

People's Democratic Republic of Algeria
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Faculty of Mathematics, Informatics, and Sciences of Matter
Department of Mathematics
Domiciliation Laboratory of Applied Mathematics and Modelling (LMAM)

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Presented by:
Roumaissa KHALFALLAOUI
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Before the jury composed of:

Full name	Rank	University	
Mr. Fateh ELLAGGOUNE	Prof	Univ. of Guelma	President
Mr. Abderrezak CHAOUI	Prof	Univ. of Guelma	Supervisor
Mr. Ali BOUSSAYOUD	Prof	Univ. of Jijel	Examiner
Mr. Rabah DEBBAR	Prof	Univ. of Guelma	Examiner
Mr. Rabah KHEMIS	Prof	Univ. of Skikda	Examiner

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Roumaissa KHALFALLAOUI

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To my parents



Remerciements

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ملخص

نهدف من خلال هذه الأطروحة إلى إجراء دراسات تحليلية وأخرى تقريبية (عددية) على معادلات تفاضلية جزئية ذوات درجات عليا.

في العمل الأول، نقوم بدراسة فئة من المعادلات التكاملية-التفاضلية غير الخطية شبه المكافئة ذات التدفق غير المعروف على جزء من حافة ديريشليه، ونعالجها تحليليًا وعدديًا. لتحقيق ذلك، نستخدم طريقة روث.

أما العمل الثاني، فهو نظري بالدرجة الأولى، حيث ندرس معادلة $p(\cdot)$ -ثنائية التوافقية الزائدية مع شرط عدم وجود تدفق على الحدود. نثبت وجود الحل الضعيف وسلوكه الانفجاري باستخدام طريقة غالركين.

أخيرًا، في العمل الثالث، نقدم دراسات تقريبية لمسألة التطور p -ثنائية التوافقية باستخدام طريقة العناصر المنتهية المختلطة مقرونة مع طريقة روث.

الكلمات المفتاحية: تقدير الخطأ المسبق، الحل الضعيف، المسألة المتقطعة بالكامل، طريقة العناصر المنتهية المختلطة، طريقة روث، طريقة غالركين، انفجار الحل، الوجود الشامل، معادلة قطع مكافئ، معادلة قطع زائد، معادلة p -لابلاس، معادلة $p(\cdot)$ -ثنائية التوافقية، معادلة تكاملية تفاضلية، شرط ديريشليه غير معلوم، طاقة ابتدائية سالبة.

Abstract

In this thesis, we aim to conduct analytical and approximate (numerical) studies on higher-order partial differential equations.

In the first work, we investigate a class of nonlinear parabolic integro-differential equations with an unknown flux on a part of the Dirichlet boundary, treating it both analytically and numerically. To this end, we employ the Rothe method.

The second work is primarily theoretical, we study hyperbolic $p(\cdot)$ -biharmonic equation with no flux boundary condition. We prove the existence and blow-up behavior of the weak solution using the Galerkin method.

Finally, in the third work, we show the approximation studies to evolution p -biharmonic problem employing the mixed finite element method combined with the Rothe method.

Key-words : A priori error estimation, weak solution, fully discretized problem, mixed finite element method, Rothe method, Galerkin method, blow-up of the solution, global existence, parabolic equation, hyperbolic equation, p -Laplace equation, $p(\cdot)$ -biharmonic equation, integro-differential equation, unknown Dirichlet condition, negative initial energy.

Résumé

Dans cette thèse, nous visons à mener des études analytiques et approximatives (numériques) sur les équations aux dérivées partielles d'ordre élevé.

Dans le premier travail, nous étudions une classe d'équations intégró-différentielles paraboliques non linéaires avec un flux inconnu sur une partie de la frontière de Dirichlet, en la traitant à la fois analytiquement et numériquement. À cette fin, nous utilisons la méthode de Rothe.

Le deuxième travail est principalement théorique, nous étudions l'équation hyperbolique $p(\cdot)$ -biharmonique avec une condition de non-flux au bord. Nous prouvons l'existence et le comportement d'explosion de la solution faible en utilisant la méthode de Galerkin.

Enfin, dans le troisième travail, nous montrons les études d'approximation du problème d'évolution p -biharmonique en utilisant la méthode des éléments finis mixtes combinée à la méthode de Rothe .

Mots-Clés: Estimation a priori de l'erreur, solution faible, problème complètement discrétisé, méthode d'élément fini mixte, méthode de Rothe, méthode de Galerkin, explosion de la solution, existence globale, équation parabolique, équation du p -Laplacien, équation $p(\cdot)$ -biharmonique, équation intégró-différentielle, énergie initiale négative.

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Introduction

Partial differential equations (PDEs) are fundamental in engineering and the physical sciences, as they model complex systems and phenomena, particularly those that evolve continuously over time. High-order PDEs arise in various applications, such as elastic beam deformations [30, 35], thin film theory [41, 14], non-Newtonian fluid motions [43], and image inpainting and restoration [44]. Among these, parabolic p -Laplace equations have received significant attention due to their relevance in modeling non-linear diffusion processes.

For $1 < p < 2$, the parabolic p -Laplace equation appears in the study of non-Newtonian fluid theory [3]. Furthermore, when $\beta(u) = u$, the problem under investigation (2.1)

$$\left\{ \begin{array}{ll} \frac{\partial \beta(u)}{\partial t} - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = f(t, x) + \int_0^t M(s, u(s, x)) ds & \text{in } I \times \Omega, \\ -\frac{\partial u}{\partial \eta} = \alpha(t, x) & \text{on } I \times \Gamma_N, \\ u = \rho(t) & \text{on } I \times \Gamma_D, \\ u(0, x) = u_0 & \text{on } \Omega. \end{array} \right. \quad (2.1)$$

can describe heat propagation, where the heat flux depends on the process history [5]. In the special case of $p = 2$, problem (2.1) models reaction contaminant transport in saturated zones, where $u(t)$ represents the contaminant concentration, $\rho = u|_{\Gamma_D}$ describes contact with a winer medium through fast diffusion, and $\sigma(t)$ denotes the total contaminant mass in Ω .

There are many methods for solving partial differential equations, both theoretically and numerically. In this thesis, we discuss some of these methods, such as the Rothe method, the Galerkin method, the finite element method and the mixed finite element method.

Rothe method is a numerical technique for solving partial differential equations (PDEs). It is also known as Rothe's time-discretization method. In this approach, the time domain is divided into discrete intervals, transforming the original PDE into a sequence of elliptic problems. The discretization process proceeds as follows:

Time Discretization:

The time interval $[0, T]$ is divided into n subintervals (t_{i-1}, t_i) , where $i = 1, \dots, n$.

The discrete time steps are defined as $t_i = i\tau$, where $\tau = \frac{T}{n}$ represents the time step size.

Approximation of the Time Derivative :

The time derivative $\frac{\partial w}{\partial t}$ is approximated by

$$\frac{w_i - w_{i-1}}{\tau}$$

where $w_i = w(t_i, x)$ for all $i = 1, \dots, n$.

Transformation into a Discrete System:

This discretization leads to a system of n equations in x , where $w_i(x)$ represents the unknown at each time step.

At each time t_i , the continuous problem is approximated by a discrete problem, leading to the Rothe function w_n , which approximates the solution w using a linear polynomial on each subinterval (t_{i-1}, t_i) .

The Approximate Solution:

The approximate solution is given by:

$$w_n = w_{i-1} + (t_i - t_{i-1}) \frac{w_i - w_{i-1}}{\tau}, \quad t \in [t_{i-1}, t_i], \quad i = 1, \dots, n$$

Step Function:

The discrete function $w_n(t)$ can be expressed using step functions:

$$\bar{w}_n = \begin{cases} w_i & t \in [t_{i-1}, t_i], \\ w_0 & t \in [-\tau, 0], \end{cases}$$

The Galerkin Method is a powerful numerical technique used to solve differential equations, especially partial differential equations (PDEs) which is based on approximating the solution within a finite-dimensional subspace by projecting the problem onto a set of basis functions. This approach transforms the original partial differential equation into a system of algebraic equations, facilitating its numerical and theoretical analysis.

In this context, evolution p -biharmonic equations have attracted considerable research interest. For instance, in [12, 15], the authors established the existence and

uniqueness of weak solutions and provided numerical results for parabolic and degenerate parabolic p –biharmonic equations with a constant exponent using the mixed finite element method combined with the Backward-Euler scheme. Cömert et al. [21] analyzed the global existence and exponential decay of solutions for a higher-order parabolic equation with logarithmic nonlinearity. More recently, authors in [39] investigated the blow-up behavior of a $p(x)$ –biharmonic heat equation with a variable exponent. Additional relevant results can be found in [6, 23, 33].

The thesis consists of an introduction and four chapters.

Chapter 1 contains fundamental notions of functional analysis, along with key definitions, lemmas, and fundamental theorems that will be used in subsequent chapters. Additionally, it provides an overview of mixed finite element methods.

In chapter 2, We study a nonlinear degenerate parabolic p –Laplace equation with an unknown Dirichlet boundary condition and a memory term. An additional integral measurement allows us to eliminate the unknown Dirichlet condition ρ , which leads to the weak formulation of the problem. We apply Rothe’s method for time discretization and use the monotone operator theorem to solve the semi-discretized problem at each time step. Furthermore, we show some a priori estimates and establish the existence and uniqueness of the weak solution. Finally, we conduct computational experiments to support the theoretical results.

Chapter 3 is dedicated to studying the hyperbolic $p(x)$ –biharmonic problem under homogeneous Neumann boundary conditions using the Galerkin method. We establish the global existence of a weak solution and derive some key a priori estimates. Finally, we investigate conditions leading to finite-time blow-up in the case of negative

initial functional energy.

In Chapter 4, we study the initial boundary value problem for a high-order parabolic p -biharmonic equation with a memory term. We propose a numerical scheme based on the Rothe method and the mixed finite element method to approximate the solution. Furthermore, we establish the existence and uniqueness of weak solutions under suitable conditions .

Finally, we end with a conclusion that encompasses the main results of this work and some future perspectives.

1

Preliminaries

1.1 Introduction

The aim of this chapter is to recall fundamental concepts, basic notations, definitions, theorems, lemmas, and key properties that will be used in the subsequent chapters.

1.2 Functional Spaces

Let Ω be a bounded open domain of $\mathbb{R}^n$ with $n \geq 1$ and $T > 0$.

1.2.1 Lebesgue Spaces

1.2.1.1 Lebesgue Spaces with Constant Exponent $L^p(\Omega)$

Definition 1.2.1. [13] For a constant exponent p such that $1 \leq p \leq \infty$, the Lebesgue space $L^p(\Omega)$ is defined as:

$$L^p(\Omega) = \{\psi : \Omega \longrightarrow \mathbb{R} \text{ measurable, such that } \int_{\Omega} |\psi(x)|^p dx < \infty\},$$

with the norm

$$\|\psi\|_{L^p(\Omega)} = \left(\int_{\Omega} |\psi(x)|^p dx \right)^{\frac{1}{p}}.$$

For $p = \infty$, the norm is defined as the essential supremum:

$$\|\psi\|_{\infty} = \text{ess sup}_{x \in \Omega} |\psi(x)| = \inf \left\{ C \geq 0 : |\psi(x)| \leq C \text{ pp over } \Omega \right\}.$$

The space $L^p(\Omega)$ is a Banach space for all $1 \leq p \leq \infty$, and when $p = 2$, it becomes a Hilbert space with the inner product:

$$\langle \psi, \phi \rangle = \int_{\Omega} \psi(x) \phi(x) dx,$$

Definition 1.2.2. A function $\psi \in L^2(\Omega)$ is said to be weakly differentiable in $L^2(\Omega)$ if there exist functions $\omega_i \in L^2(\Omega)$ for each $i = 1, \dots, n$, such that for all test functions $\phi \in C_c^\infty(\Omega)$, the following integral identity holds:

$$\int_{\Omega} \psi \frac{\partial \phi}{\partial x_i} = - \int_{\Omega} \omega_i \phi, \quad \forall \phi \in C_c^{\infty}(\Omega),$$

In this case, the functions ω_i are called the weak derivatives of ψ , denoted as $\frac{\partial \psi}{\partial x_i} = \omega_i$. The space $C_c^{\infty}(\Omega)$ consists of infinitely continuously differentiable functions with compact support in Ω .

1.2.1.2 Lebesgue Spaces with Variable Exponent $L^{p(\cdot)}(\Omega)$

Let $p \in C(\overline{\Omega}, (1, \infty))$ satisfies the growth condition

$$|p(\eta) - p(\xi)| \leq \frac{c}{|\ln|\eta - \xi||} \quad \forall |\eta - \xi| < \frac{1}{2},$$

for some $c > 0$.

Definition 1.2.3. [22] The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined as follows

$$L^{p(\cdot)}(\Omega) = \left\{ \psi : \Omega \rightarrow \mathbb{R}, \psi \text{ is measurable} : \int_{\Omega} |\psi(x)|^{p(x)} dx < \infty \right\},$$

equipped with the following Luxemburg norm

$$\|\psi\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{\psi(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

Note that $(L^{p(x)}(\Omega), \|\cdot\|_{L^{p(x)}(\Omega)})$ is a separable Banach space.

1.2.2 Sobolev Spaces

Let $\beta = (\beta_1, \dots, \beta_n)$ be a multi-index, meaning a vector with n components, where each β_i is a non-negative integer ($\beta_i \geq 0$).

We define the length of β as:

$$|\beta| = \sum_{i=1}^n \beta_i.$$

For a function f , we denote its partial derivative with respect to β as:

$$\frac{\partial^{|\beta|} f}{\partial^{\beta_1} x_1 \dots \partial^{\beta_n} x_n}$$

1.2.2.1 Sobolev Spaces with Constant Exponent $W^{k,p}(\Omega)$

Definition 1.2.4. [4, 15] For an integer $k \in \mathbb{N}$, and $1 \leq p < \infty$ the Sobolev spaces with constant exponent $W^{k,p}(\Omega)$ is defined as follows

$$W^{k,p}(\Omega) = \left\{ \psi \in L^p(\Omega) \text{ such that } D^\beta \psi \in L^p(\Omega), \forall \beta \in \mathbb{N}^n \text{ and } |\beta| \leq k \right\},$$

where $D^\beta \psi$ denotes the weak derivative of ψ . Equipped with the norm

$$\|\psi\|_{W^{k,p}(\Omega)} = \left[\sum_{|\alpha| \leq k} \|D^\alpha \psi\|_{L^p(\Omega)}^p \right]^{\frac{1}{p}}.$$

Definition 1.2.5. For $k=1$, We define the Sobolev space $W^{1,p}(\Omega)$ as follows

$$W^{1,p}(\Omega) = \left\{ \psi \in L^p(\Omega) \text{ such that } \frac{\partial \psi}{\partial x_i} \in L^p(\Omega), i = 1, \dots, n \right\},$$

equipped with the following norm

$$\|\psi\|_{W^{1,p}(\Omega)}^p = \|\psi\|_{L^p(\Omega)}^p + \|\nabla \psi\|_{L^p(\Omega)}^p.$$

The Sobolev space with zero boundary values is defined as:

$$W_0^{1,p}(\Omega) = \overline{C_c^\infty(\Omega)} \text{ in } W^{1,p}(\Omega),$$

and we write

$$W_0^{1,p}(\Omega) = \{\psi \in W^{1,p}(\Omega) \text{ such that } \psi|_{\Gamma} = 0\},$$

where Γ denotes the boundary of the open set $\Omega \subset \mathbb{R}^n$.

Remark 1.2.6. The norm $\|\cdot\|_{W^{1,p}(\Omega)}$ and the semi norm $\|\nabla(\cdot)\|_{L^p(\Omega)}$ are equivalent over the space $W_0^{1,p}(\Omega)$.

