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Analytical and Numerical Treatment of a Fractional Model for the PANTOGRAPH

Option: **PDE and numerical analysis**

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Directed by: **Dida Ridha**

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Mémoire

Présenté en vue de l'obtention du diplôme de
Master en Mathématiques

Intitulé

Traitement Analytique et Numérique d'un Modèle Fractionnaire pour le PANTOGRAPHE

Option : **EDP et analyse numérique**

Par : **Tabib Meryem**

Dirigé par : **Dida Ridha**

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بسم الله الرحمن الرحيم

ملخص

تشكل معادلات التفاضل ذات الكسر الجزئي و التأخير إطارا رياضيا قويا لنمذجة الظواهر الديناميكية المعقدة، والتي تظهر تأثيرات الذاكرة والتأخير.

في هذه الدراسة، نبحث في فئة من معادلات التفاضل ذات الكسر الجزئي والتأخير والتي تتضمن مشتقات كابوتو وريمان-ليوفيل الكسرية مع حد زمني متأخر. وعلى خلاف المناهج السابقة، نثبت وجود ووحدانية الحل التحليلي تحت شروط ليبشيتز المخففة على الحدود غير الخطية، دون الحاجة إلى افتراضات الانكماش. وباستخدام تقنيات تكرار بيكارد، نظهر تقارب الطريقة المقترحة، مما يبرز فائدتها في التطبيقات العلمية. تقدم نتائجنا رؤى جديدة في نمذجة وتحليل الأنظمة الديناميكية المعقدة، مع آثار مهمة لمختلف التخصصات العلمية والهندسية.

الكلمات المفتاحية: معادلة تفاضل كسرية، معامل التأخير، طريقة بيكارد، التكامل العددي، شروط ليبشيتز.

تصنيف موضوعات الرياضيات: $65L20$، $34K28$، $34A08$.

Abstract

Fractional delay differential equations constitute a powerful mathematical framework for modeling complex dynamical phenomena exhibiting memory and delay effects. In this study, we investigate a class of fractional delay differential equations incorporating Caputo and Riemann-Liouville fractional derivatives with a delay term. Unlike previous approaches, we establish the existence and uniqueness of the analytical solution under relaxed Lipschitz conditions on the nonlinear terms, without requiring contraction assumptions. Utilizing Picard iteration techniques, we demonstrate convergence of the numerical method under these Lipschitz conditions, thereby broadening the applicability of our model to a wider range of real-world scenarios. Additionally, numerical tests are conducted to validate the effectiveness and accuracy of the proposed method, further highlighting its utility in practical applications. Our findings offer new insights into the modeling and analysis of complex dynamical systems, with implications for various scientific and engineering disciplines.

Keywords: Fractional differential equation, delay term, Picard method, numerical integration, Lipschitz conditions.

Mathematics Subject Classification: 34A08, 34K28, 65L20.

Résumé

L'objectif de cette thèse est d'étudier une classe d'équations différentielles fractionnaires à retard, intégrant les dérivées de Caputo et de Riemann-Liouville avec un terme de mémoire. Ces équations permettent de modéliser des phénomènes dynamiques complexes où les effets de mémoire et de délai jouent un rôle crucial. Contrairement aux approches classiques, nous établissons l'existence et l'unicité de la solution analytique sous des conditions de Lipschitz assouplies, sans recourir à l'hypothèse de contraction. En appliquant l'itération de Picard, nous démontrons la convergence de la méthode numérique dans ce cadre. Des simulations numériques viennent appuyer la validité et l'efficacité de l'approche proposée, confirmant son intérêt pour de nombreuses applications scientifiques et techniques.

Mots clés : équation différentielle fractionnaire, terme de retard, méthode de Picard, intégration numérique, conditions de Lipschitz.

Classification des sujets mathématiques :34A08, 34K28, 65L20.

Notations

$\mathbb{R}$: Set of real numbers.

$\mathbb{C}$: Set of complex numbers.

$\mathbb{N}$: Set of naturel numbers.

$C^0[a,b]$: The Banach space of continuous functions.

$C^1[a,b]$: The Banach space of continuously differentiable functions.

$L^1[a,b]$: The Lebesgue space of integrable functions.

$L^p[a,b]$: The Lebesgue space of p-integrable functions.

$AC^n[a,b]$: The space of n-times absolutely continuous functions.

I^α: The Riemann-Liouville fractional integral of order α.

D^α: The Riemann-Liouville fractional derivative.

${}^C D^\alpha$: The Caputo fractional derivative.

$\mathfrak{I}_1^\alpha$: The Hadamard fractional integral.

$\mathfrak{D}_1^\alpha$: The Caputo-Hadamard fractional derivative.

Thanks

Firstly, all praise is due to **Allah** for his countless blessing and for granting me the strength and guidance to complete this thesis. To him belongs all praise, in the beginning and at the end.

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A sincere thanks to the esteemed jury members **Mr. Segni Sami** and

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Finally, I extend my sincere thanks and deep gratitude to my dear parents, who, after **Allah**'s grace, have been the foundation of this success. Thank you for being there for me, always and forever.

Dedication

I dedicate this work to:

My dear parents, whose love, prayers and support have been behind every step I take. This work is a small gift in return.

My family, each and every one, young and old, thank you for being a constant part of my life.

My loyal friends, thank you for always standing by me, and being my safe place through it all.

In the end, to my "**Master 02 Mathematics"** classmates, I'm grateful for every moment we shared together.

May **Allah** bless you all with peace, happiness, and endless goodness. You will remain in my prayers, today and always.

Contents

Introduction

Fractional differential equations (FDEs) have gained increasing attention in recent decades due to their ability to accurately model complex systems exhibiting memory and hereditary properties. Among various types of fractional derivatives, the Caputo-Hadamard derivative stands out for combining the advantages of the Caputo derivative's treatment of initial conditions with the logarithmic kernel structure of the Hadamard derivative, making it suitable for processes evolving over a multiplicative time scale.
Pantograph differential equations are characterized by functional arguments of the form $y(qt)$ with $0 < q < 1$, introduce a proportional delay component that reflects systems where the present state depends not only on current values but also on values at compressed time scales. These equations are relevant in diverse fields such as control theory, biology, and mathematical finance, where scaling and time-delay effects are intrinsic to the system dynamics.
The synthesis of Pantograph-type delays with the Caputo-Hadamard fractional derivative leads to a novel class of equations known as Pantograph Caputo-Hadamard Fractional Differential Equations. These equations provide a powerful modeling framework for systems that simultaneously exhibit scaling delays and fractional-order memory effects.
Physical Description: In electric trains, the *Pantograph* is a mechanical arm that maintains continuous contact with the overhead power line (catenary). As the train moves at high speeds, the height of the overhead line changes due to stretching, shrinking, or oscillations. This requires the Pantograph to react quickly and keep the electrical connection constant.

