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Intitulé

Finite Difference Method for Solving Nonlinear Fredholm Integro-Differential Equations

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Dedication

In the Name of Allah, with Love and Gratitude**

Praise be to Allah, with love, thanks, and endless gratitude. Were it not for His grace, I would never have reached this moment.

To My Father…
To the one whose name I carry with pride,
Whom Allah crowned with dignity and honor.
You plucked the thorns from my path and planted ease in their place.
Your back never bent beneath life's weight—only to lift me higher.
You saw my hidden desires before I spoke them.
Thank you… for being my father.

To My Mother…
You taught me kindness before I could even speak.
You are my bridge to Paradise,
The unseen hand that cleared obstacles from my way.
Your prayers carried my name through every night and day.
To my beloved, my inspiration… my mother.

To My Siblings…
The ones Allah blessed me with,
My strength, my solid ground, the unshakable walls of my heart.
You are my roots and my wings.

To My Friends…
When the world felt small, your steps widened it.
When I stumbled, your words lifted me before my knees could touch the ground.
You walked beside me in spirit long before we shared the road.

To Every Teacher…
To those who shaped my mind and soul,

Who guided me to this achievement—
My deepest respect and sincerest gratitude.

To My Mentor…

How can I bow enough to honor all you’ve given me?
You were more than an advisor—a guiding light, a second father.
You never hesitated with your wisdom, patience, or time.
When I doubted myself, you believed.
When I faltered, you steadied me.
This milestone is as much yours as it is mine.

Today, I turn the page on exhaustion,
Etching pride into my story’s spine.
The clouds of struggle have parted—
The horizon smiles back at me now.

Every uncertain step I once took
Has found its place atop this summit.

And my final words, as always:

All praise is due to Allah, Lord of all worlds.

Thank you speech

In the name of Allah, the Most Gracious, the Most Merciful*
Praise be to Allah who has guided us to this, and we would not have been guided if Allah had not guided us.

To my esteemed professor and supervisor, Dr. Segni Sami:
You have been my greatest guide and my spiritual father...
Words fail to express my gratitude to the one who gave me his time and patience—qualities found only in exceptional beings.
You were my support when I faltered, my light when the path grew dark.
I will never forget any of the precious moments of your enlightened advice.
May Allah reward you abundantly for all that you have done for me.

To my dear family:
You who shaped my success through your sacrifices...
You who stayed awake so many nights for me...
An entire lifetime would not be enough to repay your kindness.
Mother... Father... You will forever remain the light of my life and my strongest support.

To my dearest friends:
You who shared laughter and tears with me...
You who helped me through the toughest times...
This achievement is also yours.
To all those who extended a helping hand:
Accept my most sincere prayers.
I will never forget those who stood by my side.
"My success comes only from Allah. In Him I place my trust, and to Him I turn." (Quran 11:88)

Abstract

This thesis presents a novel numerical approach for solving nonlinear Fredholm integro-differential equations involving both the unknown function and its derivative. The proposed method combines the backward finite difference technique with the Nyström method to significantly reduce the size of the resulting nonlinear system, compared to the classical approach introduced by Bounaya et al. [1]. This reduction enhances computational efficiency, especially for large integration intervals where traditional methods become impractical. To guarantee convergence and stability, we construct an appropriate norm on $\mathbb{R}^{N+1}$. Numerical experiments confirm the effectiveness of the proposed scheme, demonstrating improved accuracy and reduced execution time. This work contributes to the development of reliable numerical tools for solving nonlinear integro-differential equations, with potential applications in various fields of science and engineering.

Key words: Fredholm integral-differential equations, nonlinearity, Nyström method, backward finite differences.

Résumé

Cette mémoire présente une nouvelle approche numérique pour la résolution des équations intégrales-différentielles non linéaires de Fredholm faisant intervenir à la fois la fonction inconnue et sa dérivée. La méthode proposée combine la technique des différences finies rétrogrades avec la méthode de Nyström, ce qui permet de réduire de manière significative la taille du système non linéaire obtenu, en comparaison avec l'approche classique introduite par Bounaya et al. [1]. Cette réduction améliore l'efficacité du calcul, notamment pour les grands intervalles d'intégration où les méthodes traditionnelles deviennent peu pratiques. Afin de garantir la convergence et la stabilité de la méthode, une norme appropriée est construite dans l'espace $\mathbb{R}^{N+1}$. Les expériences numériques confirment l'efficacité du schéma proposé, en montrant une meilleure précision et une réduction du temps de calcul. Ce travail contribue au développement d'outils numériques fiables pour la résolution des équations intégrales-différentielles non linéaires, avec des applications potentielles dans divers domaines scientifiques et techniques.

Mots clé : Équations intégrales-différentielles de Fredholm, non-linéarité, méthode de Nyström, différences finies rétrogrades.

Contents

Introduction

Integro-differential equations arise naturally in a wide range of scientific and engineering applications, including physics, biology, control theory, and finance. These equations combine the features of both differential and integral equations, and are used to model systems in which the current rate of change depends not only on the present state but also on the history or spatial distribution of the system.

In particular, nonlinear Fredholm integro-differential equations involving both the unknown function and its derivative are of significant interest due to their capacity to describe complex dynamic behaviors. However, the analytical solutions of such equations are rarely obtainable, which motivates the development of reliable and efficient numerical methods.

The study of these equations poses several challenges: the presence of both the function and its derivative inside the integral introduces additional complexity, and the global nature of the Fredholm integral operator makes the computational cost higher, especially over large domains. Therefore, the construction of accurate and stable numerical schemes is crucial.

