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Theme

Stability Analysis of a System of Sturm-Liouville and Langevin Fractional Differential Equations

Directed by: Dr. Tabouche Nora

Before the jury

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I dedicate this work:

To my dear parents, to whom I owe everything. This work is the result of their love, encouragement, and sacrifices.

To my dear brothers.

To all my dear friends from the "Master 02 Mathematics" class, each one by name.

A special dedication to my wonderful friends who encouraged me throughout my university journey.

May God grant you health, happiness, courage, and above all,

success.

Thanks

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الـملخص

تندرج هذه المذكرة ضمن إطار دراسة المعادلات التفاضلية الكسرية. هذا الموضوع مستوحى من أعمال أ. برحايل، ن. طابوش، م. م. مطر و ج. الزابوت، المقالة التي استند إليها بلعادي وبن كاموش لدراسة وجود و وحدانية الحل لنظام عام من ستورم- ليوفيل ولانجفين، باستخدام المشتقة الكسرية هيلفر-كاتيغامبولا تحت شرط ابتدائي، مستوحين من نتائج هذه الدراسات، سعينا لإكمال التحليل الرياضي من خلال استكشاف جانب الاستقرار بشكل أعمق.

ف هذا السياق ركزنا على تحليل استقرار حلول النظام المدروس وفقا لعدة مفاهيم كلاسيكية ومعممة للاستقرار، وهي: استقرار أولام -هايرز، استقرار أولام -هايرز المعمم، استقرار أولام -هايرز -راسياس، واستقرار أولام-هايرز-راسياس المعمم. استخدمنا تقنيات تحليلية دقيقة لتحديد شروط كافية تضمن صحة كل نوع من الاستقرار في إطار النظام الكسري المدروس.

في نهاية المذكرة، قدمنا مثالا علميا لتوضيح الجوانب النظرية و إبراز فعالية النتائج التي تم الحصول عليها.

الكلمات المفتاحية: نظام ستورم-ليوفيل و لانجف ي المعمم، المشتقة الكسرية من نوع هيلفر-كاتيغامبولا، مبرهنة أرزولا-أسكولي، مبرهنة النقطة الثابتة لشودر، مبدأ الإنكماش لباناخ، الاستقرار بمعنى أولام -هايرز، الاستقرار بمعنى أولام-هايرز المعمم، الاستقرار بمعنى أولام - هايرز-راسياس، الاستقرار بمعنى أولام-هايرز-رسياس المعمم.

Abstract

This thesis falls within the framework of studying fractional differential equations. This subject was inspired by the work of A. Berhail, N. Tabouche, M.M. Matar and J. Alzabut, article [9] on which Belaadi, and Benkamouche [7] based their work to study of the existence and uniqueness of the solution to a generalized system of Sturm-Liouville and Langevin type, using the Hilfer-Katugampola fractional derivative under an initial condition. Inspired by the results of these studies, we sought to complete the mathematical analysis by further exploring the aspect of stability.
In this context, we focused on the stability analysis of the solutions of the studied system according to several classical and generalized notions of stability, namely Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability, and generalized Ulam-Hyers-Rassias stability. We used rigorous analytical techniques to establish sufficient conditions that guarantee the validity of each type of stability within the framework of the studied fractional system.
At the end of this thesis, we presented a practical example to illustrate the theoretical aspects and highlight the effectiveness of the obtained results .

Key words: *Generalized Sturm-Liouville and Langevin system, Hilfer-Katugampola fractional derivative, Arzela-Ascoli theorem, Schauder fixed point theorem, Banach contraction principle, Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability, generalized Ulam-Hyers-Rassias stability.*

Résumé

Ce mémoire s'inscrit dans le cadre de l'étude des équations différentielles fractionnaires. Ce sujet a été inspiré du travail de A. Berhail, N. Tabouche, M.M. Matar et J. Alzabut, article [9] sur lequel se sont appuyés Belaadi et Benkamouche [7] pour étudier l'existence et l'unicité de la solution d'un système généralisé de type Sturm-Liouville et Langevin, en utilisant la dérivée fractionnaire de Hilfer-Katugampola, avec condition initiale. En s'inspirant des résultats de ces travaux, on a cherché à compléter l'analyse mathématique en approfondissant l'aspect de la stabilité.

Dans ce contexte, on s'est concentré sur l'analyse de la stabilité des solutions du système étudié selon plusieurs notions classiques et généralisées de stabilité, à savoir la stabilité d'Ulam-Hyers, la stabilité d'Ulam-Hyers généralisée, la stabilité d'Ulam-Hyers-Rassias, et la stabilité d'Ulam-Hyers-Rassias généralisée. On a utilisé des techniques analytiques rigoureuses afin d'établir les conditions suffisantes garantissant la vérification de chacune de ces formes de stabilité dans le cadre du système fractionnaire étudié.

À la fin du mémoire, on a présenté un exemple pratique afin d'illuster l'aspect théorique et de mettre en évidence l'efficacité des résultats obtenus.

Mots clés: *Système généralisé de Sturm-Liouville et Langevin, Dérivé fractionnaire de Hilfer-Katugampola, Théorème du Arzela-Ascoli, Théorème du point fixe de Schauder, Principe de contraction de Banach, Stabilité au sens de Ulam-Hyers, Stabilité au sens de Ulam-Hyers généralisé, Stabilité au sens de Ulam-Hyers-Rassias, Stabilité au sens de Ulam-Hyers-Rassias généralisé.*

Contents

0.1 *Introduction*

Fractional calculus is a branch of mathematical analysis that generalizes the classical concepts of differentiation and integration to non-integer orders. Unlike traditional calculus, which is limited to derivatives and integrals of integer order, fractional calculus allows for operators of arbitrary real or complex order. These operators are typically classified within the broader framework of pseudo-differential operators.

As a natural extension of conventional calculus retains many of its foundational properties while enabling new levels of mathematical modeling and analysis. It offers a unified framework for integration and differentiation of any order whether integer, fractional (e.g $0.5, 0.3, 0.7$), or even complex-valued thus providing powerful tools for studying complex phenomena across physics, engineering and mathematics.

The historical roots of fractional calculus trace back to the late 17th century. In a 1695 letter to l'Hospital, Leibniz posed the now-famous question: What does a derivative of non-integer order mean? His response "This may seem paradoxical at first, but it might one day lead to useful results" set the stage for a centuries-long exploration into derivatives of arbitrary order.

The first rigorous formalization of fractional derivatives can be traced to Liouville's work between 1832 and 1837, followed by Riemann's contributions that culminated in what is now known as the "Riemann-Liouville approach". Subsequently, additional theories emerged, including those of Grunwald-Leitnikov, Weyl, and Caputo see [24]. During that period, practical applications of these theories were almost nonexistent, which led to it being regarded as abstract and consisting mainly of mathematical manipulations with little practical use.

Since then, the field has undergone accelerated development, driven by foundational contributions such as the Riemann-Liouville, Caputo, Hadamard, and Grunwald-Letnikov formulations [12][14][36], and more recently, the Hilfer-Katugampola derivative. The latter is described in great detail in references [19][20][21][22].

At this time, fractional derivatives find applications across a wide range of disciplines including biology, mechanics, economics, and systems engineering thanks to their capacity to capture long-memory dynamics and non-local interactions. One particularly impactful application is in the modeling of random physical processes via the Langevin equation, originally introduced in [25], and further developed to describe systems evolving in stochastic or disordered environments [2][9][10][26][40].

Another cornerstone of applied mathematics, the Sturm-Liouville problem, has been successfully extended into the fractional domain, given its broad applicability in solving boundary value problems across science and engineering [3][23]. The fusion of the fractional Langevin equation with the fractional Sturm-Liouville framework allows for a more nuanced representation of complex dynamic systems, especially those influenced by memory and spatial non-locality [19][21][22].

A further research by Kiataramkul, Ntouyas, Tariboon, and Kijjahathankorn has proposed models incorporating Hadamard derivatives into the fractional Langevin-Sturm-Liouville system under periodic boundary conditions [23]. Other studies have further explored the existence and uniqueness of solutions for generalized Sturm-Liouville-Langevin systems with anti-periodic boundary conditions [27].

Another crucial and notably significant area of research that has recently garnered increased attention is dedicated to the stability analysis of differential equations of both integer and noninteger order. The initial work was started by Ulam in 1940 and subsequently validated by Hyers. This type of stability is referred to as Ulam-Hyers (UH) stability, generalized UH stability. The stability introduced by Rassias is referred Ulam-Hyers-Rassias (UHR) stability. Despite this, Obloza [29] was the first mathematician who introduced the UH stability for differential equations.

This research work is organized into three chapters, each addressing a fundamental aspect of the study on the stability of a generalized Sturm-Liouville and Langevin system of Hilfer-Katugampola fractional differential equations. The aim of this work is to provide a comprehensive analysis that moves from theoretical foundations to applied stability results.

- **First chapter**
  As a starting point, we introduce the essential mathematical tools and concepts used throughout the thesis. We review special functions such as the Gamma, Beta, and Mittag-Leffler functions, which play a crucial role in fractional calculus. Next, we present the main ideas of fractional integration and differentiation, focusing on

well-known definitions, including those of Riemann-Liouville, Hadamard, and Caputo. These concepts form the foundation for the analysis presented in the following chapters.

- **Second chapter**
  Here, we study the following generalized system of fractional differential equations of the Sturm-Liouville and Langevin type involving the Hilfer-Katugampola fractional derivative

$$\begin{cases} {}^{\rho}D_{a^+}^{\alpha_i,\beta_i}\left[p_i(t){}^{\rho}D_{a^+}^{\alpha'_i,\beta'_i}+r_i(t)\right]u_i(t)=f_i\left(t,u_1(t),u_2(t)\right) \quad t\in[a,T],\ a>0,\ i=1,2, \\ u_i(a)=0, \end{cases} \tag{0.1}$$

  where, $0<\alpha_i,\alpha'_i<1$ and $0\leq\beta_i,\beta'_i\leq 1$. ${}^{\rho}\mathcal{D}^{\alpha,\beta}$ is the Hilfer-Katugampola derivative of order α, $(0<\alpha<1)$ and type β, $(0\leq\beta\leq 1)$.
  $f_i:[a,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are a continuous functions, $p_i\subset C([a,T],\mathbb{R}\backslash\{0\})$ and
  $r_i\in C([a,T],\mathbb{R})$ for $i=1,2$.

  First, we start by introducing this type of fractional derivative and discussing its main properties. Then, we present some useful lemmas that will support the theoretical development. After that, we discuss the main results regarding the existence and uniqueness of solutions of the previous system.

- **Third chapter**
  In this chapter, we examine the stability of the system using the concepts of Ulam-Hyers and Ulam-Hyers-Rassias stability. Different types of stability are analyzed, and sufficient conditions are established within the framework of the generalized fractional system introduced earlier, supported by rigorous mathematical proofs.

To conclude, an example is provided to demonstrate the results.

Chapter 1

Preliminaries

This chapter aims to provide a thorough overview of the core principles of fractional calculus. It covers the essential properties of key functions, fractional integrals and derivatives, along with several fixed point theorems that are considered crucial for the advancement of the remaining components of our study.

1.1 Special functions of fractional calculus

In this section, we outline the basic properties of certain special functions that applied in other chapters. Specifically, we focus on the Gamma, Beta and Mittag-Leffler function, which are crucial in the study of fractional derivatives and fractional differential equations.

1.1.1 The Gamma function

Certainly, Euler's Gamma function is one of the fundamental functions in fractional calculus, as it extends the concept of the factorial $n!$ to non-integer and even complex values of n.

Definition 1.1 [24, 32] The Gamma function $\Gamma(z)$ is defined by the integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad z \in \mathbb{C}. \tag{1.1}$$

such that : $\mathrm{Re}(z) > 0$.

Properties of Gamma Function

- One of the basic properties of the Gamma function is that it satisfies the following functional equation:
  For all z with $\mathrm{Re}(z) > 0$,
  $$\Gamma(z+1) = z\Gamma(z). \tag{1.2}$$
  which can be easily proved by integration by parts:
  $$\Gamma(z+1) = \int_0^\infty e^{-t}t^z dt = \left[-e^{-t}t^z\right]_{t=0}^{t=\infty} + z\int_0^\infty e^{-t}t^{z-1}dt = z\Gamma(z)$$
  .

- Obviously, $\Gamma(1) = 1$, and using (1.2) we obtain for $z = 1, 2, 3, \cdots$:
  $$\Gamma(2) = 1\Gamma(1) = 1 = 1\,!,$$
  $$\Gamma(3) = 2\Gamma(2) = 2 \cdot 1\,! = 2\,!,$$
  $$\Gamma(2) = 3\Gamma(3) = 3 \cdot 2\,! = 3\,!,$$
  $$\vdots$$
  $$\Gamma(n+1) = n\Gamma(n) = n \cdot (n-1)\,! = n\,!.$$

- $\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-t^2}dt = \sqrt{\pi}, \quad$ (Gaussian intergral).

- $\Gamma(n+\frac{1}{2}) = \dfrac{(2n\,!)}{2^{2n}n\,!}\sqrt{\pi}.$

- $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{sin(\pi z)}, \quad z \in \mathbb{C}, \quad 0 < \mathrm{Re}(z) < 1.$

1.1.2 The Beta function

The Beta function is one of the fundamental functions in fractional calculus, and is particulary significant when used in conjunction with the Gamma function.

Definition 1.2 [24, 32] The Beta function represents a type of Euler integral, defined for all complex numbers x and y with strictly positive real parts by

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x, y \in \mathbb{C}, \tag{1.3}$$

such that : $(\mathrm{Re}(x) > 0, \mathrm{Re}(y) > 0)$.

Proposition 1.1 *The Beta function can be written in terms of Gamma function as follow*

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \mathrm{Re}(x) > 0, \ \mathrm{Re}(y) > 0.$$

Some Properties of Beta Function

The most important properties of the Beta function are :

- The Beta function is symmetric, i.e.,

$$\beta(x,y) = \beta(y,x), \quad \mathrm{Re}(x) > 0, \ \mathrm{Re}(y) > 0.$$

- Recursive relation:

$$\beta(x+1,y) = \frac{x}{x+y}\beta(x,y),$$
$$\beta(x,y+1) = \frac{y}{x+y}\beta(x,y).$$

- $\beta(x+1,y) + \beta(x,y+1) = \beta(x,y)$.

- Special value:

$$\beta(1,y) = \frac{1}{y}, \quad \beta(x,1) = \frac{1}{x}.$$

- Connection with binomial coefficients:

$$\beta(x,y) = \frac{(x+y-1)\,!}{(x-1)\,!(y-1)\,!}.$$

1.1.3 Mittag-Leffler function

The basic Mittag-Leffler function, defined and studied by the Swedish mathematician in 1903, is a generalization of the exponential function e^x. This function is of significant importance in the theory of fractional calculus.

