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Controllability for a class of \psi-Caputo fractional system

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Before the jury

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Abstract

In this work, we investigate the controllability of a class of fractional systems in the Ψ-Caputo sense with nonlocal initial conditions.
First, we establish the existence and uniqueness of solutions by applying the Schauder fixed point theorem and the Banach contraction principle. Subsequently, we analyze the controllability of the nonlinear problem. Finally, we present a numerical example to elucidate the theoretical framework and demonstrate the obtained results.

Key words : Ψ-Caputo fractional derivative, Mittag Leffler function, controllability, nonlocal initial condition, Schauder fixed point theorem, Banach contraction principle.

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Résumé

Dans cette étude, nous examinons la contrôlabilité d'une classe de systèmes fractionnaires avec des conditions aux limites non locales au sens de la dérivée de Ψ-Caputo.

Dans un premier temps, nous établissons l'existence et l'unicité des solutions en appliquant le théorème du point fixe de Schauder et le principe de contraction de Banach. Ensuite, nous analysons la contrôlabilité du problème non linéaires. Enfin, nous présentons un exemple numérique pour éclaircir le cadre théorique et illustrer les résultats obtenus.

Mots clés: Dérivée fractionnaire de Ψ-Caputo, fonction Mettag Leffler, contrôlabilité, condition initiale non locale, Théorème point fixe de Schauder, Théorème point fixe de Banach.

Contents

Notations and Abbreviations

- $\mathbb{N}$	Set of integers numbers.
- $\mathbb{R}$	Set of real numbers.
- $\mathbb{C}$	Set of complex numbers.
- $\Gamma(.)$	Gamma function.
- $B(.,.)$	Beta function.
- $Re(z)$	Real part of the complex number z.
- $E_{\alpha,\beta}(z)$	Mittag Leffler function.
- $I_a^\alpha f$	Riemann-Liouville fractional integral of order α of function f.
- ${}^cD_a^\alpha f$	Caputo fractional derivative of order α of function f.
- ${}^{RL}D_a^\alpha f$	Riemann-Liouville fractional derivative of order α of a function f.
- $\mathscr{I}_{a^+}^{\alpha;\psi} f$	The ψ-Caputo integral of order α for a function f.
- ${}^c\mathscr{D}_{a^+}^{\alpha;\psi} f$	The ψ-Caputo derivative of order α of a function f.
- $\mathscr{L}(.)$	Laplace Transform.
- $\mathscr{L}^{-1}(.)$	Inverse Laplace Transform.
- X, Y	Normed vectorial space.
- $C(X,Y)$	Space of continuous functions from X to Y.

Introduction

The concept of fractional analysis has a long historical precedent, dating back to the early 1700s, when monumental figures such as L'Hopital and Leibniz laid down the fundamental principles of differential and integral calculus. This concept, referring to a function possessing an arbitrary real or complex order, has gained substantial prominence within the academic community over the last three decades. It has unlocked new insights into understanding complex and intermittent systems that classical calculus cannot handle, contributing to numerous advances in various fields of science and engineering. Despite the complexity involved in studying non-integer derivatives, researchers have persistently pursued this area of study and have made significant progress in understanding and applying this concept.

The origins of fractional calculus can be traced back to a discourse between prominent mathematicians L'Hopital and Leibniz in 1695. L'Hopital inquired about the significance of the $\frac{d^{1/2}f}{dt}$ derivative, which inspired Leibniz to contemplate the plausibility of derivatives of non-integer order. Leibniz described this concept as a paradox with potential practical utilities. However, noteworthy advancements in this domain were not realized until the 1990s.

Then, Leibniz still wrote about derivatives of general order and in 1730, Euler investigated the result of the derivative when the order n is a fraction (In his article on the Gamma function, a mathematical concept closely connected to the factorial function, Euler presented a quandary concerning rational numbers. This may have contributed to the adoption of the term "fractional" in contemporary calculus). But, only in 1819, with Lacroix, appeared the first definition of fractional derivative based on the expression for the nth derivative of the power function. Considering $y = x^m$, with m a positive integer, Lacroix developed the nth derivative

$$\frac{d^n y}{dx^n} = \frac{m!}{(m-n)!} x^{m-n}, \quad m \geq n,$$

and using the definition of Gamma function, for the generalized factorial, he got

$$\frac{d^n y}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}.$$

Lacroix also studied the following example, for $n = \frac{1}{2}$ and $m = 1$:

$$\frac{d^{1/2} x}{dx^{1/2}} = \frac{\Gamma(2)}{\Gamma(3/2)} x^{\frac{1}{2}} = \frac{2\sqrt{x}}{\sqrt{\pi}}. \tag{1.1}$$

During the 1830s, Liouville and Riemann, in their own distinct manners, established the concept of fractional derivatives, a method subsequently referred to as the "Riemann-Liouville" approach. Over time, additional hypotheses, including the Grunwald-Letnikov, Weyl, and Caputo conjectures, were developed ([9, 12, 15]).

Additionally, control theory studies the behavior of dynamical systems based on their parameters. It can be seen as a strategy to select the appropriate input for a system so that the output matches the desired response. The goal is thus to drive the system from a given initial state to a certain final state, while possibly respecting specific constraints [8, 10, 17].

In practice, controllability problems arise in numerous disciplines, such as: parking a car, piloting an aircraft or guiding a satellite into orbit, optimizing information flow in a network, controlling an epidemic, performing laser-assisted surgical procedures, and many others. For several decades, extensive research has been conducted on controllability problems for fractional differential equations [2, 3, 11, 19].

In this work we focus our attention on the controllability problem of a nonlinear fractional system where the derivatives are taken in the $\psi-$ Caputo sense .

This manuscript is organized as follow

In the first Chapter, we introduce the fundamental concepts of fractional calculus, which will prove instrumental for our study. Specifically, we will highlight special functions such as the Gamma function and Beta function, along with established techniques for fractional derivatives and integrals. Additionally, we present several fixed-point theorems, which are necessary for analyzing the existence and uniqueness of the solution.

In the second chapter, we focus on control theory for distributed parameter systems, where we present fundamental concepts including controllability, Kalman's Controllability Criterion, optimal control, and related notions.

In the final chapter we establish the necessary conditions to ensure the exacte controllability of our problem. This follows the proof of existence and uniqueness for solutions to a nonlinear fractional differential equation in the Ψ-Caputo sense, using Banach's contraction principle and Schauder's fixed-point theorem. Finally, we provide an illustrative example to demonstrate the obtained results.

Chapter 1

Preliminaries

In this chapter, we present some fundamental theories related to fractional calculus. In this context, the focus will intentionally be on different approaches to generalizing the concepts of differentiation and integration for a fractional order.

1.1 Special functions

1.1.1 Gamma Function

One of the fundamental functions for fractional calculus is the Gamma function, which extends the factorial function to the set of complex numbers.

Definition 1.1 [9] The Gamma function $\Gamma(z)$ is defined by the following integral

$$\Gamma(z) = \int_0^{+\infty} x^{z-1}e^{-x}\mathrm{d}x,\ Re(z) > 0, \tag{1.1}$$

where the improper integral converges absolutely in the complex half-plane where the real part is strictly positive. Obviously, $\Gamma(1) = 1$ and $\Gamma(n+1) = n!$.

Remark 1.1 :
1) The Gamma function is strictly decreasing for $0 < z < 1$, and moreover, we have

$$\Gamma(z+1) = z\Gamma(z), \quad \forall z \in \mathbb{C}.$$

In general, we have

$$\Gamma(z+n) = z(z+1)(z+2)......(z+n-1)\Gamma(z),\ \forall n \geq 1.$$

2) The Gamma function is verified

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{sin(\pi z)}.$$

We will obtain this formula under the condition $0 < Re(z) < 1$ and then show that it holds for $z \neq 0, \ \pm 1, \ \pm 2...$

Special values

$$\Gamma(1/2) = \sqrt{\pi}, \ \Gamma(3/2) = \frac{\sqrt{\pi}}{2}, \ \Gamma(5/2) = \frac{3\sqrt{\pi}}{4}, ...,$$

In general, we have

$$\Gamma(n + \frac{1}{2}) = \frac{(2n)!}{2^{2n}n!}\sqrt{\pi}, \ \forall n \in \mathbb{N}.$$

Legendre's duplication formula:

$$\Gamma(1+z)\Gamma(z+\frac{1}{2}) = 2^{-2z}\sqrt{\pi}\Gamma(2z+1).$$

1.1.2 Beta Function

The so-called Beta function, which is an Euler type integral, insteacl of a certain combination of values of the Gamma function. In some cases the Beta function is more favorable than the Gamma function. Since it is convenient to use it in fractional derivatives of the Power function.