Definition 1.2.7. [4] For $p = 2$, the Sobolev space $W^{k,2}(\Omega)$ is denoted as $H^k(\Omega)$. This space consists of functions whose weak derivatives up to order k are square-integrable over Ω , making it a Hilbert space with inner product

$$(\psi, \phi)_{H^k(\Omega)} = \sum_{|\beta| \leq k} (D^\beta \psi, D^\beta \phi),$$

equipped with the norm

$$\|\psi\|_{H^k(\Omega)} = \left(\sum_{|\beta| \leq k} \|D^\beta \psi\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.$$

1.2.2.2 Sobolev Spaces with Variable Exponent $W^{k,p(\cdot)}(\Omega)$

Definition 1.2.8. For some $p \in C(\overline{\Omega}, (1, \infty))$ and $k \in \mathbb{N}^*$, we define the generalized variable exponent sobolev space $W^{k,p(\cdot)}(\Omega)$ as follows

$$W^{k,p(\cdot)}(\Omega) = \left\{ \psi \in L^{p(\cdot)}(\Omega); D^\beta \psi \in L^{p(\cdot)}(\Omega), \forall \beta \in \mathbb{N}^n \text{ such that } |\beta| \leq k \right\},$$

endowed with the norm

$$\|\psi\|_{W^{k,p(\cdot)}(\Omega)} = \sum_{|\beta| \leq k} \|D^\beta \psi\|_{L^{p(\cdot)}(\Omega)}^p.$$

Definition 1.2.9. We introduce the space $W_0^{2,p(\cdot)}(\Omega)$ as follows

$$W_0^{2,p(\cdot)}(\Omega) = \left\{ \psi \in W^{2,p(\cdot)}(\Omega); \psi|_{\partial\Omega} = 0 \text{ and } \nabla\psi|_{\partial\Omega} = 0 \right\},$$

Over the space $W_0^{2,p(\cdot)}(\Omega)$, the semi norm $\|\Delta(\cdot)\|_{L^{p(\cdot)}(\Omega)}$ and the norm $\|\cdot\|_{W^{2,p(\cdot)}(\Omega)}$ are equivalent.

Remark 1.2.10. Let $p, q \in C(\overline{\Omega}, (1, \infty))$, such that $q(\cdot)$ the conjugate of $p(\cdot)$ i.e.,

$$\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1$$

1) If $p(x) \leq q(x)$ in $\overline{\Omega}$, then

$$L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega). \quad (1.1)$$

2) For all $\psi \in L^{p(x)}(\Omega)$, we have

$$\min\{\|\psi\|_{L^{p(x)}(\Omega)}^{p^-}, \|\psi\|_{L^{p(x)}(\Omega)}^{p^+}\} \leq \int_{\Omega} |\psi|^{p(x)} dx \leq \max\{\|\psi\|_{L^{p(x)}(\Omega)}^{p^-}, \|\psi\|_{L^{p(x)}(\Omega)}^{p^+}\}. \quad (1.2)$$

3) Let $m(x) : \overline{\Omega} \rightarrow \mathbb{R}$ be a measurable function and $p(x) : \overline{\Omega} \rightarrow \mathbb{R}$ be a Lipschitz continuous function such that $1 < p^- \leq p^+ < \frac{n}{k}$. Then, the embedding $W^{k,p(x)}(\Omega) \hookrightarrow L^{m(x)}(\Omega)$ is continuous if

$$p(x) \leq m(x) \leq \frac{np(x)}{n - kp(x)} \text{ a.e.} \quad \forall x \in \overline{\Omega}$$

1.2.3 Bochner Spaces

Let Ω be a subset of $\mathbb{R}^n$ and $[0, T]$ is a real interval. We define the following spaces :

- $C((0, T), L^p(\Omega)) = \{\psi : (0, T) \longrightarrow L^p(\Omega) \text{ is continuous} \}$ equipped with the norm

$$\|\psi\|_{C((0,T),L^p(\Omega))} = \max_{t \in [0,T]} \|\psi(t)\|_{L^p(\Omega)}.$$

- $L^2((0, T), L^2(\Omega)) = \{\psi : (0, T) \longrightarrow L^2(\Omega) \text{ square-integrable} \}$ equipped with the norm

$$\|\psi\|_{L^2((0,T),L^2(\Omega))}^2 = \int_0^T \|\psi(t)\|_{L^2(\Omega)}^2 dt.$$

- $L^2((0, T), H_0^1(\Omega)) = \{\psi : (0, T) \longrightarrow H_0^1(\Omega) \text{ square-integrable} \}$ equipped with the norm

$$\|\psi\|_{L^2((0,T),H_0^1(\Omega))}^2 = \int_0^T \|\psi(t)\|_{H_0^1(\Omega)}^2 dt.$$

- $L^p((0, T), L^p(\Omega)) = \{\psi : (0, T) \longrightarrow L^p(\Omega) \text{ } p\text{-integrable} \}$ equipped with the norm

$$\|\psi\|_{L^p((0,T),L^p(\Omega))}^p = \int_0^T \|\psi(t)\|_{L^p(\Omega)}^p dt.$$

- $L^p((0, T), W_0^{2,p}(\Omega)) = \{\psi : (0, T) \longrightarrow W_0^{2,p}(\Omega) \text{ } p\text{-integrable} \}$ equipped with the norm

$$\|\psi\|_{L^p((0,T),W_0^{2,p}(\Omega))}^p = \int_0^T \|\psi(t)\|_{W_0^{2,p}(\Omega)}^p dt.$$

- $L^\infty((0, T), L^2(\Omega)) = \{\psi : (0, T) \longrightarrow L^2(\Omega) \text{ essentially bounded} \}$ equipped with the norm

$$\|\psi\|_{L^\infty((0,T),L^2(\Omega))} = \text{ess sup}_{t \in [0,T]} \|\psi(t)\|_{L^2(\Omega)}.$$

- $L^\infty((0, T), W^{1,p}(\Omega)) = \{\psi : (0, T) \longrightarrow W^{1,p}(\Omega) \text{ essentially bounded} \}$ equipped with the norm

$$\|\psi\|_{L^\infty((0,T),W^{1,p}(\Omega))} = \text{ess sup}_{t \in [0,T]} \|\psi(t)\|_{W^{1,p}(\Omega)}.$$

- $L^\infty((0, T), W_0^{2,p(x)}(\Omega)) = \{\psi : (0, T) \longrightarrow W_0^{2,p(x)}(\Omega) \text{ essentially bounded} \}$ equipped with the norm

$$\|\psi\|_{L^\infty((0, T), W_0^{2,p(x)}(\Omega))} = \text{ess sup}_{t \in [0, T]} \|\psi(t)\|_{W_0^{2,p(x)}(\Omega)}.$$

Note that $W^{-2,q}(\Omega)$ is the dual space of $W_0^{2,p}(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

- $C((0, T), W^{-2,q}(\Omega)) = \{\psi : (0, T) \longrightarrow W^{-2,q}(\Omega) \text{ is continuous} \}$ equipped with the norm

$$\|\psi\|_{C((0, T), W^{-2,q}(\Omega))} = \max_{t \in [0, T]} \sup_{\substack{\varphi \in W_0^{2,p}(\Omega) \\ \|\varphi\|_{W^{2,p}(\Omega)} \leq 1}} |\langle \psi(t), \varphi \rangle|.$$

- $L^q((0, T), W^{-2,q}(\Omega)) = \{\psi : (0, T) \longrightarrow W^{-2,q}(\Omega) \text{ } q\text{-integrable} \}$ equipped with the norm

$$\|\psi\|_{L^q((0, T), W^{-2,q}(\Omega))}^q = \int_0^T \|\psi(t)\|_{W^{-2,q}(\Omega)}^q dt,$$

where

$$\|\psi(t)\|_{W^{-2,q}(\Omega)} = \sup_{\substack{\varphi \in W_0^{2,p}(\Omega) \\ \|\varphi\|_{W^{2,p}(\Omega)} \leq 1}} |\langle \psi(t), \varphi \rangle|.$$

- $L^2((0, T), (W_0^{2,p(x)}(\Omega))') = \{\psi : (0, T) \longrightarrow (W_0^{2,p(x)}(\Omega))' \text{ square-integrable} \}$ equipped with the norm

$$\|\psi\|_{L^2((0, T), (W_0^{2,p(x)}(\Omega))')}^2 = \int_0^T \|\psi(t)\|_{(W_0^{2,p(x)}(\Omega))'}^2 dt.$$

- $L^\infty((0, T), (W_0^{2,p(x)}(\Omega))') = \{\psi : (0, T) \longrightarrow (W_0^{2,p(x)}(\Omega))' \text{ essentially bounded} \}$ equipped with the norm

$$\|\psi\|_{L^\infty((0, T), (W_0^{2,p(x)}(\Omega))')} = \text{ess sup}_{t \in [0, T]} \|\psi(t)\|_{(W_0^{2,p(x)}(\Omega))'},$$

with

$$\|\psi(t)\|_{(W_0^{2,p(x)}(\Omega))'} = \sup_{\substack{\varphi \in W_0^{2,p(x)}(\Omega) \\ \|\varphi\|_{W^{2,p(x)}(\Omega)} \leq 1}} |\langle \psi(t), \varphi \rangle|.$$

1.3 Some Properties and Fundamental Theorems

Theorem 1.3.1. [13](Cauchy-Schwarz Inequality)

Let Ω be a subset of $\mathbb{R}^n$, for all $\psi, \phi \in L^2(\Omega)$, we define the continuous form and the discrete form respectively

$$\left| \int_{\Omega} \psi(x)\phi(x)dx \right| \leq \left(\int_{\Omega} |\psi(x)|^2 \right)^{\frac{1}{2}} \left(\int_{\Omega} |\phi(x)|^2 \right)^{\frac{1}{2}},$$

and

$$\left| \sum_{i=1}^n \psi_i \phi_i dx \right| \leq \left(\sum_{i=1}^n |\psi_i|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^n |\phi_i|^2 \right)^{\frac{1}{2}}.$$

Theorem 1.3.2. [13](Hölder Inequality)(continuous form)

Let $\psi \in L^p(\Omega)$ and $\phi \in L^q(\Omega)$, for $1 \leq p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\int_{\Omega} |\psi(x)\phi(x)|dx \leq \left(\int_{\Omega} |\psi(x)|^p \right)^{\frac{1}{p}} \left(\int_{\Omega} |\phi(x)|^q \right)^{\frac{1}{q}}.$$

Theorem 1.3.3. [13](Hölder Inequality)(discrete form)

Let $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$ be two sequences in $\mathbb{R}$, for $1 \leq p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\sum_{i=1}^n |a_i b_i| \leq \left(\sum_{i=1}^n |a_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |b_i|^q \right)^{\frac{1}{q}}.$$

Lemma 1.3.4. (Young's Inequality)

For $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ and for all $\alpha, \beta \geq 0$, we have

$$\alpha\beta \leq \frac{1}{p}\alpha^p + \frac{1}{q}\beta^q.$$

Lemma 1.3.5. (The ϵ -Young's Inequality)

Let $\alpha, \beta \geq 0$ and $\forall \epsilon \geq 0$, we have

$$2\alpha\beta \leq \epsilon\alpha^2 + C_\epsilon\beta^2,$$

where $C_\epsilon = \frac{C}{\epsilon}$ with ϵ is small.

Lemma 1.3.6. (Poincaré Inequality)

There exists a positive constant $C(\Omega)$, such that

$$\|\psi\|_{L^2(\Omega)} \leq C(\Omega) \left(\sum_{i=1}^n \left\| \frac{\partial \psi}{\partial x_i} \right\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \quad \forall \psi \in H_0^1(\Omega).$$

Lemma 1.3.7. [36] Let $x, y \in \mathbb{R}^n$, with $x \neq y$

1. For $p \geq 2$ there exists $C_1(p)$ such that

$$\left(|x|^{p-2}x - |y|^{p-2}y, x - y \right)_{\mathbb{R}^n} \geq C_1(p) |x - y|^p.$$

2. For $1 < p \leq 2$ there exists $C_2(p)$ such that

$$\left| |x|^{p-2}x - |y|^{p-2}y \right| \leq C_2(p) |x - y|^{p-1}.$$

Lemma 1.3.8. (Abel's Summation Formula)

For all $\alpha_i, \beta_i \in \mathbb{R}$, we have

$$\sum_{i=1}^j \alpha_i (\beta_i - \beta_{i-1}) = \alpha_j \beta_j - \alpha_0 \beta_0 - \sum_{i=1}^j (\alpha_i - \alpha_{i-1}) \beta_{i-1}.$$

Theorem 1.3.9. (Trace Inequality)

Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with Lipschitz boundary Γ , for all $\psi \in H^1(\Omega)$, and $0 \leq \epsilon < \epsilon_0$, we have

$$\|\psi\|_{L^2(\Gamma)}^2 \leq \epsilon \|\nabla \psi\|_{L^2(\Omega)}^2 + C_\epsilon \|\psi\|_{L^2(\Omega)}^2$$

Theorem 1.3.10. [4](Green's Formula)

Let Ω be a regular bounded open set with C^2 -smooth boundary Γ . Denote by $\eta(x)$ the outward unit normal vector to Γ . For $\psi, \phi \in H^2(\Omega)$, we have

$$\int_{\Omega} \Delta \psi \phi dx = - \int_{\Omega} \nabla \psi \nabla \phi dx + \int_{\Gamma} \frac{\partial \psi}{\partial \eta} \phi d\sigma.$$

where $\frac{\partial \psi}{\partial \eta} = \nabla \psi \cdot \eta$ is the normal derivative of ψ on Γ and $d\sigma$ represents the surface measure on Γ .

Lemma 1.3.11. [19](Gronwall Inequality)(continuous form)

Let ψ , ϕ , and ζ be real valued functions defined on an interval $[a, b]$, where

- $\psi(t)$ and $\zeta(t)$ are continuous functions on $[a, b]$
- $\psi(t) \geq 0$ for all $t \in [a, b]$
- ϕ is nondecreasing on $[a, b]$
- The function $\zeta(t)$ satisfies the integral inequality:

$$\zeta(t) \leq \phi(t) + \int_a^t \psi(s) \zeta(s) ds, \quad \forall t \in [a, b],$$

then

$$\zeta(t) \leq \phi(t) \exp \left(\int_a^t \psi(s) ds \right), \quad \forall t \in [a, b].$$

Lemma 1.3.12. [19](Gronwall Inequality)(discrete form)

Let $\{\psi_n\}$, $\{\phi_n\}$ and $\{\zeta_n\}$ be sequences of real numbers satisfying:

- $\psi_n \geq 0$ and $\zeta_n \geq 0$ for all n
- ϕ_n is nondecreasing
- The sequence ζ_n satisfies the recurrence inequality:

$$\zeta_n \leq \phi_n + \sum_{i=0}^{n-1} \psi_i \zeta_i, \quad \forall n \geq 0,$$

then

$$\zeta_n \leq \phi_n \exp \left(\sum_{i=0}^{n-1} \psi_i \right), \quad \forall n \geq 0.$$

Definition 1.3.13. (Hemicontinuous Operator)

We say that the operator $A : X \rightarrow X'$ is hemicontinuous in X if for $v \in X$, $u + t_n v \in X$ where t_n is a sequence of positive numbers such that $t_n \rightarrow 0$, imply

$$A(u + t_n v) \rightharpoonup A(u)$$

Definition 1.3.14. (Continuity of a bilinear form $\alpha(.,.)$)

A function $\alpha(.,.)$ is called a continuous bilinear form if there exists a constant $C > 0$, such that for all $\psi, \phi \in X$, the following inequality holds:

$$|\alpha(\psi, \phi)| \leq C \|\psi\| \|\phi\|.$$

Definition 1.3.15. (Coercivity of a bilinear form $\alpha(.,.)$)

The bilinear form $\alpha(.,.)$ is coercive if

$$\exists M > 0 \quad \forall \phi \in X : |\alpha(\phi, \phi)| \geq M \|\phi\|^2.$$

Theorem 1.3.16. (Lax-Milgram)

Let X be a Hilbert space, and let $\alpha(.,.) : X \times X \rightarrow \mathbb{R}$ be a continuous and coercive bilinear form. Let $L(.) : X \rightarrow \mathbb{R}$ be a continuous linear form. Then, there exists a unique $\psi \in X$ such that

$$\forall \phi \in X, \quad a(\psi, \phi) = L(\phi).$$

1.4 The Mixed Finite Element Method

Consider two Hilbert spaces X and Y . The goal is to find a solution pair $(\psi, \varphi) \in X \times Y$ that satisfies:

$$\begin{cases} \alpha(\psi, \phi) + \beta(\varphi, \phi) = L_X(\phi) & \forall \phi \in X \\ \beta(\psi, \zeta) = 0 & \forall \zeta \in Y. \end{cases}$$

where $\alpha(.,.)$ and $\beta(.,.)$ are bilinear forms on $X \times X$, $X \times Y$ respectively, and $L_X(.)$ is a linear functional on X .

