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Stress Analysis in Straight and Inclined Sections of Continuous Beams and Frames Using the Force Method: A Comparative Study with the Finite Element Method

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بسم الله الرحمن الرحيم

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I would like to express my heartfelt gratitude to my dear parents for their unwavering moral support and the strength they gave me to overcome the many challenges I faced throughout my years of study. Their encouragement was instrumental in enabling me to complete this modest project.

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Dedications

To my dear parents, for all their sacrifices, love, tenderness, constant support, and prayers throughout my years of study.

To my dear brothers, for their unwavering encouragement and support.

To all my family, for standing by me throughout this academic journey.

May this work be the fulfillment of your cherished hopes and a reflection of your unfailing support.

Thank you for always being there for me.

Abstract

This Master’s thesis presents a detailed comparative study between two essential approaches used in the analysis of statically indeterminate structures: the classical Force Method and the Finite Element Method (FEM). The investigation focuses on continuous beams and planar frame structures, exploring their theoretical principles, step-by-step procedures, and practical implementation. Through a series of carefully developed case studies and numerical evaluations, the Force Method is highlighted for its exact analytical solutions and pedagogical value, while the Finite Element Method is distinguished by its computational versatility and its capacity to handle structures with complex geometries and varied loading scenarios. The results of this study demonstrate the complementary nature of both methods: the Force Method provides precise, physically meaningful outcomes, whereas FEM offers highly accurate approximations supported by detailed graphical outputs of internal forces, stress and structural deformations. By combining traditional analytical techniques with modern numerical tools, this work underscores the value of integrating classical and contemporary methods to enhance structural analysis, promote deeper understanding, and support more robust engineering decisions.

الملخص

تقدم أطروحة الماجستير هذه دراسة مقارنة مفصلة بين منهجين أساسيين يستخدمان في تحليل الهياكل غير المحددة استاتيكيًا: طريقة القوة الكلاسيكية وطريقة العناصر المحدودة (FEM). يركز البحث على الأعمدة المتصلة وهياكل الإطارات المستوية، ويستكشف مبادئها النظرية وإجراءاتها التدريجية وتطبيقها العملي. من خلال سلسلة من دراسات الحالة المطورة بعناية والتقييمات العددية، يتم تسليط الضوء على طريقة القوة لحلولها التحليلية الدقيقة وقيمتها التربوية، في حين تتميز طريقة العناصر المحدودة بتعدد استخداماتها الحسابية وقدرتها على التعامل مع الهياكل ذات الأشكال الهندسية المعقدة وسيناريوهات التحميل المتنوعة. تُظهر نتائج هذه الدراسة الطبيعة التكاملية لكلتا الطريقتين: توفر طريقة القوة نتائج دقيقة وذات مغزى فيزيائيًا، بينما تقدم طريقة العناصر المحدودة تقديرات تقريبية دقيقة للغاية مدعومة بمخرجات بيانية مفصلة للقوى الداخلية والإجهاد والتشوهات الهيكلية. من خلال الجمع بين الأساليب التحليلية التقليدية والأدوات العددية الحديثة، يؤكد هذا العمل على قيمة دمج الأساليب الكلاسيكية والمعاصرة لتعزيز التحليل الإنشائي وتعزيز الفهم الأعمق ودعم قرارات هندسية أكثر قوة..

Résumé

Ce mémoire de Master présente une étude comparative approfondie entre deux approches fondamentales utilisées pour l’analyse des structures hyperstatiques : la méthode classique des forces et la méthode des éléments finis (MEF). L’étude porte sur les poutres continues et les portiques plans, en examinant leurs fondements théoriques, les étapes méthodologiques, ainsi que leur mise en œuvre pratique. À travers une série d'études de cas rigoureusement développées et d’évaluations numériques, la méthode des forces se distingue par la précision de ses solutions analytiques et sa valeur pédagogique, tandis que la méthode des éléments finis se caractérise par sa souplesse computationnelle et sa capacité à modéliser des structures ayant des géométries complexes sous diverses conditions de chargement. Les résultats de cette étude mettent en évidence la complémentarité des deux approches : la méthode des forces fournit des résultats exacts et physiquement interprétables, alors que la MEF offre des approximations fiables enrichies par des représentations graphiques détaillées des efforts internes, des contraintes et des déformations structurelles. En combinant les techniques analytiques traditionnelles avec les outils numériques modernes, ce travail souligne l’importance d’intégrer les méthodes classiques et contemporaines pour enrichir l’analyse structurelle, approfondir la compréhension et soutenir une prise de décision plus robuste en ingénierie des structures.

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General Introduction

Structural engineering plays a vital role in ensuring the safety, reliability, and efficiency of buildings, infrastructure, and mechanical systems. A central challenge in this field is the accurate analysis of statically indeterminate structures—those whose internal forces cannot be determined using only the basic equations of static equilibrium. To overcome this challenge, engineers and researchers have developed various analytical and numerical techniques. Among the most prominent and widely used are the Force Method and the Finite Element Method (FEM).

The Force Method, also known as the Method of Consistent Deformations, is a classical analytical approach that transforms a statically indeterminate structure into a determinate one by removing selected redundant forces. These redundants are then computed by applying compatibility conditions derived from material behavior and deformation theory. This method provides exact analytical solutions and is particularly well-suited for educational purposes and relatively simple structural systems, offering deep physical insight into internal force behavior and support reactions.

