrow{\nabla} f.$$

Then, using the integration with respect to x, *we obtain,*

$$\int \frac{\partial f(x, y, z)}{\partial x} dx = \int \sin(y) dx \Rightarrow f(x, y, z) = \sin(y)x + K(y, z), K : \mathbb{R}^2 \to \mathbb{R} \ \ \textit{is a function.}$$

Now, we differentiate $f(x, y, z)$ *with respect to* y *to have,*

$$\frac{\partial f(x, y, z)}{\partial y} = x\cos(y) + \frac{\partial K(y, z)}{\partial y} = x\cos(y) + K_y(y, z).$$

While f is the potential function of $\overrightarrow{F}$, *we have* $x\cos(y) + K_y(y, z) = x\cos(y) + \cos(z)$, *then,*

$$K_y(y, z) = \cos(z),$$

which gives by integrating with respect to y,

$$K(y, z) = \cos(z)y + C(z), C : \mathbb{R} \to \mathbb{R} \ \textit{is a function.}$$

Therefore,

$$f(x,y,z) = \sin(y)x + \cos(z)y + C(z), \quad C : \mathbb{R} \to \mathbb{R} \textit{ is a function.}$$

Now, we differentiate $f(x,y,z)$ with respect to z, we get

$$\frac{\partial f(x,y,z)}{\partial z} = -y\sin(z) + C'(z).$$

As $\overrightarrow{F} = \overrightarrow{\nabla} f$, we have $-y\sin(z) + C'(z) = -y\sin(z)$, then,

$$C'(z) = 0$$

which implies,

$$C(z) = U, \quad U \textit{ is constant.}$$

Hence,

$$f(x,y,z) = \sin(y)x + \cos(z)y + U, \quad U \textit{ is constant.}$$

If we take $U = 0$,

$$f(x,y,z) = \sin(y)x + \cos(z)y.$$

- *Computing* $\displaystyle\int_C \overrightarrow{F}.d\overrightarrow{r}$.

We have $\overrightarrow{r}(t) = \langle \sin(t), t, 2t\rangle, 0 \leq t \leq \frac{\pi}{2}$. Then, using the Fundamental theorem of line integral, we obtain,

$$\begin{aligned}
\int_C \overrightarrow{F}.d\overrightarrow{r} &= \int_C \overrightarrow{\nabla} f.d\overrightarrow{r} \\
&= f\left(\overrightarrow{r}(\pi/2)\right) - f(\overrightarrow{r}(0)) \\
&= f(1,\pi/2,\pi) - f(0,0,0) \\
&= (\sin(\pi/2).1 + \cos(\pi).\frac{\pi}{2}) - (\sin(0).0 + \cos(0).0) \\
&= (1 - \frac{\pi}{2}) - (0) \\
&= \frac{2-\pi}{2}.
\end{aligned}$$

Solution of exercise 4.

Reminder.

Green's theorem

• **Circulation form** *(Theorem)*

Let D be an open, simply connected region with a boundary curve C that is a piecewise smooth, simple closed curve oriented counterclockwise. Let $\overrightarrow{F} = \langle F_1, F_2\rangle$ be a vector field with component functions that have continuous partial derivatives on D. Then,

$$\oint_C \overrightarrow{F}.d\overrightarrow{r} = \oint_C F_1 dx + F_2 dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

• **Flux form** *(theorem)*

Let D be an open, simply connected region with a boundary curve C that is a piecewise smooth, simple closed curve that is oriented counterclockwise. Let $\overrightarrow{F} = \langle F_1, F_2\rangle$ be a vector field with component functions that have continuous partial derivatives on an open region containing D. Then,

$$\oint_C \overrightarrow{F}.\overrightarrow{N} ds = \iint_D \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} dA.$$

where $\overrightarrow{N}$ is the unit normal vector.

Solution.

1. *We have $\overrightarrow{F} = \langle F_1, F_2\rangle = \langle y + e^{\sqrt{x}}, 2x + \cos(y^2)\rangle$, then,*

$$\frac{\partial F_2}{\partial x} = 2, \frac{\partial F_1}{\partial y} = 1.$$

In other side, as C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$,

$$D = \left\{(x, y) \in \mathbb{R}^2, 0 \leq x \leq 1, x^2 \leq y \leq \sqrt{x}\right\},$$

Therefore, from the circulation form of Green's theorem, we obtain,

$$\int_C (y + e^{\sqrt{x}})dx + (2x + \cos(y^2))dy = \oint_C (y + e^{\sqrt{x}})dx + (2x + \cos(y^2))dy$$

$$
\begin{aligned}
&= \iint_D (2-1)dA = \iint_D dA \\
&= \int_0^1 \int_{x^2}^{\sqrt{x}} dydx \\
&= \int_0^1 \sqrt{x} - x^2 dx \\
&= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{3} \right]_0^1 \\
&= \frac{1}{3}.
\end{aligned}
$$

2. *We have* $\overrightarrow{F} = \langle F_1, F_2 \rangle = \langle y^4, 2xy^3 \rangle$, *then,*

$$\frac{\partial F_2}{\partial x} = 2y^3, \frac{\partial F_1}{\partial y} = 4y^3.$$

From the circulation form of Green's theorem, we obtain,

$$
\begin{aligned}
\int_C y^4 dx + 2xy^3 dy &= \oint_C y^4 dx + 2xy^3 dy \\
&= \iint_D (2y^3 - 4y^3) dA \\
&= -\iint_D 2y^3 dA \\
&= -\int_0^{2\pi} \int_0^1 2(r\sin(\theta))^3 \sqrt{2} r dr d\theta \\
&= -2\sqrt{2} \int_0^{2\pi} \int_0^1 r^4 \sin^3(\theta) dr d\theta \\
&= 2\sqrt{2} \int_0^{2\pi} (1-\cos^2(\theta)). -\sin(\theta) d\theta \int_0^1 r^4 dr \\
&= 2\sqrt{2} \left[\cos(\theta) - \frac{\cos^3(\theta)}{3} \right]_0^{2\pi} . \left[\frac{r^5}{5} \right]_0^1 \\
&= 0.
\end{aligned}
$$

Here, we used the polar coordinates of the ellipse $\left(\frac{x}{\sqrt{2}}\right)^2 + \left(\frac{y}{1}\right)^2 = 1$, *with,*

$$
\begin{cases}
x = \sqrt{2} r \cos(\theta), & 0 \le r \le 1, \\
y = r\sin(\theta), & 0 \le \theta \le 2\pi.
\end{cases}
$$

3. *We have* $\overrightarrow{F} = \langle F_1, F_2 \langle xy, x^2y^3 \rangle$, *then,*

$$\frac{\partial F_1}{\partial x} = y, \frac{\partial F_2}{\partial y} = 3x^2y^2.$$

As C *is the triangle with vertices* $(0,0), (1,0)$ *and* $(1,2)$,

$$D = \left\{ (x,y) \in \mathbb{R}^2, 0 \leq x \leq 1, 0 \leq y \leq 2x \right\}.$$

Therefore, from the flux form of Green's theorem, we obtain,

$$\begin{aligned}
\oint_C \langle xy, x^2y^3 \rangle . \overrightarrow{N} ds &= \iint_D (y + 3x^2y^2) dA \\
&= \int_0^1 \int_0^{2x} (y + 3x^2y^2) dy dx \\
&= \int_0^1 \left[\frac{y^2}{2} + x^2y^3 \right]_0^{2x} dx \\
&= \int_0^1 (2x^2 + 8x^5) dx \\
&= \left[\frac{2x^3}{3} + \frac{8x^6}{6} \right]_0^1 \\
&= \frac{2}{3} + \frac{4}{3} \\
&= 2.
\end{aligned}$$

4. *We have* $\overrightarrow{F} = \langle F_1, F_2 \rangle = \langle y - \cos(y), y \rangle$, *then,*

$$\frac{\partial F_1}{\partial x} = 0, \frac{\partial F_2}{\partial y} = 1.$$

From the flux form of Green's theorem, we obtain,

$$\begin{aligned}
\oint_C \langle y - \cos(y), x\sin(y) \rangle . \overrightarrow{N} ds &= \iint_D 1 dA \\
&= \textit{Area of } D \\
&= 4\pi.
\end{aligned}$$

Solution of exercise 5.

Reminder.

Divergence

• *If* $\overrightarrow{F} = \langle F_1, F_2 \rangle$ *is a vector field in* $\mathbb{R}^2$, *and* $\frac{\partial F_1}{\partial x}$ *and* $\frac{\partial F_2}{\partial y}$ *both exist, then the* **divergence** *of* $\overrightarrow{F}$ *is defined similarly as*

$$div\ \overrightarrow{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = \overrightarrow{\nabla}.\overrightarrow{F}.$$

• *If* $\overrightarrow{F} = \langle F_1, F_2, F_3 \rangle$ *is a vector field in* $\mathbb{R}^3$ *and* $\frac{\partial F_1}{\partial x}, \frac{\partial F_2}{\partial y}$ *and* $\frac{\partial F_3}{\partial z}$ *all exist, then the* **divergence** *of* $\overrightarrow{F}$ *is defined by*

$$div\ \overrightarrow{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \overrightarrow{\nabla}.\overrightarrow{F}.$$

Curl

• *If* $\overrightarrow{F} = \langle F_1, F_2, F_3 \rangle$ *is a vector field in* $\mathbb{R}^3$, *and* $\frac{\partial F_1}{\partial x}, \frac{\partial F_2}{\partial y}$ *and* $\frac{\partial F_3}{\partial z}$ *all exist, then the curl of* $\overrightarrow{F}$ *is defined by*

$$curl\ \overrightarrow{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\overrightarrow{\mathbf{i}} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\overrightarrow{\mathbf{j}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\overrightarrow{\mathbf{k}}.$$

Or,

$$curl\ \overrightarrow{F} = \begin{vmatrix} \overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.$$

which gives,

$$\begin{aligned} curl\ \overrightarrow{F} &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\overrightarrow{\mathbf{i}} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\overrightarrow{\mathbf{j}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\overrightarrow{\mathbf{k}} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\overrightarrow{\mathbf{i}} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\overrightarrow{\mathbf{j}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\overrightarrow{\mathbf{k}} \end{aligned}$$

• *If* $\overrightarrow{F} = \langle F_1, F_2 \rangle$ *is a vector field in* $\mathbb{R}^2$, *then, the curl of* $\overrightarrow{F}$ *is,*

$$curl\ \overrightarrow{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\overrightarrow{\mathbf{k}}.$$

Solution.

1. $\overrightarrow{F}(x,y,z) = \langle xy^2z^2, y^4z^3, x^2y^2z\rangle$
   - **The divergence of $\overrightarrow{F}$.**

$$\overrightarrow{\nabla}.\overrightarrow{F} = \frac{\partial}{\partial x}(xy^2z^2) + \frac{\partial}{\partial y}(y^4z^3) + \frac{\partial}{\partial z}(x^2y^2z) = y^2z^2 + 4z^3y^3 + x^2y^2.$$

   - **The curl of $\overrightarrow{F}$.**

$$\begin{aligned} curl\ \overrightarrow{F} &= \begin{vmatrix} \overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z^2 & y^4z^3 & x^2y^2z \end{vmatrix} \\ &= \overrightarrow{\mathbf{i}}(2x^2zy - 3y^4z^2) - \overrightarrow{\mathbf{j}}(2y^2zx - 2xy^2z) + \overrightarrow{\mathbf{k}}(0 - 2xyz^2) \\ &= \overrightarrow{\mathbf{i}}\, zy(2x^2 - 3y^3z) - \overrightarrow{\mathbf{k}}\, 2xyz^2. \end{aligned}$$

2. $\overrightarrow{F}(x,y,z) = \langle \ln(2y+3z), \ln(x+3z), \ln(x+2y)\rangle$
   - **The divergence of $\overrightarrow{F}$.**

$$\overrightarrow{\nabla}.\overrightarrow{F} = \frac{\partial}{\partial x}(\ln(2y+3z)) + \frac{\partial}{\partial y}(\ln(x+3z)) + \frac{\partial}{\partial z}(\ln(x+2y)) = 0+0+0 = 0.$$

   - **The curl of $\overrightarrow{F}$.**

$$\begin{aligned} curl\ \overrightarrow{F} &= \begin{vmatrix} \overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \ln(2y+3z) & \ln(x+3z) & \ln(x+2y) \end{vmatrix} \\ &= \overrightarrow{\mathbf{i}}\left(\frac{2}{x+2y} - \frac{3}{x+3z}\right) - \overrightarrow{\mathbf{j}}\left(\frac{1}{x+2y} - \frac{3}{2y+3z}\right) + \overrightarrow{\mathbf{k}}\left(\frac{1}{x+3z} - \frac{2}{2y+3z}\right). \end{aligned}$$

3. $\overrightarrow{F}(x,y,z) = e^x\sin(y), e^y\sin(z), e^z\sin(x)\rangle$
   - **The divergence of $\overrightarrow{F}$.**

$$\overrightarrow{\nabla}.\overrightarrow{F} = \frac{\partial}{\partial x}(e^x\sin(y)) + \frac{\partial}{\partial y}(e^y\sin(z)) + \frac{\partial}{\partial z}(e^z\sin(x)) = \sin(y)e^x + \sin(z)e^y + \sin(x)e^z.$$

• **The curl of $\overrightarrow{F}$.**

$$\begin{aligned}
curl\ \overrightarrow{F} &= \begin{vmatrix} \overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \sin(y) & e^y \sin(z) & e^z \sin(x) \end{vmatrix} \\
&= \overrightarrow{\mathbf{i}}\,(0 - e^y \cos(z)) - \overrightarrow{\mathbf{j}}\,(e^z \cos(x) - 0) + \overrightarrow{\mathbf{k}}\,(0 - e^x \cos(y)) \\
&= -\overrightarrow{\mathbf{i}}\,e^y \cos(z) - \overrightarrow{\mathbf{j}}\,e^z \cos(x) - \overrightarrow{\mathbf{k}}\,e^x \cos(y).
\end{aligned}$$

4. $\overrightarrow{F}(x, y, z) = \langle \arctan(xy), \arctan(yz), \arctan(zx) \rangle$

• **The divergence of $\overrightarrow{F}$.**

$$\overrightarrow{\nabla}.\overrightarrow{F} = \frac{\partial}{\partial x}(\arctan(xy)) + \frac{\partial}{\partial y}(\arctan(yz)) + \frac{\partial}{\partial z}(\arctan(zx)) = \frac{y}{1 + x^2y^2} + \frac{z}{1 + y^2z^2} + \frac{x}{1 + z^2x^2}.$$

• **The curl of $\overrightarrow{F}$.**

$$\begin{aligned}
curl\ \overrightarrow{F} &= \begin{vmatrix} \overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \arctan(xy) & \arctan(yz) & \arctan(zx) \end{vmatrix} \\
&= \overrightarrow{\mathbf{i}}\left(0 - \frac{y}{1 + y^2z^2}\right) - \overrightarrow{\mathbf{j}}\left(\frac{z}{1 + z^2x^2} - 0\right) + \overrightarrow{\mathbf{k}}\left(0 - \frac{x}{1 + x^2y^2}\right) \\
&= -\overrightarrow{\mathbf{i}}\,\frac{y}{1 + y^2z^2} - \overrightarrow{\mathbf{j}}\,\frac{z}{1 + z^2x^2} - \overrightarrow{\mathbf{k}}\,\frac{x}{1 + x^2y^2}.
\end{aligned}$$

Solution of exercise 6.

Reminder.

Surface integral.

• **Surface integral of a scalar valued function.**

Let S be a piecewise smooth surface with parameterization,

$$\overrightarrow{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle,$$

with parameter domain D and let $f(x,y,z)$ be a function with a domain that contains S. To calculate scalar surface integrals, we have

$$\iint_S f(x,y,z)dS = \iint_D f(\overrightarrow{r}(u,v))||\overrightarrow{t}_u \times \overrightarrow{t}_v||dA,$$

with,

$$\overrightarrow{t}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \quad and \quad \overrightarrow{t}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle.$$

$\overrightarrow{t}_u$*and* $\overrightarrow{t}_u$

- **Surface integral of a vector valued function.**

Let $\overrightarrow{F}$ be a continuous vector field with a domain that contains oriented surface S with unit normal vector $\overrightarrow{N}$. The surface integral of $\overrightarrow{F}$ over S is

$$\iint_S \overrightarrow{F}.dS = \iint_S \overrightarrow{F}.\overrightarrow{N}dS = \iint_D \left(\overrightarrow{F}(\overrightarrow{r}(u,v)).(\overrightarrow{t}_u \times \overrightarrow{t}_v)\right) dA.$$

with, $\overrightarrow{N} = \frac{\overrightarrow{t}_u \times \overrightarrow{t}_v}{||\overrightarrow{t}_u \times \overrightarrow{t}_v||}$, $\overrightarrow{t}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$ *and* $\overrightarrow{t}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$.

Solution.

1. *We have $f(x,y,z) = x+y+z$, we have,*

$$\overrightarrow{r}(u,v) = \langle u+v, u-v, 1+2u+v \rangle, 0 \leq u \leq 2, 0 \leq v \leq 1.$$

The tangent vectors are,

$$\overrightarrow{t}_u = \langle 1,1,2 \rangle \quad and \quad \overrightarrow{t}_v = \langle 1,-1,1 \rangle.$$

Therefore,

$$\begin{aligned} \overrightarrow{t}_u \times \overrightarrow{t}_v &= \begin{vmatrix} \overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}} \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{vmatrix} \\ &= \overrightarrow{\mathbf{i}}(1+2) - \overrightarrow{\mathbf{j}}(1-2) + \overrightarrow{\mathbf{k}}(-1-1) \\ &= \langle 3, 1, -2 \rangle. \end{aligned}$$

The magnitude of this vector is,

$$||\overrightarrow{t}_u \times \overrightarrow{t}_v|| = \sqrt{3^2 + 1^2 + (-2)^2} = \sqrt{9+1+4} = \sqrt{14}.$$

The surface integral is,

$$\begin{aligned}
\iint_S (x+y+z)dS &= \int_0^2 \int_0^1 ((u+v)+(u-v)+(1+2u+v))\sqrt{14}dvdu \\
&= \sqrt{14}\int_0^2 \int_0^1 (4u+v+1)dvdu \\
&= \sqrt{14}\int_0^2 \left[4uv + \frac{v^2}{2} + v\right]_0^1 du \\
&= \sqrt{14}\int_0^2 \left(4u + \frac{3}{2}\right) du \\
&= \sqrt{14}\left[2u^2 + \frac{3}{2}u\right]_0^2 \\
&= \sqrt{14}(8+3) \\
&= 11\sqrt{14}.
\end{aligned}$$

2. *We have* $f(x,y,z) = x^2yz$, *with*

$$\overrightarrow{r}(u,v) = \langle u, v, 1+2u+3v\rangle, 0 \le u \le 3, 0 \le v \le 2.$$

The tangent vectors are,

$$\overrightarrow{t}_u = \langle 1,0,2\rangle \quad \textit{and} \quad \overrightarrow{t}_v = \langle 0,1,3\rangle .$$

Therefore,

$$\begin{aligned}
\overrightarrow{t}_u \times \overrightarrow{t}_v &= \begin{vmatrix} \overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}} \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} \\
&= \overrightarrow{\mathbf{i}}(0-2) - \overrightarrow{\mathbf{j}}(3-0) + \overrightarrow{\mathbf{k}}(1-0) \\
&= \langle -2, -3, 1\rangle.
\end{aligned}$$

and,

$$||\overrightarrow{t}_u \times \overrightarrow{t}_v|| = \sqrt{(-2)^2 + (-3)^2 + 1^2} = \sqrt{4+9+1} = \sqrt{14}.$$

The surface integral is,

$$\begin{aligned}
\iint_S x^2yzdS &= \int_0^2\int_0^3 u^2v(1+2u+3v)\sqrt{14}dudv \\
&= \sqrt{14}\int_0^2\int_0^3 u^2v(1+2u+3v)dudv \\
&= \sqrt{14}\int_0^2\int_0^3 (u^2v+2u^3v+3u^2v^2)dudv \\
&= \sqrt{14}\int_0^2 \left[v\frac{u^3}{3}+v\frac{u^4}{2}+v^2u^3\right]_0^3 dv \\
&= \sqrt{14}\int_0^2 \left(9v+\frac{81}{2}v+27v^2\right)dv \\
&= \sqrt{14}\left[9\frac{v^2}{2}+\frac{81}{2}\frac{v^2}{2}+9v^3\right]_0^2 \\
&= \sqrt{14}\,(18+81+72) \\
&= 171\sqrt{14}.
\end{aligned}$$

3. *We have* $\overrightarrow{F} = \langle -x, -y, z^3\rangle$, *with,*

$$\overrightarrow{r}(u,v) = \langle u\cos(v), u\sin(v), u\rangle, 1 \le u \le 3, 0 \le v \le 2\pi.$$

Then,

$$\overrightarrow{t}_u = \langle \cos(v), \sin(v), 1\rangle \quad \textit{and} \quad \overrightarrow{t}_v = \langle -u\sin(v), u\cos(v), 0\rangle.$$

Therefore,

$$\begin{aligned}
\overrightarrow{t}_u \times \overrightarrow{t}_v &= \begin{vmatrix} \overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}} \\ \cos(v) & \sin(v) & 1 \\ -u\sin(v) & u\cos(v) & 0 \end{vmatrix} \\
&= \overrightarrow{\mathbf{i}}\,(0-u\cos(v)) - \overrightarrow{\mathbf{j}}\,(0+u\sin(v)) + \overrightarrow{\mathbf{k}}\,(u\cos^2(v)+u\sin^2(v)) \\
&= \langle -u\cos(v), -u\sin(v), u\rangle.
\end{aligned}$$

The integral surface is,

$$\begin{aligned}
\iint_S \langle -x,-y,z^3\rangle . dS &= \iint_S \langle -x,-y,z^3\rangle . \overrightarrow{N} dS \\
&= \int_0^{2\pi}\int_1^3 \langle -u\cos(v), -u\sin(v), u^3\rangle . \langle -u\cos(v), -u\sin(v), u\rangle dudv \\
&= \int_0^{2\pi}\int_1^3 (u^2\cos^2(v) + u^2\sin^2(v) + u^4) dudv \\
&= \int_0^{2\pi} dv \int_1^3 (u^2+u^4)du \\
&= 2\pi\left[\frac{u^3}{3}+\frac{u^5}{5}\right]_1^3 \\
&= 2\pi\left(9+\frac{243}{5}-\frac{1}{3}-\frac{1}{5}\right) \\
&= 2\pi\frac{856}{15} \\
&= \frac{1712}{15}\pi.
\end{aligned}$$

4. *We have* $\overrightarrow{F} = \langle y, -x, 2z\rangle$, *with,*

$$\overrightarrow{r}(\phi,\theta) = \langle 2\cos(\theta)\sin(\phi), 2\sin(\theta)\sin(\phi), 2\cos(\phi)\rangle, 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{2}.$$

Then,

$$\overrightarrow{t}_\theta = \langle -2\sin(\phi)\sin(\theta), 2\sin(\phi)\cos(\theta), 0\rangle$$

and,

$$\overrightarrow{t}_\phi = \langle 2\cos(\theta)\cos(\phi), 2\sin(\theta)\cos(\phi), -2\sin(\phi)\rangle.$$

Therefore,

$$\overrightarrow{t}_\theta \times \overrightarrow{t}_\phi = \begin{vmatrix} \overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}} \\ 2\cos(\theta)\cos(\phi) & 2\sin(\theta)\cos(\phi) & -2\sin(\phi) \\ -2\sin(\phi)\sin(\theta) & 2\sin(\phi)\cos(\theta) & 0 \end{vmatrix}$$

$$= \overrightarrow{\mathbf{i}}(0+4\sin^2(\phi)\cos(\theta)) - \overrightarrow{\mathbf{j}}(0-4\sin^2(\phi)\sin(\theta)) + \overrightarrow{\mathbf{k}}(4\sin(\phi)\cos(\phi)\cos^2(\theta)$$

$$
\begin{aligned}
& +4\sin(\phi)\cos(\phi)\sin^2(\theta)) \\
= \ & \langle 4\sin^2(\phi)\cos(\theta)), 4\sin^2(\phi)\sin(\theta), 4\sin(\phi)\cos(\phi)\rangle.
\end{aligned}
$$

$$
\begin{aligned}
\iint_S \langle -x,-y,z^3\rangle.dS &= \iint_S \langle y,-x,2z\rangle.\overrightarrow{N}dS \\
&= \int_0^{\frac{\pi}{2}}\int_0^{2\pi} \langle 2\sin(\theta)\sin(\phi), -2\cos(\theta)\sin(\phi), 4\cos(\phi)\rangle \\
&\quad .\langle 4\sin^2(\phi)\cos(\theta)), 4\sin^2(\phi)\sin(\theta), 4\sin(\phi)\cos(\phi)\rangle d\theta d\phi \\
&= \int_0^{\frac{\pi}{2}}\int_0^{2\pi} 8\sin(\theta)\cos(\theta)\sin^3(\phi) - 8\sin(\theta)\cos(\theta)\sin^3(\phi) \\
&\quad +16\sin(\phi)\cos^2(\phi)d\theta d\phi \\
&= \int_0^{\frac{\pi}{2}}\int_0^{2\pi} 16\sin(\phi)\cos^2(\phi)d\theta d\phi \\
&= 16\int_0^{\frac{\pi}{2}} \sin(\phi)\cos^2(\phi)d\phi \int_0^{2\pi} d\theta \\
&= 16.2\pi\left[-\frac{\cos^3(\phi)}{3}\right]_0^{\frac{\pi}{2}} \\
&= 32\pi\frac{1}{3} \\
&= \frac{32}{3}\pi.
\end{aligned}
$$

Solution of exercise 7.

Reminder.

The divergence theorem

Let S be a piecewise, smooth closed surface that encloses solid E in space. Assume that S is oriented outward, and let $\overrightarrow{F}$ be a vector field with continuous partial derivatives on an open region containing E. Then,

$$\iiint_E div\ \overrightarrow{F}dV = \iint_S \overrightarrow{F}.d\overrightarrow{S}.$$

Solution.

1. *We have* $\overrightarrow{F}(x,y,z) = \langle xye^z, xy^2z^3, -ye^z\rangle$. *The divergence of* $\overrightarrow{F}$ *is,*

$$\overrightarrow{\nabla}.\overrightarrow{F} = \frac{\partial}{\partial x}(xye^z) + \frac{\partial}{\partial y}(xy^2z^3) + \frac{\partial}{\partial z}(-ye^z) = ye^z + 2xyz^3 + -ye^z = 2xyz^3.$$

Then,

$$\begin{aligned}
\iint_S \langle xye^z, xy^2z^3, -ye^z\rangle.d\overrightarrow{S} &= \iiint_E \overrightarrow{\nabla}.\overrightarrow{F}dV \\
&= \iiint_E 2xyz^3 dV \\
&= \int_0^3\int_0^2\int_0^1 2xyz^3 dzdydx \\
&= 2\int_0^1 z^3dz \int_0^2 ydy \int_0^3 xdx \\
&= 2\left[\frac{z^4}{4}\right]_0^1 \left[\frac{y^2}{2}\right]_0^2 \left[\frac{x^2}{2}\right]_0^3 \\
&= 2.\frac{1}{4}.2.\frac{9}{2} \\
&= \frac{9}{2}.
\end{aligned}$$

2. *We have* $\overrightarrow{F} = \langle x^3+y^3, y^3+z^3, z^3+x^3\rangle$. *The divergence of* $\overrightarrow{F}$ *is,*

$$\overrightarrow{\nabla}.\overrightarrow{F} = \frac{\partial}{\partial x}(x^3+y^3) + \frac{\partial}{\partial y}(y^3+z^3) + \frac{\partial}{\partial z}(z^3+x^3) = 3x^2+3y^2+3z^2 = 3(x^2+y^2+z^2).$$

Then,

$$\begin{aligned}
\iint_S \langle x^3+y^3, y^3+z^3, z^3+x^3\rangle.d\overrightarrow{S} &= \iiint_E \overrightarrow{\nabla}.\overrightarrow{F}dV \\
&= \iiint_E 3(x^2+y^2+z^2)dV
\end{aligned}$$

To compute this triple integral, we use the spherical coordinates,

$$\begin{cases} x = \rho\cos(\theta)\sin(\phi), & 0 \le \phi \le \pi, \\ y = \rho\sin(\theta)\sin(\phi), & 0 \le \theta \le 2\pi, \\ z = \rho\cos(\phi), & 0 \le \rho \le 2. \end{cases}$$

to obtain,

$$3\iiint_E (x^2+y^2+z^2)dV$$

$$\begin{aligned}
&= 3\int_0^{2\pi}\int_0^{\pi}\int_0^{2}(\rho^2\cos^2(\theta)\sin^2(\phi)+\rho^2\sin^2(\theta)\sin^2(\phi)+\rho^2\cos^2(\phi))\rho^2\sin(\phi)d\rho d\phi d\theta\\
&= 3\int_0^{2\pi}\int_0^{\pi}\int_0^{2}\rho^2\rho^2\sin(\phi)d\rho d\phi d\theta\\
&= 3\int_0^{2\pi}d\theta\int_0^{\pi}\sin(\phi)d\phi\int_0^{2}\rho^4 d\rho\\
&= 3.2\pi\left[-\cos(\phi)\right]_0^{\pi}\left[\frac{\rho^5}{5}\right]_0^2\\
&= 3.2\pi.2\frac{32}{5}\\
&= \frac{384}{5}\pi.
\end{aligned}$$

Therefore,

$$\iint_S \langle x^3+y^3, y^3+z^3, z^3+x^3\rangle.d\overrightarrow{S} = \frac{384}{5}\pi.$$

3. *We have* $\overrightarrow{F} = \langle xe^y, z-e^y, xy\rangle$. *The divergence of* $\overrightarrow{F}$ *is,*

$$\overrightarrow{\nabla}.\overrightarrow{F} = \frac{\partial}{\partial x}(xe^y)+\frac{\partial}{\partial y}(z-e^y)+\frac{\partial}{\partial z}(xy) = e^y-e^y+0=0.$$

Then,

$$\begin{aligned}
\iint_S \langle xe^y, z-e^y, xy\rangle.d\overrightarrow{S} &= \iiint_E \overrightarrow{\nabla}.\overrightarrow{F}dV\\
&= \iiint_E 0dV\\
&= 0.
\end{aligned}$$

Solution of exercise 8.

Reminder.

Stokes'theorem.

Let S be a piecewise smooth oriented surface with a boundary that is a simple closed curve C with positive orientation. If $\overrightarrow{F}$ is a vector field with component functions that have continuous partial derivatives on an open region containing S, then

$$\int_C \overrightarrow{F}.d\overrightarrow{r} = \iint_S curl\ \overrightarrow{F}.d\overrightarrow{S}.$$

Stokes'theorem gives

$$\int_C \overrightarrow{F}.d\overrightarrow{r} = \iint_S curl\ \overrightarrow{F}.\overrightarrow{N}dA.$$

where $\overrightarrow{N}$ is the unit normal vector.

Solution.

1. *We have $\overrightarrow{F} = \langle -y, x, -2\rangle$, S is the cone $z^2 = x^2 + y^2, 0 \leq z \leq 4$, oriented downward.*

   To verify Stoke's theorem, we should to compute both of Integrals in the theorem.

   - $\displaystyle\int_C \overrightarrow{F}.d\overrightarrow{r}$.

   The parameterization of S is given by,

   $$\overrightarrow{r}(t) = \langle 4\cos(t), 4\sin(t), 4\rangle, \quad 0 \leq t \leq 2\pi.$$

   Then,

   $$\overrightarrow{r}'(t) = \langle -4\sin(t), 4\cos(t), 0\rangle.$$

   Therefore,

   $$\begin{aligned}
   \int_C \langle -y, x, -2\rangle.d\overrightarrow{r} &= \int_0^{2\pi} \langle -4\sin(t), 4\cos(t), -2\rangle.\langle -4\sin(t), 4\cos(t), 0\rangle dt \\
   &= \int_0^{2\pi} (16\sin^2(t) + 16\cos^2(t))dt \\
   &= 16\int_0^{2\pi} dt \\
   &= 16.2\pi \\
   &= 32\pi.
   \end{aligned}$$

- $\displaystyle\iint_S curl\ \overrightarrow{F}.d\overrightarrow{S}.$

The curl of $\overrightarrow{F}$ *is,*

$$\begin{aligned} curl\ \overrightarrow{F} &= \begin{vmatrix} \overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & -2 \end{vmatrix} \\ &= \overrightarrow{\mathbf{i}}(0-0) - \overrightarrow{\mathbf{j}}(0-0) + \overrightarrow{\mathbf{k}}(1+1) \\ &= 2\overrightarrow{\mathbf{k}}. \end{aligned}$$

If we take

$$\begin{cases} x = r\cos(\theta), & 0 \le r \le 4, \\ y = r\sin(\theta), & 0 \le \theta \le 2\pi, \\ z = r, \end{cases}$$

we obtain,

$$\iint_S curl\ \overrightarrow{F}.d\overrightarrow{S} = \iint_S 2d\overrightarrow{S} = 2\int_0^{2\pi}\int_0^4 rdrd\theta = 2\int_0^{2\pi} d\theta \int_0^4 rdrd\theta = 2\pi[r^2]_0^4 = 32\pi.$$

Then,

$$\int_C \overrightarrow{F}.d\overrightarrow{r} = \iint_S curl\ \overrightarrow{F}.d\overrightarrow{S}.$$

2. *We have* $\overrightarrow{F} = \langle -2yz, y, 3x\rangle$. *S is the part of the paraboloid* $z = 5 - x^2 - y^2$ *that lies above the plane* $z = 1$, *oriented upward.*

- $\displaystyle\int_C \overrightarrow{F}.d\overrightarrow{r}$

The parameterization of S is given by,

$$\overrightarrow{r}(t) = \langle 2\cos(t), 2\sin(t), 1\rangle, \quad 0 \le t \le 2\pi.$$

Then,

$$\overrightarrow{r'}(t) = \langle -2\sin(t), 2\cos(t), 0\rangle.$$

Therefore,

$$\int_C \langle -2yz, y, 3x\rangle.d\overrightarrow{r} = \int_0^{2\pi} \langle -2.2\sin(t).1, 2\sin(t), 3\cos(t)\rangle.\langle -2\sin(t), 2\cos(t), 0\rangle dt$$

$$\begin{aligned}
&= \int_0^{2\pi} \langle -4\sin(t), 2\sin(t), 3\cos(t)\rangle . \langle -2\sin(t), 2\cos(t), 0\rangle dt \\
&= \int_0^{2\pi} (8\sin^2(t) + 4\sin(t)\cos(t)) dt \\
&= \int_0^{2\pi} (4(1-\cos(2t)) + 2\sin(2t)) dt \\
&= [4t - 2\sin(2t) - \cos(2t)]_0^{2\pi} \\
&= 8\pi.
\end{aligned}$$

- $\displaystyle\iint_S curl\ \overrightarrow{F}.d\overrightarrow{S}$

The curl of $\overrightarrow{F}$ *is,*

$$\begin{aligned}
curl\ \overrightarrow{F} &= \begin{vmatrix} \overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2yz & y & 3x \end{vmatrix} \\
&= \overrightarrow{\mathbf{i}}(0-0) - \overrightarrow{\mathbf{j}}(3+2y) + \overrightarrow{\mathbf{k}}(0+2z) \\
&= -\overrightarrow{\mathbf{j}}(3+2y) + \overrightarrow{\mathbf{k}}\,2z.
\end{aligned}$$

Since S *can be parameterized as,*

$$\overrightarrow{r}(x,y) = \langle x, y, 5 - x^2 - y^2\rangle,$$

we have,

$$\overrightarrow{t}_x = \langle 1, 0, -2x\rangle,\ \overrightarrow{t}_y = \langle 0, 1, -2y\rangle.$$

Then,

$$\begin{aligned}
\overrightarrow{t}_x \times \overrightarrow{t}_y &= \begin{vmatrix} \overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} \\
&= \overrightarrow{\mathbf{i}}(0+2x) - \overrightarrow{\mathbf{j}}(-2y-0) + \overrightarrow{\mathbf{k}}(1-0) \\
&= \overrightarrow{\mathbf{i}}\,2x + \overrightarrow{\mathbf{j}}\,2y + \overrightarrow{\mathbf{k}}.
\end{aligned}$$

Therefore, using the polar coordinates, we get,

$$\begin{aligned}
\iint_S curl\ \overrightarrow{F}.d\overrightarrow{S} &= \iint_S curl\ \overrightarrow{F}(\overrightarrow{r}(x,y)).(\overrightarrow{t}_x \times \overrightarrow{t}_y)dA \\
&= \iint_S \langle 0, -(3+2y), 2(5-x^2-y^2)\rangle.\langle 2x, 2y, 1\rangle dA \\
&= \iint_S (-2y(3+2y) + 2(5-x^2-y^2))dA \\
&= \iint_S (-2x^2 - 6y^2 - 6y + 10)dA \\
&= \int_0^{2\pi}\int_0^2 (-2r^2\cos^2(\theta) - 6r^2\sin^2(\theta) - 6r\sin(\theta) + 10)rdrd\theta \\
&= \int_0^{2\pi}\int_0^2 (-2r^3\cos^2(\theta) - 6r^3\sin^2(\theta) - 6r^2\sin(\theta) + 10)drd\theta \\
&= \int_0^{2\pi}\left[-2\frac{r^4}{4}\cos^2(\theta) - 6\frac{r^4}{4}\sin^2(\theta) - 6\frac{r^3}{3}\sin(\theta) + 10r\right]_0^2 d\theta \\
&= \int_0^{2\pi}\big(-8\cos^2(\theta) - 24\sin^2(\theta) - 16\sin(\theta) + 20\big)\,d\theta \\
&= \int_0^{2\pi}(-4(1+\cos(2\theta)) - 12(1-\cos(2\theta)) - 16\sin(\theta) + 20)\,d\theta \\
&= [-4\theta - 2\sin(2\theta) - 12\theta + 6\sin(2\theta) + 16\cos(\theta) + 20\theta]_0^{2\pi} \\
&= -8\pi - 24\pi + 16 + 40\pi - 16 \\
&= 8\pi.
\end{aligned}$$

Then,

$$\int_C \overrightarrow{F}.d\overrightarrow{r} = \iint_S curl\ \overrightarrow{F}.d\overrightarrow{S}.$$

3. $\overrightarrow{F}(x,y,z) = \langle y, z, x\rangle$. *S is the hemisphere* $x^2+y^2+z^2=1$, $y \geq 0$, *oriented in the direction of the positive y-axis.*
   - $\int_C \overrightarrow{F}.d\overrightarrow{r}$

The parameterization of S is given by,

$$\overrightarrow{r}(t) = \langle \cos(t), 0, \sin(t) \rangle, \quad 0 \leq t \leq 2\pi.$$

Then,

$$\overrightarrow{r'}(t) = \langle -\sin(t), 0, \cos(t) \rangle.$$

Therefore,

$$\begin{aligned} \int_C \langle y, z, x \rangle . d\overrightarrow{r} &= \int_0^{2\pi} \langle 0, \cos(t), \cos(t) \rangle . \langle -\sin(t), 0, \cos(t) \rangle dt \\ &= \int_0^{2\pi} \cos^2(t) dt \\ &= \int_0^{2\pi} \frac{1}{2}(1 + \cos(2t)) dt \\ &= \frac{1}{2}\left[t + \frac{\sin(2t)}{2}\right]_0^{2\pi} \\ &= \pi. \end{aligned}$$

- $\displaystyle\iint_S curl\ \overrightarrow{F} . d\overrightarrow{S}$

The curl of $\overrightarrow{F}$ is,

$$\begin{aligned} curl\ \overrightarrow{F} &= \begin{vmatrix} \overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} \\ &= \overrightarrow{\mathbf{i}}(0-1) - \overrightarrow{\mathbf{j}}(1-0) + \overrightarrow{\mathbf{k}}(0-1) \\ &= -\overrightarrow{\mathbf{i}} - \overrightarrow{\mathbf{j}} - \overrightarrow{\mathbf{k}}. \end{aligned}$$

Since S can be parameterized as,

$$\overrightarrow{r}(\phi, \theta) = \langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi) \rangle, 0 \leq \phi\pi, 0 \leq \theta \leq \pi.$$

we have,

$$\overrightarrow{t}_\phi = \langle \cos(\theta)\cos(\phi), \sin(\theta)\cos(\phi), -\sin(\phi) \rangle,$$

$$\overrightarrow{t}_{\theta} = \langle -\sin(\phi)\sin(\theta), \sin(\phi)\cos(\theta), 0\rangle.$$

Then,

$$\begin{aligned}
\overrightarrow{t}_x \times \overrightarrow{t}_y &= \begin{vmatrix} \overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}} \\ \cos(\theta)\cos(\phi) & \sin(\theta)\cos(\phi) & -\sin(\phi) \\ -\sin(\phi)\sin(\theta) & \sin(\phi)\cos(\theta) & 0 \end{vmatrix} \\
&= \overrightarrow{\mathbf{i}}(0+\sin^2(\phi)\cos(\theta)) - \overrightarrow{\mathbf{j}}(0-\sin^2(\phi)\sin(\theta)) \\
&\quad + \overrightarrow{\mathbf{k}}(\cos^2(\theta)\cos(\phi)\sin(\phi) + \sin^2(\theta)\cos(\phi)\sin(\phi)) \\
&= \langle \sin^2(\phi)\cos(\theta), \sin^2(\phi)\sin(\theta), \cos(\phi)\sin(\phi)\rangle
\end{aligned}$$

Therefore, we get,

$$\begin{aligned}
\iint_S curl\ \overrightarrow{F}.d\overrightarrow{S} &= \iint_S curl\ \overrightarrow{F}(\overrightarrow{r}(x,y)).(\overrightarrow{t}_x \times \overrightarrow{t}_y)dA \\
&= \iint_S \langle -1,-1,-1\rangle.\langle \sin^2(\phi)\cos(\theta), \sin^2(\phi)\sin(\theta), \cos(\phi)\sin(\phi)\rangle dA \\
&= -\iint_S (\sin^2(\phi)\cos(\theta) + \sin^2(\phi)\sin(\theta) + \cos(\phi)\sin(\phi))dA \\
&= -\int_0^{\pi}\int_0^{\pi} \left(\frac{1}{2}(1-\cos(2\phi))\cos(\theta) + \frac{1}{2}(1-\cos(2\phi))\sin(\theta) + \frac{1}{2}\sin(2\phi)\right) \\
&\quad d\phi d\theta \\
&= -\int_0^{\pi} \left[\frac{1}{2}\left(\phi - \frac{\sin(2\phi)}{2}\right)\cos(\theta) + \frac{1}{2}\left(\phi - \frac{\sin(2\phi)}{2}\right)\sin(\theta) - \frac{1}{2}\frac{\cos(2\phi)}{2}\right]_0^{\pi} \\
&\quad d\theta \\
&= -\int_0^{\pi} \left(\frac{\pi}{2}\cos(\theta) + \frac{\pi}{2}\sin(\theta)\right) d\theta \\
&= -\frac{\pi}{2}\left[\sin(\theta) - \cos(\theta)\right]_0^{\pi} \\
&= -\frac{\pi}{2}(1+1) \\
&= -\pi
\end{aligned}$$

Explication.