Definition 1.2 Let $z, w \in \mathbb{C}$ such that $Re(z) > 0$ and $Re(w) > 0$, the function beta is defined by

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt. \tag{1.2}$$

Proposition 1.1 *Let $z, w \in \mathbb{C}$ such that $Re(z) > 0, \ Re(w) > 0$. The Beta function B satisfies the following properties:*

- *The Beta function is symmetric, i.e. $B(z,w) = B(w,z)$.*
- $B(z,w) = B(z+1,w) + B(z,w+1)$.
- $B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$.
  This relation provides the continuation of the beta function for the entire complex plane, if we have the continued gamma function.

1.1.3 Mittag-Leffler Function

The Mittag-Leffler function (denoted $E_{\alpha,\beta}$, named after the Swedish mathematician Gösta Mittag-Leffler (1903)) is a special function - meaning it cannot be computed from rational equations - that operates in the complex plane and depends on two parameters. This function is a direct generalization of the exponential function, and it plays a major role in fractional calculus.

Definition 1.3 [9] Let $z \in \mathbb{C}$ such that $Re(z) > 0$. We define the Mittag-Leffler function as follows:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, \tag{1.3}$$

In particular, when $\alpha = 1$ we recover the exponential function $E_1(z) = e^z$.
More generally, the two-parameter Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \quad \alpha, \beta > 0, \tag{1.4}$$

If A is a $n \times n$ matrix, we get

$$E_{\alpha,\beta}(At^\alpha) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha + \beta)}. \tag{1.5}$$

Remark 1.2 :
1) If $\beta = 1$, we get the relation (1.3) because

$$E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)} = E_\alpha(z).$$

2) If $\alpha = \beta = 1$, then

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k + 1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z.$$

3) If $\alpha = 2, \ \beta = 1$, then

$$E_{2,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(2k + 1)} = cosh(\sqrt{z}).$$

4) If $\alpha = n, \ n \in \mathbb{N}$, the Mettag Effler function satisfies the following formulas

$$\left(\frac{d}{dz}\right)^n E_n(\lambda z^n) = \lambda E_n(\lambda z^n),$$

$$\left(\frac{d}{dz}\right)^n [z^{\beta-1} E_{n,\beta}(\lambda z^n)] = z^{\beta-n-1} E_{n,\beta-n}(\lambda z^n).$$

In general,

$$\left(\frac{d}{dz}\right)^n E_{\alpha,\beta}(z) = n! E^{n+1}_{\alpha,\beta+\alpha n}(z),$$

1.2 Fractional Calculus

This section will be devoted to basic definitions of some fractional integrals and derivative.

1.2.1 Riemann-Liouville fractional derivative

The notion of fractional integral of order $\alpha \in \mathbb{C}$ ($Re(\alpha) > 0$), according to the Riemann-Liouville approach, generalizes the famous formula (attributed to Cauchy) of repeated integral n times. Let f be a continuous function on the interval $[0, T]$, $T > 0$. A primitive of f is given by the expression:

$$I^1 f(t) = \int_0^t f(\tau)\, d\tau.$$

For a second primitive, we have:

$$\begin{aligned} I^2 f(t) &= \int_0^t I^1 f(u)\, du = \int_0^t \left(\int_0^u f(s)\, ds \right) du \\ &= \int_0^t \left(\int_s^t du \right) f(s)\, ds = \int_0^t (t-s) f(s) ds. \end{aligned}$$

By repeating n times, we arrive at the nth primitive of the function f in the form:

$$\begin{aligned} I^n f(t) &= \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots\ldots \int_0^{t_{n-1}} f(t_n) dt_n \\ &= \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f(s) ds, \quad n \in \mathbb{N}^*. \end{aligned}$$

This formula is called the Cauchy formula, and since the generalization of the factorial by the Gamma function: $\Gamma(n) = (n-1)!$ Riemann realized that this formula could make sense even when n takes a non-integer value, and he defined the fractional integral as follows:

Definition 1.4 [9] Let $\Omega = [a, b]$ be a finite interval on the real axis $\mathbb{R}$ and f is a continuous function in Ω. The Riemann-Liouville fractional integral $I_a^\alpha f(t)$ of order $\alpha > 0$ of the function f is defined by

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad t > a, \ n < \alpha < n+1. \tag{1.6}$$

Definition 1.5 [9] The Riemann-Liouville fractional derivative (noted by ${}^{RL}D^\alpha$) of order $\alpha > 0$ of the function $f \in L^1(\Omega)$ is defined by

$$\begin{aligned} {}^{RL}D_a^\alpha f(t) &= \left(\frac{d}{dt} \right)^n (I_a^{n-\alpha} f)(t) && (1.7) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s) ds, && (1.8) \end{aligned}$$

with $n-1<\alpha<n,\ n\in\mathbb{N}^*$.

When $\alpha=n$, then ${}^{RL}D_a^\alpha f(t)=f^{(n)}(t)$, where $f^{(n)}$ is the usual derivative of f of order n.

Example 1.1 :

We consider the function f defined by $f(x)=(x-a)^\beta, \quad \beta\in\mathbb{R}$. We have

$$\begin{aligned} I_a^\alpha f(x) &= I_a^\alpha (x-a)^\beta = \frac{1}{\Gamma(\alpha)}\int_a^t (x-s)^{\alpha-1}(s-a)^\beta ds \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(x-a)^{\alpha+\beta}. \end{aligned}$$

Moreover, we have

$${}^{RL}D_a^\alpha (t-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}.$$

Proposition 1.2 *Let $f:[a,b]\to\mathbb{R}$ be a continuous function, the Riemann-Liouville fractional integral has the following property:*

$$I_a^\alpha\left[I_a^\beta f(t)\right]=I_a^{\alpha+\beta}f(t), \quad \alpha,\beta>0.$$

Moreover, we have

$$\frac{d}{dt}\left(I_a^\alpha f\right)(t)=I_a^{\alpha-1}f(t), \quad \alpha>0.$$

1.2.2 Hadamard Fractional Integral and Derivative

Definition 1.6 [12] Let (a,b), $(0<a\le b\le\infty)$ be a finite interval. The Hadamard fractional integral of order α of a function x is defined by

$$I_a^\alpha x(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{x(s)}{s}\,ds, \quad a\le t\le b. \tag{1.18}$$

Definition 1.7 [12] Let (a,b), $(0<a\le b\le\infty)$ be a finite interval. The Hadamard fractional derivative of order α of a function x is defined as follows:

$$D_a^\alpha x(t)=\frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n\int_a^t\left(\log\frac{t}{s}\right)^{n-\alpha-1}\frac{x(s)}{s}\,ds, \quad n=[\alpha]+1, \quad a\le t\le b. \tag{1.19}$$

Lemma 1.1 *If $a>0$ and $\beta>\alpha>0$, then*

$$D_a^\alpha\left(\log\frac{x}{a}\right)^{\beta-1}(t)=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log\frac{x}{a}\right)^{\beta-\alpha-1}, \tag{1.9}$$

and

$$I_a^\alpha\left(\log\frac{x}{a}\right)^{\beta-1}(t)=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(\log\frac{t}{a}\right)^{\beta+\alpha-1}. \tag{1.10}$$

1.2.3 Caputo fractional derivative

The Riemann-Liouville formulation involves initial conditions that incorporate the boundary values of its fractional derivatives at the lower limit. Although these initial value problems can be addressed mathematically, their solutions lack practical utility due to the absence of a clear physical interpretation for such conditions. To resolve this issue, M. Caputo introduced an alternative approach.