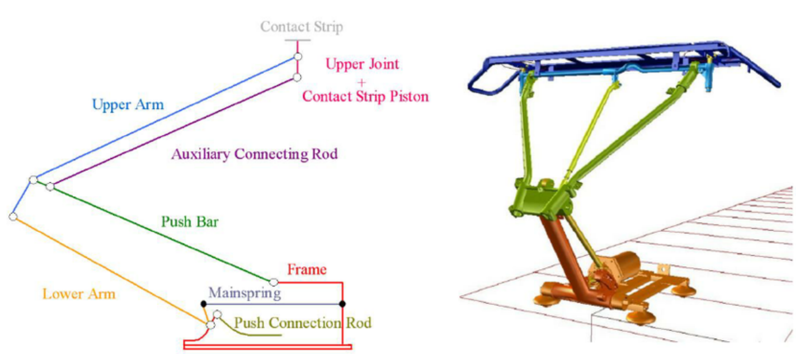

Figure 1: A typical Pantograph mechanism used in electric locomotives.

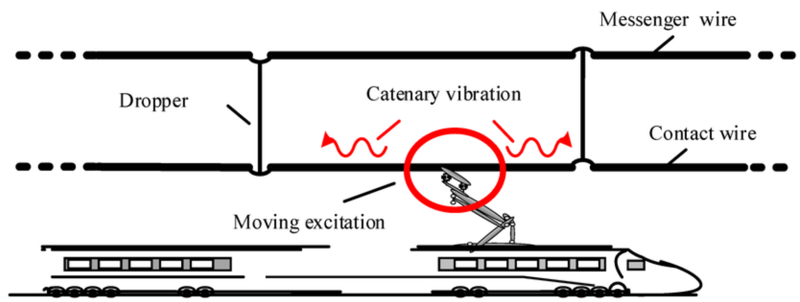

Figure 2: Pantograph-Catenary System Schematic.

Mathematical Model (Derived from a Mechanical System): Assume the Pantograph's head moves vertically under the influence of the following forces:

- **Elastic force** from the spring (Hooke's law): $-ky(t)$
- **Damping force** (internal friction): $-c\dfrac{dy}{dt}$
- **External force/triggered by contact with the overhead wire**: $F(t)$

In the classical case, the equation is:

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky(t) = F(t)$$

However, since the system exhibits **memory and history effects**, we use fractional derivatives to better model these effects.
Corresponding Fractional Equation: We replace the second derivative with a fractional derivative of order $\alpha \in (1,2)$:

$$m\,\mathfrak{D}^{\alpha}y(t) + c\,\mathfrak{D}^{\alpha-1}y(t) + ky(t) = F(t)$$

Where:

- $\mathfrak{D}^{\alpha}$ is the fractional derivative (e.g., Caputo or Caputo-Hadamard),
- m is the equivalent mass of the moving head,
- c is the damping coefficient,
- k is the spring constant,
- $F(t)$ is the excitation force due to the oscillation of the overhead wire or sudden movements.

Why Use Fractional Derivatives? Fractional derivatives are beneficial because they account for:

- **Memory effects**: The past movements influence the current state.
- **Non-local behavior**: The system's state is influenced by values over time, not just the immediate past.
- **Improved accuracy**: Especially for physical systems with irregular or non-smooth signals.

In this thesis, we analyze a fractional differential equation that models a system with memory effects and delayed feedback. The equation is given by:

$$\begin{cases} \mathfrak{D}_1^{\alpha}\varphi(t) - \mathfrak{D}_1^{\alpha-1}g(t,\varphi(1+\lambda t)) = f(t,\varphi(t),\varphi(1+\lambda t)), \quad t \geq 1, \\ \varphi(1) = x_0, \quad \varphi'(1) = x_1, \end{cases} \tag{0.1}$$

where:

- $\lambda \in (0,1), 1 < \alpha \leq 2,$
- $x_0, x_1 \in \mathbb{R},$
- $\mathfrak{D}_1^{\alpha}$ is the Caputo-Hadamard fractional derivative.

Description of the Equation

This equation represents a **fractional differential equation (FDE)** that incorporates both **memory effects** and **delayed feedback**, which are common in many physical, mechanical, and biological systems.
- The term $\mathfrak{D}_1^{\alpha}\varphi(t)$ represents the **fractional derivative** of order α, where $1 < \alpha \leq 2$, and it captures the system's **memory**. In contrast to traditional integer-order derivatives, fractional derivatives account for past states, allowing for more accurate modeling of systems where the current state depends not only on the present but also on its historical states.
- The term $g(t,\varphi(1+\lambda t))$ introduces **delayed feedback**, where the state of the system at time t is influenced by its value at a previous time $(1+\lambda t)$. Here, λ is a parameter that controls the amount of delay in the system's response. This term is particularly important for modeling systems where **delayed reactions** play a significant role, such as in mechanical systems with friction, electrical circuits with inductive elements, and biological systems with feedback mechanisms.
- The function $f(t,\varphi(t),\varphi(1+\lambda t))$ represents the **external forcing function**, which influences the system. This function could represent forces acting on a physical object, electrical inputs to a circuit, or external environmental factors.
- The **initial conditions** $\varphi(1) = x_0$ and $\varphi'(1) = x_1$ specify the state of the system at time $t = 1$.
The goal of this work is to study the behavior of solutions to this equation both analytically and numerically.

CHAPTER 1

Preliminaries

1.1 Euler's Functions

In this section several special functions used in the follow-up of the book are presented briefly. More details about these functions can be found in [20, 3, 4, 14].

1.1.1 The Gamma Function

We start by considering the Gamma function, or second order Euler integral, denoted $\Gamma(\cdot)$. For more details see for example the references [21, 2, 18].

Definition 1.1.1. [31] For real $p > 0$, the Gamma function denoted (Γ) is defined by:

$$\Gamma(p) = \int_0^\infty e^{-x} x^{p-1} dx \tag{1.1}$$

Theorem 1.1.1. *[11] The function $\Gamma(p)$ is convergent for $p > 0$*

Proof. The integral can be written as :

$$\Gamma(p) = \int_0^1 e^{-x} x^{p-1} dx + \int_1^\infty e^{-x} x^{p-1} dx = I_1 + I_2,$$

where, $I_1 = \int_0^1 e^{-x} x^{p-1} dx$ is convergent. Since e^{-x} is decreasing on the intervall $[0, 1]$, from $x = 0$ we have :

$$\int_0^1 e^{-x} x^{p-1} dx < \int_0^1 x^{p-1} dx = \frac{1}{p}.$$

Moreover, $I_2 = \int_1^\infty e^{-x} x^{p-1} dx$ is also convergent. we obtain:

$$1 \leq x \;\Rightarrow\; x^{p-1} e^{-x} \;\leq e^{-x/2} \;\Leftrightarrow\; x^{p-1} \leq e^{x/2} \;\Leftrightarrow\; \frac{x^{p-1}}{e^{x/2}} \leq 1.$$