Definition 1.3 [24] The Mittag-Leffler function is defined by the following power series

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha+1)}, \quad x \in \mathbb{C}, \ \alpha > 0.$$

Definition 1.4 [24] The generalized Mittag-Leffler function is given by

$$E_{\alpha,\beta}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k\alpha+\beta)},\quad x\in\mathbb{C},\ \alpha>0,\ \beta>0.$$

Some relations with classical functions

Now, we present some relations with classical functions:

- $E_{\alpha,1}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k\alpha+1)}=E(x),$

- $E_{1,1}(x)=E_1(x)=e^x,$

- $E_{2,1}(x)=\cosh(\sqrt{x}),$

- $E_{1,2}(x)=\frac{e^x-1}{x},$

- $E_{1,3}(x)=\frac{e^x-1-x}{x^2},$

- $E_{2,2}(x)=\frac{\sinh(x)}{2}.$

1.2 Fractional Integrals and Fractional Derivatives

This section contains definitions and some properties of fractional integrals and fractional derivatives of different type.

1.2.1 Riemann-Liouville Fractional Integral and Derivative

In this section, we give the definition of the Riemann-Liouville fractional integrals and fractional derivatives.

Definition 1.5 [24] Let $I = [a, b]$ be a finite interval on the real axis $\mathbb{R}$. The Riemann-Liouville fractional integrals, $I^{\alpha}_{a^+}f$ and $I^{\alpha}_{b^-}f$ of order $\alpha \in \mathbb{R}$, the left and right-sided are defined, for $f \in C([a, b], \mathbb{R})$, respectively, by

$$I^{\alpha}_{a^+}f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \qquad \alpha > 0, \qquad t > a, \tag{1.4}$$

and

$$I^{\alpha}_{b^-}f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s)ds, \qquad \alpha > 0, \qquad b > t. \tag{1.5}$$

Definition 1.6 [24] Let $\alpha > 0$ and $n \in \mathbb{N}^*$ such that $n - 1 < \alpha < n$. The right Riemann-Liouville fractional derivative of order α, of a function $f \in C([a, b], \mathbb{R})$ is defined as follows

$${}^{RL}D^{\alpha}_{a^+}f(t) = \left(\frac{d}{dt}\right)^n I^{n-\alpha}_{a^+}f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} f(s)ds, \tag{1.6}$$

The left Riemann-Liouville derivative of order α of f is defined by

$${}^{RL}D^{\alpha}_{b^-}f(t) = \left(\frac{-d}{dt}\right)^n I^{n-\alpha}_{b^-}f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{-d}{dt}\right)^n \int_t^b (s-t)^{n-\alpha-1} f(s)ds. \tag{1.7}$$

Properties

- Linearity
$$ {}^{RL}D^{\alpha}_{a^+}(\lambda f(t) + \mu g(t)) = \lambda {}^{RL}D^{\alpha}_{a^+}f(t) + \mu {}^{RL}D^{\alpha}_{a^+}g(t). \tag{1.8}$$

  In general, we have

$${}^{RL}D^{\alpha}_{a^+}({}^{RL}D^{\beta}_{a^+}f)(t) \neq {}^{RL}D^{\beta}_{a^+}({}^{RL}D^{\alpha}_{a^+}f)(t) \neq {}^{RL}D^{\alpha+\beta}_{a^+}f(t). \tag{1.9}$$

- The Riemann-Liouville fractional derivative of a constant C is given by

$${}^{RL}D^{\alpha}_{a^+}C = \frac{C(t-a)^{-\alpha}}{\Gamma(1-\alpha)}, \quad t > a.$$

- The Riemann-Liouville fractional derivative of a power function $(t-a)^{\nu}$ for $\nu > -1$ and $\alpha > 0$
$$^{RL}D^{\alpha}_{a^+}(t-a)^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)}(t-a)^{\nu-\alpha}, \quad t > a.$$

- Composition formulas
Let $m-1 \leq \alpha < m$ and $n-1 \leq \beta < n$

$$^{RL}D^{\alpha}_{a^+}(^{RL}D^{\beta}_{a^+}f)(t) =^{RL} D^{\alpha+\beta}_{a^+}f(t) - \sum_{j=1}^{n}[^{RL}D^{\beta-j}_{a^+}f(t)]_{t=a}\frac{(t-a)^{-\alpha-j}}{\Gamma(-\alpha-j+1)}. \quad (1.10)$$

$$^{RL}D^{\beta}_{a^+}(^{RL}D^{\alpha}_{a^+}f)(t) =^{RL} D^{\alpha+\beta}_{a^+}f(t) - \sum_{j=1}^{m}[^{RL}D^{\alpha-j}_{a^+}f(t)]_{t=a}\frac{(t-a)^{-\beta-j}}{\Gamma(-\beta-j+1)}. \quad (1.11)$$

1.2.2 Hadamard Fractional Integral and Derivative

Here, we present the Hadamard fractional integrals and derivatives, we outline some key properties of these operators.

Definition 1.7 [24] Let $\alpha > 0$ and let $(a, b), (a \leq a \leq b \leq \infty)$ be a finite or infinite interval. The Hadamard fractional integral of order α for a function f is defined by

$$I^{\alpha}_{a}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (log\frac{t}{s})^{\alpha-1}\frac{f(s)}{s}ds, \quad (a \leq t \leq b). \quad (1.12)$$

Definition 1.8 [24] Let $(a, b), (a \leq a \leq b \leq \infty)$ be a finite or infinite interval. The Hadamard fractional derivative of order α for a function f is defined by

$$D^{\alpha}_{a^+}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n \int_a^t (log\frac{t}{s})^{n-\alpha-1}\,\frac{f(s)}{s}\,ds, \quad n = [\alpha]+1, \quad a \leq t \leq b. \quad (1.13)$$

Properties

- If $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$, then

$$\left(I_a^\alpha \left(\log \frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \left(\log \frac{x}{a}\right)^{\beta+\alpha-1}.$$

- Let $\alpha, \beta \in \mathbb{R}$, $0 < a < b < \infty$ such that $\alpha > 0$, $\beta > n$ and $n = [\alpha] + 1$,

$$\left(D_a^\alpha \left(\log \frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left(\log \frac{x}{a}\right)^{\beta-\alpha-1}.$$

Caputo Fractional Derivatives

Now, we present the definition of the Caputo fractional derivative [24].

Definition 1.9 The Caputo fractional derivative of order $\alpha \in \mathbb{R}_+$ of a function $f \in C^n([a,b])$ is defined by

$$^C D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds \qquad t > a$$

with $n - 1 < \alpha < n$.

Some Properties

- Linearity: Let $n - 1 < \alpha < n$ and $\lambda, \mu \in \mathbb{C}$

$$^C D_a(\lambda f(t) + \mu g(t)) = \lambda ^C D_a f(t) + \mu ^C D_a g(t).$$

- $^C D_a C = 0, \quad (Constant)$.

- Interpolation [17]

$$\lim_{\alpha \longrightarrow n} {}^C D^\alpha f(t) = f^{(n)}(t),$$
$$\lim_{\alpha \longrightarrow n-1} {}^C D^\alpha f(t) = f^{(n-1)}(t) - f^{(n-1)}(0).$$

1.3 Theorems

In this section, we present several theorems that will be used in subsequent development of our work.

1.3.1 Shauder fixed point theorem

Theorem 1.1 *[13] Let F be a nonempty closed subset of a Banach space E and $T : F \to F$ be a continuous mapping such that $T(F) \subset E$ is relatively compact. Then T has at least one fixed point in F.*

1.3.2 Krasnoselskii fixed point theorem

Theorem 1.2 *[13] Let E be a Banach space, let F be a bounded closed convex subset of E and let T_1, T_2 be two mappings from F into E such that $T_1x + T_2y \in F$ for every pair $x, y \in F$. If T_1 is contraction and T_2 is completely continuous, then the equation $T_1x + T_2x = x$ has a solution on F.*

1.3.3 Banach contraction principle

Theorem 1.3 *[13] Let Ω be a nonempty closed subset of a Banach space and $T : \Omega \to$ a conraction operator. Then, there exists a unique $\omega \in E$ such that $T(\omega) = \omega$.*

1.3.4 Arzela-Ascoli theorem

Theorem 1.4 *[13] Let $X = C([a, b])$ equipped with the norm*

$$\|f\| = \max_{t\in[a,b]} | f(t) | .$$

If M is a subset of X such that:

- *M is uniformely bounded, i.e $\exists\, c > 0, \|f\| \leq c, \forall\ f \in M$.*

- *M is equicontinuous, that is,*

  $\forall\, \varepsilon > 0, \exists\, \delta > 0, \forall\, t_1, t_2 \in [a, b]$ *such that* $| t_1 - t_2 | < \delta$ *et* $u \in M \Rightarrow | u(t_1) - u(t_2) | < \varepsilon$.

Then, M is relatively compact.

Chapter 2

Study of generalized Sturm-Liouville system and Langevin of Hilfer-Katugampola fractional differential equations

In this chapter, we focus on the existence and uniqueness of solutions for a generalized system of Sturm-Liouville and Langevin fractional differential equations involving the Hilfer-Katugampola derivative, in the weighted spaces. First, we will present the Hilfer-Katugampola fractional derivatives which was introduced by Oliveira [30, 31]. This new formulation is a Hilfer-type fractional differentiation operator, this is, an integer order derivative performing between generalized fractional integrals according to Katugampola [19]. This new fractional derivatives interpolates the Hilfer, Hilfer-Hadamard, Riemann-Liouville, Hadamard, Caputo, Caputo-Hadamard, generalized and generalized Caputo-type fractional derivatives, as well as the Weyl and Liouville fractional derivatives for particular cases of integration extremes.

This chapter is organized as follows: In Section 2.1, we provide some definitions and lemmas that will be utilized in subsequent sections. In Section 2.2, we present our problem and estabish a key lemma that demonstrates the equivalence between the inital value problem and the integral equation. In Section 2.3, we present and prove the existence and uniqueness theorem for the initial value problem presented in the previous section by using Schauder's fixed point and Banach's contraction principle.

2.1 Preliminary

In this section, we will conduct a rigorous examination of fractional calculus within the Hilfer-Katugampola framework, accompanied by a comprehensive and detailed analysis designed to facilate a deeper comprehension and appreciation of the foundational principles underlying this concept.

2.1.1 Function spaces

Definition 2.1 [22, 30, 31] Let $I = [a, b]$ $(0 < a < b < \infty)$ be a finite interval on the half-axis $\mathbb{R}^+$. We denote by $C[a, b]$ the Banach space of continuous functions g from I to $\mathbb{R}$ with the norm

$$\|g\|_C = \max_{t\in I} |g(t)|.$$

1) The weighted space $C_{\gamma,\rho}[a, b]$ of functions g on $(a, b]$ is defined by

$$C_{\gamma,\rho}[a,b] = \left\{ g : (a,b] \to \mathbb{R} : \left(\frac{t^\rho - a^\rho}{\rho}\right)^\gamma g(t) \in C(I) \right\},$$

where $0 \leq \gamma < 1,\ \rho > 0$ equipped with the norm

$$\|g\|_{C_{\gamma,\rho}} = \left\| \left(\frac{t^\rho - a^\rho}{\rho}\right)^\gamma g(t) \right\|_C = \max_{t\in I} \left| \left(\frac{t^\rho - a^\rho}{\rho}\right)^\gamma g(t) \right|.$$

We have $C_{0,\rho}[a, b] = C[a, b]$.

2) Let $\delta_\rho = \left(t^\rho \frac{d}{dt}\right)$. For $n \in \mathbb{N}$, we denote by $C^n_{\delta_\rho,\gamma}([a, b])$ the Banach space of functions g which are continuously differentiable with operetor δ_ρ on $[a, b]$ up to order $(n - 1)$ and the derivative $\delta^n_\rho g$ of order n on $(a, b]$ such that $\delta^n_\rho g \in C_{\gamma,\rho}[a, b]$, this is

$$C^n_{\delta_\rho,\gamma}[a,b] = \left\{ g : (a,b] \to \mathbb{R} \,:\, \delta^k_\rho \in C(I),\ k = 0, 1 \ldots, n-1,\ \delta^n_\eta g \in C_{\gamma,\rho}(I) \right\},$$

where $n \in \mathbb{N}$, with the norms

$$\|g\|_{C^n_{\delta_\rho,\gamma}} = \sum_{k=0}^{n-1} \left\|\delta^k_\rho g\right\|_C + \left\|\delta^n_\rho g\right\|_{C_{\gamma,\rho}},$$

and

$$\|g\|_{C^n_{\delta_\rho}} = \sum_{k=0}^{n} \max_{t\in I} \left|\delta^k_\rho g(x)\right|.$$

For $n = 0$, we have

$$C^0_{\delta_\rho,\gamma}[a,b] = C_{\gamma,\rho}[a,b].$$

2.1.2 Generalized Fractional Integrals and Derivatives

In order to generalize the Riemann-Liouville and Hadamard fractional integrals, Katugampola [19] introduced the generalized fractional integral. Subsequently, the author defined the generalized fractional derivatives associated with the generalized integral operators, constructed so that these differential operators extend the Riemann-Liouville and Hadamard fractional derivatives [20].

Generalized Fractional Integrals

Definition 2.2 [19, 21] Let $\alpha, \rho \in \mathbb{R}$ with $\alpha > 0$ and $\rho > 0$. The generalized left-sided fractional integral ${}^{\rho}I_{a^+}^{\alpha} f(\cdot)$ of order α is defined by

$$\left({}^{\rho}\mathcal{I}_{a^+}^{\alpha} f\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \left(t^{\rho} - s^{\rho}\right)^{\alpha-1} s^{\rho-1} f(s) ds, \quad t > a. \tag{2.1}$$

Similarly, the generalized right-sided fractional integral ${}^{\rho}I_{b^-}^{\alpha} f(\cdot)$ is defined by

$$\left({}^{\rho}\mathcal{I}_{b^-}^{\alpha} f\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_t^b \left(t^{\rho} - s^{\rho}\right)^{\alpha-1} s^{\rho-1} f(s) ds, \quad t < b. \tag{2.2}$$

Generalized Fractional Derivatives

Now, we introduce the generalized fractional derivatives corresponding respectively to the fractional integrals (2.1) and (2.2).