Definition 1.8 [9] Let $[a,b]$ be a finite interval of the real line $\mathbb{R}$ and let f be a function of class $C^n([a,b])$. The fractional Caputo derivative ${}^cD_a^\alpha$ of order $\alpha>0$ of the function f is defined through the Riemann-Liouville fractional derivative, that is to say

$$ {}^cD_a^\alpha f(t) = {}^{RL}D_a^\alpha \left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k \right], \tag{1.11} $$

with $n-1<\alpha<n, \quad n\in\mathbb{N}^*$.
We deduce that if $f^k(a)=0$ for $k=0,1,2,...,n-1$, we get

$$ {}^cD_a^\alpha f(t) = {}^{RL}D_a^\alpha f(t). $$

Definition 1.9 [9] The Caputo derivative of order $\alpha>0$ of the function f of class $C^n([a,b])$ is defined by

$$ {}^cD_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds, \quad t>a, $$

with $n-1<\alpha<n$, $n\in\mathbb{N}^*$.
When $\alpha=n\in\mathbb{N}$, then

$$ {}^cD_a^\alpha f(t) = f^{(n)}(t), $$

where $f^{(n)}$ is the usual derivative of f of order n.

Remark 1.3 One difference between the Riemann-Liouville definition and the Caputo definition is that the Caputo derivative of a constant is zero, where as the Riemann-Liouville fractional derivative of a constant C is

$$ {}^{RL}D_a^\alpha C = \frac{Ct^{-\alpha}}{\Gamma(1-\alpha)} \neq 0. $$

Lemma 1.2 *[9]:*
1) Let $f\in C([a,b])$, $\forall t\in[a,b]$, $\forall\alpha\in]n-1,n[$, we have the following propertie

$$ {}^cD_a^\alpha I_a^\alpha f(t) = f(t). $$

2) Let $f \in C^n([a,b])$, $\forall t \in [a,b]$, $\forall \alpha \in]n-1, n[$, we have the following propertie

$$I^{\alpha}({}^{c}D^{\alpha})f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k. \tag{1.12}$$

Theorem 1.1 *[12] Let $Re(\alpha) > 0$. If $f \in C^n[a,b]$ then the Caputo fractional derivative ${}^{c}D_a^{\alpha} f$ exist almost every where on $[a,b]$ and we have*

$$ {}^{c}D_a^{\alpha} f(t) = I^{n-\alpha} D^n f(t), \quad t > a. \tag{1.13}$$

1.2.4 Ψ-Caputo fractional derivative

Some primitive notions, definitions and notations about $\Psi-$Caputo derivative are recalled in this section. Let $[a,b]$ be a finite interval of the real line $\mathbb{R}$ and $\alpha > 0$. Let $\Psi \in C^1([a,b], \mathbb{R})$ be an increasing function having a continuous derivative such $\Psi'(t) \neq 0$ on $[a,b]$.

Definition 1.10 [9] The α^{th} Ψ- integral for an integrable function $\mathrm{x} : [a,b] \to \mathbb{R}$ with respect to Ψ is given as

$$\mathscr{I}_{a^+}^{\alpha;\Psi} \mathrm{x}(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (\Psi(t) - \Psi(s))^{\alpha-1} \Psi'(s) \mathrm{x}(s)\, \mathrm{d}s, \ t > a, \tag{1.14}$$

in which $n < \alpha < n+1, n \in \mathbb{N}$.

Remark 1.4 :

- We can write $\mathscr{I}_{a^+}^{\alpha;\Psi} \mathrm{x}(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \breve{\Phi}_s^{\alpha}(t) \mathrm{x}(s)\, \mathrm{d}s$, where $\breve{\Phi}_s^{\alpha}(t) = (\Psi(t) - \Psi(s))^{\alpha-1} \Psi'(s)$.
- When $\Psi(t) = t$, the Ψ-Caputo fractional derivative corresponds to the classical Caputo fractional derivative.
- When $\Psi(t) = \ln t$, the Ψ-Caputo fractional derivative corresponds to the Hadamard fractional derivative.

Definition 1.11 [9] Let $n \in \mathbb{N}$, $\mathrm{x} \in C^n([a,b], \mathbb{R})$, and $\Psi \in C^n([a,b], \mathbb{R})$. The α^{th} Ψ-Caputo derivative of x is defined by

$$ {}^{c}\mathscr{D}_{a^+}^{\alpha;\Psi} \mathrm{x}(t) = \mathscr{I}_{a^+}^{n-\alpha;\Psi}\ \partial_{\Psi}^{n} \mathrm{x}(t),$$

in which $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$, $n = \alpha$ for $\alpha \in \mathbb{N}$ and $\partial_{\Psi} = \left(\frac{1}{\Psi'(t)} \frac{\mathrm{d}}{\mathrm{d}t} \right)$.
In other words,

$$ {}^{c}\mathscr{D}_{a^+}^{\alpha;\Psi} \mathrm{x}(t) = \begin{cases} \int_a^t \frac{\Psi'(s)(\Psi(t)-\Psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)} \partial_{\Psi}^{n} \mathrm{x}(s)\, \mathrm{d}s, & \alpha \notin \mathbb{N}, \\ \partial_{\Psi}^{n} \mathrm{x}(t), & \alpha = n \in \mathbb{N}. \end{cases}$$

Remark 1.5 :

If $\mathrm{x} \in C^{n-1}([a,b],\mathbb{R})$, the α^{th} Ψ-Caputo derivative of x is specified as

$$^{c}\mathscr{D}_{a^+}^{\alpha;\Psi}\mathrm{x}(t) = \mathscr{D}_{a^+}^{\alpha;\Psi}\left(\mathrm{x}(t) - \sum_{k=0}^{n-1}\frac{\partial_{\Psi}^{k}\mathrm{x}(a)}{k!}(\Psi(t)-\Psi(a))^{k}\right).$$

The composition rules for above Ψ-operators are given in this lemma.

Lemma 1.3 *[9] Let $n-1<\alpha<n$, $\Psi \in C^{n}([a,b],\mathbb{R})$, and $\mathrm{x} \in C^{n-1}([a,b],\mathbb{R})$. Then the following holds*

$$\mathscr{I}_{a^+}^{\alpha;\Psi}\ {}^{c}\mathscr{D}_{a^+}^{\alpha;\Psi}\mathrm{x}(t) = \mathrm{x}(t) - \sum_{k=0}^{n-1}\frac{\partial_{\Psi}^{k}\mathrm{x}(a)}{k!}\left[\Psi(t)-\Psi(a)\right]^{k},$$

for all $t \in [a,b]$. Moreover, if $m \in \mathbb{N}$ and $\mathrm{x} \in C^{n+m-1}([a,b],\mathbb{R})$, then, the following holds:

$$\partial_{\Psi}^{m}\left({}^{c}\mathscr{D}_{a^+}^{\alpha;\Psi}\mathrm{x}\right)(t) = {}^{c}\mathscr{D}_{a^+}^{\alpha+m;\Psi}\mathrm{x}(t) + \sum_{k=0}^{n-1}\frac{[\Psi(t)-\Psi(a)]^{k+n-\alpha-m}}{\Gamma(k+n-\alpha-m+1)}\partial_{\Psi}^{k+m}\mathrm{x}(a).$$

Observe that if $\partial_{\Psi}^{k}\mathrm{x}(a)=0$, $\forall\, k=n,n+1,\ldots,n+m-1$, we can get the following relation

$$\partial_{\Psi}^{m}\left({}^{c}\mathscr{D}_{a^+}^{\alpha;\Psi}\mathrm{x}\right)(t) = {}^{c}\mathscr{D}_{a^+}^{\alpha+m;\Psi}\mathrm{x}(t), \qquad t \in [a,b].$$

Lemma 1.4 *[12] Let $\alpha, l > 0$, and $x \in C([a,b],\mathbb{R})$. Then $\forall t \in [a,b]$ and we suppose that $\mathscr{Q}_a(t) = \Psi(t)-\Psi(a)$, we have*

1. $\mathscr{I}_{a}^{\alpha;\Psi}\mathscr{I}_{a}^{l;\Psi}x(t) = \mathscr{I}_{a}^{\alpha+l;\Psi}x(t)$.

2. ${}^{c}\mathscr{D}_{a^+}^{\alpha;\Psi}\mathscr{I}_{a}^{\alpha;\Psi}x(t) = x(t)$.

3. $\mathscr{I}_{a}^{\alpha;\Psi}(\mathscr{Q}_a(t))^{l-1} = \dfrac{\Gamma(l)}{\Gamma(l+\alpha)}(\mathscr{Q}_a(t))^{l+\alpha-1}$.