The $(-)$ *between the computing of* $\iint_S curl\ \overrightarrow{F}.d\overrightarrow{S}$ *and* $\int_C \overrightarrow{F}.d\overrightarrow{r}$ *should be the orientation of* C*, which was in clockwise direction instead of counterclockwise direction.*

Solution of exercise 9.

1. - *The work done along the parabola* $x = y^2$.
     We note that the parameterization of $x = y^2$ *from* $(0,0)$ *to* $(1,1)$ *is* $\overrightarrow{r}(t) = \langle t^2, t\rangle$, $0 \leq t \leq 1$, *then,*
$$\begin{aligned}\int_C \overrightarrow{F} d\overrightarrow{r} &= \int_C (x^2 - y^2 + x)dx - (2xy + y)dy \\ &= \int_0^1 (t^4 - t^2 + t^2).2tdt - (2t^3 + t)dt \\ &= \int_0^1 (2t^5 - 2t^3 + t)dt \\ &= \left[\frac{t^6}{3} - \frac{t^4}{2} - \frac{t^2}{2}\right]_0^1 \\ &= -\frac{2}{3}.\end{aligned}$$
   - *The work done along the parabola* $x = y^2$.
     We note that the parameterization of $x = y$ *from* $(0,0)$ *to* $(1,1)$ *is* $\overrightarrow{r}(t) = \langle t, t\rangle$, $0 \leq t \leq 1$, *then,*
$$\begin{aligned}\int_C \overrightarrow{F} d\overrightarrow{r} &= \int_C (x^2 - y^2 + x)dx - (2xy + y)dy \\ &= \int_0^1 (t^2 - t^2 + t)dt - (2t^2 + t)dt \\ &= \int_0^1 -2t^2 dt \\ &= \left[\frac{-2t^3}{3}\right]_0^1 \\ &= -\frac{2}{3}.\end{aligned}$$
     The work done does not depend on the path followed.

2. *The parameterization of* S *is,*
$$\overrightarrow{r}(u,v) = \langle a\sin(u)\cos(v), a\sin(u)\sin(v), a\cos(u)\rangle, 0 \leq u \leq \frac{\pi}{2}, 0 \leq v \leq 2\pi.$$

The tangent vectors is,

$$\begin{aligned}
\overrightarrow{r}_u \times \overrightarrow{r}_v &= \begin{vmatrix} \overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}} \\ a\cos(u)\cos(v) & a\sin(v)\cos(u) & -a\sin(u) \\ -a\sin(u)\sin(v) & a\sin(u)\cos(v) & 0 \end{vmatrix} \\
&= \langle a^2\sin^2(u)\cos(v), -a^2\sin^2(u)\sin(v), a^2\cos(u)\sin(u)\rangle
\end{aligned}$$

which gives,

$$||\overrightarrow{r}_u \times \overrightarrow{r}_v|| = a^2\sin(u).$$

Then,

$$\begin{aligned}
U &= \iint_S \frac{\sigma}{\sqrt{x^2+y^2+(z+a)^2}} dA = \sigma\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \frac{1}{2a\cos\left(\frac{u}{2}\right)} a^2\sin(u) du dv \\
&= a\sigma\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \frac{2\sin\left(\frac{u}{2}\right)\cos\left(\frac{u}{2}\right)}{2a\cos\left(\frac{u}{2}\right)} du dv = 2\pi a\sigma\int_0^{\frac{\pi}{2}} \sin\left(\frac{u}{2}\right) du \\
&= 2\pi a\sigma\left[-2\cos\left(\frac{u}{2}\right)\right]_0^{\frac{\pi}{2}} = 2\pi a\sigma(2-\sqrt{2}).
\end{aligned}$$

Solution of exercise 10.

1. - *The Green's theorem.*
     We have $P(x,y) = 3x - 5y$ *and* $Q(x,y) = x - 6y$*, then,* $\frac{\partial Q}{\partial x} = 1$, $\frac{\partial P}{\partial y} = -6$. *Then, we apply Green's theorem where* D *is the interior of* $C : \frac{x^2}{4} + y^2 \leq 1$.

$$\begin{aligned}
\int_C (3x-5y)dx + (x+6y)dy &= \iint (1-(-5))dxdy \\
&= 6 \times (\textit{Area of the ellipse }) \\
&= 6.2\pi \\
&= 12\pi.
\end{aligned}$$

   - *The parameterization of* C *is given by,*

$$\overrightarrow{r}(t) = \langle 2\cos(t), \sin(t)\rangle, 0 \leq t \leq 2\pi.$$

Then,

$$\int_C (3x-5y)dx + (x+6y)dy$$

$$\begin{aligned}
&= \int_0^{2\pi} (6\cos(t) - 5\sin(t)).-2\sin(t)dt + (2\cos(t) - 6\sin(t)).2\cos(t)dt \\
&= \int_0^{2\pi} (-18\cos(t)\sin(t) + 10\sin^2(t) + 2\cos^2(t))dt \\
&= \int_0^{2\pi} (-9\sin(2t) + 5(1-\cos(2t)) + (1+\cos(2t)))dt \\
&= \left[\frac{9}{2}\cos(2t) + 5t - \frac{5}{2}\sin(2t) + t + \frac{1}{2}\sin(2t)\right]_0^{2\pi} \\
&= 6.2\pi \\
&= 12\pi.
\end{aligned}$$

2. *Let* $P(x,y,z) = 2x-4y-5z, Q(x,y,z) = -4x+2y, R(x,y,z) = -5x+6z$. *We have,*

$$\frac{\partial P}{\partial y} = -4 = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = 0 = \frac{\partial R}{\partial y}, \frac{\partial R}{\partial x} = -5 = \frac{\partial P}{\partial z}.$$

Then, $\overrightarrow{F}$ *is conservative. Now, suppose* $\overrightarrow{\nabla} f = \overrightarrow{F}$, *therefore,*

$$f_x = 2x-4y-5z, f_y = -4x+2y, f_z = -5x+6z.$$

If we integrate f_x *with respect to* x, *we get,*

$$f(x,y,z) = \int (2x-4y-5z)dx = x^2 - 4yx - 5zx + h(y,z),$$

with $K : \mathbb{R}^2 \to \mathbb{R}$ *is a function. Since* $\overrightarrow{F}$ *is conservative, we have,*

$$f_y = -4x+2y \Rightarrow -4x+2y = -4x + h_y(x,y) \Rightarrow h_y(y,z) = 2y \Rightarrow h(y,z) = y^2 + C(z),$$

with $C : \mathbb{R} \to \mathbb{R}$ *is a function. Then,*

$$f(x,y,z) = x^2 - 4yx - 5zx + y^2 + C(z).$$

Since $\overrightarrow{F}$ *is conservative, we have,*

$f_z = -5x+6z \Rightarrow -5x+6z = -5x+C'(z) \Rightarrow C'(z) = 6z \Rightarrow C(z) = 3z^2+K.$ *K is a constant.*

By taking $K = 0$, *we obtain,*

$$f(x, y, z) = x^2 - 4yx - 5zx + y^2 + 3z^2.$$

Solution of exercise 11.

1. • *Since* $\overrightarrow{\nabla} f = \overrightarrow{F}$, *we have*

$$\overrightarrow{F} = \langle 12x^2 + 3y^2 + 5y, 6xy - 3y^2 + 5x\rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}. \right\rangle = \overrightarrow{\nabla} f.$$

Then, using the integrating with respect to x, we obtain

$$\int \frac{\partial f}{\partial x} dx = \int 12x^2 + 3y^2 + 5ydx \Rightarrow f(x, y) = 4x^3 + 3y^2x + 5yx + K(y),$$

with $K : \mathbb{R} \to \mathbb{R}$ *is a function. Now, w differentiate* $f(x, y)$ *with respect to y to have,*

$$\frac{\partial f(x, y)}{\partial y} = 6xy + 5x + K'(y).$$

While f is the potential function of $\overrightarrow{F}$, *we have* $6xy + 5x + K'(y) = 6xy - 3y^2 + 5x$, *then,*

$$K'(y) = -3y^2,$$

which gives by integrating with respect to y,

$$K(y) = -y^3 + C, \ C \text{ is a constant.}$$

Therefore,

$$f(x, y) = 4x^3 + 3y^2x + 5yx - y^3 + C, \ C \text{ is a constant.}$$

If we take $C = 0$,

$$f(x, y) = 4x^3 + 3y^2x + 5yx - y^3.$$

- *We have* $\overrightarrow{r}(t) = \langle t, \frac{1}{t}\rangle, 1 \leq t \leq 4$. *Then, using the Fundamental theorem of line integral, we obtain,*

$$\int_C \overrightarrow{F}.d\overrightarrow{r} = \int_C \overrightarrow{\nabla} f.d\overrightarrow{r} = f(\overrightarrow{r}(4)) - f(\overrightarrow{r}(1)) = f(4,1/4) - f(1,1) = \frac{16047}{64}.$$

2. *Use Stokes theorem. We parametrize the surface as*

$$\overrightarrow{r}(u,v) = \langle u, v, 12 - 6u - 3v\rangle.$$

We have $\overrightarrow{r}_u \times \overrightarrow{r}_v = \langle 6, 3, 1\rangle$. *The curl of the vector field is* $\langle 0, 0, 7e^u\rangle$. *Integrating gives*

$$\int_0^2 \int_0^{4-2u} 7e^u dvdu = -42 + 14e^2.$$

3. *Since* $div\overrightarrow{F} = y^2 + z^2 + x^2$, *the surface integral is equal to the triple integral*

$$\iint_B (y^2 + z^2 + x^2)dV$$

where B *is ball of radius* 3.

To evaluate the triple integral , we can change variables to spherical coordinates. In spherical coordinates, the ball is

$$\begin{cases} x = \rho\sin(\phi)\cos(\theta), & 0 \leq \rho \leq 3, \\ y = \rho\sin(\phi)\sin(\theta), & 0 \leq \theta \leq 2\pi, \\ z = \rho\cos(\phi), & 0 \leq \phi \leq \pi. \end{cases}$$

Therefore, the integral is

$$\begin{aligned} \iint_B (y^2 + z^2 + x^2)dV &= \int_0^3 \int_0^{2\pi} \int_0^{\pi} \rho^2.\rho^2 \sin(\phi)d\phi d\theta d\rho \\ &= \int_0^3 \int_0^{2\pi} \left[-\rho^4\cos(\phi)\right]_0^{\pi} d\theta d\rho \\ &= \int_0^3 \int_0^{2\pi} 2\rho^4 d\theta d\rho \\ &= \int_0^3 4\pi\rho^4 d\rho \\ &= \left[\frac{4\pi\rho^5}{5}\right]_0^3 \\ &= \frac{972}{5}\pi. \end{aligned}$$

Chapitre 2
Series.

Exercise 1: *Compute the value of the following series*

1. $\displaystyle\sum_{n=0}^{\infty} \frac{4}{(-3)^n} - 3^{1-n}$,
2. $\displaystyle\sum_{n=0}^{\infty} \frac{3}{2^n} + \frac{4}{5^n}$,
3. $\displaystyle\sum_{n=0}^{\infty} \frac{4^{n+1}}{5^n}$,
4. $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n+1}}{7^{n+1}}$,
5. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n$,
6. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{5^{n+1}}$.

Exercise 2: *Use Comparison test or limit comparison test to determine whether each series converges or diverges,*

1. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{2n^2+3n-5}$,
2. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+4}{2n^4+3n+5}$,
3. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{3^n}$,
4. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^3}$,
5. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$,
6. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{2^n+3^n}$.

Exercise 3: *Use Integral test to determine whether each series converges or diverges*

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{\pi}{4}}}$,
2. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$,
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n}$,
4. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$,
5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$,
6. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$.

Exercise 4: *Determine whether each alternating series converges or diverges.*

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+5}$,
2. $\displaystyle\sum_{n=4}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n-3}}$,
3. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\frac{n}{3n-2}$,
4. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\frac{\ln(n)}{n}$,
5. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{n}{n+1}$,
6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

Exercise 5: *Determine whether each series converges absolutely, converges conditionally or diverges.*

1. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3n^2+4}{2n^2+3n+5}$,
2. $\sum_{n=0}^{\infty}(-1)^{n}\frac{3^n}{2^n+5^n}$,
3. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln(n)}{n^3}$,
4. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln(n)}{n}$,
5. $\sum_{n=1}^{\infty}\frac{\cos(n)}{n^2}$,
6. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\arctan(n)}{n}$.

Exercise 6: *Use the ratio or root test to determine whether each series converges or diverges.*

1. $\sum_{n=0}^{\infty}(-1)^{n}\frac{3^n}{5^n}$,
2. $\sum_{n=1}^{\infty}\frac{n!}{n^n}$,
3. $\sum_{n=1}^{\infty}\frac{n^5}{n^n}$,
4. $\sum_{n=1}^{\infty}\frac{(n!)^2}{n^n}$,
5. $\sum_{n=1}^{\infty}\frac{(n^2+3n)^n}{(4n^2+5)^n}$,
6. $\sum_{n=1}^{\infty}\frac{n^n}{(\ln(n))^n}$.

Exercise 7: *Find the radius and interval of convergence for each power series*

1. $\sum_{n=0}^{\infty}nx^n$,
2. $\sum_{n=0}^{\infty}\frac{x^n}{n!}$,
3. $\sum_{n=1}^{\infty}\frac{n!}{n^n}x^n$,
4. $\sum_{n=1}^{\infty}\frac{n!}{n^n}(x-2)^n$,
5. $\sum_{n=1}^{\infty}\frac{(n!)^2}{n^n}(x-2)^n$,
6. $\sum_{n=1}^{\infty}\frac{(x+5)^n}{n(n+1)}$.

Exercise 8: *For each function, find the Maclaurin series or Taylor series centered at a, and the radius of convergence.*

1. $\cos(x), a=0$,
2. e^x,
3. $\frac{1}{x}, a=5$,
4. $\ln(x), a=2$,
5. $\frac{1}{x^2}, a=1$,
6. $\frac{1}{\sqrt{1-x}}$.

Exercise 9: *Using the power series, resolve the following differential equations*

1. $(x^2-1)y''+6xy'+4y=-4$,
2. $y''-2xy'+y=0, y(0)=0, y'(0)=1$,

3. $(x-3)y' + 2y = 0.$

Exercice 10. *(Exam)*

Study the nature of the following series

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$, *2.* $\displaystyle\sum_{n=1}^{\infty} \frac{4^{n+1}((n+1)!)^2}{(2n-1)!}$, *3.* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}}$, *4.* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$.

Solution of exercise 1.

Reminder.

Geometric series : *A geometric series is a series of the form*

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots .$$

If $|r| < 1$, *the series* **converges**, *and*

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad for \quad |r| < 1.$$

If $|r| \geq 1$, *The series* **diverges**.

Solution.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{4}{(-3)^n} - 3^{1-n} = 4\sum_{n=0}^{\infty}\left(\frac{-1}{3}\right)^n - 3\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n = 4\frac{1}{1-\frac{-1}{3}} - 3\frac{1}{1-\frac{1}{3}} = 3 - \frac{9}{2} = -\frac{3}{2}.$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{3}{2^n} + \frac{4}{5^n} = 3\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n + 4\sum_{n=0}^{\infty}\left(\frac{1}{5}\right)^n = 3\frac{1}{1-\frac{1}{2}} + 4\frac{1}{1-\frac{1}{5}} = 6 + 5 = 11.$

3. $\displaystyle\sum_{n=0}^{\infty} \frac{4^{n+1}}{5^n} = 4\sum_{n=0}^{\infty}\left(\frac{4}{5}\right)^n = 4\frac{1}{1-\frac{4}{5}} = 20.$

4. $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n+1}}{7^{n+1}} = \frac{3}{7}\sum_{n=0}^{\infty}\left(\frac{3}{7}\right)^n = \frac{3}{7}.\frac{1}{1-\frac{3}{7}} = \frac{3}{4}.$

5. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^n = \frac{\frac{3}{5}}{1-\frac{3}{5}} = \frac{3}{2}.$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{5^{n+1}} = \frac{1}{5}\sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^n = \frac{1}{5}.\frac{3}{2} = \frac{3}{10}.$

Solution of exercise 2.

Reminder.

- **Comparison test :**

  i. *Suppose there exist an integer N such that $0 \leq a_n \leq b_n$ for all $n \geq N$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*

  ii. *Suppose there exist an integer N such that $a_n \geq b_n \geq 0$ for all $n \geq N$. If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.*

- **Limit comparison test :** *Let $a_n, b_n \geq 0$ for all $n \geq 1$.*

  i. *If $\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.*

  ii. *If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*

  iii. *$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.*

Solution.

1. $\sum_{n=2}^{\infty} \frac{1}{2n^2+3n-5}$. *Compare to* $\sum_{n=2}^{\infty} \frac{1}{2n^2}$. *Since* $\sum_{n=2}^{\infty} \frac{1}{n^2}$ *is a p-series with $p = 2$, it converges, then,* $\sum_{n=2}^{\infty} \frac{1}{2n^2}$ *converges. Further,*

$$\frac{1}{2n^2+3n-5} < \frac{1}{2n^2}, \quad \forall n \in \mathbb{N}.$$

Therefore, we can conclude that $\sum_{n=2}^{\infty} \frac{1}{2n^2+3n-5}$ *converges.*

2. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+4}{2n^4+3n+5}$. *Compare this series with* $\displaystyle\sum_{n=1}^{\infty} \frac{3}{2n^2}$. *We see that*

$$\lim_{n\to\infty} \frac{\frac{3n^2+4}{2n^4+3n+5}}{\frac{3}{2n^2}} = \lim_{n\to\infty} \frac{3n^2+4}{2n^4+3n+5}.\frac{2n^2}{3} = 1.$$

By the limit comparison test, since $\displaystyle\sum_{n=1}^{\infty} \frac{3}{2n^2}$ *converges, then* $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+4}{2n^4+3n+5}$ *converges.*

3. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{3^n}$. *Compare with* $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$. *We see that*

$$\lim_{n\to\infty} \frac{\frac{2^n+1}{3^n}}{\left(\frac{2}{3}\right)^n} = \lim_{n\to\infty} \frac{2^n+1}{3^n}.\frac{3^n}{2^n} = \lim_{n\to\infty} \left[1+\left(\frac{1}{2}\right)^n\right] = 1.$$

Therefore, since $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ *is a geometric series with* $r = \frac{2}{3}$ *and* $\left|\frac{2}{3}\right| < 1$, *it converges, we conclude that* $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{3^n}$ *converges.*

4. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^3}$. *Compare with* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$. *Since* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ *is p-series with* $p = 2$, *it converges. Further,*

$$\frac{\ln(n)}{n^3} < \frac{1}{n^2}, \quad n \in \mathbb{N}.$$

Therefore, we can conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^3}$ *converges.*

5. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$. *Compare to* $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$. *Since*

$$\frac{1}{\ln(n)} > \frac{1}{n}, \quad n \geq 2,$$

and $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ *diverges, we have then* $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ *diverges.*

6. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{2^n+3^n}$. *Compare with* $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$. *We see that*

$$\lim_{n\to\infty} \frac{\frac{4^n}{2^n+3^n}}{\left(\frac{4}{3}\right)^n} = \lim_{n\to\infty} \frac{3^n}{2^n+3^n} = \lim_{n\to\infty} \frac{1}{\left(\frac{2}{3}\right)^n+1} = 1.$$

Therefore, since $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$ *a geometric series with* $r = \frac{4}{3}$ *and* $\left|\frac{4}{3}\right| > 1$, *it diverges, we conclude that* $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{2^n+3^n}$.

Solution of exercise 3.

Reminder.

Integral test. *(Theorem)*

Suppose $\displaystyle\sum_{n=1}^{\infty} a_n$ *is a series with positive terms* a_n. *Suppose there exists a function* f *and a positive integer* N *such that the following three conditions are satisfied:*

i. f *is continuous,*

ii. f *is decreasing,*

iii. $f(n) = a_n$ *for all integers* $n \geq N$.

Then, $\displaystyle\sum_{n=1}^{\infty} a_n$ *and* $\displaystyle\int_N^{\infty} f(x)dx$ *both converge or both diverge.*

Solution.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{\pi}{4}}}$. *Compare* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{\pi}{4}}}$ *and* $\displaystyle\int_1^{\infty} \frac{1}{x^{\frac{\pi}{4}}}dx$ *because* $\frac{1}{n^{\frac{\pi}{4}}}$ *is continuous and decreasing for* $n \geq 1$. *Then, We have*

$$\int_1^{\infty} \frac{1}{x^{\frac{\pi}{4}}}dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x^{\frac{\pi}{4}}}dx = \lim_{b\to\infty} \left[\frac{4}{4-\pi}\frac{1}{x^{\frac{\pi-4}{4}}}\right]_1^b = \frac{4}{4-\pi}\lim_{b\to\infty}\left[\frac{1}{b^{\frac{\pi-4}{4}}} - 1\right] = \infty.$$

Thus, the integral $\displaystyle\int_1^\infty \frac{1}{x^{\frac{\pi}{4}}}dx$ *diverges, and therefore, the series* $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{\frac{\pi}{4}}}$ *diverges.*

2. $\displaystyle\sum_{n=1}^{\infty}\frac{\ln(n)}{n^2}$. *Compare* $\displaystyle\sum_{n=2}^{\infty}\frac{\ln(n)}{n^2}$ *to* $\displaystyle\int_2^\infty \frac{\ln(x)}{x^2}dx$ *because* $\frac{\ln(n)}{n^2}$ *is continuous and decreasing for* $n \geq 2$. *Then, we have*

$$\begin{aligned}
\int_2^\infty \frac{\ln(x)}{x^2}dx &= \lim_{b\to\infty}\int_2^b \frac{\ln(x)}{x^2}dx = \lim_{b\to\infty}\left[\left.\frac{-\ln(x)}{x}\right|_2^b + \int_2^b \frac{1}{x^2}dx\right] \\
&= \lim_{b\to\infty}\left[-\frac{\ln(b)}{b} + \frac{\ln(2)}{2} - \left.\frac{1}{x}\right|_2^b\right] = \lim_{b\to\infty}\left[-\frac{\ln(b)}{b} + \frac{\ln(2)}{2} - \frac{1}{b} + \frac{1}{2}\right] \\
&= \frac{\ln(2)+1}{2} < \infty.
\end{aligned}$$

which implies that $\displaystyle\int_2^\infty \frac{\ln(x)}{x^2}$ *converges. Therefore,* $\displaystyle\sum_{n=1}^{\infty}\frac{\ln(n)}{n^2}$ *converges.*

3. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{e^n}$. *Compare* $\displaystyle\sum_{n=1}^{\infty}\frac{1}{e^n}$ *to* $\displaystyle\int_1^\infty \frac{1}{e^x}dx$. *We have*

$$\int_1^\infty \frac{1}{e^x}dx = \lim_{b\to\infty}\int_1^b e^{-x}dx = \lim_{b\to\infty}\left[-e^{-x}\right]_1^b = \lim_{b\to\infty}\left[-e^{-b} + \frac{1}{e}\right] = \frac{1}{e} < \infty.$$

which implies $\displaystyle\sum_{n=1}^{\infty}\frac{1}{e^n}$ *converges.*

4. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\ln(n)}$. *Compare* $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\ln(n)}$ *to* $\displaystyle\int_2^\infty \frac{1}{x\ln(x)}dx$. *We have*

$$\begin{aligned}
\int_2^\infty \frac{1}{x\ln(x)}dx &= \lim_{b\to\infty}\int_2^b \frac{1}{x\ln(x)}dx = \lim_{b\to\infty}\int_2^b \frac{\frac{1}{x}}{\ln(x)}dx \\
&= \lim_{b\to\infty}\left[\ln(\ln(x))\right]_2^b = \lim_{b\to\infty}\left[\ln(\ln(b)) - \ln(\ln(2))\right] \\
&= \infty.
\end{aligned}$$

Then, $\displaystyle\int_2^\infty \frac{1}{x\ln(x)}dx$ *diverges and we conclude that* $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\ln(n)}$.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$. *Compare* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ *and* $\displaystyle\int_1^{\infty} \frac{1}{x^2+1}dx$. *We have*

$$\int_1^{\infty} \frac{1}{x^2+1}dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x^2+1}dx = \lim_{b\to\infty} [\arctan(x)]_1^b = \lim_{b\to\infty} \left[\arctan(b) - \frac{\pi}{4}\right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

Since $\displaystyle\int_1^{\infty} \frac{1}{x^2+1}dx$ *converges, then,* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ *converges.*

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$. *Compare* $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ *to* $\displaystyle\int_1^{\infty} \frac{x}{x^2+1}dx$. *We have*

$$\int_1^{\infty} \frac{x}{x^2+1}dx = \lim_{b\to\infty} \int_1^{b} \frac{x}{x^2+1}dx = \lim_{b\to\infty} \left[\frac{\ln(x^2+1)}{2}\right]_1^b = \frac{1}{2}\lim_{b\to\infty} [\ln(b^2+1) - \ln(2)] = \infty.$$

Then, $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ *is divergent because* $\displaystyle\int_1^{\infty} \frac{x}{x^2+1}dx$ *diverges.*

Solution of exercise 4.

Reminder.

Leibniz's test

An alternating series of the form $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ *or* $\displaystyle\sum_{n=1}^{\infty} (-1)^{n} b_n$ *converges if*

i. $0 \leq b_{n+1} \leq b_n$ *for all* $n \geq 1$.

ii. $\displaystyle\lim_{n\to\infty} b_n = 0$.

Solution.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+5}$. *Since* $\frac{1}{2n+7} < \frac{1}{2n+5}$ *and* $\frac{1}{2n+5} \to 0$, *as* $n \to \infty$, *the series converges.*

2. $\displaystyle\sum_{n=4}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n-3}}$. *Since* $\frac{1}{\sqrt{n-2}} < \frac{1}{\sqrt{n-3}}$ *and* $\frac{1}{\sqrt{n-3}} \to 0$, *as* $n \to \infty$, *the series converges.*

3. $\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{n}{3n-2}$. *Since* $\frac{n}{3n-2}\to 1\neq 0$ *as* $n\to\infty$, *we cannot apply the alternating series test. Now, we use the* n^{th} *term test for divergence. Since* $\lim_{n\to\infty}\dfrac{n}{3n-2}=1\neq 0$, *the series diverges.*

4. $\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{\ln(n)}{n}$. *Since* $\frac{\ln(n)}{n}$ *is decreasing for* $n\geq e$ *and* $\frac{\ln(n)}{n}\to 0$, *as* $n\to\infty$, *the series converges.*

5. $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{n}{n+1}$. *Since* $\frac{n}{n+1}\to 1\neq 0$ *as* $n\to\infty$, *we cannot apply the alternating series test. Then, we use the* n^{th} *term test for divergence. Since* $\lim_{n\to\infty}\dfrac{n}{n+1}=1\neq 0$, *the series diverges.*

6. $\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n^2}$. *Since* $\frac{1}{(n+1)^2}<\frac{1}{n^2}$ *and* $\frac{1}{n^2}\to 0$, *as* $n\to\infty$, *the series converges.*

Solution of exercise 5.

Reminder.

- *A series* $\sum_{n=1}^{\infty}a_n$ *exhibits* **absolute convergence** *if* $\sum_{n=1}^{\infty}|a_n|$ *converges. A series* $\sum_{n=1}^{\infty}a_n$ *exhibits* **conditional convergence** *if* $\sum_{n=1}^{\infty}a_n$ *converges but* $\sum_{n=1}^{\infty}|a_n|$ *diverges.*

- *If* $\sum_{n=1}^{\infty}|a_n|$ *converges, then* $\sum_{n=1}^{\infty}a_n$ *converges.*

Solution.

1. $\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{3n^2+4}{2n^2+3n+5}$. *Since* $\lim_{n\to\infty}\dfrac{3n^2+4}{2n^2+3n+5}=\dfrac{3}{2}\neq 0$, *by the divergence test, the series* $\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{3n^2+4}{2n^2+3n+5}$ *diverges, therefore,* $\sum_{n=1}^{\infty}\left|(-1)^{n-1}\dfrac{3n^2+4}{2n^2+3n+5}\right|$ *diverges.*

2. $\sum_{n=0}^{\infty}(-1)^n\frac{3^n}{2^n+5^n}$. *We have*

$$\left|\sum_{n=0}^{\infty}(-1)^n\frac{3^n}{2^n+5^n}\right| = \sum_{n=0}^{\infty}\frac{3^n}{2^n+5^n}.$$

We compare $\sum_{n=0}^{\infty}\frac{3^n}{2^n+5^n}$ *to* $\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n$. *We see that*

$$\lim_{n\to\infty}\frac{\frac{3^n}{2^n+5^n}}{\left(\frac{3}{5}\right)^n} = \lim_{n\to\infty}\frac{5^n}{2^n+5^n} = \lim_{n\to\infty}\frac{1}{\left(\frac{2}{5}\right)^n+1} = 1.$$

Therefore, since $\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n$ *a geometric series with* $r=\frac{3}{5}$ *and* $\left|\frac{3}{5}\right|<1$ *converges, we conclude that* $\sum_{n=0}^{\infty}\frac{3^n}{2^n+5^n}$, *hence,* $\sum_{n=0}^{\infty}(-1)^n\frac{3^n}{2^n+5^n}$ *is absolutely convergent.*

3. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln(n)}{n^3}$. *We have*

$$\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{\ln(n)}{n^3}\right| = \sum_{n=1}^{\infty}\frac{\ln(n)}{n^3}.$$

We compare $\sum_{n=1}^{\infty}\frac{\ln(n)}{n^3}$ *to* $\sum_{n=1}^{\infty}\frac{1}{n^2}$. *Since* $\frac{\ln(n)}{n^3}<\frac{1}{n^2}$, $\forall n\in\mathbb{N}^*$ *and by comparison test,* $\sum_{n=1}^{\infty}\frac{1}{n^2}$ *converges, then,* $\sum_{n=1}^{\infty}\frac{\ln(n)}{n^3}$ *converges, therefore,* $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln(n)}{n^3}$ *converges absolutely.*

4. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln(n)}{n}$. *We have*

$$\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{\ln(n)}{n}\right| = \sum_{n=1}^{\infty}\frac{\ln(n)}{n}.$$

We compare $\sum_{n=e}^{\infty} \frac{\ln(n)}{n}$ *to* $\int_e^{\infty} \frac{\ln(x)}{x} dx$. *We have*

$$\int_e^{\infty} \frac{\ln(x)}{x} dx = \lim_{b\to\infty} \int_e^{b} \frac{\ln(x)}{x} dx = \frac{1}{2} \lim_{b\to\infty} \left[(\ln(x))^2\right]_e^b = \infty.$$

Then, $\int_e^{\infty} \frac{\ln(x)}{x} dx$ *diverges, therefore* $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ *diverges and hence,* $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\ln(n)}{n}$ *is not absolutely convergent. Now, we study the nature of* $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\ln(n)}{n}$. *For that, we apply Leibniz's test. Since* $\frac{\ln(n)}{n}$ *is decreasing for* $n \geq e$ *and* $\frac{\ln(n)}{n} \to 0$ *as* $n \to \infty$, *then,* $\sum_{n=3}^{\infty} (-1)^{n-1} \frac{\ln(n)}{n}$ *converges, therefore,* $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\ln(n)}{n}$ *converges. Hence,* $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\ln(n)}{n}$ *is conditionally convergent.*

5. $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$. *Noting that* $|\cos(n)| \leq 1$, *to determine whether the series converges absolutely, compare* $\sum_{n=1}^{\infty} \left|\frac{\cos(n)}{n^2}\right|$ *with the series* $\sum_{n=1}^{\infty} \frac{1}{n^2}$. *Since* $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges, by the comparison test,* $\sum_{n=1}^{\infty} \left|\frac{\cos(n)}{n^2}\right|$ *converges, and therefore* $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$ *converges absolutely.*

6. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\arctan(n)}{n}$. *We have*

$$\sum_{n=1}^{\infty} \left|(-1)^{n-1} \frac{\arctan(n)}{n}\right| = \sum_{n=1}^{\infty} \frac{\arctan(n)}{n}.$$

We compare $\sum_{n=1}^{\infty} \frac{\arctan(n)}{n}$ *to* $\sum_{n=1}^{\infty} \frac{\frac{\pi}{2}}{n}$. *We see that*

$$\lim_{n\to\infty} \frac{\frac{\arctan(n)}{n}}{\frac{\frac{\pi}{2}}{n}} = \frac{2}{\pi} \lim_{n\to\infty} \arctan(n) = 1.$$

Therefore, since $\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ *is divergent, then,* $\sum_{n=1}^{\infty}\frac{\arctan(n)}{n}$ *diverges. Therefore, the series* $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\arctan(n)}{n}$ *does not converge absolutely. Now, we study the nature of* $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\arctan(n)}{n}$ *and we apply the Leibniz's test. Since* $\frac{\arctan(n)}{n}$ *is decreasing for* $n \geq 1$ *and* $\frac{\arctan(n)}{n} \to 0$, *as* $n \to \infty$, *then,* $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\arctan(n)}{n}$ *converges. Thus* $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\arctan(n)}{n}$ *is conditionally convergent.*

Solution of exercise 6.

Reminder.

- **Ratio test :** *Let* $\sum_{n=1}^{\infty} a_n$ *be a series with nonzero terms. let*

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|.$$

Then,

  i. If $0 \leq \rho < 1$, *then* $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*

  ii. If $\rho > 1$ *or* $\rho = \infty$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

  iii. If $\rho = 1$, *the test does not provide any information.*

- **Root test :** *Consider the series* $\sum_{n=1}^{\infty} a_n$. *Let*

$$\rho = \lim_{n\to\infty}\sqrt[n]{|a_n|}.$$

Then,

  i. If $0 \leq \rho < 1$, *then* $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*

ii. *If* $\rho > 1$ *or* $\rho = \infty$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

iii. *If* $\rho = 1$, *the test does not provide any information.*

Solution.

1. $\sum_{n=0}^{\infty}(-1)^n \frac{3^n}{5^n}$. *From the ratio test, we can see*

$$\rho = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{3^{n+1}}{5^{n+1}}}{(-1)^n\frac{3^n}{5^n}}\right| = \lim_{n\to\infty}\frac{3^{n+1}}{5^{n+1}}.\frac{5^n}{3^n} = \frac{3}{5}.$$

Since $\rho < 1$, *the series converges.*

2. $\sum_{n=1}^{\infty}\frac{n!}{n^n}$. *From the ratio test, we can see*

$$\rho = \lim_{n\to\infty}\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty}\frac{(n+1)!}{n!}.\frac{n^n}{(n+1)^{n+1}} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^n = \frac{1}{e}.$$

Since $\rho < 1$, *the series converges.*

3. $\sum_{n=1}^{\infty}\frac{n^5}{n^n}$. *From the root test, we can see*

$$\rho = \lim_{n\to\infty}\sqrt[n]{\frac{n^5}{n^n}} = \lim_{n\to\infty}\frac{n^{\frac{5}{n}}}{n} = 0.$$

Since $\rho < 1$, *the series converges.*

4. $\sum_{n=1}^{\infty}\frac{(n!)^2}{n^n}$. *From the ratio test, we can see*

$$\rho = \lim_{n\to\infty}\frac{\frac{((n+1)!)^2}{(n+1)^{n+1}}}{\frac{(n!)^2}{n^n}} = \lim_{n\to\infty}\frac{(n+1)!(n+1)!}{n!n!}.\frac{n^n}{(n+1)^{n+1}} = \lim_{n\to\infty}(n+1)\left(\frac{n}{n+1}\right)^n = \infty.$$

Since $\rho = \infty$, *the series diverges.*

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(n^2+3n)^n}{(4n^2+5)^n}$. *From the root test, we can see*

$$\rho = \lim_{n\to\infty} \sqrt[n]{\frac{(n^2+3n)^n}{(4n^2+5)^n}} = \lim_{n\to\infty} \frac{n^2+3n}{4n^2+5} = \frac{1}{4}.$$

Since $\rho < 1$, the series converges.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{(\ln(n))^n}$. *From the root test, we can see*

$$\rho = \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{(\ln(n))^n}} = \lim_{n\to\infty} \frac{n}{(\ln(n))} = \infty.$$

Since $\rho = \infty$, the series diverges.