Because $\lim_{x\to\infty} \frac{x^{p-1}}{e^{x/2}} = 0$, we have :

$$\int_1^\infty e^{-x} x^{p-1} dx \leq \int_1^\infty e^{-x/2} dx = 2e^{-1/2}.$$

The integral (1.1) is convergent for $p > 0$ and divergent for $p \leq 0$. □

Properties 1.1.1. *Some of the most important properties of the Gamma function are:*

1. *The function $\Gamma(p)$ is continuous for $p > 0$.*
2. *The function $\Gamma(p)$ obeys the property:*

$$\Gamma(p+1) = p\Gamma(p). \tag{1.2}$$

Proof. By integration by parts we obtain:

$$\Gamma(p+1) = \int_0^\infty e^{-x} x^p dx = -[e^{-x} x^p]_0^\infty + p \int_0^\infty e^{-x} x^{p-1} dx = p\Gamma(p).$$ □

3. *The following relation are also valid :*

$$\Gamma(p+n) = (p+n-1)\ldots(p+1)\Gamma(p) \tag{1.3}$$

$$\Gamma(1) = 1,$$
$$\Gamma(n+1) = n!,$$
$$\Gamma(0) = +\infty.$$

4. *For $p = -n$, we have:*

$$\Gamma(-n) = \frac{\Gamma(-n+1)}{-n}$$
$$= \frac{\Gamma(-n+2)}{n(n-1)} = \frac{\Gamma(-n+3)}{n(n-1)(n-2)} = \ldots = (-1)^n \frac{\Gamma(0)}{n!} = (-1)^n \infty.$$

1.1.2 The Beta Function

Here we consider the Beta function, denoted (B) .
The Beta function, or the first order Euler function, can be defined as [30, 5]:

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx,$$

where $Re(p) > 0$ and $Re(q) > 0$.

Properties 1.1.2. *In the following we will enumerate the fundamental properties of the Beta function:*

1. *For every $p > 0$ and $q > 0$, we have :*

$$B(p,q) = B(q,p).$$

2. *For every $p > 0$ and $q > 1$, the Beta function (B) satisfies the property :*

$$B(p,q) = \frac{q-1}{p+q-1}B(p,q-1).$$

Proof.

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$$

$$x^p(1-x)^{q-2} = x^{p-1}(1-x)^{q-2} - x^{p-1}(1-x)^{q-1},$$

$$\begin{aligned} B(p,q) &= \int_0^1 (1-x)^{q-1}\frac{dx^p}{p} = \frac{x^p(1-x)^{q-1}}{p}\bigg|_0^1 + \frac{q-1}{p}\int_0^1 x^p(1-x)^{q-2}dx \\ &= \frac{q-1}{p}\int_0^1 x^{p-1}(1-x)^{q-2}dx - \frac{q-1}{p}\int_0^1 x^{p-1}(1-x)^{q-1}dx \\ &= \frac{q-1}{p}B(p,q-1) - \frac{q-1}{p}B(p,q). \end{aligned}$$

□

3. *For every $p > 0$ and $q > 0$, it is valid the identity :*

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Proof. The product $\Gamma(p)\Gamma(q)$ can be written as :

$$\Gamma(p)\Gamma(q) = \int_0^{+\infty} e^{-t}t^{p-1}dt \int_0^{+\infty} e^{-s}s^{q-1}ds = \int_0^{+\infty}\int_0^{+\infty} e^{-(t+s)}t^{p-1}s^{q-1}dtds,$$

$$\Gamma(p+q) = \int_0^{+\infty}\int_0^{+\infty} e^{-(t+s)}t^{p-1}s^{q-1}dtds.$$

We use the notation $t = xy$ and $s = x(1-y)$, then $s+t = x$ and $\dfrac{t}{t+s} = y$, for $0 < t < \infty$ and $0 < s < \infty$.
The Jacobian is :

$$\frac{D[t,s]}{D[x,y]} = \left\| \begin{matrix} y & x \\ 1-y & -x \end{matrix} \right\| = -xy - x + xy = -x$$

hence :

$$dtds = \left| \frac{D[t,s]}{D[x,y]} \right| dxdy = xdxdy,$$

$$\begin{aligned} \Gamma(p)\Gamma(q) &= \int_0^{\infty}\int_0^1 e^{-x}(xy)^{p-1}x^{q-1}(1-y)^{q-1}xdxdy \\ &= \int_0^{\infty} e^{x}x^{p+q+1}dx \int_0^1 y^{p-1}(1-y)^{q-1}dy, \end{aligned}$$

$$\Gamma(p)\Gamma(q) = \Gamma(p+q)B(p,q).$$

□

4. *For every $p > 0$, and for the natural number n, it can be proved*

$$B(p,n) = B(n,p) = \frac{1\cdot 2\cdot 3\ldots(n-1)}{p(p+1)\ldots(p+n)},$$

and also :

$$B(p,1) = \frac{1}{p}.$$

For any natural number m, n we obtain :

$$B(m,n) = \frac{(n-1)!(m-1)!}{(m+n-1)!}.$$

1.2 Fractional integrals and derivatives

In this section, we review some basic properties of fractional integrals and derivatives, which we will need later in the analysis of concrete problems. This section contains results from various books and papers([1, 25, 16]).

1.2.1 Riemann-Liouville fractional integrals and derivatives

There are many possible generalizations of the notion of a derivative of a function that would lead to the answer of the question: what is $\frac{d^n}{dx^n}y(x)$ when n is any real number? We start from the Cauchy formula for an n-fold primitive of a function f given as

$$ {}_aI_t^\alpha f(t) = \frac{1}{(n-1)!}\int_a^t (t-\tau)^{n-1} f(\tau)d\tau, \quad t \in [a,b], \quad n \in \mathbb{N}, $$

Where it is assumed that $f(t) = 0$, for $t < a$. Note that $(n-1)! = \Gamma(n)$, where Γ is the Euler Gamma function (see definition 1.1.1).

Definition 1.2.1. [29] The left Riemann-Liouville fractional integral of order $\alpha \in \mathbb{C}$ is formally given by

$$ {}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \tag{1.4} $$

where $t \in [a,b]$, $\operatorname{Re}\alpha > 0$ and $\Gamma(\alpha)$ is the Gamma function.