4. ${}^{c}\mathscr{D}_{a^+}^{\alpha;\Psi}(\mathscr{Q}_a(t))^{l-1} = \dfrac{\Gamma(l)}{\Gamma(l-\alpha)}(\mathscr{Q}_a(t))^{l-\alpha-1}$.

5. ${}^{c}\mathscr{D}_{a^+}^{\alpha;\Psi}(\mathscr{Q}_a(t))^{k} = 0,\ k \in \{0,...,n-1\},\ n \in \mathbb{N},\ \alpha \in (n-1,n)$.

Lemma 1.5 *[9] If $\gamma > 0$ and $\lambda \in \mathbb{C}$, then*

$$^{c}D_{a}^{\gamma}(E_{\gamma}(\lambda(t-a)^{\gamma})(x)) = \lambda E_{\gamma}(\lambda(x-a)^{\gamma}), \tag{1.15}$$

and

$$^{c}D_{a}^{\gamma}t^{\gamma-1}(E_{\gamma}(\lambda t^{-\gamma})(x)) = \frac{1}{x}E_{\gamma,1-\gamma}(\lambda x^{-\gamma}), \tag{1.16}$$

where E_{γ} is the Mettag Leffler function.

1.3 Laplace Transform

As in the integer-order case, the Laplace transform is used to solve fractional-order differential equations. It is a tool that converts a differential equation into a linear equation where derivative forms disappear.

Definition 1.12 The Laplace transform of a function $f(t)$ of a real positive variable $t \in (0, +\infty)$ is the function $F(\lambda)$ defined by:

$$F(\lambda) = (\mathscr{L}f)(\lambda) = \mathscr{L}\{f(t)\}(\lambda) = \int_0^{+\infty} e^{-\lambda t} f(t)\, dt, \quad \lambda \in \mathbb{C}. \tag{1.17}$$

The Laplace transform of f exists if the integral (1.17) converges.

Properties of the Laplace Transform:

1. The Laplace transform is a linear operator, meaning that for any functions f and g with Laplace transforms and for any real numbers α, β we have:

$$\mathscr{L}\{\alpha f + \beta g\} = \alpha \mathscr{L}\{f\} + \beta \mathscr{L}\{g\}.$$

2. Let $F(\lambda)$ and $G(\lambda)$ be the Laplace transforms of $f(t)$ and $g(t)$, respectively. Then, the convolution product $(f * g)$ is given by:

$$\mathscr{L}\{(f * g)(t)\}(\lambda) = F(\lambda) \cdot G(\lambda) = \mathscr{L}\left\{\int_0^t f(t - z) g(z)\, dz\right\}. \tag{1.18}$$

3. The Laplace transform of the n-th derivative of a function f is:

$$\mathscr{L}\{f^{(n)}(t)\}(\lambda) = \lambda^n F(\lambda) - \sum_{k=0}^{n-1} \lambda^k f^{(n-k-1)}(0).$$

Definition 1.13 [9] The inverse Laplace transform of a function $g(\lambda)$ is given by:

$$(\mathscr{L}^{-1}g)(t) = \mathscr{L}^{-1}\{g(\lambda)\}(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} g(s)\, ds, \tag{1.19}$$

where γ is chosen such that the integral converges.

Definition 1.14 [5] The Laplace transform formulas for the fractional integral and the Caputo derivative are given by:

$$\begin{cases} \mathscr{L}\left\{{}^cD^\alpha f(t)\right\}(\lambda) = \lambda^\alpha F(\lambda) - \sum_{k=0}^{n-1} f^{(k)}(0)\lambda^{\alpha-k-1}, & (n - 1 < \alpha \le n), \\ \mathscr{L}\left\{I_0^{1-\alpha} f(t)\right\}(\lambda) = \lambda^{\alpha-1} F(\lambda). \end{cases} \tag{1.20}$$

Moreover, if A is a $n \times n$ matrix, then

$$\begin{cases} \mathscr{L}^{-1}\left\{\lambda^{\alpha-1}(\lambda^{\alpha}I-A)^{-1}\right\}(t)=E_{\alpha}(At^{\alpha}), \\ \mathscr{L}^{-1}\left\{(\lambda^{\alpha}-A)^{-1}\right\}(t)=t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha}). \end{cases} \tag{1.21}$$

Lemma 1.6 *[11] Let $y \in C^n(a,b)$ and $y^{(n)} \in L^1(a,b)$. The Laplace transform of the Ψ-Caputo fractional derivative is given by the following relation*

$$\mathscr{L}\left\{{}^{c}\mathscr{D}_{a^+}^{\alpha;\Psi}y(t)\right\}(\lambda)=\lambda^{\alpha}\mathscr{L}(y(t))-\sum_{k=0}^{n-1}\lambda^{\alpha-k-1}y^{(k)}(a). \tag{1.22}$$

with $n-1<\alpha\leq n,(n\in\mathbb{N}^*)$.

Lemma 1.7 *[9] Let $\gamma>0$, $y \in C^n(a,b)$ and $y^{(n)} \in L^1(a,b)$. Then the following relations holds*

$$\begin{cases} \mathscr{L}^{-1}\left\{\lambda^{\gamma-1}(\lambda^{\gamma}I-A)^{-1}\right\}(t)=E_{\gamma}(A(\mathscr{Q}_a(t))^{\gamma}), \\ \\ \mathscr{L}^{-1}\left\{(\lambda^{\gamma}I-A)^{-1}\right\}(t)=\Phi_s^{\gamma}(t)E_{\gamma,\gamma}(A(\mathscr{Q}_a(t))^{\gamma}), \end{cases} \tag{1.23}$$

where $\mathscr{L}^{-1}$ is the inverse Laplace transform.

1.4 Point fixed Theorems

Definition 1.15 [14] A set U of a normed space X is relatively compact if the closure $\overline{U}$ is compact it means if every sequence of points in U has a cluster point in X.

Remark 1.6 Relatively compact sets are granting some compactness properties. They are commonly used to study the convergence and properties of sequences and continuous functions.

Definition 1.16 [14] Let (E,d) be a metric space. We say that (E,d) is a compact space if and only if, for every open covering of E, we can extract a finite open subcovering.

A space is compact if it is relatively compact in itself.

Definition 1.17 [14] A bounded linear operator Φ acting from a Banach space X into another Banach space Y is completely continuous if it transforms weakly-convergent sequences in X to norm-convergent sequences in Y.
Equivalently, the operator Φ is completely continuous if it maps every relatively weakly compact subset of X into a relatively compact subset of Y.

Remark 1.7 :

1) It is easy to see that every completely continuous operator is compact, however the converse is false.

2) If X is reflexive, the two classes of operators (completely continuous operator and compact) do coincide.

1.4.1 Ascoli-Arzela Theorem

Arzela-Ascoli theorem, demonstrated by Italian mathematicians Giulio Ascoli and Cesare Arzela, characterizes, using the notion of equicontinuity, the relatively compact subsets of the space of continuous functions from a compact space into a metric space.

Theorem 1.2 *[4] Let $(X, \|.\|_X)$ be a compact normed space and $(Y, \|.\|_Y)$ a complete normed space, a subset A of $C(X, Y)$ is relatively compact if and only if:*

1) A is equicontinuous if it is equicontinuous at all $a \in X$ i.e., for all $\epsilon > 0$, $\exists \delta > 0$ such that

$$\forall x \in X, \|x - a\|_X < \delta \Rightarrow \forall \Phi \in A, \|\Phi(x) - \Phi(a)\|_Y < \epsilon, .$$

2) The set A is uniformly bounded, i.e., there exists a constant $K > 0$ such that

$$\|\Phi(x)\|_X \leqslant K, \ \forall \ x \in X \text{ and } \Phi \in A.$$

1.4.2 Banach fixed-point theorem

Theorem 1.3 *[4] Let $X = (X, \|.\|_X)$ be a Banach space, and let $\Phi : X \to X$ be a contraction mapping on X i.e. such that*

$$\exists \ 0 < k < 1, \ \|\Phi u - \Phi v\| \leq k\|u - v\|, \ \forall \ u, v \in X.$$

Then, Φ admits a unique fixed point u in X, i.e $\Phi u = u$.