In contrast, the Finite Element Method is a powerful numerical technique that discretizes a structure into smaller elements and assembles a global system of equations using matrix operations. FEM is especially effective for analyzing structures with complex geometries, mixed boundary conditions, and heterogeneous material properties. When implemented through computational software such as ANSYS, ABAQUS, or RDM6, FEM provides highly detailed graphical and numerical outputs for internal forces, displacements, and stress fields with a high degree of accuracy.

This thesis presents a comparative study of the Force Method and the Finite Element Method, as applied to continuous beams and planar frame structures subjected to various external loads. The aim is to demonstrate the consistency, advantages, and limitations of each approach through theoretical exposition, detailed worked examples, and graphical interpretation.

The structure of the document is as follows: Abstract, General Introduction, General Conclusion, Bibliographic References, and three main chapters, each progressively addressing more complex structural configurations and deepening the comparative analysis.

Chapter 1: Fundamental Concepts and Theoretical Framework

This introductory chapter reviews the mathematical foundations and theoretical principles of both the Force Method and FEM. It discusses the formulation of statically indeterminate problems, the selection of redundants, and the derivation of compatibility equations in the Force Method. It also introduces the core concepts of FEM, including element discretization, stiffness matrix formulation, application of boundary conditions, and numerical solution techniques. The chapter concludes with a bibliographic overview of established works in structural analysis and outlines the objectives of this comparative study.

Chapter 2: Application of the Force Method and FEM to Continuous Beams

This chapter analyzes continuous beams subjected to various external loads. The Force Method is first applied to determine support reactions, internal forces, and stress distributions. The same beam is then re-analyzed using FEM via RDM6 software. Numerical and graphical results from both methods are compared to assess their consistency and reliability. These examples illustrate the basic implementation and validation of classical and numerical methods.

Chapter 3: Comparative Analysis of Planar Frame Structures

The final chapter extends the analysis to planar frames composed of multiple members and various load configurations. It provides a comprehensive application of the Force Method, including the construction of canonical equations, use of moment-area techniques, and superposition of effects. The same frames are then evaluated using FEM, with a detailed comparison of internal force diagrams and stress distributions. The chapter emphasizes FEM's strength in handling geometric and loading complexity, while reaffirming the accuracy of the Force Method when applied correctly. The discussion concludes with practical reflections on the implementation and integration of both methods in engineering design.

Conclusion of the Introduction

The combined use of the Force Method and FEM not only enables cross-verification of structural results but also illustrates the complementarity of traditional analytical and modern numerical approaches. While the Force Method offers clarity, analytical rigor, and physical insight, FEM provides flexibility, scalability, and computational efficiency—qualities essential to modern structural engineering.

This study aims to provide students and engineers with a robust understanding of both methodologies, helping them select the most appropriate technique based on structural complexity, available tools, and the required level of precision. Moreover, it lays the groundwork for future studies involving dynamic loading, nonlinear material behavior, and hybrid computational strategies.

Chapter 1

General Principles of the Force Method and the Finite Element Method Accompanied by Bibliographic Research

Chapter 1: *General Principles of the Force Method and the Finite Element Method Accompanied by Bibliographic Research*

1. Introduction

The analysis of stresses in structural elements such as beams and frames is a cornerstone of engineering mechanics and structural engineering. These elements are fundamental in civil, mechanical, and aerospace structures, as they are designed to safely withstand various loads while maintaining stability and functionality. In practice, structures rarely consist of isolated members; they are usually continuous systems with interconnected elements that experience complex internal force distributions. Moreover, loading and geometry are often not aligned with the principal axes, making it essential to analyze stress states in both straight and inclined sections.

This chapter introduces the fundamental concepts and techniques required to analyze internal forces and corresponding stresses in continuous beams and frames, with a special focus on inclined sections, which frequently arise in real-world structural systems. Inclined sections are particularly relevant in the design of sloped roofs, bridge girders, trusses, and off-axis loadings in mechanical structures. Stress analysis in these sections requires careful treatment of both normal and shear forces due to the inclination, making it a more advanced and insightful case for engineering students.

The force method, also known as the method of consistent deformations, is emphasized as the primary analytical approach in this chapter. It is a classical yet powerful technique rooted in equilibrium and compatibility conditions. The method is especially suitable for statically indeterminate structures, where internal forces cannot be determined using equilibrium equations alone. By selecting appropriate redundant forces and applying compatibility conditions through flexibility relationships, the Force Method allows for an exact and systematic solution of internal stresses in complex structures.

To enhance and reinforce the practical relevance of this method, a comparative study with the finite element method (FEM) is included. FEM has become a standard tool in engineering practice due to its flexibility, generality, and ability to handle arbitrary geometries, boundary conditions, and material properties. However, its application often lacks the physical insight provided by classical methods. This comparative study aims to:

- ✓ Highlight the underlying assumptions of both methods;
- ✓ Identify potential sources of discrepancies;
- ✓ Emphasize the importance of validation through independent analytical solutions;
- ✓ And most importantly, show how classical techniques complement numerical methods in professional practice.

Through this dual perspective, we will not only gain proficiency in both techniques but also develop critical thinking in selecting appropriate methods for different engineering problems. Additionally, the work will show how stress analysis is deeply connected to real-world engineering design, including examples from bridge structures, building frameworks, and mechanical assemblies. Emphasis will be placed on using graphical representations, structural interpretation, and detailed worked-out examples to illustrate the mechanics and behavior of structures under various loading and support conditions.