Theorem 1.4.1. (Inf-sup Stability) There exist $\kappa, C \geq 0$ such that for $\psi \in X$, we have

$$\kappa \leq C \inf_{0 \neq \zeta \in Y} \sup_{0 \neq \psi \in X} \frac{\beta(\psi, \zeta)}{\|\psi\|_X \|\zeta\|_Y}.$$

This theorem ensures that the mixed problem above is well posed. In other words,

there exists a unique solution $(\psi, \varphi) \in X \times Y$ that satisfies the stability estimate

$$\|\psi\|_X^2 + \|\varphi\|_Y^2 \leq C_1 \|f\|_X^2$$

where C_1 is a positive constant.

Approximate Solution Using Finite-Dimensional Subspaces: Since the function spaces X and Y are often infinite dimensional, finding an exact solution may not be feasible. To obtain an approximate solution, we replace these spaces with finite dimensional subspaces $X_h \subset X$ and $Y_h \subset Y$. Thus, instead of solving the original problem, we seek an approximate solution $(\psi_h, \varphi_h) \in X_h \times Y_h$ that satisfies a corresponding mixed problem:

$$\begin{cases} \alpha(\psi_h, \phi_h) + \beta(\varphi_h, \phi_h) = L_X(\phi_h) & \forall \phi_h \in X_h \\ \beta(\psi_h, \zeta_h) = 0 & \forall \zeta_h \in Y_h. \end{cases}$$

Thus, we can define the approximate mixed problem in a matrix representation

$$\begin{pmatrix} A_h & B_h^T \\ B_h & 0 \end{pmatrix} \begin{pmatrix} \psi_h \\ \varphi_h \end{pmatrix} = \begin{pmatrix} f_h \\ 0 \end{pmatrix}$$

where the operators $A_h : X_h \rightarrow X_h$ and $B_h : X_h \rightarrow Y_h$ are given by

$$(A_h \psi_h, \phi_h)_X = \alpha(\psi_h, \phi_h) \quad \forall \psi_h, \phi_h \in X_h$$

$$(B_h \zeta_h, \phi_h)_Y = b(\zeta_h, \phi_h) \quad \forall \phi_h \in X_h, \forall \zeta_h \in Y_h$$

$$(f_h, \phi_h)_X = L(\phi_h) \quad \forall \phi_h \in X_h.$$

Convergence: The bilinear form $\beta(.,.)$ satisfies the Inf-Sup condition on the finite-dimensional spaces $X_h \times Y_h$, ensuring that the approximate mixed problem remains well-posed. This guarantees both the existence and uniqueness of the solution (ψ_h, φ_h) while ensuring its convergence to the exact solution (ψ, φ) .

The following theorem establishes the rate of convergence:

Theorem 1.4.2. *There exists a constant $C > 0$ independent of h such that*

$$\|\psi - \psi_h\|_X + \|\varphi - \varphi_h\|_Y \leq \left\{ \inf_{\phi_h \in X_h} \|\psi - \phi_h\|_X + \inf_{\zeta_h \in Y_h} \|\varphi - \zeta_h\|_Y \right\}.$$

1.5 Conclusion

In this chapter, we reviewed the fundamental concepts and mathematical tools needed for our study. We introduced the main functional spaces, discussed key theoretical results, and presented the mixed finite element method that will be used in the next chapters.

2

On the Solution of Evolution p -Laplace Equation with a Memory Term and Unknown Boundary Dirichlet Condition

2.1 Introduction

In this chapter, we investigate a class of nonlinear parabolic integro-differential equations with an unknown flux on a part of the Dirichlet boundary. To address this

inverse problem, we employ a non-standard technique that transforms it into a direct problem by eliminating the unknown Dirichlet condition $\rho(t)$ using prescribed measurement. We introduce a discrete-time approximation scheme based on Rothe's method and establish the existence and uniqueness of a weak solution at each time step. Furthermore, we use a priori estimates to demonstrate the convergence of the approximate solution to the weak solution. Finally, we present computational experiments to support and validate the theoretical findings.

2.2 Position of the Problem

We consider a connected bounded open domain Ω of $\mathbb{R}^n$, with a Lipschitz-continuous boundary $\Gamma = \overline{\Gamma_N} \cup \overline{\Gamma_D}$ and $I = [0, T]$, $T > 0$.

Here $\Gamma_N \cap \Gamma_D = \emptyset$, $|\Gamma_D| > 0$, where Γ_N and Γ_D are two open subsets of Γ . We denote η the outward unit normal vector on Γ . We address the following problem:

$$\left\{ \begin{array}{ll} \frac{\partial \beta(u)}{\partial t} - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = f(t, x) + \int_0^t M(s, u(s, x)) ds & \text{in } I \times \Omega, \\ -\frac{\partial u}{\partial \eta} = \alpha(t, x) & \text{on } I \times \Gamma_N, \\ u = \rho(t) & \text{on } I \times \Gamma_D, \\ u(0, x) = u_0 & \text{on } \Omega. \end{array} \right. \quad (2.1)$$

Where $2 < p < \infty$, M is globally Lipschitz continuous function satisfying the growth condition

$$|M(t, s)| \leq C(1 + |s|^{p-1}), \quad \forall s \in \mathbb{R}$$

β is a monotonically increasing continuous function verifying the growth condition

$$|\beta(s)| \leq C(1 + |s|^{p-1}), \quad \forall s \in \mathbb{R} \quad (2.2)$$

with

$$\beta(0) = 0 \text{ and } 0 < \lambda \leq \beta'(s).$$

The addition integral measurement is given by

$$\int_{\Omega} \beta(u(t, x)) dx = \sigma(t). \quad (2.3)$$

2.3 The Variational Formulation

Let us introduce the space

$$V = C + W_{\Gamma_D}^{1,p}(\Omega) = \{u \mid u \in W^{1,p}(\Omega), u|_{\Gamma_D} = C\},$$

where

$$W_{\Gamma_D}^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) \text{ such that } u|_{\Gamma_D} = 0\},$$

which is equipped with the norm of $W^{1,p}(\Omega)$.

Remark 2.3.1. V is Banach space because it is closed subspace of the Banach space $W^{1,p}(\Omega)$.

2. (Poincaré-Wirtinger inequality) For all $p \in [1, \infty]$, there exists a constant $C = C(\Omega, p) > 0$ such that

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W_{\Gamma_D}^{1,p}(\Omega).$$

3. The norm $\|\cdot\|_{W^{1,p}(\Omega)}$ and the semi norm $\|\nabla(\cdot)\|_{L^p(\Omega)}$ are equivalent over the space $W_{\Gamma_D}^{1,p}(\Omega)$.

Multiplying (2.1) by $v \in V$ and integrating over Ω , we have

$$\begin{aligned} (\partial_t \beta(u), v) + (|\nabla u|^{p-2} \nabla u, \nabla v) + (\alpha, v)_{\Gamma_N} - \nu|_{\Gamma_D} (\partial_\eta u, 1)_{\Gamma_D} &= (f, v) \\ &+ \left(\int_0^t M(s, u(s)) ds, v \right), \quad \forall v \in V. \end{aligned} \quad (2.4)$$

Taking $v = 1$ in (2.4) and using (2.3), we get

$$(\partial_\eta u, 1)_{\Gamma_D} = \sigma' + (\alpha, 1)_{\Gamma_N} - (f, 1) - \left(\int_0^t M(s, u(s)) ds, 1 \right).$$

Hence, the variational formulation becomes

$$\begin{aligned} (\partial_t \beta(u), v) + (|\nabla u|^{p-2} \nabla u, \nabla v) + (\alpha, v)_{\Gamma_N} &= (f, v) + \left(\int_0^t M(s, u(s)) ds, v \right) \\ &+ \nu|_{\Gamma_D} \left[\sigma' + (\alpha, 1)_{\Gamma_N} - (f, 1) - \left(\int_0^t M(s, u(s)) ds, 1 \right) \right], \quad \forall v \in V. \end{aligned} \quad (2.5)$$

2.4 Semi-Discretized Problem

We subdivide the time interval $I = [0, T]$ to n subintervals, $\tau = \frac{T}{n}$, $i = 1, 2, \dots, n$ where $u_i = u(t_i, x)$, $t_i = i\tau$ and let

$$\delta u_i = \frac{u_i - u_{i-1}}{\tau}, \quad \delta \beta(u_i) = \frac{\beta(u_i) - \beta(u_{i-1})}{\tau}, \quad \rho_i = \rho(t_i).$$

For $i = 1, \dots, n$, the recurrent semi discretized problem is

$$\left\{ \begin{array}{l} \text{Find } (u_i, \rho_i) \cong (u(t_i, \cdot), \rho(t_i)), i = 1, 2, \dots, n \text{ such that ,} \\ (\beta(u_i) - \beta(u_{i-1}), v) + \tau(|\nabla u_i|^{p-2} \nabla u_i, \nabla v) = -\tau(\alpha_i, v)_{\Gamma_N} + \tau(f_i, v) \\ + v|_{\Gamma_D} \tau \left[\sigma'_i + (\alpha_i, 1)_{\Gamma_N} - (f_i, 1) - \left(\sum_{j=0}^{i-1} M(t_j, u_j) \tau, 1 \right) \right], \forall v \in V \\ + \tau \left(\sum_{j=0}^{i-1} M(t_j, u_j) \tau, v \right). \end{array} \right. \quad (2.6)$$

Theorem 2.4.1. *Let $\alpha(t) \in L^q(\Gamma_N)$, $f(t) \in L^q(\Omega)$ and $\sigma'(t) \in \mathbb{R}$, for all $t \in I$, the problem (2.6) admits a unique weak solution $u_i \in V, \forall i = 1, \dots, n$.*

Proof. Let us define the operator $A : V \rightarrow V'$ by

$$\langle A(u), v \rangle = (\beta(u), v) + \tau(|\nabla u|^{p-2} \nabla u, \nabla v), \quad \forall v \in V$$

We apply monotone operators theory (see [45]), we must prove that the operator A is bounded, hemicontinuous, monotone and coercive.

We have

$$\begin{aligned} \langle A(u_i), v \rangle &= (\beta(u_{i-1}), v) - \tau(\alpha_i, v)_{\Gamma_N} + \tau(f_i, v) + \tau \left(\sum_{j=0}^{i-1} M(t_j, u_j) \tau, v \right) \\ &\quad + v|_{\Gamma_D} \tau \left[\sigma'_i + (\alpha_i, 1)_{\Gamma_N} - (f_i, 1) - \left(\sum_{j=0}^{i-1} M(t_j, u_j) \tau, 1 \right) \right]. \end{aligned}$$

Using the growth condition of the functions β and M , the Hölder, Poincaré and trace inequalities, we successively deduce that

$$\begin{aligned}
|\langle A(u_i), v \rangle| &\leq |(\beta(u_{i-1}), v)| + \tau |(\alpha_i, v)_{\Gamma_N}| + \tau |(f_i, v)| + \tau \left| \left(\sum_{j=0}^{i-1} M(t_j, u_j) \tau, v \right) \right| \\
&\quad + |v|_{\Gamma_D} |\tau| \left[|\sigma'_i| + |(\alpha_i, 1)_{\Gamma_N}| + |(f_i, 1)| + \left| \left(\sum_{j=0}^{i-1} M(t_j, u_j) \tau, 1 \right) \right| \right] \\
&\leq (C(1 + |u_{i-1}|^{p-1}), |v|) + C\tau \|\alpha_i\|_{L^q(\Gamma_N)} \|v\|_{W^{1,p}(\Omega)} + \tau \|f_i\|_{L^q(\Omega)} \|v\|_{L^p(\Omega)} \\
&\quad + \tau^2 \sum_{j=0}^{i-1} (C|u_j|^{p-1}, |v|) + \tau^2 \sum_{j=0}^{i-1} (C, |v|) + \tau \|v\|_{L^p(\Gamma_D)} \left[|\sigma'_i| + \|\alpha_i\|_{L^q(\Gamma_N)} \|1\|_{L^p(\Gamma_N)} \right. \\
&\quad \left. + \|f_i\|_{L^q(\Omega)} \|1\|_{L^p(\Omega)} + \left(\sum_{j=0}^{i-1} (C + C|u_j|^{p-1}) \tau, 1 \right) \right] \\
&\leq C \|u_{i-1}\|_{L^p(\Omega)}^{\frac{p}{q}} \|v\|_{W^{1,p}(\Omega)} + C \|v\|_{W^{1,p}(\Omega)} + C\tau \|\alpha_i\|_{L^q(\Gamma_N)} \|v\|_{W^{1,p}(\Omega)} \\
&\quad + \tau \|f_i\|_{L^q(\Omega)} \|v\|_{W^{1,p}(\Omega)} + C\tau^2 \sum_{j=0}^{i-1} \|u_j\|_{L^p(\Omega)}^{\frac{p}{q}} \|v\|_{W^{1,p}(\Omega)} \\
&\quad + C \|v\|_{W^{1,p}(\Omega)} + \tau \|v\|_{W^{1,p}(\Omega)} \left[|\sigma'_i| + \|\alpha_i\|_{L^q(\Gamma_N)} \|1\|_{L^p(\Gamma_N)} \right. \\
&\quad \left. + \|f_i\|_{L^q(\Omega)} \|1\|_{L^p(\Omega)} + C + C \sum_{j=0}^{i-1} \tau \|u_j\|_{L^p(\Omega)}^{\frac{p}{q}} \|1\|_{L^p(\Omega)} \right],
\end{aligned}$$

which implies

$$|\langle A(u_i), v \rangle| \leq C(i) \|v\|_{W^{1,p}(\Omega)}.$$

For the hemicontinuity of A , we take $u_n \rightarrow u$ in V . Then for $v \in V$, we have

$$\begin{aligned}
|\langle A(u_n) - A(u), v \rangle| &= |(\beta(u_n) - \beta(u), v) + \tau (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u, \nabla v)| \\
&\leq C \|\beta(u_n) - \beta(u)\|_{L^q(\Omega)} \|\nabla v\|_{L^p(\Omega)} \\
&\quad + \tau \| |\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u \|_{L^q(\Omega)} \|\nabla v\|_{L^p(\Omega)} \\
&\leq C \|\beta(u_n) - \beta(u)\|_{L^q(\Omega)} \|v\|_{W^{1,p}(\Omega)} \\
&\quad + \tau \| |\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u \|_{L^q(\Omega)} \|v\|_{W^{1,p}(\Omega)}.
\end{aligned}$$

It follows from the continuity of the operator $u \mapsto |\nabla u|^{p-2} \nabla u$ in $L^p(\Omega)$ into $L^{\frac{p}{p-1}}(\Omega)$

and the continuity of β that

$$|\langle A(u_n) - A(u), v \rangle| \longrightarrow_{n \rightarrow \infty} 0, \quad \forall v \in V.$$

Finally, to show the monotony of A , using the fact that $u - v \in W_{\Gamma_D}^{1,p}(\Omega)$, we have

$$\begin{aligned} \langle A(u) - A(v), u - v \rangle &= (\beta u - \beta v, u - v) + \tau (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v, \nabla u - \nabla v) \\ &\geq \lambda \|u - v\|_{L^2(\Omega)}^2 + \tau \|\nabla u - \nabla v\|_{L^p(\Omega)}^p \\ &\geq \tau \|u - v\|_{W^{1,p}(\Omega)}^p, \end{aligned}$$

by virtue of Remark (2.3.1).