1.4.3 Schauder fixed point theorem

Theorem 1.4 *[4] Let $X = (X, \|.\|_X)$ be a Banach space and let U be a closed convex subset of X. Let $\Phi : U \to U$ be a continuous and compact mapping. Then Φ admits a fixed point belonging to U.*

Chapter 2

Controllability of distributed system

Controllability is a fundamental concept in the analysis of dynamic systems. It refers to the ability to impose a desired behaviour on a system, meaning to move a system from an arbitrary initial state to a desired state in finite time using a control.

A control system is a dynamic system depending on a parameter called the control, usually subject to constraints. Among the main objectives of control theory, which will be discussed in this work, is the notion of controllability.

2.1 Controllability

Let $T > 0$, consider a linear differential system defined on $[0, T]$ by

$$\begin{cases} x'(t) = Ax(t) + Bu(t), \\ x(0) = x_0, \end{cases} \tag{2.1}$$

where A is a square matrix $(n \times n)$ called the state matrix, and B is a matrix $(n \times m)$ called the control matrix, $x(t)$ is the state of the system, and x_0 is the initial condition. The solution of (2.1) is given by:

$$x(t, x_0, u) = e^{tA}x_0 + \int_0^t e^{(t-s)A}Bu(s)ds. \tag{2.2}$$

Several notions of controllability can be defined. The most important ones are exact controllability, approximate controllability, and zero controllability.

2.1.1 Exacte Controllability

Definition 2.1 [8] The system (2.1) is exactly controllable at time T if, for all states x_0, x_1 in the state space, there exists an admissible control u such that

$$x_1 = x(T, x_0, u). \tag{2.3}$$

Remark 2.1 If $x_1 = 0$, the system (2.1) is said to be "zero-controllable" or have "zero controllability" .

Definition 2.2 We define the *controllability space* (or *reachable space*), denoted $\mathscr{C}$, as the set of all states reachable from the initial state x_0. Mathematically:

$$\mathscr{C} = \{x \in \mathbb{R}^n : \exists u \in L^2(0, T; \mathbb{R}^m) \text{ such that } x(T) = x_1\}. \tag{2.4}$$

2.1.2 Approximate Controllability

Approximate controllability is a property of a dynamical system that ensures it can be steered arbitrarily close to any desired state within a given time frame, even if exact controllability (reaching the target state precisely) is not guaranteed.

Definition 2.3 [8] Let $T > 0$. The control system (2.1) is approximately (weakly) controllable at time T if, for all x_0, x_1in the state space and for all $\varepsilon > 0$, there exists a control u such that the solution of the system satisfies:

$$\|x(T, x_0, u) - x_1\| \leq \varepsilon. \tag{2.5}$$

Example: Heat equation is approximately controllable but not exactly controllable due to smoothing effects (because heat diffuses everywhere, allowing the state to be steered close to any target profile).

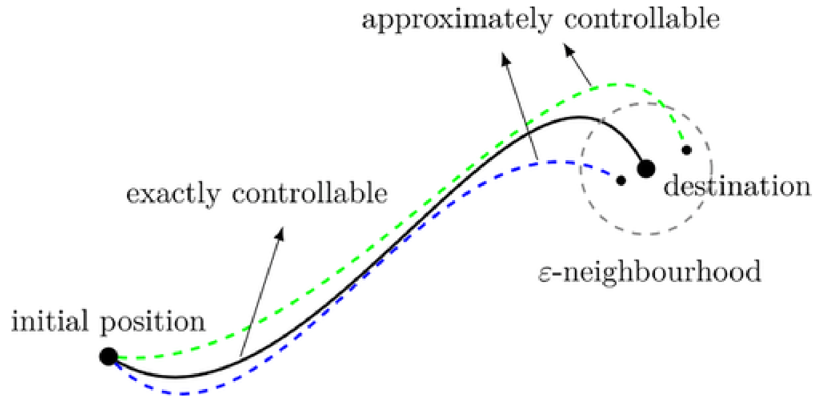

Figure :Exacte and approximate controllability

Remark 2.2 :

1) Weak or approximate controllability is easier to verify in applications.

2) Exacte controllability is the strongest of the three notions.

Example:

Steering the rudder of an airplane will change its direction (yaw), but it will not affect its altitude. Conversely, adjusting the angle of the wings to ascend or descend will change the altitude but not the direction. This illustrates that certain variables in a system cannot be influenced by specific inputs, which means the system is not fully controllable from those inputs.

2.1.3 Kalman's Controllability Criterion

Here, we will give the famous Kalman controllability condition of a finite-dimensional linear system, this condition is based on an algebraic condition, it is the rank of a matrix by block formed by A and B as follows.

Definition 2.4 Controllability Matrix:

The matrix

$$W = (B, AB, \cdots, A^{n-1}B), \tag{2.6}$$

is called the Kalman controllability matrix.

The following theorem provides a necessary and sufficient condition for controllability when A and B do not depend on time and there are no constraints on the control.

Theorem 2.1 *[17] The linear system (2.1) is controllable if and only if:*

$$rank(W) = n. \tag{2.7}$$

We also say that the pair (A, B) is controllable.

Remark 2.3 :

1) The matrix W is called the kalman matrix and the condition is referred to as the "kalman condition" this condition depends on the initial data, in other words, ifx_0 is a system ,the autonomous linear system is controllable in time T since it is controllable at all times from any points.

2) If the matrix A defining the system (2.1 is diagonal with distinct elements two by two, then the system is controllable if and only if the matrix B has no null columns.

Example 2.1 Consider a dynamic system described by:

$$\begin{cases} x'(t) = Ax(t) + Bu(t), \\ x(0) = x_0 \end{cases}$$

where the matrices A and B are given by:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This system is controllable because the controllability matrix W has full rank. Indeed,

$$AB = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \Rightarrow W = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$$

$$det(W) = -1 \Rightarrow rankW = 2.$$

we give here a result that allows us to provide a characterization of controllability.

2.1.4 Characterization of Controllability

For all $t \in [0, T]$, we can write the solution (2.2) of the system (2.1) in the form:

$$x(t, x_0, u) = X_0 + L_t u, \tag{2.8}$$

where $L_t u$ is the bounded linear operator defined by:

$$L_t : \begin{cases} L_2(0, T, U) \to \mathbb{R}^n \\ u \to \int_0^t e^{(t-s)A} Bu(s)ds, \end{cases} \tag{2.9}$$

and

$$X_0 = e^{tA} x_0.$$

For simplicity, let us take $X_0 = 0$.

We consider its adjoint L_t^* given by:

$$L_t^* : \begin{cases} \mathbb{R}^n \to L^2(0, T, U) \\ z \to L_t^*(z) = B^* e^{A^*(T-t)} z, \end{cases}$$

such that

$$(L_t^*(z), u) = (z, L_t u), \forall u \in L^2(0, T, U), \quad \forall z \in \mathbb{R}^n. \tag{2.10}$$

Proposition 2.1 *[17] The system (2.1) is exactly controllable at time $T > 0$ if and only if the operator L_T is surjective, i.e.,*

$$\forall x_0, x_1 \in \mathbb{R}^n,\ \exists u \in U_{ad},\ x(x_0, u)(T) = x_1.$$

Proof. Let $x_0, x_1 \in \mathbb{R}^n$ be two arbitrary states. The equation $x(T, x_0, u) = x_1$, has a solution in $L^2(0, T, U)$ if and only if the equation

$$L_T u = x_1 - e^{TA} x_0,$$

has a solution in $L^2(0, T, U)$. The equivalence of the two equations leads to the proposition. ∎

Remark 2.4 From the previous proposition, we can say that the system (2.1) is controllable if and only if $Im(L_T) = \mathbb{R}^n$.

The matrix W has full rank if and only if the linear operator L_t is surjective.

Proposition 2.2 *[1] There is an equivalence between*

- *The system (2.1) is weakly controllable,*
- $\overline{Im}(L_T) = \mathbb{R}^n$.
- $\overline{Im}(L_T L_T^*) = \mathbb{R}^n$,
- $Ker(L_T^*) = \{0\}$.

2.2 Controllability Gramian

We introduce the controllability matrix called the "Controllability Gramian" by:

$$W = L_T L_T^* = \int_0^T e^{sA} BB^* e^{sA^*} ds, \tag{2.11}$$

where A^* and B^* denote the transpose matrices of A and B.