A direct computation yields

$$ {}_aI_t^\alpha (t-a)^p = \frac{\Gamma(1+p)}{\Gamma(1+p+\alpha)}(t-a)^{p+\alpha}. $$

In the special case of positive real $\alpha(\alpha \in \mathbb{R}_+)$ and $f \in L^1(a,b)$, the integral ${}_aI_t^\alpha f$ exists for almost all $t \in [a,b]$. Also ${}_aI_t^\alpha f \in L^1(a,b)$ (see [17]p.13). For $\alpha = 0$, we define ${}_aI_t^0 f = f$. This definition is motivated by the following reasoning.
Suppose that $f \in C^1([a,b])$. Then, after integration by parts, from (1.4), we have

$$ {}_aI_t^\alpha f(t) = \frac{(t-a)^\alpha}{\Gamma(\alpha+1)}f(a) + \frac{1}{\Gamma(\alpha+1)}\int_a^t (t-\tau)^\alpha f^{(1)}(\tau)d\tau, $$

so that

$$ \lim_{\alpha\to 0} {}_aI_t^\alpha f(t) = f(a) + \int_a^t f^{(1)}(\tau)d\tau = f(t). $$

Definition 1.2.2. [29] The right Riemann-Liouville fractional integral of order $\alpha \in \mathbb{C}$ is formally given by

$$ {}_tI_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (\tau-t)^{\alpha-1} f(\tau)d\tau, \quad t \in [a,b], \quad \operatorname{Re}\alpha > 0. \tag{1.5} $$

The existance is the same as in the case of the left Riemann-Liouville fractional integral given above.
In the special case when $f(t) = (t-a)^{\beta-1}$ and $g(t) = (b-t)^{\beta-1},\ t \in [a,b],\ \alpha, \beta \in \mathbb{C}$, we have

$$ {}_aI_t^\alpha (t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(t-a)^{\beta+\alpha-1}, \quad \operatorname{Re}\alpha > 0,\ \operatorname{Re}\beta > 0, $$

$$ {}_tI_b^\alpha (b-t)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(b-t)^{\beta+\alpha-1}, \quad \mathrm{Re}\,\alpha > 0,\ \mathrm{Re}\,\beta > 0. $$

Operators ${}_aI_t^\alpha$ and ${}_tI_b^\alpha$ with $\mathrm{Re}\,\alpha > 0$ are bounded operators from $L^p(a,b)$ into $L^p(a,b)$, $p \geq 1$. The following estimates hold :

$$ \|{}_aI_t^\alpha f\|_{L^p(a,b)} \leq \frac{(b-a)^{\mathrm{Re}\,\alpha}}{\mid \Gamma(\alpha) \mid \mathrm{Re}\,\alpha}\|f\|_{L^p(a,b)}, \quad \|{}_tI_b^\alpha f\|_{L^p(a,b)} \leq \frac{(b-a)^{\mathrm{Re}\,\alpha}}{\mid \Gamma(\alpha) \mid \mathrm{Re}\,\alpha}\|f\|_{L^p(a,b)}, $$

(see [28]p.48). If $\alpha \in (0,1)$ and $1 < p < \dfrac{1}{\alpha}$, then the operators ${}_aI_t^\alpha$ and ${}_tI_b^\alpha$ are bounded from $L^p(a,b)$ into $L^q(a,b)$ for $q = \dfrac{p}{1-\alpha p}$ (see [28]p.66).

Definition 1.2.3. [29] The left and right Riemann-Liouville fractional derivatives ${}_aD_t^\alpha f$ and ${}_tD_b^\alpha f$ of the order $\alpha \in \mathbb{C}$, $\mathrm{Re}\,\alpha \geq 0$, $n-1 \leq \mathrm{Re}\,\alpha < n$, $n \in \mathbb{N}$, with the appropriate assumptions on f (see below), are defined as

$$ {}_a\mathrm{D}_t^\alpha f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\big({}_a\mathrm{I}_t^{n-\alpha} f(t)\big) = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau, \quad t \in (a,b). \tag{1.6} $$

$$ {}_t\mathrm{D}_b^\alpha f(t) = (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}t^n}\big({}_t\mathrm{I}_b^{n-\alpha} f(t)\big) = (-1)^n\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_t^b \frac{f(\tau)}{(\tau-t)^{\alpha-n+1}}d\tau,\ t \in (a,b). \tag{1.7} $$

If $f \in AC^n([a,b])$ and $n-1 \leq \mathrm{Re}\,\alpha < n,\ \ n \in \mathbb{N}$, then ${}_a\mathrm{D}_t^\alpha f$ and ${}_t\mathrm{D}_b^\alpha f$ exist almost everywhere on $[a,b]$ and

$$ {}_a\mathrm{D}_t^\alpha f(t) = \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(1+k-\alpha)}(t-a)^{k-\alpha} + \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau, \tag{1.8} $$

$$ {}_t\mathrm{D}_b^\alpha f(t) = \sum_{k=0}^{n-1}(-1)^k\frac{f^{(k)}(b)}{\Gamma(1+k-\alpha)}(b-t)^{k-\alpha} + \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^b \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau, \tag{1.9} $$

see [1]. From the definitions, it follows that in the special case when $f(t) = (t-a)^{\beta-1}$, $t > a$, and $f(t) = (b-t)^{\beta-1},\ t < b,\ \beta \in \mathbb{C}$, we have

$$ \begin{aligned} {}_a\mathrm{D}_t^\alpha (t-a)^{\beta-1} &= \frac{\Gamma(\beta)}{(\beta-\alpha)}(t-a)^{\beta-\alpha-1}, \\ {}_t\mathrm{D}_b^\alpha (b-t)^{\beta-1} &= \frac{\Gamma(\beta)}{(\beta-\alpha)}(t-a)^{\beta-\alpha-1}. \end{aligned} \tag{1.10} $$

Again, from (1.10), for constant function $f = C$, we have

$$ {}_a\mathrm{D}_t^{\alpha} C = \frac{C}{(1-\alpha)}(t-a)^{-\alpha} \quad \text{and} \quad {}_t\mathrm{D}_b^{\alpha} C = \frac{C}{(1-\alpha)}(b-t)^{-\alpha} $$

Also, ${}_a\mathrm{D}_t^{\alpha} f(t) = 0$ and ${}_t\mathrm{D}_b^{\alpha} g(t) = 0$, $n-1 \leq \mathrm{Re}(\alpha) < n$, if and only if, respectively,

$$ f(t) = \sum_{k=1}^{n} c_k (t-a)^{\alpha-k} \quad \text{and} \quad g(t) = \sum_{k=1}^{n} d_k (b-t)^{\alpha-k} \tag{1.11} $$

where c_k et d_k, $k = 1, \dots, n$, are arbitrary constants. Thus, functions f and g in (1.11)play the role of constants for the left and right Riemann-Liouville fractional derivatives, respectively.