Solution of exercise 7.

Reminder.

Radius of convergence.

Consider the power series $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$. The set of real numbers x where the series converges is the interval of convergence. If there exists a real number $R > 0$ such that the series converges for $|x-a| < R$ and diverges for $|x-a| > R$, then R is the radius of convergence. If the series converges only at $x = a$, we say the radius of convergence is $R = 0$. If the series converges for all real numbers x, we say the radius of convergence is $R = \infty$.

Solution.

1. $\displaystyle\sum_{n=0}^{\infty} nx^n$. *To check for convergence, we apply the ratio test. We have*

$$\rho = \lim_{n\to\infty} \left|\frac{(n+1)x^{n+1}}{nx^n}\right| = \lim_{n\to\infty} |x|\frac{n+1}{n} = |x|.$$

The ratio $\rho < 1$ if $|x| < 1$. Since $|x| < 1$ implies that $-1 < x < 1$, the series converges absolutely if $-1 < x < 1$. The ratio $\rho > 1$ if $|x| > 1$. Therefore, the series diverges if $x < -1$

or $x > 1$. *Now, we need to test the convergence at the endpoints. For* $x = -1$, *the series is given by* $\sum_{n=0}^{\infty} n(-1)^n$. *Since this series is alternating series which diverges by the divergence test. Thus, the series diverges at* $x = -1$. *For* $x = 1$, *the series is given by* $\sum_{n=0}^{\infty} n$, *which is divergent. Therefore, the power series diverges at* $x = \pm 1$. *We conclude that the interval of convergence is* $(-1, 1)$ *and the radius of convergence is* 1.

2. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. *To check for convergence, we apply the ratio test. We have*

$$\rho = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} . \frac{n!}{x^n} \right| = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1.$$

for all values of x. *Therefore, the series converges for all real numbers* x. *The interval of convergence is* $(-\infty, \infty)$ *and the radius of convergence is* $R = \infty$.

3. $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$. *To check for convergence, we apply the ratio test. We have*

$$\rho = \lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{(n+1)^{(n+1)}} x^{n+1}}{\frac{n!}{n^n} x^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!}{(n+1)^{(n+1)}} . \frac{n^n}{n!} . \frac{x^{n+1}}{x^n} \right| = |x| \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^n = \frac{|x|}{e}.$$

Therefore, the series converges for $|x| < e$. *At the endpoints,* $x = \pm e$, *the series is divergent. Then, the interval of convergence is* $(-e, e)$ *and the radius of convergence is* $R = e$.

4. $\sum_{n=1}^{\infty} \frac{n!}{n^n} (x-2)^n$. *To check for convergence, we apply the ratio test. We have*

$$\begin{aligned} \rho = \lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{(n+1)^{(n+1)}} (x-2)^{(n+1)}}{\frac{n!}{n^n} (x-2)^n} \right| &= \lim_{n\to\infty} \left| \frac{(n+1)!}{(n+1)^{(n+1)}} . \frac{n^n}{n!} . \frac{(x-2)^{(n+1)}}{(x-2)^n} \right| \\ &= |x-2| \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^n \\ &= |x-2| \frac{1}{e}. \end{aligned}$$

The series converges for $|x-2| < e$. *At the endpoints,* $x = 2 \pm e$, *the series is divergent.Therefore, the interval of convergence is* $2-e < x < 2+e$ *and the radius of convergence is* e.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{n^n}(x-2)^n$. *To check for convergence, we apply the ratio test. We have*

$$\begin{aligned}
\rho = \lim_{n\to\infty} \left| \frac{\frac{((n+1)!)^2}{(n+1)^{(n+1)}}(x-2)^{(n+1)}}{\frac{(n!)^2}{n^n}(x-2)^n} \right| &= \lim_{n\to\infty} \left| \frac{((n+1)!)^2}{(n+1)^{(n+1)}} . \frac{n^n}{(n!)^2} . \frac{(x-2)^{(n+1)}}{(x-2)^n} \right| \\
&= |x-2| \lim_{n\to\infty} (n+1) \left(1 - \frac{1}{n+1}\right)^n \\
&= \infty.
\end{aligned}$$

Therefore, the series diverges for all $x \neq 2$. *Since the series is centred at* $x = 2$, *it must converge there, so the series diverges only for* $x \neq 2$. *The interval of convergence is the single value* $x = 2$ *and the radius of convergence is* $R = 0$.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(x+5)^n}{n(n+1)}$. . *To check for convergence, we apply the ratio test. We have*

$$\rho = \lim_{n\to\infty} \left| \frac{\frac{(x+5)^{(n+1)}}{(n+1)(n+2)}}{\frac{(x+5)^n}{n(n+1)}} \right| = \lim_{n\to\infty} \left| \frac{n(n+1)}{(n+1)(n+2)} . \frac{(x+5)^{n+1}}{(x+5)^n} \right| = |x+5| \lim_{n\to\infty} \frac{n}{n+2} = |x+5|.$$

The ratio $\rho < 1$ *if* $|x+5| < 1$. *Since* $|x+5| < 1$ *implies that* $-6 < x < -4$, *the series converges absolutely if* $-6 < x < -4$. *The ratio* $\rho > 1$ *if* $|x+5| > 1$. *Therefore, the series diverges if* $x < -6$ *or* $x > -4$. *Now, we need to test the convergence at the endpoints of* x *separately. For* $x = -6$, *the series is given by* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}$. *Since this is the alternating series which converges. Thus, the series converges at* $x = -6$. *For* $x = -4$, *the series is given by* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. *Since this series is telescopic series convergent. therefore power series converges at* $x = -4$. *We conclude that the interval of convergence is* $[-6, -4]$ *and the radius of convergence is* $R = 1$.

Solution of exercise 8.

Reminder.

Maclaurin and Taylor series.

- *If f has derivatives of all orders at $x = a$, then the* **Taylor series** *for the function f at a is,*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

The Taylor series for f at 0 is known as the **Maclaurin series** *for f.*

- *If f has n derivatives at $x = a$, then, the n^{th} degree Taylor polynomial of f at a is,*

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Then, n^{th} degree Taylor polynomial for f at 0 is known as the n^{th} **degree Maclaurin polynomial** *for f.*

Solution.

1. - *For $f(x) = \cos(x)$, the values of the function and its first four derivatives at $x = 0$ are given as follows,*

$$\begin{array}{rcll} f(x) &=& \cos(x) & \Rightarrow f(0) = 1, \\ f'(x) &=& -\sin(x) & \Rightarrow f'(0) = 0, \\ f''(x) &=& -\cos(x) & \Rightarrow f''(0) = -1, \\ f'''(x) &=& \sin(x) & \Rightarrow f'''(0) = 0, \\ f^{(4)}(x) &=& \cos(x) & \Rightarrow f^{(4)}(0) = 1. \end{array}$$

Since the fourth derivatives is $\cos(x)$, the pattern repeats. That is, $f^{(2m)}(0) = (-1)^m$ and $f^{(2m+1)}(0) = 0$, for $m \geq 0$. Thus, we have,

$$p_0(x) = 1,$$

$$\begin{aligned}
p_1(x) &= 1+0=1,\\
p_2(x) &= 1+0-\frac{1}{2!}x^2 = 1-\frac{1}{2!}x^2,\\
p_3(x) &= 1+0-\frac{1}{2!}x^2+0 = 1-\frac{1}{2!}x^2,\\
p_4(x) &= 1+0-\frac{1}{2!}x^2+0+\frac{1}{4!}x^4 = 1-\frac{1}{2!}x^2+\frac{1}{4!}x^4,\\
p_5(x) &= 1+0-\frac{1}{2!}x^2+0+\frac{1}{4!}x^4+0 = 1-\frac{1}{2!}x^2+\frac{1}{4!}x^4,
\end{aligned}$$

and for $m \geq 0$,

$$p_{2m}(x) = 1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\cdots+(-1)^m\frac{x^{2m}}{(2m)!} = \sum_{k=0}^{m}(-1)^k\frac{x^{2k}}{(2k)!}.$$

Then,

$$\cos(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}.$$

- *To find the radius of convergence, we apply the ratio test. We find*

$$\begin{aligned}
\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}}{(2n+2)!}x^{2n+2}}{\frac{(-1)^n}{(2n)!}x^{2n}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(-1)^n}.\frac{(2n)!}{(2n+2)!}.\frac{x^{2n+2}}{x^{2n}}\right|\\
&= |x^2|\lim_{n\to\infty}\frac{1}{(2n+1)(2n+2)} = 0 < 1.
\end{aligned}$$

Therefore, the series is convergent for all real numbers x. The interval of convergence is $(-\infty,\infty)$ and the radius of convegrence is $R=\infty$.

2. - *Since $f(x)=e^x$, we know that*

$$f(x) = f'(x) = f''(x) = \cdots = f^{(n)}(x) = e^x, \forall n \in \mathbb{N}.$$

Therefore,

$$f(0) = f'(0) = f''(0) = \cdots = f^{(n)}(0) = 1, \forall n \in \mathbb{N}.$$

Therefore, we have,

$$p_0(x) = f(0) = 1,$$

$$
\begin{aligned}
p_1(x) &= f(0) + f'(0)x = 1 + x, \\
p_2(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + x + \frac{1}{2}x^2, \\
p_3(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3, \\
&\vdots \\
p_n(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \\
&= 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{x^n}{n!} \\
&= \sum_{k=0}^{n} \frac{x^k}{k!}.
\end{aligned}
$$

Which gives,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

- *To find the radius of convergence, we apply the ratio test. We find*

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \lim_{n\to\infty} \left|\frac{n!}{(n+1)!}.\frac{x^{n+1}}{x^n}\right| = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

Therefore, the series is convergent for all real numbers x. The interval of convergence is $(-\infty, \infty)$ and the radius of convegrence is $R = \infty$.

3. - *For $f(x) = \frac{1}{x}$, the values of the function and its first four derivatives at $x = 5$ are given as follows,*

$$
\begin{aligned}
f(x) &= \frac{1}{x} &&\Rightarrow f(5) = \frac{1}{5}, \\
f'(x) &= -\frac{1}{x^2} &&\Rightarrow f'(5) = -\frac{1}{5^2}, \\
f''(x) &= \frac{2}{x^3} &&\Rightarrow f''(5) = \frac{2}{5^3}, \\
f'''(x) &= -\frac{6}{x^4} &&\Rightarrow f'''(5) = -\frac{6}{5^4}, \\
f^{(4)}(x) &= \frac{24}{x^5} &&\Rightarrow f^{(4)}(5) = \frac{24}{5^5}.
\end{aligned}
$$

Therefore, we have,

$$
\begin{aligned}
p_0(x) &= f(5) = \frac{1}{5}, \\
p_1(x) &= f(5) + f'(5)(x-5) = \frac{1}{5} - \frac{1}{5^2}(x-5), \\
p_2(x) &= f(5) + f'(5)(x-5) + \frac{f''(5)}{2!}(x-5)^2 = \frac{1}{5} - \frac{1}{5^2}(x-5) + \frac{1}{5^3}(x-5)^2, \\
p_3(x) &= f(5) + f'(5)(x-5) + \frac{f''(5)}{2!}(x-5)^2 + \frac{f'''(5)}{3!}(x-5)^3 \\
&= \frac{1}{5} - \frac{1}{5^2}(x-5) + \frac{1}{5^3}(x-5)^2 - \frac{1}{5^4}(x-5)^3, \\
p_4(x) &= f(5) + f'(5)(x-5) + \frac{f''(5)}{2!}(x-5)^2 + \frac{f'''(5)}{3!}(x-5)^3 + \frac{f^{(4)}(5)}{4!}(x-5)^4 \\
&= \frac{1}{5} - \frac{1}{5^2}(x-5) + \frac{1}{5^3}(x-5)^2 - \frac{1}{5^4}(x-5)^3 + \frac{1}{5^5}(x-5)^4, \\
\vdots \qquad & \qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
p_n(x) &= f(5) + f'(5)(x-5) + \frac{f''(5)}{2!}(x-5)^2 + \frac{f'''(5)}{3!}(x-5)^3 + \cdots + \frac{f^{(n)}(5)}{n!}(x-5)^n \\
&= \frac{1}{5} - \frac{1}{5^2}(x-5) + \frac{1}{5^3}(x-5)^2 - \frac{1}{5^4}(x-5)^3 + \cdots + (-1)^n\frac{1}{5^{n+1}}(x-5)^n \\
&= \sum_{k=0}^{n}(-1)^k\frac{(x-5)^k}{5^{k+1}}.
\end{aligned}
$$

Which gives,

$$\frac{1}{x} = \sum_{n=0}^{\infty}(-1)^n\frac{(x-5)^n}{5^{n+1}}.$$

- *To find the radius of convergence, we apply the ratio test. We find*

$$
\begin{aligned}
\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{(x-5)^{n+1}}{5^{n+2}}}{(-1)^n\frac{(x-5)^n}{5^{n+1}}}\right| = \lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{(x-5)^n}.\frac{5^{n+1}}{5^{n+2}}\right| \\
&= |x-5|\lim_{n\to\infty}\frac{1}{5} = \frac{|x-5|}{5}.
\end{aligned}
$$

This series is convergent if $|x-5| < 5$. Therefore, the radius of convergence is $R = 5$.

4. - *For $f(x) = \ln(x)$, the values of the function and its first four derivatives at $x = 2$ are* *given as follows,*

$$
\begin{aligned}
f'(x) &= \frac{1}{x} && \Rightarrow f'(2) = \frac{1}{2}, \\
f''(x) &= -\frac{1}{x^2} && \Rightarrow f''(2) = -\frac{1}{4}, \\
f'''(x) &= \frac{2}{x^3} && \Rightarrow f'''(2) = \frac{1}{4}, \\
f^{(4)}(x) &= -\frac{6}{x^4} && \Rightarrow f^{(4)}(2) = -\frac{3}{8}.
\end{aligned}
$$

Therefore, we have,

$$
\begin{aligned}
p_1(x) &= f'(2)(x-2) = \frac{1}{2}(x-2), \\
p_2(x) &= f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 = \frac{1}{2}(x-2) - \frac{1}{2^2.2}(x-2)^2, \\
p_3(x) &= f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 \\
&= \frac{1}{2}(x-2) - \frac{1}{2^2.2}(x-2)^2 + \frac{1}{2^3.3}(x-2)^3, \\
p_4(x) &= f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \frac{f^{(4)}(2)}{4!}(x-2)^4 \\
&= \frac{1}{2}(x-2) - \frac{1}{2^2.2}(x-2)^2 + \frac{1}{2^3.3}(x-2)^3 - \frac{1}{2^4.4}(x-2)^4, \\
\vdots & \qquad \vdots \qquad \vdots \\
p_n(x) &= f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n \\
&= \frac{1}{2}(x-2) - \frac{1}{2^2.2}(x-2)^2 + \frac{1}{2^3.3}(x-2)^3 - \frac{1}{2^4.4}(x-2)^4 + \cdots + \\
&\quad (-1)^{n+1}\frac{1}{2^n.n}(x-2)^n \\
&= \sum_{k=1}^{n}(-1)^{k+1}\frac{1}{2^k.k}(x-2)^k.
\end{aligned}
$$

Which gives,

$$
\ln(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2^n.n}(x-2)^n
$$

- *To find the radius of convergence, we apply the ratio test. We find*

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}\frac{1}{2^{n+1}(n+1)}(x-2)^{n+1}}{(-1)^{n+1}\frac{1}{2^n n}(x-2)^n}\right| = |x-2|\lim_{n\to\infty}\frac{n}{2(n+1)} = \frac{|x-2|}{2}.$$

This series is convergent if $|x-2|<2$. *Therefore, the radius of convergence is* $R=2$.

5. - *For* $f(x)=\frac{1}{x^2}$, *the values of the function and its first four derivatives at* $x=1$ *are given as follows,*

$$\begin{aligned} f(x) &= \frac{1}{x^2} \quad \Rightarrow f(1)=1,\\ f'(x) &= -\frac{2}{x^3} \quad \Rightarrow f'(1)=-2,\\ f''(x) &= \frac{6}{x^4} \quad \Rightarrow f''(1)=6,\\ f'''(x) &= -\frac{24}{x^5} \quad \Rightarrow f'''(1)=-24,\\ f^{(4)}(x) &= \frac{120}{x^6} \quad \Rightarrow f^{(4)}(1)=120. \end{aligned}$$

Therefore, we have,

$$\begin{aligned} p_0(x) &= f(1)=1,\\ p_1(x) &= f(1)+f'(1)(x-1)=1-2(x-1),\\ p_2(x) &= f(1)+f'(1)(x-1)+\frac{f''(1)}{2!}(x-1)^2 = 1-2(x-5)+3(x-5)^2,\\ p_3(x) &= f(1)+f'(1)(x-1)+\frac{f''(1)}{2!}(x-1)^2+\frac{f'''(1)}{3!}(x-1)^3\\ &= 1-2(x-1)+3(x-1)^2-4(x-1)^3,\\ p_4(x) &= f(1)+f'(1)(x-1)+\frac{f''(1)}{2!}(x-1)^2+\frac{f'''(1)}{3!}(x-1)^3+\frac{f^{(4)}(1)}{4!}(x-1)^4\\ &= 1-2(x-1)+3(x-1)^2-4(x-1)^3+5(x-1)^4,\\ &\vdots\\ p_n(x) &= f(1)+f'(1)(x-1)+\frac{f''(1)}{2!}(x-1)^2+\frac{f'''(1)}{3!}(x-1)^3+\cdots+\frac{f^{(n)}(1)}{n!}(x-1)^n\\ &= 1-2(x-1)+3(x-1)^2-4(x-1)^3+5(x-1)^4+\cdots+(-1)^n(n+1)(x-1)^n \end{aligned}$$

$$= \sum_{k=0}^{n}(-1)^k(k+1)(x-1)^k.$$

Which gives,

$$\frac{1}{x^2} = \sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n.$$

- *To find the radius of convergence, we apply the ratio test. We find*

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+2)(x-1)^{n+1}}{(-1)^n(n+1)(x-1)^n}\right| = |x-1|\lim_{n\to\infty}\frac{n+2}{n+1} = |x-1|.$$

This series is convergent if $|x-1| < 1$. *Therefore, the radius of convergence is* $R = 1$.

6. - *For* $f(x) = \frac{1}{\sqrt{1-x}}$, *the values of the function and its first four derivatives at* $x = 0$ *are given as follows,*

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{1-x}} &&\Rightarrow f(0) = 1,\\
f'(x) &= \frac{1}{2(1-x)^{\frac{3}{2}}} &&\Rightarrow f'(0) = \frac{1}{2},\\
f''(x) &= \frac{3}{4(1-x)^{\frac{5}{2}}} &&\Rightarrow f''(0) = \frac{3}{4},\\
f'''(x) &= \frac{15}{8(1-x)^{\frac{7}{2}}} &&\Rightarrow f'''(0) = \frac{15}{8},\\
f^{(4)}(x) &= \frac{105}{16(1-x)^{\frac{9}{2}}} &&\Rightarrow f^{(4)}(0) = \frac{105}{16}.
\end{aligned}$$

Therefore, we have,

$$\begin{aligned}
p_0(x) &= f(0) = 1,\\
p_1(x) &= f(0) + f'(0)x = 1 + \frac{1}{2}x,\\
p_2(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + \frac{1}{2}x + \frac{3}{8}x^2,\\
p_3(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3,\\
p_4(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4
\end{aligned}$$

$$\begin{aligned}
&= 1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\frac{35}{128}x^4,\\
\vdots & \qquad\qquad \vdots \qquad\qquad \vdots\\
p_n(x) &= f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots+\frac{f^{(n)}(0)}{n!}x^n\\
&= 1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\frac{35}{128}x^4+\cdots+\frac{(2n)!}{(2^n n!)^2}x^n\\
&= \sum_{k=0}^{n}\frac{(2k)!}{(2^k k!)^2}x^k.
\end{aligned}$$

Which gives,

$$\frac{1}{\sqrt{1-x}}=\sum_{n=0}^{\infty}\frac{(2n)!}{(2^n n!)^2}x^n$$

- *To find the radius of convergence, we apply the ratio test. We find*

$$\rho=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{(2n+2)!}{(2^{n+1}(n+1)!)^2}x^{n+1}}{\frac{(2n)!}{(2^n n!)^2}x^n}\right|=|x|\lim_{n\to\infty}\frac{2n+1}{2(n+1)}=|x|.$$

This series is convergent if $|x|<1$. Therefore, the radius of convergence is $R=1$.

Solution of exercise 9.

1. $(x^2-1)y''+6xy'+4y=-4.$

Suppose that there exists a power series solution ,

$$y(x)=\sum_{n=0}^{\infty}c_nx^n=c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+C_5x^5+\cdots.$$

Differentiating this series term by term, we obtain,

$$\begin{aligned}
y'(x) &= \sum_{n=1}^{\infty}nc_nx^{n-1}=c_1+2c_2x+3c_3x^2+4c_4x^3+5c_5x^4+\cdots.\\
y''(x) &= \sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}=2c_2+6c_3x+12c_4x^2+20c_5x^3+\cdots.
\end{aligned}$$

If y satisfies the differential equation, then,

$$\begin{aligned}
x^2y''-y''+6xy'+4y &= -4\\
x^2\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}-\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}+6x\sum_{n=1}^{\infty}nc_nx^{n-1}+4\sum_{n=0}^{\infty}c_nx^n &= -4\\
\sum_{n=2}^{\infty}n(n-1)c_nx^{n}-\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}+6\sum_{n=1}^{\infty}nc_nx^{n}+4\sum_{n=0}^{\infty}c_nx^n &= -4\\
\sum_{n=2}^{\infty}\left(n(n-1)+6n+4\right)c_nx^{n}-\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}+(6c_1+4c_1)x+4c_0 &= -4\\
\sum_{n=2}^{\infty}\left(n^2+5n+4\right)c_nx^{n}-\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}+10c_1x+4c_0 &= -4\\
\sum_{n=2}^{\infty}\left(n^2+5n+4\right)c_nx^{n}-\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^{n}+10c_1x+4c_0 &= -4\\
\sum_{n=2}^{\infty}\left(n^2+5n+4\right)c_nx^{n}-\sum_{n=2}^{\infty}(n+2)(n+1)c_{n+2}x^{n}+(10c_1-6c_3)x+4c_0-2c_2 &= -4.
\end{aligned}$$

Using the uniqueness of power series representations, we know that these series can only be equal if their coefficients are equal. Therefore,

$$\begin{aligned}
4c_0-2c_2 &= -4,\\
\left(n^2+5n+4\right)c_n &= (n+2)(n+1)c_{n+2}, \forall n\geq 1.
\end{aligned}$$

which gives,

$$\begin{aligned}
c_2 &= 2c_0+2,\\
c_{n+2} &= \frac{n^2+5n+4}{(n+2)(n+1)}c_n, \forall n\geq 1.
\end{aligned}$$

That is,

$$c_2=2c_0+2, c_3=\frac{5}{3}c_1, n=1, c_4=\frac{3}{2}c_2=3c_0+3, n=2, \cdots .$$

Therefore,

$$y(x)=c_0+c_1x+(2c_0+2)x^2+\frac{5}{3}c_1x^3+(3c_0+3)x^4+\cdots .$$

2. $y'' - 2xy' + y = 0, y(0) = 0, y'(0) = 1.$

Suppose that there exists a power series solution ,

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots .$$

Differentiating this series term by term, we obtain,

$$\begin{aligned} y'(x) &= \sum_{n=1}^{\infty} n c_n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots . \\ y''(x) &= \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = 2c_2 + 6c_3 x + 12c_4 x^2 + \cdots . \end{aligned}$$

If y satisfies the differential equation, then,

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - 2x \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n &= 0 \\ \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - 2 \sum_{n=1}^{\infty} n c_n x^{n} + \sum_{n=0}^{\infty} c_n x^n &= 0 \\ \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} (-2n+1) c_n x^{n} + c_0 &= 0, \end{aligned}$$

which gives,

$$(2c_2 + 6c_3 x + 12c_4 x^2 + \cdots) + c_0 = (c_1 x + 3c_2 x^2 + 5c_3 x^3 + 7c_4 x^4 + 9c_5 x^5 + \cdots).$$

Using the uniqueness of power series representations, we know that these series can only be equal if their coefficients are equal. Therefore,

$$\begin{aligned} 2c_2 + c_0 &= 0, \\ 6c_3 &= c_1 \\ 12c_4 &= 3c_2 \\ 20c_5 &= 5c_3 \\ \vdots &= \vdots \end{aligned}$$

Using the initial conditions , we obtain,

$$\begin{aligned} c_0 &= 0, \\ c_1 &= 1. \end{aligned}$$

Then,

$$\begin{aligned} c_2 &= 0, \\ c_3 &= \frac{1}{6}, \\ c_4 &= 0, \\ c_5 &= \frac{1}{24}. \end{aligned}$$

Therefore,

$$y(x) = x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \cdots .$$

3. $(x-3)y' + 2y = 0.$

Suppose that there exists a power series solution ,

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots .$$

Differentiating this series term by term, we obtain,

$$y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots .$$

If y satisfies the differential equation, then,

$$\begin{aligned} (x-3)\sum_{n=1}^{\infty} n c_n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n &= 0 \\ \sum_{n=1}^{\infty} n c_n x^n - 3\sum_{n=1}^{\infty} n c_n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n &= 0 \\ \sum_{n=1}^{\infty}(n+2) c_n x^n - 3\sum_{n=1}^{\infty} n c_n x^{n-1} + 2c_0 &= 0, \end{aligned}$$

which gives,

$$(3c_1x + 4c_2x^2 + 5c_3x^3 + 6c_4x^4 + \cdots) + 2c_0 = 3c_1 + 6c_2x + 9c_3x^2 + 12c_4x^3 + \cdots .$$

Using the uniqueness of power series representations, we know that these series can only be equal if their coefficients are equal. Therefore,

$$\begin{aligned} 2c_0 &= 3c_1, \\ 3c_1 &= 6c_2, \\ 4c_2 &= 9c_3, \\ 5c_3 &= 12c_4, \\ \vdots &= \vdots \end{aligned}$$

Then,

$$\begin{aligned} c_1 &= \frac{2}{3}c_0, \\ c_2 &= \frac{1}{2}c_1 = \frac{1}{3}c_0, \\ c_3 &= \frac{4}{9}c_2 = \frac{4}{27}c_0, \\ c_4 &= \frac{5}{12}c_3 = \frac{5}{81}c_0 \\ \vdots &= \vdots \end{aligned}$$

Therefore,

$$y(x) = c_0 + \frac{2}{3}c_0x + \frac{1}{3}c_0x^2 + \frac{4}{27}c_0x^3 + \frac{5}{81}c_0x^4 + \cdots = c_0\left(1 + \frac{2}{3}x + \frac{1}{3}x^2 + \frac{4}{27}x^3 + \frac{5}{81}x^4 + \cdots\right).$$

Solution of exercise 10.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$. *Since* $0 \leq \frac{1}{(n+1)^2} \leq \frac{1}{n^2}$ *and* $\frac{1}{n^2} \to 0$, *by Leibniz's test the series converges.*

2. $\displaystyle\sum_{n=1}^{\infty} \frac{4^{n+1}((n+1)!)^2}{(2n-1)!}$. *the ratio test don't say anything about the nature of the series because* $\rho = 1$. *Using comparison test with* $\frac{1}{n}$, *we find the series is divergent.*

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}}$. *We compare our series with* $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{2x-1}} dx$. *Since,*

$$\int_1^{\infty} \frac{1}{\sqrt{2x-1}} dx = \lim_{b \to \infty} \int_1^{b} \frac{1}{\sqrt{2x-1}} dx = \lim_{b \to \infty} \sqrt{2x-1}\,\Big|_1^b = \infty.$$

As the integral diverges, the series diverges too.

4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$. *We compare* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$ *to* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. *As* $\displaystyle\lim_{n \to \infty} \frac{\frac{1}{\sqrt{n}+1}}{\frac{1}{\sqrt{n}}} = 1$, *by the limit comparison test,* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ *diverges, then,* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$ *diverges.*

Chapitre 3

Fourier series.

Exercise 1.

1. *Find the Fourier series for the function* $f(x) = x + x^2$ *in the interval* $-\pi < x < \pi$*. Hence show that*

   - $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$.
   - $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6} + \frac{\pi}{4}$.

2. *If* $f(x) = \left(\frac{\pi - x}{2}\right)^2$ *in the interval* $0 < x < 2\pi$*, show that* $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$*. Hence obtain the following relations:*

   - $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$.
   - $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$.
   - $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$.

3. *Given* $f(x) = \begin{cases} -x + 1 & for \quad -\pi \le x \le 0 \\ x + 1 & for \quad 0 \le x \le \pi \end{cases}$ *.Is the function even or odd? Find the Fourier series for* $f(x)$ *and deduce the value of*

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

4. *Expand the function* $f(x) = x\sin(x)$ *as a Fourier series in the interval* $-\pi \le x \le \pi$*.. Deduce that*

$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \cdots = \frac{\pi - 2}{4}.$$

Exercise 2.

1. *Find the Fourier series expansion for* $f(x)$*, if* $f(x) = \begin{cases} -\pi, & -\pi < x < 0, \\ x, & 0 < x < \pi. \end{cases}$

   Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

2. *Obtain Fourier series of the function* $f(x) = \begin{cases} x, & -\pi < x < 0, \\ -x, & 0 < x < \pi. \end{cases}$

   Show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

3. *Find the Fourier series for the function*

$$f(x) = \begin{cases} -1, & -\pi < x < -\frac{\pi}{2}, \\ 0, & -\frac{\pi}{2} < x < \frac{\pi}{2}, \\ 1, & \frac{\pi}{2} < x < \pi. \end{cases}$$

Exercise 3.

1. *Find a Fourier series for* $f(t) = 1 - t^2$ *when* $-1 \leq t \leq 1$.

2. *Develop* $f(x)$ *in a Fourier series in the interval* $(-2, 2)$ *when* $f(x) = \begin{cases} 0, & -2 < x < 0, \\ 1, & 0 < x < 2. \end{cases}$

3. *Expand* $f(x) = e^{-x}$ *as a Fourier series in the interval* $(-l, l)$.

4. *Expand* $f(t) = \begin{cases} t, & 0 < t < 1, \\ 1 - t, & 1 < x < 2. \end{cases}$

Exercise 4.

1. *Find the half range sine series for* $f(x) = \frac{l}{2} - x$, $0 < x < l$.

2. *Find the half range cosine series for* $f(x) = x\sin(x)$, $0 < x < \pi$.

3. *Obtain a half range cosine series for* $f(x) = \begin{cases} kx, & 0 \leq x \leq \frac{l}{2}, \\ k(l - x), & \frac{l}{2} \leq x \leq l. \end{cases}$
   Deduce the sum of the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

4. *Let* $f(x) = \begin{cases} wx, & 0 \leq x \leq \frac{l}{2}, \\ w(l - x), & \frac{l}{2} \leq x \leq l. \end{cases}$ *. Show that* $f(x) = \frac{4\omega l}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin\left(\frac{(2n+1)\pi x}{l}\right)$.
   Hence, obtain the sum of the series $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$.

Exercise 5.

1. *Prove that in the range* $(0, l)$, $x = \frac{l}{2} - \frac{4l}{\pi^2} \sum_{1}^{\infty} \frac{1}{(2n-1)^2} \cos\left(\frac{(2n-1)\pi}{l} x\right)$ *and deduce that* $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96}$.

2. *Show that for* $0 < x < \pi$, $x(\pi - x) = \frac{\pi^2}{6} - \left(\frac{\cos(2x)}{1^2} + \frac{\cos(4x)}{2^2} + \frac{\cos(6x)}{3^2} + \cdots\right)$ *and hence evaluate* $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

3. *Find the Fourier sine series for unity in* $0 < x < \pi$ *and hence show that* $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$.

4. *Find Fourier series of* x^2 *in* $(-\pi, \pi)$. *Use Parseval's identity to prove that* $\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$

Exercise 6. *(Exam)*

Consider the $\pi-$*periodic function defined by* $f(x) = |\cos(x)|$.

1. *Represent the graph of* f *on* $[-3\pi, 3\pi]$.
2. *Apply the Dirichlet Theorem to* f.
3. *Find the general form of Fourier series of* f.
4. *Deduce that for* $x \in \mathbb{R}$, $|\cos(x)| = \frac{4-\pi}{\pi} + \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}}{(4n^2 - 1)} \cos^2(nx)$.
5. *Deduce the Fourier series of* $g(x) = |\sin(x)|$.

Exercise 7. *(Exam)*

Consider the $\pi-$*periodic function defined by* $f(x) = |\sin(x)|$.

1. *Represent the graph of* f *on* $[-3\pi, 3\pi]$.
2. *Apply the Dirichlet Theorem to* f.
3. *Find the general form of Fourier series of* f.
4. *Deduce that for* $x \in \mathbb{R}$, $|\sin(x)| = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin^2(nx)}{4n^2 - 1}$.
5. *Deduce the Fourier series of* $g(x) = |\cos(x)|$.

Exercice 8. *(Exam)*

Let f be a function defined by $f(t) = \max(\cos(t), 0), t \in \mathbb{R}$.

1. *Sketch the function of f.*
2. *Apply Dirichlet theorem to the function f*
3. *Determine a_0, a_n, b_n the coefficients of Fourier series of f.*
4. *Give the form general of Fourier series of f.*
5. *Deduce the sum of*
   - $S = \displaystyle\sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{4p^2 - 1}$,
   - $T = \displaystyle\sum_{p=1}^{\infty} \frac{1}{4p^2 - 1}$.
6. *Calculate the sum of* $V = \displaystyle\sum_{p=1}^{\infty} \frac{1}{(4p^2 - 1)^2}$.
7. *Deduce the Fourier series of g where $g(t) = \max(\sin(t), 0), t \in \mathbb{R}$.*

Exercice 9. *(Exam)*

Let f be a function defined by $f(t) = \max(\sin(t), 0), t \in \mathbb{R}$.

1. *Sketch the function of f.*
2. *Apply Dirichlet theorem to the function f*
3. *Determine a_0, a_n, b_n the coefficients of Fourier series of f.*
4. *Give the form general of Fourier series of f.*
5. *Deduce the sum of*

- $S = \displaystyle\sum_{p=1}^{\infty} \frac{1}{4p^2 - 1}$,
- $T = \displaystyle\sum_{p=1}^{\infty} \frac{(-1)^p}{4p^2 - 1}$.

6. *Deduce the Fourier series of* g *where* $g(t) = \max(\cos(t), 0), t \in \mathbb{R}$.

Solution of exercise 1.

Reminder.

- *The Fourier series for a function* $f(x)$ *in the interval* $c < x < c + 2\pi$ *is given by*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

where,

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)dx, \quad a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos(nx)dx, \quad b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin(nx)dx.$$

- *The Fourier series for an even function* $f(x)$ *in the interval* $-\pi < x < \pi$ *is given by*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

where,

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} f(x)dx. \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx. \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = 0. \end{aligned}$$

- *The Fourier series for an odd function* $f(x)$ *in the interval* $-\pi < x < \pi$ *is given by*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

where,

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = 0. \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = 0. \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx. \end{aligned}$$

Solution.