In this work, we consider a nonlinear Fredholm integro-differential equation of the form

$$\varphi(t) = g(t) + \int_a^b F(t, s, \varphi(s), \varphi'(s))\, ds, \quad t \in [a, b],$$

where the kernel function F and the source term g are given and assumed to satisfy suitable regularity conditions. Under appropriate assumptions, the existence and uniqueness of a solution $\varphi \in C^1[a, b]$ are guaranteed.

The aim of this thesis is twofold. First, we recall a known numerical method based on the Nyström technique [2, 3, 8, 10, 13, 14], as introduced by Bounaya et al. [1], for solving such equations by discretizing both the original equation and its derivative. Although effective, this method generates a large linear system that can be computationally expensive.

Second, we propose an alternative numerical approach that incorporates finite difference approximations to reduce the overall system size, thereby improving efficiency without compromising accuracy. We study the convergence and stability of the proposed scheme, and provide theoretical error estimates under standard assumptions.

This thesis is organized as follows: after presenting the mathematical formulation and assumptions, we describe the classical Nyström method and its limitations. We then introduce our new scheme, followed by a detailed error analysis and a set of numerical experiments to illustrate the performance and reliability of the proposed method.

CHAPTER 1

Preliminaries

1.1 Integral and Integro-Differential Equations

This section is devoted to present real integral equations to be studied in our work integro-differential equations.

1.1.1 Integral Equations

An integral equation is an equation that contains at least one integral operator acting on the unknown function. Let $[a,b] \subset \mathbb{R}$ A real fredholm integral equations with an integral operator has the following general form:

$$g(x)\varphi(x) = f(x) + \int_a^b F(x,s,\varphi(s))\,ds, \quad x \in [a,b].$$

where f, $g : [a,b] \to \mathbb{R}$, and $F : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$.

An integral equation is said to be:

- **Linear**, if the kernel K is linear with respect to the third variable, i.e., $F(x,s,u(s)) = F_0(x,s)\varphi(s)$; otherwise, it is nonlinear.
- **Of the first (second) kind**, if $\forall x \in [a,b]$, $g(x) = 0(g(x) = \text{const})$.
- **Homogeneous (nonhomogeneous)**, if $\forall x \in [a,b]$, $f(x) = 0(f(x) \neq 0)$.

Example 1.1.1. We give the following

$$\forall\, t \in [a,b], \quad \begin{cases} \displaystyle\int_a^b K(t,s)\varphi(s)ds = f(t), \quad \text{(first kind)}, \\ \displaystyle\varphi(t) = f(t) + \lambda \int_a^b F(t,s)\varphi(s)ds, \quad \text{(second kind)}. \end{cases}$$

and for example the following nonliear equation

$$\forall\, t \in [0,1];\ \lambda\varphi(t) = \int_0^1 \frac{e^{ts}}{10+\varphi^2(s)} ds + f(t)$$

This work is dedicated to presenting the study conducted in the article [1] of the following equation:

$$\forall\, t \in [a,b]; \varphi(t) = g(t) + \int_a^b F(t,s,\varphi(s),\varphi'(s))\, ds, \tag{1.1}$$

where the kernel F and the function g satisfy some assumption that will be devlopped in the next section :

1.2 Theoretical Study

To begin, we investigate the conditions that guarantee the existence and uniqueness of a solution to equation (1.1).
We start by considering the space $X = C^1([a,b])$, which comprises all functions that are continuously differentiable on the interval $[a,b]$. This space is endowed with the norm:

$$\|\varphi\|_X = \max_{t\in[a,b]} |\varphi(t)| + \max_{t\in[a,b]} |\varphi'(t)| = \|\varphi\|_\infty + \|\varphi'\|_\infty,$$

under which X forms a Banach space.
Bounaya et al [1] have studied the existence and uniqueness of the exact solution of equation (1.1), we assume that the kernel verifies the below hypotheses which are sufficient for ensuring this.

- $(A_1)\, \frac{\partial F}{\partial t} \in C^0\left([a,b]^2 \times \mathbb{R}^2\right)$,
- $(A_2)\, g \in C^1[a,b]$,
- $(A_3)\, \mid F(t,s,x,y) - F(t,s,\bar{x},\bar{y}) \mid < l_1 \mid x-\bar{x} \mid + l_2 \mid y - \bar{y} \mid$.
- $(A_4)\, \mid \frac{\partial F}{\partial t}(t,s,x,y) - \frac{\partial F}{\partial t}(t,s,\bar{x},\bar{y}) \mid < l_3 \mid x-\bar{x} \mid + l_4 \mid y-\bar{y} \mid$.
- $(A_5)\, \Omega := \max(l_1+l_2, l_3+l_4) < (2\,(b-a))^{-1}$.

Theorem 1.2.1. *Under the hypotheses* $(A_1 - A_5)$, *the equation* (1.1) *has a unique solution in* $C^1[a,b]$.

Proof. Details can be found in [1] □

1.3 General principle of finite difference methods

1.3.1 Introduction to the Finite Difference Method

The principle of finite difference methods is closely related to the numerical schemes used to solve ordinary differential equations (ODEs). The simplest approach to numerically solving partial differential equations (PDEs) consists in setting up a regular grid in space (and possibly time) and computing approximate solutions at the grid points.
A key aspect of this method is **discretization**, which involves approximating the differential operators by replacing the derivatives in the equation with *difference quotients*. The difference between the exact solution and the numerical one is determined by the error made in this replacement. This error is called the **discretization error** or **truncation error**.
The term truncation error reflects the fact that only a finite part of a Taylor series expansion is used in the approximation. For simplicity, we will consider only the one-dimensional case.
The main idea behind any finite difference scheme is based on the definition of the derivative of a smooth function φ at a point $x \in \mathbb{R}$:

$$\varphi'(x) = \lim_{h \to 0} \frac{\varphi(x+h) - \varphi(x)}{h}.$$

When h is small but non-zero, the quotient on the right-hand side provides a good approximation of the derivative. In other words, if h is sufficiently small, then:

$$\frac{\varphi(x+h) - \varphi(x)}{h} \approx \varphi'(x).$$

The quality of this approximation is judged by how the error behaves as $h \to 0$. More precisely, the approximation is said to be **consistent** if the error tends to zero as h goes to zero. If the function u is sufficiently smooth in a neighborhood of x, this error can be quantified using a Taylor expansion:

$$\varphi(x+h) = \varphi(x) + h\varphi'(x) + \frac{h^2}{2}\varphi''(x) + \cdots,$$

which implies that:

$$\frac{\varphi(x+h) - \varphi(x)}{h} = \varphi'(x) + \frac{h}{2}\varphi''(x) + \mathcal{O}(h^2).$$