In similar way we get

$$\langle A(u), u \rangle \geq \tau \|u\|_{W^{1,p}(\Omega)}^p,$$

this achieves the prove. ■

2.5 Necessary A Priori Estimates

Now we are going to show some a priori estimates which will help us in the existence part.

Lemma 2.5.1. *There exists a positive constant C , such that*

$$\sum_{i=1}^m \tau \|\delta u_i\|_{L^2(\Omega)}^2 \leq C, \quad \sum_{i=1}^m \|\nabla u_i\|_{L^p(\Omega)}^p \leq C.$$

Proof. Choosing in (2.6), $v = u_i - u_{i-1}$ and summing over $i = 1, \dots, m$ with $1 \leq m \leq n$, we obtain

$$\begin{aligned}
 & \sum_{i=1}^m \tau(\delta\beta(u_i), \delta u_i) + \sum_{i=1}^m (|\nabla u_i|^{p-2} \nabla u_i, \nabla u_i) = \sum_{i=1}^m (|\nabla u_i|^{p-2} \nabla u_i, \nabla u_{i-1}) \\
 & + \sum_{i=1}^m \tau(f_i, \delta u_i) - \sum_{i=1}^m \tau(\alpha_i, \delta u_i)_{\Gamma_N} + \tau \sum_{i=1}^m \sum_{j=0}^{i-1} (M(t_j, u_j) \tau, \delta u_i) \\
 & + \tau \sum_{i=1}^m \delta u_i|_{\Gamma_D} [\sigma'_i + (\alpha_i, 1)_{\Gamma_N} - (f_i, 1) - \sum_{j=0}^{i-1} (M(t_j, u_j) \tau, 1)]. \tag{2.7}
 \end{aligned}$$

By the mean value theorem, we get

$$\begin{aligned}
 & \lambda \sum_{i=1}^m \tau \|\delta u_i\|_{L^2(\Omega)}^2 + \sum_{i=1}^m \|\nabla u_i\|_{L^p(\Omega)}^p \\
 & \leq \sum_{i=1}^m \tau(\delta\beta(u_i), \delta u_i) + \sum_{i=1}^m (|\nabla u_i|^{p-2} \nabla u_i, \nabla u_i).
 \end{aligned}$$

Hölder and Young inequalities implies

$$\begin{aligned}
 \left| \sum_{i=1}^m (|\nabla u_i|^{p-2} \nabla u_i, \nabla u_{i-1}) \right| & \leq \sum_{i=1}^m (|\nabla u_i|^{p-1}, |\nabla u_{i-1}|) \\
 & \leq \sum_{i=1}^m \|\nabla u_i\|_{L^p(\Omega)}^{\frac{p}{q}} \|\nabla u_{i-1}\|_{L^p(\Omega)} \\
 & \leq \varepsilon \sum_{i=1}^m \|\nabla u_i\|_{L^p(\Omega)}^p + C_\varepsilon \sum_{i=1}^m \|\nabla u_{i-1}\|_{L^p(\Omega)}^p.
 \end{aligned}$$

To estimate the second term in the RHS of (2.7), we use Hölder, Poincare and Young inequalities to get

$$\left| \sum_{i=1}^m \tau(f_i, \delta u_i) \right| \leq C \left(\frac{1}{\varepsilon} + \varepsilon \sum_{i=1}^m \tau \|\nabla \delta u_i\|_{L^p(\Omega)}^p \right).$$

The memory term can be estimated as follows

$$\left| \tau \sum_{i=1}^m \sum_{j=0}^{i-1} (M(t_j, u_j) \tau, \delta u_i) \right| \leq C \left(\frac{1}{\varepsilon} + \varepsilon \sum_{i=1}^m \tau \|\nabla \delta u_i\|_{L^p(\Omega)}^p \right).$$

Thanks to Abel's summation formula, Hölder, Young and trace inequalities, we obtain

$$\begin{aligned} \left| \sum_{i=1}^m \tau (\alpha_i, \delta u_i)_{\Gamma_N} \right| &= \left| (\alpha_m, u_m)_{\Gamma_N} - \tau \sum_{i=1}^m (\delta \alpha_i, u_{i-1})_{\Gamma_N} - (\alpha_0, u_0)_{\Gamma_N} \right| \\ &\leq \|\alpha_m\|_{L^q(\Gamma_N)} \|u_m\|_{L^p(\Gamma_N)} + \tau \sum_{i=1}^m \|\delta \alpha_i\|_{L^q(\Gamma_N)} \|u_{i-1}\|_{L^p(\Gamma_N)} + C \\ &\leq C_\varepsilon \|\alpha_m\|_{L^q(\Gamma_N)}^q + \varepsilon \|u_m\|_{L^p(\Gamma_N)}^p + C_\varepsilon \sum_{i=1}^m \tau \|\delta \alpha_i\|_{L^q(\Gamma_N)}^q \\ &\quad + \sum_{i=1}^m \tau \varepsilon \|u_{i-1}\|_{L^p(\Gamma_N)}^p + C \\ &\leq \varepsilon \|\nabla u_m\|_{L^p(\Omega)}^p + C. \end{aligned}$$

For the last term in the RHS of (2.7), we have

$$\begin{aligned} &\left| \tau \sum_{i=1}^m \delta u_{i|\Gamma_D} \left[\sigma'_i + (\alpha_i, 1)_{\Gamma_N} - (f_i, 1) - \sum_{j=0}^{i-1} (M(t_j, u_j) \tau, 1) \right] \right| \\ &\leq \left| u_{m|\Gamma_D} \left[\sigma'_m + (\alpha_m, 1)_{\Gamma_N} - (f_m, 1) - \sum_{j=0}^{m-1} (M(t_j, u_j) \tau, 1) \right] \right| \\ &\quad + \left| \tau \sum_{i=1}^m u_{i-1|\Gamma_D} \left[\delta \sigma'_i + (\delta \alpha_i, 1)_{\Gamma_N} - (\delta f_i, 1) - (M(t_{i-1}, u_{i-1}) \tau, 1) \right] \right| \\ &\quad + \left| u_{0|\Gamma_D} \left[\sigma'_0 + (\alpha_0, 1)_{\Gamma_N} - (f_0, 1) \right] \right| \\ &\leq C \|u_m\|_{L^p(\Gamma_D)} \left(|\sigma'_m| + \|\alpha_m\|_{L^q(\Gamma_N)} + \|f_m\|_{L^q(\Omega)} + 1 \right) \\ &\quad + C \sum_{i=1}^m \tau \|u_{i-1}\|_{L^p(\Gamma_D)} \left(|\delta \sigma'_i| + \|\delta \alpha_i\|_{L^q(\Gamma_N)} + \|\delta f_i\|_{L^q(\Omega)} + 1 \right) + C \\ &\leq \varepsilon \|\nabla u_m\|_{L^p(\Omega)}^p + C. \end{aligned}$$

Putting all the above considerations together, yields

$$\begin{aligned}
 & \lambda \sum_{i=1}^m \tau \|\delta u_i\|_{L^2(\Omega)}^2 + \sum_{i=1}^m \|\nabla u_i\|_{L^p(\Omega)}^p \\
 & \leq \varepsilon \sum_{i=1}^m \|\nabla u_i\|_{L^p(\Omega)}^p + C_\varepsilon \sum_{i=1}^m \|\nabla u_{i-1}\|_{L^p(\Omega)}^p + C \left(\frac{1}{\varepsilon} + \varepsilon \sum_{i=1}^m \tau \|\nabla \delta u_i\|_{L^p(\Omega)}^p \right) \\
 & + \varepsilon \|\nabla u_m\|_{L^p(\Omega)}^p + C.
 \end{aligned}$$

Choosing $\varepsilon > 0$ small enough and using Gronwall lemma we conclude the proof. ■

Lemma 2.5.2. *There exists a positive constant C such that*

$$\|\delta\beta(u_i)\|_{V'} \leq C.$$

Proof. From (2.6), we have

$$\begin{aligned}
 (\delta\beta(u_i), v) &= (f_i, v) - (|\nabla u_i|^{p-2} \nabla u_i, \nabla v) - (\alpha_i, v)_{\Gamma_N} + \left(\sum_{j=0}^{i-1} M(t_j, u_j) \tau, v \right) \\
 &+ v|_{\Gamma_D} \left[\sigma'_i + (\alpha_i, 1)_{\Gamma_N} - (f_i, 1) - \left(\sum_{j=0}^{i-1} M(t_j, u_j) \tau, 1 \right) \right]
 \end{aligned}$$

By invoking standard estimates, we arrive at

$$|(\delta\beta(u_i), v)| \leq C \|v\|_{W^{1,p}(\Omega)}$$

Thus

$$\|\delta\beta(u_i)\|_{V'} = \sup_{v \in V, v \neq 0} \frac{|(\delta\beta(u_i), v)|}{\|v\|_V} \leq C. \quad \blacksquare$$

2.6 Existence

In this part, we will show the existence of a weak solution to the problem (2.1). To this end, let us introduce Rothe functions

$$\begin{cases} u_n(0) = u_0 \\ u_n(t) = u_{i-1} + (t - t_{i-1})\delta u_i, \quad 1 \leq i \leq n, \quad \forall t \in (t_{i-1}, t_i], \end{cases}$$

$$\begin{cases} \beta_n(0) = \beta_0 \\ \beta_n(t) = \beta(u_{i-1}) + (t - t_{i-1})\delta\beta(u_i), \quad 1 \leq i \leq n, \quad \forall t \in (t_{i-1}, t_i], \end{cases}$$

Together with step functions,

For $1 \leq i \leq n$ and $\forall t \in (t_{i-1}, t_i]$,

$$\begin{cases} \overline{u_n}(0) = u_0, \overline{u_n}(t) = u_i, \\ \overline{f_n}(0) = f(0), \overline{f_n}(t) = f(t_i), \\ \overline{\alpha_n}(0) = \alpha(0), \overline{\alpha_n}(t) = \alpha(t_i), \\ \overline{M_n}(0, \overline{u_n}(0)) = M(0, u_0), \overline{M_n}(t, \overline{u_n}(t)) = M(t_i, u_i), \\ \overline{\sigma'_n}(0) = \sigma'(0), \overline{\sigma'_n}(t) = \sigma'(t_i). \end{cases}$$

Then, (2.6) may be written as

$$\begin{aligned} (\partial_t \beta_n(t), v) + (|\nabla \overline{u_n}|^{p-2} \nabla \overline{u_n}, \nabla v) + (\overline{\alpha_n}(t), v)_{\Gamma_N} &= (\overline{f_n}(t), v) + \left(\int_0^{t_i} \overline{M_n}(s, \overline{u_n}(s)) ds, v \right) \\ + v|_{\Gamma_D} \left[\overline{\sigma'_n}(t) + (\overline{\alpha_n}(t), 1)_{\Gamma_N} - (\overline{f_n}(t), 1) - \left(\int_0^{t_i} \overline{M_n}(s, \overline{u_n}(s)) ds, 1 \right) \right]. \end{aligned} \quad (2.8)$$

Theorem 2.6.1. *There exists $u \in C(I, L^p(\Omega)) \cap L^\infty(I, W^{1,p}(\Omega))$ and $\rho \in L^p(0, T)$ such that $\{u, \rho\}$ solve (2.1).*

Proof. The proof is split into two parts

Step1:

In view of Lemma (2.5.1) and Lemma 1.3.13 in [32], we deduce that there exist a function $u \in C(I, L^p(\Omega)) \cap L^\infty(I, W^{1,p}(\Omega))$ and a subsequence of $\{u_n\}_{n \in \mathbb{N}}$ denote $\{u_n\}_{n \in \mathbb{N}}$, such that

$$\begin{aligned} u_n &\rightarrow u && \text{in } C(I, L^p(\Omega)) \\ \partial_t u_n &\rightharpoonup \partial_t u && \text{in } L^q(I, W^{-1,q}(\Omega)) \\ \bar{u}_n(t) &\rightharpoonup u(t) && \text{in } W^{1,p}(\Omega) \end{aligned} \quad (2.9)$$

For all $t \in [t_{i-1}, t_i]$, we have

$$\begin{aligned} \|\bar{u}_n(t) - u_n(t)\|_{L^p(\Omega)} &= \|(t - t_i)\delta u_i\|_{L^p(\Omega)} \\ &\leq \left\| \int_{t_i}^t \partial_s u_n \right\|_{L^p(\Omega)} \\ &\leq (t - t_i)^{\frac{1}{q}} \left\| \left(\int_{t_i}^t (\partial_s u_n)^p \right)^{\frac{1}{p}} \right\|_{L^p(\Omega)} \\ &\leq \tau^{\frac{1}{q}} \left\| \left(\int_0^t (\partial_s u_n)^p \right)^{\frac{1}{p}} \right\|_{L^p(\Omega)} \end{aligned} \quad (2.10)$$

(2.9) together with (2.10) implies that

$$\bar{u}_n \rightarrow u \text{ in } C(I, L^p(\Omega))$$

Taking into account (2.2), the continuity of β and DCT gives us

$$\begin{aligned} \beta(u_n) &\rightarrow \beta(u) && \text{in } C(I, L^p(\Omega)) \\ \beta(\bar{u}_n) &\rightarrow \beta(u) && \text{in } C(I, L^p(\Omega)) \end{aligned}$$

By integration of (2.8) from 0 to t and taking the limit ($n \rightarrow \infty$), then Lemma (2.5.2)

implies

$$\begin{aligned}
 & (\beta(u(t)) - \beta(u_0), v) + \int_0^t (|\nabla u|^{p-2} \nabla u, \nabla v) ds + \int_0^t (\alpha(s), v)_{\Gamma_N} ds = \\
 & \int_0^t (f(s), v) ds + \int_0^t \left(\int_0^{s_i} M(z, u(z)) dz, v \right) ds + \\
 & \int_0^t v|_{\Gamma_D} \left[\sigma'(s) + (\alpha(s), 1)_{\Gamma_N} - (f(s), 1) - \left(\int_0^{s_i} M(z, u(z)) dz, 1 \right) \right] ds. \quad (2.11)
 \end{aligned}$$

We differentiate (2.11) with respect to t we conclude the existence of a weak solution to (2.5).

Step2:

Let us introduce the function $\rho_n = u_n|_{\Gamma_D}$ and $\rho = u|_{\Gamma_D}$.

In this step we prove that ρ exist and satisfying (2.1).