Definition 2.3 [19, 21] Let $\alpha \in \mathbb{R}$, such that $\alpha \notin \mathbb{N}$, $\alpha > 0$, $n = [\alpha] + 1$ and $\rho > 0$. The generalized fractional derivatives ${}^{\rho}D_{a^+}^{\alpha} f(\cdot)$ (left-sided) and ${}^{\rho}D_{b^-}^{\alpha} f(\cdot)$ (right-sided) corresponding to the generalized integrals (2.1) and (2.2) are defined for
$0 \leq a < t < b \leq \infty$, by :

$$\left({}^{\rho}\mathcal{D}_{a^+}^{\alpha} f\right)(t) = \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)} \left(t^{1-\rho} \frac{d}{dt}\right)^n \int_a^t \left(t^{\rho} - s^{\rho}\right)^{n-\alpha-1} s^{\rho-1} f(s)\, ds, \tag{2.3}$$

and

$$\left({}^{\rho}\mathcal{D}_{b^-}^{\alpha} f\right)(t) = \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)} \left(-t^{1-\rho} \frac{d}{dt}\right)^n \int_t^b \left(t^{\rho} - s^{\rho}\right)^{n-\alpha-1} s^{\rho-1} f(s)\, ds. \tag{2.4}$$

2.1.3 Hilfer-Katugampola Fractional Derivative

In this subsection, we present the definition of the Hilfer-Katugampola fractional derivatives introduced by Oliveira [30, 31].

Definition 2.4 [30, 31] Let order α and type β satisfy $0 < \alpha < 1$ and $0 \leq \beta \leq 1$, the Hilfer-Katugampola fractional derivative (left-sided/right-sided), with respect to t, with $\rho > 0$ of a function $f \in C_{1-\gamma,\rho}(I)$ is defined by

$$\begin{aligned}\left({}^{\rho}\mathcal{D}_{a\pm}^{\alpha,\beta} f\right)(t) &= \left(\pm{}^{\rho}\mathcal{I}_{a\pm}^{\beta(1-\alpha)} \left(t^{\rho-1}\frac{d}{dt}\right) {}^{\rho}\mathcal{I}_{a\pm}^{(1-\beta)(1-\alpha)} f\right)(t) \\ &= \left(\pm{}^{\rho}\mathcal{I}_{a\pm}^{\beta(1-\alpha)} \delta_{\rho} {}^{\rho}\mathcal{I}_{a\pm}^{(1-\beta)(1-\alpha)} f\right)(t),\end{aligned} \tag{2.5}$$

where $\mathcal{I}$ is the generalized fractional integral given in Definition 2.2.

We present and discuss our results involving the Hilfer-Katugampola fractional derivative using only the left-sided operator ${}^{\rho}\mathcal{D}_{a+}^{\alpha,\beta}$.

Properties 1 *[19, 30, 31]*

P1) The operator ${}^{\rho}\mathcal{D}_{a+}^{\alpha,\beta}$ can be written as

$${}^{\rho}\mathcal{D}_{a+}^{\alpha,\beta} = {}^{\rho}\mathcal{I}_{a+}^{\beta(1-\alpha)} \delta_{\rho} {}^{\rho}\mathcal{I}_{a+}^{1-\gamma} = {}^{\rho}\mathcal{I}_{a+}^{\beta(1-\alpha)} {}^{\rho}\mathcal{D}_{a+}^{\gamma}, \qquad \gamma = \alpha + \beta(1-\alpha).$$

Proof. *From Definition 2.2 of the generalized fractional integral, we have*

$$\begin{aligned}({}^{\rho}\mathcal{D}_{a+}^{\alpha,\beta} f)(x) &= {}^{\rho}\mathcal{I}_{a+}^{\beta(1-\alpha)} \left(x^{1-\rho}\frac{d}{dx}\right) \left[\frac{\rho^{1-(1-\beta)(1-\alpha)}}{\Gamma[(1-\beta)(1-\alpha)]} \int_a^x \frac{t^{\rho-1}}{(x^{\rho}-t^{\rho})^{1-(1-\beta)(1-\alpha)}} f(t)dt\right] \\ &= \left[{}^{\rho}\mathcal{I}_{a+}^{\beta(1-\alpha)} \frac{\rho^{1+\alpha+\beta-\alpha\beta}}{\Gamma[(1-\beta)(1-\alpha)-1]} \int_a^x \frac{t^{\rho-1}}{(x^{\rho}-t^{\rho})^{1+\alpha+\beta-\alpha\beta}} f(t)dt\right](x) \\ &= ({}^{\rho}\mathcal{I}_{a+}^{\beta(1-\alpha)} {}^{\rho}\mathcal{D}_{a+}^{\gamma} f)(x),\end{aligned}$$

where operator $\mathcal{D}$ is the generalized fractional derivetive given in Definition 2.3. This completes the proof. ■

P2) The fractional derivative ${}^{\rho}\mathcal{D}_{a+}^{\alpha,\beta}$ is an interpolator of the following fractional derivatives : Hilfer ($\rho \to 1$), Hilfer-Hadamard ($\rho \to 0$), generalized fractional derivative ($\beta = 0$), generalized Caputo-type ($\beta = 1$), Riemann-Liouville ($\beta = 0$, $\rho \to 1$), Hadamard ($\beta = 0$, $\rho \to 0$), Caputo ($\beta = 1$, $\rho \to 1$), Caputo-Hadamard ($\beta = 1$, $\rho \to 0$), Liouville ($\beta = 0$, $\rho \to 1$, $a = 0$) and Weyl ($\beta = 0$, $\rho \to 1$, $a = -\infty$).

P3) We consider the following parameters α, β, γ, μ satisfying
$\gamma = \alpha + \beta(1-\alpha)$, $0 \leq \alpha, \beta, \gamma < 1$, $0 \leq \mu < 1$.
Thus, we define the spaces

$$C^{\alpha,\beta}_{1-\gamma,\mu}[a,b] = \left\{ f \in C_{1-\gamma,\rho}[a,b], {}^{\rho}\mathcal{D}^{\alpha,\beta}_{a^+} f(\cdot) \in C_{\mu,\rho}[a,b] \right\},$$

and

$$C^{\gamma}_{1-\gamma,\rho}[a,b] = \left\{ f \in C_{1-\gamma,\rho}[a,b], {}^{\rho}\mathcal{D}^{\gamma}_{a^+} f(\cdot) \in C_{1-\gamma,\rho}[a,b] \right\},$$

where $C_{\mu,\rho}[a,b]$ and $C_{1-\gamma,\rho}[a,b]$ are weighted spaces of continuous functions on $(a,b]$ defined in Definition 2.1 .
It is abvious that

$$C^{\gamma}_{1-\gamma}[a,b] \subset C^{\alpha,\beta}_{1-\gamma}[a,b].$$

Lemma 2.1 *[19, 35] Let α, β and $\rho > 0$. Then, for $f \in C^{\gamma}_{1-\gamma,\rho}(a,b)$, we have*

$$\left({}^{\rho}\mathcal{I}^{\alpha}_{a^+} {}^{\rho}I^{\beta}_{a^+} f \right)(\cdot) = \left({}^{\rho}\mathcal{I}^{\alpha+\beta}_{a^+} f \right)(\cdot),$$

$$\left({}^{\rho}\mathcal{D}^{\alpha}_{a^+} {}^{\rho}\mathcal{I}^{\dot{\alpha}}_{a^+} f \right)(\cdot) = f(\cdot),$$

$$\left({}^{\rho}\mathcal{D}^{\alpha}_{a^+} {}^{\rho}\mathcal{D}^{\beta}_{a^+} f \right)(\cdot) = \left({}^{\rho}\mathcal{D}^{\alpha+\beta}_{a^+} f \right)(\cdot).$$

Lemma 2.2 *[4] Let ${}^{\rho}\mathcal{I}^{\alpha}_{a^+}$ and ${}^{\rho}\mathcal{D}^{\alpha}_{a^+}$, respectively according to equations* (2.1) *and* (2.3)*. Then,*

$$ {}^{\rho}\mathcal{I}^{\alpha}_{a^+} \left(\frac{t^{\rho} - a^{\rho}}{\rho} \right)^{\beta-1}(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \left(\frac{x^{\rho} - a^{\rho}}{\rho} \right)^{\alpha+\beta-1}, \quad \alpha \geq 0,\ \beta > 0,$$

$$ {}^{\rho}\mathcal{D}^{\alpha}_{a^+} \left(\frac{t^{\rho} - a^{\rho}}{\rho} \right)^{\beta-1}(x) = 0, \quad 0 < \alpha < 1.$$

Lemma 2.3 *[30, 31] Let $0 < \alpha < 1$, $0 \leq \gamma < 1$. If $f \in C_{\gamma}[a,b]$ and ${}^{\rho}\mathcal{I}^{1-\alpha}_{a+} f(\cdot) \in C^{1}_{\gamma}[a,b]$, then*

$$ {}^{\rho}\mathcal{I}^{\alpha}_{a^+} {}^{\rho}\mathcal{D}^{\alpha}_{a^+}(f)(x) = f(x) - \frac{\left({}^{\rho}\mathcal{I}^{1-\alpha}_{a^+} f \right)(a)}{\Gamma(\alpha)} \left(\frac{x^{\rho} - a^{\rho}}{\rho} \right)^{\alpha-1},$$

for all $x \in I = (a,b)$.

Lemma 2.4 *[30, 31] Let $0 < a < b < \infty$, $\alpha > 0$, $0 \leq \gamma < 1$ and $f \in C_{\gamma,\rho}[a,b]$. If $\alpha > \gamma$, then ${}^{\rho}\mathcal{I}^{\alpha}_{a^+}(f)$ is continuous on $[a,b]$ and*

$$\left({}^{\rho}\mathcal{I}^{\alpha}_{a^+} f \right)(a) = \lim_{t \to a^+} \left({}^{\rho}\mathcal{I}^{\alpha}_{a^+} \right) f(t) = 0.$$

Proof. Since $f \in C_{\gamma,\rho}[a,b]$, then $\left(\dfrac{t^\rho - a^\rho}{\rho}\right)^\gamma f(t)$ is continuous on [a,b], and

$$\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^\gamma f(t)\right| \leq M, \quad t \in [a,b],$$

for some positive constant M. Consequently,

$$|({}^\rho\mathcal{I}^\alpha_{a^+} f)(t)| \leq M \left[{}^\rho\mathcal{I}^\alpha_{a^+} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{-\gamma}\right](t),$$

and by Lemma 2.2, we can write

$$|({}^\rho\mathcal{I}^\alpha_{a^+} f)(t)| \leq M \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-\gamma+1)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-\gamma}.$$

As $\alpha > \gamma$, the right-hand side of the last equation goes to zero when $t \to a^+$.
The proof is finished. ■

Lemma 2.5 *[30, 31] Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta - \alpha\beta$. If $f \in C^\gamma_{1-\gamma}[a,b]$, then*

$${}^\rho\mathcal{I}^\gamma_{a^+}\, {}^\rho\mathcal{D}^\gamma_{a^+} f(\cdot) = {}^\rho\mathcal{I}^\alpha_{a^+}\, {}^\rho\mathcal{D}^{\alpha,\beta}_{a^+} f(\cdot), \tag{2.6}$$

and

$${}^\rho\mathcal{D}^\gamma_{a^+}\, {}^\rho\mathcal{I}^\alpha_{a^+} f(\cdot) = {}^\rho\mathcal{D}^{\beta(1-\alpha)}_{a^+} f(\cdot). \tag{2.7}$$

Proof. We first prove Equation (2.6), using Property P1, we can write

$${}^\rho\mathcal{I}^\gamma_{a^+}\, {}^\rho\mathcal{D}^\gamma_{a^+} f = {}^\rho\mathcal{I}^\gamma_{a^+}\, {}^\rho\mathcal{I}^{-\beta(1-\alpha)}_{a^+}\, {}^\rho\mathcal{D}^{\alpha,\beta}_{a^+} f = {}^\rho\mathcal{I}^{\alpha+\beta-\alpha\beta}_{a^+}\, {}^\rho\mathcal{I}^{-\beta+\alpha\beta}_{a^+}\, {}^\rho\mathcal{D}^{\alpha,\beta}_{a^+} f = {}^\rho\mathcal{I}^\alpha_{a^+}\, {}^\rho\mathcal{D}^{\alpha,\beta}_{a^+} f.$$

To prove Equation (2.7), we use Definition 2.4 to get

$${}^\rho\mathcal{D}^\gamma_{a^+}\, {}^\rho\mathcal{I}^\alpha_{a^+} f = \delta_\rho\, {}^\rho\mathcal{I}^{1-\gamma}_{a^+}\, {}^\rho\mathcal{I}^\alpha_{a^+} f = \delta_\rho\, {}^\rho\mathcal{I}^{1-\beta+\alpha\beta}_{a^+} f = \delta_\rho\, {}^\rho\mathcal{I}^{1-\beta(1-\alpha)}_{a^+} f = {}^\rho\mathcal{D}^{\beta(1-\alpha)}_{a^+} f.$$

The proof is finished. ■

Lemma 2.6 *[30, 31] Let $f \in L(a,b)$. If ${}^\rho\mathcal{D}^{\beta(1-\alpha)}_{a^+} f$ exists on $L(a,b)$, Then*

$${}^\rho\mathcal{D}^{\alpha,\beta}_{a^+}\, {}^\rho\mathcal{I}^\alpha_{a^+} f(\cdot) = {}^\rho\mathcal{I}^{\beta(1-\alpha)}_{a^+}\, {}^\rho\mathcal{D}^{\beta(1-\alpha)}_{a^+} f(\cdot).$$

Proof. From Lemma 2.2, Definition 2.3 and Definitin 2.4, we get

$$\begin{aligned}
{}^\rho\mathcal{D}_{a^+}^{\alpha,\beta}\,{}^\rho\mathcal{I}_{a^+}^{\alpha} f &= {}^\rho\mathcal{I}_{a^+}^{\beta(1-\alpha)}\,{}^\rho\mathcal{D}_{a^+}^{\gamma}\,{}^\rho\mathcal{I}_{a^+}^{\alpha} f \\
&= {}^\rho\mathcal{I}_{a^+}^{\beta(1-\alpha)}\delta_\rho\,{}^\rho\mathcal{I}_{a^+}^{(1-\gamma)}\,{}^\rho\mathcal{I}_{a^+}^{\alpha} f \\
&= {}^\rho\mathcal{I}_{a^+}^{\beta(1-\alpha)}\delta_\rho\,{}^\rho\mathcal{I}_{a^+}^{1-\beta(1-\alpha)} f \\
&= {}^\rho\mathcal{I}_{a^+}^{\beta(1-\alpha)}\,{}^\rho\mathcal{D}_{a^+}^{\beta(1-\alpha)} f.
\end{aligned}$$

This completes the proof. ■

Lemma 2.7 *[30, 31] Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta(1-\alpha)$. If $f \in C_{1-\gamma}[a,b]$ and ${}^\rho\mathcal{I}_{a^+}^{1-\beta(1-\alpha)} f \in C^1_{1-\gamma}[a,b]$, then ${}^\rho\mathcal{D}_{a^+}^{\alpha,\beta}\,{}^\rho\mathcal{I}_{a^+}^{\alpha} f$ exists on $(a,b]$ and*

$${}^\rho\mathcal{D}_{a^+}^{\alpha,\beta}\cdot{}^\rho\mathcal{I}_{a^+}^{\alpha} f(\cdot) = f(\cdot).$$

Proof. Using Lemma 2.3, Lemma 2.2 and Lemma 2.6, we obtain

$$\begin{aligned}
\left({}^\rho\mathcal{D}_{a^+}^{\alpha,\beta}\,{}^\rho\mathcal{I}_{a^+}^{\alpha} f\right)(t) &= \left({}^\rho\mathcal{I}_{a^+}^{\beta(1-\alpha)}\,{}^\rho\mathcal{D}_{a^+}^{\beta(1-\alpha)} f\right)(t) \\
&= f(t) - \frac{({}^\rho\mathcal{I}_{a^+}^{\beta(1-\alpha)} f)(a)}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta(1-\alpha)-1} \\
&= f(t), \qquad\qquad t \in (a,b].
\end{aligned}$$

The proof is finished. ■

2.2 Problem Statement

We cosider the following generalized system of fractional differential equations of the Sturm-Liouville and Langevin involving the Hilfer-Katugampola fractional derivative with initial value

$$\begin{cases} {}^{\rho}D_{a^+}^{\alpha_i,\beta_i}\left[p_i(t)^{\rho}D_{a^+}^{\alpha_i',\beta_i'}+r_i(t)\right]u_i(t)=f_i\left(t,u_1(t),u_2(t)\right) \quad t\in[a,T],\ a>0,\ i=1,2,\\ u_i(a)=0, \end{cases} \tag{2.8}$$

where, $0<\alpha_i,\alpha_i'<1$ and $0\leq\beta_i,\beta_i'\leq 1$. ${}^{\rho}\mathcal{D}^{\alpha,\beta}$ is the Hilfer-Katugampola derivative of order α, $(0<\alpha<1)$ and type β, $(0\leq\beta\leq 1)$.
$f_i:[a,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are a continuous functions, $p_i\in C([a,T],\mathbb{R}\backslash\{0\})$ and
$r_i\in C([a,T],\mathbb{R})$ for $i=1,2$.