Thus, the truncation error is $\frac{h}{2}\varphi''(x) + \mathcal{O}(h^2)$, which tends to zero as $h \to 0$. This approach forms the foundation of more advanced schemes used to approximate solutions of ordinary or partial differential equations in practice.

1.3.2 Taylor series

Given that the function φ is C^2 (twice continuously differentiable) in the neighborhood of x, the general Taylor expansion for $\varphi(x+h)$ around x up to the second order is:

$$\varphi(x+h) = \varphi(x) + h\varphi'(x) + \frac{h^2}{2}\varphi''(x) + O(h^3)$$

Here:

- $\varphi(x)$ is the value of the function at x,
- $\varphi'(x)$ is the first derivative of φ at x,
- $\varphi''(x)$ is the second derivative of φ at x,
- $O(h^3)$ denotes the remainder term, which is of order h^3 as $h \to 0$.

1.3.3 Forward Finite Difference

The forward finite difference is given by the formula:

$$\varphi'(x) \approx \frac{\varphi(x+h) - \varphi(x)}{h}.$$

It is clear that this formula converges if $\varphi \in C^1[a,b]$. Moreover, if $\varphi \in C^2[a,b]$, a Taylor expansion shows that:

$$\left| \frac{\varphi(x+h) - \varphi(x)}{h} - \varphi'(x) \right| \leq Ch,$$

where the constant C is given by:

$$C = \frac{1}{2} \cdot \sup_{y \in [x, x+h_0]} \left| \varphi''(y) \right|.$$

This means that the error is of order $\mathcal{O}(h)$, and the method is first-order accurate.

1.3.4 Backward Finite Difference

The backward finite difference approximation of the derivative is given by:

$$\varphi'(x) \approx \frac{\varphi(x) - \varphi(x-h)}{h}.$$

Similarly, if $\varphi \in C^2[a,b]$, then:

$$\left| \frac{\varphi(x) - \varphi(x-h)}{h} - \varphi'(x) \right| \leq Ch,$$

where:

$$C = \frac{1}{2} \cdot \sup_{y \in [x-h_0, x]} \left| \varphi''(y) \right|.$$

This also results in a first-order approximation.

1.3.5 Central Finite Difference

The central finite difference approximation provides a more accurate estimate of the first derivative by using points on both sides of x. It is given by:

$$\varphi'(x) \approx \frac{\varphi(x+h) - \varphi(x-h)}{2h}.$$

Assuming $\varphi \in C^3[a,b]$, a Taylor expansion around the point x shows that:

$$\left| \frac{\varphi(x+h) - \varphi(x-h)}{2h} - \varphi'(x) \right| \leq Ch^2,$$

where:

$$C = \frac{1}{6} \cdot \sup_{y \in [x-h_0, x+h_0]} \left| \varphi^{(3)}(y) \right|.$$

Thus, the central difference method is **second-order accurate**, meaning the truncation error is of order $\mathcal{O}(h^2)$, and it provides better accuracy compared to forward and backward finite differences for the same step size.

CHAPTER 2

Numerical Solution of Nonlinear Fredholm Integro-Differential Equations

In this chapter, we focuses onthe numerical study of the equation :

$$\varphi(t) = g(t) + \int_a^b F(t, s, \varphi(s), \varphi'(s)\, ds, \quad \forall t \in [a, b],$$

where F, and g are given real functions that satisfy

$$\frac{\partial F}{\partial t} \in C^0([a, b]^2 \times \mathbb{R}^2, \mathbb{R}),\, g \in C^1([a, b], \mathbb{R}).$$

Under the assumptions $(A_1) - (A_5)$ presented previousely. The soltion φ exists and unique in $C^1[a, b]$. Firstly, we recal the method used by Bounaya et al [1]. Secoundly, we explain our new approch based on finite difference method. Additionally, they derived the form ofits derivative $\varphi'(t)$:

$$\varphi'(t) = \int_a^b \frac{\partial F}{\partial t}(t, s, \varphi(s), \varphi'(s))\, ds + g'(t). \tag{2.1}$$

2.1 Classical Nyström method

In [1], Bounaya et al. used a new Nyström method to deal with $\varphi(t)$ and $\varphi'(t)$ jointly. They obtained a big linear system, which made their method difficult to apply in the context of large integration intervals.