By the end of this chapter, our work should be able to present the following advantages:

- ✓ Perform stress analysis in both straight and inclined sections of continuous beams and frames using the Force Method;
- ✓ Understand the role and limitations of statical determinacy and redundancy;
- ✓ Interpret results from FEM in comparison with classical methods;
- ✓ And appreciate the role of both analytical and numerical methods in modern structural engineering.

2. Stress analysis in straight and inclined sections

Stress analysis is a fundamental task in the design and verification of structural components [1]. It provides insights into how internal forces are distributed within a structure and how these forces translate into stresses, which are critical in determining structural safety and performance. While straight sections are often used as the basis for introductory analysis, inclined sections are equally important in real-world applications and demand a more nuanced approach.

2.1. Straight sections

A straight (or vertical/horizontal) section is a cross-section that is perpendicular to the longitudinal axis of a beam or structural member. These sections are typically aligned with the global coordinate axes, making the analysis relatively straightforward [2]. For such sections:

- ✓ Normal stress due to axial force is given by $\sigma = N/A$;
- ✓ Bending stress is evaluated using $\sigma = My/I$;
- ✓ Shear stress from transverse forces is computed via $\tau = VQ/Ib$.

where N is the axial force, M is the bending moment, V is the shear force, A is the cross-sectional area, I is the moment of inertia, Q is the first moment of area, and b is the width of the section at the point of interest.

These relations are widely used and provide clear interpretations of how loads result in internal stresses under the assumption of linear elastic behavior and small deformations.

2.2. Inclined sections

In practice, engineers frequently encounter inclined sections, either because the structure itself is inclined (e.g., sloped beams, diagonal braces, roof rafters), or because an analysis requires studying stress behavior along an arbitrary plane [2]. Stress analysis in these sections is essential for:

- ✓ Understanding failure along non-orthogonal planes (e.g., cracks that develop at an angle);
- ✓ Analyzing shear connectors, welded joints, or composite materials;
- ✓ Evaluating sections of beams subject to torsion or off-axis loads.

Inclined sections pose additional complexity because internal forces must be transformed into components acting along the inclined plane. The analysis relies on the transformation of stress using Mohr’s circle or analytical expressions.

Let an inclined plane be oriented at an angle θ with respect to the horizontal axis. The normal stress σ_θ and shear stress τ_θ acting on that inclined plane are given by:

$$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta),$$
$$\tau_\theta = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta),$$

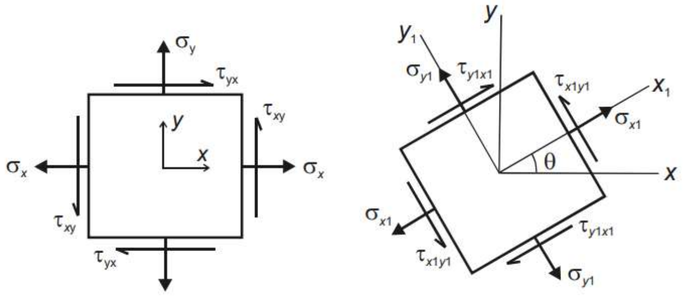


Fig. 1: Representation of the stress state in straight and inclined sections

where σ_x, σ_y are normal stresses along the principal axes and τ_{xy} is the shear stress on the original plane.

This transformation allows engineers to determine:

- ✓ The maximum normal stress, which governs yielding in materials obeying normal stress-based failure theories;
- ✓ The maximum shear stress, which is critical in ductile failure and in evaluating joint performance.

2.3.Applications and importance

Stress analysis in inclined sections is particularly important in:

- ✓ Failure prediction, since cracks and fractures often propagate along planes of maximum shear or tensile stress;
- ✓ Design of connections, where inclined welds or fasteners must carry combined loads;
- ✓ Composite materials and anisotropic structures, where strength properties vary with direction.

In structural elements like continuous beams and frames, internal forces and moments can vary significantly along the length and direction of the members. Analyzing inclined sections reveals how these forces translate into stress components that influence deformation and failure.

2.4. Educational and practical benefits

For practicing engineers, this type of analysis ensures reliable design, particularly in areas of high stress concentration, complex geometries, or multi-directional loading.

- ✓ Reinforces their understanding of the tensorial nature of stress;

- ✓ Builds competence in graphical and analytical methods (e.g., Mohr's circle, stress transformation equations);
- ✓ Prepares them for advanced studies in material mechanics, finite element analysis, and structural design.

2.5. Worked examples

Example 1: straight section in a beam

A simply supported beam of length ***L=4 m*** carries a uniform distributed load of ***w=5 kN/m***. Determine the normal and shear stresses at the mid-span in a vertical section of the beam, at a point located ***y=30 mm*** from the neutral axis. The beam has a rectangular cross-section of ***b=100 mm*** and ***h=200 mm***.

Solution:

1. Reactions: $R_A = R_B = \frac{wL}{2} = 10\,\mathrm{kN}$
2. Shear force at mid-span: $V = R_A - w \cdot \frac{L}{2} = 0$
3. Moment at mid-span:
   $M = \frac{wL^2}{8} = \frac{5 \cdot 4^2}{8} = 10\,\mathrm{kN \cdot m}$
4. Moment of inertia:
   $I = \frac{bh^3}{12} = \frac{100 \cdot 200^3}{12} = 66.67 \times 10^6\,\mathrm{mm}^4$
5. Bending stress:
   $\sigma = \frac{My}{I} = \frac{10^7 \cdot 30}{66.67 \times 10^6} = 4.5\,\mathrm{MPa}$
6. **Shear stress** is zero at mid-span due to zero shear force.