In order to prove the main theorem of existence and uniqueness of the solution of Problem (2.8), we present the following key lemma, which describes the corresponding integral equation.

Lemma 2.8 *[7] Let $\alpha_i+\beta_i(1-\alpha_i)$, where $0<\alpha_i<1$ and $0\leq\beta_i\leq 1$ if $f_i:J\times\mathbb{R}^2\to\mathbb{R}$ is a function such that*

$$f_i(\cdot,u_1(\cdot),u_2(\cdot))\in C_{1-\gamma_i,\rho}[a,T],\quad \forall u_i\in C_{1-\gamma_i,\rho}[a,T], i=1,2.$$

A function $u=(u_1,u_2)$ with $u_i\in C^{\gamma_i}_{1-\gamma_i,\rho}[a,T]$ is the solution of Problem (2.8) if and only if u_i satisfies the following integral equation

$$u_i(t)={}^{\rho}\mathcal{I}_{a^+}^{\alpha_i'}\left(\frac{1}{p_i(t)}{}^{\rho}\mathcal{I}_{a^+}^{\alpha_i}f_i\left(t,u_1(t),u_2(t)\right)\right)-{}^{\rho}\mathcal{I}_{a^+}^{\alpha_i'}\left(\frac{r_i(t)}{p_i(t)}u_i(t)\right). \tag{2.9}$$

Proof. We start by showing the implication in this direction $(\Rightarrow)$.
We apply ${}^{\rho}\mathcal{I}_{a^+}^{\alpha_i}$ on both sides of Problem (2.8) and using Lemma 2.3 and Lemma 2.5, we obtain

$${}^{\rho}\mathcal{I}_{a^+}^{\alpha_i}\,{}^{\rho}\mathcal{D}_{a^+}^{\alpha_i,\beta_i}\left[p_i(t)^{\rho}\mathcal{D}_{a^+}^{\alpha_i',\beta_i'}+r_i(t)\right]u_i(t)={}^{\rho}\mathcal{I}_{a^+}^{\alpha_i}f_i(t,u_1(t),u_2(t)),$$

$${}^{\rho}\mathcal{I}_{a^+}^{\gamma_i}\,{}^{\rho}\mathcal{D}_{a^+}^{\gamma_i}\left[p_i(t)^{\rho}\mathcal{D}_{a^+}^{\alpha_i',\beta_i'}+r_i(t)\right]u_i(t)={}^{\rho}\mathcal{I}_{a^+}^{\alpha_i}f_i(t,u_1(t),u_2(t)),$$

$$\left[p_i(t)^\rho\mathcal{D}_{a^+}^{\alpha_i',\beta_i'}+r_i(t)\right]u_i(t)-{}^\rho\mathcal{I}_{a^+}^{1-\gamma_i}\frac{\left[p_i(a)^\rho\mathcal{D}_{a^+}^{\alpha_i',\beta_i'}+r_i(a)\right]u_i(a)}{\Gamma(\gamma_{i)}}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma_i-1}$$

$$={}^\rho\mathcal{I}_{a^+}^{\alpha_i}f_i(t,u_1(t),u_2(t)),$$

$$\left[p_i(t)^\rho\mathcal{D}_{a^+}^{\alpha_i',\beta_i'}+r_i(t)\right]u_i(t)={}^\rho\mathcal{I}_{a^+}^{\alpha_i}f_i(t,u_1(t),u_2(t)),$$

it follows that

$$^\rho\mathcal{D}_{a^+}^{\alpha_i',\beta_i'}u_i(t)=\frac{1}{p_i(t)}\left({}^\rho\mathcal{I}_{a^+}^{\alpha_i}f_i(t,u_1(t),u_2(t))\right)-\frac{r_i(t)}{p_i(t)}u_i(t). \tag{2.10}$$

Next, we apply the integral ${}^\rho\mathcal{I}_{a^+}^{\alpha_i'}$ on both sides of Equation (2.10), then we use the Lemma 2.3 an Lemma 2.5, we obtain

$$^\rho\mathcal{I}_{a^+}^{\alpha_i'}\,{}^\rho\mathcal{D}_{a^+}^{\alpha_i',\beta_i'}u_i(t)={}^\rho\mathcal{I}_{a^+}^{\alpha_i'}\left[\frac{1}{p_i(t)}\left({}^\rho\mathcal{I}_{a^+}^{\alpha_i}f_i(t,u_1(t),u_2(t))\right)-\frac{r_i(t)}{p_i(t)}u_i(t)\right],$$

$$^\rho\mathcal{I}_{a^+}^{\gamma_i'}\,{}^\rho\mathcal{D}_{a^+}^{\gamma_i'}u_i(t)={}^\rho\mathcal{I}_{a^+}^{\alpha_i'}\left[\frac{1}{p_i(t)}\left({}^\rho\mathcal{I}_{a^+}^{\alpha_i}f_i(t,u_1(t),u_2(t))\right)-\frac{r_i(t)}{p_i(t)}u_i(t)\right],$$

$$u_i(t)-{}^\rho\mathcal{I}_{a^+}^{1-\gamma_i'}\frac{u_i(a)}{\Gamma(\gamma_i)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma_i'-1}={}^\rho\mathcal{I}_{a^+}^{\alpha_i'}\left[\frac{1}{p_i(t)}\left({}^\rho\mathcal{I}_{a^+}^{\alpha_i}f_i(t,u_1(t),u_2(t))\right)-\frac{r_i(t)}{p_i(t)}u_i(t)\right],$$

$$u_i(t)={}^\rho\mathcal{I}_{a^+}^{\alpha_i'}\left(\frac{1}{p_i(t)}{}^\rho\mathcal{I}_{a^+}^{\alpha_i}f_i(t,u_1(t),u_2(t))\right)-{}^\rho\mathcal{I}_{a^+}^{\alpha_i'}\left(\frac{r_i(t)}{p_i(t)}u_i(t)\right). \tag{2.11}$$

Now, we show the implication in the opposite direction ($\Leftarrow$).
We apply ${}^\rho\mathcal{D}_{a^+}^{\alpha_i',\beta_i'}$ on both sides of Equation (2.11) and by the Lemma 2.7, it follows that

$$^\rho\mathcal{D}_{a^+}^{\alpha_i',\beta_i'}u_i(t)={}^\rho\mathcal{D}_{a^+}^{\alpha_i',\beta_i'}{}^\rho\mathcal{I}_{a^+}^{\alpha_i'}\left(\frac{1}{p_i(t)}{}^\rho\mathcal{I}_{a^+}^{\alpha_i}f_i(t,u_1(t),u_2(t))\right)-{}^\rho\mathcal{D}_{a^+}^{\alpha_i',\beta_i'}{}^\rho\mathcal{I}_{a^+}^{\alpha_i'}\left(\frac{r_i(t)}{p_i(t)}u_i(t)\right),$$

$$^\rho\mathcal{D}_{a^+}^{\alpha_i',\beta_i'}u_i(t)=\frac{1}{p_i(t)}{}^\rho\mathcal{I}_{a^+}^{\alpha_i}f_i(t,u_1(t),u_2(t))-\frac{r_i(t)}{p_i(t)}u_i(t),$$

$$p_i(t)^\rho\mathcal{D}_{a^+}^{\alpha_i',\beta_i'}u_i(t) = {}^\rho\mathcal{I}_{a^+}^{\alpha_i}f_i(t,u_1(t),u_2(t)) - r_i(t)u_i(t),$$

$$p_i(t)^\rho\mathcal{D}_{a^+}^{\alpha_i',\beta_i'}u_i(t) + r_i(t)u_i(t) = {}^\rho\mathcal{I}_{a^+}^{\alpha_i}f_i(t,u_1(t),u_2(t)),$$

$$\left[p_i(t)^\rho\mathcal{D}_{a^+}^{\alpha_i',\beta_i'} + r_i(t)\right]u_i(t) = {}^\rho\mathcal{I}_{a^+}^{\alpha_i}f_i(t,u_1(t),u_2(t)). \tag{2.12}$$

Now, by applying the operator ${}^\rho\mathcal{D}_{a^+}^{\alpha_i,\beta_i}$ on both sides of Equation (2.12) and using the Lemma 2.7, we get

$${}^\rho\mathcal{D}_{a^+}^{\alpha_i,\beta_i}\left[p_i(t)^\rho\mathcal{D}_{a^+}^{\alpha_i',\beta_i'} + r_i(t)\right]u_i(t) = {}^\rho\mathcal{D}_{a^+}^{\alpha_i,\beta_i}{}^\rho\mathcal{I}_{a^+}^{\alpha_i}f_i(t,u_1(t),u_2(t)),$$

$${}^\rho\mathcal{D}_{a^+}^{\alpha_i,\beta_i}\left[p_i(t)^\rho\mathcal{D}_{a^+}^{\alpha_i',\beta_i'} + r_i(t)\right]u_i(t) = f_i(t,u_1(t),u_2(t)).$$

By Equation (2.11), we have

$$u_i(t) = \frac{{}^\rho\mathcal{I}_{a^+}^{\alpha_i'}\left(\frac{1}{p_i(t)}{}^\rho\mathcal{I}_{a^+}^{\alpha_i}f_i(t,u_1(t),u_2(t))\right)}{1 + {}^\rho\mathcal{I}_{a^+}^{\alpha_i'}\left(\frac{r_i(t)}{p_i(t)}\right)},$$

and for $t = a$,

$$u_i(a) = \frac{{}^\rho\mathcal{I}_{a^+}^{\alpha_i'}\left(\frac{1}{p_i(a)}{}^\rho\mathcal{I}_{a^+}^{\alpha_i}f_i(a,u_1(a),u_2(t))\right)}{1 + {}^\rho\mathcal{I}_{a^+}^{\alpha_i'}\left(\frac{r_i(a)}{p_i(a)}\right)}.$$

Hence, by Lemma 2.4, we have $u_i(a) = 0$. The proof is fulfilled. ■

2.3 Existence and uniqueness

For study existence and uniqueness of solution of the problem (2.8), we consider the following hypotheses and notations.
In all the sequel, we consider the operator

$$\mathcal{A} : C_{1-\gamma_1,\rho}(J) \times C_{1-\gamma_2,\rho}(J) \to C_{1-\gamma_1,\rho}(J) \times C_{1-\gamma_2,\rho}(J),$$

$(J = [a, T])$ defined by

$$\mathcal{A}u = \mathcal{A}(u_1, u_2) = (\mathcal{A}_1 u_1, \mathcal{A}_2 u_2)$$

with

$$|||\mathcal{A}u||| = \max\{\|\mathcal{A}_1 u_1\|, \|\mathcal{A}_2 u_2\|\},$$

where

$$\mathcal{A}_i u_i(t) = {}^{\rho}\mathcal{I}_{a^+}^{\alpha_i'} \left(\frac{1}{p_i} {}^{\rho}\mathcal{I}_{a^+}^{\alpha_i} f_i \right)(t, u_1(t), u_2(t)) - {}^{\rho}\mathcal{I}_{a^+}^{\alpha_i'} \left(\frac{r_i}{p_i} u_i \right)(t). \tag{2.13}$$

Notations We denote

$$p_i^* = \inf_t |p_i(t)|\,,\ r_i^* = \sup_t |r_i(t)|\,,$$

and we assume that

$$L_i = 2C_i M_i + D_i < 1, \quad i = 1, 2. \tag{2.14}$$

with

$$C_i = \left(\frac{T^\rho - a^\rho}{\rho} \right)^{1-\gamma_i+\alpha_i+\alpha_i'} \frac{1}{p_i^* \rho^2 \Gamma(\alpha_i + 1)\Gamma(\alpha_i' + 1)},$$

and

$$D_i = \left(\frac{T^\rho - a^\rho}{\rho} \right)^{\alpha_i'} \frac{r_i^*}{p_i^* \rho \Gamma(\alpha_i' + 1)}.$$

Hypotheses

$(H1)$ For $i = 1, 2$, there exists a constant $M_i > 0$ such that :

$$|f_i(t, u_1, u_2) - f_i(t, v_1, v_2)| \leq M_i (\|u_1 - v_1\| + \|u_2 - v_2\|)\,,\ \forall\, t \in [a, T],\ \forall\, u_i, v_i \in \mathbb{R} \quad i = 1, 2.$$

$(H2)$ For all $u = (u_1, u_2)$, there exists $\sigma_i > 0$ such that

$$|f_i(t, u)| \leq \sigma_i, \quad \forall\ t \in J,\ (i = 1, 2).$$

2.3.1 Existence

Within this section, we prove the existence of the solution of Problem (2.8), which is based on Shauder's fixed point theorem.

Theorem 2.1 *Assume that* $(H2)$ *is satisfied. Then Problem* (2.8) *has at least one solution in* $C_{1-\gamma_1,\rho} \times C_{1-\gamma_2,\rho}$.