Theorem 2.1.1. *Under assumptions $(A_1 - A_5)$, the equation (1.1) has a unique solution in X*

Proof. Bounaya et al ([1]) used Banach fixed point to obtain this result □

The approach begins by reducing the continuous equations (1.1) and (2.1) to a finite-dimensional problem using the well-known Nyström method. Then, the resulting algebraic system is solved using the method of successive approximations.
Let $N \in \mathbb{N}^*$, and consider the uniform subdivision

$$\Delta_N = \left\{ t_i = a + ih,\, h = \frac{b-a}{N},\, i = 0, 1, \ldots, N \right\}.$$

Accordingly, we have:

$$\varphi(t_i) = g(t_i) + \int_a^b F(t_i, s, \varphi(s), \varphi'(s))\, ds, \tag{2.2}$$

$$\varphi'(t_i) = g'(t_i) + \int_a^b \frac{\partial F}{\partial t}(t_i, s, \varphi(s), \varphi'(s))\, ds. \tag{2.3}$$

By applying a quadrature formula, equations (2.2) and (2.3) are approximated as:

$$\varphi_i = g_i + \sum_{j=0}^{N} w_j F(t_i, t_j, \varphi_j, \varphi'_j) + R_1(h, i), \tag{2.4}$$

$$\varphi'_i = g'_i + \sum_{j=0}^{N} w_j \frac{\partial F}{\partial t}(t_i, t_j, \varphi_j, \varphi'_j) + R_2(h, i), \tag{2.5}$$

where $\varphi(t_i) = \varphi_i$, $g(t_i) = g_i$, and the weights w_j as well as the remainder terms $R_1(h,i)$, $R_2(h,i)$ depend on the quadrature rule used.
The quadrature rule is said to be *consistent* if it satisfies:

$$\lim_{N\to\infty} \left(\max_{0\le i\le N} |R_1(h,i)| \right) = \lim_{N\to\infty} \left(\max_{0\le i\le N} |R_2(h,i)| \right) = 0.$$

Assuming the local consistency errors $R_1(h,i)$ and $R_2(h,i)$ are negligible, we obtain the following algebraic system for all $i \in \{0, 1, \ldots, N\}$:

$$\rho_i = g_i + \sum_{j=0}^{N} w_j F(t_i, t_j, \rho_j, \sigma_j), \tag{2.6}$$

$$\sigma_i = g'_i + \sum_{j=0}^{N} w_j \frac{\partial F}{\partial t}(t_i, t_j, \rho_j, \sigma_j). \tag{2.7}$$

Let $\eta = (\rho_0, \sigma_0, \rho_1, \sigma_1, \ldots, \rho_N, \sigma_N) \in \mathbb{R}^{2N+2}$ denote the vector of unknowns. Then, the system can be written as a fixed-point equation:

$$\eta = \Phi(\eta).$$

From this formulation, the following existence theorem can be established.

Theorem 2.1.2. *Under the hypothesis $(A_1 - A_5)$ and if one of the following conditions*

$$\alpha = \max\{l_1 + l_3, l_2 + l_4\} < \frac{1}{b-a}, \tag{2.8}$$

$$\beta = \max\left\{l_4, l_1 + \frac{l_2 l_3 (b-a)}{1 - l_4(b-a)}\right\} < \frac{1}{b-a}, \tag{2.9}$$

is satisfied, then, the system of (2.6), (2.7) has a unique solution.

Proof. Bounaya et al [1] used the classical Banach fixed point in $\mathbb{R}^{2N+2}$ to obtain the result. □

The next theorem proves the convergence of the classical Nyström method:

Theorem 2.1.3. *Denote $R(h) = \max_{i=0,\ldots,N} |R_1(h,i)| + \max_{i=0,\ldots,N} |R_2(h,i)|$ and*

$$\|v\|_{\mathbb{R}^{N+1}} = \|v\|_\infty = \max_{i=0,\ldots,N} |v_i|, \quad C = \max\left(1, \frac{l_2(b-a)}{1 - l_4(b-a)}\right).$$

So, under the hypothesis $(A_1 - A_5)$, (2.8) and (2.9), we have the following error estimates :

$$\|\varphi - \rho\|_\infty + \|\varphi' - \sigma\|_\infty \leq \frac{R(h)}{1 - \alpha(b-a)}, \tag{2.10}$$

$$\|\varphi - \rho\|_\infty \leq \frac{C R(h)}{1 - \beta(b-a)}, \tag{2.11}$$

$$\|\varphi' - \sigma\|_\infty \leq \left(\frac{l_1(b-a)C + 1 - \beta(b-a)}{[1 - \beta(b-a)]^2}\right) R(h). \tag{2.12}$$

Proof. $(A_1 - A_5)$ give as

$$\|\varphi - \rho\|_\infty \leq \sum_{j=0}^{N} |w_j| (l_1 |\varphi_j - \rho_j| + l_2 |\varphi'_j - \sigma_j|) + \max_{i=0,\ldots,N} |R_1(h,i)|, \tag{2.13}$$

$$\|\varphi' - \sigma\|_\infty \leq \sum_{j=0}^{N} |w_j| (l_3 |\varphi_j - \rho_j| + l_4 |\varphi'_j - \sigma_j|) + \max_{i=0,\ldots,N} |R_2(h,i)|. \tag{2.14}$$

The summation of (2.13) and (2.14) gives us

$$\|\varphi-\rho\|_\infty+\|\varphi'-\sigma\|_\infty \leq (\|\varphi-\rho\|_\infty+\|\varphi'-\sigma\|_\infty)\alpha\sum_{j=0}^{N}|w_j|+R(h).$$

Since $\sum_{j=0}^{N}|w_j|=b-a$, then, the inequality (2.10) is satisfied. Otherwise, from (2.14), we get

$$\|\varphi'-\sigma\|_\infty \leq \frac{(b-a)l_3}{1-l_4(b-a)}\|\varphi-\rho\|_\infty+\frac{\max_{i=0,\dots,N}|R_2(h,i)|}{1-l_4(b-a)}. \tag{2.15}$$

After that, by substituting (2.15) in (2.13), we obtain

$$\|\varphi-\rho\|_\infty \leq (b-a)\beta\|\varphi-\rho\|_\infty+CR(h).$$

Hence, (2.11) is satisfied.
Finally, since $1-\beta(b-a)\leq 1-l_4(b-a)$, then, substituting (2.11) in (2.15) implies (2.12). □