1. • *The Fourier series of $f(x)$.*

• a_0 :

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)dx = \frac{1}{\pi}\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\frac{2}{3}\pi^3 = \frac{2}{3}\pi^2.$$

• a_n :

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos(nx)dx = \frac{1}{\pi}\left[(x+x^2)\frac{\sin(nx)}{n}\right]_{-\pi}^{\pi} - \frac{1}{\pi n}\int_{-\pi}^{\pi}(1+2x)\sin(nx)dx \\ &= -\frac{1}{\pi n}\left[-\frac{\cos(nx)}{n}\right]_{-\pi}^{\pi} - \frac{2}{\pi n}\left[-x\frac{\cos(nx)}{n}\right]_{-\pi}^{\pi} - \frac{2}{\pi n^2}\int_{-\pi}^{\pi}\cos(nx)dx \\ &= \frac{4}{n^2}\cos(n\pi) - \frac{2}{\pi n^2}\left[\frac{\sin(nx)}{n}\right]_{-\pi}^{\pi} = \frac{4}{n^2}(-1)^n. \end{aligned}$$

• b_n :

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin(nx)dx = \frac{1}{\pi}\left[-(x+x^2)\frac{\cos(nx)}{n}\right]_{-\pi}^{\pi} + \frac{1}{\pi n}\int_{-\pi}^{\pi}(1+2x)\cos(nx)dx \\ &= -\frac{2}{n}\cos(n\pi) + \frac{1}{\pi n}\left[\frac{\sin(nx)}{n}\right]_{-\pi}^{\pi} + \frac{2}{\pi n}\left[x\frac{\sin(nx)}{n}\right]_{-\pi}^{\pi} - \frac{2}{\pi n}\int_{-\pi}^{\pi}\sin(nx)dx \\ &= -\frac{2}{n}(-1)^n - \frac{2}{\pi n}\left[-\frac{\cos(nx)}{n}\right]_{-\pi}^{\pi} = -\frac{2}{n}(-1)^n. \end{aligned}$$

Then,

$$f(x) = (x+x^2) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx) - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(nx).$$

• *Putting $x=0$, we get,*

$$0 = \frac{\pi^2}{3} - 4\left(\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots\right) \Rightarrow \frac{\pi^2}{12} = \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$$

• *Putting $x=\pi$, we get,*

$$\pi+\pi^2 = \frac{\pi}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2} \Rightarrow \pi+\frac{2\pi}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \frac{\pi}{4}+\frac{\pi^2}{6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$$

2. • *The Fourier series of $f(x)$.*

• a_0 :

$$a_0 = \frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2dx = \frac{1}{4\pi}\left[-\frac{(\pi-x)^3}{3}\right]_0^{2\pi} = \frac{1}{4\pi}\left[\frac{\pi^3}{3}+\frac{\pi^3}{3}\right] = \frac{\pi^2}{6}.$$

• a_n :

$$\begin{aligned} a_n &= \frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2\cos(nx)dx = \frac{1}{4\pi}\left[(\pi-x)^2\frac{\sin(nx)}{n}\right]_0^{2\pi} + \frac{1}{2\pi n}\int_0^{2\pi}(\pi-x)\sin(nx)dx \\ &= \frac{1}{2\pi n}\left[-(\pi-x)\frac{\cos(nx)}{n}\right]_0^{2\pi} - \frac{1}{2\pi n^2}\int_0^{2\pi}\cos(nx)dx \\ &= \frac{1}{2\pi n^2}\left[\pi\cos(2n\pi)+\pi\right] - \frac{1}{2\pi n^2}\left[\frac{\sin(nx)}{n}\right]_0^{2\pi} \\ &= \frac{1}{n^2}. \end{aligned}$$

• b_n :

$$\begin{aligned} b_n &= \frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2\sin(nx)dx = \frac{1}{4\pi}\left[-(\pi-x)^2\frac{\cos(nx)}{n}\right]_0^{2\pi} - \frac{1}{2\pi n}\int_0^{2\pi}(\pi-x)\cos(nx)dx \\ &= \frac{1}{4\pi}.0 - \frac{1}{2\pi n}\left[(\pi-x)\frac{\sin(nx)}{n}\right]_0^{2\pi} - \frac{1}{2\pi n^2}\int_0^{2\pi}\sin(nx)dx \\ &= -\frac{1}{2\pi n^2}\left[-\frac{\cos(nx)}{n}\right]_0^{2\pi} = -\frac{1}{2\pi n^3}\left[-\frac{1}{n}+\frac{1}{n}\right] = 0. \end{aligned}$$

Then,

$$f(x) = \left(\frac{\pi-x}{2}\right)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{\cos(nx)}{n^2}.$$

• *Putting $x=0$, we get*

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \frac{2\pi^2}{12} = \sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \frac{\pi^2}{6} = 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$$

• *Putting $x=\pi$, we get*

$$0 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} \Rightarrow \frac{-\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} \Rightarrow \frac{\pi^2}{12} = 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$$

- *we do the addition of the first and the second sum to get*

$$\frac{\pi^2}{6}+\frac{\pi^2}{12}=2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)\Rightarrow\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$$

3. • $f(-x)=f(x)$, *for* $-\pi\leq x\leq\pi$. *Then,* $f(x)$ *is even. and its Fourier series is*

$$f(x)=x\sin(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(nx).$$

- a_0 *:*

$$a_0=\frac{2}{\pi}\int_0^{\pi}(x+1)dx=\frac{2}{\pi}\left[\frac{(x+1)^2}{2}\right]_0^{\pi}=\frac{2}{\pi}\left[\frac{(\pi+1)^2}{2}-\frac{1}{2}\right]=\frac{2}{\pi}\left[\frac{\pi^2}{2}+\pi\right]=\pi+2.$$

- a_n *:*

$$\begin{aligned}a_n &= \frac{2}{\pi}\int_0^{\pi}(x+1)\cos(nx)dx=\frac{2}{\pi}\left[(x+1)\frac{\sin(nx)}{n}\right]_0^{\pi}-\frac{2}{\pi n}\int_0^{\pi}\sin(nx)dx\\ &= -\frac{2}{\pi n}\left[-\frac{\cos(nx)}{n}\right]_0^{\pi}=\frac{2}{\pi n^2}\left[\cos(n\pi)-1\right]=\begin{cases}0, & \text{if } n \text{ is even},\\ -\frac{4}{\pi n^2}, & \text{if } n \text{ is odd}.\end{cases}\end{aligned}$$

Then,

$$f(x)=\frac{\pi}{2}+1-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos((2n-1)x)}{(2n-1)^2}.$$

- *Putting* $x=0$*, we get*

$$\begin{aligned}1 &= \frac{\pi}{2}+1-\frac{4}{\pi}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)\\ \Rightarrow\frac{\pi^2}{8} &= \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots.\end{aligned}$$

4. • *The Fourier series of* $f(x)=x\sin(x)$.

Since $x\sin(x)$ *is an even function* $(-x\sin(-x)=-x.-\sin(x)=x\sin(x))$, $b_n=0$ *and*

$$f(x)=x\sin(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(nx).$$

- a_0:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\sin(x)dx = \frac{2}{\pi}\left[-x\cos(x)\right]_0^{\pi} + \frac{\pi}{2}\int_0^{\pi}\cos(x)dx = 2 + \frac{\pi}{2}\left[\sin(x)\right]_0^{\pi} = 2.$$

- a_n:

$$\begin{aligned} a_1 &= \frac{2}{\pi}\int_0^{\pi} x\sin(x)\cos(x)dx = \frac{1}{\pi}\int_0^{\pi} x\sin(2x)dx = \frac{1}{\pi}\left[-x\frac{\cos(2x)}{2}\right]_0^{\pi} + \frac{1}{2\pi}\int_0^{\pi}\cos(2x)dx \\ &= -\frac{1}{2} + \frac{1}{2\pi}\left[\frac{\sin(2x)}{2}\right]_0^{\pi} = -\frac{1}{2}. \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} x\sin(x)\cos(nx)dx = \frac{1}{\pi}\int_0^{\pi} x(2\cos(nx)\sin(x))dx, \qquad n>1 \\ &= \frac{1}{\pi}\int_0 \pi x\left[\sin((n+1)x) - \sin((n-1)x)\right]dx \\ &= \frac{1}{\pi}\left[x\left(-\frac{\cos((n+1)x)}{n+1} + \frac{\cos((n-1)x)}{n-1}\right)\right]_0^{\pi} + \frac{1}{\pi}\int_0^{\pi}\left(\frac{\cos((n+1)x)}{n+1} - \frac{\cos((n-1)x)}{n-1}\right)dx \\ &= -\frac{\cos((n+1)\pi)}{n+1} + \frac{\cos((n-1)\pi)}{n-1} + \frac{1}{\pi}\left[\frac{\sin((n+1)x)}{(n+1)^2} - \frac{\sin((n-1)x)}{(n-1)^2}\right]_0^{\pi} \\ &= -\frac{\cos((n+1)\pi)}{n+1} + \frac{\cos((n-1)\pi)}{n-1}, \quad n\neq 1. \\ &= \begin{cases} \frac{1}{n-1} - \frac{1}{n+1}, & n \text{ is odd,} \\ \frac{-1}{n-1} + \frac{1}{n+1}, & n \text{ is even,} \end{cases} \\ &= \begin{cases} \frac{2}{n^2-1}, & n \text{ is odd,} \\ \frac{-2}{n^2-1}, & n \text{ is even.} \end{cases} \end{aligned}$$

Then,

$$f(x) = 1 - \frac{1}{2}\cos(x) - 2\left(\frac{\cos(2x)}{2^2-1} - \frac{\cos(3x)}{3^2-1} + \frac{\cos(4x)}{4^2-1} - \frac{\cos(5x)}{5^2-1} + \cdots\right).$$

- *Putting* $x = \frac{\pi}{2}$, *we get*

$$\frac{\pi}{2} = 1 - 2\left(-\frac{1}{2^2-1} + \frac{1}{4^2-1} - \frac{1}{6^2-1} + \cdots\right)$$

$$\begin{aligned}
\Rightarrow \frac{\pi-2}{4} &= 2\left(\frac{1}{2^2-1}-\frac{1}{4^2-1}+\frac{1}{6^2-1}+\cdots\right)\\
\Rightarrow \frac{\pi-2}{4} &= \frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}+\cdots
\end{aligned}$$

Solution of exercise 2.

Reminder.

let $f(x)$ be defined by $f(x)=\begin{cases} f_1(x), & c<x<x_0,\\ f_2(x), & x_0<x<c+2\pi,\end{cases}$ *where x_0 is the point of finite discontinuity in the interval $(c,c+2\pi)$. The value of a_0, a_n, b_n are given by*

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)dx+\int_{x_0}^{c+2\pi} f_2(x)dx\right],\\
a_n &= \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\cos(nx)dx+\int_{x_0}^{c+2\pi} f_2(x)\cos(nx)dx\right],\\
b_n &= \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\sin(nx)dx+\int_{x_0}^{c+2\pi} f_2(x)\sin(nx)dx\right].
\end{aligned}$$

At $x=x_0$, there is a finite jump in the graph of the function. The limits $f(x_0-0)$ and $f(x_0+0)$ exist but are unequal. The sum of the Fourier series is equal to $\frac{1}{2}\left[f(x_0+0)+f(x_0-0)\right]$.

Solution.

1. • *The Fourier series of $f(x)$.*

• *a_0:*

$$a_0=\frac{1}{\pi}\int_{-\pi}^{0}-\pi dx+\frac{1}{\pi}\int_0^{\pi}xdx=-\left[x\right]_{-\pi}^{0}+\frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}=-\pi+\frac{\pi}{2}=-\frac{\pi}{2}.$$

• *a_n:*

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{0}-\pi\cos(nx)dx+\frac{1}{\pi}\int_0^{\pi}x\cos(nx)dx=-\left[\frac{\sin(nx)}{n}\right]_{-\pi}^{0}+\frac{1}{\pi}\left[x\frac{\sin(nx)}{n}\right]_0^{\pi}\\
&\quad -\frac{1}{\pi n}\int_0^{\pi}\sin(nx)dx
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{\pi n}\left[-\frac{\cos(nx)}{n}\right]_0^\pi = -\frac{1}{\pi n^2}\left[(-1)^{n+1}+1\right] \\
&= \begin{cases} 0, & \textit{if } n \textit{ is even} \\ -\frac{2}{\pi n^2}, & \textit{if } n \textit{ is odd} \end{cases}.
\end{aligned}$$

- b_n:

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{0} -\pi\sin(nx)dx + \frac{1}{\pi}\int_0^\pi x\sin(nx)dx = \left[\frac{\cos(nx)}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-x\frac{\cos(nx)}{n}\right]_0^\pi \\
&\quad + \frac{1}{\pi n}\int_0^\pi \cos(nx)dx \\
&= \frac{1}{n}[1-(-1)^n] + \frac{1}{n}(-1)^{n+1} + \frac{1}{\pi n}\left[\frac{\sin(nx)}{n}\right]_0^\pi = \frac{1}{n}[1-(-1)^n] + \frac{1}{n}(-1)^{n+1} \\
&= \frac{1}{n} + 2\frac{(-1)^{n+1}}{n} = \begin{cases} \frac{3}{n}, & \textit{if } n \textit{ is odd} \\ -\frac{1}{n}, & \textit{if } n \textit{ is even} \end{cases}.
\end{aligned}$$

Then,

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos((2n-1)x)}{(2n-1)^2} + \sum_{n=1}^{\infty}\left[3\frac{\sin((2n-1)x)}{(2n-1)} - \frac{\sin((2n)x)}{2n}\right].$$

- *Putting $x = 0$, we get*

$$\begin{aligned}
\frac{1}{2}\left[f(0-0)+f(0+0)\right] &= -\frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} \\
\Rightarrow -\frac{\pi}{2}\left(-\frac{\pi}{2}+\frac{\pi}{4}\right) &= \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} \\
\Rightarrow \frac{\pi^2}{8} &= \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}.
\end{aligned}$$

2. • *The Fourier series of $f(x)$.*

- a_0 :

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} xdx + \frac{1}{\pi}\int_0^\pi -xdx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} - \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^\pi = -\frac{\pi}{2} - \frac{\pi}{2} = -\pi.$$

- a_n :

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{0} x\cos(nx)dx - \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)dx \\
&= \frac{1}{\pi}\left[x\frac{\sin(nx)}{n}\right]_{-\pi}^{0} - \frac{1}{\pi}\left[x\frac{\sin(nx)}{n}\right]_{0}^{\pi} - \frac{1}{\pi n}\int_{-\pi}^{0}\sin(nx)dx + \frac{1}{\pi n}\int_{0}^{\pi}\sin(nx)dx \\
&= -\frac{1}{\pi n}\left[-\frac{\cos(nx)}{n}\right]_{-\pi}^{0} + \frac{1}{\pi n}\left[-\frac{\cos(nx)}{n}\right]_{0}^{\pi} \\
&= -\frac{1}{\pi n^2}\left[-1+(-1)^n\right] + \frac{1}{\pi n^2}\left[-(-1)^n+1\right] \\
&= -\frac{2}{\pi n^2}[-1+(-1)^n] \\
&= -\frac{2}{\pi n^2}\begin{cases} 0, & \textit{if } n \textit{ is even} \\ -2, & \textit{if } n \textit{ is odd} \end{cases}.
\end{aligned}$$

- b_n:

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{0} x\sin(nx)dx - \frac{1}{\pi}\int_{0}^{\pi} x\sin(nx)dx = \frac{1}{\pi}\left[-x\frac{\cos(nx)}{n}\right]_{-\pi}^{0} - \frac{1}{\pi}\left[-x\frac{\cos(nx)}{n}\right]_{0}^{\pi} \\
&\quad + \frac{1}{\pi n}\int_{-\pi}^{0}\cos(nx)dx - \frac{1}{\pi n}\int_{0}^{\pi}\cos(nx)dx \\
&= -\frac{(-1)^n}{n} + \frac{(-1)^n}{n} + \frac{1}{\pi n}\left[\frac{\sin(nx)}{n}\right]_{-\pi}^{0} - \frac{1}{\pi n}\left[\frac{\sin(nx)}{n}\right]_{0}^{\pi} \\
&= 0.
\end{aligned}$$

Note: *As $f(x)$ is even, we can calculate a_0 and b_n with the formulas in the reminder of exercise 1 with $b_n = 0$.*
Then,

$$f(x) = -\frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos((2n-1)x)}{(2n-1)^2}.$$

- *Putting $x = 0$, we get*

$$\frac{\pi}{2}[f(0-0)+f(0+0)] = -\frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$\begin{aligned}\Rightarrow \frac{\pi}{2}.\frac{\pi}{4} &= \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\\ \Rightarrow \frac{\pi^2}{8} &= \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}.\end{aligned}$$

3. • *The Fourier series of $f(x)$.*

• a_0:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{-\frac{\pi}{2}} -1dx + \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 0dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} 1dx = \frac{1}{\pi}\left[\frac{\pi}{2}-\pi\right] + 0 + \frac{1}{\pi}\left[\pi - \frac{\pi}{2}\right] = 0.$$

• a_n :

$$\begin{aligned}a_n &= \frac{1}{\pi}\int_{-\pi}^{-\frac{\pi}{2}} -\cos(nx)dx + \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 0.\cos(nx)dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}\cos(nx)dx\\ &= \frac{1}{\pi}\left[-\frac{\sin(nx)}{n}\right]_{-\pi}^{-\frac{\pi}{2}} + \frac{1}{\pi}\left[\frac{\sin(nx)}{n}\right]_{\frac{\pi}{2}}^{\pi} = \frac{\sin(n\pi/2)}{\pi n} - \frac{\sin(n\pi/2)}{\pi n} = 0.\end{aligned}$$

• b_n :

$$\begin{aligned}b_n &= \frac{1}{\pi}\int_{-\pi}^{-\frac{\pi}{2}} -\sin(nx)dx + \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 0.\sin(nx)dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}\sin(nx)dx\\ &= \frac{1}{\pi}\left[\frac{\cos(nx)}{n}\right]_{-\pi}^{-\frac{\pi}{2}} + \frac{1}{\pi}\left[-\frac{\cos(nx)}{n}\right]_{\frac{\pi}{2}}^{\pi}\\ &= \frac{1}{\pi n}\left[\cos(n\pi/2) - \cos(n\pi)\right] + \frac{1}{\pi n}\left[-\cos(n\pi) + \cos(n\pi/2)\right]\\ &= \begin{cases}\frac{2}{\pi n}, & \textit{if } n \textit{ is odd},\\ 0, & \textit{else if } n \textit{ is even and multiple of } 4,\\ -\frac{4}{\pi n}, & \textit{else}.\end{cases}\end{aligned}$$

Then,

$$f(x) = \frac{2}{\pi}\sin(x) - \frac{2}{\pi}\sin(2x) + \frac{2}{3\pi}\sin(3x) + \frac{2}{5\pi}\sin(5\pi) - \frac{2}{3\pi}\sin(6x) + \cdots.$$

Solution of exercise 3.

Reminder.

- *The Fourier series for $f(x)$ in the interval $c < x < 2l + c$*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}x\right),$$

where,

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)dx, a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi}{l}x\right)dx, b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi}{l}x\right)dx.$$

- *If we put $c = 0$, the interval becomes $0 < x < 2l$ and the above result reduce to*

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx, a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi}{l}x\right)dx, b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi}{l}x\right)dx.$$

- *If we put $c = -l$, the interval becomes $-l < x < l$ and the above result reduce to*

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx, a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi}{l}x\right)dx, b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi}{l}x\right)dx.$$

  - *If $f(x)$ is an* **even** *function, we have*

$$a_0 = \frac{2}{l}\int_0^{l} f(x)dx, \quad a_n = \frac{2}{l}\int_0^{l} f(x)\cos\left(\frac{n\pi}{l}x\right)dx, b_n = 0.$$

  - *If $f(x)$ is an* **odd** *function, we have*

$$a_0 = 0, \quad a_n = 0, \quad b_n = \frac{2}{l}\int_0^{l} f(x)\sin\left(\frac{n\pi}{l}x\right)dx.$$

Solution

1. *We note that $f(x)$ is even, then, $b_n = 0$, and, we have*

- a_0 *:*

$$a_0 = \int_{-1}^{1}(1-t^2)dt = 2\int_0^1(1-t^2)dt = 2\left[t - \frac{t^3}{3}\right]_0^1 = 2.\frac{2}{3} = \frac{4}{3}.$$

- a_n :

$$\begin{aligned}
a_n &= \int_{-1}^{1}(1-t^2)\cos(n\pi t)\,dt = 2\int_0^1 (1-t^2)\cos(n\pi t)dt\\
&= 2\left[(1-t^2)\frac{\sin(n\pi t)}{n\pi}\right]_0^1 + \frac{4}{n\pi}\int_0^1 t\sin(n\pi t)dt\\
&= \frac{4}{n\pi}\left[-t\frac{\cos(n\pi t)}{n\pi}\right]_0^1 + \frac{4}{n^2\pi^2}\int_0^1 \cos(n\pi t)dt\\
&= -\frac{4}{n^2\pi^2}\cos(n\pi) + \frac{4}{n^2\pi^2}\left[\frac{\sin(n\pi t)}{n\pi}\right]_0^1\\
&= \frac{4}{n^2\pi^2}(-1)^{n+1}.
\end{aligned}$$

Then,

$$f(t) = 1-t^2 = \frac{2}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos(n\pi t).$$

2.

- a_0

$$a_0 = \frac{1}{2}\int_{-2}^{0} 0dx + \frac{1}{2}\int_0^2 1dx = 1.$$

- a_n

$$a_n = \frac{1}{2}\int_{-2}^{0} 0\cos\left(\frac{n\pi}{2}x\right)dx + \frac{1}{2}\int_0^2 1\cos\left(\frac{n\pi}{2}x\right)dx = \frac{1}{2}\int_0^2 \cos\left(\frac{n\pi}{2}x\right)dx = \frac{1}{2}\left[\frac{\sin\left(\frac{n\pi}{2}x\right)}{\frac{n\pi}{2}}\right]_0^2 = 0.$$

- b_n

$$\begin{aligned}
b_n &= \frac{1}{2}\int_{-2}^{0} 0\sin\left(\frac{n\pi}{2}x\right)dx + \frac{1}{2}\int_0^2 1\sin\left(\frac{n\pi}{2}x\right)dx = \frac{1}{2}\int_0^2 \sin\left(\frac{n\pi}{2}x\right)dx = \frac{1}{2}\left[-\frac{\cos\left(\frac{n\pi}{2}x\right)}{\frac{n\pi}{2}}\right]_0^2\\
&= \frac{1}{n\pi}\left[-\cos(n\pi)+1\right] = \frac{1}{n\pi}\left[(-1)^{n+1}+1\right] = \begin{cases} 0, & \text{if } n \text{ is even,}\\ \frac{2}{n\pi}, & \text{if } n \text{ is odd.}\end{cases}
\end{aligned}$$

Then,

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{(2n-1)\pi}{2}x\right)}{(2n-1)}.$$

3.

- a_0:

$$a_0 = \frac{1}{l}\int_{-l}^{l} e^{-x}dx = \frac{1}{l}\left[-e^{-x}\right]_{-l}^{l} = \frac{1}{l}\left[-e^{-l}+e^{l}\right] = \frac{2\sinh(l)}{l}.$$

- a_n:

$$\begin{aligned} a_n &= \frac{1}{l}\int_{-l}^{l} e^{-x}\cos\left(\frac{n\pi}{l}x\right)dx = \frac{1}{l}\left[e^{-x}\frac{\sin\left(\frac{n\pi}{l}x\right)}{\frac{n\pi}{l}}\right]_{-l}^{l} + \frac{1}{n\pi}\int_{-l}^{l} e^{-x}\sin\left(\frac{n\pi}{l}x\right)dx \\ &= \frac{1}{n\pi}\left[-e^{-x}\frac{\cos\left(\frac{n\pi}{l}x\right)}{\frac{n\pi}{l}}\right]_{-l}^{l} - \frac{l}{n^2\pi^2}\int_{-l}^{l} e^{-x}\cos\left(\frac{n\pi}{l}x\right)dx. \end{aligned}$$

Then,

$$\begin{aligned} \Rightarrow \left(\frac{1}{l}+\frac{l}{n^2\pi^2}\right)\int_{-l}^{l} e^{-x}\cos\left(\frac{n\pi}{l}x\right)dx &= \frac{l}{n^2\pi^2}\left[-e^{-l}\cos(n\pi)+e^{l}\cos(n\pi)\right] \\ \Rightarrow \int_{-l}^{l} e^{-x}\cos\left(\frac{n\pi}{l}x\right)dx &= \frac{l^2}{n^2\pi^2+l^2}\cos(n\pi)(e^{l}-e^{-l}) \\ \Rightarrow \int_{-l}^{l} e^{-x}\cos\left(\frac{n\pi}{l}x\right)dx &= \frac{2l^2}{n^2\pi^2+l^2}(-1)^n\sinh(l). \end{aligned}$$

Hence,

$$a_n = \frac{2l}{n^2\pi^2+l^2}(-1)^n\sinh(l).$$

- b_n:

$$\begin{aligned} b_n &= \frac{1}{l}\int_{-l}^{l} e^{-x}\sin\left(\frac{n\pi}{l}x\right)dx = \frac{1}{l}\left[-e^{-x}\frac{\cos\left(\frac{n\pi}{l}x\right)}{\frac{n\pi}{l}}\right]_{-l}^{l} - \frac{1}{n\pi}\int_{-l}^{l} e^{-x}\cos\left(\frac{n\pi}{l}x\right)dx \\ &= \frac{1}{n\pi}\left[-e^{-l}\cos(n\pi)+e^{l}\cos(n\pi)\right] - \frac{1}{n\pi}\left[e^{-x}\frac{\sin\left(\frac{n\pi}{l}x\right)}{\frac{n\pi}{l}}\right]_{-l}^{l} - \frac{l}{n^2\pi^2}\int_{-l}^{l} e^{-x}\sin\left(\frac{n\pi}{l}x\right)dx. \end{aligned}$$

Then,

$$\begin{aligned}\Rightarrow\left(\frac{1}{l}+\frac{l}{n^2\pi^2}\right)\int_{-l}^{l}e^{-x}\sin\left(\frac{n\pi}{l}x\right)dx &= \frac{1}{n\pi}\cos(n\pi)\left(-e^{-l}+e^{l}\right)\\ \Rightarrow\int_{-l}^{l}e^{-x}\sin\left(\frac{n\pi}{l}x\right)dx &= \frac{2ln\pi}{n^2\pi^2+l^2}(-1)^n\sinh(l).\end{aligned}$$

Hence,

$$b_n=\frac{2n\pi}{n^2\pi^2+l^2}(-1)^n\sinh(l).$$

Finally,

$$f(x)=e^{-x}=\frac{\sinh(l)}{l}+2\sinh(l)\sum_{n=1}^{\infty}\left[\frac{l(-1)^n}{n^2\pi^2+l^2}\cos\left(\frac{n\pi}{l}x\right)+\frac{n\pi(-1)^n}{n^2\pi^2+l^2}\sin\left(\frac{n\pi}{l}x\right)\right]$$

4.

- a_0:

$$a_0 = \int_0^1 t\,dt+\int_1^2(1-t)dt=\left[\frac{t^2}{2}\right]_0^1+\left[-\frac{(1-t)^2}{2}\right]_1^2=\frac{1}{2}-\frac{1}{2}=0.$$

- a_n:

$$\begin{aligned}a_n &= \int_0^1 t\cos(n\pi t)dt+\int_1^2(1-t)\cos(n\pi t)dt=\left[t\frac{\sin(n\pi t)}{n\pi}\right]_0^1-\frac{1}{n\pi}\int_0^1\sin(n\pi t)dt\\ &\quad+\left[(1-t)\frac{\sin(n\pi t)}{n\pi}\right]_1^2+\frac{1}{n\pi}\int_1^2\sin(n\pi t)dt\\ &= -\frac{1}{n\pi}\left[-\frac{\cos(n\pi t)}{n\pi}\right]_0^1+\frac{1}{n\pi}\left[-\frac{\cos(n\pi t)}{n\pi}\right]_1^2\\ &= -\frac{1}{n^2\pi^2}\left[-(-1)^n+1\right]+\frac{1}{n^2\pi^2}\left[-1+(-1)^n\right]\\ &= \frac{2}{n^2\pi^2}\left[-1+(-1)^n\right]=\begin{cases}0, & \textit{if } n \textit{ is even},\\ -\frac{4}{n^2\pi^2}, & \textit{if } n \textit{ is odd}.\end{cases}\end{aligned}$$

- b_n:

$$b_n = \int_0^1 t\sin(n\pi t)dt+\int_1^2(1-t)\sin(n\pi t)dt=\left[-t\frac{\cos(n\pi t)}{n\pi}\right]_0^1+\frac{1}{n\pi}\int_0^1\cos(n\pi t)dt$$

$$
\begin{aligned}
&+\left[-(1-t)\frac{\cos(n\pi t)}{n\pi}\right]_1^2-\frac{1}{n\pi}\int_1^2\cos(n\pi t)dt\\
=\;&\frac{1}{n\pi}(-1)^{n+1}+\frac{1}{n\pi}\left[\frac{\sin(n\pi t)}{n\pi}\right]_0^1+\frac{1}{n\pi}-\frac{1}{n\pi}\left[\frac{\sin(n\pi t)}{n\pi}\right]_1^2\\
=\;&\frac{1}{n\pi}(-1)^{n+1}+\frac{1}{n\pi}=\begin{cases}0, & \textit{if } n \textit{ is even},\\ \frac{2}{n\pi}, & \textit{if } n \textit{ is odd}.\end{cases}
\end{aligned}
$$

Then,

$$f(t)=-\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos((2n-1)\pi t)}{(2n-1)^2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin((2n-1)\pi t)}{2n-1}.$$

Solution of exercise 4.

Reminder.

If the range is $0<x<\pi$, *then,*

- *The half-range cosine series is* $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(nx)$, *where*

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx \qquad , \qquad a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos(nx)dx.$$

- *The half-range sine series is* $f(x)=\sum_{n=1}^{\infty}b_n\sin(nx)$, *where*

$$b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin(nx)dx.$$

If the range is $0<x<l$, *then,*

- *The half-range cosine series is* $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi}{l}x\right)$, *where*

$$a_0=\frac{2}{l}\int_0^{l}f(x)dx \qquad , \qquad a_n=\frac{2}{l}\int_0^{l}f(x)\cos\left(\frac{n\pi}{l}x\right)dx.$$

- *The half-range sine series is* $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}x\right)$, *where*

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi}{l}x\right)dx.$$

Solution.

1. *The half range sine for* $f(x)$.

- b_n :

$$\begin{aligned} b_n &= \frac{2}{l}\int_0^l \left(\frac{l}{2}-x\right)\sin\left(\frac{n\pi}{l}x\right)dx = \frac{2}{l}\left[-\left(\frac{l}{2}-x\right)\frac{\cos\left(\frac{n\pi}{l}x\right)}{\frac{n\pi}{l}}\right]_0^l - \frac{2}{n\pi}\int_0^l \cos\left(\frac{n\pi}{l}x\right)dx \\ &= \frac{2}{n\pi}\left[\frac{l}{2}\cos(n\pi)+\frac{l}{2}\right] - \frac{2}{n\pi}\left[\frac{\sin\left(\frac{n\pi}{l}x\right)}{\frac{n\pi}{l}}\right]_0^l = \frac{l}{n\pi}\left[(-1)^n+1\right] = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \frac{2l}{n\pi}, & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Then,

$$f(x) = \sum_{n=1}^{\infty} \frac{2l}{2n\pi}\sin\left(\frac{2n\pi}{l}x\right).$$

2. *The half range cosine for* $f(x)$.

- a_0 :

$$a_0 = \frac{2}{\pi}\int_0^\pi x\sin(x)dx = \frac{2}{\pi}\left[-x\cos(x)\right]_0^\pi + \frac{2}{\pi}\int_0^\pi \cos(x)dx = -2\cos(\pi) + \frac{2}{\pi}\left[\sin(x)\right]_0^\pi = 2.$$

- a_1 :

$$\begin{aligned} a_1 &= \frac{2}{\pi}\int_0^\pi x\sin(x)\cos(x)dx = \frac{1}{\pi}\int_0^\pi x\sin(2x)dx = \frac{1}{\pi}\left[-x\frac{\cos(2x)}{2}\right]_0^\pi + \frac{1}{2\pi}\int_0^\pi \cos(2x)dx \\ &= -\frac{1}{2} + \frac{1}{2\pi}\left[\frac{\sin(2x)}{2}\right]_0^\pi = -\frac{1}{2}. \end{aligned}$$

- $a_n, (n>1)$:

$$a_n = \frac{2}{\pi}\int_0^\pi x\sin(x)\cos(nx)dx = \frac{1}{\pi}\int_0^\pi x\left[\sin((1+n)x)+\sin((1-n)x)\right]dx$$

$$\begin{aligned}
&= -\frac{1}{\pi}\left[x\left(\frac{\cos((1+n)x)}{(1+n)}+\frac{\cos((1-n)x)}{(1-n)}\right)\right]_0^{\pi}+\frac{1}{\pi}\int_0^{\pi}\left(\frac{\cos((1+n)x)}{(1+n)}\right.\\
&\quad\left.+\frac{\cos((1-n)x)}{(1-n)}\right)dx\\
&= \left[\frac{(-1)^{2+n}}{(1+n)}+\frac{(-1)^{2-n}}{(1-n)}\right]+\frac{1}{\pi}\left[\frac{\sin((1+n)x)}{(1+n)^2}+\frac{\sin((1-n)x)}{(1-n)^2}\right]_0^{\pi}\\
&= \begin{cases}\frac{2}{1-n^2}, & \textit{if } n \textit{ is even},\\ -\frac{2}{1-n^2}, & \textit{if } n \textit{ is odd.}\end{cases}
\end{aligned}$$

Then,

$$f(x) = 1-\frac{1}{2}\cos(x)+2\sum_{n=2}^{\infty}\frac{(-1)^n}{1-n^2}\cos(nx).$$

3. *The half range cosine series for $f(x)$.*

- a_0:

$$a_0 = \frac{2}{l}\left[\int_0^{\frac{l}{2}}kxdx+\int_{\frac{l}{2}}^{l}k(l-x)dx\right]=\frac{2}{l}\left[\frac{kx^2}{2}\right]_0^{\frac{l}{2}}+\frac{2}{l}\left[-\frac{k(l-x)^2}{2}\right]_{\frac{l}{2}}^{l}=\frac{kl}{4}+\frac{kl}{4}=\frac{kl}{2}.$$

- a_n:

$$\begin{aligned}
a_n &= \frac{2}{l}\left[\int_0^{\frac{l}{2}}kx\cos\left(\frac{n\pi}{l}x\right)dx+\int_{\frac{l}{2}}^{l}k(l-x)\cos\left(\frac{n\pi}{l}x\right)dx\right]\\
&= \frac{2k}{l}\left[x\frac{\sin\left(\frac{n\pi}{l}x\right)}{\frac{n\pi}{l}}\right]_0^{\frac{l}{2}}-\frac{2k}{n\pi}\int_0^{\frac{l}{2}}\sin\left(\frac{n\pi}{l}x\right)dx+\frac{2}{l}\left[k(l-x)\frac{\sin\left(\frac{n\pi}{l}x\right)}{\frac{n\pi}{l}}\right]_{\frac{l}{2}}^{l}+\frac{2k}{n\pi}\int_{\frac{l}{2}}^{l}\sin\left(\frac{n\pi}{l}x\right)dx\\
&= \frac{kl}{n\pi}\sin\left(\frac{n\pi}{2}\right)-\frac{2k}{n\pi}\left[-\frac{\cos\left(\frac{n\pi}{l}x\right)}{\frac{n\pi}{l}}\right]_0^{\frac{l}{2}}-\frac{lk}{n\pi}\sin\left(\frac{n\pi}{2}\right)+\frac{2k}{n\pi}\left[-\frac{\cos\left(\frac{n\pi}{l}x\right)}{\frac{n\pi}{l}}\right]_{\frac{l}{2}}^{l}\\
&= -\frac{2kl}{n^2\pi^2}\left[-\cos\left(\frac{n\pi}{2}\right)+1\right]+\frac{2kl}{n^2\pi^2}\left[-\cos(n\pi)+\cos\left(\frac{n\pi}{2}\right)\right]\\
&= \frac{2kl}{n^2\pi^2}\left[2\cos\left(\frac{n\pi}{2}\right)-1-\cos(n\pi)\right]
\end{aligned}$$

$$= \begin{cases} 0, & \textit{if } n \textit{ is odd,} \\ 0, & \textit{else if } n \textit{ is even and it is multiple of 4,} \\ -\frac{8kl}{n^2\pi^2}, & \textit{else.} \end{cases}$$

Then,

$$f(x) = \frac{kl}{4} - \frac{8kl}{\pi^2}\left[\frac{1}{2^2}\cos\left(\frac{2\pi}{l}x\right) + \frac{1}{6^2}\cos\left(\frac{6\pi}{l}x\right) + \frac{1}{10^2}\cos\left(\frac{10\pi}{l}x\right) + \cdots\right]$$

- *Putting $x = l$, we get*

$$\begin{aligned} 0 &= \frac{kl}{4} - \frac{8kl}{\pi^2}\left[\frac{1}{2^2} + \frac{1}{6^2} + \frac{1}{10^2} + \cdots\right] \\ \Rightarrow 0 &= \frac{kl}{4} - \frac{8kl}{2^2\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] \\ \Rightarrow \frac{\pi^2}{8} &= \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \end{aligned}$$

4. *The half range sine series for $f(x)$.*

- b_n*:*

$$\begin{aligned} b_n &= \frac{2}{l}\left[\int_0^{\frac{l}{2}} wx\sin\left(\frac{n\pi}{l}x\right)dx + \int_{\frac{l}{2}}^{l} w(l-x)\sin\left(\frac{n\pi}{l}x\right)dx\right] \\ &= \frac{2}{l}\left[-wx\frac{\cos\left(\frac{n\pi}{l}x\right)}{\frac{n\pi}{l}}\right]_0^{\frac{l}{2}} + \frac{2w}{n\pi}\int_0^{\frac{l}{2}}\cos\left(\frac{n\pi}{l}x\right)dx + \frac{2}{l}\left[-w(l-x)\frac{\cos\left(\frac{n\pi}{l}x\right)}{\frac{n\pi}{l}}\right]_{\frac{l}{2}}^{l} \\ &\quad -\frac{2w}{n\pi}\int_{\frac{l}{2}}^{l}\cos\left(\frac{n\pi}{l}x\right)dx \\ &= -\frac{wl}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{2w}{n\pi}\left[\frac{\sin\left(\frac{n\pi}{l}x\right)}{\frac{n\pi}{l}}\right]_0^{\frac{l}{2}} + \frac{wl}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{2w}{n\pi}\left[\frac{\sin\left(\frac{n\pi}{l}x\right)}{\frac{n\pi}{l}}\right]_{\frac{l}{2}}^{l} \\ &= \frac{2wl}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) + \frac{2wl}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \\ &= \frac{4wl}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right). \end{aligned}$$

Then,

$$f(x) = \frac{4wl}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin\left(\frac{(2n+1)\pi}{l} x\right).$$

- *Putting $x = \frac{l}{2}$, we get*

$$\begin{aligned} \frac{wl}{2} &= \frac{4wl}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] \\ \Rightarrow \frac{\pi^2}{8} &= \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \end{aligned}$$

Solution of exercise 5.

Reminder.

- *Theorem : If the Fourier series of $f(x)$ over an interval $c < x < c + 2l$ is given as*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi}{l} x\right) + b_n \sin\left(\frac{n\pi}{l} x\right) \right\}$$

  then,

$$\frac{1}{2l} \int_c^{c+2l} (f(x))^2 dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

- *If $c = 0$, the interval becomes $0 < x < 2l$ and Parseval's identify reduces to*

$$\frac{1}{2l} \int_0^{2l} (f(x))^2 dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

- *If $c = -l$, the interval becomes $-l < x < l$ and Parseval's identify reduces to*

$$\frac{1}{2l} \int_{-l}^{l} (f(x))^2 dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

- *If $f(x)$ is an* **even** *function in $(-l, l)$, then* $\frac{2}{l} \int_0^l (f(x))^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2.$

- *If* $f(x)$ *is an* **odd** *function in* $(-l,l)$, *then* $\frac{2}{l}\int_0^l (f(x))^2dx = \sum_{n=1}^{\infty} b_n^2$.
- *If* $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}x\right)$ *in* $(0,l)$, *then* $\frac{2}{l}\int_0^l (f(x))^2dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$.
- *If* $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}x\right)$ *in* $(0,l)$, *then* $\frac{2}{l}\int_0^l (f(x))^2dx = \sum_{n=1}^{\infty} b_n^2$.

Solution.

1. *The half range cosine series for* x.

- a_0:

$$a_0 = \frac{2}{l}\int_0^l xdx = \frac{2}{l}\left[\frac{x^2}{2}\right]_0^l = \frac{2}{l}\frac{l^2}{2} = l.$$

- a_n:

$$\begin{aligned} a_n &= \frac{2}{l}\int_0^l x\cos\left(\frac{n\pi}{l}x\right)dx = \frac{2}{l}\left[x\frac{\sin\left(\frac{n\pi}{l}x\right)}{\frac{n\pi}{l}}\right]_0^l - \frac{2}{n\pi}\int_0^l \sin\left(\frac{n\pi}{l}x\right)dx = -\frac{2}{n\pi}\left[-\frac{\cos\left(\frac{n\pi}{l}x\right)}{\frac{n\pi}{l}}\right]_0^l \\ &= -\frac{2l}{n^2\pi^2}\left[-\cos(n\pi)+1\right] = -\frac{2l}{n^2\pi^2}\left[-(-1)^n+1\right] = \begin{cases} -\frac{4l}{n^2\pi^2}, & \textit{if } n \textit{ is odd,} \\ 0, & \textit{if } n \textit{ is even.} \end{cases} \end{aligned}$$

Then,

$$x = \frac{l}{2} - \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{(2n-1)\pi}{l}x\right)}{(2n-1)^2}$$

- *Applying the Parseval's on half range cosine series for* x *in* $(0,l)$

$$\frac{2}{l}\int_0^l x^2dx = \frac{l^2}{2} + \sum_{n=1}^{\infty}\frac{16l^2}{(2n-1)^4\pi^4} \Rightarrow \left[\frac{2l^2}{3} - \frac{l^2}{2}\right]\frac{\pi^4}{16l^2} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} \Rightarrow \frac{\pi^4}{96} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}.$$

2. *The half range cosine series for* $x(\pi - x)$.