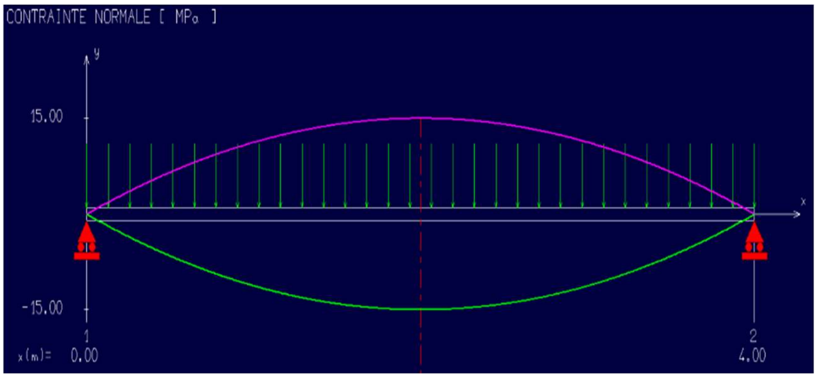


Fig. 2. Stress distribution along the entire beam shown in the previous example

Example 2: Inclined section in a beam

In the same beam considered previously, examine an inclined section at mid-span that forms an angle $\theta = 45°$ with the horizontal axis. Use the normal stress $\sigma_x = 4.5\ MPa$ and assume $\sigma_y = 0$ and $\tau_{xy} = 0$. Compute the normal and shear stresses on the inclined plane.

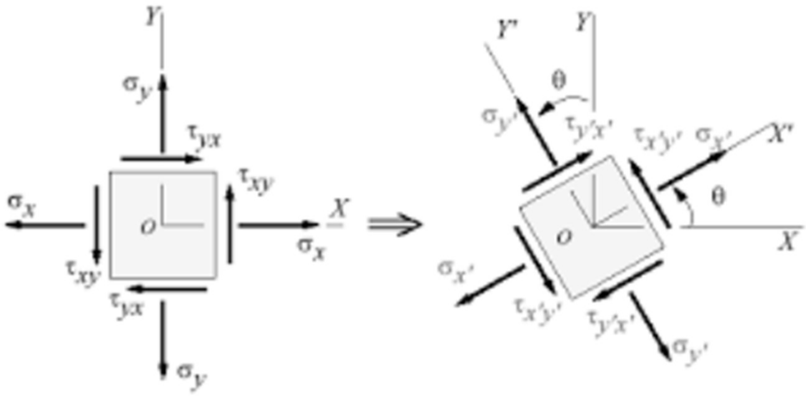


Fig 3. Representation of the stress on elements oriented at 0° and 45°

Solution:

$$
\begin{aligned}
\sigma_\theta &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta) \\
&= \frac{4.5}{2} + \frac{4.5}{2}\cos(90°) + 0 \cdot \sin(90°) = 2.25\,\text{MPa} \\
\tau_\theta &= -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta) \\
&= -\frac{4.5}{2} \cdot \sin(90°) = -2.25\,\text{MPa}
\end{aligned}
$$

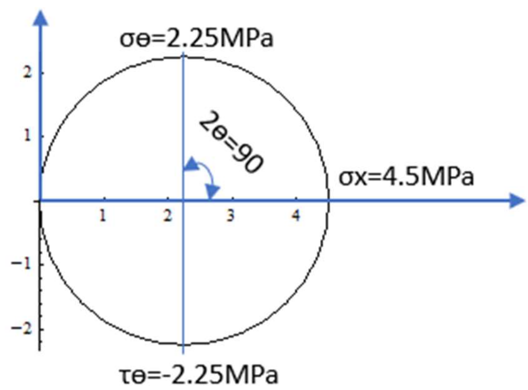


Fig. 4: Mohr's circle representation of the stress distribution at the section

3. The force method (method of consistent deformations)

3.1.Definition

The Force Method, also known as the Method of Consistent Deformations or the Flexibility Method, is one of the most classical and insightful approaches used in the analysis of statically indeterminate structures [3]. Unlike the Displacement Method (e.g., the Finite Element Method), which starts with assumed displacements, the Force Method works by considering equilibrium first, making it physically intuitive and especially useful in educational contexts and for hand calculations.

It is particularly effective in analyzing beams and frames with moderate degrees of static indeterminacy and is a critical tool for understanding the deeper mechanics of internal force redistribution, redundancy, and compatibility in structural systems.\

3.2.Fundamental concepts

3.2.1. Static determinacy

Statically Determinate: A structure is considered statically determinate when all internal forces and support reactions can be determined solely using the equations of static equilibrium [4].

Statically Indeterminate: A structure is statically indeterminate when the number of unknown internal forces and reactions exceeds the number of available equilibrium equations. In such cases, additional compatibility conditions must be introduced to solve the system.

The degree of static indeterminacy (DoSI) is defined as:

$$\mathrm{DoSI} = r + m - 3j$$

where:

r: Number of support reactions;

m: Number of internal members;

j: Number of joints (for trusses) or number of critical sections (for beams and frames).