2.2 Backward finite difference method

For $N\geq 2$, we introduce a subdivision of $[a,b]$ given by:

$$h=\frac{b-a}{N-1},\ x_j=a+(j-1)\,h,\ 1\leq j\leq N,$$

and a numerical integration formula given for all $\varphi\in C^0\,[a,b]$ *by*

$$\int_a^b\varphi\,(t)\,dt\approx\sum_{j=1}^{N}\rho_j\varphi\,(t_j)\,,$$

where, the weights $\{\rho_j\}_{j=1}^{N}$ are supposed to verify,

$$\Theta:=\sup_{N\geq 2}\sum_{j=1}^{N}\mid\rho_j\mid<\infty,$$

$$\forall\varphi\in C^0\,[a,b]\,,\ \lim_{N\to+\infty}\mid\int_a^b\varphi\,(t)\,dt-\sum_{j=1}^{N}\rho_j\varphi\,(t_j)\mid=0. \tag{2.16}$$

Applying the Nyström method[11, 12, 13], with the previous numerical integration formula, to equation (1.1), we obtain, $\forall 1\leq j\leq N$,

$$\varphi_i=\sum_{j=1}^{N}\rho_jF(t_i,t_j,\varphi_j,\varphi'_j)+g\,(t_i)\,, \tag{2.17}$$

where, $\varphi_j \approx \varphi(t_j)$ and $\varphi'_j \approx \varphi'(t_j)$. Now, we use the Backward finite difference derivative [14] to deal with φ'_j, $2 \leq j \leq N$, which gives us

$$\varphi'_j \approx h^{-1}(\varphi_j - \varphi_{j-1}). \tag{2.18}$$

Including the last approximation in system (2.17) to obtain, for all $1 \leq i \leq N$,

$$\varphi_i = \sum_{j=2}^{N} \rho_j F(t_i, t_j, \varphi_j, h_{-1}^{1}(\varphi_j - \varphi_{j-1})) + \rho_1 F(t_i, t_j, \varphi_1, \varphi'_1) + g(t_i). \tag{2.19}$$

At this stage, we understand that we have to add an equation for φ'_1. For that, we apply Nyström method with the same numerical integration formula to equation (2.1), which gives us

$$\varphi'_1 = \sum_{j=1}^{N} \rho_j \frac{\partial F}{\partial t}(t_1, t_j, \varphi_j, \varphi'_j) + g'(t_1).$$

Using (2.18) with the same previous steps, we obtain

$$\varphi'_1 = \sum_{j=2}^{N} \rho_j \frac{\partial F}{\partial t}(t_1, t_j, \varphi_j, h_{-1}^{1}(\varphi_j - \varphi_{j-1})) + \rho_1 \frac{\partial F}{\partial t}(t_1, t_1, \varphi_1, \varphi'_1) + g'(t_1). \tag{2.20}$$

Our new method involves solving the linear system (2.19)- (2.20) of size $N+1$, which is much better compared to the one developed by Bounaya et al.[1], which is of size $2N$.

2.3 Numerical study

In this section, we are going to show that our new method based on backward finite difference and Nyström techniques is perfectly converging. For that, we equip $\mathbb{R}^{N+1}$ with a special norm given, for all $V \in \mathbb{R}^{N+1}$ and $h > 0$, by

$$\| V \|_h = \max\left\{ \| V \|_\infty, \max_{2 \leq i \leq N+1} | h^{-1}(V_i - V_{i-1}) | \right\},$$

where, $\| \,.\, \|_\infty$ is the usual norm of $\mathbb{R}^{N+1}$ given for all $V \in \mathbb{R}^{N+1}$ as

$$\| V \|_\infty = \max_{1 \leq i \leq N+1} | V_i |.$$

Theorem 2.3.1. *Supposing that $(A_1 - A_5)$, then for all $h > 0$, the system (2.19) - (2.20) has a unique solution $(\varphi_1, \varphi_2, ..., \varphi_N, \varphi'_1) \in \mathbb{R}^{N+1}$.*

Proof. For $N \geq 1$, we introduce :

$$\begin{array}{cccc} \Phi_N & : & \mathbb{R}^{N+1} & \longrightarrow & \mathbb{R}^{N+1} \\ & & V = \begin{pmatrix} V_1 \\ V_2 \\ . \\ . \\ . \\ . \\ V_{n+1} \end{pmatrix} & \mapsto & \begin{pmatrix} \Phi_{N,1} \\ \Phi_{N,2} \\ . \\ . \\ . \\ . \\ \Phi_{N,N+1} \end{pmatrix} \end{array}$$

by

$$\forall\, V \in \mathbb{R}^{N+1}, \Phi_N(V) = (\Phi_{N,1}(V), \Phi_{N,2}(V), ..., \Phi_{N,N+1}(V)) \in \mathbb{R}^{N+1}$$

where: $\forall\, 1 \leq i \leq N$,

$$\Phi_{N,i}(V) \;=\; \sum_{j=2}^{N} \rho_j F\left(t_i, t_j, V_j, h^{-1}(V_j - V_{j-1})\right) + \rho_1 F(t_i, t_1, V_N, V_{N+1}) + g(t_i)$$

and,

$$\Phi_{N,N+1}(V) \;=\; \sum_{j=2}^{N} \rho_j \frac{\partial F}{\partial t}\left(t_1, t_j, V_j, h^{-1}(V_j - V_{j-1})\right) + \rho_1 \frac{\partial F}{\partial t}(t_1, t_1, V_N, V_{N+1}) + g'(t_1)$$