Corollary 2.1 *[8] The following properties are equivalent:*

1. *The pair (A, B) is controllable at time $T > 0$.*
2. *The operator L_t is surjective.*
3. *The operator L_t^* is injective.*
4. *The matrix W is invertible.*

Remark 2.5 : The controllability matrix W is always positive because

$$\langle Wx, x\rangle = \int_0^T |B^* e^{sA^*} x|^2 ds = \|L_T^* x\|^2 \geqslant 0. \tag{2.12}$$

2.3 Optimal Control

In this section, we will determine the optimal control that allows reaching a given target.
In the case where the system is controllable, there will generally be an infinite number of controls that answer the question.
Optimization is used to find the control that ensures controllability with a minimal cost given by a function. It is interesting to construct one that "consumes the least energy"
The energy functional we choose here is

$$J(u) = \int_0^T \|u(s)\|^2 ds.$$

We denote

$$U_{ad} = \{u \in U, x(T, x_0, u) = x_1\}.$$

Thus, we seek the solution to the optimization problem with the following constraint:

$$(P)\ \{\min J(u), \quad u \in U_{ad}\}. \tag{2.13}$$

Two questions arise:

- Find the existence of an optimal control.
- Find a way to compute it, that is, describe a method to calculate the control in terms of the various parameters of the problem.

Theorem 2.2 *[17] The problem (P) has a unique solution if and only if :*
• J is continuous, coercive and strictly convex,
• U_{ad} is convex, closed and non-empty

the following theorem defines the unique control u that minimizes the functional J over the set U_{ad}

Theorem 2.3 *[17] the control u transfers x_0 in $x_1 = x(T, x_0, u)$ is given by:*

$$u(s) = B^* e^{(T-s)A^*} W_T^{-1}(x_1 - e^{TA} x_0). \tag{2.14}$$

Chapter 3

Controllability for a class of Ψ-caputo fractionnal system

In this chapter, we investigate the controllability of a nonlinear fractional problem with nonlocal initial conditions in the sense of the Ψ-Caputo derivative but firstly, we establish the existence and uniqueness of solutions to the problem. To achieve this, we employ two key mathematical tools: Schauder's fixed-point theorem and the Banach contraction principle.

3.1 Problem statement

We consider the following fractional differential problem

$$\begin{cases} {}^{c}\mathscr{D}^{\alpha;\Psi}\ x(t) = Ax(t) + f(t, x(t)) + Bu(t), \quad t \in J = (0, T], \\ x(0) = h(x), \end{cases} \tag{3.1}$$

where

- ${}^{c}\mathscr{D}^{\alpha;\Psi}$ is the fractional derivative in the sense of Ψ-Caputo of ordre α such that $0 < \alpha < 1$.
- $x(t) \in \mathbb{R}^n$ is the state vector.
- $u(t) \in \mathbb{R}^m$ is the control.
- A is (n, n) square matrix ans B is (n, m) matrix.
- $h : \mathbb{R}^n \to \mathbb{R}^n$ and $f : J \times \mathbb{R}^n \to \mathbb{R}^n$ are a given functions.
- $x(0)$ is the initial condition.

3.2 Integral equation

In order to prove the main theorems of existence and uniqueness of the solution of the fractional problem (3.1), we present the following key lemma, which describes the corresponding integral equation. We prove an equivalence result between the differential equation of our problem and the integral equation.

Lemma 3.1 *Let η be a continuous function. Consider the following linear system*

$$\begin{cases} {}^{c}\mathscr{D}^{\alpha;\Psi}x(t) = Ax(t) + Bu(t) + \eta(t), \quad t \in J \\ x(0) = h(x). \end{cases} \tag{3.2}$$

Then, x is the solution of the problem (3.2) if and only if x satisfies the following integral equation

$$x(t) = h(x) + \frac{1}{\Gamma(\alpha)}\int_0^t (\mathscr{Q}_s(t))^{\alpha-1}\Psi'(s)(Ax(s) + Bu(s) + \eta(s))ds. \tag{3.3}$$

One can rewrite the solution in terms of the Mittag-Leffler function

$$x(t) = E_\alpha(A(\mathscr{Q}_0(t))^\alpha)h(x) + \int_0^t \breve{\Phi}_s^\alpha(t)E_{\alpha,\alpha}(A(\mathscr{Q}_s(t)^\alpha)(Bu(s) + \eta(s))\,ds, \tag{3.4}$$

Proof. :

1) Performing $\mathscr{I}^{\alpha;\Psi}$ to both sides of (3.2) and using definition (1.10), one can get the following integral equation:

$$\mathscr{I}^{\alpha;\Psi}\left[{}^{c}\mathscr{D}^{\alpha;\Psi}x(t)\right] \quad = \quad \mathscr{I}^{\alpha;\Psi}\left[Ax(t) + Bu(t) + \eta(t))\right].$$

Appling the Lemme 1.3, it means

$$\mathscr{I}^{\alpha;\Psi}\ {}^{c}\mathscr{D}^{\alpha;\Psi}x(t) = \mathrm{x}(t) - \sum_{k=0}^{n-1}\frac{\partial_\Psi^k x(0)}{k!}\left[\Psi(t) - \Psi(0)\right]^k, \quad n-1 < \alpha < n,$$

we obtain

$$x(t) - x(0) = \frac{1}{\Gamma(\alpha)}\int_0^t (\mathscr{Q}_s(t))^{\alpha-1}\Psi'(s)\left[Ax(s) + Bu(s) + \eta(s))\right]ds.$$

2) To obtain the formula (3.4), applying the Laplace transform to both sides of the first equation of system (3.2), we get

$$\mathscr{L}\{\ {}^{c}\mathscr{D}^{\alpha;\Psi}x(t)\}(\lambda) = \mathscr{L}\{Ax(t) + Bu(t) + \eta(t)\}(\lambda),$$

by using Lemma (1.6), we obtain

$$\lambda^{\alpha}\mathscr{L}(x(t))(\lambda) - \lambda^{\alpha-1}x(0) = A\mathscr{L}(x(t))(\lambda) + \mathscr{L}\{Bu(t) + \eta(t)\}(\lambda).$$

Therefore,

$$(\lambda^{\alpha}I - A)\mathscr{L}(x(t))(\lambda) = \lambda^{\alpha-1}x(0) + \mathscr{L}\{Bu(t) + \eta(t)\}(\lambda).$$

We find

$$\mathscr{L}(x(t))(\lambda) = \lambda^{\alpha-1}(\lambda^{\alpha}I - A)^{-1}h(x) + (\lambda^{\alpha}I - A)^{-1}\mathscr{L}\{Bu(t) + \eta(t)\}(\lambda).$$

Taking the inverse Laplace transform of the above side equation, we obtain

$$x(t) = \mathscr{L}^{-1}\{\lambda^{\alpha-1}(\lambda^{\alpha}I - A)^{-1}\}h(x) + \mathscr{L}^{-1}\{(\lambda^{\alpha}I - A)^{-1}\mathscr{L}\{Bu(t) + \eta(t)\}(\lambda)\}.$$

By virtue of Lemma (1.7), we get

$$x(t) = E_{\alpha}(A(\mathscr{Q}_0(t))^{\alpha})(h(x)) + \int_0^t \breve{\Phi}_s^{\alpha}(t)E_{\alpha,\alpha}(A(\mathscr{Q}_s(t)^{\alpha})(Bu(s) + \eta(s))\,ds.$$

Alternatively, by applying the derivative ${}^{c}\mathscr{D}^{\alpha;\Psi}$ to equation (3.4), one obtains problem (3.2). The proof is complete. ■

Remark 3.1 :
Lemme (3.1) establishes the equivalence between the nonlinear problem (3.1) and the following integral equation

$$x(t) = h(x) - \frac{1}{\Gamma(\alpha)}\int_0^t (\mathscr{Q}_s(t))^{\alpha-1}\Psi'(s)(Ax(s) + Bu(s) + f(s, x(s)))ds. \tag{3.5}$$

Or

$$x(t) \;=\; E_{\alpha}(A(\mathscr{Q}_0(t))^{\alpha})h(x) + \int_0^t \breve{\Phi}_s^{\alpha}(t)E_{\alpha,\alpha}(A(\mathscr{Q}_s(t)^{\alpha})(Bu(s) + f(s, x(s))\,ds.$$

In the next section, we formulate and prove the existence and the uniqueness for solution of the nonlinear fractional system (3.1) in the Ψ-Caputo sense.