According to the imbedding $C((0, T), L^p(\Omega)) \hookrightarrow L^p((0, T), L^p(\Gamma))$, we have

$$\begin{aligned}
 \|\rho_n - \rho\|_{L^p((0, T), L^p(\Gamma_D))}^p &= \|u_n - u\|_{L^p((0, T), L^p(\Gamma_D))}^p \\
 &\leq \|u_n - u\|_{L^p((0, T), L^p(\Gamma))}^p \\
 &\leq \|u_n - u\|_{C((0, T), L^p(\Omega))}^p
 \end{aligned}$$

When we pass to the limit ($n \rightarrow \infty$), we obtain that ρ_n converges to ρ in $L^p((0, T), L^p(\Gamma_D))$. ■

2.7 Uniqueness of Weak Solution

Now we will show the uniqueness of a weak solution to (2.1) in the case

$$m(x, t) = \int_0^t M(s, u(x, s)) ds = \int_0^t k(t-s) \Delta_p u(x, s) ds,$$

where

$$k \in C^1(0, T), \quad \Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u), \quad 2 < p < \infty.$$

The first equation of the problem (2.1) becomes

$$\partial_t \beta(u) - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = f(x, t) + \int_0^t k(t-s) \Delta_p u(x, s) ds$$

Simple calculation gives us

$$\begin{aligned} \int_0^t k(t-s) \Delta_p u(x, s) ds &= - \int_0^t k(t-s) \int_0^s k(s-z) \Delta_p u(x, z) dz ds \\ &\quad + k(0) \beta(u(x, t)) - k(t) \beta(u_0(x)) + \int_0^t k'(t-s) \beta(u(x, s)) ds \\ &\quad - \int_0^t k(t-s) f(x, s) ds, \end{aligned}$$

which proves that m can be written as

$$m(x, t) = - \int_0^t k(t-s) m(x, s) ds + g(x, t, u),$$

such that

$$g(x, t, u) = \int_0^t k'(t-s) \beta(u(x, s)) ds - \int_0^t k(t-s) f(x, s) ds + k(0) \beta(u(x, t)) - k(t) \beta(u_0(x)).$$

Therefore, (2.1) is equivalent to the following system

$$\left\{ \begin{array}{l} \text{Find } (u, m) \text{ such that ,} \\ (\partial_t \beta(u), v) + (|\nabla u|^{p-2} \nabla u, \nabla v) = -(\alpha, v)_{\Gamma_N} + (f, v) \\ + (m, v) + v|_{\Gamma_D} \left[\sigma' + (\alpha, 1)_{\Gamma_N} - (f, 1) - (m, 1) \right] \\ (m, v) = (- \int_0^t k(t-s) m(x, s) ds, v) + (g, v) \end{array} \right. , \forall v \in H_0^1(\Omega) \quad (2.12)$$

Theorem 2.7.1. *The problem (2.12) has a unique weak solution.*

Proof. Let (u_1, m_1) and (u_2, m_2) be two solution to problem (2.12). Rewriting the second equation of the problem (2.12) for m_1 and m_2 respectively, subtracting the second from the first, putting $v = m_1 - m_2$, we have

$$\begin{aligned} \|m_1 - m_2\|_{L^2(\Omega)}^2 &\leq C \int_0^t \|m_1 - m_2\|_{L^2(\Omega)}^2 ds + C \|u_1 - u_2\|_{L^2(\Omega)}^2 \\ &\quad + C \int_0^t \|u_1 - u_2\|_{L^2(\Omega)}^2 ds. \end{aligned}$$

Then, applying Gronwall's lemma, we obtain

$$\|m_1 - m_2\|_{L^2(\Omega)}^2 \leq C \int_0^t \|u_1 - u_2\|_{L^2(\Omega)}^2 ds + C \|u_1 - u_2\|_{L^2(\Omega)}^2. \quad (2.13)$$

By the same way, we obtain

$$(\partial_t \beta(u_1) - \partial_t \beta(u_2), u_1 - u_2) \leq C \int_0^t \|u_1 - u_2\|_{L^2(\Omega)}^2 ds + C \|u_1 - u_2\|_{L^2(\Omega)}^2.$$

Integrating it from 0 to ξ , $\xi \in [0, T]$, we have

$$(\beta(u_1(\xi)) - \beta(u_2(\xi)), u_1(\xi) - u_2(\xi)) \leq C(1 + \xi) \int_0^\xi \|u_1(s) - u_2(s)\|_{L^2(\Omega)}^2 ds.$$

Integrating it once more from 0 to t , $t \in [0, T]$ and using the mean value theorem, we arrive at

$$\lambda \int_0^t \|u_1(\xi) - u_2(\xi)\|_{L^2(\Omega)}^2 d\xi \leq \int_0^t C(1 + \xi) \int_0^\xi \|u_1 - u_2\|_{L^2(\Omega)}^2 ds d\xi.$$

Then, applying Gronwall's lemma, we obtain

$$\int_0^t \|u_1(\xi) - u_2(\xi)\|_{L^2(\Omega)}^2 d\xi = 0, \quad \forall \xi \in [0, T]$$

which means that $u_1 = u_2$ in $L^2((0, t), \Omega)$, $\forall t \in [0, T]$, replacing this in (2.13), we get $\|m_1(t) - m_2(t)\|_{L^2(\Omega)}^2 = 0$, $\forall t \in [0, T]$.

Which prove that $u_1 = u_2$ and $m_1 = m_2$ a.e in $I \times \Omega$. ■

2.8 Numerical Experiment

In this section we summaries numerical tests validating the obtained results in previous sections. We begin with Rothe's method for the time discretization. For the spatial discretization, we apply the finite element method. Thus, we solve this system iteratively relying on the Newton-Raphson method.

Here, the unknown function $u(x, t)$ is approximated by a linear polynomial. For the test example, the computational domain Ω is taken as $\Omega = (0, 1)$ and the time interval $I = (0, 1)$ i.e. $T = 1$.

The source function f and $\rho(t)$ are chosen according :
to the exact solution

$$u(x, t) = \frac{1}{50} \left(t(x - x^2 + \frac{1}{2}) + \frac{x}{2} \right).$$

and the memory term β is given as

$$\beta(u) = u^2$$

Further, we prescribe the nonlinear function M as

$$M(s, u) = us^2.$$

The numerical error of u in $W^{1,p}(0, 1)$ at final time level $t_i = 1$ is calculated by taking a step length of space discretization intervals as follows $h \in \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}\}$ and $\tau = 2^{-5}$. likewise, for the numerical error of ρ in $L^p(0, 1)$ at final space $x = 1$, it is calculated by choosing $\tau = \frac{1}{2^k}$ such that $k = 1, \dots, 6$ and $h = 2^{-6}$.

In Figure 1, we plot the error results for u and ρ for $p = 2.2, 2.5$ in loglog-plot.

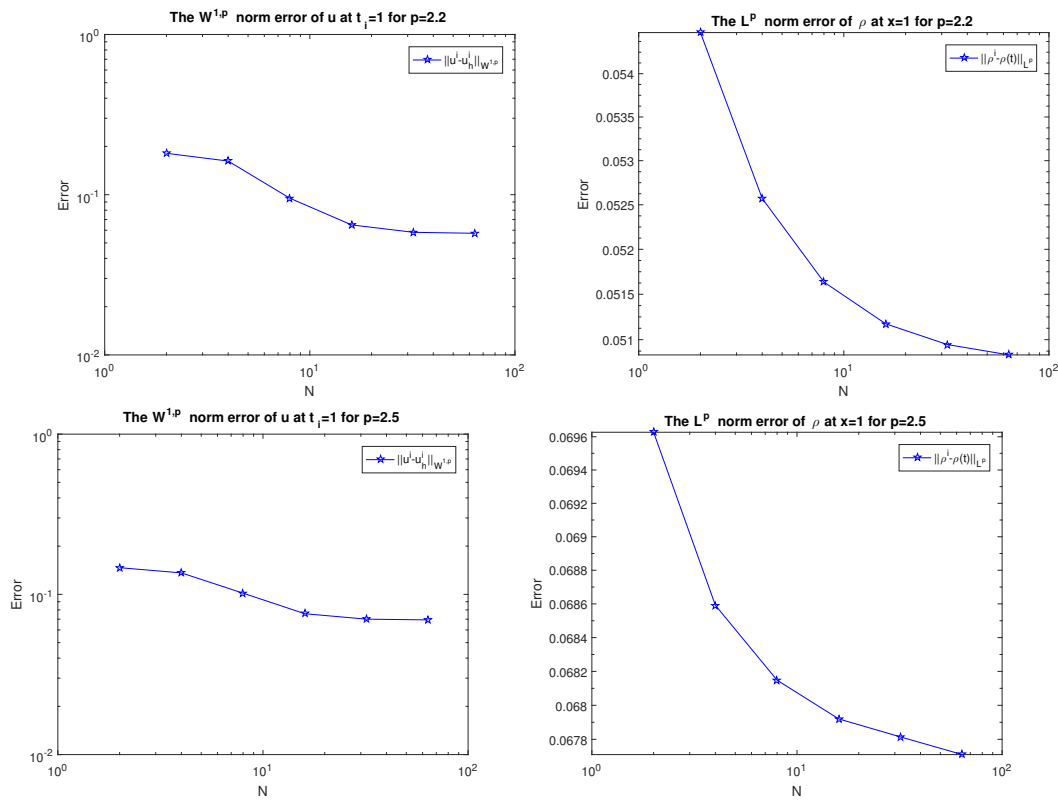


Figure 1 : The results of error for u and ρ in log log-plot for $p = 2.2, 2.5$ respectively.

2.9 Conclusion

In this chapter, we considered a degenerate parabolic problem with a memory term and an unknown Dirichlet boundary condition. The Dirichlet boundary condition ρ , initially unknown in the inverse problem, was eliminated through an appropriate integral measurement. By applying Rothe's method together with the theory of monotone operators, we established the existence and uniqueness of the weak solution to the problem. Finally, we supported our theoretical results with a numerical experiment.

3

Advanced Qualitative Results on Hyperbolic $p(\cdot)$ -Biharmonic Equation with No-Flux Boundary Condition

3.1 Introduction

In this chapter, we focus on studying the hyperbolic $p(x)$ -biharmonic problem with no-flux boundary condition in variable exponent Sobolev spaces. We will prove the

global existence of a weak solution using the Galerkin method and discuss the critical criterion for identifying finite-time blow-up when the initial functional energy is negative.

3.2 Position of the Problem

We consider a bounded open domain Ω of $\mathbb{R}^n$, with a Lipschitz-continuous boundary Γ and $I = [0, T]$, $T \in \mathbb{R}$. We address the following hyperbolic $p(x)$ -biharmonic problem subject to homogenous Newman boundary condition

$$\left\{ \begin{array}{ll} \frac{\partial^2 u}{\partial t^2} + \Delta(|\Delta u|^{p(x)-2} \Delta u) - \sigma \Delta u_t = \phi(x, u) + f(x, t) & \text{in } I \times \Omega, \\ u = c, \Delta u = 0 & \text{on } \Gamma, \\ \int_{\Gamma} \frac{\partial}{\partial \eta} (|\Delta u|^{p(x)-2} \Delta u) d\theta = 0 & \text{on } \Gamma, \\ u(0, x) = u_0, \frac{\partial u}{\partial t}(0, x) = U_1 & \text{on } \Omega. \end{array} \right. \quad (3.1)$$

Where σ is a positive constant. ϕ is a continuous function satisfying

$$|\phi(x, u)| \leq \beta |u|^{m(x)-1} + \gamma, \quad (3.2)$$

for some $\gamma, \beta > 0$, $1 < m(x) < \infty$ if $n \leq p(x)$ and $1 < m(x) < \frac{np(x)}{n-p(x)}$ if $n > p(x)$,

and

$$\Phi(x, u) = \int_0^u \phi(x, s) ds.$$

Here $p \in C(\overline{\Omega}, (1, \infty))$ such that

$$1 < p^- \leq p(x) \leq p^+ < \infty,$$

with

$$p^- = \operatorname{ess\,inf}_{x \in \overline{\Omega}} p(x), \quad p^+ = \operatorname{ess\,sup}_{x \in \overline{\Omega}} p(x).$$

3.3 Existence

In this section we will show the existence of a weak solution to the problem (3.1) with the aid of Galerkin method.

Theorem 3.3.1. *Let $u_0 \in W^{2,p(\cdot)}(\Omega) \cap W_0^{1,2}(\Omega)$, $U_1 \in L^2(\Omega)$ be given functions and assume that either $(m^+ \leq 2)$ or $(2 < m^+ \leq p^-)$. Then the problem (3.1) admits weak solution $u : I \times \Omega \longrightarrow \mathbb{R}$, $T < \infty$, such that*

1. $u \in L^\infty((0, T), V)$, $V = \{u \in W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega) : |u|^{\frac{m}{2}} \in L^2(\Omega)\}$,
2. $u' \in L^\infty((0, T), L^2(\Omega)) \cap L^\infty((0, T), V)$,
3. $u_{tt} + \Delta(|\Delta u|^{p(x)-2} \Delta u) - \sigma \Delta u_t = \phi(x, u) + f(x, t)$ a.e. in $L^2((0, T), V')$.

Proof. We will demonstrate the existence of a weak solution $u \in V$. The weak solution of our problem will be $\tilde{u} = u + c$.

Let $\{e_i\}_{i=1}^\infty$ be the orthonormal basis of $W_0^{S,2}(\Omega)$ that satisfies

$$\begin{cases} -\Delta e_i = \lambda_i e_i, & \text{if } x \in \Omega, \\ -\Delta^2 e_i = -\lambda_i^2 e_i, & \text{if } x \in \Omega, \\ e_i = 0, & \text{if } x \in \Gamma. \end{cases}$$

Let $E_l = \langle e_1, \dots, e_l \rangle$ and

$$\langle Mu, v \rangle = (u_{tt}, v) + (|\Delta u|^{p(x)-2} \Delta u, \Delta v) + \sigma (\nabla u_t, \nabla v) - (\phi, v) - (f, v), \quad \forall v \in E_l$$

For $l \in \mathbb{N}^*$, we consider Galerkin approximate solution

$$u_l = \sum_{i=1}^l C_i(t) e_i(x),$$

such that

$$\begin{cases} \langle Mu_l, e_i \rangle = 0, & i = 1, \dots, l, \\ u_l(0) = u_{0l}, & u_{lt}(0) = u_{1l}, \end{cases} \quad (3.3)$$

where

$$u_{0l} = \sum_{i=1}^l (u_0, e_i) e_i, \quad u_{1l} = \sum_{i=1}^l (U_1, e_i) e_i, \quad C_i = (u(x, t), e_i(x)),$$

and

$$u_{0l} \longrightarrow u_0 \text{ in } W^{2,p(\cdot)}(\Omega) \cap W^{1,2}(\Omega), \quad u_{1l} \longrightarrow U_1 \text{ in } L^2(\Omega).$$

From problem (3.1) we can derive the following system of l differential equations

$$\begin{cases} u_i''(t) = -(|\Delta u_l|^{p(x)-2} \Delta u_l, \Delta e_i) - \sigma(\nabla \partial_t u_l, \nabla e_i) + (\phi(u_l), e_i) + (f, e_i), \\ u_i(0) = (u_0, e_i), \quad u_i'(0) = (U_1, e_i), & i = 1 \dots l, \end{cases} \quad (3.4)$$

where $(\cdot, \cdot)$ is L^2 inner product. Note that the solution u_l of (3.4) is assured by standard theory of ODE in $[0, t_l]$. A priori estimates that will be prove bellow, prove that the solution $u_l(t)$ can be extended to $[0, T]$, $T > 0$. multiplying each equations in system (3.4) by $C_i'(t)$ and summing with respect to $i = 1, \dots, l$, we obtain

$$\begin{aligned} \frac{d}{dt} \left[\int_{\Omega} \frac{|\Delta u_l(t)|^{p(x)}}{p(x)} dx + \frac{1}{2} \int_{\Omega} |\partial_t u_l(t)|^2 dx - \int_{\Omega} \Phi(x, u_l(t)) dx \right] \\ + \sigma \|\nabla \partial_t u_l(t)\|_{L^2(\Omega)}^2 = \int_{\Omega} f \partial_t u_l(t) dx, \end{aligned}$$

which implies

$$J'(t) + \sigma \|\nabla \partial_t u_l\|_{L^2(\Omega)}^2 = \int_{\Omega} f \partial_t u_l dx,$$

where

$$J(t) = \int_{\Omega} \frac{|\Delta u_l(t)|^{p(x)}}{p(x)} dx + \frac{1}{2} \int_{\Omega} |\partial_t u_l(t)|^2 dx - \int_{\Omega} \Phi(x, u_l(t)) dx,$$

is the energy functional.

To prove this theorem we need some a priori estimates which is given in this lemma.