Which means that the system (2.19) - (2.20) is equivalent to : Find $\Lambda \in \mathbb{R}^{N+1}$ such that $\Lambda = \Phi_N(\Lambda)$. We have, for all $V, W \in \mathbb{R}^{N+1}$ and all $1 \leq i \leq N$,

$$\begin{aligned} \mid \Phi_{N,i}(V) - \Phi_{N,i}(W) \mid \;&\leq\; l_1 \sum_{j=2}^{N} \mid \rho_j \mid\mid V_j - W_j \mid \\ &+\; l_2 \sum_{j=2}^{N} \mid \rho_j \mid\mid h_{-1}^{1}(V_j - V_{j-1}) - h_{-1}^{1}(W_j - W_{j-1}) \mid \\ &+\; \rho_1 l_1 \mid V_N - W_N \mid + \rho_1 l_2 \mid V_{N+1} - W_{N+1} \mid \end{aligned}$$

Then, for all $1 \leq i \leq N$,

$$\| \Phi_{N,i}(V) - \Phi_{N,i}(W) \| \leq 2\Theta\Omega \| V - W \|_h .$$

Now, for all $V, W \in \mathbb{R}^{N+1}$, we obtain

$$\begin{aligned} \mid \Phi_{N,N+1}(V) - \Phi_{N,N+1}(W) \mid \;&\leq\; \sum_{j=2}^{N} \mid \rho_j \mid (l_1 \mid V_j - W_j \mid \\ &+\; l_2 \mid h^{-1}(V_j - V_{j-1}) - h^{-1}(W_j - W_{j-1}) \mid \\ &+\; \rho_1 l_3 \mid V_N - W_N \mid + \rho_1 l_4 \mid V_{N+1} - W_{N+1} \mid) \\ &\leq\; 2(b-a)\Omega \| V - W \|_h . \end{aligned}$$

Using the mean value theorem, we conclude the existence of $\{\xi_{ij}\}$ and $\left\{\widehat{\xi}_{ij}\right\}$ where $1 \le i, j \le N, i \ne j$ such that,

$$\frac{\partial F}{\partial t}(\xi_{ij}, t_j, V_j, W_j) = h^{-1}\left(F(t_i, t_j, V_j, W_j) - F(t_{i-1}, t_j, V_j, W_j)\right)$$

Then, for all $2 \le i \le N$,

$$| \, h^{-1}(\Phi_{N,i}(V) - \Phi_{N,i-1}(V)) - h^{1}_{-1}(\Phi_{N,i}(W) - \Phi_{N,i-1}(W)) \, | = \sum_{j=2}^{N} \rho_j \frac{\partial F}{\partial t}(t_i, t_j, V_i, W_i) -$$
$$\frac{\partial F}{\partial t}(t_{i-1}, t_j, V_{i-1}, W_{i-1}) + \rho_1 F(t_i, t_1, V_i, W_i) - \rho_1 F(t_{i-1}, t_1, V_{i-1}, W_{i-1} \le 2(b-a)\Omega \, || \, V - W \, ||_h \, .$$

We obtain, $\forall V, W \in \mathbb{R}^{N+1}$,

$$|| \, \Phi_N(V) - \Phi_N(W) \, ||_h \le 2(b-a)\Omega \, || \, V - W \, ||_h \, .$$

Which means that Φ_N is a contraction, because $2\Theta\Omega < 1$, and using Banach fixed point theorem, we conclude that the system 2.19 - 2.20 has a unique solution

$$\left(\varphi_1, \varphi_2, ..., \varphi_N, \varphi_1'\right) \in \mathbb{R}^{N+1}$$

for all $N \ge 2$. □

Just take the trapezoidal rule [14] with $\rho_1 = \rho_N = \dfrac{h}{2}$ and $\rho_j = h, 2 \le j \le N-1$.
To study the error of our method, we introduce the discretization error $\{\nu_i\}_{1 \le i \le N+1}$ as

$$\begin{aligned} \nu_i &= \varphi(x_i) - \varphi_i, \; 1 \le i \le N, \\ \nu_{N+1} &= \varphi'(x_1) - \varphi_1', \end{aligned}$$

and the consistence errors as

$$\begin{aligned} \varpi_1(h) &= \max_{a \le x \le b} \left| \int_a^b F\left(t, s, \varphi(s), \varphi'(s)\right) ds - \sum_{j=1}^{N} \rho_j F\left(t, s, \varphi(t_j), \varphi'(t_j)\right) \right|, \\ \varpi_2(h) &= \max_{2 \le i \le N} \left| \varphi'(t_i) - h^{-1}\left(\varphi(t_i) - \varphi(t_{i-1})\right) \right| \\ \varpi_3(h) &= \max_{a \le x \le b} \left| \int_a^b \frac{\partial F}{\partial t}\left(t, s, \varphi(s), \varphi'(s)\right) ds - \sum_{j=1}^{N} \rho_j \frac{\partial F}{\partial t}\left(t, s, \varphi(t_j), \varphi'(t_j)\right) \right| \end{aligned}$$

Theorem 2.3.2. *For $N \ge 1$,*

$$|| \, \nu \, ||_h \le \left(1 - 2(b-a)\Omega\right)^{-1}\left(\max \varpi_1(h) + (b-a)\Omega\varpi_2(h)\right).$$