Proof. To demonstrate the existence of a solution, we reformulate Problem (2.8) as a fixed point problem. Indeed, since Problem (2.8) is equivalent to the integral equation (2.13), the fixed points of $\mathcal{A}$ are the solutions of the problem (2.8). Then, we will verify the assumptions of Shauder's fixed point theorem.
Consequently, the proof will be devided into several steps.
First Step
We prove that the operator $\mathcal{A}$ is continuous. For every bounded $\Omega \subset C_{1-\gamma_1,\rho} \times C_{1-\gamma_2,\rho}$ there exists $\omega > 0$ such that

$$\Omega = \{u \in C_{1-\gamma_1,\rho} \times C_{1-\gamma_2,\rho} : \|u\|_{C_{1-\gamma_i,\rho}} \leq \omega\},$$

where $\omega = \max(\omega_1, \omega_2)$ and $\gamma = \max(\gamma_1, \gamma_2)$.
Let $(u_n)_{n\in\mathbb{N}} \in \Omega$ a sequence such that $\lim\limits_{n\to\infty} \|u_{ni} - u_i\|_{C_{1-\gamma_i,\rho}} = 0$. Then for every $t \in J$ we have

$$\left| (\mathcal{A}_i u_{ni}(t) - \mathcal{A}_i u_i(t)) \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} \right|$$

$$\leq \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} {}^\rho\mathcal{I}_{a^+}^{\alpha_i'} \left(\frac{1}{p_i^*} {}^\rho\mathcal{I}_{a^+}^{\alpha_i} (|f_i\,(s, u_{n1}(s), u_{n2}(s)) - f_i(s, u_1(s), u_2(s))|)\right)$$

$$+ \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} {}^\rho\mathcal{I}_{a+}^{\alpha_i'} \left(\frac{r_i^*}{p_i^*} (|u_{ni}(s) - u_i(s)|)\right)(t)$$

$$\leq \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} {}^\rho\mathcal{I}_{a^+}^{\alpha_i'} \left(\frac{1}{p_i^*} {}^\rho\mathcal{I}_{a^+}^{\alpha_i} (|f_i\,(s, u_{n1}(s), u_{n2}(s)) - f_i(s, u_1(s), u_2(s))|)\right)$$

$$+ {}^\rho\mathcal{I}_{a+}^{\alpha_i'} \left(\frac{r_i^*}{p_i^*} (\|u_{ni} - u_i)\|_{C_{1-\gamma_i,\rho}}\right)(t)$$

$$\leq \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} \frac{1}{p_i^*\Gamma(\alpha_i)\Gamma(\alpha_i')} \frac{(T^\rho - a^\rho)^{\alpha_i+\alpha_i'}}{\rho^{\alpha_i+\alpha_i'+2}\alpha_i\alpha_i'} (|f_i\,(s, u_{n1}(s), u_{n2}(s)) - f_i(s, u_1(s), u_2(s))|)$$

$$+\frac{r_i^*}{p_i^*\Gamma(\alpha_i')}\frac{(T^\rho-a^\rho)^{\alpha_i'}}{\rho^{\alpha_i'+1}\alpha_i\prime}\left(\|u_{ni}-u_i\|_{C_{1-\gamma_i,\rho}}\right)$$

$$\leq\left(\frac{T^\rho-a^\rho}{\rho}\right)^{1+\alpha_i+\alpha_i'-\gamma_i}\frac{1}{p_i^*\rho^2\Gamma(\alpha_i+1)\Gamma(\alpha_i'+1)}(|f_i\left(s,u_{n1}(s),u_{n2}(s)\right)-f_i(s,u_1(s),u_2(s))|)$$

$$+\frac{r_i^*}{p_i^*\rho\Gamma(\alpha_i'+1)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{\alpha_i'}\left(\|u_{ni}-u_i\|_{C_{1-\gamma_i,\rho}}\right)$$

Since f is continuous and $\lim_{n\to\infty}\|u_{ni}-u_i\|_{C_{1-\gamma_i,\rho}}=0$. Then

$$\|\|(\mathcal{A}u_{ni})(t)-(\mathcal{A}u_i)(t)\|\|_{n\to\infty}\to 0.$$

Consequently, $\mathcal{A}$ is continuous.

Second Step

We will show that $\mathcal{A}(\Omega)\subset\Omega$. We choose ω a positive real number such that

$$\omega\geq\max\left(\frac{C_1\sigma_1}{1-D_1},\frac{C_2\sigma_2}{1-D_2}\right).$$

By the assumption $(H2)$, for all $t\in J$ and for all $u\in\Omega$, we have

$$\left|\mathcal{A}_iu_i(t)\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma_i}\right|$$

$$\leq\left|\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma_i}{}^\rho\mathcal{I}_{a^+}^{\alpha_i'}\left(\frac{1}{p_i}{}^\rho\mathcal{I}_{a^+}^{\alpha_i}f_i(s,u_1(s),u_2(s))\right)(t)\right|$$

$$+\left|\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma_i}{}^\rho\mathcal{I}_{a^+}^{\alpha_i'}\left(\frac{r_i}{p_i}u_i\right)(t)\right|$$

$$\leq\left(\frac{T^\rho-a^\rho}{\rho}\right)^{1-\gamma_i}{}^\rho\mathcal{I}_{a^+}^{\alpha_i'}\left(\frac{1}{p_i^*}{}^\rho\mathcal{I}_{a^+}^{\alpha_i}\left|f_i(s,u_1(s),u_2(s))\right|\right)(t)$$

$$+\left(\frac{T^\rho-a^\rho}{\rho}\right)^{1-\gamma_i}{}^\rho\mathcal{I}_{a^+}^{\alpha_i'}\left(\frac{r_i^*}{p_i^*}\left|u_i(s)\right|\right)(t)$$

$$\leq\left(\frac{T^\rho-a^\rho}{\rho}\right)^{1-\gamma_i}{}^\rho\mathcal{I}_{a^+}^{\alpha_i'}(\frac{1}{p_i^*}{}^\rho\mathcal{I}_{a^+}^{\alpha_i}\sigma_i)(t)+{}^\rho\mathcal{I}_{a^+}^{\alpha_i'}(\frac{r_i^*}{p_i^*}\|u_i\|)(t)$$

$$\leq \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} \frac{1}{p_i^*\Gamma(\alpha_i)\Gamma(\alpha_i')} \frac{(T^\rho - a^\rho)^{(\alpha_i+\alpha_i')}}{\rho^{\alpha_i+\alpha_i'+2}\alpha_i\alpha_i'}\sigma_i + \frac{r_i^*}{p_i^*\Gamma(\alpha_i')} \frac{(T^\rho - a^\rho)^{\alpha_i'}}{\rho^{\alpha_i'+1}\alpha_i'} \|u_i\|$$

$$\leq \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i+\alpha_i+\alpha_i'} \frac{\sigma_i}{p_i^*\rho^2\Gamma(\alpha_i+1)\Gamma(\alpha_i'+1)} + \frac{r_i^*}{p_i^*\rho\Gamma(\alpha_i'+1)} \left(\frac{T^\rho - a^\rho}{\rho}\right)^{\alpha_i'} \omega_i$$

$$\leq C_i\sigma_i + D_i \|u_i\|$$

$$\leq C_i\sigma_i + D_i\omega_i.$$

We obtain $|||\mathcal{A}u||| \leq \omega$, from which it follows that $\mathcal{A}(\Omega) \subset \Omega$.

Third Step

We show that $\mathcal{A}u$ is uniformly bounded. For all $u \in \Omega$, we have

$$\left|\mathcal{A}_i u_i(t) \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma_i}\right|$$

$$\leq \left| \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} {}^\rho\mathcal{I}_{a^+}^{\alpha_i'} \left(\frac{1}{p_i^*} {}^\rho\mathcal{I}_{a^+}^{\alpha_i} f_i(s, u_1(s), u_2(s))\right)(t)\right|$$

$$+ \left| \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} {}^\rho\mathcal{I}_{a^+}^{\alpha_i'} \frac{r_i}{p_i} u_i(s)(t)\right|$$

$$\leq \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} {}^\rho\mathcal{I}_{a^+}^{\alpha_i'} \left(\frac{1}{p_i^*} {}^\rho\mathcal{I}_{a^+}^{\alpha_i} |f_i(s, u_1(s), u_2(s))|\right)(t)$$

$$+ {}^\rho\mathcal{I}_{a^+}^{\alpha_i'} \left(\frac{r_i^*}{p_i^*} \|u_i(s)\|\right)(t)$$

$$\leq \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i+\alpha_i+\alpha_i'} \frac{1}{p_i^*\rho^2\Gamma(\alpha_i+1)\Gamma(\alpha_i'+1)}\sigma_i$$

$$+ \frac{r_i^*}{p_i^*\rho\Gamma(\alpha_i'+1)} \left(\frac{T^\rho - a^\rho}{\rho}\right)^{\alpha_i'} \omega_i$$

$$\leq (C_i\sigma_i + D_i\omega_i).$$

Thus, $|||\mathcal{A}u||| \leq \max\{C_1\sigma_1 + D_1\omega_1,\ C_2\sigma_2 + D_2\omega_2\}$. Then, $\mathcal{A}u$ is uniformly bounded.

Fourth Step

Finally, we show that $\mathcal{A}u$ is equicontinuous. Let $u \in \Omega$ and $t_1, t_2 \in J$ with $t_1 < t_2$, we have

$$\left|(\mathcal{A}_i u_i(t_2) - \mathcal{A}_i u_i(t_1))\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma_i}\right|$$

$$\leq \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} \frac{1}{p_i^* \Gamma(\alpha_i + \alpha_i')} \int_a^{t_1} \left(\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha_i + \alpha_i' - 1} - \left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha_i + \alpha_i' - 1}\right) s^{\rho-1}$$

$$\times |f_i(s, u_1(s), u_2(s))| ds$$

$$+ \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} \frac{1}{p_i^* \Gamma(\alpha_i + \alpha_i')} \int_{t_1}^{t_2} \left(\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha_i + \alpha_i' - 1} s^{\rho-1}\right) |f_i(s, u_1(s), u_2(s))| ds$$

$$+ \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} {}^\rho\mathcal{I}_{a^+}^{\alpha_i'} \left(\frac{r_i^*}{p_i^*} |u_i(s)(t_2) - u_i(s)(t_1)|\right).$$

It follows that

$$\left|(\mathcal{A}_i u_i(t_2) - \mathcal{A}_i u_i(t_1))\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma_i}\right|$$

$$\leq \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} \frac{\sigma_i}{p_i^* \Gamma(\alpha_i + \alpha_i')} \int_a^{t_1} \left(\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha_i + \alpha_i' - 1} - \left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha_i + \alpha_i' - 1}\right) s^{\rho-1} ds$$

$$+ \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} \frac{\sigma_i}{p_i^* \Gamma(\alpha_i + \alpha_i')} \int_{t_1}^{t_2} \left(\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha_i + \alpha_i' - 1} s^{\rho-1}\right) ds$$

$$+ {}^\rho\mathcal{I}_{a+}^{\alpha_i'} \left(\frac{r_i^*}{p_i^*} \|u_i(s)\| \, |(t_2 - t1)|\right)$$

$$\leq \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} \frac{\sigma_i}{p_i^* \rho \Gamma(\alpha_i + \alpha_i' + 1)} \left(\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{\alpha_i + \alpha_i'} - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{\alpha_i + \alpha_i'}\right)$$

$$+ \frac{r_i^* \omega}{p_i^* \Gamma(\alpha_i')} \left(\int_a^{t_2} \left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha_i' - 1} s^{\rho-1} ds - \int_a^{t_1} \left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha_i' - 1} s^{\rho-1} ds\right)$$

$$\leq \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} \frac{\sigma_i}{p_i^* \rho \Gamma(\alpha_i + \alpha_i' + 1)} \left(\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{\alpha_i + \alpha_i'} - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{\alpha_i + \alpha_i'}\right)$$

$$+ \frac{r_i^* \omega}{\rho p_i^* \Gamma(\alpha_i' + 1)} \left(\frac{t_2^\rho - t_1^\rho}{\rho}\right)^{\alpha_i'}.$$

Consequently, we have

$$\|\|\mathcal{A}u\,(t_2) - \mathcal{A}u\,(t_1)\,\|\| \to 0 \text{ as } t_1 \to t_2,$$

which implies that $\mathcal{A}u$ is equicontinuous.
Thus, by the Arzela-Ascoli theorem, $\mathcal{A}u$ is completely continuous, and by Shauder's fixed point theorem, the operator $\mathcal{A}$ indeed has a fixed point. The proof is finished. ■

2.3.2 Uniqueness

Theorem 2.2 *Suppose that assumption* $(H1)$ *and* (2.14) *are satisfied, the problem* (2.8) *has a unique solution in* $C_{1-\gamma_1,\rho} \times C_{1-\gamma_2,\rho}$.

Proof. For all $u_i, v_i \in C_{1-\gamma_i,\rho}(J)$ and for $t \in J$, we have

$$\left|(\mathcal{A}_i u_i(t) - \mathcal{A}_i v_i(t)) \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma_i}\right|$$

$$\leq \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} {}^\rho\mathcal{I}_{a^+}^{\alpha_i'} \left(\frac{1}{p_i^*} {}^\rho\mathcal{I}_{a^+}^{\alpha_i} |f_i\,(s, u_1(s), u_2(s)) - f_i\,(s, v_1(s), v_2(s))\,|\right)(t)$$

$$+ \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} {}^\rho\mathcal{I}_{a^+}^{\alpha_i'} \left(\frac{r_i^*}{p_i^*} |u_i(s) - v_i(s)|\right)(t)$$

$$\leq \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} {}^\rho\mathcal{I}_{a^+}^{\alpha_i'} \left(\frac{1}{p_i^*} {}^\rho\mathcal{I}_{a^+}^{\alpha_i} (M_i(\|u_1 - v_1\| + \|u_2 - v_2\|))\right)(t)$$

$$+ {}^\rho\mathcal{I}_{a^+}^{\alpha_i'} \left(\frac{r_i^*}{p_i^*} \|u_i - v_i\|\right)(t)$$

$$\leq \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} \frac{1}{p_i^* \Gamma(\alpha_i) \Gamma(\alpha_i')} \frac{(T^\rho - a^\rho)^{\alpha_i + \alpha_i'}}{\alpha_i \alpha_i' \rho^{\alpha_i + \alpha_i' + 2}} (M_i(\|u_1 - v_1\| + \|u_2 - v_2\|))$$

$$+ \frac{r_i^*}{p_i^* \Gamma(\alpha_i')} \frac{(T^\rho - a^\rho)^{\alpha_i'}}{\alpha_i' \rho^{\alpha_i' + 1}} (\|u_i - v_i\|)$$

$$\leq \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i+\alpha_i+\alpha'_i} \frac{1}{p_i^* \rho^2 \Gamma(\alpha_i+1)\Gamma(\alpha'_i+1)} (M_i(\|u_1 - v_1\| + \|u_2 - v_2\|))$$

$$+ \frac{r_i^*}{p_i^* \rho \Gamma(\alpha'_i+1)} \left(\frac{T^\rho - a^\rho}{\rho}\right)^{\alpha'_i} (\|u_i - v_i\|)$$

$$\leq (2M_i C_i + D_i)\|u - v\|.$$

Hence, $|||\mathcal{A}u - \mathcal{A}v||| \leq L\|u - v\|$ where $L = \max\{L_1, L_2\}$. Since $L < 1$, the operator $\mathcal{A}$ is a contraction. Thus, according to Banach's contraction principle, Problem (2.8) has a unique solution. This completes the proof. ■

Chapter 3

Stability in the sense of Ulam-Hyers and Ulam-Hyers-Rassias

Stability analysis has got too much importance on research side. it is very important from numerical and optimization point of view in investigating various problems of physics, mathematical biology, biophysics, economics, where the actual solution is almost difficult. In literature, we come across different approaches towards stability analysis. However, in this paper we discuss the Ulam-Hyers (UH) stability approach which is comparatively the most simple and easy way of investigation the stability of systems.
Its history goes back to the middle of the 19th century. In 1940, Ulam gave a wide ranging talk before the mathematics club of the Wisconsin university in which he discussed several important unsolved problems [37] and he raised the question concerning the homomorphism stability "Under what conditions does there exist an additive mapping near an approximately additive mapping?". In the next year, Hyers positively answered the question of Ulam's partially [15]. Moreover, a generalized was obtained by Rassias for the Hyers results [33]. And since 1940, the mathematics authors have been interested to studied the Ulam-type stability problems [5, 28, 34]. In last few decades, Jung extended the previously mentioned stability results, see [18]. Obloza [29] established the UH stability of linear differential equations.
The Ulam stability analysis is very useful in a lot of applications, such as optimization, numerical analysis, etc..., for more detailed study of Ulam-type stability with different approaches see [16, 39]. Some recent work on stability of differential equations (or their solutions) with variety of initial or boundary conditions are reported in [1, 6, 8, 11, 38]. In this chapter, we aim to study the stability of solution to problem(2.8) based on Ulam-Hyers, generalized Ulam-Hyers, Ulam-Hyers-Rassias, and generalized Ulam-Hyers-Rassias

principles and define its fundamental criteria.