- a_0:

$$a_0 = \frac{2}{\pi}\int_0^\pi x(\pi - x)dx = \frac{2}{\pi}\int_0^\pi (\pi x - x^2)dx = \frac{2}{\pi}\left[\pi\frac{x^2}{2} - \frac{x^3}{3}\right]_0^\pi = \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right] = \frac{2}{\pi}\frac{\pi^3}{6} = \frac{\pi^2}{3}.$$

- a_n:

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi x(\pi - x)\cos(nx)\,dx = \frac{2}{\pi}\left[x(\pi - x)\frac{\sin(nx)}{n}\right]_0^\pi - \frac{2}{n\pi}\int_0^\pi (\pi - 2x)\sin(nx)dx \\ &= -\frac{2}{n\pi}\left[-(\pi - 2x)\frac{\cos(nx)}{n}\right]_0^\pi + \frac{4}{n^2\pi}\int_0^\pi \cos(nx)dx \\ &= -\frac{2}{n^2}(-1)^n - \frac{2}{n^2} + \frac{4}{n^2\pi}\left[\frac{\sin(nx)}{n}\right]_0^\pi \\ &= -\frac{2}{n^2}(-1)^n - \frac{2}{n^2} \\ &= \begin{cases} -\frac{4}{n^2}, & \textit{if } n \textit{ is even,} \\ 0, & \textit{if } n \textit{ is odd.} \end{cases} \end{aligned}$$

Then

$$x(\pi - x) = \frac{\pi^2}{6} - 4\sum_{n=1}^{\infty}\frac{\cos(2nx)}{(2n)^2} = \frac{\pi^2}{6} - \sum_{n=1}^{\infty}\frac{\cos(2nx)}{n^2}.$$

- *Applying the Parseval's theorem on Fourier constants with* $c = 0, l = \frac{\pi}{2}$ *for* $x(\pi - x)$ *in* $(0, \pi)$

$$\begin{aligned} \frac{1}{\pi}\int_0^\pi [x(\pi - x)]^2\,dx &= \frac{\pi^4}{36} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4} \\ \Rightarrow \frac{\pi^4}{30} - \frac{\pi^4}{36} &= \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4} \\ \Rightarrow 2\frac{6\pi^4}{1080} &= \sum_{n=1}^{\infty}\frac{1}{n^4} \\ \Rightarrow \frac{\pi^4}{90} &= \sum_{n=1}^{\infty}\frac{1}{n^4}. \end{aligned}$$

3. *The Fourier sine for unity in* $0 < x < \pi$.

- b_n:

$$b_n = \frac{2}{\pi}\int_0^{\pi} 1\sin(nx)dx = \frac{2}{\pi}\left[-\frac{\cos(nx)}{n}\right]_0^{\pi} = \frac{2}{n\pi}(1-(-1)^n).$$

Now,

$$b_n = \begin{cases} 0, & n \text{ is even,} \\ \frac{4}{n\pi}, & n \text{ is odd.} \end{cases}$$

Then,

$$1 = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi}\sin((2n-1)x).$$

- *Applying the Parseval's theorem on Fourier constants with* $c = 0, l = \frac{\pi}{2}$ *for 1 in* $(0, \pi)$

$$\begin{aligned} \int_0^{\pi} (1)^2 dx &= \pi\left(\frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{(2n-1)^2\pi^2}\right) \\ \Rightarrow \pi &= \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} \\ \Rightarrow \frac{\pi^2}{8} &= \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}. \end{aligned}$$

4. *The Fourier series of* x^2 *in* $(-\pi, \pi)$*. Since* x^2 *is an even function, then,* $b_n = 0$*.*

- a_0:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{3}\pi^2.$$

- a_n:

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)dx = \frac{2}{\pi}\left[x^2\frac{\sin(nx)}{n}\right]_0^{\pi} - \frac{4}{\pi n}\int_0^{\pi} x\sin(nx)dx \\ &= \frac{2}{\pi}.0 - \frac{4}{\pi n}\left[-x\frac{\cos(nx)}{n}\right]_0^{\pi} - \frac{4}{\pi n^2}\int_0^{\pi}\cos(nx)dx = \frac{4}{n^2}\cos(n\pi) - \frac{4}{\pi n^2}\left[\frac{\sin(nx)}{n}\right]_0^{\pi} \\ &= \frac{4}{n^2}(-1)^n. \end{aligned}$$

Then,

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx).$$

- *Using Parseval's identity, we get*

$$\begin{aligned}
\int_{-\pi}^{\pi} x^4 dx &= 2\pi\left(\frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4}\right)\\
\Rightarrow \frac{2}{5}\pi^5 &= \frac{2}{9}\pi^5 + 16\pi\sum_{n=1}^{\infty}\frac{1}{n^4}\\
\Rightarrow \frac{8}{45}\pi^5 &= 16\pi\sum_{n=1}^{\infty}\frac{1}{n^4}\\
\Rightarrow \frac{\pi^4}{90} &= \sum_{n=1}^{\infty}\frac{1}{n^4}.
\end{aligned}$$

Solution of exercise 6.

1. *Easy.*
2. *As $f(x) = |\cos(x)|$ is piecewise continuous and $\pi-$periodic, then, the Fourier series of f converges to f.*
3. *The Fourier coefficients of f.*

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^{\pi}|\cos(x)|dx = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}}\cos(x)dx - \int_{\frac{\pi}{2}}^{\pi}\cos(x)dx\right] = \frac{2}{\pi}\left[\sin(x)\right]_0^{\frac{\pi}{2}} - \frac{2}{\pi}\left[\sin(x)\right]_{\frac{\pi}{2}}^{\pi} = \frac{4}{\pi}.\\
a_1 &= \frac{2}{\pi}\int_0^{\pi}|\cos(x)|\cos(x)dx = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}}\cos(x)\cos(x)dx - \int_{\frac{\pi}{2}}^{\pi}\cos(x)\cos(x)dx\right]\\
&= \frac{2}{\pi}\left[\frac{1}{2}\int_0^{\frac{\pi}{2}}(1+\cos(2x))dx - \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}(1+\cos(2x))dx\right]\\
&= \frac{1}{\pi}\left[x + \frac{\sin(2x)}{2}\right]_0^{\frac{\pi}{2}} - \frac{1}{\pi}\left[x + \frac{\sin(2x)}{2}\right]_{\frac{\pi}{2}}^{\pi}\\
&= \frac{1}{2} - 1 + \frac{1}{2}\\
&= 0.\\
a_n &= \frac{2}{\pi}\int_0^{\pi}|\cos(x)|\cos(nx)dx = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}}\cos(x)\cos(nx)dx - \int_{\frac{\pi}{2}}^{\pi}\cos(x)\cos(nx)dx\right], n > 1
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}} (\cos((1+n)x)+\cos((1-n)x))\,dx - \int_{\frac{\pi}{2}}^{\pi} (\cos((1+n)x)+\cos((1-n)x))\,dx\right] \\
&= \frac{1}{\pi}\left[\frac{\sin((1+n)x)}{1+n}+\frac{\sin((1-n)x)}{1-n}\right]_0^{\frac{\pi}{2}} - \frac{1}{\pi}\left[\frac{\sin((1+n)x)}{1+n}+\frac{\sin((1-n)x)}{1-n}\right]_{\frac{\pi}{2}}^{\pi} \\
&= \frac{2}{\pi}\left(\frac{\sin\left((1+n)\frac{\pi}{2}\right)}{1+n}+\frac{\sin\left((1-n)\frac{\pi}{2}\right)}{1-n}\right) \\
&= \frac{1}{\pi}\begin{cases} \frac{4(-1)^{(n+1)}}{n^2-1}, & n \text{ is even}, \\ 0, & n \text{ is odd}. \end{cases}
\end{aligned}$$

$b_n = 0.$ *$f(x)$ is an even function.*

Then,

$$|\cos(x)| = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4(-1)^{(n+1)}}{\pi(4n^2-1)}\cos(2nx). \qquad (\star)$$

4. *We put $x = 0$ in $(\star)$, then,*

$$1 = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4(-1)^{(n+1)}}{\pi(4n^2-1)} \Rightarrow 0 = \frac{2-\pi}{\pi} + \sum_{n=1}^{\infty} \frac{4(-1)^{(n+1)}}{\pi(4n^2-1)}. \qquad (\star\star)$$

Therefore, $(\star) + (\star\star)$ gives,

$$|\cos(x)| = \frac{4-\pi}{\pi} + \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}}{(4n^2-1)}\cos^2(nx).$$

with, $2\cos^2(nx) = 1 + \cos(2nx)$.

5. *We have, $g(x) = |\sin(x)| = |\cos\left(\frac{\pi}{2}-x\right)| = f\left(\frac{\pi}{2}-x\right)$. Then, we put $x = \frac{\pi}{2} - x$ in $(\star)$,*

$$\begin{aligned}
|\cos(x)| &= \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4(-1)^{(n+1)}}{\pi(4n^2-1)}\cos\left(2n\left(\frac{\pi}{2}-x\right)\right) \\
&= \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4(-1)^{(n+1)}}{\pi(4n^2-1)}\cos(n\pi - 2nx) \\
&= \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4(-1)^{(n+1)}}{\pi(4n^2-1)}\cos(n\pi)\cos(2nx)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4(-1)^{(n+1)}}{\pi(4n^2-1)}(-1)^n \cos(2nx) \\
&= \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(4n^2-1)} \cos(2nx).
\end{aligned}
$$

Solution of exercise 7.

1. *Easy.*

2. *As $f(x) = |\sin(x)|$ is piecewise continuous and $\pi-$periodic, then, the Fourier series of f converges to f.*

3. *The Fourier coefficients of f.*

$$
\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^{\pi} |\sin(x)|dx = \frac{2}{\pi}\int_0^{\pi} \sin(x)dx = \frac{2}{\pi}\left[-\cos(x)\right]_0^{\pi} = \frac{4}{\pi}. \\
a_1 &= \frac{2}{\pi}\int_0^{\pi} |\sin(x)|\cos(x)dx = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(x)dx = \frac{1}{\pi}\int_0^{\pi} \sin(2x)dx \\
&= \frac{1}{\pi}\left[\frac{-\cos(2x)}{2}\right]_0^{\pi} = 0. \\
a_n &= \frac{2}{\pi}\int_0^{\pi} |\sin(x)|\cos(nx)dx = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(nx)dx, n > 1 \\
&= \frac{1}{\pi}\int_0^{\pi} \left(\sin((1+n)x) + \sin((1-n)x)\right) dx \\
&= \frac{1}{\pi}\left[\frac{-\cos((1+n)x)}{1+n} + \frac{-\cos((1-n)x)}{1-n}\right]_0^{\pi} \\
&= \frac{1}{\pi}\left(\frac{(-1)^{2+n}}{1+n} + \frac{(-1)^{2-n}}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right) \\
&= \frac{1}{\pi}\left(\frac{(-1)^{2+n}}{1+n} + \frac{(-1)^{2-n}}{1-n} + \frac{2}{1-n^2}\right) \\
&= \frac{1}{\pi}\begin{cases} \frac{-4}{n^2-1}, & n \text{ is even,} \\ 0, & n \text{ is odd.} \end{cases} \\
b_n &= 0. \quad f(x) \text{ is an even function.}
\end{aligned}
$$

Then,

$$|\sin(x)| = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{-4}{\pi(4n^2-1)} \cos(2nx). \qquad (\star)$$

4. *We put $x = 0$ in $(\star)$, then,*

$$0 = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{-4}{\pi(4n^2-1)} \Rightarrow 0 = -\frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi(4n^2-1)}. \qquad (\star\star)$$

Therefore, $(\star) + (\star\star)$ gives,

$$|\sin(x)| = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin^2(nx)}{(4n^2-1)}$$

with, $2\sin^2(nx) = 1 - \cos(2nx)$.

5. *We have, $g(x) = |\cos(x)| = |\sin\left(\frac{\pi}{2} - x\right)| = f\left(\frac{\pi}{2} - x\right)$. Then, we put $x = \frac{\pi}{2} - x$ in $(\star)$,*

$$\begin{aligned} |\cos(x)| &= \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{-4}{\pi(4n^2-1)} \cos\left(2n\left(\frac{\pi}{2} - x\right)\right) \\ &= \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{-4}{\pi(4n^2-1)} \cos(n\pi - 2nx) \\ &= \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{-4}{\pi(4n^2-1)} \cos(n\pi)\cos(2nx) \\ &= \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{-4}{\pi(4n^2-1)} (-1)^n \cos(2nx) \\ &= \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(4n^2-1)} \cos(2nx). \end{aligned}$$

Solution of exercise 8.

1. *Easy.*

2. *The function f is piecewise continuous and 2π-periodic, then, the Fourier series of f converges to f.*

3. The Fourier coefficients of f.

$$
\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^{\pi}\max(\cos(t),0)dt = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\cos(t)dt + \frac{\pi}{2}\int_{\frac{\pi}{2}}^{\pi}0dt = \frac{2}{\pi}\left[\sin(x)\right]_0^{\frac{\pi}{2}} = \frac{2}{\pi}.\\
a_1 &= \frac{2}{\pi}\int_0^{\pi}\max(\cos(t),0)\cos(t)dt = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\cos^2(t)dt + \frac{\pi}{2}\int_{\frac{\pi}{2}}^{\pi}0.\cos(t)dt\\
&= \frac{1}{\pi}\int_0^{\frac{\pi}{2}}(1+\cos(2t))dt = \frac{1}{\pi}\left[t+\frac{\sin(2t)}{2}\right]_0^{\frac{\pi}{2}} = \frac{1}{\pi}.\frac{\pi}{2} = \frac{1}{2}.\\
a_n &= \frac{2}{\pi}\int_0^{\pi}\max(\cos(t),0)\cos(nt)dt = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\cos(t)\cos(nt)dt + \frac{\pi}{2}\int_{\frac{\pi}{2}}^{\pi}0.\cos(nt)dt, n>1\\
&= \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\left(\cos((n+1)t)+\cos((n-1)t)\right)dt\\
&= \left[\frac{\sin((n+1)t)}{n+1}+\frac{\sin((n-1)t)}{n-1}\right]_0^{\frac{\pi}{2}}\\
&= \frac{\sin\left((n+1)\frac{\pi}{2}\right)}{n+1}+\frac{\sin\left((n-1)\frac{\pi}{2}\right)}{n-1}\\
&= \begin{cases} 0, & \textit{if } n \textit{ is odd},\\ \frac{1}{\pi}\left(\frac{\sin\left(\frac{\pi}{2}+p\pi\right)}{2p+1}+\frac{\sin\left(-\frac{\pi}{2}+p\pi\right)}{2p-1}\right) = \frac{1}{\pi}\left(\frac{\cos(p\pi)}{2p+1}-\frac{\cos(p\pi)}{2p-1}\right) = \frac{(-1)^p}{\pi}\left(\frac{1}{2p+1}-\frac{1}{2p-1}\right), & \textit{if } n \textit{ is even}.\end{cases}\\
&= \begin{cases} 0, & \textit{if } n \textit{ is odd},\\ \frac{2(-1)^{p+1}}{\pi(4p^2-1)}, & \textit{if } n \textit{ is even}.\end{cases}\\
b_n &= 0. \quad f(x) \textit{ is an even function.}
\end{aligned}
$$

4. The Fourier series of $f(x)$ is,

$$
f(t) = \frac{1}{\pi}+\frac{\cos(t)}{2}+\frac{2}{\pi}\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{4p^2-1}\cos(2pt). \qquad (\star)
$$

5. We put $t=0$ in $(\star)$, we obtain,

$$
1 = \frac{1}{\pi}+\frac{1}{2}+\frac{2}{\pi}\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{4p^2-1} \Rightarrow \sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{4p^2-1} = \frac{\pi}{2}\left(1-\frac{1}{\pi}-\frac{1}{2}\right) = \frac{\pi}{2}\left(\frac{1}{2}-\frac{1}{\pi}\right).
$$

Then,

$$S=\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{4p^2-1}=\frac{1}{2}\left(\frac{\pi}{2}-1\right).$$

We put $t=\frac{\pi}{2}$ in $(\star)$, we obtain,

$$0=\frac{1}{\pi}+\frac{2}{\pi}\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{4p^2-1}\cos(p\pi)\Rightarrow 0=\frac{1}{\pi}+\frac{2}{\pi}\sum_{p=1}^{\infty}\frac{(-1)^{2p+1}}{4p^2-1}\Rightarrow\frac{2}{\pi}\sum_{p=1}^{\infty}\frac{1}{4p^2-1}=\frac{1}{\pi}$$

Then,

$$T=\sum_{p=1}^{\infty}\frac{1}{4p^2-1}=\frac{1}{2}.$$

6. *Using Parseval Theorem, as f^2 is an even function and equals to 0 on $\left[\frac{\pi}{2},\pi\right]$, we have,*

$$\begin{aligned}\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\cos^2(t)dt &= \frac{a_0^2}{4}+\frac{a_1^2}{2}+\frac{1}{2}\sum_{p=1}^{\infty}a_{2p}^2\\ \frac{1}{2\pi}\int_0^{\frac{\pi}{2}}(1+\cos(2t))dt &= \frac{1}{\pi^2}+\frac{1}{8}+\frac{2}{\pi^2}\sum_{p=1}^{\infty}\frac{1}{(4p^2-1)^2}\\ \frac{\pi}{4} &= \frac{1}{\pi^2}+\frac{1}{8}+\frac{2}{\pi^2}\sum_{p=1}^{\infty}\frac{1}{(4p^2-1)^2}\end{aligned}$$

Then,

$$V=\sum_{p=1}^{\infty}\frac{1}{(4p^2-1)^2}=\frac{\pi^2}{2}\left(\frac{1}{8}-\frac{1}{\pi^2}\right)=\frac{1}{2}\left(\frac{\pi^2}{8}-1\right).$$

7. *We have,*

$$g(t)=\max(\sin(t),0)=\max\left(\cos\left(t-\frac{\pi}{2}\right),0\right)=f\left(t-\frac{\pi}{2}\right).$$

Now, we put $t=t-\frac{\pi}{2}$ dans $(\star)$, we get,

$$\begin{aligned}g(t) &= \frac{1}{\pi}+\frac{1}{2}\cos\left(t-\frac{\pi}{2}\right)+\frac{2}{\pi}\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{4p^2-1}\cos(2pt-p\pi)\\ &= \frac{1}{\pi}+\frac{1}{2}\sin(t)+\frac{2}{\pi}\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{4p^2-1}(-1)^p\cos(2pt)\end{aligned}$$

$$= \frac{1}{\pi} + \frac{1}{2}\sin(t) - \frac{2}{\pi}\sum_{p=1}^{\infty}\frac{1}{4p^2-1}\cos(2pt)$$

Here, we used $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$, *where,* $a = 2pt$ *and* $b = pt$.

Solution of exercise 9.

1. *Easy.*

2. *The function* f *is piecewise continuous and* $2\pi-$*periodic, then, the Fourier series of* f *converges to* f.

3. *The Fourier coefficients of* f.

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{2\pi}\max(\sin(t),t)dt = \frac{1}{\pi}\int_0^{\pi}\sin(t)dt + \frac{1}{\pi}\int_{\pi}^{2\pi}0dt = [-\cos(t)]_0^{\pi} = \frac{2}{\pi}.\\
a_1 &= \frac{1}{\pi}\int_0^{2\pi}\max(\sin(t),t)\cos(t)dt = \frac{1}{\pi}\int_0^{\pi}\sin(t)\cos(t)dt + \frac{1}{\pi}\int_{\pi}^{2\pi}0.\cos(t)dt\\
&= \frac{1}{2}\pi\int_0^{\pi}\sin(2t)dt = \frac{1}{2\pi}\left[-\frac{\cos(2t)}{2}\right]_0^{\pi} = 0.\\
a_n &= \frac{1}{\pi}\int_0^{2\pi}\max(\sin(t),t)\cos(nt)dt = \frac{1}{\pi}\int_0^{\pi}\sin(t)\cos(nt)dt + \frac{1}{\pi}\int_{\pi}^{2\pi}0.\cos(nt)dt, n>1\\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\pi}\left(\sin(t(n+1)) + \sin(t(1-n))\right)dt\\
&= \frac{1}{2\pi}\left[-\frac{\cos(t(1+n))}{1+n} - \frac{\cos(t(1-n))}{1-n}\right]_0^{\pi}\\
&= \frac{1}{2\pi}\left[-\frac{\cos(\pi(1+n))}{1+n} - \frac{\cos(\pi(1-n))}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right]\\
&= \frac{1}{2\pi}\begin{cases}\frac{-4}{n^2-1}, & \textit{if } n \textit{ is even,}\\ 0, & \textit{if } n \textit{ is odd,}\end{cases}\\
&= \begin{cases}\frac{-2}{\pi(n^2-1)}, & \textit{if } n \textit{ is even,}\\ 0, & \textit{if } n \textit{ is odd.}\end{cases}\\
b_1 &= \frac{1}{\pi}\int_0^{2\pi}\max(\sin(t),t)\sin(t)dt = \frac{1}{\pi}\int_0^{\pi}\sin^2(t)dt + \frac{1}{\pi}\int_{\pi}^{2\pi}0.\sin(t)dt
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi}\int_0^{\pi}(1-\cos(2t))dt = \frac{1}{2\pi}\left[t-\frac{\sin(2t)}{2}\right]_0^{\pi} = \frac{1}{2\pi}.\pi = \frac{1}{2}.\\
b_n &= \frac{1}{\pi}\int_0^{2\pi}\max(\sin(t),t)\sin(nt)dt = \frac{1}{\pi}\int_0^{\pi}\sin(t)\sin(nt)dt + \frac{1}{\pi}\int_{\pi}^{2\pi}0.\sin(nt)dt\\
&= -\frac{1}{2\pi}\int_0^{\pi}\left(\cos(t(1+n))-\cos(t(1-n))\right)dt\\
&= -\frac{1}{2\pi}\left[\frac{\sin(t(1+n))}{1+n}-\frac{\sin(t(1-n))}{1-n}\right]_0^{\pi}\\
&= 0.
\end{aligned}$$

4. *The Fourier series of $f(x)$ is,*

$$f(t) = \frac{1}{\pi}+\frac{1}{2}\sin(t)-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos(2nt), \qquad (\star)$$

5. *We put $t=0$ in $(\star)$, we obtain,*

$$0 = \frac{1}{\pi}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1} \Rightarrow \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{.}\pi$$

Then,

$$S = \sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2}.$$

We put $t=\frac{\pi}{2}$ in $(\star)$, we get,

$$1 = \frac{1}{\pi}+\frac{1}{2}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos(n\pi) \Rightarrow 1 = \frac{1}{\pi}+\frac{1}{2}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} \Rightarrow \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} = \frac{1}{\pi}-\frac{1}{2}.$$

Then,

$$T = \sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} = \frac{\pi}{2}\left(\frac{1}{\pi}-\frac{1}{2}\right) = \frac{1}{2}\left(1-\frac{\pi}{2}\right).$$

6. *We have,*

$$g(t) = \max\left(\cos(t),0\right) = \max\left(\sin\left(\frac{\pi}{2}-t\right),0\right) = f\left(\frac{\pi}{2}-t\right).$$

Now, we put $t = \frac{\pi}{2} - t$ *in* $(\star)$, *we obtain,*

$$\begin{aligned} g(t) &= \frac{1}{\pi} + \frac{1}{2}\sin\left(\frac{\pi}{2} - t\right) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos(n\pi - 2nt) \\ &= \frac{1}{\pi} + \frac{1}{2}\cos(t) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}(-1)^n\cos(2nt) \\ &= \frac{1}{\pi} + \frac{1}{2}\cos(t) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}\cos(2nt) \end{aligned}$$

Chapitre 4

Fourier and Laplace transforms.

Exercise 1. *Find the Fourier integral representations of the following functions*

1. $f(x) = \begin{cases} 0, & x < 0, \\ x, & 0 \leq x < 2, \\ 0, & x \geq 2. \end{cases}$

2. $f(x) = \begin{cases} 2, & -2 \leq x < 1, \\ 1, & 1 \leq x < 3, \\ 0, & \textit{elsewhere}. \end{cases}$

3. $f(x) = \begin{cases} \cos(x), & -2 \leq x \leq 0, \\ \sin(x), & 0 < x \leq 2, \\ 0, & |x| > 2. \end{cases}$

4. $f(x) = \begin{cases} e^{-|x|}, & |x| < 1, \\ 0, & \textit{elsewhere}. \end{cases}$

Exercise 2. *Find the appropriate integral representation of the given function*

1. *Cosine representation :*
   - $f(x) = \begin{cases} x^2, & 0 \leq x \leq 5, \\ 0, & x > 5. \end{cases}$
   - $f(x) = \begin{cases} \sin(x), & 0 \leq x \leq \pi, \\ 0, & x > \pi. \end{cases}$

2. *Sine representation :*
   - $f(x) = \begin{cases} \sinh(x), & 0 \leq x \leq 3, \\ 0, & x > 3. \end{cases}$
   - $f(x) = \begin{cases} 0, & 0 < x < 1, \\ 1, & 1 < x < 2, \\ 0, & x > 2. \end{cases}$

3. *Complex form of the Fourier integral representation :*
   - $f(x) = \begin{cases} \cosh(x), & |x| \leq a, \\ 0, & |x| > a. \end{cases}$
   - $f(x) = \begin{cases} 1 + x, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$

Exercise 3. *Find the Fourier transform of the following functions*

1. $H(t-3)e^{-4t}$.
2. $e^{-a|t+1|}, a > 0$.
3. $\frac{1}{1+t^2}$.
4. $2e^{-3|t|}\sin(4t)$.

5. $\begin{cases} -e^{\alpha t}, & t < 0, \\ e^{-\alpha t}, & t > 0, \alpha > 0. \end{cases}$

6. $f(x) = \begin{cases} \frac{1}{n^2}(n+x), & -n < x < 0, \\ \frac{1}{n^2}(n-x), & 0 < x < n, \\ 0, & otherwise.(Exam) \end{cases}$

7. $f(x) = \begin{cases} 1, & |x| \leq \frac{a}{2}, a > 0, \\ 0, & otherwise.(Exam) \end{cases}$

8. $f(x) = \begin{cases} mx, & |x| \leq \frac{1}{m}, m > 0, \\ 0, & otherwise.(Exam) \end{cases}$

Exercise 4. *Find the inverse Fourier transform of the following functions*

1. $\frac{e^{-i\omega}}{2(1+i\omega)}$.
2. $\frac{e^{-(1-i\omega)}}{3+i\omega}$.
3. $\frac{2i\omega}{3+i\omega}$.
4. $\frac{e^{-2i\omega}}{4+\omega^2}$.
5. $\frac{1}{6+5i\omega-\omega^2}$.
6. $\frac{1}{a^4+\omega^4}, a > 0$.

Exercise 5. *(Exam) Using frequency convolution show that*

1. $$\int_{-\infty}^{+\infty} \frac{d\tau}{(2-i\tau+i\omega)(2+i\tau)} = \frac{2\pi}{4+i\omega}.$$
2. $$\int_{-\infty}^{+\infty} \frac{d\tau}{(4-i\omega)(4-i\tau+i\omega)} = 0.$$

Exercise 6. *Use the time convolution to find the inverse of the following functions*

1. $\frac{1}{(i\omega+k)^2}, k > 0$.
2. $\frac{1}{(i\omega+k)^3}, k > 0$.

Exercise 7. *(Exam) Using Fourier transforms, find the solution of the following differential equations.*

1. $y' - 4y = H(t)e^{-4t}$.
2. $y'' + 5y' + 4y = \delta(t-2)$.

Exercise 8. *Find the Laplace transform of the following functions.*

- *For the following, use the definition of Laplace transform.*

1. $2t - 5$.
2. $\cos(at + b)$.
3. $\cos(t) - \sin(t)$.
4. te^{2t}.
5. $t^2 e^t$.
6. $e^{-t}\sin(t)$.
7. $6 - e^{3t}$.
8. $3t^2 - 5e^{-2t} + 6$.
9. $e^{-t}\sinh(t)$.
10. $f(t) = \begin{cases} 2, & 0 \leq t < 3, \\ 0, & t \geq 3, \end{cases}$
11. $e^t \cosh(t)$.
12. $f(t) = \begin{cases} \cosh(t), & 0 \leq t < \pi, \\ 0, & t \geq \pi. \end{cases}$

- *For the following, use the formulas of Laplace transforms of derivatives.*

1. $\cos^2(2t)$.
2. te^{at}.
3. $t\sin(at)$.
4. $t\cos(at)$.
5. $t\sinh(at)$.

- *For the following, use the shifting theorem of Laplace transforms.*

1. $(t^2 - 2t - 3)e^{2t}$.
2. $t^5 e^{-4t}$.
3. $t3\cosh(3t)$.
4. $\cosh(\omega t)\sin(\omega t)$.
5. $\sinh(t)\sin(t)$.
6. $e^t \sin(5t)$.

- *For the following, use the convolution theorem of Laplace transforms.*

1. $1 * e^{-2t}$.
2. $t * e^{at}$.
3. $e^{at} * e^{bt}$.
4. $\sin(\omega t) * \cos(\omega t)$.

Exercise 9. *Find the inverse Laplace transform of the following functions.*

1. $\frac{2}{s+5}$.
2. $\frac{\pi}{s^2+\pi^2}$.
3. $\frac{6}{s^4}$.

4. $\frac{s+3}{(s-1)(s+2)}$.
5. $\frac{13}{s^2-9}$.
6. $\frac{s^2+2s+5}{(s-1)(s-2)(s-3)}$.
7. $\frac{1}{s^2+5s}$.
8. $\frac{16}{s^3+9s}$.
9. $\frac{\omega}{s^2(s^2+\omega^2)}$.
10. $\frac{s-a}{s^2(s^2+a^2)}$.
11. $\frac{1}{s^4+3s^3}$.
12. $\frac{1}{s^2+6s+15}$.
13. $\frac{s}{s^2+4s+8}$.
14. $\frac{5s+6}{(s-1)^2}$.
15. $\frac{(s+1)^2}{(s-2)^4}$.
16. $\frac{1}{(s-a)(s-b)}$.
17. $\frac{1}{s^2(s^2+16)}$.
18. $\frac{s}{(s^2+4)^2}$.
19. $\frac{6}{(s+1)^2(s+2)}$.
20. $\frac{1}{(s+a)^2(s+b)^2}, \quad a \neq b$.

Exercise 10. *Solve the following integral equations using the convolution theorem*

1. $f(t) = t + 6\int_0^t f(\tau)e^{(t-\tau)}d\tau.$
2. $f(t) + \int_0^t f(\tau)\cos(t-\tau)d\tau = e^{-t}.$ *(Exam)*
3. $f(t) = \cos(t) + e^{-t}\int_0^t f(\tau)e^{\tau}d\tau.$

Exercise 11. *Solve the following initial value problems.*

1. $y' + 2y = \sin(t), \quad y(0) = 1.$
2. $y'' - 6y' + 5y = e^{2t}, \quad y(0) = 1, \quad y'(0) = -1.$
3. $y' + 6y + 5\int_0^t y(\tau)d\tau = 1 + t, \quad y(0) = 1.$ *(Exam)*
4. $y' + y = f(t), \quad y(0) = 3, \quad f(t) = \begin{cases} 1, & 0 \leq t < 1, \\ -1, & t \geq 1. \end{cases}$
5. $y'' + 5y' + 6y = 1 - u_3(t) - u_5(t), \quad y(0) = 0, \quad y'(0) = 0.$ *(Exam)*
6. $y'' + 4y' + 5y = \delta(t-3), \quad y(0) = 0, \quad y'(0) = 0.$ *(Exam)*
7. $y'' + 4y = 8\delta\left(t - \frac{\pi}{6}\right), \quad y(0) = 2, \quad y'(0) = 0.$

Solution of exercise 1.

Reminder : *If f satisfies :*

(P_1) : *$f(x)$ is piecewise continuous on every interval $[-l, l]$.*

(P_2) : *$f(x)$ is absolutely integrable on the $x-$axis, that is $\int_{-\infty}^{+\infty} |f(x)| dx$ converges.*

(P_3) : *At every x on the real line, $f(x)$ has left and right hand derivatives.*

Then, the Fourier integral of f is given by,

$$f(x) = \frac{1}{\pi} \int_0^{\infty} [A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)]\, d\omega \tag{4.1}$$

where,

$$A(\omega) = \int_{-\infty}^{\infty} f(t)\cos(\omega t) dt, \tag{4.2}$$

$$B(\omega) = \int_{-\infty}^{\infty} f(t)\sin(\omega t) dt. \tag{4.3}$$

Otherwise, if x is a point of discontinuity, $f(x)$ in (4.1) is replaced by $(f(x^+) + f(x^-))/2$.

Solution.

1. *$f(x)$ satisfies $(P_1) - (P_3)$. So, we have,*

$$\begin{aligned}
A(\omega) &= \int_{-\infty}^{0} 0.\cos(\omega t) dt + \int_0^2 t.\cos(\omega t) dt + \int_2^{+\infty} 0.\cos(\omega t) dt \\
&= \int_0^2 t\cos(\omega t) dt = \left[t\frac{\sin(\omega t)}{\omega} \right]_0^2 - \frac{1}{\omega} \int_0^2 \sin(\omega t) dt \\
&= \frac{2}{\omega}\sin(2\omega) - \frac{1}{\omega}\left[\frac{-\cos(\omega t)}{\omega} \right]_0^2 = \frac{2}{\omega}\sin(2\omega) + \frac{1}{\omega^2}(\cos(2\omega) - 1). \\
B(\omega) &= \int_{-\infty}^{0} 0.\sin(\omega t) dt + \int_0^2 t.\sin(\omega t) dt + \int_2^{+\infty} 0.\sin(\omega t) dt \\
&= \int_0^2 t\sin(\omega t) dt = \left[t\frac{-\cos(\omega t}{\omega} \right]_0^2 + \frac{1}{\omega} \int_0^2 \cos(\omega t) dt
\end{aligned}$$

$$= -\frac{2}{\omega}\cos(2\omega) + \frac{1}{\omega}\left[\frac{\sin(\omega t)}{\omega}\right]_0^2 = -\frac{2}{\omega}\cos(2\omega) + \frac{1}{\omega^2}\sin(2\omega).$$

Therefore,

$$\begin{aligned} f(x) &= \frac{1}{\pi\omega^2}\int_0^\infty \left[(2\omega\sin(2\omega) + (\cos(2\omega)-1))\cos(\omega x)\right. \\ &\qquad \left. + (\sin(2\omega) - 2\omega\cos(2\omega))\sin(\omega x)\right] d\omega. \end{aligned}$$

2. *$f(x)$ satisfies $(P_1)-(P_3)$. So, we have,*

$$\begin{aligned} A(\omega) &= \int_{-\infty}^{-2} 0.\cos(\omega t)dt + \int_{-2}^{1} 2.\cos(\omega t)dt + \int_1^3 1.\cos(\omega t)dt + \int_3^{+\infty} 0.\cos(\omega t)dt \\ &= 2\int_{-2}^{1}\cos(\omega t)dt + \int_1^3 \cos(\omega t)dt = 2\left[\frac{\sin(\omega t)}{\omega}\right]_{-2}^{1} + \left[\frac{\sin(\omega t)}{\omega}\right]_1^3 \\ &= \frac{2}{\omega}\left(\sin(\omega) + \sin(2\omega)\right) + \frac{1}{\omega}\left(\sin(3\omega) - \sin(\omega)\right) \\ &= \frac{1}{\omega}\left(\sin(\omega) + 2\sin(2\omega) + \sin(3\omega)\right). \\ B(\omega) &= \int_{-\infty}^{-2} 0.\sin(\omega t)dt + \int_{-2}^{1} 2.\sin(\omega t)dt + \int_1^3 1.\sin(\omega t)dt + \int_3^{+\infty} 0.\sin(\omega t)dt \\ &= 2\int_{-2}^{1}\sin(\omega t)dt + \int_1^3 \sin(\omega t)dt = 2\left[\frac{-\cos(\omega t)}{\omega}\right]_{-2}^{1} + \left[\frac{-\cos(\omega t)}{\omega}\right]_1^3 \\ &= \frac{2}{\omega}\left(\cos(2\omega) - \cos(\omega)\right) + \frac{1}{\omega}\left(\cos(\omega) - \cos(3\omega)\right) \\ &= \frac{1}{\omega}\left(2\cos(2\omega) - \cos(\omega) - \cos(3\omega)\right). \end{aligned}$$

Therefore,

$$\begin{aligned} f(x) &= \frac{1}{\pi\omega}\int_0^\infty \left[(\sin(\omega) + 2\sin(2\omega) + \sin(3\omega))\cos(\omega x)\right. \\ &\qquad \left. + (2\cos(2\omega) - \cos(\omega) - \cos(3\omega))\sin(\omega x)\right] d\omega. \end{aligned}$$

3. *$f(x)$ satisfies $(P_1)-(P_3)$. So, we have,*

$$A(\omega) = \int_{-\infty}^{-2} 0.\cos(\omega t)dt + \int_{-2}^{0} \cos(t).\cos(\omega t)dt + \int_0^2 \sin(t).\cos(\omega t)dt$$

$$
\begin{aligned}
& + \int_{2}^{+\infty} 0.\cos(\omega t)dt \\
&= \int_{-2}^{0} \cos(t)\cos(\omega t)dt + \int_{0}^{2} \sin(t)\cos(\omega t)dt = (I) + (II). \\
(I) &= \int_{-2}^{0} \cos(t)\cos(\omega t)dt = \left[\cos(t)\frac{\sin(\omega t)}{\omega}\right]_{-2}^{0} - \frac{1}{\omega}\int_{-2}^{0} -\sin(t)\sin(\omega t)dt \\
&= \frac{1}{\omega}\cos(2)\sin(2\omega) + \frac{1}{\omega}\left[\left[\sin(t)\frac{-\cos(\omega t)}{\omega}\right]_{-2}^{0} + \frac{1}{\omega}\int_{-2}^{0}\cos(t)\cos(\omega t)dt\right] \\
&= \frac{1}{\omega}\cos(2)\sin(2\omega) + \frac{1}{\omega^2}\left[-\sin(2)\cos(2\omega) + \int_{-2}^{0}\cos(t)\cos(\omega t)dt\right].
\end{aligned}
$$