Let $\alpha = k + \gamma$, $k \in \mathbb{N}_0$, $\gamma \in [0,1)$.Then, ${}_0\mathrm{D}_t^{\alpha}$ and ${}_tD_b^{\alpha}$ may be written as

$$ {}_0\mathrm{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\gamma)} \frac{\mathrm{d}^{k+1}}{\mathrm{d}t^{k+1}} \int_0^t \frac{f(\tau)}{(t-\tau)^{\gamma}} d\tau, \quad t > 0, $$
$$ {}_t\mathrm{D}_b^{\alpha} f(t) = (-1)^{k+1} \frac{1}{\Gamma(1-\gamma)} \frac{\mathrm{d}^{k+1}}{\mathrm{d}t^{k+1}} \int_t^b \frac{f(\tau)}{(\tau-t)^{\gamma}} d\tau, \quad t < b. $$

Sometimes, in short, it is written ${}_a\mathrm{D}_t^{\alpha} f = f^{(\alpha)}$.
Let $\alpha \in [0,1)$. Then, for $t > a$ and $t < b$, we have

$$ \begin{aligned} {}_a\mathrm{D}_t^{\alpha} f(t) &= \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha}} \mathrm{d}\tau, \\ {}_t\mathrm{D}_b^{\alpha} f(t) &= -\frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_t^b \frac{f(\tau)}{(\tau-t)^{\alpha}} \mathrm{d}\tau. \end{aligned} \tag{1.12} $$

In the case when α is purely imaginary, i.e. $\alpha = \mathrm{i}\theta$, the left Riemann-Liouville fractional derivative is defined as

$$ {}_a\mathrm{D}_t^{\mathrm{i}\theta} f(t) = \frac{1}{\Gamma(1-\mathrm{i}\theta)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \frac{f(\tau)}{(t-\tau)^{\mathrm{i}\theta}} \mathrm{d}\tau, \quad t \geq a. $$

Consider the problem of determining $\lim\limits_{\alpha \to 1^-} {}_a\mathrm{D}_t^{\alpha} f$. Then, we have the following proposition.

Proposition 1.2.1. *[6] Suppose that $f \in C^1([0,T])$. Then, $\lim\limits_{\alpha \to 1^-} {}_aD_t^{\alpha} f = f^{(1)}$*

We put $\dfrac{\mathrm{d}^n}{\mathrm{d}t^n}(.) = \mathrm{D}^n(.)$.The index rule holds for the integer-order integrals and derivatives

$$ \begin{aligned} ({}_a\mathrm{I}_t^n {}_a\mathrm{I}_t^m) f(t) = ({}_a\mathrm{I}_t^m {}_a\mathrm{I}_t^n) f(t) = ({}_a\mathrm{I}_t^{n+m}) f(t), \quad n,\ m\ \in\ \mathbb{N}_0 \\ ({}_a\mathrm{D}_t^n {}_a\mathrm{D}_t^m) f(t) = ({}_a\mathrm{D}_t^m {}_a\mathrm{D}_t^n) f(t) = ({}_a\mathrm{D}_t^{n+m}) f(t), \quad n,\ m\ \in\ \mathbb{N}_0 \end{aligned} \tag{1.13} $$

The semi-group property (1.13) holds for fractinal integrals only.

Proposition 1.2.2. *[17] The fractional integral ${}_aI_t^\alpha$ as a mapping from $L^1(a,b) \to L^1(a,b)$ forms a commutative semi-group with respect to orders of integrals. The identity operator ${}_aI_t^0$ is the neuteral elememt. Thus, if $Re\,\alpha$, $Re\,\beta > 0$*

$$({}_aI_t^\alpha\, {}_aI_t^\beta)f(t) = ({}_aI_t^\beta\, {}_aI_t^\alpha)f(t) = ({}_aI_t^{\alpha+\beta})f(t),$$
$$({}_tI_b^\alpha\, {}_tI_b^\beta)f(t) = ({}_tI_b^\beta\, {}_tI_b^\alpha)f(t) = ({}_tI_b^{\alpha+\beta})f(t),$$

holds for almost all $t \in [a,b]$ (almost everywhere (a.e.) in $[a,b]$) if $f \in L^p(a,b)$, $1 \leq p \leq \infty$

Also, it can be shown that for $\mathrm{Re}\,\alpha > 0$, $f \in L^p(a,b)$, $1 \leq p \leq \infty$, the composition of fractional derivatives and fractinal integrals holds, for almost all $t \in (a,b)$ (see [28]p.44),

$$({}_a\mathrm{D}_t^\alpha\, {}_a\mathrm{I}_t^\alpha)f(t) = f(t), \quad \text{and} \quad ({}_t\mathrm{D}_b^\alpha\, {}_t\mathrm{I}_b^\alpha)f(t) = f(t).$$

Showing that ${}_a\mathrm{D}_t^\alpha$, ${}_t\mathrm{D}_b^\alpha$ are the left inverses of ${}_a\mathrm{I}_t^\alpha$, ${}_t\mathrm{I}_b^\beta$, respectively. However by applying ${}_a\mathrm{D}_t^\alpha$ and ${}_t\mathrm{D}_b^\alpha$ to the right of ${}_a\mathrm{I}_t^\alpha$, and ${}_t\mathrm{I}^\alpha$,we have different situation. To examine the resulting relations, we define the following spaces:

$$\begin{aligned} {}_a\mathrm{I}_t^\alpha(L^p) &= \{f \mid f = {}_a\mathrm{I}_t^\alpha \varphi,\ \ \varphi \in L^p(a,b)\} \text{ and} \\ {}_t\mathrm{I}_b^\alpha(L^p) &= \{g \mid g = {}_t\mathrm{I}_b^\alpha \psi,\ \ \psi \in L^p(a,b)\}. \end{aligned} \tag{1.14}$$

Proposition 1.2.3. *[1] Let $Re\ \alpha > 0$, $n-1 < Re\ \alpha < n$. Then the following holds :*

i) *If $f \in {}_aI_t^\alpha(L^p)$, $1 \leq p \leq \infty$, then*

$$({}_aI_t^\alpha\, {}_aD_t^\alpha)f(t) = f(t), a.e., \ in\ [a,b]. \tag{1.15}$$

ii) *If $f \in L^1(a,b)$, ${}_aI_t^{n-\alpha}f \in AC^n([a,b])$, then*

$$({}_aI_t^\alpha\, {}_aD_t^\alpha)f(t) = f(t) - \sum_{j=1}^{n} \frac{(t-a)^{\alpha-j}}{\Gamma(\alpha-j+1)} \Big[\frac{d^{n-j}}{dt^{n-j}}\big({}_aI_t^{n-\alpha}\big)\Big]_{t=a}. \tag{1.16}$$

holds for almost all $t \in [a,b]$.

We state the results about the index rule for the fractional derivatives.