Lemma 3.3.2. *For all $T < \infty$, there exists a positive constant C , such that*

$$\sup_{t \in [0, T]} \left[\|u_l'\|_{L^2(\Omega)}^2 + \int_{\Omega} |\Delta u_l(t)|^{p(x)} dx \right] + \sigma \int_0^T \|\nabla u_l'\|_{L^2(\Omega)}^2 ds \leq C$$

Proof. Substituting e_i in (3.3) by $u_l'(t)$, we get

$$(u_l'', u_l') + (|\Delta u_l|^{p(x)-2} \Delta u_l, \Delta u_l') + \sigma \|\nabla u_l'\|_{L^2(\Omega)}^2 - (\phi, u_l') \leq \|f\|_{L^2(\Omega)} \|u_l'\|_{L^2(\Omega)}.$$

Integrating it over $(0, t)$, we obtain

$$\begin{aligned} & \frac{1}{2} \|u_l'(s)\|_{L^2(\Omega)}^2 \Big|_0^t + \int_{\Omega} \frac{|\Delta u_l(s)|^{p(x)}}{p(x)} dx \Big|_0^t + \int_0^t \sigma \|\nabla u_l'(s)\|_{L^2(\Omega)}^2 ds \\ & - \int_{\Omega} \Phi(x, u_l(s)) dx \Big|_0^t \leq \int_0^t \|f\|_{L^2(\Omega)} \|u_l'(s)\|_{L^2(\Omega)} ds. \end{aligned}$$

This implies

$$\begin{aligned} & \frac{1}{2} \|u_l'(t)\|_{L^2(\Omega)}^2 + \int_{\Omega} \frac{|\Delta u_l(t)|^{p(x)}}{p(x)} dx + \int_0^t \sigma \|\nabla u_l'(s)\|_{L^2(\Omega)}^2 ds + \int_{\Omega} \Phi(x, u_{0l}) dx \\ & \leq \int_0^t \|f\|_{L^2(\Omega)} \|u_l'(s)\|_{L^2(\Omega)} ds + \frac{1}{2} \|u_{1l}\|_{L^2(\Omega)}^2 + \int_{\Omega} \frac{|\Delta u_{0l}|^{p(x)}}{p(x)} dx + \int_{\Omega} \Phi(x, u_l(t)) dx. \quad (3.5) \end{aligned}$$

Taking into account (3.2) and for $m^+ \leq 2$, we have

$$\begin{aligned}
 \int_{\Omega} |\Phi(x, u_l(t))| dx &= \int_{\Omega} \left| \int_0^{u_l} \phi(x, s) ds \right| dx \\
 &\leq \int_{\Omega} \int_0^{u_l} (\beta |s|^{m(x)-1} + \gamma) ds dx \\
 &\leq \beta \int_{\Omega} \frac{|u_l|^{m(x)}}{m(x)} dx + \gamma \|u_l\|_{L^1(\Omega)} \\
 &\leq C(\|u_l\|_{L^2(\Omega)}^2 + 1) \\
 &\leq 2C \left(t \int_0^t \|u'_l\|_{L^2(\Omega)}^2 ds + \|u_{0l}\|_{L^2(\Omega)}^2 + 1 \right) \\
 &\leq c \left(t \int_0^t \left[\frac{1}{2} \|u'_l\|_{L^2(\Omega)}^2 + \int_{\Omega} \frac{|\Delta u_l(s)|^{p(x)}}{p(x)} dx \right] ds + 1 \right).
 \end{aligned}$$

Assume that

$$\mathcal{J}(t) = \frac{1}{2} \|u'_l\|_{L^2(\Omega)}^2 + \int_{\Omega} \frac{|\Delta u_l(t)|^{p(x)}}{p(x)} dx.$$

Then, the inequality (3.5) becomes

$$\begin{aligned}
 \mathcal{J}(t) + \int_0^t \sigma \|\nabla u'_l(s)\|_{L^2(\Omega)}^2 ds + c \left(t \int_0^t \left[\frac{1}{2} \|u_{1l}\|_{L^2(\Omega)}^2 \right. \right. \\
 \left. \left. + \int_{\Omega} \frac{|\Delta u_{0l}(s)|^{p(x)}}{p(x)} dx \right] ds + 1 \right) \leq \int_0^t \|f\|_{L^2(\Omega)}^2 ds + c \\
 + (1 + ct) \int_0^t \mathcal{J}(s) ds + \frac{1}{2} \|u_{1l}\|_{L^2(\Omega)}^2 + \int_{\Omega} \frac{|\Delta u_{0l}|^{p(x)}}{p(x)} dx.
 \end{aligned} \tag{3.6}$$

Now, for $2 < m^+ \leq p^-$, we have

$$\begin{aligned}
 \int_{\Omega} |\Phi(x, u_l(t))| dx &\leq \beta \int_{\Omega} \frac{|u_l|^{m(x)}}{m(x)} dx + \gamma \|u_l\|_{L^1(\Omega)} \\
 &\leq C \int_{\Omega} \frac{|\Delta u_l|^{m(x)}}{m(x)} dx + C \|u_l\|_{L^{p(x)}(\Omega)} \\
 &\leq C \int_{\Omega} \frac{|\Delta u_l|^{p^-}}{m^-} dx + C \|\Delta u_l\|_{L^{p(x)}(\Omega)}
 \end{aligned}$$

Then

$$\begin{aligned}
 \int_{\Omega} |\Phi(x, u_l(t))| dx &\leq C \int_{\Omega} |\Delta u_l|^p dx + C \|u_l\|_{W^{2,p(x)}(\Omega)} \\
 &\leq C \max\{\|\Delta u_l\|_{L^{p(x)}(\Omega)}^{p^-}, \|\Delta u_l\|_{L^{p(x)}(\Omega)}^{p^+}\} + C \|u_l\|_{W^{2,p(x)}(\Omega)} \\
 &\leq C \max\{\|u_l\|_{W^{2,p(x)}(\Omega)}^{p^-}, \|u_l\|_{W^{2,p(x)}(\Omega)}^{p^+}, \|u_l\|_{W^{2,p(x)}(\Omega)}\}.
 \end{aligned}$$

Thus

$$\begin{aligned}
 &\frac{1}{2} \|u'_l(t)\|_{L^2(\Omega)}^2 + \int_{\Omega} \frac{|\Delta u_l(t)|^{p(x)}}{p(x)} dx + \int_0^t \sigma \|\nabla u'_l(s)\|_{L^2(\Omega)}^2 ds \\
 &+ C \max\{\|u_{0l}\|_{W^{2,p(x)}(\Omega)}^{p^-}, \|u_{0l}\|_{W^{2,p(x)}(\Omega)}^{p^+}, \|u_{0l}\|_{W^{2,p(x)}(\Omega)}\} \\
 &\leq \int_0^t \|f\|_{L^2(\Omega)} \|u'_l(s)\|_{L^2(\Omega)} ds + \frac{1}{2} \|u_{1l}\|_{L^2(\Omega)}^2 + \int_{\Omega} \frac{|\Delta u_{0l}|^{p(x)}}{p(x)} dx \\
 &+ C \max\{\|u_l\|_{W^{2,p(x)}(\Omega)}^{p^-}, \|u_l\|_{W^{2,p(x)}(\Omega)}^{p^+}, \|u_l\|_{W^{2,p(x)}(\Omega)}\} dx. \tag{3.7}
 \end{aligned}$$

Applying Gronwall's lemma in (3.6) and (3.7) we achieves the proof of lemma. ■

Return to the proof of theorem (3.3.1). Using Lions-Aubin lemma, the precedent a priori estimates enable us to deduce the existence of a subsequence of $\{u_l\}$ denote $\{u_l\}$ and a function u such that

$$\begin{aligned}
 u_l &\rightarrow u \text{ in } L^2((0, T), L^2(\Omega)) \\
 \partial_t u_l &\rightarrow \partial_t u \text{ in } L^2((0, T), L^2(\Omega)) \\
 u_l &\rightarrow u \text{ in } L^\infty((0, T), W_0^{2,p(x)}(\Omega)) \\
 \partial_t u_l &\rightarrow \partial_t u \text{ in } L^\infty((0, T), L^2(\Omega)) \\
 \partial_t u_l &\rightarrow \partial_t u \text{ in } L^2((0, T), H_0^1(\Omega)) \\
 |\Delta u_l|^{p(\cdot)-2} \Delta u_l &\rightarrow \zeta \text{ in } L^\infty((0, T), (W_0^{2,p(x)}(\Omega))')
 \end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\phi(x, u_l) &\rightarrow \phi(x, u) \text{ in } L^{m'}((0, T), L^{m'}(\Omega)) \text{ with } \frac{1}{m} + \frac{1}{m'} = 1 \\ \partial_{tt} u_l &\rightarrow u_{tt} \text{ in } L^2((0, T), (W_0^{2,p(x)}(\Omega))').\end{aligned}\quad (3.9)$$

Let $w = \sum_{i=1}^l k_i(t) e_i$, with $k_i(t) \in C^2[0, T]$. Taking equation (3.3), multiplying it by $k_i(t)$ and summing over i , we get

$$(\partial_{tt} u_l, w) + (|\Delta u_l|^{p(x)-2} \Delta u_l, \Delta w) + \sigma (\nabla \partial_t u_l, \nabla w) - (\phi(u_l), w) - (f, w) = 0.$$

Then, integrating it over $[0, T]$, we obtain

$$\begin{aligned}- \int_0^T (\partial_t u_l, \partial_t w) ds + (\partial_t u_l, w) \Big|_0^T + \int_0^T (|\Delta u_l|^{p(x)-2} \Delta u_l, \Delta w) ds \\ + \sigma \int_0^T (\nabla \partial_t u_l, \nabla w) ds - \int_0^T (\phi(u_l), w) ds - \int_0^T (f, w) ds = 0.\end{aligned}\quad (3.10)$$

Taking the limit ($m \rightarrow \infty$) and using (3.8) and (3.9), we arrive at

$$\begin{aligned}- \int_0^T (\partial_t u, \partial_t w) ds + (\partial_t u, w) \Big|_0^T + \int_0^T (\zeta, \Delta w) ds + \sigma \int_0^T (\nabla \partial_t u, \nabla w) ds \\ - \int_0^T (\phi(u), w) ds - \int_0^T (f, w) ds = 0 \text{ a.e.}\end{aligned}\quad (3.11)$$

It remains to prove that $\zeta = |\Delta u|^{p(x)-2} \Delta u$. Let us choose $w = u_l$ in (3.10) and $w = u$ in (3.11), we get

$$\begin{aligned}- \int_0^T (\partial_t u_l, \partial_t u_l) ds + (\partial_t u_l, u_l) \Big|_0^T + \int_0^T (|\Delta u_l|^{p(x)-2} \Delta u_l, \Delta u_l) ds \\ + \frac{\sigma}{2} (\nabla u_l, \nabla u_l) \Big|_0^T - \int_0^T (\phi(u_l), u_l) ds - \int_0^T (f, u_l) ds = 0.\end{aligned}\quad (3.12)$$

And

$$\begin{aligned} & - \int_0^T (\partial_t u, \partial_t u) ds + (\partial_t u, u) \Big|_0^T + \int_0^T (\zeta, \Delta u) ds + \frac{\sigma}{2} (\nabla u, \nabla u) \Big|_0^T \\ & - \int_0^T (\phi(u), u) ds - \int_0^T (f, u) ds = 0 \text{ a.e.} \end{aligned} \quad (3.13)$$

By virtue of the monotonicity of the operator $\Delta_{p(x)}^2 := \Delta(|\Delta u|^{p(x)-2} \Delta u)$, we have

$$\int_0^T (|\Delta u_l|^{p(x)-2} \Delta u_l - |\Delta v|^{p(x)-2} \Delta v, \Delta u_l - \Delta v) ds \geq 0. \quad (3.14)$$

According to (3.8) – (3.9) and (3.12) – (3.14), we can conclude that

$$\int_0^T (\zeta - |\Delta v|^{p(x)-2} \Delta v, \Delta u - \Delta v) \geq 0, \quad \forall v \in C_0^\infty((0, T) \times \Omega). \quad (3.15)$$

Substituting v in (3.15) by $u - \kappa\varphi$ and by $u + \kappa\varphi$ respectively and passing to the limit ($\kappa \rightarrow 0$), we arrive at

$$\begin{aligned} & \int_0^T (\zeta - |\Delta v|^{p(x)-2} \Delta v, \Delta\varphi) \geq 0, \\ & \int_0^T (\zeta - |\Delta v|^{p(x)-2} \Delta v, \Delta\varphi) \leq 0. \end{aligned}$$

For some $\kappa > 0$ and $\varphi \in C_0^\infty((0, T) \times \Omega)$ and from the density of $C_0^\infty((0, T) \times \Omega)$, we conclude the proof. ■

3.4 Blow-Up

In this section we will show the finite time Blow-up of the solution to the problem (3.1) in the case

$$f = 0, J(0) \leq 0, (u_0, U_1) > 0, 2 \leq p^- \leq p^+ < \kappa < m. \quad (3.16)$$

Assume that

$$2 \leq p(x) \leq \frac{2n}{n-2}, \quad n > 2,$$

and

$$\phi(x, u) = \beta |u|^{m(x)-2} u, \quad \beta > 0.$$

Let us define the energy function

$$J(t) = \int_{\Omega} \frac{|\Delta u|^{p(x)}}{p(x)} dx + \frac{1}{2} \int_{\Omega} |u_t|^2 dx - \beta \int_{\Omega} \frac{|u|^{m(x)}}{m(x)} dx. \quad (3.17)$$

It easy to see that

$$J(t) + \sigma \int_0^T \int_{\Omega} |\nabla u_t|^2 dx ds \leq J(0) \leq 0. \quad (3.18)$$

Theorem 3.4.1. *The weak solution u of (3.1) which satisfies (3.17) and (3.18) Blows-up in finite time $T^* = \frac{C}{(\rho-1)(H'(0))^{\rho-1}}$.*

Proof. To show blow-up of solution u , we need to define the following function

$$H(t) = \int_{\Omega} |u(t)|^2 dx + \sigma \int_0^T \int_{\Omega} |\nabla u|^2 dx ds.$$

Thus

$$H'(t) = 2 \int_{\Omega} u_t u dx + \sigma \int_{\Omega} |\nabla u|^2 dx. \quad (3.19)$$

Then

$$\begin{aligned} H''(t) &= 2 \int_{\Omega} u_{tt} u dx + 2\sigma \int_{\Omega} \nabla u_t \nabla u dx + 2 \int_{\Omega} |u_t|^2 dx \\ &= 2\beta \int_{\Omega} |u|^{m(x)} dx - 2 \int_{\Omega} |\Delta u|^{p(x)} dx + 2 \int_{\Omega} |u_t|^2 dx. \end{aligned}$$

Consequently by virtue of (3.18) and for $\kappa > 2$, $p^+ < \kappa < m^-$, we have

$$\begin{aligned}
 H''(t) &\geq H''(t) + 2\kappa \left(J(t) + \sigma \int_0^T \int_{\Omega} |\nabla u_t|^2 dx ds \right) \\
 &\geq 2\beta \int_{\Omega} |u|^{m(x)} dx - 2 \int_{\Omega} |\Delta u|^{p(x)} dx + 2 \int_{\Omega} |u_t|^2 dx \\
 &\quad + 2\kappa \int_{\Omega} \frac{|\Delta u|^{p(x)}}{p(x)} dx + \frac{1}{2} \int_{\Omega} |u_t|^2 dx - 2\kappa\beta \int_{\Omega} \frac{|u|^{m(x)}}{m(x)} dx \\
 &\quad + 2\kappa\sigma \int_0^T \int_{\Omega} |\nabla u_t|^2 dx ds \\
 &\geq 2\beta \int_{\Omega} \left(1 - \frac{\kappa}{m(x)} \right) |u|^{m(x)} dx + 2 \int_{\Omega} \left(\frac{\kappa}{p(x)} - 1 \right) |\Delta u|^{p(x)} dx \\
 &\quad + (2 + \kappa) \int_{\Omega} |u_t|^2 dx + 2\kappa\sigma \int_0^T \int_{\Omega} |\nabla u_t|^2 dx ds > 0.
 \end{aligned} \tag{3.20}$$