3.1 Stability analysis

In this section, we describe the various stability definitions that are essential to understanding our problem.

3.1.1 Ulam-Hyers stability

Let Φ_i defined in $C_{1-\gamma_1,\rho}(J) \times C_{1-\gamma_2,\rho}(J)$ such that

$$\Phi_i(u_1,u_2)(t) ={}^\rho D_{a^+}^{\alpha_i,\beta_i}\left[p_i(t)^\rho D_{a^+}^{\alpha_i',\beta_i'}+r_i(t)\right]u_i(t)-f_i\left(t,u_1(t),u_2(t)\right), t\in[a,T], i=1,2.$$

Definition 3.1 The problem (2.8) is Ulam-Hyers (UH) stable if there exists a real number $\zeta > 0$ such that for each pair $(\varepsilon_1,\varepsilon_2) \in \mathbb{R}^+ \times \mathbb{R}^+$ and for each solution $u = (u_1, u_2) \in C_{1-\gamma_i,\rho}(J) \times C_{1-\gamma_i,\rho}(J)$ satisfying the inequality

$$\begin{cases} |\Phi_i(u_1,u_2)(t)| \leq \varepsilon_i \quad t\in[a,T],\ i=1,2, \\ \\ u_i(a)=0, \end{cases} \tag{3.1}$$

there exists a unique solution $v = (v_1, v_2) \in C_{1-\gamma_1,\rho}(J) \times C_{1-\gamma_2,\rho}(J)$ of (2.8) satisfying the inequality

$$\|u-v\|_{C_{1-\gamma,\rho}} \leq \zeta\varepsilon,$$

where $\varepsilon = \max(\varepsilon_1, \varepsilon_2)$.

3.1.2 Generalized Ulam-Hyers stability

Definition 3.2 The problem (2.8) is generalized Ulam-Hyers (GUH) stable if there exists $\xi \in C(\mathbb{R}^+,\mathbb{R}^+)$ with $\xi(0) = 0$ such that for each pair $(\varepsilon_1,\varepsilon_2) \in \mathbb{R}^+ \times \mathbb{R}^+$ and for each solution $u = (u_1, u_2) \in C_{1-\gamma_i,\rho}(J) \times C_{1-\gamma_i,\rho}(J)$ satisfying (3.1), there exists a unique solution $v = (v_1, v_2) \in C_{1-\gamma_1,\rho}(J) \times C_{1-\gamma_2,\rho}(J)$ of (2.8) satisfying the inequality

$$\|u-v\|_{C_{1-\gamma,\rho}} \leq \xi(\varepsilon).$$

3.1.3 Ulam-Hyers-Rassias stability

Definition 3.3 The problem (2.8) is Ulam-Hyers-Rassias (UHR) stable with respect to $(\psi_1, \psi_2) \in C([a,T], \mathbb{R}^+) \times C([a,T], \mathbb{R}^+)$ if there exists a real number $\mu > 0$ such that for each pair $(\varepsilon_1, \varepsilon_2) \in \mathbb{R}^+ \times \mathbb{R}^+$ and for each solution $u = (u_1, u_2) \in C_{1-\gamma_1,\rho}(J) \times C_{1-\gamma_2,\rho}(J)$ satisfying the inequality

$$\begin{cases} |\Phi_i(u_1, u_2)(t)| \leq \varepsilon_i \psi_i(t) \quad t \in [a,T],\ i = 1,2, \\ \\ u_i(a) = 0, \end{cases} \tag{3.2}$$

there exists a unique solution $v = (v_1, v_2) \in C_{1-\gamma_1,\rho}(J) \times C_{1-\gamma_2,\rho}(J)$ of (2.8) satisfying the inequality

$$\|u - v\|_{C_{1-\gamma,\rho}} \leq \mu\varepsilon\psi(t),$$

where $\varepsilon = \max(\varepsilon_1, \varepsilon_2)$.

3.1.4 Generalized Ulam-Hyers-Rassias stability

Definition 3.4 The problem (2.8) is generalized Ulam-Hyers-Rassias (GUHR) stable with respect to $(\psi_1, \psi_2) \in C([a,T], \mathbb{R}^+) \times C([a,T], \mathbb{R}^+)$ if there exists a real number $\mu > 0$ such that for each solution $u = (u_1, u_2) \in C_{1-\gamma_1,\rho}(J) \times C_{1-\gamma_2,\rho}(J)$ of (3.2), there exists a unique solution $v = (v_1, v_2) \in C_{1-\gamma_1,\rho}(J) \times C_{1-\gamma_2,\rho}(J)$ of (2.8) satisfying the inequality

$$\|u - v\|_{C_{1-\gamma,\rho}} \leq \mu\psi(t).$$

Remark 3.1 A function $u = (u_1, u_2) \in C_{1-\gamma_i,\rho} \times C_{1-\gamma_i,\rho}$ is a solution of (3.1) if and only if there exists a function $g_i \in C([a,T], \mathbb{R})$ such that

1) $|g_i(t)| \leq \varepsilon_i$, $t \in [a,T]$, $i = 1,2$.
2) ${}^\rho D_{a^+}^{\alpha_i,\beta_i} \left[p_i(t) {}^\rho D_{a^+}^{\alpha_i',\beta_i'} + r_i(t) \right] u_i(t) = f_i\left(t, u_1(t), u_2(t)\right) + g_i(t), \quad t \in [a,T],\ i = 1,2.$

Remark 3.2 A function $u = (u_1, u_2) \in C_{1-\gamma_i,\rho} \times C_{1-\gamma_i,\rho}$ is a solution of (3.2) if and only if there exists a function $g_i \in C([a,T], \mathbb{R})$ such that

1) $|g_i(t)| \leq \varepsilon_i \psi_i(t), \quad t \in [a,T],\ i = 1,2.$
2) ${}^\rho D_{a^+}^{\alpha_i,\beta_i} \left[p_i(t) {}^\rho D_{a^+}^{\alpha_i',\beta_i'} + r_i(t) \right] u_i(t) = f_i\left(t, u_1, u_2(t)\right) + g_i(t), \quad t \in [a,T],\ i = 1,2.$

3.2 Stability results

Theorem 3.1 *Assume that hypotheses* $(H1)$ *and condition* (2.14) *are satisfied. Then, solution of Problem* (2.8) *is stable in the sense of Ulam-Hyers (UH) and generalized Ulam-Hyers (GUH).*

Proof. Let $\varepsilon_i \in \mathbb{R}^+$, $i = 1, 2$, and since we have assumed that $u = (u_1, u_2) \in C_{1-\gamma_1,\rho}(J) \times C_{1-\gamma_2,\rho}(J)$ is a solution of (3.1). By Remark 3.1, there exists a function $g_i \in C([a, T], \mathbb{R})$ such that $|g_i(t)| \leq \varepsilon_i$, $t \in [a, T]$, $i = 1, 2$, and satisfying the problem

$$\begin{cases} {}^{\rho}D_{a^+}^{\alpha_i,\beta_i}\left[p_i(t)\,{}^{\rho}D_{a^+}^{\alpha_i',\beta_i'} + r_i(t)\right]u_i(t) = f_i\left(t, u_1(t), u_2(t)\right) + g_i(t), \quad t \in [a, T],\ i = 1, 2, \\ \\ u_i(a) = 0, \end{cases} \tag{3.3}$$

Using Lemma 2.8, the problem (3.3) has a solution given as

$$u_i(t) = {}^{\rho}\mathcal{I}_{a^+}^{\alpha_i'}\left(\frac{1}{p_i(t)}\,{}^{\rho}\mathcal{I}_{a^+}^{\alpha_i}\left[f_i\left(t, u_1(t), u_2(t)\right) + g_i(t)\right]\right) - {}^{\rho}\mathcal{I}_{a^+}^{\alpha_i'}\left(\frac{r_i(t)}{p_i(t)}u_i(t)\right).$$

Theorem 2.2 ensures the existence of a unique solution $v = (v_1, v_2) \in C_{1-\gamma_1,\rho}(J) \times C_{1-\gamma_2,\rho}(J)$ of Problem (2.8) and satisfies the integral equation

$$v_i(t) = {}^{\rho}\mathcal{I}_{a^+}^{\alpha_i'}\left(\frac{1}{p_i(t)}\,{}^{\rho}\mathcal{I}_{a^+}^{\alpha_i} f_i\left(t, v_1(t), v_2(t)\right)\right) - {}^{\rho}\mathcal{I}_{a^+}^{\alpha_i'}\left(\frac{r_i(t)}{p_i(t)}v_i(t)\right).$$

On the other hand, for any $i = 1, 2$, $t \in [a, T]$, we establish

$$\begin{aligned} &\left|(u_i(t) - v_i(t))\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma_i}\right| \\ &\leq \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} {}^{\rho}\mathcal{I}_{a^+}^{\alpha_i'}\left(\frac{1}{p_i^*}\,{}^{\rho}\mathcal{I}_{a^+}^{\alpha_i}|f_i\left(s, u_1(s), u_2(s)\right) - f_i\left(s, v_1(s), v_2(s)\right)|\right)(t) \\ &+ \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} {}^{\rho}\mathcal{I}_{a^+}^{\alpha_i'}\left(\frac{1}{p_i^*}\,{}^{\rho}\mathcal{I}_{a^+}^{\alpha_i}\,|g_i(s)|\right)(t) \\ &+ \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} {}^{\rho}\mathcal{I}_{a^+}^{\alpha_i'}\left(\frac{r_i^*}{p_i^*}|u_i(s) - v_i(s)|\right)(t) \end{aligned}$$

$$
\begin{aligned}
&\leq \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} {}^\rho\mathcal{I}_{a^+}^{\alpha_i'} \left(\frac{1}{p_i^*} {}^\rho\mathcal{I}_{a^+}^{\alpha_i}(M_i(\|u_1 - v_1\| + \|u_2 - v_2\|))\right)(t)\\
&+ \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} {}^\rho\mathcal{I}_{a^+}^{\alpha_i'} \left(\frac{1}{p_i^*} {}^\rho\mathcal{I}_{a^+}^{\alpha_i}\varepsilon_i\right)(t)\\
&+ {}^\rho\mathcal{I}_{a^+}^{\alpha_i'} \left(\frac{r_i^*}{p_i^*}\|u_i - v_i\|\right)(t)\\
&\leq \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} \frac{1}{p_i^*\Gamma(\alpha_i)\Gamma(\alpha_i')} \frac{(T^\rho - a^\rho)^{\alpha_i+\alpha_i'}}{\alpha_i\alpha_i'\rho^{\alpha_i+\alpha_i'+2}}(M_i(\|u_1 - v_1\| + \|u_2 - v_2\|))\\
&+ \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} \frac{1}{p_i^*\Gamma(\alpha_i)\Gamma(\alpha_i')} \frac{(T^\rho - a^\rho)^{\alpha_i+\alpha_i'}}{\alpha_i\alpha_i'\rho^{\alpha_i+\alpha_i'+2}}\varepsilon_i\\
&+ \frac{r_i^*}{p_i^*\Gamma(\alpha_i')} \frac{(T^\rho - a^\rho)^{\alpha_i'}}{\alpha_i'\rho^{\alpha_i'+1}}(\|u_i - v_i\|)\\
&\leq \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i+\alpha_i+\alpha_i'} \frac{1}{p_i^*\rho^2\Gamma(\alpha_i+1)\Gamma(\alpha_i'+1)}(M_i(\|u_1 - v_1\| + \|u_2 - v_2\|)\\
&+ \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i+\alpha_i+\alpha_i'} \frac{1}{p_i^*\rho^2\Gamma(\alpha_i+1)\Gamma(\alpha_i'+1)}\varepsilon_i\\
&+ \frac{r_i^*}{p_i^*\rho\Gamma(\alpha_i'+1)} \left(\frac{T^\rho - a^\rho}{\rho}\right)^{\alpha_i'}(\|u_i - v_i\|)\\
&\leq C_iM_i(\|u_1 - v_1\| + \|u_2 - v_2\|) + C_i\varepsilon_i + D_i(\|u_i(s) - v_i\|)\\
&\leq C_iM_i2\|u - v\| + C_i\varepsilon_i + D_i\|u - v\|\\
&\leq (2C_iM_i + D_i)\|u - v\| + C_i\varepsilon_i\\
&\leq L_i\|u - v\| + C_i\varepsilon_i.
\end{aligned}
$$

By condition (2.14), we deduce that

$$\|u - v\|_{C_{1-\gamma,\rho}} \leq \frac{C}{1-L}\varepsilon$$

with $C = \max(C_1, C_2)$ and $\varepsilon = \max(\varepsilon_1, \varepsilon_2)$. This prove the existence of positive real number $\zeta = \frac{C}{1-L}$, so that by Definition 3.1, the solution of Problem (2.8) is Ulam-Hyers (UH) stable.