1.2.2 Caputo fractinal derivatives

We present the definition of fractional derivative from Caputo [22] and Caputo and Mainardi [23]. The left Caputo fractional derivative of a function of order α, denoted by ${}_a^C\mathrm{D}_t^\alpha f$, is :

$${}_a^C\mathrm{D}_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} \mathrm{d}\tau, & n-1 \leq \alpha \leq n \\ \dfrac{\mathrm{d}^n}{\mathrm{d}t^n} f(t) & \alpha = n, \end{cases} \quad t \in [a,b] \tag{1.17}$$

Similarly, the right Caputo derivative is defined as

$$ {}_{t}^{C}\mathrm{D}_{b}^{\alpha} f(t) = \begin{cases} (-1)^n \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_t^b \dfrac{f^{(n)}(\tau)}{(\tau - t)^{\alpha+1-n}} \mathrm{d}\tau, & n-1 \leq \alpha \leq n \\ (-1)^n \dfrac{\mathrm{d}^n}{\mathrm{d}t^n} f(t) & \alpha = n, \quad n \in \mathbb{N} \end{cases} \quad t \in [a,b] \tag{1.18} $$

It is easy to see that

$$ {}_{a}^{C}\mathrm{D}_{t}^{\alpha} f(t) = {}_{a}\mathrm{I}_{t}^{n-\alpha}\Big(\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t)\Big) \quad \text{and} \quad {}_{t}^{C}\mathrm{D}_{b}^{\alpha} f(t) = (-1)^n \, {}_{t}\mathrm{I}_{b}^{n-\alpha}\Big(\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t)\Big), $$

where ${}_{a}\mathrm{I}_{t}^{n-\alpha}$ and ${}_{t}\mathrm{I}_{b}^{n-\alpha}$ are the Riemann-Liouville fractional integrals (1.4), (1.5) respectively. Observe that (1.17) for $a = 0$, can be written as

$$ {}_{0}^{C}\mathrm{D}_{t}^{\alpha} f(t) = \frac{t^{n-1-\alpha}}{\Gamma(n-\alpha)} * \frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t), \quad t > 0, \quad n-1 \leq \operatorname{Re} \alpha < n. \tag{1.19} $$

Note that the Caputo derivative of a constant function is zero

$$ {}_{a}^{C}\mathrm{D}_{t}^{\alpha} C = 0 \quad \text{and} \quad {}_{t}^{C}\mathrm{D}_{b}^{\alpha} C = 0. \tag{1.20} $$

1.3 Fixed point theory

Banach fixed point theorem

Recall that problem of initial value

$$ x' = f(t,x), \; x(t_0) = x_0, \tag{1.21} $$

can be expressed as an integral equation

$$ x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds, \tag{1.22} $$

from which a sequence of functions $\{x_n\}$ can be defined by

$$ x_0(t) = x_0, \; x_1(t) = x_0 + \int_{t_0}^{t} f(s, x_0)\, ds, $$

and, in general

$$ x_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, x_n(s))\, ds. \tag{1.23} $$

This is called Picard's method of successive approximations, and under generous conditions on f, it can be shown that $\{x_n\}$ converges uniformly over an interval $|t - t_0| \leq k$

to a continuous function, say x. By taking the limit in the defining equation x_{n+1}, we pass the limit through the integral and we get

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds,$$

So that $x(t_0) = x_0$ and after differentiation, we obtain $x'(t) = f(t, x(t))$. Thus, x is a solution of the initial values problem.

Banach discovered that it was, in fact a fixed-point theorem with a broad application. By defining an operator B on a complet metric space $C([t_0, t_0 + k], \mathbb{R})$ using the supremum norme $\|.\|$ for any $x \in C$ implies

$$(Bx)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds. \tag{1.24}$$

If a fixed point of the operator B, denoted as $B\phi = \phi$ is found, it corresponds to a solution of the initial value problem. This approach had two remarkable characteristics.
Firstly, it found applications in problems across a wide range of mathematical disciplines involving complete metric spaces. Secondly, it introduced a level of clarity and rigor. For example, the often complex and uncertain proofs of the implicit functions theorems have been simplified and strengthened by the application of fixed-point theory. In this context, we will use this theory to demonstrate the existence of solutions for various types of differential equations.

Definition 1.3.1. [26] Let (E, ρ) be a complete metric space and $B : E \to E$. The operator B is said to be a contraction if there exists a $\lambda \in (0, 1)$ such that $x, y \in E$ implies

$$\rho(Bx, By) \leq \lambda\rho(x, y).$$

Theorem 1.3.1. *[26][The Contraction Mapping Principal] Let (E, ρ) be a complete metric space and $B : E \to E$ a contracting operator. Then, there exist a unique $x \in E$ such that $Bx = x$. Moreover, if $y \in E$ and if $\{y_n\}$ is defined inductively by $y_1 = By_0$ and $y_{n+1} = By_n$, then $y_n \to x$, the unique fixed point. In particular, the equation $Bx = x$ has one and only one solution.*

Proof. Let $x_0 \in E$ and define a sequence $\{x_n\}$ in E by:
$x_1 = Bx_0,\ x_2 = Bx_1 = B^2x_0, \dots, x_n = Bx_{n-1} = B^n x_0$.
To see that $\{x_n\}$ is a Couchy sequence, note that if $m > n$, then

$$
\begin{aligned}
\rho(x_n, y_m) &= \rho(B^n x_0, B^m x_0) \\
&\leq \lambda \rho(B^{n-1} x_0, B^{m-1} x_0) \\
&\vdots \\
&\leq \lambda^n \rho(x_0, x_{m-n}) \\
&\leq \lambda^n \{\rho(x_0, x_1) + \rho(x_1, x_2) + \ldots + \rho(x_{m-n-1}, x_{m-n})\} \\
&\leq \lambda^n \left\{\rho(x_0, x_1) + \lambda\rho(x_0, x_1) + \ldots + \alpha^{m-n-1}\rho(x_0, x_1)\right\} \\
&= \lambda^n \rho(x_0, x_1) \left\{1 + \lambda + \ldots + \lambda^{m-n-1}\right\} \\
&\leq \lambda^n \rho(x_0, x_1) \left\{\frac{1}{(1-\lambda)}\right\}.
\end{aligned}
$$

Since $\lambda < 1$, the right side tends towards 0 as $n \to \infty$. Thus, $\{x_n\}$ is a Cauchy sequence and (E, ρ) is complete, so it has a limit $x \in E$. Now, B is certainly continuous, so

$$
Bx = B\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} (Bx_n) = \lim_{n\to\infty} x_{n+1} = x,
$$

and x is a fixed point. To see that x is the unique fixed point, let's assume that $Bx = x$ and $By = y$. Then

$$
\rho(x, y) = \rho(Bx, By) \leq \lambda\rho(x, y),
$$

and, for $\lambda < 1$, we calculate $\rho(x, y) = 0$, so that $x = y$. This concludes the proof.
By applying this result in (1.21), adisturbing event occurred, which we will briefly describe. Suppose that f is continuous and satisfies a global Lipschitz condition at x, say

$$
|f(t, x_1) - f(t, x_2)| \leq L\,|x_1 - x_2|\,,
$$

for $t \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{R}^n$. Then, by (1.24), we obtain (for $t \geq t_0$)

$$
\begin{aligned}
|Bx_1(t) - Bx_2(t)| &= \left|\int_{t_0}^{t} [f(s, x_1(s)) - f(s, x_2(s))]\,ds\right| \\
&\leq \int_{t_0}^{t} L\,|x_1(s) - x_2(s)|\,ds,
\end{aligned}
$$