3.3 Existence and uniqueness

Throughout this work, we denote $\Xi = C(J, \mathbb{R}^n)$ the Banach space of continuous functions in J which equipped by the norm

$$\|x\|_{\Xi} = \sup_{t\in J}|x(t)|.$$

We define the operator $\Lambda : \Xi \to \Xi$ by

$$\Lambda x(t) = h(x) + \frac{1}{\Gamma(\alpha)}\int_0^t (\mathscr{Q}_s(t))^{\alpha-1}\Psi'(s)(Ax(s) + Bu(s) + f(s, x(s)))ds. \tag{3.6}$$

Having established the necessary groundwork, we can now proceed to derive the existence criteria for the solution of the nonlinear fractional problem (3.1).

The following assumptions will be considered:

(A_1) For any $t \in J$, the function $x \to f(t, x)$ satisfies the Lipschitz condition:

$\forall x_1, x_2 \in \Xi,\ \exists L_f > 0$, such that

$$\|f(t, x_1) - f(t, x_2)\|_{\Xi} \leq L_f \|x_1 - x_2\|_{\Xi}. \tag{3.7}$$

(A_2) There exist constant $M > 0$ such that : $\sup_{t\in J}\|f(t, x)\| \leq M, \quad \forall x \in \Xi.$

(A_3) The function h is continuous and there exist a positive constant μ_1 such that

$$\|h(x)\| \leq \mu_1\|x\|. \tag{3.8}$$

and $\forall x, y \in \Xi,\ \exists k > 0$, such that

$$\|h(x) - h(y)\| \leq k\|x - y\|. \tag{3.9}$$

(A_4) There exist $\theta \in \mathbb{R}^+$, with

$$\theta = k + \frac{\|A\| + L_f}{\Gamma(\alpha + 1)}\|\mathscr{Q}(T)\|^{\alpha} < 1.$$

Theorem 3.1 *Given the conditions $(A_1) - (A_3)$, the fractional system (3.1) has at least one solution over the interval J.*

Proof. We shall show that the map $\Lambda : \Xi \to \Xi$ has a fixed point based on Schauder fixed point theorem.

First Step:

we prove that Λ is continuous operator.

Let $(x_n)_{n\in N}$ a sequcence of real numbers of Ξ, such that $\|x_n - x\| \to 0 \quad as \quad n \to +\infty$.

We define the norm between Λx_n and Λx as follows:

$$\|\Lambda x_n - \Lambda x\|_\Xi = \sup_{t\in[0,T]} |\Lambda x_n(t) - \Lambda x(t)|.$$

Then, $\forall t \in [0, T]$, we have

$$\begin{aligned}
|\Lambda x_n(t) - \Lambda x(t)| &\leq |h(x_n) - h(x)| + \frac{1}{\Gamma(\alpha)} \int_0^t |\mathscr{Q}_s(t)|^{\alpha-1} |\Psi'(s)| \big(|A(x_n(s) - x(s))| \\
&+ |f(s, x_n(s)) - f(s, x(s)|\big) ds \\
&\leq k\|x_n - x\| + \frac{1}{\Gamma(\alpha)} (\|A\| + L_f)\|x_n - x\| \int_0^T |\mathscr{Q}_s(t)|^{\alpha-1} |\Psi'(s)| ds \\
&\leq \left(k + \frac{\|A\| + L_f}{\Gamma(\alpha+1)} \|\mathscr{Q}_0(T)\|^\alpha\right) \|x_n - x\|.
\end{aligned}$$

Since $\|x_n - x\|_\Xi \to 0$ as $n \to \infty$, this implies that: $\|\Lambda x_n - \Lambda x\|_\Xi \to 0 \quad \text{as} \quad n \to \infty$.

Consequently, Λ is continuous.

Second Step:

we show that Λ transfers bounded sets into bounded sets in Ξ.

Suppose that there exists ρ such that

$$\rho > \frac{(\|Bu\| + M)|\mathscr{Q}_0(T)|^\alpha}{(1 - \mu_1)\Gamma(\alpha + 1) - \|A\||\mathscr{Q}_0(T)|^\alpha} > 0.$$

For any x in the subset B_ρ of the space Ξ given by

$$B_\rho = \{x \in \Xi, \|x\|_\Xi \leq \rho\}.$$

We have

$$\begin{aligned}
|\Lambda x(t)| &\leq |h(x)| + \frac{1}{\Gamma(\alpha)} \int_0^t |\mathscr{Q}_s(T)|^{\alpha-1} |\Psi'(s)| |(Ax(s) + Bu(s) + f(s, x(s))| ds \\
&\leq \mu_1\|x\| + \frac{1}{\Gamma(\alpha)} \int_0^t |\mathscr{Q}_s(T)|^{\alpha-1} |\Psi'(s)| |(Ax(s) + Bu(s) + f(s, x(s))| ds \\
&\leq \mu_1\|x\| + \frac{\|A\|\|x\| + \|Bu\| + M}{\Gamma(\alpha+1)} |\mathscr{Q}_0(T)|^\alpha \\
&\leq \mu_1\rho + \frac{\|A\|\rho + \|Bu\| + M}{\Gamma(\alpha+1)} |\mathscr{Q}_0(T)|^\alpha.
\end{aligned}$$

Therefore $\Lambda(B_\rho) \subset B_\rho$.

Third Step: Show that $\Lambda(B_\rho)$ is equicontinuous. Let $x \in \Xi$, for any $t_1, t_2 \in J$, such that $0 \leq t_1 < t_2 \leq T$, we have

$$\begin{aligned}
|\Lambda x(t_2) - \Lambda x(t_1)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_2} |\mathscr{Q}_s(t_2)|^{\alpha-1} \Psi'(s) |Ax(s) + Bu(s) + f(s, x(s))| ds \\
&- \frac{1}{\Gamma(\alpha)} \int_0^{t_1} |\mathscr{Q}_s(t_1)|^{\alpha-1} \Psi'(s) |Ax(s) + Bu(s) + f(s, x(s))| ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left(|\mathscr{Q}_s(t_2)|^{\alpha-1} - |\mathscr{Q}_s(t_1))|^{\alpha-1} \right) \Psi'(s) \\
&\times |Ax(s) + Bu(s) + f(s, x(s))| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} |\mathscr{Q}_s(t_2)|^{\alpha-1} \Psi'(s) |Ax(s) + Bu(s) + f(s, x(s))| ds \\
&\leq \frac{1}{\Gamma(\alpha)} (\|A\| \|x\| + \|Bu\| + M) \\
&. \left(\int_0^{t_1} (|\mathscr{Q}_s(t_2)|^{\alpha-1} - |\mathscr{Q}_s(t_1)|^{\alpha-1}) \Psi'(s) ds + \frac{|\mathscr{Q}_{t_1}(t_2)|^\alpha}{\alpha} \right),
\end{aligned}$$

which yields that

$$|\Lambda x(t_2) - \Lambda x(t_1)| \to 0 \ as \ t_2 \to t_1.$$

This implies that $\Lambda(B_\rho)$ is equicontinuous.
Consequently, by the Arzela-Ascoli theorem, the operator Λ is completely continuous. According to Shauder fixed point Theorem, the map Λ has at least one fixed point in Ξ, which complete the proof. ■

In the next theorem, the uniqueness of solutions for problem (3.1) is established using the Banach contraction principle.