Let $\xi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a continuous function such that

$$\xi(\varepsilon) = \zeta\varepsilon,$$

where $\xi(0) = 0$. According to the above, we can write

$$\|u - v\|_{C_{1-\gamma,\rho}} \leq \xi(\varepsilon) = \frac{C}{1-L}\varepsilon.$$

Hence, by Definition 3.2, the solution of Problem (2.8) is generalized Ulam-Hyers (GUH) stable. This concludes the proof. ■

Theorem 3.2 *Assume that hypotheses* $(H1)$ *and condition* (2.14) *are satisfied and the following hypotheses hold:*
$(H3)$ *There exists an increasing function* $\psi_i \in C([a,T], \mathbb{R}^+)$ *and there exists* $\eta_i > 0$ *such that for any* $t \in [a,T]$

$$^{\rho}\mathcal{I}_{a^+}^{\alpha_i'+\alpha_i}\psi_i(t) \leq \eta_i\psi_i(t), \ \ i = 1, 2.$$

Then, Problem (2.8) *is Ulam-Hyers-Rassias(UHR) stable and consequently generalized Ulam-Hyers-Rassias(GUHR) stable.*

Proof. Let $\varepsilon_i > 0$, $i = 1, 2$, and since we have assumed that $u \in C_{1-\gamma_1,\rho}(J) \times C_{1-\gamma_2,\rho}(J)$ is solution of (3.2). Hence by Remark 3.2, there exists a function $g_i \in C([a,T], \mathbb{R})$ such that $|g_i(t)| \leq \varepsilon_i\psi_i(t)$, $t \in [a,T]$, $i = 1, 2$, and satisfying the problem

$$\begin{cases} ^{\rho}D_{a^+}^{\alpha_i,\beta_i}\left[p_i(t)^{\rho}D_{a^+}^{\alpha_i',\beta_i'} + r_i(t)\right]u_i(t) = f_i\left(t, u_1(t), u_2(t)\right) + g_i(t), \quad t \in [a,T], \ a > 0, \ i = 1, 2, \\ \\ u_i(a) = 0, \end{cases} \tag{3.4}$$

By Lemma 2.8, the solution to Problem (3.4) is given by

$$u_i(t) = {}^{\rho}\mathcal{I}_{a^+}^{\alpha_i'}\left(\frac{1}{p_i(t)}{}^{\rho}\mathcal{I}_{a^+}^{\alpha_i}\left[f_i\left(t, u_1(t), u_2(t)\right) + g_i(t)\right]\right) - {}^{\rho}\mathcal{I}_{a^+}^{\alpha_i'}\left(\frac{r_i(t)}{p_i(t)}u_i(t)\right), \ \ i = 1, 2.$$

According to Theorem 2.2, Problem (2.8) has a unique solution $v = (v_1, v_2) \in C_{1-\gamma_1,\rho}(J) \times C_{1-\gamma_2,\rho}(J)$ that satisfies the integral equation

$$v_i(t) = {}^{\rho}\mathcal{I}_{a^+}^{\alpha_i'}\left(\frac{1}{p_i(t)}{}^{\rho}\mathcal{I}_{a^+}^{\alpha_i}f_i\left(t, v_1(t), v_2(t)\right)\right) - {}^{\rho}\mathcal{I}_{a^+}^{\alpha_i'}\left(\frac{r_i(t)}{p_i(t)}v_i(t)\right).$$

Therefore, for any $t \in [a,T]$, we have

$$\left|(u_i(t) - v_i(t))\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma_i}\right|$$

$$\leq \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} {}^\rho\mathcal{I}_{a^+}^{\alpha_i'} \left(\frac{1}{p_i^*} {}^\rho\mathcal{I}_{a^+}^{\alpha_i} |f_i\left(s, u_1(s), u_2(s)\right) - f_i\left(s, v_1(s), v_2(s)\right)|\right)(t)$$

$$+ \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} {}^\rho\mathcal{I}_{a^+}^{\alpha_i'} \left(\frac{1}{p_i^*} {}^\rho\mathcal{I}_{a^+}^{\alpha_i} |g_i(s)|\right)(t)$$

$$+ \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} {}^\rho\mathcal{I}_{a^+}^{\alpha_i'} \left(\frac{r_i^*}{p_i^*} |u_i(s) - v_i(s)|\right)(t)$$

$$\leq \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} {}^\rho\mathcal{I}_{a^+}^{\alpha_i'} \left(\frac{1}{p_i^*} {}^\rho\mathcal{I}_{a^+}^{\alpha_i} (M_i(\|u_1 - v_1\| + \|u_2 - v_2\|))\right)(t)$$

$$+ \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} {}^\rho\mathcal{I}_{a^+}^{\alpha_i'} \left(\frac{1}{p_i^*} {}^\rho\mathcal{I}_{a^+}^{\alpha_i} \varepsilon_i \psi_i(s)\right)(t) + {}^\rho\mathcal{I}_{a^+}^{\alpha_i'} \left(\frac{r_i^*}{p_i^*} \|u_i - v_i\|\right)(t)$$

$$\leq \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} \frac{1}{p_i^* \Gamma(\alpha_i)\Gamma(\alpha_i')} \frac{(T^\rho - a^\rho)^{\alpha_i + \alpha_i'}}{\alpha_i \alpha_i' \rho^{\alpha_i + \alpha_i' + 2}} (M_i(\|u_1 - v_1\| + \|u_2 - v_2\|))$$

$$+ \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i} \frac{1}{p_i^* \Gamma(\alpha_i)\Gamma(\alpha_i')} \frac{(T^\rho - a^\rho)^{\alpha_i + \alpha_i'}}{\alpha_i \alpha_i' \rho^{\alpha_i + \alpha_i' + 2}} \varepsilon_i \eta_i \psi_i(t)$$

$$+ \frac{r_i^*}{p_i^* \Gamma(\alpha_i')} \frac{(T^\rho - a^\rho)^{\alpha_i'}}{\alpha_i' \rho^{\alpha_i' + 1}} (\|u_i - v_i\|)$$

$$\leq \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i + \alpha_i + \alpha_i'} \frac{1}{p_i^* \rho^2 \Gamma(\alpha_i + 1)\Gamma(\alpha_i' + 1)} (M_i(\|u_1 - v_1\| + \|u_2 - v_2\|)$$

$$+ \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_i + \alpha_i + \alpha_i'} \frac{1}{p_i^* \rho^2 \Gamma(\alpha_i + 1)\Gamma(\alpha_i' + 1)} \varepsilon_i \eta_i \psi_i(t)$$

$$+ \frac{r_i^*}{p_i^* \rho \Gamma(\alpha_i' + 1)} \left(\frac{T^\rho - a^\rho}{\rho}\right)^{\alpha_i'} (\|u_i - v_i\|)$$

$$\leq C_i M_i(\|u_1 - v_1\| + \|u_2 - v_2\|) + C_i \varepsilon_i \eta_i \psi_i(t) + D_i(\|u_i - v_i\|)$$

$$\leq 2C_i M_i \|u - v\| + C_i \varepsilon_i \eta \psi_i(t) + D_i \|u - v\|$$

$$\leq (2C_i M_i + D_i)\|u - v\| + C_i \varepsilon_i \eta_i \psi_i(t)$$

$$\leq L_i \|u - v\| + C_i \varepsilon_i \eta_i \psi_i(t).$$

By condition (2.14), we deduce that

$$\|u - v\|_{C_{1-\gamma,\rho}} \leq \frac{C\eta}{1-L}\varepsilon\psi(t).$$

Where $\varepsilon = \max(\varepsilon_1, \varepsilon_2)$ and $C = \max(C_1, C_2)$. This demonstrates the existence of positive real number

$$\mu = \frac{C\eta}{1-L},$$

with $\eta = \max(\eta_1, \eta_2)$, $\psi(t) = \max_{t\in[a,T]}(\psi_1(t), \psi_2(t))$. Consequently, as by Definition 3.3, the solution of Problem (2.8) is Ulam-Hyers-Rassias (UHR) stable.
Furthermore, letting $\varepsilon = 1$, the problem (2.8) is generalized Ulam-Hyers-Rassias (GUHR) stable. This completes the proof. ■

3.3 Numerical example

We consider the following fractionnal differential problem

$$\begin{cases} {}^{0.3}D^{0.4,0.5}\left[\frac{100}{t}{}^{0.3}D^{0.3,0.4}+0.01t\right]u_1(t)=\frac{t}{8}(cos|u_1|-cos|u_2|) \quad t\in[\frac{\pi}{3},\frac{5\pi}{6}] \\ \\ {}^{0.3}D^{0.3,0.6}\left[\frac{200}{t}{}^{0.3}D^{0.6,0.3}+0.1t\right]u_2(t)=\frac{t}{15}(sin|u_1|-sin|u_2|) \\ \\ u_1(\frac{\pi}{3})=u_2(\frac{\pi}{3})=0, \end{cases} \tag{3.5}$$

with

$$(\alpha_1=0.4,\alpha_1'=0.3,\alpha_2=0.3,\alpha_2'=0.6)\in(0,1),$$
$$(\beta_1=0.5,\beta_1'=0.4,\beta_2=0.6,\beta_2'=0.3)\in[0,1],$$
$$(\gamma_1=0.7,\gamma_1'=0.5,\gamma_2=0.9,\gamma_2'=0.6)\in[0,1).$$

and

$$f_1(t,u_1,u_2)=\frac{t}{8}(cos|u_1|-cos|u_2|)$$
$$f_2(t,u_1,u_2)=\frac{t}{15}(sin|u_1|-sin|u_2|)$$

Since $\left(\dfrac{t^{0.3}-(\frac{\pi}{3})^{0.3}}{0.3}\right)^{0.3} f_1(t,u_1,u_2)\in C([\frac{\pi}{3},\frac{5\pi}{6}])$, we have $f_1\in C_{0.3,0.3}([\frac{\pi}{3},\frac{5\pi}{6}])$,

and since $\left(\dfrac{t^{0.3}-(\frac{\pi}{3})^{0.3}}{0.3}\right)^{0.1} f_1(t,u_1,u_2)\in C([\frac{\pi}{3},\frac{5\pi}{6}])$, we conclude that $f_2\in C_{0.1,0.3}([\frac{\pi}{3},\frac{5\pi}{6}])$.

By using all the data provided in our problem (3.5), we obtain

$$|f_1(t,u_1,u_2)-f_1(t,v_1,v_2)|\le\frac{5\pi}{48}(\|u_1-u_2\|+\|v_1-v_2\|),$$
$$|f_2(t,u_1,u_2)-f_2(t,v_1,v_2)|\le\frac{\pi}{18}(\|u_1-u_2\|+\|v_1-v_2\|),$$

$$|f_1(t,u_1,u_2)|\le\frac{5\pi}{24},$$
$$|f_2(t,u_1,u_2)|\le\frac{\pi}{9},$$

for all $t \in [\frac{\pi}{3}, \frac{5\pi}{6}]$. Consequently, hypoyheses (H1) and (H2) are satisfied. with

$$\begin{cases} p_1(\frac{\pi}{3}) = 100(\frac{3}{\pi}) = \frac{300}{\pi} = 95.492965, \\ \\ p_1(\frac{5\pi}{6}) = 100(\frac{6}{5\pi}) = \frac{600}{5\pi} = 38.197186, \end{cases}$$

where $p_1^* = \inf \left\{ p_1(\frac{\pi}{3}), p_1(\frac{5\pi}{6}) \right\} = 38.197186.$

$$\begin{cases} p_2(\frac{\pi}{3}) = 200(\frac{3}{\pi}) = \frac{600}{\pi} = 190.985931, \\ \\ p_2(\frac{5\pi}{6}) = 200(\frac{6}{5\pi}) = \frac{1200}{5\pi} = 76.394372, \end{cases}$$

where $p_2^* = \inf \left\{ p_2(\frac{\pi}{3}), p_2(\frac{5\pi}{6}) \right\} = 76.394372.$

Also,

$$\begin{cases} r_1(\frac{\pi}{3}) = 0.01(\frac{\pi}{3}) = 0.010471, \\ \\ r_1(\frac{5\pi}{6}) = 0.01(\frac{5\pi}{6}) = 0.026179, \end{cases}$$

where $r_1^* = \sup \left\{ r_1(\frac{\pi}{3}), r_1(\frac{5\pi}{6}) \right\} = 0.026179.$

$$\begin{cases} r_2(\frac{\pi}{3}) = 0.1(\frac{\pi}{3}) = 0.104719, \\ \\ r_2(\frac{5\pi}{6}) = 0.1(\frac{5\pi}{6}) = 0.261799, \end{cases}$$

where $r_2^* = \sup \left\{ r_2(\frac{\pi}{3}), r_2(\frac{5\pi}{6}) \right\} = 0.261799.$

Since

$$\begin{cases} M_1 = \frac{5\pi}{48} \\ \\ M_2 = \frac{\pi}{18} \end{cases}$$

$$\begin{cases} C_1 = \left(\dfrac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_1+\alpha_1+\alpha_1'} \dfrac{1}{p_1^*\rho^2\Gamma(\alpha_1+1)\Gamma(\alpha_1'+1)} \\ = (1.069299)^1 \dfrac{1}{38.197186 \times (0.3)^2 \times 0.4 \times 2.218 \times 0.3 \times 2.991} \\ = \dfrac{1.069299}{2.736737} = 0.390720, \\ \\ C_2 = \left(\dfrac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma_2+\alpha_2+\alpha_2'} \dfrac{1}{p_2^*\rho^2\Gamma(\alpha_2+1)\Gamma(\alpha_2'+1)} \\ = (1.069299)^1 \dfrac{1}{76.394372 \times (0.3)^2 \times 0.3 \times 2.991 \times 0.6 \times 1.489} \\ = \dfrac{1.069299}{5.511724} = 0.194004, \end{cases}$$

and

$$\begin{cases} D_1 = \left(\dfrac{T^\rho - a^\rho}{\rho}\right)^{\alpha_1'} \dfrac{r_1^*}{p_1^*\rho\Gamma(\alpha_1'+1)} = (1.069299)^{0.3} \dfrac{0.026179}{38.197186 \times 0.3 \times 0.3 \times 2.991} \\ = (1.069299)^{0.3} \dfrac{0.026179}{10.285738} = 0.002597, \\ \\ D_2 = \left(\dfrac{T^\rho - a^\rho}{\rho}\right)^{\alpha_2'} \dfrac{r_2^*}{p_2^*\rho\Gamma(\alpha_2'+1)} = (1.069299)^{0.6} \dfrac{0.261799}{76.394372 \times 0.3 \times 0.6 \times 1.489} \\ = (1.069299)^{0.6} \dfrac{0.261799}{20.475219} = 0.013310, \end{cases}$$

we obtain

$$\begin{cases} L_1 = 2C_1M_1 + D_1 = 2(0.390720)\dfrac{5\pi}{48} + 0.002597 = 0.258322 < 1, \\ \\ L_2 = 2C_2M_2 + D_2 = 2(0.194004)\dfrac{\pi}{18} + 0.013310 = 0.081030 < 1. \end{cases}$$

Therefore, condition (2.14) is verified. Since the assumption $(H1)$ of Theorem 2.2 is satisfied, we deduce that Problem (3.5) has a unique solution in $C_{0.3,0.3}([\frac{\pi}{3}, \frac{5\pi}{6}]) \times C_{0.1,0.3}([\frac{\pi}{3}, \frac{5\pi}{6}])$.