Therefore if $\|.\|$ is the sup norm on continuous functions on $[t_0, t_0 + k]$, then

$$
\|Bx_1 - Bx_2\| \leq Lk\,\|x_1 - x_2\|\,.
$$

It is a contraction where $Lk = \lambda < 1$. Now, L is fixed, and we choose k small enough for $Lk < 1$. This gives a fixed point, which is a solution to (1.21) on the interval $[t_0, t_0 + k]$. $\square$

1.4 Integral Equations

1.4.1 Volterra Equation

An equation of the form

$$f(t) - \int_a^t K(t,s,f(s))ds = g(t), \ \ a \leq t \leq T \tag{1.25}$$

is a Volterra equation of the second kind. Here the unknown is $f(s)$. The right-hand side $g(t)$ and the kernel $K(t,s,u)$ are assumed to be known.
Equation (1.25) is one of several forms in which a Volterra equation can be written. More generally, one might consider the form

$$F\left(f(t), t, \int_a^t K(t,s,f(s))ds, g(t)\right) = 0, \tag{1.26}$$

but we will limit our attention to the more common form (1.25).

For our purposes we assume that T is finite. In many practical applications, the behavior of the solution on the whole real axis of intrest. In this situation the limiting behavior of the solution is usually found from its behavior for large, but finite T. In numerical computations it is necessary in any case to use a finite T.
For notational simplicity w can, without loss of generality, choose the range of the independent variable so that the lower limit is zero and consider only the equation

$$f(t) - \int_0^t K(t,s,f(s))ds = g(t), \ \ 0 \leq t \leq T \tag{1.27}$$

In our subsequent discussion, wherever the domain of the equation is unspecified, we will assume it to be $0 \leq t \leq T < \infty$.

Of special interest is the linear case in which

$$K(t,s,f(s)) = k(t,s)f(s). \tag{1.28}$$

Linearity somewhat simplifies the treatment of the equation, although when the nonlinearity is suitably restricted it introduces few essential complications.
There are many applications where the kernel of the equation is unbounded, that is, the equation is (in our terminology) singular. Where possible, we will write the kernels of such equations as

$$K(t,s,f(s)) = p(t,s)H(t,s,f(s)),$$

where $p(t,s)$ represents the singular part, that is, it is chosen so that $H(t,s,f(s))$ is bounded.

A fundamentally different kind of equation is the Volterra equation of the first kind

$$\int_0^t k(t,s,f(s))ds = g(t). \tag{1.29}$$

Although formally one can often reduce such an equation to one of the second kind (e.g.,by differentiation), we will see in subsequent discussions that equations of the first kind present some serious practical difficulties.

Historically, one of the earliest integral equations to be studied was Abel's equation

$$\int_0^t \frac{f(s)}{\sqrt{t-s}} ds = g(t), \tag{1.30}$$

which is an example of a singular equation of the first kind. Nowadays it is fairly common practice to call the equation

$$\int_0^t p(t,s)h(t,s)f(s)ds = g(t), \tag{1.31}$$

with $h(t,s)$ bounded and $p(t,s)$ unbounded (but restricted to guarantee existence and uniqueness of the solution) a generalized Abel equation.

Formally, one can immediately extend the classification to systems of equations by interpreting f, K and g as vectors. Thus (1.27) becomes

$$f(t) - \int_0^t K(t,s,f(s))ds = g(s), \tag{1.32}$$

where

$$f(s) = \begin{pmatrix} f_1(s) \\ f_2(s) \\ \vdots \\ f_m(s) \end{pmatrix},$$

and

$$K(t,s,f(s)) = \begin{pmatrix} K_1(t,s,f_1(s),f_2(s),\ldots,f_m(s)) \\ K_2(t,s,f_1(s),f_2(s),\ldots,f_m(s)) \\ \vdots \\ K_m(t,s,f_1(s),f_2(s),\ldots,f_m(s)) \end{pmatrix}.$$

Equation (1.32) is then a system of the second kind.

Volterra integro-differential equations involve derivatives of the unknown as well as integral terms. The presence of both derivatives and integrals allows for a profusion of different forms, but there does not exist any commonly used convention for classifying them. Fortunately, most of the equations arising in practice have a fairly simple form and can usually be reduced to integral equations.

1.5 Numerical Integration

We have various methods for approximating the integral of a bounded function f defined on an interval $[a,b]$,

$$\int_a^b f(x)dx$$

There are several reasons why such approximations are useful. First, not all functions can be integrated analytically. Second, even when an antiderivative exists, it may not be the most efficient way to compute the integral. Furthermore, in some cases, we may need to integrate an unknown function for which only a few sample values are available.
To gain a better understanding of numerical integration, it is natural to revisit Riemann integration, a framework that can be regarded as an approach to integral approximation.

1.5.1 Rectangle method

In this method, the integrated function f is replaced by a piecewise constant function $g(x)$ on each elementary subinterval $[x_{i-1}, x_i]$, either by

- **Left rectangles:** $g(x) = f(x_{i-1})$ for $x \in [x_{i-1}, x_i]$
$$\int_a^b f(x)dx \simeq \sum_{i=1}^{n}(x_i - x_{i-1})f(x_{i-1}),$$
- **Right rectangles:** $g(x) = f(x_i)$ for $x \in [x_{i-1}, x_i]$
$$\int_a^b f(x)dx \simeq \sum_{i=1}^{n}(x_i - x_{i-1})f(x_i).$$
- **Midpoint rectangles:** $g(x) = f\left(\frac{x_{i-1}+x_i}{2}\right)$ for $x \in [x_{i-1}, x_i]$
$$\int_a^b f(x)dx \simeq \sum_{i=1}^{n}(x_i - x_{i-1})f\left(\frac{x_{i-1}+x_i}{2}\right),$$

If the subdivision is uniform with step size $h = \frac{b-a}{n}$,
$$x_i = a + ih, \;\; i = 0, \ldots, n,$$
we obtain the rectangle formulas:

- On the left
$$\int_a^b f(x)dx \simeq h\sum_{i=1}^{n} f(x_{i-1}) = h(f(x_0) + f(x_1) + \cdots + f(x_{n-1})).$$
- On the right
$$\int_a^b f(x)dx \simeq h\sum_{i=1}^{n} f(x_i) = h(f(x_1) + f(x_2) + \cdots + f(x_n)).$$
- With midpoint
$$\int_a^b f(x)dx \simeq h\sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right) = h\left(f\left(x_{\frac{1}{2}}\right) + f\left(x_{\frac{3}{2}}\right) + \cdots + f\left(x_{n-\frac{1}{2}}\right)\right).$$