Theorem 3.2 *Given the conditions $(A_1) - (A_4)$, the fractional equation (3.1) is ensured to have a unique solution over the interval J.*

Proof. For all $x, y \in \Xi$ and for $t \in J$, we have

$$\begin{aligned}
|\Lambda x(t) - \Lambda y(t)| &\leq |h(x) - h(y)| \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t |\mathscr{Q}_s(t))^{\alpha-1} || \Psi'(s)| \\
&. \left(|Ax(s) - Ay(s)| + |f(s, x(s)) - f(s, y(s))| \right) ds.
\end{aligned}$$

According to hypothesis (A1)-(A3), we have

$$\begin{aligned}
|\Lambda x(t) - \Lambda y(t)| &\leq k\|x-y\| + \frac{1}{\Gamma(\alpha)}(\|A\| + L_f)\|x-y\| \int_0^T |\mathscr{Q}_s(t))^{\alpha-1}||\Psi'(s)|ds \\
&\leq k\|x-y\| + \frac{1}{\Gamma(\alpha+1)}(\|A\| + L_f)\|x-y\||\mathscr{Q}_0(T)|^{\alpha} \\
&\leq \left(k + \frac{\|A\| + L_f}{\Gamma(\alpha+1)}|\mathscr{Q}_0(T)|^{\alpha}\right)\|x-y\|.
\end{aligned}$$

By using (A4), we get

$$\|\Lambda x - \Lambda y\| \leq \theta\|x-y\|.$$

Since $\theta < 1$, this implies that the operator Λ is contraction. According to the Banach contraction principle, the problem (3.1) has a unique solution in Ξ. ■

3.4 Controllability of a Fractional System

Definition 3.1 (Exact Controllability) :

The system (3.1) is said to be exactly (or completely) controllable on J if for every desired final state x_1, there exists a control u such that x satisfies

$$x(T,u) = x_1.$$

Theorem 3.3 *System (3.1) is exactly controllable on J if and only if the Gramian matrix*

$$W = \int_0^T \breve{\Phi}_s^\alpha(T) E_{\alpha,\alpha}(A(\mathscr{Q}_s(t)^\alpha) BB^* E_{\alpha,\alpha}(A^*(\mathscr{Q}_s(T)^\alpha)\, ds. \tag{3.10}$$

is nonsingular.

Proof.

Sufficiency. Assume that the matrix W is nonsingular, then W^{-1} exists. Set the control $u(t)$ as

$$u(t) = B^* E_{\alpha,\alpha}(A(\mathscr{Q}_s(t)^\alpha) W^{-1} \left(x_1 - E_\alpha(A(\mathscr{Q}_0(t))^\alpha h(x) - \int_0^t \breve{\Phi}_s^\alpha(t) E_{\alpha,\alpha}(A(\mathscr{Q}_s(t)^\alpha) f(s,x(s)\, ds \right).$$

Then

$$\begin{aligned}
x(T,u) &= E_\alpha(A(\mathscr{Q}_0(T))^\alpha h(x) \\
&+ \int_0^T \breve{\Phi}_s^\alpha(T) E_{\alpha,\alpha}(A(\mathscr{Q}_s(T)^\alpha)(Bu(s) + f(s,x(s))\, ds \\
&= E_\alpha(A(\mathscr{Q}_0(T))^\alpha h(x) \\
&+ \int_0^T \breve{\Phi}_s^\alpha(T) E_{\alpha,\alpha}(A(\mathscr{Q}_s(T)^\alpha)(BB^* E_{\alpha,\alpha}(A(\mathscr{Q}_s(T)^\alpha) W^{-1} \Big(x_1 - E_\alpha(A(\mathscr{Q}_0(T))^\alpha h(x) \\
&- \int_0^T \breve{\Phi}_s^\alpha(t) E_{\alpha,\alpha}(A(\mathscr{Q}_s(T)^\alpha) f(s,x(s) \Big) + f(s,x(s))\, ds \\
&= x_1.
\end{aligned}$$

We get $x(T,u) = x_1$, then system (3.1) is controllable on J.

Necessity. Assuming system (3.1) is controlled on J, we shall demonstrate that the Gramian matrix W is nonsingular. In fact, if W is singular, then a nonzero vector z exists, such that

$$z^* W z = 0.$$

That is,

$$\int_0^t z^* \breve{\Phi}_s^\alpha(t) E_{\alpha,\alpha}(A(\mathscr{Q}_s(t)^\alpha) BB^* E_{\alpha,\alpha}(A^*(\mathscr{Q}_s(t)^\alpha) z\, ds = 0.$$

which yields

$$z^* \breve{\Phi}_s^\alpha(t) E_{\alpha,\alpha}(A(\mathscr{Q}_s(t)^\alpha) B = 0. \tag{3.11}$$

Suppose that system (3.1) is controllable on J, and we choose control functions $u_1(t), u_2(t)$ such that

$$\begin{aligned} x(t) &= E_\alpha(A(\mathscr{Q}_0(t))^\alpha h(x) \\ &+ \int_0^t \breve{\Phi}_s^\alpha(t) E_{\alpha,\alpha}(A(\mathscr{Q}_s(t)^\alpha)(Bu_1(s) + f(s, x(s))\, ds = 0, \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} z &= E_\alpha(A(\mathscr{Q}_0(t))^\alpha h(x) \\ &+ \int_0^t \breve{\Phi}_s^\alpha(t) E_{\alpha,\alpha}(A(\mathscr{Q}_s(t)^\alpha)(Bu_2(s) + f(s, x(s))\, ds \neq 0 \end{aligned} \tag{3.13}$$

Inserting (3.12) into (3.13), one can get

$$z = \int_0^t \breve{\Phi}_s^\alpha(t) E_{\alpha,\alpha}(A(\mathscr{Q}_s(t)^\alpha) B(u_2(s) - u_1(s))\, ds.$$

Therefore,

$$z^* z = \int_0^t \breve{\Phi}_s^\alpha(t) z^* E_{\alpha,\alpha}(A(\mathscr{Q}_s(t)^\alpha) B\,(u_2(s) - u_1(s))\, ds,$$

according to (3.11) we get $z^*z = 0$, which leads to $z = 0$. This result contradicts $z \neq 0$. The proof is finished. ■

3.5 Example

Consider the following fractional nonlinear system:

$$\begin{cases} {}^C D_0^{0.3} x(t) = Ax(t) + Bu(t) + f(t, x(t)), \quad t \in [0, 0.5] \\ x(0) = h(x) \end{cases}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad h(x) = \begin{pmatrix} x-1 \\ x^2 \\ 0 \end{pmatrix}.$$

And the nonlinear term is :

$$f(t, x(t)) = \begin{pmatrix} \sin x \\ t \\ x - t \end{pmatrix}.$$

We have

$$A^2 = A \cdot A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow A^3 = 0.$$

Hence, $A^k = 0, \ \forall \, k \geq 3$. Then the Mittag-Leffler function, is given by:

$$E_\alpha(AT^\alpha) = I + \frac{AT^\alpha}{\Gamma(1+\alpha)} + \frac{A^2T^{2\alpha}}{\Gamma(1+2\alpha)}.$$

Then:

$$E_{0.3}(AT^{0.3}) \approx I + 0.905A + 0.628A^2 = \begin{pmatrix} 1 & 0.905 & 0.628 \\ 0 & 1 & 0.905 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore,he controllability Gramian is given by:

$$W = \int_0^T \breve{\Phi}_s^\alpha(T) E_{\alpha,\alpha}(A(T-s)^\alpha) BB^T E_{\alpha,\alpha}(A^T(T-s)^\alpha)\, ds$$

Using:

$$BB^T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

$$W = \int_0^T T^{-0.7} E_{0.3,0.3}(AT^{0.3}) BB^\top E_{0.3,0.3}(A^\top T^{0.3})\, d\tau$$

the Mittag-Leffler Function is

$$E_{0.3,0.3}(AT^{0.3}) = \frac{I}{\Gamma(0.3)} + \frac{AT^{0.3}}{\Gamma(0.6)} + \frac{A^2T^{0.6}}{\Gamma(0.9)}$$

then

$$W \approx \begin{pmatrix} 0.675 & 0.246 & 0.489 \\ 0.246 & 0.268 & 0.246 \\ 0.489 & 0.246 & 0.489 \end{pmatrix}$$

$$\det(W) \approx 0.0131 \neq 0,$$

then, the controllability Gramian W is non-singular. Therefore, the system is **controllable**.

General Conclusion

During this work, an analytical method was proposed to study the controllability of a Ψ-Caputo fractional system with nonlocal initial conditions. Firstly, we prove the existence and uniqueness for solution of the system by using the Schauder fixed point theorem and the contraction principal of Banach.

To demonstrate exact controllability, we constructed the controllability Gramian matrix using the Mittag-Leffler function. Moreover, example was included to verify the effectiveness of the results.

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