Proof. For, $1 \leq i \leq N$,

$$
\begin{aligned}
\mid \nu_i \mid &= \mid \varphi\left(x_i\right) - \varphi_i \mid \\
&= \mid \int_a^b F\left(t, s, \varphi(s), \varphi'(s)\right) ds - \sum_{j=2}^{N} \rho_j F(t_i, t_j, \varphi_j, h_{-1}^{1}(\varphi_j - \varphi_{j-1})) \\
&- \rho_1 F(t_i, t_1, \varphi_1, \varphi_1') \mid \\
&\leq \varpi_1\left(h\right) + l_1 \sum_{j=1}^{N} \mid \rho_j \mid\mid \varphi\left(x_j\right) - \varphi_j \mid + l_2 \sum_{j=2}^{N} \mid \rho_j \mid\mid \varphi'\left(t_j\right) - h^{-1}\left(\varphi_j - \varphi_{j-1}\right) \mid \\
&+ l_2 \mid \rho_1 \mid \varphi'\left(t_1\right) - \varphi_1' \mid \\
&\leq \varpi_1\left(h\right) + \Xi\Omega\varpi_2\left(h\right) + l_1 \sum_{j=1}^{N} \mid \rho_j \mid\mid \varphi(t_j) - \varphi_j \mid + l_2 \sum_{j=2}^{N} \mid \rho_j \mid \\
&\times \mid h^{-1}\left(\varphi\left(t_j\right) - \varphi\left(t_{j-1}\right)\right) - h^{-1}\left(\varphi_j - \varphi_{j-1}\right) \mid + l_2 \mid \rho_1 \mid\mid \varphi'\left(t_1\right) - \varphi_1' \mid .
\end{aligned}
$$

Then, for all $1 \leq i \leq N$,

$$
\mid \nu_i \mid \leq \varpi_1\left(h\right) + (b-a)\Omega\varpi_2\left(h\right) + (b-a)\Omega \mid\mid \nu \mid\mid_h . \tag{2.21}
$$

We have,

$$
\begin{aligned}
\mid \nu_{N+1} \mid &= \mid \varphi'\left(x_1\right) - \varphi_1' \mid \\
&= \mid \int_a^b \frac{\partial F}{\partial t}\left(t, s, \varphi(s), \varphi'(s)\right) ds - \sum_{j=1}^{N} \rho_j \frac{\partial F}{\partial t}\left(t_1, t_j, \varphi_j, h^{-1}(\varphi_j - \varphi_{j-1})\right) \\
&- \rho_1 \frac{\partial F}{\partial t}\left(t_1, t_1, \varphi_1, \varphi_1'\right) \mid \\
&\leq \varpi_3\left(h\right) + l_3 \sum_{j=1}^{N} \mid \rho_j \mid \varphi\left(t_j\right) - \varphi_j \mid \\
&+ l_4 \sum_{j=2}^{N} \mid \rho_j \mid\mid \varphi'\left(t_j\right) - h^{-1}\left(\varphi_j - \varphi_{j-1}\right) \mid + \rho_1 l_2 \mid \varphi'\left(x_1\right) - \varphi_1' \mid \\
&\leq \varpi_3\left(h\right) + (b-a)\Omega\varpi_2\left(h\right) + l_3 \sum_{j=1}^{N} \mid \rho_j \mid\mid \varphi\left(t_j\right) - t_j \mid \\
&+ l_4 \sum_{j=2}^{N} \mid \rho_j \mid\mid h^{-1}\left(\varphi\left(t_j\right) - \varphi\left(t_{j-1}\right)\right) - h^{-1}\left(\varphi_j - \varphi_{j-1}\right) \mid \\
&+ l_4 \mid \rho_1 \mid\mid \varphi'\left(t_1\right) - \varphi_1' \mid .
\end{aligned}
$$

Then,

$$
\mid \nu_{N+1} \mid \leq \varpi_3(h) + (b-a)\Omega\varpi_2\left(h\right) + 2(b-a)\Omega \mid\mid \nu \mid\mid_h . \tag{2.22}
$$

Using the same sequences $\{\xi_{ij}\}_{\substack{1\leq i,j\leq N \\ i\neq 1}}$ and $\left\{\widehat{\xi}_{ij}\right\}_{\substack{1\leq i,j\leq N \\ i\neq 1}}$ as in the previous theorem's proof, we obtain for all $2 \leq i \leq N$,

$$\begin{aligned} \mid h^{-1}(v_i - v_{i-1}) \mid &= \mid h^{-1}(\varphi(t_i) - \varphi(t_{i-1})) - h^{-1}(\varphi_i - \varphi_{i-1}) \mid \\ &\leq \mid \int_a^b h^{-1}\left(F\left(t_i, s, \varphi(s), \varphi'(s)\right) - F\left(t_{i-1}, s, \varphi(s), \varphi'(s)\right)\right) ds \\ &- l_3 \sum_{j=1}^{N} \rho_j \xi_{ij} F_j - l_4 \sum_{j=2}^{N} \rho_j \widehat{\xi_{ij}} h^{-1}(\varphi_j - \varphi_{j-1}) \\ &- \rho_1 l_4 \widehat{\xi}_{11}\left(\varphi'(x_1) - \varphi_1'\right) \mid \\ &\leq I_i + J_i + (b-a)\Omega\varpi_3(h) + l_3 \sum_{j=1}^{N} \mid \rho_j \mid\mid \varphi(t_j) - \varphi_j \mid \\ &+ l_4 \sum_{j=2}^{N} \mid \rho_j \mid\mid h^{-1}\left(\varphi(x_j) - \varphi(x_{j-1})\right) - h^{-1}(\varphi_j - \varphi_{j-1}) \mid \\ &+ l_4 \mid \rho_1 \mid\mid \left(\varphi'(x_1) - \varphi_1'\right) \mid, \end{aligned}$$

Therefore, for all $2 \leq i \leq N$,

$$\mid h^{-1}(v_i - v_{i-1}) \mid \leq \varpi_3(h) + (b-a)\Omega\varpi_2(h) + 2(b-a)\Omega \mid\mid v \mid\mid_h \tag{2.23}$$

Using (2.20), (2.21) and (2.22), we obtain

$$\mid\mid v \mid\mid_h \leq \max(\varpi_1(h), \varpi_3(h)) + (b-a)\Omega\varpi_2(h) + 2(b-a)\Omega \mid\mid v \mid\mid_h .$$

Then,

$$\mid\mid v \mid\mid_h \leq (1 - 2(b-a)\Omega)^{-1} (\max(\varpi_1(h)), \varpi_3(h)) + (b-a)\Omega\varpi_2(h)) .$$

□

From (2.16) and (A_1), we can conclude that, for all $i \in \{1, 2, 3\}$,

$$\lim_{h\to 0} \varpi_i(h) = 0.$$

But, using $(A_1) - (A_2)$, we obtain,

$$\varpi_2(h) \underset{h\to 0}{\to} 0 .$$

And this proves that our method is perfectly converging under assumptions $(A_1), (A_2), (A_3), (A_4)$ and (A_5).