Then, we obtain,

$$
\begin{aligned}
(I) &= \int_{-2}^{0} \cos(t)\cos(\omega t)dt = \frac{1}{\omega^2 - 1}\left[\omega\cos(2)\sin(2\omega) - \sin(2)\cos(2\omega)\right]. \\
(II) &= \int_{0}^{2} \sin(t)\cos(\omega t)dt = \left[\sin(t)\frac{\sin(\omega t)}{\omega}\right]_{0}^{2} - \frac{1}{\omega}\int_{0}^{2}\cos(t)\sin(\omega t)dt \\
&= \frac{1}{\omega}\sin(2)\sin(2\omega) - \frac{1}{\omega}\left[\left[\cos(t)\frac{-\cos(\omega t)}{\omega}\right]_{0}^{2} - \frac{1}{\omega}\int_{0}^{2}\sin(t)\cos(\omega t)dt\right] \\
&= \frac{1}{\omega}\sin(2)\sin(2\omega) + \frac{1}{\omega^2}\left[\cos(2)\cos(2\omega) - 1 + \int_{0}^{2}\sin(t)\cos(\omega t)dt\right].
\end{aligned}
$$

Then, we obtain,

$$
(II) = \int_{0}^{2} \sin(t)\cos(\omega t)dt = \frac{1}{\omega^2 - 1}\left[\omega\sin(2)\sin(2\omega) + \cos(2)\cos(2\omega) - 1\right].
$$

Therefore,

$$
\begin{aligned}
A(\omega) &= \frac{1}{\omega^2 - 1}\left[\omega\cos(2)\sin(2\omega) - \sin(2)\cos(2\omega) + \omega\sin(2)\sin(2\omega)\right. \\
&\quad \left. + \cos(2)\cos(2\omega) - 1\right] \\
&= \frac{1}{2(\omega^2 - 1)}\left[\omega\left(\sin(2(1+\omega)) - \sin(2(1-\omega)) - \cos(2(1+\omega))\right.\right. \\
&\quad \left. + \cos(2(1-\omega))\right) + \sin(2(1+\omega)) + \sin(2(1-\omega)) + \cos(2(1+\omega)) \\
&\quad \left. + \cos(2(1+\omega)) - 2\right].
\end{aligned}
$$

$$\begin{aligned}
B(\omega) &= \int_{-\infty}^{-2} 0.\sin(\omega t)dt + \int_{-2}^{0} \cos(t).\sin(\omega t)dt + \int_{0}^{2} \sin(t).\sin(\omega t)dt \\
&\quad + \int_{2}^{+\infty} 0.\sin(\omega t)dt \\
&= \int_{-2}^{0} \cos(t)\sin(\omega t)dt + \int_{0}^{2} \sin(t)\sin(\omega t)dt = (III) + (IV). \\
(III) &= \int_{-2}^{0} \cos(t)\sin(\omega t)dt = \left[\cos(t)\frac{-\cos(\omega t)}{\omega}\right]_{-2}^{0} - \frac{1}{\omega}\int_{-2}^{0} \sin(t)\cos(\omega t)dt \\
&= \frac{1}{\omega}\left(-1+\cos(2)\cos(2\omega)\right) - \frac{1}{\omega}\left[\left[\sin(t)\frac{\sin(\omega t)}{\omega}\right]_{-2}^{0} - \frac{1}{\omega}\int 0_{-2}\cos(t)\sin(\omega t)dt\right] \\
&= \frac{1}{\omega}\left(-1+\cos(2)\cos(2\omega)\right) + \frac{1}{\omega^2}\left[\sin(2)\sin(2\omega) + \int_{-2}^{0} \cos(t)\sin(\omega t)dt\right]
\end{aligned}$$

Then, we obtain,

$$\begin{aligned}
(III) &= \int_{-2}^{0} \cos(t)\sin(\omega t)dt = \frac{1}{\omega^2-1}\left[\omega\left(\cos(2)\cos(2\omega)-1\right) + \sin(2)\sin(2\omega)\right]. \\
(IV) &= \int_{0}^{2} \sin(t)\sin(\omega t)dt = \left[\sin(t)\frac{-\cos(\omega t)}{\omega}\right]_{0}^{2} + \frac{1}{\omega}\int_{0}^{2} \cos(t)\cos(\omega t)dt \\
&= -\frac{1}{\omega}\sin(2)\cos(2\omega) + \frac{1}{\omega}\left[\left[\cos(t)\frac{\sin(\omega t)}{\omega}\right]_{0}^{2} + \frac{1}{\omega}\int_{0}^{2} \sin(t)\sin(\omega t)dt\right] \\
&= -\frac{1}{\omega}\sin(2)\cos(2\omega) + \frac{1}{\omega^2}\left[\cos(2)\sin(2\omega) + \int_{0}^{2} \sin(t)\sin(\omega t)dt\right].
\end{aligned}$$

Then, we obtain,

$$(IV) = \int_{0}^{2} \sin(t)\sin(\omega t)dt = \frac{1}{\omega^2-1}\left[-\omega\sin(2)\cos(2\omega) + \cos(2)\sin(2\omega)\right].$$

Therefore,

$$\begin{aligned}
B(\omega) &= \frac{1}{\omega^2-1}\left[\omega\left(\cos(2)\cos(2\omega) - \sin(2)\cos(2\omega) - 1\right) + \sin(2)\sin(2\omega)\right. \\
&\quad \left. + \cos(2)\sin(2\omega)\right]. \\
&= \frac{1}{2(\omega^2-1)}\left[\omega\left(\cos(2(1+\omega)) + \cos(2(1-\omega)) - \sin(2(1+\omega))\right)\right.
\end{aligned}$$

$$-\sin(2(1-\omega))-2)-\cos(2(1+\omega))+\cos(2(1-\omega))+\sin(2(1+\omega))$$
$$-\sin(2(1-\omega))].$$

Finally, we get,

$$\begin{aligned}
f(x) &= \frac{1}{2\pi(\omega^2-1)}\int_0^{\infty}[[\omega\,(\sin(2(1+\omega))-\sin(2(1-\omega))-\cos(2(1+\omega))\\
&\quad+\cos(2(1-\omega)))+\sin(2(1+\omega))+\sin(2(1-\omega))+\cos(2(1+\omega))\\
&\quad+\cos(2(1+\omega))-2]\cos(\omega x)+[\omega\,(\cos(2(1+\omega))+\cos(2(1-\omega))\\
&\quad-\sin(2(1+\omega))+\sin(2)\sin(2\omega)-\sin(2(1-\omega))-2)-\cos(2(1+\omega))\\
&\quad+\cos(2(1-\omega))+\sin(2(1+\omega)-\sin(2(1-\omega))]\sin(\omega x)]\,d\omega,\omega\neq 1.
\end{aligned}$$

4. $f(x)$ satisfies $(P_1)-(P_3)$. So, we have,

$$\begin{aligned}
A(\omega) &= \int_{-\infty}^{-1} 0.\cos(\omega t)dt+\int_{-1}^{0} e^t.\cos(\omega t)dt+\int_0^1 e^{-t}.\cos(\omega t)dt\\
&\quad+\int_1^{+\infty} 0.\cos(\omega t)dt\\
&= \int_{-1}^{0} e^t\cos(\omega t)dt+\int_0^1 e^{-t}\cos(\omega t)dt=(1)+(2).\\
(1) &= \int_{-1}^{0} e^t\cos(\omega t)dt=\left[e^t\cos(\omega t)\right]_{-1}^{0}+\omega\int_{-1}^{0} e^t\sin(\omega t)dt\\
&= 1-\frac{\cos(\omega)}{e}+\omega\left[\left[e^t\sin(\omega t)dt\right]_{-1}^{0}-\omega\int_{-1}^{0} e^t\cos(\omega t)dt\right]\\
&= 1-\frac{\cos(\omega)}{e}+\omega\left[\frac{\sin(\omega)}{e}-\omega\int_{-1}^{0} e^t\cos(\omega t)dt\right].
\end{aligned}$$

Then, we obtain,

$$\begin{aligned}
(1) &= \int_{-1}^{0} e^t\cos(\omega t)dt=\frac{1}{1+\omega^2}\left[1-\frac{\cos(\omega)}{e}+\omega\frac{\sin(\omega)}{e}\right].\\
(2) &= \int_0^1 e^{-t}\cos(\omega t)dt=\left[-e^{-t}\cos(\omega t)\right]_0^1-\omega\int_0^1 e^{-t}\sin(\omega t)dt\\
&= \frac{-\cos(\omega)}{e}+1-\omega\left[\left[-e^{-t}\sin(\omega t)\right]_0^1+\omega\int_0^1 e^{-t}\cos(\omega t)dt\right]
\end{aligned}$$

$$= \frac{-\cos(\omega)}{e} + 1 - \omega\left[\frac{-\sin(\omega)}{e} + \omega\int_0^1 e^{-t}\cos(\omega t)dt\right].$$

Then, we obtain,

$$(2) = \int_0^1 e^{-t}\cos(\omega t)dt = \frac{1}{\omega^2+1}\left[1 - \frac{\cos(\omega)}{e} + \omega\frac{\sin(\omega)}{e}\right]$$

Therefore,

$$A(\omega) = \frac{2}{\omega^2+1}\left[1 - \frac{\cos(\omega)}{e} + \omega\frac{\sin(\omega)}{e}\right]$$

$$\begin{aligned} B(\omega) &= \int_{-\infty}^{-1} 0.\sin(\omega t)dt + \int_{-1}^{0} e^t.\sin(\omega t)dt + \int_0^1 e^{-t}.\sin(\omega t)dt \\ &\quad + \int_1^{+\infty} 0.\sin(\omega t)dt \\ &= \int_{-1}^0 e^t\sin(\omega t)dt + \int_0^1 e^{-t}\sin(\omega t)dt = (3) + (4). \end{aligned}$$

$$\begin{aligned} (3) &= \int_{-1}^0 e^t\sin(\omega t)dt = \left[e^t\sin(\omega t)\right]_{-1}^0 - \omega\int_{-1}^0 e^t\cos(\omega t)dt \\ &= \frac{\sin(\omega)}{e} - \omega\left[\left[e^t\cos(\omega t)\right]_{-1}^0 + \omega\int_{-1}^0 e^t\sin(\omega t)dt\right] \\ &= \frac{\sin(\omega)}{e} - \omega\left[1 - \frac{\cos(\omega)}{e} + \omega\int_{-1}^0 e^t\sin(\omega t)dt\right] \end{aligned}$$

which gives,

$$(3) = \int_{-1}^0 e^t\sin(\omega t)dt = \frac{1}{\omega^2+1}\left[\frac{\sin(\omega)}{e} + \omega\frac{\cos(\omega)}{e} - \omega\right].$$

$$\begin{aligned} (4) &= \int_0^1 e^{-t}\sin(\omega t)dt = \left[-e^{-t}\sin(\omega t)\right]_0^1 + \omega\int_0^1 e^{-t}\cos(\omega t)dt \\ &= -\frac{\sin(\omega)}{e} + \omega\left[\left[-e^{-t}\cos(\omega t)\right]_0^1 - \omega\int_0^1 e^{-t}\sin(\omega t)dt\right] \\ &= -\frac{\sin(\omega)}{e} + \omega\left[1 - \frac{\cos(\omega)}{e} - \omega\int_0^1 e^{-t}\sin(\omega t)dt\right] \end{aligned}$$

Then, we obtain,

$$(4) \quad = \quad \int_0^1 e^{-t}\sin(\omega t)dt = \frac{1}{\omega^2+1}\left[-\frac{\sin(\omega)}{e} - \omega\frac{\cos(\omega)}{e} + \omega\right].$$

Therefore,

$$B(\omega) = 0.$$

Finally, we get,

$$f(x) \quad = \quad \frac{2}{\pi(\omega^2+1)}\int_0^\infty \left[1 - \frac{\cos(\omega)}{e} + \omega\frac{\sin(\omega)}{e}\right]\cos(\omega x)d\omega.$$

Solution of exercise 2.

Reminder : *If f satisfies :*

(P_1) : *$f(x)$ is piecewise continuous on every interval $[0,l]$.*

(P_2) : *$f(x)$ is absolutely integrable on $[0,\infty[$, that is $\int_{-\infty}^{+\infty}|f(x)|dx$ converges.*

(P_3) : *At every $x \in (0,\infty)$, $f(x)$ has left and right hand derivatives.*

- *The Fourier cosine integral representation of an even function $f(x)$ satisfied $(P_1)-(P_3)$ on $[0,\infty)$ is given by,*
$$f(x) = \frac{1}{\pi}\int_0^\infty A(\omega)\cos(\omega x)d\omega, \tag{4.4}$$
where $A(\omega)$ is defined by,
$$A(\omega) = 2\int_0^\infty f(t)\cos(\omega t)dt. \tag{4.5}$$

- *The Fourier sine integral representation of an odd function $f(x)$ satisfied $(P_1)-(P_3)$ on $[0,\infty)$ is given by,*
$$f(x) = \frac{1}{\pi}\int_0^\infty B(\omega)\sin(\omega x)d\omega, \tag{4.6}$$
where $B(\omega)$ is defined by,
$$B(\omega) = 2\int_0^\infty f(t)\sin(\omega t)dt. \tag{4.7}$$

- *The Complex form of Fourier integral representation of a function f is given by,*

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} c(\omega)e^{-i\omega x}d\omega, \tag{4.8}$$

where $c(\omega)$ is defined by,

$$c(\omega) = \int_{-\infty}^{+\infty} f(t)e^{i\omega t}dt. \tag{4.9}$$

Otherwise, if x is a point of discontinuity, $f(x)$ in (4.4), (4.6) and (4.8) is replaced by $(f(x^+) + f(x^-))/2$.

Solution.

1. *We have,*

$$\int_0^{\infty} |f(x)|\, dx = \int_0^5 x^2 dx = \left[\frac{x^3}{3}\right]_0^5 = \frac{125}{3}.$$

Therefore, the improper integral $\displaystyle\int_0^{\infty}|f(x)|dx$ exists and $f(x)$ satisfies the previous hypothesis. To obtain the Fourier cosine integral representation, we compute $A(\omega)$ as,

$$\begin{aligned}
A(\omega) &= 2\int_0^{\infty} f(t)\cos(\omega t)dt = 2\int_0^5 t^2\cos(\omega t)dt \\
&= 2\left[t^2\frac{\sin(\omega t)}{\omega}\right]_0^5 - \frac{4}{\omega}\int_0^5 t\sin(\omega t)dt \\
&= 50\frac{\sin(5\omega)}{\omega} - \frac{4}{\omega}\left[\left[t\frac{-\cos(\omega t)}{\omega}\right]_0^5 + \frac{1}{\omega}\int_0^5 \cos(\omega t)dt\right] \\
&= 50\frac{\sin(5\omega)}{\omega} - \frac{4}{\omega}\left[-5\frac{\cos(5\omega)}{\omega} + \frac{1}{\omega}\left[\frac{\sin(\omega t)}{\omega}\right]_0^5\right] \\
&= 50\frac{\sin(5\omega)}{\omega} + 20\frac{\cos(5\omega)}{\omega^2} - 4\frac{\sin(5\omega)}{\omega^3} \\
&= \frac{2}{\omega^3}\left[\left(25\omega^2 - 2\right)\sin(5\omega) + 10\omega\cos(5\omega)\right].
\end{aligned}$$

Therefore,

$$f(t) = \frac{2}{\pi}\int_0^{\infty}\left[\left(25\omega^2 - 2\right)\sin(5\omega) + 10\omega\cos(5\omega)\right]\frac{\cos(\omega x)}{\omega^3}d\omega.$$

2. *We have,*

$$\int_0^{\infty} |f(x)|\, dx = \int_0^{\pi} sin(x)dx = [-\cos(x)]_0^{\pi} = 2.$$

Therefore, the improper integral $\int_0^{\infty} |f(x)|dx$ *exists and* $f(x)$ *satisfies the previous hypothesis. To obtain the Fourier cosine integral representation, we compute* $A(\omega)$ *as,*

$$\begin{aligned}
A(\omega) &= 2\int_0^{\infty} f(t)\cos(\omega t)dt = 2\int_0^{\pi} \sin(t)\cos(\omega t)dt \\
&= 2\left[-\cos(t)\cos(\omega t)\right]_0^{\pi} - 2\omega \int_0^{\pi} \cos(t)\sin(\omega t)dt \\
&= 2\left[\cos(\omega\pi)+1\right] - 2\omega\left[\left[\sin(t)\sin(\omega t)\right]_0^{\pi} - \omega\int_0^{\pi}\sin(t)\cos(\omega t)dt\right] \\
&= \frac{2}{1-\omega^2}\left[\cos(\omega\pi)+1\right].
\end{aligned}$$

Therefore,

$$f(t) = \frac{2}{\pi}\int_0^{\infty} \frac{\cos(\omega\pi)+1}{1-\omega^2}\cos(\omega x)d\omega, \omega \neq 1.$$

3. *We have,*

$$\begin{aligned}
\int_0^{\infty} |f(x)|\, dx &= \int_0^{3} |\sinh(x)|\, dx = \int_0^{3} \left|\frac{e^x - e^{-x}}{2}\right| dx = \int_0^{3} \frac{e^x - e^{-x}}{2} dx \\
&= \left[\frac{e^x + e^{-x}}{2}\right]_0^3 = \cosh(3) - 1.
\end{aligned}$$

Therefore, the improper integral $\int_0^{\infty} |f(x)|dx$ *exists and* $f(x)$ *satisfies the previous hypothesis. To obtain the Fourier sine integral representation, we compute* $B(\omega)$ *as,*

$$\begin{aligned}
B(\omega) &= 2\int_0^{\infty} f(t)\sin(\omega t)dt = 2\int_0^{3}\sinh(t)\sin(\omega t)dt \\
&= 2\left[\sinh(t)\frac{-\cos(\omega t}{\omega}\right]_0^3 + \frac{2}{\omega}\int_0^3 \cosh(t)\cos(\omega t)dt \\
&= -\frac{2}{\omega}\sinh(3)\cos(3\omega) + \frac{2}{\omega}\left[\left[\cosh(t)\frac{\sin(\omega t)}{\omega}\right]_0^3 - \frac{1}{\omega}\int_0^3 \sinh(t)\sin(\omega t)dt\right]
\end{aligned}$$

$$\begin{aligned}
&= -\frac{2}{\omega}\sinh(3)\cos(3\omega) + \frac{2}{\omega^2}\cosh(3)\sin(3\omega) - \frac{2}{\omega^2}\int_0^3 \sinh(t)\sin(\omega t)dt \\
&= \frac{2}{\omega^2+1}\left[\cosh(3)\sin(3\omega) - \omega\sinh(3)\cos(3\omega)\right].
\end{aligned}$$

Therefore,

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{1}{\omega^2+1}\left[\cosh(3)\sin(3\omega) - \omega\sinh(3)\cos(3\omega)\right]\sin(\omega x)d\omega.$$

4. *We have,*

$$\int_0^\infty |f(x)|\,dx = \int_1^2 1.dx = [x]_1^2 = 1.$$

Therefore, the improper integral $\displaystyle\int_0^\infty |f(x)|dx$ *exists and* $f(x)$ *satisfies the previous hypothesis. To obtain the Fourier sine integral representation, we compute* $B(\omega)$ *as,*

$$\begin{aligned}
B(\omega) &= 2\int_0^\infty f(t)\sin(\omega t)dt = 2\int_1^2 \sin(\omega t)dt = 2\left[-\frac{\cos(\omega t)}{\omega}\right]_1^2 \\
&= -\frac{2}{\omega}\left[\cos(2\omega) - \cos(\omega)\right].
\end{aligned}$$

Therefore,

$$f(x) = -\frac{2}{\pi}\int_0^\infty \frac{\cos(2\omega)-\cos(\omega)}{\omega}\sin(\omega x)d\omega.$$

5. *We have,*

$$\begin{aligned}
c(\omega) &= \int_{-\infty}^{+\infty} f(t)e^{i\omega t}dt = \int_{-a}^{a}\cosh(t)e^{i\omega t}dt \\
&= \left[\cosh(t)\frac{e^{i\omega t}}{i\omega}\right]_{-a}^{a} - \frac{1}{i\omega}\int_{-a}^{a}\sinh(t)e^{i\omega t}dt \\
&= \frac{\cosh(a)}{i\omega}\left(e^{i\omega a} - e^{-i\omega a}\right) - \frac{1}{i\omega}\left[\left[\sinh(t)\frac{e^{i\omega t}}{i\omega}\right]_{-a}^{a} - \frac{1}{i\omega}\int_{-a}^{a}\cosh(t)e^{i\omega t}dt\right] \\
&= \frac{2\cosh(a)}{\omega}\left(\frac{e^{i\omega a}-e^{-i\omega a}}{2i}\right) + \frac{1}{\omega^2}\left[2\sinh(a)\left(\frac{e^{i\omega a}+e^{-i\omega a}}{2}\right) - \int_{-a}^{a}\cosh(t)e^{i\omega t}dt\right] \\
&= \frac{2}{\omega}\cosh(a)\sin(\omega a) + \frac{2}{\omega^2}\sinh(a)\cos(\omega a) - \frac{1}{\omega^2}\int_{-a}^{a}\cosh(t)e^{i\omega t}dt
\end{aligned}$$

$$= \frac{2}{\omega^2+1}\left[\omega\cosh(a)\sin(\omega a)+\sinh(a)\cos(\omega a)\right].$$

Therefore,

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\omega\cosh(a)\sin(\omega a)+\sinh(a)\cos(\omega a)}{\omega^2+1}e^{-i\omega x}d\omega.$$

6. *We have,*

$$\begin{aligned} c(\omega) &= \int_{-\infty}^{+\infty} f(t)e^{i\omega t}dt = \int_{-1}^{1}(1+t)e^{i\omega t}dt = \left[(1+t)\frac{e^{i\omega t}}{i\omega}\right]_{-1}^{1} - \frac{1}{i\omega}\int_{-1}^{1}e^{i\omega t}dt \\ &= 2\frac{e^{\omega i}}{i\omega} + \frac{1}{\omega^2}\left[e^{i\omega t}\right]_{-1}^{1} = 2\frac{e^{i\omega}}{i\omega} + \frac{2i}{\omega^2}\left[\frac{e^{i\omega}-e^{-i\omega}}{2i}\right] = \frac{2i}{\omega^2}\left[\sin(\omega)-\omega e^{i\omega}\right]. \end{aligned}$$

Therefore,

$$f(x) = \frac{i}{\pi}\int_{-\infty}^{+\infty}\frac{\sin(\omega)-\omega e^{i\omega}}{\omega^2}e^{-i\omega x}d\omega.$$

Solution of exercise 3.

Reminder :

- *Let $f(t)$ be piecewise continuous on $(-\infty,+\infty)$. Assume that $f(t)$ is absolutely convergent, that is $\int_{-\infty}^{+\infty}|f(t)|dt$ converges. The Fourier transform of $f(t)$ denoted by $\mathscr{F}[f(t)]$ is defined as,*

$$\mathscr{F}[f(t)] = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt = F(\omega). \tag{4.10}$$

- *Some useful properties of Fourier transform :*

  – ***Linearity.***
  If $\mathscr{F}[f(t)]$ and $\mathscr{F}[g(t)]$ are the Fourier transforms of $f(t)$ and $g(t)$ respectively, then,

$$\mathscr{F}[af(t)+bg(t)] = a\mathscr{F}[f(t)] + b\mathscr{F}[g(t)], \tag{4.11}$$

  where, a and b are constants.

- ***Time shifting.***
  If $\mathscr{F}[f(t)] = F(\omega)$ *and* t_0 *is any real number then,*

$$\mathscr{F}[f(t-t_0)] = F(\omega)e^{-i\omega t_0} \tag{4.12}$$

- ***Modulation.***
  If $\mathscr{F}[f(t)] = F(\omega)$ *and* ω_0 *is any real number, then,*

$$\begin{aligned} \mathscr{F}[f(t)\sin(\omega_0 t)] &= \frac{i}{2}\left[F(\omega+\omega_0) - F(\omega-\omega_0)\right], && (4.13)\\ \mathscr{F}[f(t)\cos(\omega_0 t)] &= \frac{1}{2}\left[F(\omega+\omega_0) + F(\omega-\omega_0)\right]. && (4.14) \end{aligned}$$

- ***Symmetry property of Fourier transforms.***
  Let $\mathscr{F}[f(t)] = F(\omega)$. *Then,*

$$\mathscr{F}[F(t)] = 2\pi f(-\omega). \tag{4.15}$$

Solution.

1. $H(t-3)e^{-4t}$. *We have,*

$$H(t-3) = u_{-3}(t) = \begin{cases} 1, & t \geq 3, \\ 0, & t < 3, \end{cases}, \textit{then,} \quad H(t-3)e^{-4t} = \begin{cases} e^{-4t}, & t \geq 3, \\ 0, & t < 3, \end{cases}$$

and, we obtain,

$$\begin{aligned} \mathscr{F}[H(t-3)e^{-4t}] &= \int_{-\infty}^{+\infty} H(t-3)e^{-4t}e^{-i\omega t}dt = \int_{3}^{+\infty} e^{-4t}e^{-i\omega t}dt \\ &= \int_{3}^{+\infty} e^{-(4+i\omega)t}dt = \left[-\frac{e^{-(4+i\omega)t}}{4+i\omega}\right]_3^{+\infty} = \frac{e^{-(4+i\omega)3}}{4+i\omega}. \end{aligned}$$

2. $e^{-a|t+1|}, \quad a > 0.$

$$\begin{aligned} \mathscr{F}[e^{-a|t+1|}] &= \int_{-\infty}^{+\infty} e^{-a|t+1|}e^{-i\omega t}dt = \int_{-\infty}^{-1} e^{a(t+1)}e^{-i\omega t}dt + \int_{-1}^{+\infty} e^{-a(t+1)}e^{-i\omega t}dt \\ &= e^{a}\int_{-\infty}^{-1} e^{(a-i\omega)t}dt + e^{-a}\int_{-1}^{+\infty} e^{-(a+i\omega)t}dt \end{aligned}$$

$$\begin{aligned} &= e^{a}\left[\frac{e^{(a-i\omega)t}}{a-i\omega}\right]_{-\infty}^{-1}+e^{-a}\left[-\frac{e^{-(a+i\omega)t}}{a+i\omega}\right]_{-1}^{+\infty},\, a-i\omega>0 \\ &= \frac{e^{i\omega}}{a-i\omega}+\frac{e^{i\omega}}{a+i\omega}=\frac{2ae^{i\omega}}{a^2+\omega^2}. \end{aligned}$$

3. $\frac{1}{1+t^2}$. *We know that* $\frac{1}{1+t^2}=\frac{1/2}{1+it}+\frac{1/2}{1-it}$, *then, by the linearity* (4.11) *of Fourier transform, we obtain,*

$$\mathscr{F}\left[\frac{1}{1+t^2}\right] = \mathscr{F}\left[\frac{1/2}{1+it}+\frac{1/2}{1-it}\right]=\frac{1}{2}\mathscr{F}\left[\frac{1}{1+it}\right]+\frac{1}{2}\left[\frac{1}{1-it}\right].$$

Now, we use the symmetry property (4.15) *of the Fourier transform, we know that* $\mathscr{F}[b(t)] = \mathscr{F}[H(t)e^{-t}]=\frac{1}{1+it}=B(t)$ *and* $\mathscr{F}[c(t)]=\mathscr{F}[H(-t)e^{t}]=\frac{1}{1-it}=C(t)$ *then,*

$$\begin{aligned} \mathscr{F}\left[\frac{1}{1+it}\right] &= 2\pi b(-\omega)=2\pi H(-\omega)e^{\omega}, \\ \mathscr{F}\left[\frac{1}{1-it}\right] &= 2\pi c(-\omega)=2\pi H(\omega)e^{-\omega}. \end{aligned}$$

Therefore,

$$\mathscr{F}\left[\frac{1}{1+t^2}\right]=\pi\left[H(-\omega)e^{\omega}+H(\omega)e^{-\omega}\right]=\pi e^{-|\omega|}.$$

4. $2e^{-3|t|}\sin(4t)$. *We use Frequency modulation theorem* (4.13) *with* $f(t)=2e^{-3|t|}$ *and* $\omega_0=4$, *then*

$$\begin{aligned} \mathscr{F}[2e^{-3|t|}] &= \int_{-\infty}^{0} 2e^{3t}e^{-i\omega t}dt+\int_{0}^{+\infty} 2e^{-3t}e^{-i\omega t}dt \\ &= 2\int_{-\infty}^{0} e^{(3-i\omega)t}dt+2\int_{0}^{+\infty} e^{-(3+i\omega)t}dt \\ &= 2\left[\frac{e^{(3-i\omega)t}}{3-i\omega}\right]_{-\infty}^{0}+2\left[-\frac{e^{-(3+i\omega)t}}{3+i\omega}\right]_{0}^{+\infty} \\ &= 2\left[\frac{1}{3-i\omega}+\frac{1}{3+i\omega}\right]=\frac{12}{9+\omega^2}. \end{aligned}$$

Therefore,

$$\mathscr{F}[2e^{-3|t|}\sin(4t)]=6i\left[\frac{1}{9+(\omega+4)^2}-\frac{1}{9+(\omega-4)^2}\right].$$

5. *Let* $f(t) = \begin{cases} -e^{\alpha t}, & t < 0, \\ e^{-\alpha t}, & t > 0, \alpha > 0. \end{cases}$ *Then,*

$$\begin{aligned}
\mathscr{F}[f(t)] &= \int_{-\infty}^{0} -e^{\alpha t} e^{-i\omega t} dt + \int_{0}^{+\infty} e^{-\alpha t} e^{-i\omega t} dt \\
&= -\int_{-\infty}^{0} e^{(\alpha - i\omega)t} dt + \int_{0}^{+\infty} e^{-(\alpha + i\omega)t} dt, \alpha - i\omega > 0 \\
&= -\left[\frac{e^{(\alpha - i\omega)t}}{\alpha - i\omega}\right]_{-\infty}^{0} + \left[-\frac{e^{-(\alpha + i\omega)t}}{\alpha + i\omega}\right]_{0}^{+\infty} \\
&= -\frac{1}{\alpha - i\omega} + \frac{1}{\alpha + i\omega} = -\frac{2i\omega}{\alpha^2 + \omega^2}.
\end{aligned}$$

6. *Let* $f(t) = \begin{cases} \frac{1}{n^2}(n + t), & -n < x < 0, \\ \frac{1}{n^2}(n - t), & 0 < x < n, \\ 0, & \textit{otherwise.} \end{cases}$ *Then,*

$$\begin{aligned}
\mathscr{F}[f(t)] &= \int_{-n}^{0} \frac{1}{n^2}(n + t) e^{-i\omega t} dt + \int_{0}^{n} \frac{1}{n^2}(n - t) e^{-i\omega t} dt \\
&= \frac{1}{n^2}\left[(n + t)\frac{e^{-i\omega t}}{-i\omega} \Big|_{-n}^{0} + \frac{1}{i\omega}\int_{-n}^{0} e^{-i\omega t} dt + (n - t)\frac{e^{-i\omega t}}{-i\omega} \Big|_{0}^{n} - \frac{1}{i\omega}\int_{0}^{n} e^{-i\omega t} dt\right] \\
&= \frac{1}{n^2}\left[-\frac{n}{i\omega} + \frac{1}{i\omega}\frac{e^{-i\omega t}}{-i\omega} \Big|_{-n}^{0} + \frac{n}{i\omega} - \frac{1}{i\omega}\frac{e^{-i\omega t}}{-i\omega} \Big|_{0}^{n}\right] \\
&= \frac{1}{i\omega n^2}\left[-\frac{1}{i\omega} + \frac{e^{i\omega n}}{i\omega} + \frac{e^{-i\omega n}}{i\omega} - \frac{1}{i\omega}\right] \\
&= \frac{1}{\omega^2 n^2}\left[1 - e^{i\omega n} - e^{-i\omega n} + 1\right] \\
&= \frac{2}{\omega^2 n^2}[1 - \frac{e^{i\omega n} + e^{-i\omega n}}{2}] \\
&= \frac{2}{\omega^2 n^2}[1 - \cos(\omega n)] \\
&= \frac{4}{\omega^2 n^2}\sin^2\left(\frac{\omega n}{2}\right).
\end{aligned}$$

7. $f(x) = \begin{cases} 1, & |t| \leq \frac{a}{2}, a > 0, \\ 0, & \textit{otherwise.} \end{cases}$ *Then,*

$$\begin{aligned} \mathscr{F}[f(t)] &= \int_{-\frac{a}{2}}^{\frac{a}{2}} 1.e^{-i\omega t}dt = \left[\frac{e^{-i\omega t}}{-i\omega}\right]_{-\frac{a}{2}}^{\frac{a}{2}} = \frac{1}{i\omega}\left[-e^{-i\omega\frac{a}{2}} + e^{i\omega\frac{a}{2}}\right] = \frac{2i}{i\omega}\frac{e^{i\omega\frac{a}{2}} - e^{-i\omega\frac{a}{2}}}{2i} \\ &= = \frac{2}{\omega}\sin\left(\frac{\omega a}{2}\right). \end{aligned}$$

8. $f(x) = \begin{cases} mt, & |t| \leq \frac{1}{m}, m > 0, \\ 0, & \textit{otherwise.} \end{cases}$ *Then,*

$$\begin{aligned} \mathscr{F}[f(t)] &= \int_{-\frac{1}{m}}^{\frac{1}{m}} mt.e^{-i\omega t}dt = mt\frac{e^{-i\omega t}}{-i\omega}\Big|_{-\frac{1}{m}}^{\frac{1}{m}} + \frac{m}{i\omega}\int_{-\frac{1}{m}}^{\frac{1}{m}} e^{-i\omega t}dt \\ &= \frac{-2}{i\omega}\left(\frac{e^{-i\omega\frac{1}{m}} + e^{i\omega\frac{1}{m}}}{2}\right) - \frac{2im}{\omega^2}\left(\frac{-e^{-i\omega\frac{1}{m}} + e^{i\omega\frac{1}{m}}}{2i}\right) \\ &= \frac{-2}{i\omega}\cos\left(\frac{\omega}{m}\right) - \frac{2im}{\omega^2}\sin\left(\frac{\omega}{m}\right). \end{aligned}$$

Solution of exercise 4.

Reminder :

- *Let $f(t)$ be piecewise continuous on $(-\infty, +\infty)$. Assume that $f(t)$ is absolutely convergent, that is $\int_{-\infty}^{+\infty} |f(t)|dt$ converges. The inverse Fourier transform of $F(\omega)$ denoted by $\mathscr{F}^{-1}[F(\omega)]$ is defined as,*

$$\mathscr{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{i\omega t}d\omega = f(t). \tag{4.16}$$

Solution.

1. $\frac{e^{-i\omega}}{2(1+i\omega)}$. *We have $\mathscr{F}[e^{-\alpha t}H(t)] = \frac{1}{\alpha + i\omega}, \alpha > 0$. Here $\alpha = 1$, then,*

$$\mathscr{F}^{-1}\left[\frac{1}{1+i\omega}\right] = e^{-t}H(t),$$

using the shift Theorem with $t_0 = 1$, we obtain,

$$\mathscr{F}^{-1}\left[\frac{e^{-i\omega}}{2(1+i\omega)}\right] = \frac{1}{2}\mathscr{F}^{-1}\left[\frac{e^{-i\omega}}{1+i\omega}\right] = \frac{1}{2}e^{-(t-1)}H(t-1).$$

2. $\frac{e^{-(1-i\omega)}}{3+i\omega}$. *Similar to the previous problem, with $\alpha = 3$ and by the shift theorem with $t_0 = -1$, we get,*

$$\mathscr{F}^{-1}\left[\frac{e^{-(1-i\omega)}}{3+i\omega}\right] = e^{-1}\mathscr{F}^{-1}\left[\frac{e^{i\omega}}{3+i\omega}\right] = e^{-1}e^{-3(t+1)}H(t+1) = e^{-(3t+4)}H(t+1).$$

3. $\frac{2i\omega}{3+i\omega}$. *We have $\frac{2i\omega}{3+i\omega} = 2 - \frac{6}{3+i\omega}$, then,*

$$\begin{aligned}\mathscr{F}^{-1}\left[\frac{2i\omega}{3+i\omega}\right] &= \mathscr{F}^{-1}\left[2 - \frac{6}{3+i\omega}\right] = 2\mathscr{F}^{-1}[1] - 6\mathscr{F}^{-1}\left[\frac{1}{3+i\omega}\right] \\ &= 2\delta(t) - 6e^{-3t}H(t),\end{aligned}$$

where $\delta(t)$ is defined as,

$$\delta(t) = \lim_{k\to 0}\left[u_0(t) - u_k(t)\right] = \lim_{k\to 0}[H(t) - H(t-k)] = \begin{cases} 0, & t < 0, \\ 1, & 0 \leq t < k, \\ 0, & t \geq k. \end{cases}$$

With $\mathscr{F}[\delta(t)] = 1$.