Moreover, since the assumption $(H1)$ and condition (2.14) of Theorem 3.1 are satisfied, this confirms that the solution is stable in the sense of Ulam-Hyers (UH) and generalized Ulam-Hyers (GUH).

Now, let us define the following two increasing functions on the interval $[\frac{\pi}{3}, \frac{5\pi}{6}]$:

$$\psi_1(t) = t^2, \quad \psi_2(t) = e^t.$$

For the first function $\psi_1(t)$, we have

$$I(t) = \int_a^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} s^2 ds = \frac{t^{\rho\alpha+2}}{\rho} \int_{(\frac{a}{t})^\rho}^1 (1-u)^{\alpha-1} u^{\frac{\rho+2}{\rho}} du.$$

by the following change of variable

$$u = \frac{s^\rho}{t^\rho} \Longrightarrow s = tu^{\frac{1}{\rho}} \Longrightarrow ds = t(\frac{1}{\rho}) u^{\frac{1}{\rho}-1} du.$$

Let us note that the integral is a Beta-type function such that

$$\beta_x(p,q) = \int_0^x t^{p-1}(1-t)^{q-1} dt.$$

An upper bound of the integral which denoted by $J(t)$

$$\begin{aligned} J(t) &= \int_{(\frac{a}{t})^\rho}^1 (1-u)^{\alpha-1} u^{\frac{\rho+2}{\rho}} du \\ &\leq \int_0^1 (1-u)^{\alpha-1} u^{\frac{\rho+2}{\rho}} du \\ &\leq \beta\left(\alpha, \frac{\rho+2}{\rho}\right). \end{aligned}$$

On a bounded interval $[a, T]$, we have

$$\begin{aligned} I(t) &\leq \frac{1}{\rho} T^{\rho\alpha} t^2 \beta\left(\alpha, \frac{\rho+2}{\rho}\right) \\ &\leq Ct^2, \end{aligned}$$

with

$$C = \beta\left(\alpha, \frac{\rho+2}{\rho}\right) \frac{1}{\rho} T^{\rho\alpha},$$

where

$$\beta\left(\alpha, \frac{\rho+2}{\rho}\right) = \frac{\Gamma(\alpha)\Gamma(\frac{\rho+2}{\rho})}{\Gamma(\alpha + \frac{\rho+2}{\rho})}$$

Substituting the values $\alpha = \alpha_1 + \alpha_1' = 0.7$, $\rho = 0.3$, and $T = \dfrac{5\pi}{6}$, we get

$$C \simeq 1.2901$$

Then, for $t \in [\dfrac{\pi}{3}, \dfrac{5\pi}{6}]$, we have

$$I(t) \leq 1.2901 t^2, \quad \psi_1(t) = t^2.$$

Since our integral is given by

$$^{\rho}\mathcal{I}_{a^+}^{\alpha_1+\alpha_1'}\psi_1(t) = \frac{0.3^{0.3}}{\Gamma(0.7)} I(t).$$

Then, we obtain

$$^{\rho}\mathcal{I}_{a^+}^{\alpha_1+\alpha_{1'}}\psi_1(t) \leq \eta_1\psi_1(t).$$

with $\eta_1 = 0.6926$.

Now, we move on to the second function $\psi_2(t)$, we have

$$J(t) = {}^{\rho}\mathcal{I}_{a^+}^{\alpha_2+\alpha_2'}\psi_2(t) = C_0\int_a^t (t^\rho - a^\rho)^{\alpha-1} s^{\rho-1} e^s ds,$$

such that

$$C_0 = \frac{\rho^{1-(\alpha_2+\alpha_2')}}{\Gamma(\alpha_2+\alpha_2')}.$$

For $s \in [a, t]$, as $s \leq t$, we have $e^s \leq e^t$ then

$$J(t) \leq C_0 e^t \int_a^t (t^\rho - a^\rho)^{\alpha-1} s^{\rho-1} ds.$$

Let us set

$$K(t) = \int_a^t (t^\rho - a^\rho)^{\alpha-1} s^{\rho-1} ds.$$

We evaluate the integral $K(t)$ using the same change of variable as before and we obtain

$$K(t) = \frac{t^{\rho\alpha}}{\rho}\int_{\frac{a\rho}{t^\rho}}^1 (1-u)^{\alpha-1} du.$$

Let us make a second change of variable

$$x = 1 - u \Longrightarrow dx = -du,$$

we get

$$K(t) = \frac{t^{\rho\alpha}}{\rho\alpha}\left(1 - \frac{a^{\rho}}{t^{\rho}}\right)^{\alpha}.$$

Hence,

$$J(t) \leq C_0 e^t \frac{t^{\rho\alpha}}{\rho\alpha}\left(1 - \frac{a^{\rho}}{t^{\rho}}\right)^{\alpha}, \quad \text{with} \ \left(1 - \frac{a^{\rho}}{t^{\rho}}\right)^{\alpha} \leq 1.$$

It follows that

$$J(t) \leq \frac{C_0}{\rho\alpha} T^{\rho\alpha} e^t.$$

Substituting the values $\alpha = \alpha_2 + \alpha_2' = 0.9$, $\rho = 0.3$, $T = \frac{5\pi}{6}$, and $C_0 = \frac{0.3^{0.1}}{\Gamma(0.9)}$, we obtain

$$J(t) \leq 3.9845 e^t.$$

However, $J(t) = {}^{\rho}\mathcal{I}_{a^+}^{\alpha_2+\alpha_2'}\psi_2(t)$, this yields

$${}^{\rho}\mathcal{I}_{a^+}^{\alpha_2+\alpha_2'}\psi_2(t) \leq \eta_2 \psi_2(t), \quad \text{with } \eta_2 = 3.9845.$$

So, hypotheses $(H3)$ is satisfied.

Consequently, the hypotheses of Theorem 3.2 are satisfied, which ensures the stability of solution in both Ulam-Hyers-Rassias (UHR) and generalized Ulam-Hyers-Rassias (GUHR).

Conclusion

This research makes a significant contribution, mainly covering the following aspects:
New extension to the study of Sturm-Liouville and Langevin fractional differential equations involving the Hilfer-Katugampola fractional derivative. By means of rigorous analysis, we have successfully derived an integral representation of our problem. This has facilated the effective reformulation of the problem as a fixed point theorem. The existence of a solution to our problem (see (2.8)) was demonstrated via Schauder fixed point theorem, while uniqueness was ensured by imposing an additional constraint using Banach contraction principle.
Furthermore, under the same conditions ensuring the existence and uniqueness of the solution, we are able to analyze and establish its stability in the sense of Ulam-Hyers and generalized Ulam-Hyers. Moreover, by imposing only one additional condition, we extended the stabiity results to the senses of Ulam-Hyers-Rassias and generalized Ulam-Hyers-Rassias.
Finally, a numerical example was introduced to substantiate and confirm the theoretical derived results.

Bibliography

[1] Abbas, S., Benchohra, M., Lazreg, J. E., Alsaedi, A., Zhou, Y., Existence and Ulam stability for fractional differential equations of Hilfer-Hadamard type, Adv. Difference Equ., 2017 (2017), 14 pages.

[2] Ahmad, B., Alsaedi, A., Salem, S., On a nonlocal integral boundary value problem of nonlinear Langevin equation with different fractional orders, Adv Difference Equ, 2019, 57, https://doi.org/10.1186/s13662-019-2003-x.

Nonlinear Anal, (2012), 599-606.

[3] Ahmad, B., Ntouyas, S. K., A fully Hadamard type integral boundary value problem of a coupled system of fractional differential equations, Fract. Calc. Appl. Anal., 17(2014), 348-360.

[4] Almeida, R., Malinowska, A. B., Odzijewicz, T., Fractional differential equations with dependence on the CaputoKatugampola derivative, J. Comput. Nonlinear Dyn., 11(6), 11 (2016).

[5] Andràs, Sz., Kolumbàn, J. J., On the Ulam-Hyers stability of first order differential systems with nonlocal initial conditions, Nonlinear Anal. Theory Methods Appl., 82, (2013), 1-11.

[6] Baitiche, Z., Derbazi, Ch. & Matar, M. M., Ulam stability for nonlinear-Langevin fractional differential equations involving two fractional orders in the Ψ-Caputo sense, Applicable Analysis, DOI: 10.1080/00036811.2021.1873300.

[7] Belaadi, I., Benkamouche, N., Etude d'un système d'équations difféerentielles fractionnaires généralisé de Sturm-Liouville et Langevin, Mémoire: Master Académique en Mathématiques, https://dspace.univ-guelma.dz.

[8] Benchohra, M., Lazreg, J. E., On stability for nonlinear fractional implicit differential equations, Matematiche (Catania), 70 (2015), 49-61.

[9] Berhail, A., Tabouche, N., Matar, M. M., Alzabut, J., Boundary value problem defined by system of generalized Sturm-Liouville and Langevin Hadamard fractional differentail equations, Math. Meth. Appl. Sci., (2020), 1-13.

[10] Berhail, A., Tabouche, N., Matar, M. M., Alzabut, J., On nonlocal integral and derivative boundary value problem of nonlinear Hadamard Langevin equation with three different fractional orders, Boletín de la Sociedad Mathemática Mexicana, (2019), 1-16.

[11] Bhairat, S. P., Existence and stability of fractional differential equations involving generalized Katugampola derivative, Stud. Univ. Babes-Bolyai. Math., 65 (2020), 29-46.

integration operators and the semigroup property, J. Math. Anal. Appl., 269 (2002).

[12] Butzer, P. L., Kilbas, A. A., Trujillo, J. J., Fractional calculus in the Mellin setting and Hadamard-type fractional integrals, J. Math. Anal. Appl., 269 (2002), 127.

by parts for Hadamard-type fractional integrals, J. Math. Anal. Appl., 270 (2002), 115.

[13] Granas, A., Dugundji, J., Fixed Point Theory., Springer, New York, (2003).

[14] Hadamard, J., Essai sur l'étude des fonctions données par leur development de Taylor, J. Mat. Pure Appl. Ser., 8 (1892), 101-186.

[15] Hyers, D. H., On the stability of the linear functional equation, Proc. Nat. Acad. Sci., U.S.A., 27 (1941), 222-224.

[16] Ibrahim, R. W., Generalized Ulam-Hyers stability for fractional differential equations, Int. J. Math., 23 (2012).

Langevin equations with two Hilfer-Katugampola fractional derivatives, Stud. Univ. Babes-Bolyai Math., 63 (2018), 291-302.

[17] Ishteva, M. K., Properties and Applications of the Caputo Fractional Operator, Department of Mathematics, University of Karlsruhe, Karlsruhe 5, 2005.

[18] Jung, S. -M., Hyers-Ulam stability of linear differential equations of first order, Appl. Math. Lett., 17 (2004), 1135-1140.

[19] Katugampola, U. N., New approach to a genaralized fractional integral, Appl. Math. Comput. 218, (3) (2011), 860-865.

[20] Katugampola, U. N., A new approach to generalized fractional derivatives, Bull. Math. Anal. Appl., 6, 4 (2014), 1-15.

[21] Katugampola, U. N., Existence and uniqueness results for a class of generalized fractional differential equations, Bull. Math. Anal. Appl., 6(4), 1-15 (2014).

[22] Katugampola, U. N., New fractional integral unifying six existing fractional integrals, arxiv: 1612.08596 [math.CA] (2016).

[23] Kiataramkul, C., Ntouyas, S. K., Tariboon, J., Kijjathanakorn, A., Generalized Sturm-Liouville and Langevin equations via Hadamard fractional derivatives with anti-periodic boundary conditions, Bound Value Probl., 2016, 2016, (1), 1-13.

[24] Kilbas, A., Srivastava, H. M., and Trujillo, J. J., Theory and Applications of Fractional Differential Equations, vol. 204 of North-Holland Mathematics Elsevier Science, Amestrdam, The Netherlands, 2006.

[25] Langevin, P., Sur la théorie du mouvement brownien [On the theory of Brownian motion], C R Acad. Sci. Paris., (1908) , 146, 350-533.

[26] Li, X., Sun, S., Sun, Y., Existence of solutions for fractional Langevin equation with infinite-point boundary conditions, J. Appl. Math. Comput., 2016, 53 (1), 1-10.

[27] Muensawat, T., Ntouyas, S. K., Tariboon, J., Systems of generalized Sturm-Liouville and Langevin fractional differential equations, Adv. Differ. Equ., 2017, 63 (2017).

[28] Muniyappan, P., Rajan, S., Hyers-Ulam-Rassias stability of fractional differential equation, Int. J. Pure Appl. Math., 102 (2015), 631-642.

[29] Obloza, M., Hyers stability of the linear differential equation, Rocznik Nauk.-Dydakt. Prace Mat., 13, (1993), 259-270.

[30] Oliveira, D. S., and Capelas de Oliveira, E., Hilfer-Katugampola fractional derivative, (published online). Comp. Appl. Math., (2017), 1-19.

[31] Oliveira, D.S., Capelas de Oliveira, E., Hilfer-Katugampola fractional derivative,(2017).

[32] Podlubny, I., Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential equations, to Methods of Their solution and Some of Their Applications, vol. 198 of Mathematics in Science and Engineering, Academic Press, San Diego, Calif, USA, 1993.

[33] Rassias, Th. M., On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc., 72 (1978), 297-300.

[34] Rus, I. A., Ulam stabilities of ordinary differential equations in a Banach space, Carpathian J. Math., 26 (2010), 103-107.

[35] Tarasov, V. E., Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media. Springer, HEP, New York (2011).

[36] Thiramanus, P., Ntouyas, S. K., Tariboon, J., Existence and uniqueness results for Hadamard-type fractional differential equations with nonlocal fractional integral boundary conditions, Abstr. Appl. Anal., (2014).

[37] Ulam, S. M., A collection of mathematical problems, Interscience Tracts in Pure and Applied Mathematics, Interscience Publishers, New York, (1960).

[38] Vivek, D., Kanagarajan, K., Elsayed, E. M., Some existence and stability results for hilfer fractional implicit differential equations with nonlocal conditions, Mediterr. J. Math., 15 (2018).

[39] Wang, J., Lv, L., Zhou, Y., Ulam stability and data dependence for fractional differential equations with Caputo derivative, Electron. J. Qual. Theory Differ. Equ., 2011 (2011), 10 pages.

[40] Yukunthorn, W., Ntouyas, S. K., Tariboon, J., Nonlinear fractional Caputo-Langevin equation with nonlocal Riemann-Liouville fractional integral conditions, Adv. Differ. Equ., 2014, 315.