CHAPTER 3

Numerical Examples

3.1 Numerical test

We take the same numerical example studied in [1], i.e we consider the nonlinear integro-differential equation:

$$\varphi(t) = g(t) + \int_0^1 \frac{1}{5} \sin\left[2(s + t + \varphi(s)) + (1 - s)e^s - \varphi'(s)\right] ds,$$

with known exact solution $\varphi(t) = te^t$. We apply the Nyström method using numerical quadrature and a backward finite difference scheme to approximate φ'.

Discretization

We consider a uniform subdivision of $[0, 1]$:

$$h = \frac{1}{N-1}, \quad t_j = (j-1)h, \quad 1 \leq j \leq N.$$

We use a numerical quadrature formula:

$$\int_0^1 f(s)\, ds \approx \sum_{j=1}^{N} \rho_j f(t_j),$$

where ρ_j are weights (we use the trapezoidal rule).
The derivative is approximated using the backward finite difference:

$$\varphi'_j \approx \frac{\varphi_j - \varphi_{j-1}}{h}, \quad j = 2, \ldots, N.$$

The discretized system becomes:

$$\varphi_i = \sum_{j=2}^{N} \rho_j F(t_i, t_j, \varphi_j, \frac{\varphi_j - \varphi_{j-1}}{h}) + \rho_1 F(t_i, t_1, \varphi_1, \varphi_1') + g(t_i), \quad 1 \leq i \leq N,$$

with an additional equation:

$$\varphi_1' = \sum_{j=2}^{N} \rho_j \frac{\partial F}{\partial t}(t_1, t_j, \varphi_j, \frac{\varphi_j - \varphi_{j-1}}{h}) + \rho_1 \frac{\partial F}{\partial t}(t_1, t_1, \varphi_1, \varphi_1') + g'(t_1).$$

	Backward finite difference			Full Nyström Bounaya and al.[1]		
N	Error	Matrix Conditioning	Time	Error	Matrix Conditioning	Time
20	8.23e-04	9.35e+0	2.15e-3	1.14e-04	3.41e+1	4.34e-3
50	1.35e-05	6.39e+0	1.08e-2	7.22e-04	1.75e+1	1.62e-2
100	3.27e-06	6.77e+0	1.36e-1	5.48e-06	1.68e+1	3.97e-1
200	8.19e-07	8.11e+0	4.11e-1	4.76e-06	1.68e+1	4.78e-1
500	1.32e-07	1.45e+1	2.39e+0	3.83e-06	1.68e+1	4.52e+0
750	5.96e-08	1.45e+1	5.68e+0	1.84e-07	1.68e+1	9.55e+0
1000	3.27e-08	1.45e+1	9.41e+0	8.54e-08	1.68e+1	3.22e+1

Table 3.1: Our method vs Bounaya and al. method[1]

In this example, we have studied a nonlinear Fredholm integro-differential equation with a known exact solution $\varphi(t) = te^t$, using two distinct numerical approaches:

- The **full Nyström method** as used by **Bounaya et al.**, which applies quadrature rules to both the function and its derivative within the nonlinear kernel.
- A **new approach based on the backward finite difference formula**, where the derivative $\varphi'(s)$ is approximated explicitly outside the integral using a first-order backward scheme.

Our numerical results lead to the following conclusions:

- Both methods produce accurate approximations of the exact solution, with errors on the order of 10^{-4} to 10^{-6}, depending on the discretization.
- The **backward finite difference method significantly reduces computational time** compared to the full Nyström method. This gain in efficiency is due to the explicit treatment of the derivative, which avoids its repeated nonlinear evaluation within the integral.

- From an implementation standpoint, the backward scheme is **simpler, faster, and more memory-efficient**, while still maintaining good numerical accuracy.

Therefore, the proposed **backward finite difference-based method** presents itself as an effective and computationally efficient alternative to the classical Nyström approach for solving nonlinear Fredholm integro-differential equations.

Conclusion

In this thesis, we investigated the numerical solution of a class of nonlinear Fredholm integro-differential equations involving both the unknown function and its derivative. Such equations are fundamental in modeling various real-world phenomena across physics, biology, and engineering. However, due to their inherent complexity and the presence of nonlinearities, obtaining analytical solutions is often impossible, making numerical methods essential.

We began by revisiting the classical Nyström method as applied by Bounaya et al. [1], which jointly discretizes the integro-differential equation and its derivative. Although effective, this method results in a large nonlinear system, which can be computationally intensive for problems defined over large intervals.

To address this challenge, we proposed a new numerical approach that integrates the finite difference method for approximating derivatives with the Nyström method for integral discretization. This strategy allows for a significant reduction in the size of the algebraic system, thereby improving computational efficiency while preserving accuracy.

We provided a rigorous theoretical analysis of the method, including convergence results and error estimates, under a set of standard regularity and Lipschitz conditions. Numerical experiments confirmed the theoretical predictions and demonstrated the efficiency, stability, and reliability of the proposed scheme.

Overall, the method developed in this thesis offers a practical and efficient tool for solving a broad class of nonlinear integro-differential equations. It opens the door to further research, such as extensions to systems of equations, treatment of singular kernels, or applications to inverse problems and control theory. Future work may also explore adaptive mesh techniques and higher-order approximations to further enhance accuracy and computational performance.

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