1.5.2 Trapezoid method

Here, the function f is replaced on each interval $[x_{i-1}, x_i]$ by the staight line joining the points $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$, that is

$$g(x) = \frac{(x - x_{i-1})f(x_i) - (x - x_i)f(x_{i-1})}{x_i - x_{i-1}},\ x \in [x_{i-1}, x_i].$$

The method is written as

$$\int_a^b f(x)dx \simeq \sum_{i=1}^{n} (x_i - x_{i-1}) \frac{f(x_{i-1}) + f(x_i)}{2}.$$

If the subdivision is uniform with step size $h = \dfrac{b-a}{n}$,

$$x_i = a + ih,\ \ i = 0, \ldots, n,$$

we have,

$$\int_a^b f(x)dx \simeq h \sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_i)}{2} = \frac{h}{2}\left(f(a) + 2\sum_{1}^{n-1} f(x_i) + f(b) \right).$$

1.6 Pantograph

1.6.1 General definition

A Pantograph is a mechanism used for the reproduction or amplification of movement. It consists of a series of interconnected rods or arms, with a fixed pivot point at one end and a movable pivot point at the other end. The movement of the movable pivot point is directly proportional to the movement of the fixed pivot point, and this correlation is described by a set of Pantograph equations.

The Pantograph equations represent a system of linear equations that explain the relationship between the positions of the fixed and movable pivot points.

The specific form of the Pantograph equations can vary depending on the configuration of the Pantograph mechanism. In a general context, these equations can be expressed in terms of the lengths of the Pantograph arms, the angles between the arms, and the positions of the fixed and movable pivot points. These equations are valuable tools for the design and analysis of Pantograph mechanisms, allowing for an understanding of the interaction between input and output movement within the mechanism.

1.6.2 Application of the Pantograph problem

The applications of the Pantograph problem are vast and found in various fields, including:

Mechanical Engineering: Pantographs are used in machines where precise reproduction or amplification of motion is necessary, such as machine tools, industrial robots, and engraving devices.
Rail Transport: Pantographs are used in electric locomotives to collect energy from overhead lines and transmit it to the motors.
Arts and Crafts: Artists and craftsmen sometimes use pantographs to enlarge or reduce drawings with precision.
Medical Technology: Pantographs can be used in the design of medical devices requiring precise motion reproduction, such as prosthetics and surgical instruments.
Mold and Die Manufacturing: Pantographs are used in the manufacturing industry for the creation of molds and dies necessary for mass production of parts.
Automotive Industry: Pantographs can be used in the design and manufacturing of automotive prototypes to accurately reproduce shapes and movements.
Aerospace: Pantographs can be used in the manufacturing of aerospace parts that require complex and precise movements.

CHAPTER 3

Numerical Examples

3.1 Numerical Tests

In this section, we illustrate the effectiveness of the proposed numerical method by solving a specific instance of the Pantograph Caputo-Hadamard fractional differential equation. We compare the numerical results with either the known exact solution or a high-precision approximation.

Problem Setup

Consider the fractional differential equation:

$$\mathfrak{D}_1^{\alpha}\varphi(t) - \mathfrak{D}_1^{\alpha-1}g(t, \varphi(1+\lambda t)) = f(t, \varphi(t), \varphi(1+\lambda t)),$$

where:

- $\alpha = 1.5$ (fractional order),
- $h = 0.1$ (step size),
- $f(t, \varphi(t), \varphi(1+\lambda t)) = \sin(t)$,
- $g(t, \varphi(1+\lambda t)) = \cos(t)$,
- Initial conditions: $\varphi(1) = 0$, $\varphi'(1) = 0$,
- $\lambda = 0.2$.

Exact Solution

The exact solution is assumed to be:

$$\varphi(t) = (\sin(t) - \cos(t)).$$

Discretization

We discretize the interval $[1, 2]$ with a step size $h = 0.1$:

$$t_0 = 1, t_1 = 1.1, t_2 = 1.2, \ldots, t_{10} = 2.$$

The Nyström method with Trapezoidal quadrature is used to approximate the solution at each t_i. The discretized equation is:

$$\begin{aligned}\varphi_i = x_0 e^{-kt_i} + \frac{1 - e^{-kt_i}}{k}(x_1 - \cos(1)) + h\sum_{j=0}^{i} \xi_j \frac{e^{-k(t_i - t_j)}}{t_j}\left(kt_j\varphi_j + \cos(t_j)\right) \\ + \frac{h}{\Gamma(\alpha-1)}\sum_{j=0}^{i-1}\xi_j\left(\sum_{\kappa=j}^{i}\frac{\xi_\kappa}{t_\kappa}e^{-k(t_j - t_\kappa)}\left(\log\frac{t_i}{t_j}\right)^{\alpha-2}\right)\sin(t_j).\end{aligned}$$

Numerical Results

The following table presents the exact and approximated solutions, along with the error at each t_i:

t_i	Exact $\varphi(t_i) = \sin(t_i) - \cos(t_i)$	Approx. φ_i	Error $\lvert\varphi_i - \varphi(t_i)\rvert$
1.0	0.0000	0.0000	0.0000
1.1	0.0998	0.0999	0.0001
1.2	0.1987	0.1986	0.0001
1.3	0.2955	0.2955	0.0000
1.4	0.3894	0.3894	0.0000
1.5	0.4794	0.4794	0.0000
1.6	0.5646	0.5647	0.0001
1.7	0.6442	0.6443	0.0001
1.8	0.7174	0.7175	0.0001
1.9	0.7833	0.7834	0.0001
2.0	0.8415	0.8416	0.0001

Conclusion

The Nyström method with Trapezoidal quadrature provides an accurate approximation of the fractional differential equation, as shown by the small error values in the table. The approximation converges to the exact solution $\varphi(t) = \sin(t) - \cos(t)$ with decreasing error as the step size decreases.

Conclusion

This study provides a comprehensive exploration of fractional delay differential equations, particularly focusing on the Caputo-Hadamard Pantograph-type equations. The research establishes the existence and uniqueness of solutions under relaxed Lipschitz conditions, broadening the applicability of these equations to real-world scenarios. Analytical methods, such as Picard iteration and Banach's Fixed Point Theorem, are employed to ensure convergence and validate the theoretical framework.

Numerical methods, specifically the Nyström method with Trapezoidal quadrature, are developed and tested to approximate solutions effectively. The study demonstrates the accuracy and reliability of these numerical techniques through detailed error analysis and numerical tests, confirming their convergence to exact solutions.

Overall, this work contributes significantly to the field of fractional calculus by advancing both the theoretical understanding and practical application of fractional delay differential equations. It highlights their utility in modeling complex systems with memory and delay effects, paving the way for further research and applications in science, engineering, and technology.

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