4. $\frac{e^{-2i\omega}}{4+\omega^2}$. *We have $\mathscr{F}[e^{-a|t|}] = \frac{2a}{a^2+w^2}, a > 0$. Here $a = 2$ and by the shift theorem with $t_0 = 2$, we get,*

$$\mathscr{F}^{-1}\left[\frac{e^{-2i\omega}}{4+\omega^2}\right] = \frac{1}{4}\mathscr{F}^{-1}\left[\frac{2.2}{2^2+\omega^2}e^{-2\omega}\right] = \frac{1}{4}e^{-2|t-2|}.$$

5. $\frac{1}{6+5i\omega-\omega^2}$. *We have $\frac{1}{6+5i\omega-\omega^2} = \frac{i}{2i-\omega} - \frac{i}{3i-\omega} = \frac{1}{2+i\omega} - \frac{1}{3+i\omega}$, then,*

$$\begin{aligned}\mathscr{F}^{-1}\left[\frac{1}{6+5i\omega-\omega^2}\right] &= \mathscr{F}^{-1}\left[\frac{1}{2+i\omega}\right] - \mathscr{F}^{-1}\left[\frac{1}{3+i\omega}\right] \\ &= e^{-2t}H(t) - e^{-3t}H(t) \\ &= e^{-2t}\left(1 - e^{-t}\right)H(t).\end{aligned}$$

6. $\frac{1}{a^4+\omega^4}, a > 0$. *We have,*

$$\begin{aligned}
\frac{1}{\omega^4+a^4} &= \frac{1}{(\omega^2+a^2)^2-2\omega^2a^2} = \frac{1}{((\omega+\frac{a}{\sqrt{2}})^2+(\frac{a}{\sqrt{2}})^2)((\omega-\frac{a}{\sqrt{2}})^2+(\frac{a}{\sqrt{2}})^2)}\\
&= \frac{\frac{1}{2\sqrt{2}a^3}\omega+\frac{1}{2a^2}}{(\omega+\frac{a}{\sqrt{2}})^2+(\frac{a}{\sqrt{2}})^2}+\frac{-\frac{1}{2\sqrt{2}a^3}\omega+\frac{1}{2a^2}}{(\omega-\frac{a}{\sqrt{2}})^2+(\frac{a}{\sqrt{2}})^2}\\
&= \frac{1}{2\sqrt{2}a^3}\left[\frac{\omega+\sqrt{2}a}{(\omega+\frac{a}{\sqrt{2}})^2+(\frac{a}{\sqrt{2}})^2}+\frac{-\omega+\sqrt{2}a}{(\omega-\frac{a}{\sqrt{2}})^2+(\frac{a}{\sqrt{2}})^2}\right]\\
&= \frac{1}{2\sqrt{2}a^3}\left[\frac{\frac{a}{\sqrt{2}}+(\omega+\frac{a}{\sqrt{2}})}{(\omega+\frac{a}{\sqrt{2}})^2+(\frac{a}{\sqrt{2}})^2}+\frac{\frac{a}{\sqrt{2}}-(\omega-\frac{a}{\sqrt{2}})}{(\omega-\frac{a}{\sqrt{2}})^2+(\frac{a}{\sqrt{2}})^2}\right]\\
&= \frac{1}{2\sqrt{2}a^3}\left[\left(\frac{\frac{a}{\sqrt{2}}}{(\omega+\frac{a}{\sqrt{2}})^2+(\frac{a}{\sqrt{2}})^2}+\frac{\frac{a}{\sqrt{2}}}{(\omega-\frac{a}{\sqrt{2}})^2+(\frac{a}{\sqrt{2}})^2}\right)\right.\\
&\quad\left.+\frac{1}{2i}\left(\frac{2i(\omega+\frac{a}{\sqrt{2}})}{(\omega+\frac{a}{\sqrt{2}})^2+(\frac{a}{\sqrt{2}})^2}-\frac{2i(\omega-\frac{a}{\sqrt{2}})}{(\omega-\frac{a}{\sqrt{2}})^2+(\frac{a}{\sqrt{2}})^2}\right)\right]
\end{aligned}$$

Now, we use the modulation theorem with $\omega_0 = \frac{a}{\sqrt{2}}$ *and* $\mathscr{F}^{-1}\left[\frac{\frac{a}{\sqrt{2}}}{(\frac{a}{\sqrt{2}})^2+\omega^2}\right] = \frac{e^{-\frac{a}{\sqrt{2}}|t|}}{2}$ *to get,*

$$\begin{aligned}
&\mathscr{F}^{-1}\left[\frac{\frac{a}{\sqrt{2}}}{(\omega+\frac{a}{\sqrt{2}})^2+(\frac{a}{\sqrt{2}})^2}+\frac{\frac{a}{\sqrt{2}}}{(\omega-\frac{a}{\sqrt{2}})^2+(\frac{a}{\sqrt{2}})^2}\right]\\
&= 2\left(\frac{e^{-\frac{a}{\sqrt{2}}|t|}}{2}\cos\left(\frac{a}{\sqrt{2}}t\right)\right) = e^{-\frac{a}{\sqrt{2}}|t|}\cos\left(\frac{a}{\sqrt{2}}t\right),
\end{aligned}$$

For the second equation, we use the modulation theorem with $\omega_0 = \frac{a}{\sqrt{2}}$ *and* $\mathscr{F}^{-1}\left[\frac{-2i\omega}{(\frac{a}{\sqrt{2}})^2+\omega^2}\right] =$ $y(t)$, *where* $y(t) = \begin{cases} -e^{\frac{a}{\sqrt{2}}t}, & t<0,\\ e^{-\frac{a}{\sqrt{2}}t}, & t>0,\end{cases}$ *to obtain,*

$$\mathscr{F}^{-1}\left[\frac{i}{2}\left(\frac{-2i(\omega+\frac{a}{\sqrt{2}})}{(\omega+\frac{a}{\sqrt{2}})^2+(\frac{a}{\sqrt{2}})^2}-\frac{-2i(\omega-\frac{a}{\sqrt{2}})}{(\omega-\frac{a}{\sqrt{2}})^2+(\frac{a}{\sqrt{2}})^2}\right)\right] = y(t)\sin\left(\frac{a}{\sqrt{2}}t\right)$$

Therefore,

$$\mathscr{F}^{-1}\left[\frac{1}{\omega^4+a^4}\right] = \frac{1}{2\sqrt{2}a^3}\left(e^{-\frac{a}{\sqrt{2}}|t|}\cos\left(\frac{a}{\sqrt{2}}t\right)+y(t)\sin\left(\frac{a}{\sqrt{2}}t\right)\right)$$

$$= \begin{cases} \frac{1}{2\sqrt{2}a^3}\left(\cos\left(\frac{a}{\sqrt{2}}t\right) - \sin\left(\frac{a}{\sqrt{2}}t\right)\right)e^{\frac{a}{\sqrt{2}}t}, & t < 0, \\ \\ \frac{1}{2\sqrt{2}a^3}\left(\cos\left(\frac{a}{\sqrt{2}}t\right) + \sin\left(\frac{a}{\sqrt{2}}t\right)\right)e^{-\frac{a}{\sqrt{2}}t}, & t < 0. \end{cases}$$

Solution of exercise 5.

Reminder :

- ***Frequency convolution.***

Let $f(t)$, $g(t)$ be piecewise continuous on every interval $[-l,l]$ and let $\int_{-\infty}^{+\infty}|f(t)|dt$, $\int_{-\infty}^{+\infty}|g(t)|dt$ converge. Denote $\mathscr{F}[f(t)] = F(\omega)$ and $\mathscr{F}[g(t)] = G(\omega)$. Then,

$$\mathscr{F}[f(t)g(t)] = \frac{1}{2\pi}[F * G](\omega), \tag{4.17}$$

where the convolution $(F * G)(t)$ is defined as,

$$(F * G)(t) = \int_{-\infty}^{+\infty} F(\tau)G(t-\tau)d\tau = (G * F)(t).$$

The inverse transform is given by,

$$\mathscr{F}^{-1}[F(\omega)G(\omega)] = (f * g)(t), \tag{4.18}$$

Solution.

1. $\int_{-\infty}^{+\infty} \frac{d\tau}{(2 - i\tau + i\omega)(2 + i\tau)}$. *Let $F(\tau) = \frac{1}{2+i\tau}$ and $G(\omega-\tau) = \frac{1}{2+i(\omega-\tau)}$, that is, $G(\omega) = \frac{1}{2+i\omega}$, then,*

$$\begin{aligned}(F * G)(\omega) &= \int_{-\infty}^{+\infty} \frac{1}{2+i\tau}.\frac{1}{2+i(\omega-\tau)}d\tau = 2\pi\mathscr{F}\left[f(t)g(t)\right] \\ &= 2\pi\mathscr{F}\left[\mathscr{F}^{-1}\left[\frac{1}{2+i\tau}\right].\mathscr{F}^{-1}\left[\frac{1}{2+i\omega}\right]\right] \\ &= 2\pi\mathscr{F}\left[e^{-2t}H(t).e^{-2t}H(t)\right] \\ &= 2\pi\mathscr{F}\left[e^{-4t}H(t)\right] \\ &= \frac{2\pi}{4+i\omega}.\end{aligned}$$

2. $\displaystyle\int_{-\infty}^{+\infty} \frac{d\tau}{(4-i\omega)(4-i\tau+i\omega)}$. *Let* $F(\tau)=\frac{1}{4-i\omega}$ *and* $G(\omega-\tau)=\frac{1}{4+i(\omega-\tau)}$, *that is,* $G(\omega)=\frac{1}{4+i\omega}$, *then,*

$$\begin{aligned}
(F*G)(\omega) &= \int_{-\infty}^{+\infty} \frac{1}{4-i\tau}.\frac{1}{4+i(\omega-\tau)}d\tau = 2\pi\mathscr{F}\left[f(t)g(t)\right]\\
&= 2\pi\mathscr{F}\left[\mathscr{F}^{-1}\left[\frac{1}{4-i\tau}\right].\mathscr{F}^{-1}\left[\frac{1}{4+i\omega}\right]\right]\\
&= 2\pi\mathscr{F}\left[e^{4t}H(-t).e^{-4t}H(t)\right]\\
&= 0.
\end{aligned}$$

Solution of exercise 6.

Reminder :

- ***Time convolution.***

Let $f(t)$, $g(t)$ *be piecewise continuous on every interval* $[-l,l]$ *and let* $\int_{-\infty}^{+\infty}|f(t)|dt$, $\int_{-\infty}^{+\infty}|g(t)|dt$ *converge. Denote* $\mathscr{F}[f(t)]=F(\omega)$ *and* $\mathscr{F}[g(t)]=G(\omega)$. *Then,*

$$\mathscr{F}[(f*g)(t)] \;=\; F(\omega)G(\omega). \tag{4.19}$$

The inverse transform is given by,

$$\mathscr{F}^{-1}[F(\omega)G(\omega)] \;=\; (f*g(t). \tag{4.20}$$

Solution.

1. $\frac{1}{(i\omega+k)^2}, k>0$. *Let* $F(\omega)=\frac{1}{i\omega+k}$ *and* $G(\omega)=\frac{1}{i\omega+k}$ *with* $f(t)=\mathscr{F}^{-1}\left[\frac{1}{i\omega+k}\right]=e^{-kt}H(t)$ *and* $g(t)=\mathscr{F}^{-1}\left[\frac{1}{i\omega+k}\right]=e^{-kt}H(t)$. *Then,*

$$\begin{aligned}
\mathscr{F}^{-1}[F(\omega)(G(\omega)] &= \mathscr{F}^{-1}\left[\frac{1}{i\omega+k}.\frac{1}{i\omega+k}\right]=(f*g)(t)\\
&= \int_{-\infty}^{+\infty} e^{-k\tau}H(\tau).e^{-k(t-\tau)}H(t-\tau)d\tau
\end{aligned}$$

$$
\begin{aligned}
&= e^{-kt}\int_{-\infty}^{+\infty} H(\tau).H(t-\tau)d\tau \\
&= e^{-kt}\left[\int_{-\infty}^{0} 0d\tau + \int_{0}^{t} 1d\tau + \int_{t}^{+\infty} 0d\tau\right] \\
&= te^{-kt}H(t).
\end{aligned}
$$

2. $\frac{1}{(i\omega+k)^3}, k > 0$. *Let* $F(\omega) = \frac{1}{i\omega+k}$ *and* $G(\omega) = \frac{1}{(i\omega+k)^2}$ *with* $f(t) = \mathscr{F}^{-1}\left[\frac{1}{i\omega+k}\right] = e^{-kt}H(t)$ *and* $g(t) = \mathscr{F}^{-1}\left[\frac{1}{(i\omega+k)^2}\right] = te^{-kt}H(t)$. *Then,*

$$
\begin{aligned}
\mathscr{F}^{-1}[F(\omega)(G(\omega)] &= \mathscr{F}^{-1}\left[\frac{1}{i\omega+k}.\frac{1}{(i\omega+k)^2}\right] = (f*g)(t) \\
&= \int_{-\infty}^{+\infty} e^{-k\tau}H(\tau).(t-\tau)e^{-k(t-\tau)}H(t-\tau)d\tau \\
&= \int_{-\infty}^{+\infty}\left[e^{-k\tau}H(\tau)te^{-k(t-\tau)}H(t-\tau) - e^{-k\tau}H(\tau)\tau e^{-k(t-\tau)}H(t-\tau)\right]d\tau \\
&= te^{-kt}\int_{-\infty}^{+\infty} H(\tau)H(t-\tau)d\tau - e^{-kt}\int_{-\infty}^{+\infty}\tau H(\tau)H(t-\tau)d\tau \\
&= te^{-kt}\int_{0}^{t} 1d\tau - e^{-kt}\int_{0}^{t}\tau.1d\tau \\
&= e^{-kt}\left(t^2 - \frac{t^2}{2}\right)H(t) \\
&= \frac{t^2}{2}e^{-kt}H(t).
\end{aligned}
$$

Solution of exercise 7.

Reminder :

- ***Fourier transforms of derivatives.***

Let $f(t)$ *be continuous and* $f^{(k)}(t)$, $k = 1, 2, \ldots, n$ *be piecewise continuous on every interval* $[-l, l]$ *and* $\int_{-\infty}^{+\infty}|f^{(k-1)}(t)|dt$, $k = 1, 2, \ldots, n$ *converge. Let* $f^{(k)}(t) \to 0$ *as* $t \to +\infty$ *for* $k = 0, 1, \ldots, n-1$. *If* $\mathscr{F}[f(t)] = F(\omega)$, *then*

$$
\mathscr{F}[f^{(n)}(t)] = (i\omega)^n F(\omega), \tag{4.21}
$$

where $f(t)$ and all its derivatives vanish at infinity.

Solution.

1. $y' - 4y = H(t)e^{-4t}$. *Applying the Fourier transform to the differential equation, we get*

$$\begin{aligned}
\mathscr{F}[y'] - \mathscr{F}[4y] &= \mathscr{F}[H(t)e^{-4t}] \\
i\omega Y(\omega) - 4Y(\omega) &= \frac{1}{4+i\omega}, \quad \textit{with} \quad \mathscr{F}[y(t)] = Y(\omega), \\
(i\omega - 4)Y(\omega) &= \frac{1}{4+i\omega} \\
Y(\omega) &= \frac{-1}{16+\omega^2}
\end{aligned}$$

Therefore,

$$y(t) = \mathscr{F}^{-1}\left[\frac{-1}{16+\omega^2}\right] = -\frac{1}{8}e^{-4|t|} = \begin{cases} -\frac{1}{8}e^{-4t}, & t > 0, \\ -\frac{1}{8}e^{4t}, & t < 0. \end{cases}$$

2. $y'' + 5y' + 4y = \delta(t-2)$. *Applying the Fourier transform to the differential equation, with* $\mathscr{F}[\delta(t-a)] = e^{-i\omega a}$ *we get*

$$\begin{aligned}
\mathscr{F}[y''] + \mathscr{F}[5y'] + \mathscr{F}[4y] &= \mathscr{F}[\delta(t-2)] \\
(i\omega)^2\mathscr{F}[y] + 5i\omega\mathscr{F}[y] + 4\mathscr{F}[y] &= e^{-2i\omega} \\
(i\omega)^2 Y(\omega) + 5i\omega Y(\omega) + 4Y(\omega) &= e^{-2i\omega} \\
\left(-\omega^2 + 5i\omega + 4\right) Y(\omega) &= e^{-2i\omega} \\
Y(\omega) &= \frac{e^{-2i\omega}}{-\omega^2 + 5i\omega + 4}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
y(t) &= \mathscr{F}^{-1}\left[\frac{e^{-2i\omega}}{-\omega^2+5i\omega+4}\right] = \mathscr{F}^{-1}\left[\frac{-e^{-2i\omega}}{(\omega-i)(\omega-4i)}\right] = \mathscr{F}^{-1}\left[\frac{e^{-2i\omega}}{3}\left(\frac{-i}{\omega-i} + \frac{i}{\omega-4i}\right)\right] \\
&= \frac{1}{3}\mathscr{F}^{-1}\left[\frac{e^{-2i\omega}}{i\omega+1} - \frac{e^{-2i\omega}}{i\omega+4}\right] = \frac{1}{3}\mathscr{F}^{-1}\left[\frac{e^{-2i\omega}}{i\omega+1}\right] - \frac{1}{3}\mathscr{F}^{-1}\left[\frac{e^{-2i\omega}}{i\omega+4}\right] \\
&= \frac{1}{3}\left(e^{-(t-2)} - e^{-4(t-2)}\right) H(t-2).
\end{aligned}$$

Solution of exercise 8.

- *Part I* :

Reminder :

Let $f(t)$ be a function defined for $t \geq 0$. The Laplace transform of $f(t)$ denoted by $\mathscr{L}[f(t)]$ is defined as,

$$\mathscr{L}[f(t)] = \int_0^{+\infty} e^{-st} f(t)dt = F(s). \tag{4.22}$$

Solution.

1. $2t-5$, *we have,*

$$\begin{aligned}\mathscr{L}[2t-5] = \int_0^{+\infty} e^{-st}.(2t-5)dt &= \left[\frac{-(2t-5)e^{-st}}{s}\right]_0^{+\infty} + \frac{2}{s}\int_0^{+\infty} e^{-st}dt \\ &= -\frac{5}{s} + \frac{2}{s}\left[-\frac{e^{-st}}{s}\right]_0^{+\infty} = -\frac{5}{s} + \frac{2}{s^2}, s > 0.\end{aligned}$$

2. $\cos(at+b)$, *we have,*

$$\begin{aligned}\mathscr{L}[\cos(at+b)] &= \int_0^{+\infty} e^{-st}.\cos(at+b)dt \\ &= \left[\cos(at+b)\frac{-e^{-st}}{s}\right]_0^{+\infty} - \frac{a}{s}\int_0^{+\infty} \sin(at+b)e^{-st}dt \\ &= \frac{\cos(b)}{s} - \frac{a}{s}\left[\left[\sin(at+b)\frac{-e^{-st}}{s}\right]_0^{+\infty} + \frac{a}{s}\int_0^{+\infty} \cos(at+b)e^{-st}dt\right] \\ &= \frac{1}{s}\left[\cos(a) - \frac{a}{s}\sin(b)\right] - \frac{a^2}{s^2}\int_0^{+\infty} \cos(at+b)e^{-st}dt.\end{aligned}$$

Therefore,

$$\mathscr{L}[\cos(at+b)] = \frac{1}{s^2+a^2}\left[s\cos(a) - a\sin(b)\right].$$

3. $\cos(t) - \sin(t)$, *we have,*

$$\mathscr{L}[\cos(t) - \sin(t)] = \int_0^{+\infty} \left(\cos(t) - \sin(t)\right).e^{-st}dt$$

$$\begin{aligned}
&= \left[(\cos(t)-\sin(t))\frac{-e^{-st}}{s}\right]_0^{+\infty} - \frac{1}{s}\int_0^{+\infty}(\sin(t)+\cos(t))e^{-st}dt \\
&= \frac{1}{s} - \frac{1}{s}\left[(\sin(t)+\cos(t))\frac{-e^{-st}}{s}\right]_0^{+\infty} - \frac{1}{s^2}\int_0^{+\infty}(\cos(t)-\sin(t))e^{-st}dt \\
&= \frac{1}{s} - \frac{1}{s^2} - \frac{1}{s^2}\int_0^{+\infty}(\cos(t)-\sin(t))e^{-st}dt.
\end{aligned}$$

Therefore,

$$\mathscr{L}[\cos(t)-\sin(t)] = -\frac{1-s}{1+s^2}.$$

4. te^{2t}, *we have,*

$$\begin{aligned}
\mathscr{L}[te^{2t}] = \int_0^{+\infty} e^{-st}.te^{2t}dt &= \int_0^{+\infty} te^{-(s-2)t}dt = \left[t\frac{-e^{-(s-2)t}}{s-2}\right]_0^{+\infty} + \frac{1}{s-2}\int_0^{+\infty} e^{-(s-2)t}dt \\
&= \frac{1}{s-2}\left[-\frac{e^{-(s-2)t}}{s-2}\right]_0^{+\infty} = \frac{1}{(s-2)^2}, \quad s>2.
\end{aligned}$$

5. t^2e^t, *we have,*

$$\begin{aligned}
\mathscr{L}[t^2e^t] = \int_0^{+\infty} e^{-st}.t^2e^tdt &= \int_0^{+\infty} t^2e^{-(s-1)t}dt = \left[t^2\frac{-e^{-(s-1)t}}{s-1}\right]_0^{+\infty} + \frac{2}{s-1}\int_0^{+\infty} te^{-(s-1)t}dt \\
&= \frac{2}{s-1}\left[\left[t\frac{-e^{-(s-1)t}}{s-1}\right]_0^{+\infty} + \frac{1}{s-1}\int_0^{+\infty} e^{-(s-1)t}dt\right] \\
&= \frac{2}{(s-1)^2}\left[-\frac{e^{-(s-1)t}}{s-1}\right]_0^{+\infty} = \frac{2}{(s-1)^3}, \quad s>1.
\end{aligned}$$

6. $e^{-t}\sin(t)$, *we have,*

$$\begin{aligned}
\mathscr{L}[e^{-t}\sin(t)] &= \int_0^{+\infty} e^{-st}.e^{-t}\sin(t)dt = \int_0^{+\infty} e^{-st}e^{-t}\sin(t)dt = \int_0^{+\infty} e^{-(s+1)t}\sin(t)dt \\
&= \left[\frac{-e^{(s+1)t}}{s+1}\sin(t)\right]_0^{+\infty} + \frac{1}{s+1}\int_0^{+\infty} e^{-(s+1)t}\cos(t)dt \\
&= \frac{1}{s+1}\left[\left[\frac{-e^{-(s+1)t}}{s+1}\cos(t)\right]_0^{+\infty} - \frac{1}{s+1}\int_0^{+\infty} e^{-(s+1)t}\sin(t)dt\right]
\end{aligned}$$

$$= \frac{1}{(s+1)^2} - \frac{1}{(s+1)^2}\int_0^{+\infty} e^{-(s+1)t}\sin(t)dt.$$

Therefore,

$$\mathscr{L}[e^{-t}\sin(t)] = \frac{1}{(s+1)^2+1}.$$

7. $6 - e^{3t}$, *by the linearity of Laplace transform, we have,*

$$\begin{aligned}\mathscr{L}[6-e^{3t}] &= \mathscr{L}[6] - \mathscr{L}[e^{3t}] = 6\mathscr{L}[1] - \mathscr{L}[e^{3t}] = 6\int_0^{+\infty} 1.e^{-st}dt - \int_0^{+\infty} e^{3t}.e^{-st}dt\\ &= 6\int_0^{+\infty} e^{-st}dt - \int_0^{+\infty} e^{-(s-3)t}dt = 6\left[-\frac{e^{-st}}{s}\right]_0^{+\infty} - \left[-\frac{e^{-(s-3)t}}{s-3}\right]_0^{+\infty}\\ &= \frac{6}{s} - \frac{1}{s-3}, \quad s>3.\end{aligned}$$

8. $3t^2 - 5e^{-2t} + 6$, *we have by the linearity of Laplace transform,*

$$\begin{aligned}\mathscr{L}[3t^2 - 5e^{-2t} + 6] &= 3\mathscr{L}[t^2] - 5\mathscr{L}[e^{-2t}] + 6\mathscr{L}[1]\\ &= 3\int_0^{+\infty} t^2.e^{-st}dt - 5\int_0^{+\infty} e^{-2t}.e^{-st}dt + 6\int_0^{+\infty} 1.e^{-st}dt\\ &= 3\frac{2}{s^2}\int_0^{+\infty} e^{-st}dt - 5\int_0^{+\infty} e^{-(s+2)t}dt + 6\left[-\frac{e^{-st}}{s}\right]_0^{+\infty}\\ &= \frac{6}{s^2}\left[-\frac{e^{-st}}{s}\right]_0^{+\infty} - 5\left[-\frac{e^{-(s+2)t}}{s+2}\right]_0^{+\infty} + \frac{6}{s}\\ &= \frac{6}{s^3} - \frac{5}{s+2} + \frac{6}{s}, s>0.\end{aligned}$$

9. $e^{-t}\sinh(t)$, *we have* $\sinh(t) = \frac{e^t - e^{-t}}{2}$, *then,* $e^{-t}\sinh(t) = \frac{1}{2}(1-e^{-2t})$ *and by the linearity of Laplace transform, we obtain,*

$$\begin{aligned}\mathscr{L}\left[\frac{1}{2}(1-e^{-2t})\right] &= \frac{1}{2}\mathscr{L}[1-e^{-2t}] = \frac{1}{2}\int_0^{+\infty}(1-e^{-2t}).e^{-st}dt = \frac{1}{2}\int_0^{+\infty}[e^{-st} - e^{-(s+2)t}]dt\\ &= \frac{1}{2}\left[-\frac{e^{-st}}{s} + \frac{e^{-(s+2)t}}{2+s}\right]_0^{+\infty} = \frac{1}{2}\left[\frac{1}{s} - \frac{1}{s+2}\right] = \frac{1}{s(s+2)}, s>0.\end{aligned}$$

10. $f(t) = \begin{cases} 2, & 0 \leq t < 3, \\ 0, & t \geq 3, \end{cases}$ *, we have,*

$$\mathscr{L}[f(t)] = \int_0^{+\infty} f(t)e^{-st}dt = \int_0^3 2.e^{-st}dt = 2\left[-\frac{e^{-st}}{s}\right]_0^3 = \frac{2}{s}\left[1 - e^{-3s}\right], s > 0.$$

11. $e^t \cosh(t)$*, we have* $\cosh(t) = \frac{e^t+e^{-t}}{2}$*, then,* $e^t\cosh(t) = \frac{1}{2}(e^{2t}+1)$ *and by the linearity of Laplace transform, we obtain,*

$$\begin{aligned} \mathscr{L}\left[\frac{1}{2}(e^{2t}+1)\right] &= \frac{1}{2}\mathscr{L}[e^{2t}+1] = \frac{1}{2}\int_0^{+\infty}(e^{2t}+1).e^{-st}dt = \frac{1}{2}\int_0^{+\infty}[e^{-st}+e^{-(s-2)t}]dt \\ &= \frac{1}{2}\left[-\frac{e^{-st}}{s} - \frac{e^{-(s-2)t}}{s-2}\right]_0^{+\infty} = \frac{1}{2}\left[\frac{1}{s}+\frac{1}{s-2}\right] = \frac{s-1}{s(s-2)}, \quad s > 2. \end{aligned}$$

12. $f(t) = \begin{cases} \cosh(t), & 0 \leq t < \pi, \\ 0, & t \geq \pi. \end{cases}$ *, we know that* $\cosh(t) = \frac{e^t+e^{-t}}{2}$*. Then, we have,*

$$\begin{aligned} \mathscr{L}[f(t)] &= \int_0^{+\infty} f(t)e^{-st}dt = \int_0^{\pi}\cosh(t)e^{-st}dt = \int_0^{\pi}\frac{e^t+e^{-t}}{2}e^{-st}dt \\ &= \frac{1}{2}\int_0^{\pi}[e^{(1-s)t}+e^{-(1+s)t}]dt = \frac{1}{2}\left[\frac{e^{(1-s)t}}{1-s} - \frac{e^{-(1+s)t}}{1+s}\right]_0^{\pi} \\ &= \frac{1}{2}\left[\frac{e^{(1-s)\pi}}{1-s} - \frac{e^{-(1+s)\pi}}{1+s} - \frac{1}{1-s} + \frac{1}{1+s}\right] \\ &= \frac{1}{2}\left[\frac{e^{(1-s)\pi}}{1-s} - \frac{e^{-(1+s)\pi}}{1+s} - \frac{2s}{1+s^2}\right] \\ &= \frac{1}{2}\left[\frac{e^{(1-s)\pi}}{1-s} - \frac{e^{-(1+s)\pi}}{1+s}\right] - \frac{s}{1+s^2}, s > 1. \end{aligned}$$

- *Part II* :

Reminder :

- *Let* $f(t)$*,* $t \geq 0$ *be a continuous function and be of exponential order on* $[0, +\infty)$*. Let* $f'(t)$ *be also of exponential order and at least piecewise continuous on* $[0, +\infty)$*. Then,*

$$\mathscr{L}[f'(t)] = s\mathscr{L}[f(t)] - f(0). \tag{4.23}$$

- *Let* $f(t), f'(t), \ldots, f^{(n-1)}(t)$ *be continuous on* $[0, +\infty)$ *and be of exponential order. Let* $f^{(n)}(t)$ *be at least piecewise continuous on* $[0, +\infty)$*. Then,*

$$\mathscr{L}[f^{(n)}(t)] = s^n \mathscr{L}[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \ldots - f^{(n-1)}(0). \tag{4.24}$$

Solution.

1. *Let* $f(t) = \cos^2(2t)$*. We have* $f(0) = 1$ *and* $f'(t) = -4\cos(2t)\sin(2t) = -2\sin(4t)$*. Then,*

$$\mathscr{L}[-2\sin(4t)] = s\mathscr{L}[\cos^2(2t)] - 1.$$

Therefore,

$$\begin{aligned}\mathscr{L}[\cos^2(2t)] &= \frac{1}{s}\mathscr{L}[-2\sin(4t)] + \frac{1}{s} = -\frac{2}{s}\mathscr{L}[\sin(4t)] + \frac{1}{s} = -\frac{2}{s}.\frac{4}{s^2+16} + \frac{1}{s} \\ &= -\frac{8}{s(s^2+16)} + \frac{1}{s} = \frac{-8+s^2+16}{s(s^2+16)} = \frac{s^2+8}{s(s^2+16)}, s > 0,\end{aligned}$$

with $\mathscr{L}[\sin(at)] = \frac{a}{s^2+a^2}$.

2. *Let* $f(t) = te^{at}$*. We have* $f(0) = 0$ *and* $f'(t) = e^{at} + ate^{at}$*. Then,*

$$\mathscr{L}[e^{at} + te^{at}] = s\mathscr{L}[te^{at}] - 0 = s\mathscr{L}[te^{at}].$$

Therefore,

$$\mathscr{L}[te^{at}] = \frac{1}{s-a}\mathscr{L}[e^{at}] = \frac{1}{s-a}.\frac{1}{s-a} = \frac{1}{(s-a)^2}, s > a,$$

with $\mathscr{L}[e^{at}] = \frac{1}{s-a}$.

3. *Let* $f(t) = t\sin(at)$*. We have* $f(0) = 0$*,* $f'(t) = \sin(at) + at\cos(at)$*,* $f'(0) = 0$ *and* $f''(t) = 2a\cos(at) - a^2t\sin(at)$*. Then,*

$$\mathscr{L}[2a\cos(at) - a^2t\sin(at)] = s^2\mathscr{L}[t\sin(at)] - sf(0) - f'(0) = s^2\mathscr{L}[t\sin(at)].$$

Therefore,

$$\mathscr{L}[t\sin(at)] = \frac{2a}{s^2+a^2}\mathscr{L}[\cos(at)] = \frac{2a}{s^2+a^2}.\frac{s}{s^2+a^2} = \frac{2as}{(s^2+a^2)^2},$$

with $\mathscr{L}[\cos(at)] = \frac{s}{s^2+a^2}$.

4. *Let* $f(t) = t\cos(at)$. *We have* $f(0) = 0$, $f'(t) = \cos(at) - at\sin(at)$, $f'(0) = 1$ *and* $f''(t) = -2a\sin(at) - a^2 t\cos(at)$. *Then,*

$$\mathscr{L}[-2a\sin(at) - a^2 t\cos(at)] = s^2\mathscr{L}[t\cos(at)] - sf(0) - f'(0) = s^2\mathscr{L}[t\cos(at)] - 1.$$

Therefore,

$$\begin{aligned}\mathscr{L}[t\cos(at)] &= -\frac{2a}{s^2+a^2}\mathscr{L}[\sin(at)] + \frac{1}{s^2+a^2} = -\frac{2a}{s^2+a^2}.\frac{a}{s^2+a^2} + \frac{1}{s^2+a^2} \\ &= -\frac{2a^2}{(s^2+a^2)^2} + \frac{1}{s^2+a^2} = \frac{-2a^2+s^2+a^2}{(s^2+a^2)^2} = \frac{s^2-a^2}{(s^2+a^2)^2},\end{aligned}$$

with $\mathscr{L}[\sin(at)] = \frac{a}{s^2+a^2}$.

5. *Let* $f(t) = t\sinh(at)$. *We have* $f(0) = 0$, $f'(t) = \sinh(at) + at\cosh(at)$, $f'(0) = 0$ *and* $f''(t) = 2a\cosh(at) + a^2 t\sinh(at)$. *Then,*

$$\mathscr{L}[2a\cosh(at) + a^2 t\sinh(at)] = s^2\mathscr{L}[t\sinh(at)] - sf(0) - f'(0) = s^2\mathscr{L}[t\sinh(at)].$$

Therefore,

$$\mathscr{L}[t\sinh(at)] = \frac{2a}{s^2-a^2}\mathscr{L}[\cosh(at)] = \frac{2a}{s^2-a^2}.\frac{s}{s^2-a^2} = \frac{2as}{(s^2-a^2)^2},$$

with $\mathscr{L}[\cosh(at)] = \mathscr{L}[\frac{e^{at}+e^{-at}}{2}] = \frac{s}{s^2-a^2}$.

- *Part III* :

Reminder :

- *Let* $\mathscr{L}[f(t)] = F(s)$, $s > \alpha \geq 0$ *and* a *be a real number. Then,*

$$\mathscr{L}[e^{at}f(t)] = F(s-a), s > a + \alpha \tag{4.25}$$

- *Let* $\mathscr{L}[f(t)] = F(s)$, $s > \alpha$ *and* $a \geq 0$ *be a real number. Then,*

$$\mathscr{L}[f(t-a)u_a(t)] = e^{-as}F(s). \tag{4.26}$$

Solution :

1. $(t^2-2t-3)e^{2t}$. *We know that,*

$$\mathscr{L}[t^2]=\frac{2}{s^3},\quad \mathscr{L}[t]=\frac{1}{s^2},\quad \mathscr{L}[1]=\frac{1}{s}.$$

Then, using the shifting theorem with $a=2$, *we obtain,*

$$\begin{aligned}\mathscr{L}[(t^2-2t-3)e^{2t}] &= \mathscr{L}[t^2e^{2t}]-2\mathscr{L}[te^{2t}]-3\mathscr{L}[e^{2t}]\\ &= \frac{2}{(s-2)^3}-\frac{2}{(s-2)^2}-\frac{3}{s-2}, s>2.\end{aligned}$$

2. t^5e^{-4t}. *We know that* $\mathscr{L}[t^5]=\frac{120}{s^6}$. *Then, using the shifting theorem with* $a=-4$, *we obtain,*

$$\mathscr{L}[t^5e^{-4t}]=\frac{120}{(s+4)^6}.$$

3. $t^3\cosh(3t)$. *We know that* $\cosh(3t)=\frac{e^{3t}+e^{-3t}}{2}$, *then,* $t^3\cosh(3t)=\frac{1}{2}[t^3e^{3t}+t^3e^{-3t}]$. *Now, we have* $\mathscr{L}[t^3]=\frac{6}{t^4}$ *and using the shifting theorem, we obtain,*

$$\begin{aligned}\mathscr{L}[t^3\cosh(3t)] &= \frac{1}{2}\left[\mathscr{L}[t^3e^{3t}]+\mathscr{L}[t^3e^{-3t}]\right]\\ &= \frac{1}{2}\left[\frac{6}{(s-3)^4}+\frac{6}{(s+3)^4}\right]\\ &= 3\left[\frac{1}{(s-3)^4}+\frac{1}{(s+3)^4}\right]\\ &= \frac{6(s^4+54s^2+81)}{(s^2-9)^4}, s>3.\end{aligned}$$

4. $\cosh(\omega t)\sin(\omega t)$. *We have* $\mathscr{L}[\sin(\omega t)]=\frac{\omega}{s^2+\omega^2}$. *Then, using the shifting theorem, we obtain,*

$$\begin{aligned}\mathscr{L}[\cosh(\omega t)\sin(\omega t)] &= \frac{1}{2}\mathscr{L}\left[e^{\omega t}\sin(\omega t)+e^{-\omega t}\sin(\omega t)\right]\\ &= \frac{1}{2}\left[\mathscr{L}[e^{\omega t}\sin(\omega t)]+\mathscr{L}[e^{-\omega t}\sin(\omega t)]\right]\\ &= \frac{1}{2}\left[\frac{\omega}{(s-\omega)^2+\omega^2}+\frac{\omega}{(s+\omega)^2+\omega^2}\right]\\ &= \frac{\omega(s^2+2\omega^2)}{s^4+4\omega^4}.\end{aligned}$$

5. $\sinh(t)\sin(t)$. *We know that* $\sinh(t) = \frac{e^t - e^{-t}}{2}$, *then,* $\sinh(t)\sin(t) = \frac{1}{2}[e^t \sin(t) - e^{-t}\sin(t)]$. *Therefore,*

$$\begin{aligned}
\mathscr{L}[\sinh(t)\sin(t)] &= \frac{1}{2}[\mathscr{L}[e^t \sin(t)] - \mathscr{L}[e^{-t}\sin(t)]] \\
&= \frac{1}{2}\left[\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right] \\
&= \frac{2s}{s^4+4}.
\end{aligned}$$

6. $e^t \sin(5t)$. *Using the shifting theorem, we obtain,*

$$\mathscr{L}[e^t \sin(5t)] = \frac{5}{(s-1)^2+25} = \frac{5}{s^2-2s+26}.$$

- *Part IV* :

Reminder :

- *Let* $f(t)$, $g(t)$ *be piecewise continuous functions on* $[0,+\infty)$ *and be of exponential orders. Then,*

$$\mathscr{L}[(f * g)(t)] = \mathscr{L}[f(t)]\mathscr{L}[g(t)] = F(s)G(s). \tag{4.27}$$

Solution :

1. $1 * e^{-2t}$. *We have,*

$$\mathscr{L}[1 * e^{-2t}] = = \mathscr{L}[1].\mathscr{L}[e^{-2t}] = \frac{1}{s}.\frac{1}{s+2} = \frac{1}{s(s+2)}, s > 0.$$

2. $t * e^{at}$. *We have,*

$$\mathscr{L}[t * e^{at}] = \mathscr{L}[t].\mathscr{L}[e^{at}] = \frac{1}{s^2}.\frac{1}{s-a} = \frac{1}{s^2(s-a)}, s > a.$$

3. $e^{at} * e^{bt}$. *We have,*

$$\mathscr{L}[e^{at} * e^{bt}] = \mathscr{L}[e^{at}].\mathscr{L}[e^{bt}] = \frac{1}{s-a}.\frac{1}{s-b} = \frac{1}{(s-a)(s-b)}, s > a, s > b.$$

4. $\sin(\omega t) * \sin(\omega t)$. *We have,*

$$\mathscr{L}[\sin(\omega t) * \cos(\omega t)] = \mathscr{L}[\sin(\omega t)].\mathscr{L}[\cos(\omega t)] = \frac{\omega}{s^2+\omega^2}.\frac{s}{s^2+\omega^2} = \frac{\omega s}{(s^2+\omega^2)^2}.$$

Solution of exercise 9.

Reminder :

Some useful inverse Laplace transform :

- $\mathscr{L}^{-1}[F(s)] = f(t)$.
- $\mathscr{L}^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t f(\tau)d\tau$.
- $\mathscr{L}^{-1}[F(s-a)] = e^{at}f(t), \quad s > a+\alpha$.
- $\mathscr{L}^{-1}[e^{as}F(s)] = f(t-a)u_a(t)$.
- $\mathscr{L}^{-1}[F(s)G(s)] = (f * g)(t)$.