Hence

$$H'(t) > 0, \quad H(t) > 0, \quad \forall t > 0,$$

from which we can infer that

$$H(t) \rightarrow \infty \text{ when } t \rightarrow T^*.$$

Inequality (3.20) leads us to

$$\int_{\Omega} |u|^{m(x)} dx \leq CH''(t), \quad \int_{\Omega} |u_t|^2 dx \leq CH''(t), \quad \int_{\Omega} |\Delta u|^{p(x)} dx \leq CH''(t). \tag{3.21}$$

On the other hand, we suppose that $T^* = \infty$. Taking into account (1.1), (1.2) and (3.21), we get

$$\begin{aligned}
 \|\nabla u\|_{L^2(\Omega)} &\leq C \|\nabla u\|_{L^{p(\cdot)}(\Omega)} \\
 &\leq C \max \left[\left(\int_{\Omega} |u|^{p(\cdot)} dx \right)^{\frac{1}{p^-}}, \left(\int_{\Omega} |u|^{p(\cdot)} dx \right)^{\frac{1}{p^+}} \right] \\
 &\leq C \max \left[(H''(t))^{\frac{1}{p^-}}, (H''(t))^{\frac{1}{p^+}} \right].
 \end{aligned} \tag{3.22}$$

Furthermore

$$\begin{aligned}
 \|u\|_{L^2(\Omega)} &\leq C\|u\|_{L^{m(\cdot)}(\Omega)} \\
 &\leq C\max\left[\left(\int_{\Omega}|u|^{m(\cdot)}dx\right)^{\frac{1}{m^-}}, \left(\int_{\Omega}|u|^{m(\cdot)}dx\right)^{\frac{1}{m^+}}\right] \\
 &\leq C\max\left[(H''(t))^{\frac{1}{m^-}}, (H''(t))^{\frac{1}{m^+}}\right].
 \end{aligned} \tag{3.23}$$

Assuming that $H' \geq 1$ and $H'' \geq 1$. Using Cauchy schwarz, (3.22) and (3.23). Then (3.19) can be written as

$$\begin{aligned}
 H'(t) &\leq 2\|u_t\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \sigma \int_{\Omega} |\nabla u|^2 dx \\
 &\leq C\left(H''(t)^{\frac{2+m^-}{2m^-}} + H''(t)^{\frac{2}{p^-}}\right),
 \end{aligned} \tag{3.24}$$

from which, we get

$$C(H'(t))^{\rho} \leq H''(t), \text{ with } \frac{1}{\rho} = \max\left(\frac{2+m^-}{2m^-}, \frac{1}{p^-}\right) \text{ if } p^- > 2, m^- > 2.$$

Simple calculation gives us

$$H'(t) \geq \frac{H'(0)}{\left(1 - \frac{t(\rho-1)}{C}(H'(0))^{\rho-1}\right)^{\frac{1}{\rho-1}}} \rightarrow \infty \text{ as } t \rightarrow T^*,$$

with

$$T^* = \frac{C}{(\rho-1)(H'(0))^{\rho-1}} < \infty, \quad \rho > 1,$$

which leads us to the contradiction.

Using inequality (3.24) and for $p \geq 2$ and $m \geq 2$, we obtain

$$\begin{aligned} H'(t) &\leq 2\|u_t\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \sigma \int_{\Omega} |\nabla u|^2 dx \\ &\leq 2\|u_t\|_{L^2(\Omega)}\|u\|_{L^{m(\cdot)}(\Omega)} + C\sigma\|\nabla u\|_{L^{p(\cdot)}(\Omega)}^2. \end{aligned} \quad (3.25)$$

According to (3.18), we get

$$\int_{\Omega} |u_t|^2 dx \leq C \int_{\Omega} |u|^{m(x)} dx, \quad \int_{\Omega} |\nabla u|^{p(x)} dx \leq C \int_{\Omega} |u|^{m(x)} dx$$

Then, the inequality (3.25) becomes

$$\begin{aligned} H'(t) &\leq C \left[\left(\int_{\Omega} |u|^m dx \right)^{\frac{1}{2}} + \left(\int_{\Omega} |u|^m dx \right)^{\frac{2}{p^-}} \right] \\ &\leq C \left(\int_{\Omega} |u|^m dx \right)^{\frac{2+p^-}{2p^-}}. \end{aligned}$$

Which implies that

$$\int_{\Omega} |u|^m dx \geq C \left[\frac{H'(0)}{\left(1 - \frac{t(\rho-1)}{C} (H'(0))^{\rho-1} \right)^{\frac{1}{\rho-1}}} \right]^{\frac{2p^-}{2+p^-}} \rightarrow \infty \text{ as } t \rightarrow T^*.$$

Therefore concluding the proof. ■

Theorem 3.4.2. *Let u be a weak solution of (3.1) with $\alpha = 0$ which satisfies (3.17) and (3.18). Then $\|u\|_{L^2(\Omega)}^2$ be unbounded in finite time interval $(0, T^*)$, with*

$$T^* = \frac{2\|u_0\|_{L^2(\Omega)}^2}{\kappa(u_0 - U_1) - \psi'(0)}.$$

Proof. We define the positive function ψ as

$$\psi(t) = \int_{\Omega} |u(x, t)|^2 dx.$$

Thus

$$\psi'(t) = 2 \int_{\Omega} u_t u dx.$$

Then

$$\begin{aligned} (\psi'(t))^2 &= 4 \left(\int_{\Omega} u_t u dx \right)^2 \\ &\leq 4 \left(\int_{\Omega} |u_t|^2 dx \right) \left(\int_{\Omega} |u|^2 dx \right) \\ &\leq 4\psi(t) \left(\int_{\Omega} |u_t|^2 dx \right). \end{aligned}$$

Which gives us

$$\int_{\Omega} |u_t|^2 dx \geq \frac{(\psi'(t))^2}{4\psi(t)}. \quad (3.26)$$

Thank's to (3.18) and (3.26) and for $\kappa > 2$, we have

$$\begin{aligned} \psi''(t) &= 2 \int_{\Omega} |u_t|^2 dx + 2\beta \int_{\Omega} |u|^{m(x)} dx - 2 \int_{\Omega} |\Delta u|^{p(x)} dx \\ &\geq \psi''(t) + 2\kappa J(t) \\ &\geq (\kappa + 2) \int_{\Omega} |u_t|^2 dx \\ &\geq (\kappa + 2) \frac{(\psi'(t))^2}{4\psi(t)}. \end{aligned}$$

This implies

$$\frac{\psi''(t)}{(\psi'(t))^2} \geq \frac{\kappa + 2}{4\psi(t)} \geq 0. \quad (3.27)$$

Taking into account (3.16) and (3.27), we get

$$\psi'(t) > 0, \quad t > 0.$$

Simple calculation applies to (3.27) leads us

$$\psi(t) \geq \frac{\|u_0\|_{L^2(\Omega)}^2}{\left(1 - t^{\frac{\kappa(u_0 - U_1) - \psi'(0)}{2\|u_0\|_{L^2(\Omega)}^2}}\right)^{\frac{4}{\kappa-2}}}, \quad \kappa > 2$$

therefore concluding the proof. ■

3.5 Conclusion

In this chapter, we studied the hyperbolic $p(x)$ -biharmonic problem under homogeneous Neumann boundary conditions. Using the Galerkin method, we proved the global existence of a weak solution and derived key a priori estimates. Moreover, we identified the conditions under which the solution may blow up in finite time when the initial energy is negative.

4

Approximation Studies of the Evolution p -Biharmonic Problem

4.1 Introduction

In this chapter, we are interested in the approximation study of the evolution p -biharmonic problem using the Rothe and mixed finite element methods.

4.2 Position of the Problem

We consider a bounded open domain Ω of $\mathbb{R}^n$, with a Lipschitz-continuous boundary Γ . In this chapter we are interested in the study of the following parabolic integro-differential p -biharmonic problem

$$\begin{cases} \frac{\partial u}{\partial t} + \Delta_p^2 u(t, x) = f(t, x) + \int_0^t k(t-s) \Delta_p^2 u(s, x) ds & \text{in } I \times \Omega, \\ u = 0, \nabla u = 0 & \text{on } I \times \Gamma, \\ u(0, x) = u_0 & \text{on } \Omega. \end{cases} \quad (4.1)$$

Where $\Delta_p^2 u = \Delta(|\Delta u|^{p-2} \Delta u)$ is the p -bilaplacian operator, $p > 1$, $u_0 \in W_0^{2,p}(\Omega)$, $k \in C(I)$ and $f \in C(I, L^q(\Omega))$ is a Lipschitz continuous function.

4.3 Semi-Discretized Problem

4.3.1 The Variational Formulation

Considering $v \in W_0^{2,p}(\Omega)$ as a test function, we multiply equation (4.1) by v and integrate over Ω , we obtain

$$(u_t, v) + (|\Delta u|^{p-2} \Delta u, \Delta v) = (f, v) + \left(\int_0^t k(t-s) |\Delta u|^{p-2} \Delta u, \Delta v \right), \quad \forall v \in W_0^{2,p}(\Omega) \quad (4.2)$$

4.3.2 The Semi-Discretized Formulation

We subdivide the time interval $I = [0, T]$ to n subintervals, $\tau = \frac{T}{n}$, $i = 1, \dots, n$ where $u_i = u(t_i, x)$, $t_i = i\tau$ and let

$$\delta u_i = \frac{u_i - u_{i-1}}{\tau}, \quad f_i(x) = f(t_i, x), \quad k_{ij} = k(t_i - t_j)$$

For $i = 1, \dots, n$, the recurrent semi discretized problem is

$$\left\{ \begin{array}{l} \text{Find } u_i \cong u(t_i, \cdot), i = 1, 2, \dots, n \text{ such that ,} \\ (\delta u_i, v) + (|\Delta u_i|^{p-2} \Delta u_i, \Delta v) = (f_i, v) \\ + \tau \sum_{j=0}^{i-1} k_{ij} (|\Delta u_j|^{p-2} \Delta u_j, \Delta v), \quad \forall v \in W_0^{2,p}(\Omega) \end{array} \right. \quad (4.3)$$

Theorem 4.3.1. *The problem (4.3) admits a unique weak solution $u_i \in W_0^{2,p}(\Omega)$, $\forall i = 1, \dots, n$.*

Proof. *Using monotone operator theory, we can arrive at the desired result. ■*

4.4 Necessary A Priori Estimates

In this part, we show some a priori estimates needed to prove the existence of a weak solution.

Lemma 4.4.1. *There exists $C > 0$ independent of n such that*

$$\|u_m\|_{L^2(\Omega)} \leq C, \quad \sum_{i=1}^m \tau \|\Delta u_i\|_{L^p(\Omega)}^p \leq C, \quad m = 1, \dots, n$$

Proof. Choosing in (4.3), $v = u_i$ and summing over $i = 1, \dots, m$ with $1 \leq m \leq n$, we obtain

$$\begin{aligned} \sum_{i=1}^m (u_i - u_{i-1}, u_i) + \tau \sum_{i=1}^m \|\Delta u_i\|_{L^p(\Omega)}^p &= \tau^2 \sum_{i=1}^m \left(\sum_{j=0}^{i-1} k_{ij} |\Delta u_j|^{p-2} \Delta u_j, \Delta u_i \right) \\ &\quad + \tau \sum_{i=1}^m (f_i, u_i). \end{aligned} \quad (4.4)$$

The first term on the LHS of (4.4) can be estimates as

$$\sum_{i=1}^m (u_i - u_{i-1}, u_i) = \frac{1}{2} \sum_{i=1}^m \|u_i - u_{i-1}\|_{L^2(\Omega)}^2 + \frac{1}{2} \|u_m\|_{L^2(\Omega)}^2 - \frac{1}{2} \|u_0\|_{L^2(\Omega)}^2.$$

Hölder and Young inequalities give us

$$\begin{aligned} \left| \tau^2 \sum_{i=1}^m \left(\sum_{j=0}^{i-1} k_{ij} |\Delta u_j|^{p-2} \Delta u_j, \Delta u_i \right) \right| &\leq C \sum_{i=1}^m \sum_{j=0}^{i-1} \tau^2 (|\Delta u_j|^{p-1}, |\Delta u_i|) \\ &\leq C \sum_{i=1}^m \sum_{j=0}^{i-1} \tau^2 \|\Delta u_j\|_{L^p(\Omega)}^{\frac{p}{q}} \|\Delta u_i\|_{L^p(\Omega)} \\ &\leq C \sum_{i=1}^m \sum_{j=0}^{i-1} \tau^2 \|\Delta u_j\|_{L^p(\Omega)}^p + C\epsilon \sum_{i=1}^m \tau \|\Delta u_i\|_{L^p(\Omega)}^p. \end{aligned}$$

Similarly, the estimation for the last term in the RHS of (4.4) is given as follows

$$\tau \left| \sum_{i=1}^m (f_i, u_i) \right| \leq C + C\epsilon \sum_{i=1}^m \tau \|\Delta u_i\|_{L^p(\Omega)}^p.$$

Combining everything, we arrive at

$$\|u_m\|_{L^2(\Omega)}^2 + (1 - C\epsilon) \sum_{i=1}^m \tau \|\Delta u_i\|_{L^p(\Omega)}^p \leq C \sum_{i=1}^m \sum_{j=0}^{i-1} \tau^2 \|\Delta u_j\|_{L^p(\Omega)}^p + \|u_0\|_{L^2(\Omega)}^2 + C.$$

Since ϵ can be chosen arbitrarily small to ensure $(1 - C\epsilon) > 0$, the application of Gronwall's lemma completes the proof. ■

Lemma 4.4.2. *There exists a positive constant C , such that*

$$\|\delta u_n\|_{W^{-2,q}(\Omega)} \leq C$$

Proof. let us introduce Rothe functions

$$\begin{cases} u_n(0) = u_0 \\ u_n(t) = u_{i-1} + (t - t_{i-1})\delta u_i, \quad 1 \leq i \leq n, \quad \forall t \in (t_{i-1}, t_i], \end{cases}$$

Together with step functions,

$$\begin{cases} \overline{u_n}(0) = u_0, \quad \overline{u_n}(t) = u_i, \quad \forall t \in (t_{i-1}, t_i], i = 1, \dots, n, \\ \overline{f_n}(0) = f(0), \quad \overline{f_n}(t) = f(t_i), \quad \forall t \in (t_{i-1}, t_i], i = 1, \dots, n. \end{cases}$$

Then, the equation (4.3) becomes

$$(\delta u_n(t), v) + (\Delta_p^2 \overline{u_n}(t), v) = (\overline{f_n}(t), v) + \tau \sum_{j=0}^{i-1} k_{ij} (\Delta_p^2 u_j, v), \quad \forall v \in W_0^{2,p}(\Omega). \quad (4.5)$$

Since $\Delta_p^2 : W_0^{2,p}(\Omega) \longrightarrow W^{-2,q}(\Omega)$ is bounded for $p > 2$ (see [25]), Hölder and ϵ -Young inequalities yield:

$$\begin{aligned} |(\delta u_n(t), v)| &\leq \left| (\overline{f_n}, v) \right| + \tau \sum_{j=0}^{i-1} k_{ij} (|\Delta u_j|^{p-1}, |\Delta v|) + \left| (\Delta_p^2 \overline{u_n}(t), v) \right| \\ &\leq \left\| \overline{f_n} \right\|_{L^q(\Omega)} \|v\|_{L^p(\Omega)} + \tau C \sum_{j=0}^{i-1} \left\| \Delta u_j \right\|_{L^p(\Omega)}^{\frac{p}{q}} \|\Delta v\|_{L^p(\Omega)} \\ &\quad + C \|v\|_{L^p(\Omega)} \\ &\leq \left[C \|f\|_{C(I, L^q(\Omega))} + \left(\frac{C}{\epsilon} \sum_{j=0}^{i-1} \tau \left\| \Delta u_j \right\|_{L^p(\Omega)}^p + \sum_{j=0}^{i-1} C \epsilon \tau |\Omega| \right) \right] \|v\|_{W_0^{2,p}(\Omega)}. \end{aligned} \quad (4.6)$$

Taking into account Lemma (4.4.1), and putting $\epsilon = 1$ in (4.6), we arrive at

$$|(\delta u_n(t), v)| \leq C \|v\|_{W_0^{2,p}(\Omega)}, \quad \forall v \in W_0^{2,p}(\Omega).$$

Thus

$$\|\delta u_n\|_{W^{-2,q}(\Omega)} = \sup_{\|v\|_{W_0^{2,p}(\Omega)} \leq 1} |(\delta u_n(t), v)| \leq C.$$

Hence, the proof is accomplished. ■

4.5 Existence

Now we will show the existence of a weak solution to the problem (4.1).