3.3.Procedure of the force method

The Force Method involves systematically removing the indeterminacy by replacing some unknown reactions or internal forces with redundant [5]. The structure is reduced to a statically determinate "primary structure" and compatibility conditions are enforced to find the redundants.

Here is a step-by-step summary of the method:

Step 1: Choose redundant forces

Identify the redundants (unknown reactions or internal forces) to convert the system into a statically determinate primary structure.

Step 2: Analyze the primary structure

Apply the original loads to the primary structure (with redundants removed). Calculate the displacements at the locations of the removed redundants, denoted as δ.

Step 3: Apply unit load for each redundant

Apply a unit force in the direction of each redundant (called the "unit load method"). Calculate the resulting displacements δ_{ij},

where:

δ_{ij}: Displacement at location i due to unit load at redundant j

Step 4: Compatibility conditions

Enforce that total displacement at each redundant location must be zero (i.e., compatibility):

Solve this system of equations to determine the redundant forces.

$$\sum_{j=1}^{n} \delta_{ij} X_j = -\Delta_{iP}; \qquad i = 1, \ldots, n$$

Step 5: Superposition

Once redundants are known, apply them to the primary structure and use superposition to find total internal forces and support reactions.

3.4.Mathematical formulation

For an indeterminate structure with n redundants [3], the general compatibility equation system is:

$$\begin{bmatrix} \delta_{11} & \delta_{12} & \cdots & \delta_{1n} \\ \delta_{21} & \delta_{22} & \cdots & \delta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{n1} & \delta_{n2} & \cdots & \delta_{nn} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} = - \begin{bmatrix} \Delta_1^0 \\ \Delta_2^0 \\ \vdots \\ \Delta_n^0 \end{bmatrix}$$

This system is often written as [2]:

$$\left[\delta_{ij}\right] \left\{X_i\right\} = -\left\{\Delta_{iP}\right\}$$

where:

[δ

$[\delta_{ij}]$: Flexibility matrix;

$\{X_i\}$: Vector of redundant forces;

$\{\Delta_{ip}\}$: Vector of displacements due to external loading in the primary structure

3.5. Advantages and educational value

- Provides physical insight into equilibrium and compatibility;

- Ideal for structures with low degrees of indeterminacy;
- Teaches superposition and flexibility concepts;
- Offers an excellent complement to numerical methods like FEM.

3.6. Limitations

- Becomes algebraically intensive for high degrees of indeterminacy;
- Less suited for complex geometries or non-linear materials;
- Requires careful choice of redundants for simplification.

3.7.Applications

- Analysis of statically indeterminate beams, trusses, and frames;
- Validation of numerical models;
- Manual verification of FEM software results;
- Understanding redistribution of forces in redundant systems.

Worked example using the force method:

For the beam with a circular cross-section of diameter d, determine the transverse displacement at node **2**, as well as the reactions V_1, M_1, and V_3. Draw the shear force T and bending moment M diagrams, and calculate the maximum stress experienced by the beam.

Given data: $L = 0.8\ m$, $E = 21\times10^4\ MPa$, $F = 4\ kN$, $d = 60\ mm$.

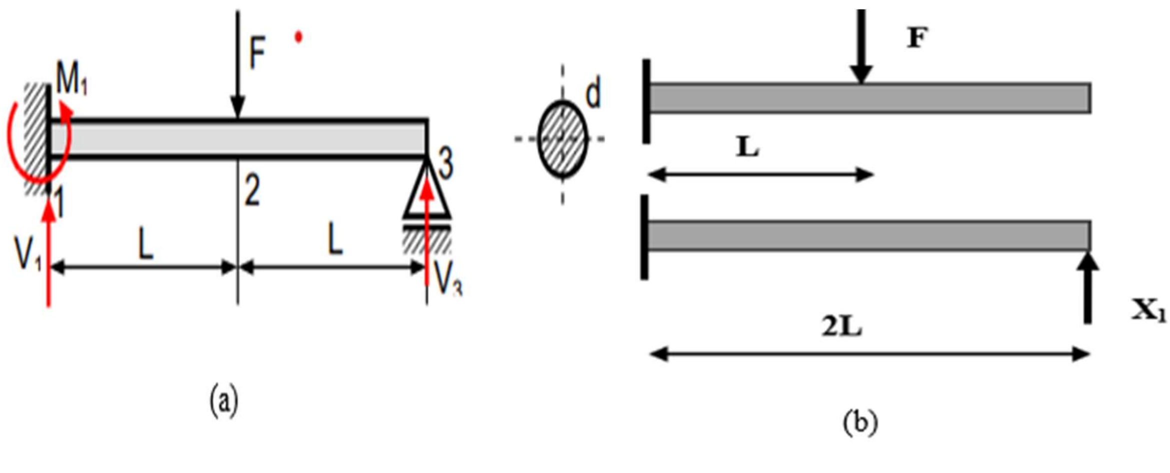


Fig. 5: Studied beam using the force method; (a): actual beam, (b): modeled beam

This study focuses on determining the support reactions, bending moments, and shear forces in the sections of the beam subjected to a concentrated load, using the force method. The objective is to derive explicit expressions for internal forces and moments as functions of the beam's geometry and loading conditions.