Solution :

1. $\frac{2}{s+5}$, *we know that* $\mathscr{L}[e^{at}] = \frac{1}{s-a}$. *Here,* $a=-5$, *then,*

$$\mathscr{L}^{-1}\left[\frac{2}{s+5}\right] = 2\mathscr{L}^{-1}\left[\frac{1}{s+5}\right] = 2e^{-5t}.$$

2. $\frac{\pi}{s^2+\pi^2}$, *we know that* $\mathscr{L}[\sin(at)] = \frac{a}{s^2+a^2}$. *Here,* $a=\pi$, *then,*

$$\mathscr{L}^{-1}\left[\frac{\pi}{s^2+\pi^2}\right] = \sin(\pi t).$$

3. $\frac{6}{s^4}$, *we know that* $\mathscr{L}[t^n] = \frac{n!}{s^{n+1}}, n = 1,2,3,\ldots$. *Here,* $n=4$, *then,*

$$\mathscr{L}^{-1}\left[\frac{6}{s^4}\right] = \mathscr{L}^{-1}\left[\frac{3!}{s^4}\right] = t^3.$$

4. $\frac{s+3}{(s-1)(s+2)}$, *we have* $\frac{s+3}{(s-1)(s+2)} = \frac{1}{3}\left[\frac{4}{s-1} - \frac{1}{s+2}\right]$, *then,*

$$\begin{aligned}\mathscr{L}^{-1}\left[\frac{s+3}{(s-1)(s+2)}\right] &= \mathscr{L}^{-1}\left[\frac{1}{3}\left[\frac{4}{s-1} - \frac{1}{s+2}\right]\right] = \frac{1}{3}\mathscr{L}^{-1}\left[\frac{4}{s-1} - \frac{1}{s+2}\right] \\ &= \frac{1}{3}\left[\mathscr{L}^{-1}\left[4\frac{1}{s-1}\right] - \mathscr{L}^{-1}\left[\frac{1}{s+2}\right]\right] \\ &= \frac{1}{3}\left[4e^t - e^{-2t}\right].\end{aligned}$$

5. $\frac{13}{s^2-9}$, *we have* $\mathscr{L}[\sinh(\omega t)] = \frac{\omega}{s^2-\omega^2}$. *Here,* $\omega = 3$, *then,*

$$\mathscr{L}^{-1}\left[\frac{13}{s^2-9}\right] = 13\mathscr{L}^{-1}\left[\frac{1}{3}\frac{3}{s^2-3^2}\right] = \frac{13}{3}\mathscr{L}^{-1}\left[\frac{3}{s^2-3^2}\right] = \frac{13}{3}\sinh(3t).$$

6. $\frac{s^2+2s+5}{(s-1)(s-2)(s-3)}$, *we have* $\frac{s^2+2s+5}{(s-1)(s-2)(s-3)} = \frac{4}{s-1} - \frac{13}{s-2} + \frac{10}{s-3}$, *then,*

$$\begin{aligned}\mathscr{L}^{-1}\left[\frac{s^2+2s+5}{(s-1)(s-2)(s-3)}\right] &= 4\mathscr{L}^{-1}\left[\frac{1}{s-1}\right] - 13\mathscr{L}^{-1}\left[\frac{1}{s-2}\right] + 10\mathscr{L}^{-1}\left[\frac{1}{s-3}\right] \\ &= 4e^t - 13e^{2t} + 10e^{3t}.\end{aligned}$$

7. $\frac{1}{s^2+5s}$, *we have* $\frac{1}{s^2+5s} = \frac{1}{s}.\frac{1}{s+5}$ *with* $\mathscr{L}[e^{-5t}] = \frac{1}{s+5}$, *then,*

$$\mathscr{L}^{-1}\left[\frac{1}{s^2+5s}\right] = \mathscr{L}^{-1}\left[\frac{1}{s}.\frac{1}{s+5}\right] = \int_0^t e^{-5\tau}d\tau = \left[-\frac{e^{-5\tau}}{5}\right]_0^t = \frac{1}{5}[1 - e^{-5t}].$$

8. $\frac{16}{s^3+9s}$, *we have* $\frac{16}{s^3+9s} = \frac{1}{s}.\frac{1}{s^2+9}$ *with* $\mathscr{L}[\sin(3t)] = \frac{3}{s^2+9}$, *then,*

$$\begin{aligned}\mathscr{L}^{-1}\left[\frac{16}{s^3+9s}\right] = \frac{16}{3}\mathscr{L}^{-1}\left[\frac{1}{s}.\frac{3}{s^2+9}\right] = \frac{16}{3}\int_0^t \sin(3\tau)d\tau &= \frac{16}{3}\left[-\frac{\cos(3\tau}{3}\right]_0^t \\ &= \frac{16}{9}[1 - \cos(3t)].\end{aligned}$$

9. $\frac{\omega}{s^2(s^2+\omega^2)}$, *we have* $\mathscr{L}^{-1}\left[\frac{\omega}{s(s^2+\omega^2)}\right] = \frac{1}{\omega}(1 - \cos(\omega t))$, *then,*

$$\begin{aligned}\mathscr{L}^{-1}\left[\frac{\omega}{s^2(s^2+\omega^2)}\right] = \mathscr{L}^{-1}\left[\frac{1}{s}.\frac{\omega}{s(s^2+\omega^2}\right] = \frac{1}{\omega}\int_0^t (1 - \cos(\omega\tau)d\tau &= \frac{1}{\omega}\left[\tau - \frac{\sin(\omega\tau)}{\omega}\right]_0^t \\ &= \frac{1}{\omega^2}[\omega t - \sin(\omega t)].\end{aligned}$$

10. $\frac{s-a}{s^2(s^2+a^2)}$, *we have* $\frac{s-a}{s^2(s^2+a^2)} = \frac{1}{s(s^2+a^2)} - \frac{a}{s^2(s^2+a^2)}$ *with* $\mathscr{L}[\sin(at)] = \frac{a}{s^2+a^2}$, *then,*

$$\begin{aligned}
\mathscr{L}^{-1}\left[\frac{s-a}{s^2(s^2+a^2)}\right] &= \mathscr{L}^{-1}\left[\frac{1}{s(s^2+a^2)}\right] - \mathscr{L}^{-1}\left[\frac{a}{s^2(s^2+a^2)}\right] \\
&= \frac{1}{a}\int_0^t \sin(a\tau)d\tau - \frac{1}{a^2}[at - \sin(at)] \\
&= \frac{1}{a}\left[-\frac{\cos(a\tau)}{a}\right]_0^t - \frac{1}{a^2}[at - \sin(at)] \\
&= \frac{1}{a^2}[1-\cos(at)] - \frac{1}{a^2}[at - \sin(at)] \\
&= \frac{1}{a^2}[1-\cos(at)+\sin(at)] - \frac{t}{a}.
\end{aligned}$$

11. $\frac{1}{s^4+3s^3}$, *we find* $\mathscr{L}^{-1}\left[\frac{1}{s^4+3s^3}\right]$ *by steps.*

- $\mathscr{L}^{-1}\left[\frac{1}{s}.\frac{1}{(s+3)}\right]$, *we have* $\mathscr{L}\left[e^{-3t}\right] = \frac{1}{s+3}$, *then,*

$$\mathscr{L}^{-1}\left[\frac{1}{s}.\frac{1}{(s+3)}\right] = \int_0^t e^{-3\tau}d\tau = \left[-\frac{e^{-3\tau}}{3}\right]_0^t = \frac{1}{3}[1-e^{-3t}].$$

- $\mathscr{L}^{-1}\left[\frac{1}{s}.\frac{1}{s(s+3)}\right]$, *we have* $\mathscr{L}\left[\frac{1}{3}(1-e^{-3t})\right] = \frac{1}{s(s+3)}$, *then,*

$$\mathscr{L}^{-1}\left[\frac{1}{s}.\frac{1}{s(s+3)}\right] = \frac{1}{3}\int_0^t (1-e^{-3\tau})d\tau = \frac{1}{3}\left[\tau + \frac{e^{-3\tau}}{3}\right]_0^t = \frac{1}{9}[3t+e^{-3t}-1].$$

Therefore, we pass to $\mathscr{L}^{-1}\left[\frac{1}{s}.\frac{1}{s^2(s+3)}\right]$, *we have* $\mathscr{L}^{-1}\left[\frac{1}{s^2(s+1)}\right] = \frac{1}{9}[3t+e^{-3t}-1]$, *then,*

$$\begin{aligned}
\mathscr{L}^{-1}\left[\frac{1}{s}.\frac{1}{s^2(s+3)}\right] &= \frac{1}{9}\int_0^t (3\tau + e^{-3\tau} - 1)d\tau = \frac{1}{9}\left[\frac{3}{2}\tau^2 - \frac{e^{-3\tau}}{3} - \tau\right]_0^t \\
&= \frac{1}{18}(3t^2-2t) + \frac{1}{27}(1-e^{-3t}).
\end{aligned}$$

12. $\frac{1}{s^2+6s+15}$, *we have* $\frac{1}{s^2+6s+15} = \frac{1}{(s+3)^2+6}$, *then,*

$$\mathscr{L}^{-1}\left[\frac{1}{s^2+6s+15}\right] = \frac{1}{(s+3)^2+6} = \frac{1}{\sqrt{6}}\frac{\sqrt{6}}{(s+3)^2+(\sqrt{6})^2} = \frac{\sin(\sqrt{6}t)e^{-3t}}{\sqrt{6}}.$$

13. $\frac{s}{s^2+4s+8}$, *we have* $\frac{s}{s^2+4s+8} = \frac{s}{(s+2)^2+4}$, *then,*

$$\mathscr{L}^{-1}\left[\frac{s}{s^2+4s+8}\right] = \mathscr{L}^{-1}\left[\frac{s}{(s+2)^2+4}\right] = \mathscr{L}^{-1}\left[\frac{s}{(s+2)^2+2^2}\right] = \cos(2t)e^{-2t}.$$

14. $\frac{5s+6}{(s-1)^2}$, *we have* $\frac{5s+6}{(s-1)^2} = \frac{5(s-1)+11}{(s-1)^2}$, *then,*

$$\begin{aligned}\mathscr{L}^{-1}\left[\frac{5s+6}{(s-1)^2}\right] = \mathscr{L}^{-1}\left[\frac{5(s-1)+11}{(s-1)^2}\right] &= 5\mathscr{L}^{-1}\left[\frac{1}{(s-1)}\right] + 11\mathscr{L}^{-1}\left[\frac{1}{(s-1)^2}\right] \\ &= 5e^t + 11te^t \\ &= (5+11t)e^t.\end{aligned}$$

15. $\frac{(s+1)^2}{(s-2)^4}$, *we have* $\left(\frac{s+1}{(s-2)^2}\right)^2 = \left(\frac{(s-2)+3}{(s-2)^2}\right)^2 = \left(\frac{1}{s-2} + \frac{3}{(s-2)^2}\right)^2 = \frac{1}{(s-2)^2} + \frac{9}{(s-2)^4} + \frac{6}{(s-2)^3}$, *then,*

$$\begin{aligned}\mathscr{L}^{-1}\left[\frac{(s+1)^2}{(s-2)^4}\right] &= \mathscr{L}^{-1}\left[\frac{1}{(s-2)^2}\right] + 9\mathscr{L}^{-1}\left[\frac{1}{(s-2)^4}\right] + 6\mathscr{L}^{-1}\left[\frac{1}{(s-2)^3}\right] \\ &= \mathscr{L}^{-1}\left[\frac{1}{(s-2)^2}\right] + \frac{9}{3!}\mathscr{L}^{-1}\left[\frac{3!}{(s-2)^4}\right] + \frac{6}{2!}\mathscr{L}^{-1}\left[\frac{2!}{(s-2)^3}\right] \\ &= te^{2t} + \frac{3}{2}t^3e^{2t} + 3t^2e^{2t} \\ &= \left(2t + 6t^2 + 3t^3\right)\frac{e^{2t}}{2}.\end{aligned}$$

16. $\frac{1}{(s-a)(s-b)}$. *Let* $F(s) = \frac{1}{s-a}$ *and* $G(s) = \frac{1}{s-b}$, *then,* $f(t) = \mathscr{L}^{-1}[F(s)] = e^{at}$ *and* $g(t) = \mathscr{L}^{-1}[G(s)] = e^{bt}$. *Therefore,*

$$\begin{aligned}\mathscr{L}^{-1}\left[\frac{1}{(s-a)(s-b)}\right] = \mathscr{L}^{-1}\left[\frac{1}{s-a}.\frac{1}{s-b}\right] &= e^{at} * e^{bt} = \int_0^t e^{a\tau}e^{b(t-\tau)}d\tau \\ &= e^{bt}\int_0^t e^{(a-b)\tau}d\tau = e^{bt}\left[\frac{e^{(a-b)\tau}}{(a-b)}\right]_0^t \\ &= \frac{e^{bt}}{a-b}\left[e^{(a-b)t} - 1\right] \\ &= \frac{1}{a-b}[e^{at} - e^{bt}], a \neq b.\end{aligned}$$

17. $\frac{1}{s^2(s^2+16)}$. *Let* $F(s) = \frac{1}{s^2}$ *and* $G(s) = \frac{1}{s^2+16}$, *then,* $f(t) = \mathscr{L}^{-1}[F(s)] = t$ *and* $g(t) = \mathscr{L}^{-1}[G(s)] = \frac{1}{4}\sin(4t)$. *Therefore,*

$$\begin{aligned}
\mathscr{L}^{-1}\left[\frac{1}{s^2(s^2+16)}\right] &= \mathscr{L}^{-1}\left[\frac{1}{s^2}.\frac{1}{s^2+16}\right] = t * \frac{1}{4}\sin(4t) = \frac{1}{4}\int_0^t \tau\sin(4(t-\tau))d\tau \\
&= \frac{1}{4}\left[\left[\tau\frac{\cos(4(t-\tau))}{4}\right]_0^t - \frac{1}{4}\int_0^t \cos(4(t-\tau))d\tau\right] \\
&= \frac{1}{4}\left[\frac{t}{4} - \frac{1}{4}\left[-\frac{\sin(4(t-\tau))}{4}\right]_0^t\right] \\
&= \frac{1}{4}\left[\frac{t}{4} - \frac{1}{16}\sin(4t)\right] \\
&= \frac{1}{64}[4t - \sin(4t)].
\end{aligned}$$

18. $\frac{s}{(s^2+4)^2}$. *Let* $F(s) = \frac{s}{s^2+4}$ *and* $G(s) = \frac{1}{s^2+4}$, *then,* $f(t) = \mathscr{L}^{-1}[F(s)] = \cos(2t)$ *and* $g(t) = \mathscr{L}^{-1}[G(t)] = \frac{\sin(2t)}{2}$. *Therefore,*

$$\begin{aligned}
\mathscr{L}^{-1}\left[\frac{s}{(s^2+4)^2}\right] = \mathscr{L}^{-1}\left[\frac{s}{s^2+4}.\frac{1}{s^2+4}\right] &= \cos(2t) * \frac{\sin(2t)}{2} \\
&= \frac{1}{2}\int_0^t \cos(2\tau)\sin(2(t-\tau))d\tau \\
&= \frac{1}{4}\int_0^t [\sin(2t) - \sin(2(2\tau-t))]d\tau \\
&= \frac{1}{4}\left[\sin(2t)\tau + \frac{\cos(2(2\tau-t))}{4}\right]_0^t \\
&= \frac{1}{4}\sin(2t)t.
\end{aligned}$$

19. $\frac{6}{(s+1)^2(s+2)}$. *Let* $F(s) = \frac{6}{(s+1)^2}$ *and* $G(s) = \frac{1}{s+2}$, *then,* $f(t) = \mathscr{L}^{-1}[F(s)] = 6te^{-t}$ *and* $g(t) = \mathscr{L}^{-1}[G(s)] = e^{-2t}$. *Therefore,*

$$\begin{aligned}
\mathscr{L}^{-1}\left[\frac{6}{(s+1)^2(s+2)}\right] = \mathscr{L}^{-1}\left[\frac{6}{(s+1)^2}.\frac{1}{(s+2)}\right] &= 6te^{-t} * e^{-2t} \\
&= 6\int_0^t \tau e^{-\tau}e^{-2(t-\tau)}d\tau
\end{aligned}$$

$$
\begin{aligned}
&= 6e^{-2t}\int_0^t \tau e^{\tau} d\tau \\
&= 6e^{-2t}[\tau e^{\tau} - e^{\tau}]_0^t \\
&= 6e^{-2t}[te^{t} - e^{t} + 1] \\
&= 6[(t-1)e^{-t} + e^{-2t}].
\end{aligned}
$$

20. $\frac{1}{(s+a)^2(s+b)^2}$, $a \neq b$. *Let* $F(s) = \frac{1}{(s+a)^2}$ *and* $G(s) = \frac{1}{(s+b)^2}$*, then,* $f(t) = \mathscr{L}^{-1}[F(s)] = te^{-at}$ *and* $g(t) = \mathscr{L}^{-1}[G(s)] = te^{-bt}$*. Therefore,*

$$
\begin{aligned}
\mathscr{L}^{-1}\left[\frac{1}{(s+a)^2(s+b)^2}\right] &= \mathscr{L}^{-1}\left[\frac{1}{(s+a)^2}.\frac{1}{(s+b)^2}\right] = te^{-at} * te^{-bt} \\
&= \int_0^t \tau e^{-a\tau}(t-\tau)e^{-b(t-\tau)} d\tau \\
&= e^{-bt}\int_0^t \tau(t-\tau)e^{-(a-b)\tau} d\tau \\
&= e^{-bt}\left[\left[\tau(t-\tau)\frac{e^{-(a-b)\tau}}{-(a-b)}\right]_0^t + \frac{1}{a-b}\int_0^t (t-2\tau)e^{-(a-b)\tau} d\tau\right] \\
&= \frac{e^{-bt}}{a-b}\left[-\frac{t-2\tau}{a-b}e^{-(a-b)\tau} + \frac{2}{(a-b)^2}e^{-(a-b)\tau}\right]_0^t \\
&= \frac{e^{-bt}}{a-b}\left[\frac{t}{a-b}e^{-(a-b)t} + \frac{2}{(a-b)^2}e^{-(a-b)t} + \frac{t}{a-b} - \frac{2}{(a-b)^2}\right] \\
&= \frac{t}{(a-b)^2}e^{-at} + \frac{2}{(a-b)^3}e^{-at} + \frac{t}{(a-b)^2}e^{-bt} - \frac{2}{(a-b)^3}e^{-bt} \\
&= \frac{1}{(a-b)^3}\left[((a-b)t+2)e^{-at} + ((a-b)t-2)e^{-bt}\right].
\end{aligned}
$$

Solution of exercise 10.

1. $f(t) = t + 6\int_0^t f(\tau)e^{(t-\tau)} d\tau.$

   - *Taking the Laplace transform to the integral equation with* $F(s) = \mathscr{L}[f(t)]$

$$
\mathscr{L}[f(t)] = \mathscr{L}[t] + \mathscr{L}\left[6\int_0^t f(\tau)e^{(t-\tau)} d\tau\right]
$$

$$\begin{aligned} F(s) &= \frac{1}{s^2} + 6\mathscr{L}\left[\int_0^t f(\tau)e^{(t-\tau)}d\tau\right] \\ F(s) &= \frac{1}{s^2} + \frac{6}{s-1}F(s). \end{aligned}$$

Therefore,

$$F(s) = \frac{1}{s^2} + \frac{6}{s^2(s-7)}. \tag{4.28}$$

- *Taking the inverse Laplace transform to (4.28).*

$$\begin{aligned} \mathscr{L}^{-1}[F(s)] &= \mathscr{L}^{-1}\left[\frac{1}{s^2}\right] + \mathscr{L}^{-1}\left[\frac{6}{s^2(s-7)}\right] \\ f(t) &= t + \frac{6}{49}[e^{7t} - 7t - 1] \\ f(t) &= \frac{1}{49}[6e^{7t} + 7t - 6]. \end{aligned}$$

⋆ *We show here how we calculate* $\mathscr{L}^{-1}\left[\frac{6}{s^2(s-7)}\right]$.

$$\begin{aligned} \mathscr{L}^{-1}\left[\frac{1}{s(s-7)}\right] &= \mathscr{L}^{-1}\left[\frac{1}{s}.\frac{1}{(s-7)}\right] = \int_0^t e^{7\tau}d\tau = \left[\frac{e^{7\tau}}{7}\right]_0^t = \frac{1}{7}[e^{7t}-1]. \\ \mathscr{L}^{-1}\left[\frac{1}{s^2(s-7)}\right] &= \mathscr{L}^{-1}\left[\frac{1}{s}.\frac{1}{s(s-7)}\right] = \frac{1}{7}\int_0^t (e^{7\tau}-1)d\tau = \frac{1}{7}\left[\frac{e^{7\tau}}{7} - \tau\right]_0^t \\ &= \frac{1}{49}[e^{7t} - 7t - 1]. \end{aligned}$$

Therefore,

$$\mathscr{L}^{-1}\left[\frac{6}{s^2(s-7)}\right] = \frac{6}{49}[e^{7t} - 7t - 1].$$

2. $f(t) + \displaystyle\int_0^t f(\tau)\cos(t-\tau)d\tau = e^{-t}$.

   - *Taking the Laplace transform to the integral equation with* $F(s) = \mathscr{L}[f(t)]$.

$$\begin{aligned}\mathscr{L}[f(t)]+\mathscr{L}\left[\int_0^t f(\tau)\cos(t-\tau)d\tau\right] &= \mathscr{L}[e^{-t}]\\ F(s)+F(s)\frac{s}{s^2+1} &= \frac{1}{s+1}.\end{aligned}$$

Therefore,

$$\begin{aligned}F(s) &= \frac{1}{s+1}-\frac{s}{(s+1)(s^2+s+1)}\\ &= \frac{1}{s+1}-\frac{1}{s^2+s+1}-\frac{1}{(s+1)(s^2+s+1)}\\ &= \frac{1}{s+1}-\frac{1}{s^2+s+1}+\frac{1}{s+1}-\frac{s}{s^2+s+1}\\ &= \frac{2}{s+1}-\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}-\frac{s}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}. \end{aligned} \tag{4.29}$$

- *Taking the inverse Laplace transform to* (4.29).

$$\begin{aligned}\mathscr{L}^{-1}[F(s)] &= \mathscr{L}^{-1}\left[\frac{2}{s+1}\right]-\mathscr{L}^{-1}\left[\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right]-\mathscr{L}^{-1}\left[\frac{s}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right]\\ f(t) &= 2\mathscr{L}^{-1}\left[\frac{1}{s+1}\right]-\frac{2}{\sqrt{3}}\mathscr{L}^{-1}\left[\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]-\mathscr{L}^{-1}\left[\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]\\ &\quad+\frac{1}{\sqrt{3}}\mathscr{L}^{-1}\left[\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]\\ f(t) &= 2\mathscr{L}^{-1}\left[\frac{1}{s+1}\right]-\frac{1}{\sqrt{3}}\mathscr{L}^{-1}\left[\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]-\mathscr{L}^{-1}\left[\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]\\ f(t) &= 2e^{-t}-e^{-\frac{1}{2}t}\left[\frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)+\cos\left(\frac{\sqrt{3}}{2}t\right)\right].\end{aligned}$$

3. $f(t)=\cos(t)+e^{-t}\int_0^t f(\tau)e^{\tau}d\tau.$

- *Taking the Laplace transform to the integral equation with* $F(s) = \mathscr{L}[f(t)]$.

$$\begin{aligned}
\mathscr{L}[f(t)] &= \mathscr{L}[\cos(t)] + \mathscr{L}\left[\int_0^t f(\tau)e^{-(t-\tau)}d\tau\right] \\
F(s) &= \frac{s}{s^2+1} + \frac{1}{s+1}F(s).
\end{aligned}$$

Therefore,

$$F(s) = \frac{s+1}{s^2+1} \tag{4.30}$$

- *Taking the inverse Laplace transform to* (4.30).

$$\begin{aligned}
\mathscr{L}^{-1}[F(s)] &= \mathscr{L}^{-1}\left[\frac{s+1}{s^2+1}\right] \\
f(t) &= \mathscr{L}^{-1}\left[\frac{s}{s^2+1}\right] + \mathscr{L}^{-1}\left[\frac{1}{s^2+1}\right] \\
f(t) &= \cos(t) + \sin(t).
\end{aligned}$$

Solution of exercise 11.

1. $y' + 2y = \sin(t), \quad y(0) = 1.$

- *Taking the Laplace transform to the integral equation with* $Y(s) = \mathscr{L}[y(t)]$.

$$\begin{aligned}
\mathscr{L}[y'] + \mathscr{L}[2y] &= \mathscr{L}[\sin(t)] \\
s\mathscr{L}[y] - y(0) + 2\mathscr{L}[y] &= \frac{1}{s^2+1} \\
sY(s) - 1 + 2Y(s) &= \frac{1}{s^2+1}
\end{aligned}$$

Therefore,

$$Y(s) = \frac{s^2+2}{(s+2)(s^2+1)} \tag{4.31}$$

- *Taking the inverse Laplace transform to* (4.31).

$$\begin{aligned}
\mathscr{L}^{-1}[Y(s)] &= \mathscr{L}^{-1}\left[\frac{s^2+2}{(s+2)(s^2+1)}\right]\\
y(t) &= \mathscr{L}^{-1}\left[\frac{6}{5(s+2)} - \frac{s}{5(s^2+1)} + \frac{2}{5(s^2+1)}\right]\\
y(t) &= \frac{6}{5}\mathscr{L}^{-1}\left[\frac{1}{(s+2)}\right] - \frac{1}{5}\mathscr{L}^{-1}\left[\frac{s}{(s^2+1)}\right] + \frac{2}{5}\mathscr{L}^{-1}\left[\frac{1}{(s^2+1)}\right]\\
y(t) &= \frac{6}{5}e^{-2t} - \frac{1}{5}\cos(t) + \frac{2}{5}\sin(t).
\end{aligned}$$

2. $y'' - 6y' + 5y = e^{2t}, \quad y(0) = 1, \quad y'(0) = -1.$

- *Taking the Laplace transform to the integral equation with* $Y(s) = \mathscr{L}[y(t)]$*.*

$$\begin{aligned}
\mathscr{L}[y''] - \mathscr{L}[6y'] + \mathscr{L}[5y] &= \mathscr{L}[e^{2t}]\\
s^2\mathscr{L}[y] - sy(0) - y'(0) - 6[s\mathscr{L}[y] - y(0)] + 5\mathscr{L}[y] &= \frac{1}{s-2}\\
s^2Y(s) - s + 1 - 6sY(s) + 6 + 5Y(s) &= \frac{1}{s-2}\\
(s^2 - 6s + 5)Y(s) - s + 7 &= \frac{1}{s-2}
\end{aligned}$$

Therefore,

$$Y(s) = \left(\frac{1}{s-2} + s - 7\right)\frac{1}{s^2 - 6s + 5}. \tag{4.32}$$

- *Taking the inverse Laplace transform to* (4.32)*.*

$$\begin{aligned}
\mathscr{L}^{-1}[Y(s)] &= \mathscr{L}^{-1}\left[\left(\frac{1}{s-2} + s - 7\right)\frac{1}{s^2 - 6s + 5}\right]\\
y(t) &= \mathscr{L}^{-1}\left[\left(\frac{1}{s-2} + s - 7\right)\frac{1}{(s-5)(s-1)}\right]\\
y(t) &= \mathscr{L}^{-1}\left[\frac{1}{(s-2)(s-5)(s-1)}\right] + \mathscr{L}^{-1}\left[\frac{s}{(s-5)(s-1)}\right]\\
&\quad -\mathscr{L}^{-1}\left[\frac{7}{(s-5)(s-1)}\right]
\end{aligned}$$

$$
\begin{aligned}
y(t) &= \mathscr{L}^{-1}\left[\frac{-1}{3(s-2)}+\frac{1}{12(s-5)}+\frac{1}{4(s-1)}\right]+\mathscr{L}^{-1}\left[\frac{5}{4(s-5)}-\frac{1}{4(s-1)}\right] \\
&\quad -\mathscr{L}^{-1}\left[\frac{7}{4(s-5)}-\frac{7}{4(s-1)}\right] \\
y(t) &= -\frac{1}{3}\mathscr{L}^{-1}\left[\frac{1}{(s-2)}\right]-\frac{5}{12}\mathscr{L}^{-1}\left[\frac{1}{(s-5)}\right]+\frac{7}{4}\mathscr{L}^{-1}\left[\frac{1}{(s-1)}\right] \\
y(t) &= -\frac{1}{3}e^{2t}-\frac{5}{12}e^{5t}+\frac{7}{4}e^{t}.
\end{aligned}
$$

3. $y' + 6y + 5\int_0^t y(\tau)d\tau = 1 + t, \quad y(0) = 1.$

- *Taking the Laplace transform to the integral equation with* $Y(s) = \mathscr{L}[y(t)]$.

$$
\begin{aligned}
\mathscr{L}[y'] + \mathscr{L}[6y] + \mathscr{L}\left[5\int_0^t y(\tau)d\tau\right] &= \mathscr{L}[1+t] \\
s\mathscr{L}[y] - y(0) + 6\mathscr{L}[y] + \frac{5}{s}\mathscr{L}[y] &= \frac{1}{s}+\frac{1}{s^2} \\
sY(s) - 1 + 6Y(s) + \frac{5}{s}Y(s) &= \frac{1}{s}+\frac{1}{s^2}.
\end{aligned}
$$

Therefore,

$$
Y(s) = \left(1+\frac{1}{s}+s\right)\frac{1}{s^2+6s+5}. \tag{4.33}
$$

- *Taking the inverse Laplace transform to* (4.33).

$$
\begin{aligned}
\mathscr{L}^{-1}[Y(s)] &= \mathscr{L}^{-1}\left[\left(1+\frac{1}{s}+s\right)\frac{1}{s^2+6s+5}\right] \\
y(t) &= \mathscr{L}^{-1}\left[\left(1+\frac{1}{s}+s\right)\frac{1}{(s+1)(s+5)}\right] \\
y(t) &= \mathscr{L}^{-1}\left[\frac{1}{(s+1)(s+5)}\right]+\mathscr{L}^{-1}\left[\frac{1}{s(s+1)(s+5)}\right]+\mathscr{L}^{-1}\left[\frac{s}{(s+1)(s+5)}\right] \\
y(t) &= \mathscr{L}^{-1}\left[\frac{1}{4(s+1)}-\frac{1}{4(s+5)}\right]+\mathscr{L}^{-1}\left[\frac{1}{5s}-\frac{1}{4(s+1)}+\frac{1}{20(s+5)}\right] \\
&\quad +\mathscr{L}^{-1}\left[-\frac{1}{4(s+1)}+\frac{5}{4(s+5)}\right]
\end{aligned}
$$

$$\begin{aligned} y(t) &= -\frac{1}{4}\mathscr{L}^{-1}\left[\frac{1}{(s+1)}\right]+\frac{1}{5}\mathscr{L}^{-1}\left[\frac{1}{s}\right]+\frac{21}{20}\mathscr{L}^{-1}\left[\frac{1}{(s+5)}\right] \\ y(t) &= -\frac{1}{4}e^{-t}+\frac{1}{5}+\frac{21}{20}e^{-5t}. \end{aligned}$$

4. $y'+y=f(t), \quad y(0)=3, \quad f(t)=\begin{cases} 1, & 0\leq t<1, \\ -1, & t\geq 1. \end{cases}$

- *Taking the Laplace transform to the integral equation with* $Y(s)=\mathscr{L}[y(t)]$.

$$\begin{aligned} \mathscr{L}[y']+\mathscr{L}[y] &= \mathscr{L}[f(t)] \\ s\mathscr{L}[y]-y(0)+\mathscr{L}[y] &= \mathscr{L}[f(t)] \\ sY(s)-3+Y(s) &= \frac{-2e^{-s}}{s}+\frac{1}{s}. \end{aligned}$$

Therefore,

$$Y(s)=\frac{-2e^{-s}}{s(s+1)}+\frac{1}{s(s+1)}+\frac{3}{(s+1)} \tag{4.34}$$

- *Taking the inverse Laplace transform to* (4.34) .

$$\begin{aligned} \mathscr{L}^{-1}[Y(s)] &= -2\mathscr{L}^{-1}\left[\frac{e^{-s}}{s(s+1)}\right]+\mathscr{L}^{-1}\left[\frac{1}{s(s+1)}\right]+3\mathscr{L}^{-1}\left[\frac{1}{(s+1)}\right] \\ y(t) &= -2\mathscr{L}^{-1}\left[\frac{e^{-s}}{s}-\frac{e^{-s}}{(s+1)}\right]+\mathscr{L}^{-1}\left[\frac{1}{s}-\frac{1}{(s+1)}\right]+3\mathscr{L}^{-1}\left[\frac{1}{(s+1)}\right] \\ y(t) &= -2\mathscr{L}^{-1}\left[\frac{e^{-s}}{s}\right]+2\mathscr{L}^{-1}\left[\frac{e^{-s}}{(s+1)}\right]+\mathscr{L}^{-1}\left[\frac{1}{s}\right]-\mathscr{L}^{-1}\left[\frac{1}{(s+1)}\right] \\ &\quad +3\mathscr{L}^{-1}\left[\frac{1}{(s+1)}\right] \\ y(t) &= -2u_1(t)+2e^{-(t-1)}u_1(t)+u_0(t)-e^{-t}u_0(t)+3e^{-t}u_0(t) \\ y(t) &= -2(1-e^{-(t-1)})u_1(t)+(1+2e^{-t})u_0(t). \end{aligned}$$

5. $y''+5y'+6y=1-u_3(t)-u_5(t), \quad y(0)=0, \quad y'(0)=0.$

- *Taking the Laplace transform to the integral equation with* $Y(s)=\mathscr{L}[y(t)]$.

$$\begin{aligned}\mathscr{L}[y''] + \mathscr{L}[5y'] + \mathscr{L}[6y] &= \mathscr{L}[1] - \mathscr{L}[u_3(t)] - \mathscr{L}[u_5(t)]\\ s^2\mathscr{L}[y] - sy(0) - y'(0) + 5[s\mathscr{L}[y] - y(0)] + 6\mathscr{L}[y] &= \frac{1}{s} - \frac{e^{-3s}}{s} - \frac{e^{-5s}}{s}\\ s^2Y(s) + 5sY(s) + 6Y(s) &= \frac{1}{s} - \frac{e^{-3s}}{s} - \frac{e^{-5s}}{s}.\end{aligned}$$

Therefore,

$$Y(s) = \left[\frac{1}{s} - \frac{e^{-3s}}{s} - \frac{e^{-5s}}{s}\right]\frac{1}{(s+3)(s+2)}. \tag{4.35}$$

- *Taking the inverse Laplace transform to* (4.35) *.*

$$\begin{aligned}\mathscr{L}^{-1}[Y(s)] &= \mathscr{L}^{-1}\left[\frac{1}{s(s+3)(s+2)}\right] - \mathscr{L}^{-1}\left[\frac{e^{-3s}}{s(s+3)(s+2)}\right] - \mathscr{L}^{-1}\left[\frac{e^{-5s}}{s(s+3)(s+2)}\right]\\ y(t) &= \mathscr{L}^{-1}\left[\frac{\frac{1}{6}}{s} + \frac{\frac{1}{3}}{(s+3)} - \frac{\frac{1}{2}}{(s+2)}\right] - \mathscr{L}^{-1}\left[\frac{\frac{e^{-3s}}{6}}{s} + \frac{\frac{e^{-3s}}{3}}{(s+3)} - \frac{\frac{e^{-3s}}{2}}{(s+2)}\right]\\ &\quad -\mathscr{L}^{-1}\left[\frac{\frac{e^{-5s}}{6}}{s} + \frac{\frac{e^{-5s}}{3}}{(s+3)} - \frac{\frac{e^{-5s}}{2}}{(s+2)}\right]\\ &= \left[\frac{1}{6} + \frac{1}{3}e^{-3t} - \frac{1}{2}e^{-2t}\right]u_0(t) - \left[\frac{1}{6} + \frac{1}{3}e^{-3(t-3)} - \frac{1}{2}e^{-2(t-3)}\right]u_3(t)\\ &\quad - \left[\frac{1}{6} + \frac{1}{3}e^{-3(t-5)} - \frac{1}{2}e^{-2(t-5)}\right]u_5(t)\\ &= \frac{1}{6}[1 + 2e^{-3t} - 3e^{-2t}]u_0(t) - \frac{1}{6}\left[1 + 2e^{-3(t-3)} - 3e^{-2(t-3)}\right]u_3(t)\\ &\quad -\frac{1}{6}\left[1 + 2e^{-3(t-5)} - 3e^{-2(t-5)}\right]u_5(t).\end{aligned}$$

6. $y'' + 4y' + 5y = \delta(t-3), \quad y(0) = 0, \quad y'(0) = 0.$

- *Taking the Laplace transform to the integral equation with* $Y(s) = \mathscr{L}[y(t)]$*.*

$$\begin{aligned}\mathscr{L}[y''] + \mathscr{L}[4y'] + \mathscr{L}[5y] &= \mathscr{L}[\delta(t-3)]\\ s^2\mathscr{L}[y] - sy(0) - y'(0) + 4[s\mathscr{L}[y] - y(0)] + 5\mathscr{L}[y] &= e^{-3s}\\ s^2Y(s) + 4sY(s) + 5Y(s) &= e^{-3s}.\end{aligned}$$

Therefore,

$$Y(s) = \frac{e^{-3s}}{s^2+4s+5}. \tag{4.36}$$

- *Taking the inverse Laplace transform to* (4.36) *.*

$$\mathscr{L}^{-1}[Y(s)] = \mathscr{L}^{-1}\left[\frac{e^{-3s}}{s^2+4s+5}\right] = \mathscr{L}^{-1}\left[\frac{e^{-3s}}{(s+2)^2+1}\right] = e^{-2(t-3)}\sin(t-3)u_3(t).$$

7. $y'' + 4y = 8\delta\left(t - \frac{\pi}{6}\right), \quad y(0) = 2, \quad y'(0) = 0.$

- *Taking the Laplace transform to the integral equation with* $Y(s) = \mathscr{L}[y(t)]$.

$$\begin{aligned}
\mathscr{L}[y''] + \mathscr{L}[4y] &= \mathscr{L}\left[8\delta\left(t - \frac{\pi}{6}\right)\right] \\
s^2\mathscr{L}[y] - sy(0) - y'(0) + 4\mathscr{L}[y] &= 8e^{-\frac{\pi}{6}s} \\
s^2Y(s) - 2s + 4Y(s) &= 8e^{-\frac{\pi}{6}s}.
\end{aligned}$$

Therefore,

$$Y(s) = \frac{8e^{-\frac{\pi}{6}s}}{s^2+4} + \frac{2s}{s^2+4}. \tag{4.37}$$

- *Taking the inverse Laplace transform to* (4.37) *.*

$$\mathscr{L}^{-1}[Y(s)] = 4\mathscr{L}^{-1}\left[\frac{2e^{-\frac{\pi}{6}s}}{s^2+4}\right] + 2\mathscr{L}^{-1}\left[\frac{s}{s^2+4}\right] = 4\sin\left(2\left(t - \frac{\pi}{6}\right)\right)u_{\frac{\pi}{6}}(t) + 2\cos(2t).$$

参考文献

[1] Med EL Amrani, Suites et séries numériques, Ellipses.

[2] Charles-Michel Marle, Philibossian, Sylvie Guerre-Delabrière, Ellipse.

[3] A. Monsouri, M,k. Belbarki. Elément d'analyse. Cours et exercises résolus.

[4] Strang, G., Herman, E. (2016). Calculus. MIT and University of Wisconsin-Stevens Point. OpenStax.

[5] Stewart, J. (2015). *Calculus (8th ed.). Cengage Learning.*

[6] Marsden, J. E., Tromba, A. J. (2020). Vector Calculus (6th ed.). W. H. Freeman and Company.

[7] Jain, R. K., Iyengar, S. R. K. (2009). Advanced Engineering Mathematics (4th ed.). Laxmi Publications.

[8] Bali N. P., Dr. A.J. George. A Textbook of Engineering Mathematics Sem-I and II, 227 Pages, Published 2011 by Laxmi Publications Pvt. Ltd.