Theorem 4.5.1. *Let $p > 2$, $u_0 \in W_0^{2,p}(\Omega)$, $k \in C(I)$ and $f \in C(I, L^q(\Omega))$ be a Lipschitz continuous function. Then, problem (4.1) admits a weak solution $u \in C(I, W^{-2,q}(\Omega)) \cap L^p(I, W_0^{2,p}(\Omega))$ with $u_t \in L^q(I, W^{-2,q}(\Omega))$.*

Proof. The precedent lemmas allow us to conclude that

$$\|\partial_t u_n\|_{L^q(I, W^{-2,q}(\Omega))} \leq C, \quad \|\overline{u_n}\|_{L^p(I, W_0^{2,p}(\Omega))}^p \leq C \text{ and } \|\overline{u_n}\|_{C(I, L^2(\Omega))} \leq C$$

Then, the ([32], Lemma 1.3.13) implies that there exists a function u and a subsequence of $\{u_n\}_n$ denote $\{u_n\}_n$, such that

$$\begin{aligned} u_n &\longrightarrow u \quad \text{in } C([0, T], W^{-2,q}(\Omega)) \\ \partial_t u_n &\rightharpoonup \partial_t u \quad \text{in } L^q([0, T], W^{-2,q}(\Omega)) \\ u_n(t) &\rightharpoonup u(t) \quad \text{in } W_0^{2,p}(\Omega) \\ \overline{u_n} &\rightharpoonup u \quad \text{in } L^p([0, T], W_0^{2,p}(\Omega)). \end{aligned}$$

Taking into account the assumptions of f , we get

$$\left\| \overline{f_n}(t) - f(t) \right\|_{L^q(\Omega)} \leq \frac{C}{n},$$

this yields

$$\overline{f_n} \rightharpoonup f \text{ in } L^q([0, T], L^q(\Omega)).$$

According the hemicontinuous of $\Delta_p^2 : W_0^{2,p}(\Omega) \longrightarrow W^{-2,q}(\Omega)$, we obtain

$$(\Delta_p^2 \overline{u_n}(t), v) \longrightarrow (\Delta_p^2 u(t), v), \quad \forall v \in W_0^{2,p}(\Omega),$$

as $n \rightarrow \infty$.

Proceeding similarly as in [25], we get

$$\tau \left(\sum_{j=0}^{i-1} k_{ij} \Delta_p^2 u_j, v \right) \rightharpoonup \int_0^t k(t-s) \Delta_p^2 u(s, x) ds \text{ in } L^q([0, T], W^{-2,q}(\Omega)) \text{ as } \tau \rightarrow 0.$$

Integrating (4.5) over $(0, T)$ and passing to the limit as $n \rightarrow \infty$, we arrive at

$$\int_0^T (u_t, v) dt + \int_0^T (\Delta_p^2 u, v) dt = \int_0^T (f, v) dt + \int_0^T \left(\int_0^t k(t-s) \Delta_p^2 u(s, x) ds, v \right) dt, \quad \forall v \in W_0^{2,p}(\Omega).$$

Differentiating with respect to t , we conclude the desired result. ■

4.6 Uniqueness of Weak Solution

In this part, we will show the uniqueness of a weak solution to (4.1).

Theorem 4.6.1. *For $1 < p \leq 2$, there exists at most one weak solution to problem (4.1).*

Proof. Let u_1 and u_2 be two solution to problem (4.1). Rewriting the equation (4.2) for u_1 and u_2 respectively, subtracting the second from the first, we obtain

$$\begin{aligned} & \left(\partial_t (u_1 - u_2), v \right) + \left(|\Delta u_1|^{p-2} \Delta u_1 - |\Delta u_2|^{p-2} \Delta u_2, \Delta v \right) \\ &= \left(\int_0^t k(t-s) (|\Delta u_1|^{p-2} \Delta u_1 - |\Delta u_2|^{p-2} \Delta u_2) ds, \Delta v \right). \end{aligned} \quad (4.7)$$

Taking into account Lemma (1.3.7) and testing (4.7) with $u_1 - u_2$, then integrating the result from 0 to ζ (with $\zeta \in [0, T]$), we get

$$\begin{aligned} & \|u_1(\zeta) - u_2(\zeta)\|_{L^2(\Omega)}^2 + C_1 \int_0^\zeta \|\Delta(u_1 - u_2)\|_{L^p(\Omega)}^p dt \\ & \leq \int_0^\zeta \left(\int_0^t k(t-s) (|\Delta u_1|^{p-2} \Delta u_1 - |\Delta u_2|^{p-2} \Delta u_2) ds, \Delta(u_1 - u_2) \right) dt, \end{aligned}$$

where $u_1(0) = u_2(0)$.

Thus

$$C_1 \int_0^\zeta \|\Delta(u_1 - u_2)\|_{L^p(\Omega)}^p dt \leq \int_0^\zeta \left(\int_0^t k(t-s) (|\Delta u_1|^{p-2} \Delta u_1 - |\Delta u_2|^{p-2} \Delta u_2) ds, \Delta(u_1 - u_2) \right) dt.$$

Using Lemma (1.3.7), together with Hölder and ϵ -Young inequalities, and choosing ϵ such that $(C_1 - C\epsilon) > 0$, we obtain

$$\begin{aligned} & (C_1 - C\epsilon) \int_0^\zeta \|\Delta(u_1 - u_2)\|_{L^p(\Omega)}^p dt \\ & \leq \frac{C}{\epsilon} \int_0^\zeta \int_\Omega \left(\int_0^t k(t-s) (|\Delta u_1|^{p-2} \Delta u_1 - |\Delta u_2|^{p-2} \Delta u_2) ds \right)^q dx dt \end{aligned}$$

Thus

$$\begin{aligned}
 & (C_1 - C\epsilon) \int_0^\zeta \|\Delta(u_1 - u_2)\|_{L^p(\Omega)}^p dt \\
 & \leq \frac{C}{\epsilon} \int_0^\zeta \int_\Omega \left(\|k\|_{C(I)} \int_0^t (|\Delta u_1|^{p-2} \Delta u_1 - |\Delta u_2|^{p-2} \Delta u_2) ds \right)^q dx dt \\
 & \leq \frac{C}{\epsilon} \int_0^\zeta \int_\Omega \left(\|k\|_{C(I)} T^{\frac{1}{p}} \left(\int_0^t |\Delta u_1|^{p-2} \Delta u_1 - |\Delta u_2|^{p-2} \Delta u_2|^q ds \right)^{\frac{1}{q}} \right)^q dx dt \\
 & \leq C \int_0^\zeta \int_\Omega \int_0^t |\Delta(u_1 - u_2)|^{(p-1)q} ds \\
 & \leq C \int_0^\zeta \int_\Omega \int_0^t |\Delta(u_1 - u_2)|^p ds dx dt.
 \end{aligned}$$

Applying Gronwall's lemma, we get

$$\int_0^\zeta \|u_1 - u_2\|_{W^{2,p}(\Omega)}^p dt = 0, \quad \forall \zeta \in [0, T],$$

which leads us to conclude that $u_1 = u_2$ in $L^p([0, T], W_0^{2,p}(\Omega))$. ■

4.7 Mixed Formulation

In this part, we analyze the mixed formulation using the observation that if $\Psi(\omega) = |\omega|^{p-2}\omega$, then its inverse satisfies

$$\Psi^{-1}(w) = \text{sgn}(w) |w|^{\frac{1}{p-1}} w = |w|^{q-2} w,$$

which motivates the introduction of the auxiliary variable

$$\lambda_i = |\Delta u_i|^{p-2} \Delta u_i$$

Let $X = W_0^{2,p}(\Omega)$ and $Y = L^q(\Omega)$. Then the problem (4.1) can be rewritten as follows

$$\begin{cases} -\Delta u_i = |\lambda_i|^{q-2} \lambda_i, \\ -\Delta \lambda_i = -f_i + \delta u_i - \tau \sum_{j=1}^{i-1} k_{ij} \Delta \lambda_j. \end{cases} \quad (4.8)$$

The mixed variational problem of (4.8) writes as follows:

$$\begin{cases} \text{Find } (u_i, \lambda_i) \in X \times Y \text{ such that} \\ \alpha(\lambda_i, v) - \beta(u_i, v) = 0 \quad \forall v \in X, \\ \beta(\lambda_i, \phi) = L_Y(\phi) \quad \forall \phi \in Y, \end{cases} \quad (4.9)$$

Where the bilinear and linear forms are defined as:

$$\begin{aligned} \alpha(\lambda_i, v) &:= (|\lambda_i|^{q-2} \lambda_i, v), \\ \beta(\lambda_i, \phi) &:= -(\Delta \lambda_i, \phi), \\ L_Y(\phi) &:= (-f_i + \delta u_i, \phi) + \tau \sum_{j=1}^{i-1} k_{ij} \beta(\lambda_j, \phi) \quad . \end{aligned}$$

Proposition 4.7.1. (*Inf-sup Condition*) *There exist positive constants κ and M , such that*

$$\kappa \leq M \inf_{0 \neq \phi \in Y} \sup_{0 \neq u_i \in X} \frac{\beta(u_i, \phi)}{\|u_i\|_X \|\phi\|_Y}.$$

Proof. For the proof see [23].

4.8 Full Discretization

Let Υ_T be a triangulation consisting of triangular elements T where the intersection of any two distinct elements is either a vertex, an edge, or empty.

This triangulation satisfies the regularity condition in the sense of Ciarlet, meaning that there exists a constant $\mu > 0$ such that:

$$\exists \mu > 0; \mu = \inf_{T \in \Upsilon_h} \frac{h_T}{\rho_T} \quad \forall T \in \Upsilon_h,$$

where ρ_T is the diameter of the largest ball contained inside T and h_T is the diameter of T .

Let $\mathbf{P}_m(\Upsilon_h)$ express the space of piecewise polynomials of degree m over the triangulation Υ_h

$$\mathbf{P}_m(\Upsilon_h) = \{\varphi : \varphi|_T \in \mathbf{P}_m(T), \quad \forall T \in \Upsilon_h\}.$$

The discrete finite spaces is given by

$$X^h = \mathbf{P}_m(\Upsilon_h) \cap C^0(\overline{\Omega}),$$

and

$$X_0^h = \{\varphi \in X^h; \varphi|_{\partial\Omega} = 0\},$$

where R is the Ritz projection operator such that

$$\int_{\Omega} \nabla(Rv) \nabla \varphi dx = \int_{\Omega} \nabla v \nabla \varphi dx, \quad \forall \varphi \in X^h \cap H_0^1(\Omega).$$

We give the mesh size h as

$$h = \max_{T \in \Upsilon_h} h_T.$$

Therefore , the fully-discrete mixed finite element scheme for (4.9) becomes

$$\left\{ \begin{array}{l} \text{Find } (u_i^h, \lambda_i^h) \in X_0^h \times X^h \text{ such that} \\ \alpha(\lambda_i^h, v) - \beta_h(u_i^h, v) = 0, \\ \beta_h(\lambda_i^h, \phi) = L(\phi) \quad \forall (v, \phi) \in X^h \times X_0^h, \end{array} \right.$$

This implies

$$\left\{ \begin{array}{l} \text{Find } (u_i^h, \lambda_i^h) \in X_0^h \times X^h \text{ such that} \\ (|\lambda_i^h|^{q-2} \lambda_i^h, v) - (\nabla u_i^h, \nabla v) = 0, \\ (\nabla \lambda_i^h, \nabla \phi) = (f_i - \delta u_i, \phi) + \sum_{j=1}^{i-1} k_{ij} (\nabla \lambda_j, \nabla \phi) \quad \forall (v, \phi) \in X^h \times X_0^h. \end{array} \right.$$

Theorem 4.8.1. *For $m \geq 2$, there exists a positive constant C such that*

$$\begin{aligned} \|u_i - u_i^h\|_{W_h^{2,p}(\Omega)}^{p-1} + \|w_i - w_i^h\|_{L^q(\Omega)} &\leq C \left(h^{\frac{q}{2}(m+1)} \|w_i\|_{W^{m+1,q}(\Omega)}^{\frac{q}{2}} + h^{m+1} \|w_i\|_{W^{m+1,q}(\Omega)} \right. \\ &\quad \left. + h^{m-1} \|u_i\|_{W^{m+1,p}(\Omega)} + h^{m+1} \|\delta u_i\|_{W^{m+1,q}(\Omega)} \right). \end{aligned}$$

proof. see [23, 43] .

4.9 Conclusion

In this work, we studied a high-order parabolic p -biharmonic equation with a memory term. By combining the Rothe method and the mixed finite element method, we established the existence and uniqueness of a weak solution and derived a fully discrete numerical scheme. The obtained a priori estimates ensured the stability and convergence of the proposed method.

Conclusion and Perspectives

In this thesis, we proposed three studies to analyze high-order parabolic and hyperbolic problem, both theoretically and numerically.

In the first work, we solved the integro-differential diffusion equation. We obtained a variational formulation in which the function u is the only unknown function because the unknown Dirichlet boundary condition ρ in the inverse problem was eliminated using the described integral measurement. We proved the existence and uniqueness of the weak solution of problem. Using Rothe's method and the theory of monotone operators.

For the second work, we investigated a high-order hyperbolic $p(x)$ -bilateral equation with a variable exponent and a damping term. We studied the existence of weak solutions and conditions that may lead to finite-time blow-up.

For the third work, We focused our attention on studying the parabolic p -biharmonic equation with a memory term, we showed the existence and uniqueness of weak solution. To this end, we employed the Rothe method and the mixed finite element method.

Then, we may focus on various methods, both theoretical and numerical, for studying and analyzing high-order PDEs. In this direction, we may also consider the role of

artificial intelligence and quantum computing, which are increasingly important for the efficient treatment of nonlinear, memory-driven, degenerate, or integro-differential PDEs that are otherwise intractable using classical techniques.

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Research Activities

INTERNATIONAL PUBLICATIONS

- Khalfallaoui, R., Chaoui, A., Djaghout, M., On the solution of evolution p -Laplace equation with memory term and unknown boundary Dirichlet condition, J Elliptic Parabol Equ, (2024). <https://doi.org/10.1007/s41808-024-00290-8>
- Khalfallaoui, R., Chaoui, A., Zennir, K., Mirgani, S.M., Advanced qualitative results on hyperbolic $p(\cdot)$ -biharmonic equation with no flux boundary condition. Accepted in AIMS Mathematics, (2025).

INTERNATIONAL COMMUNICATIONS

- Khalfallaoui, R., Chaoui, A., *High-order hyperbolic equation with variable exponent*,

the 6th International Conference on Mathematics, Istanbul, Turkey, 21-24/6/2022.

- Khalfallaoui, R., Chaoui, A., *Hyperbolic equation $p(x)$ -Laplacian: theoretical and numerical studies*, the 6Th International Workshop on Applied Mathematics and Modelling (WIMAM 2022), University of 8 May 1945, Guelma, 26-27/10/2022.

- Khalfallaoui, R., Chaoui, A., *High-order hyperbolic equation with non standard anisotropic growth condition*, the 7Th International Workshop on Applied Mathematics and Modelling (WIMAM 2023), University of 8 May 1945, Guelma, 13-14/12/2023.

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NATIONAL COMMUNICATIONS

- Khalfallaoui, R., Chaoui, A., *High-order hyperbolic equation: theoretical and numerical studies*, Third National Mathematics Seminar 2022 of the Freres Mentouri University, Constantine, 26/05/2022.