- **Assumptions and notations**

The analysis focuses on a beam of total length ***2L***, which is simply supported at one end and fixed (embedded) at the other. The beam is subjected to a concentrated load ***F*** applied at mid-span. The following symbols and notations are used throughout this study:

- ***L:*** half-length of the beam;
- **F:** applied concentrated load;
- ***E:*** Young's modulus of the material;
- ***I*:** moment of inertia of the cross-section;
- δ_{11}**:** flexibility coefficient;
- Δ_{1P}**:** displacement at the redundant location due to the applied load ***P;***
- X_1**:** redundant (unknown) reaction;
- R_a**:** vertical reaction at support ***A;***
- M_a**:** bending moment at support;

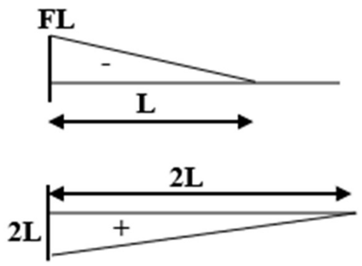


Fig. 6: Moment diagrams due to the external load and the redundant force

Calculation of flexibility coefficient and displacement

The relationship between flexibility and displacement for the redundant reaction X_1 is expressed by:

$$\delta_{11} \times X_1 = -\Delta_{1P}$$

Where:

δ_{11} is the flexibility coefficient and given by:

$$\delta_{11} = \frac{2L \times 2L \times (2/3(2L))}{2EI} = \frac{8L^3}{3EI}$$

The displacement at the redundant location produced by the load ***F*** is given by:

$$\Delta_{1P} = \frac{1}{2EI} \times FL \times L \times (L + \frac{2L}{3}) = \frac{5FL^3}{6EI}$$

Substituting into the compatibility equation and solving for X_1, we get:

$$X_1 = \frac{5F}{16}$$

Calculation of support reactions

From vertical force equilibrium, we can write:

$$F - X_1 - R_A = 0$$

By substituting the value of X_1 into the previous expression, we obtain:

$$R_A = F - \frac{5F}{16} = \frac{11F}{16}$$

Calculation of moments at the supports

By applying the moment equilibrium equation about point A, we have:

$$\sum M\Big|_A = 0$$

$$\frac{5F}{16} \times 2L - F \times L + M_A = 0 \rightarrow M_A = \frac{3FL}{8}$$

Distribution of shear forces and bending moments along the beam

The internal shear forces and bending moments along the beam are determined as follows:

Shear force: at a section located between the simple support and the point of application of the force ($0 < x < L$)

$$T = \frac{5F}{16}$$

Shear force: at a section located in the other half of the beam ($L < x < 2L$)

$$T = \frac{5F}{16} - F = \frac{-11F}{16}$$

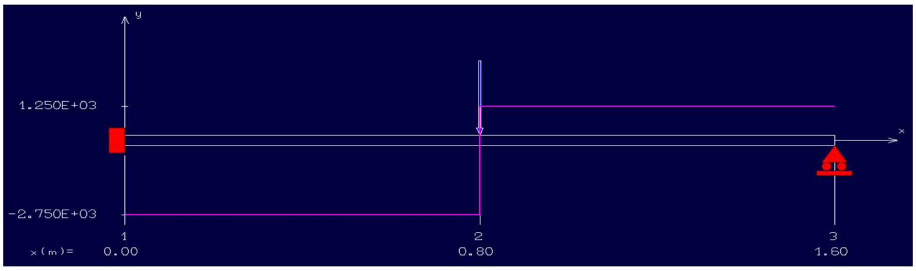


Fig. 7: Shear force diagram of the studied beam

Bending moment: In the same manner, we find the expression for the bending moment along the entire beam

$$M_x = \begin{cases} \dfrac{5F}{16}x \rightarrow 0 \le x \le L \\ \dfrac{5F}{16}(L+x) - Fx \rightarrow L \le x \le 2L \end{cases}$$

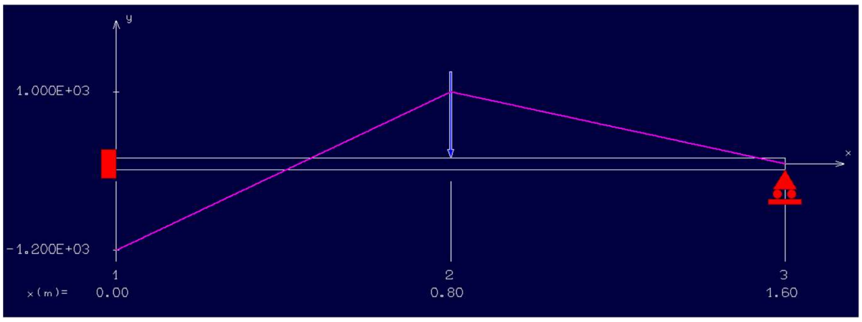


Fig. 8: Bending moment diagram of the studied beam

This analytical study clearly demonstrates the application of the force method to evaluate internal forces in a statically indeterminate beam. The approach derives explicit formulas for calculating support reactions, bending moments, and shear forces as functions of the beam's geometry, applied loading, and material properties. These results provide valuable insights into the structural response and stress distribution within the beam under a concentrated load.

4. The Finite Element Method (FEM)

4.1. Basic definitions of FEM

The Finite Element Method (FEM) is a powerful numerical technique for solving a wide range of engineering problems, particularly those involving complex geometries, loading conditions, and material behaviors [6]. Originally developed for structural mechanics problems in the mid-20th century, FEM has since become a standard tool in disciplines including solid mechanics, fluid dynamics, heat transfer, and electromagnetics.

In structural engineering and mechanics, FEM enables the approximation of displacements, strains, and stresses within structures under given loads and boundary conditions. Its strength lies in its flexibility: it can handle structures that would be analytically intractable using classical methods like the Force Method or Displacement Method.

4.2.Basic principles of FEM

The core idea behind FEM is to divide a complex structure into a finite number of simpler, discrete components called finite elements [6]. These elements are interconnected at points called nodes. Within each element, the field variables (such as displacement or temperature) are approximated by interpolation functions (shape functions) defined at the nodes. The behavior of each element is described by a set of algebraic equations derived from the governing differential equations of the problem. By assembling these local equations into a global system, the entire problem can be solved numerically.

4.3.General FEM procedure

The standard steps in applying the Finite Element Method are as follows:

Step 1: Discretization of the structure

Divide the entire domain into finite elements (triangles, quadrilaterals for 2D, tetrahedra, or hexahedra for 3D problems). Define nodes at the element corners and sometimes at mid-side points.

Step 2: Selection of interpolation functions

Choose suitable shape functions to approximate the variation of the unknown variable (e.g., displacement) within each element. The order of the shape functions (linear, quadratic, cubic) affects accuracy.

Step 3: Derivation of element equations

Use governing differential equations (e.g., equilibrium, compatibility, constitutive relations). Apply variational principles (like the Principle of Virtual Work) or direct approaches (like Galerkin’s method) to derive element stiffness matrices and load vectors.

Step 4: Assembly of global equations

Combine individual element equations into a global system of equations using connectivity information [7].

The general form of the global system is:

$$[K]\{u\} =\{F\}$$

where:

$[K]$ is the global stiffness matrix;

$\{u\}$ is the nodal displacement vector;

$\{F\}$ is the global force vector.

Step 5: Application of boundary conditions

Apply displacement and force boundary conditions to the global system by modifying the stiffness matrix and force vector accordingly.

Step 6: Solving the system of equations

Use numerical solvers (direct or iterative) to obtain nodal displacements.

Step 7: Post-processing

Compute strains and stresses at desired locations using the displacements. Visualize results using contour plots, deformed shape diagrams, or stress distribution maps.

4.4. Mathematical formulation for a beam element

1D Example

Consider a beam element of length L with nodes located at each end. The displacement field can be approximated using the corresponding shape functions as follows:

$$u(x) = N_1(x)u_1 + N_2(x)u_2$$

The element stiffness matrix $[k_e]$ for a simple axial element is:

$$[k_e] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

where:

A: cross-sectional area;

***E*:** Young's modulus;

L: length of the element.

By assembling multiple such matrices, one builds the global system.

4.5.Advantages of FEM

- Can handle complex geometries and boundary conditions;
- Suitable for linear and nonlinear problems;
- Applicable to static, dynamic, steady-state, and transient analyses;
- Automatable with commercial software like ANSYS, Abaqus, SAP2000, and COMSOL;
- Enables visualization of detailed results (stress, strain, displacement).

4.6.Limitations of FEM

- Requires substantial computational resources for large 3D or nonlinear problems;
- Depends on appropriate meshing: poor mesh quality affects accuracy;
- Demands careful selection of boundary conditions and material properties;
- Results are approximate; validation with analytical or experimental data is essential.

4.7.Application in stress analysis of beams and frames

Analyzing stresses in straight and inclined sections of continuous beams and frames FEM offers:

- Automatic calculation of internal forces and moments at any point along the structure;
- Precise evaluation of stresses at inclined sections by transforming local stress tensors;
- Capability to model real-world conditions: varying cross-sections, inclined members, complex supports, and load combinations;
- Comparison with analytical results from the Force Method to validate assumptions and identify discrepancies.

4.8.Educational and practical significance:

For students and engineers:

- FEM illustrates the power of numerical approximation in mechanics;
- Helps bridge the gap between simplified analytical methods and real-world structures;
- Enhances understanding of stress distribution, stress concentration, and deformation patterns;
- Equips professionals to tackle industrial problems where manual calculations are impractical

Worked example using the finite element method

For the beam with a circular cross-section of diameter *d*, determine the transverse displacement at node 2, as well as the reactions ***V1, M1,*** and ***V3***. Draw the shear force ***T*** and bending moment ***M*** diagrams, and calculate the maximum stress experienced by the beam.

Given data: ***L=0.8 m, E=21×104 MPa, F=4 kN, d=60 mm***.

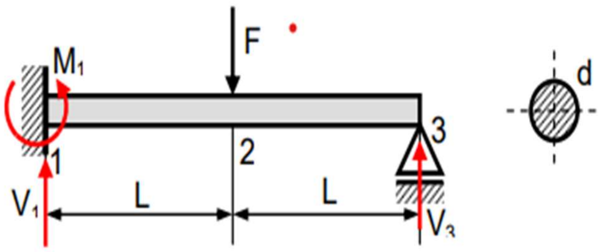


Fig. 9: Beam with fixed and simple supports subjected to a concentrated force F applied at mid-span

The analysis begins with the stiffness matrix of a beam element expressed in the local coordinate system.

$$[K^e]=\begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

In both regions 1–2 and 2–3, rows and columns 1 and 4 are omitted from the expression due to the absence of axial forces.

$$[K^{I}]=\frac{EI}{L^3}\cdot\begin{bmatrix}12 & 6L & -12 & 6L\\ 6L & 4L^2 & -6L & 2L^2\\ -12 & -6L & 12 & -6L\\ 6L & 2L^2 & -6L & 4L^2\end{bmatrix}_I=\frac{EI}{L^3}\cdot\begin{bmatrix}12 & 6L & -12 & 6L & 0 & 0\\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0\\ -12 & -6L & 12 & -6L & 0 & 0\\ 6L & 2L^2 & -6L & 4L^2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}_I$$

$$[K^{II}]=\frac{EI}{L^3}\cdot\begin{bmatrix}12 & 6L & -12 & 6L\\ 6L & 4L^2 & -6L & 2L^2\\ -12 & -6L & 12 & -6L\\ 6L & 2L^2 & -6L & 4L^2\end{bmatrix}_I=\frac{EI}{L^3}\cdot\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 12 & 6L & -12 & 6L\\ 0 & 0 & 6L & 4L^2 & -6L & 2L^2\\ 0 & 0 & -12 & -6L & 12 & -6L\\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2\end{bmatrix}_{II}$$

Which allows us to write:

$$\begin{Bmatrix}V_1\\ M_1\\ V_2^I\\ M_2^I\end{Bmatrix}_I=[K^I]\cdot\begin{Bmatrix}W_1\\ \varphi_1\\ W_2^I\\ \varphi_2^I\end{Bmatrix}_I \quad \text{et} \quad \begin{Bmatrix}V_2^{II}\\ M_2^{II}\\ V_3\\ M_3\end{Bmatrix}_{II}=[K^{II}]\cdot\begin{Bmatrix}W_2^{II}\\ \varphi_2^{II}\\ W_3\\ \varphi_3\end{Bmatrix}_{II}$$

By combining the two stiffness matrices, we get:

$$[K]=[K^I]+[K^{II}]=\begin{bmatrix}12 & 6L & -12 & 6L & 0 & 0\\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0\\ -12 & -6L & 24 & 0 & -12 & 6L\\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2\\ 0 & 0 & -12 & -6L & 12 & -6L\\ 0 & 0 & 6L & 2L^2 & -6L & 4l^2\end{bmatrix}$$

By writing the fundamental equation of the Finite Element Method (FEM), we obtain for both elements:

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2^I \\ M_2^I \\ V_3 \\ M_3 \end{Bmatrix} = \frac{EI}{L^3} \cdot \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 12 & -6L & 0 & 0 \\ 6L & 2L^2 & -6L & 4L^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{Bmatrix} W_1 \\ \varphi_1 \\ W_2 \\ \varphi_2 \\ W_3 \\ \varphi_3 \end{Bmatrix}$$

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2^{II} \\ M_2^{II} \\ V_3 \\ M_3 \end{Bmatrix} = \frac{EI}{L^3} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & 6L & -12 & 6L \\ 0 & 0 & 6L & 4L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \cdot \begin{Bmatrix} W_1 \\ \varphi_1 \\ W_2 \\ \varphi_2 \\ W_3 \\ \varphi_3 \end{Bmatrix}$$

By assembling the two element equations into a global system, the following result is obtained:

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 = -F \\ M_2 = 0 \\ V_3 \\ M_3 = 0 \end{Bmatrix} = \frac{EI}{L^3} \cdot \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4l^2 \end{bmatrix} \cdot \begin{Bmatrix} W_1 = 0 \\ \varphi_1 = 0 \\ W_2 \\ \varphi_2 \\ W_3 = 0 \\ \varphi_3 \end{Bmatrix}$$

By selecting only rows and columns 3, 4, and 6, we obtain the following:

$$\begin{Bmatrix} -F \\ 0 \\ 0 \end{Bmatrix} = \frac{EI}{L^3} \cdot \begin{bmatrix} 24 & 0 & 6L \\ 0 & 8L^2 & 2L^2 \\ 6L & 2L^2 & 4L^2 \end{bmatrix} \cdot \begin{Bmatrix} W_2 \\ \varphi_2 \\ \varphi_3 \end{Bmatrix}$$

By substituting these three values into the matrix system corresponding to rows and columns 1, 2, and 5, we obtain:

$$W_2 = -\frac{7FL^3}{96EI}, \quad \varphi_2 = -\frac{3FL^2}{96EI}, \quad \varphi_3 = \frac{12FL^2}{96EI}$$

Subsequently, solving the matrix system provides the shear forces and fixed-end moments at the supports.

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_3 \end{Bmatrix} = \frac{EI}{L^3} \cdot \begin{bmatrix} -12 & 6L & 0 \\ -6L & 2L^2 & 0 \\ -12 & -6L & -6L \end{bmatrix} \cdot \begin{Bmatrix} -\frac{7}{96}\frac{FL^3}{EI} \\ -\frac{3}{96}\frac{FL^2}{EI} \\ \frac{12}{96}\frac{FL^2}{EI} \end{Bmatrix}$$

Which gives:

$$V_1 = \frac{11F}{16}, \quad M_1 = -\frac{3FL}{8}, \quad V_3 = \frac{5F}{16}$$

Comparative example: Analyzing a two-span continuous beam using the force method and FEM

Problem statement

Consider a continuous beam ***ABC*** with two equal spans (each of length ***L = 6 m***) supported at points ***A, B***, and ***C***. The beam is subjected to a uniform distributed load ***q = 10 kN/m*** over its entire length. Assume constant flexural rigidity ***EI***.

We aim to compute the reactions at supports ***A, B***, and ***C*** using both the Force Method and FEM.

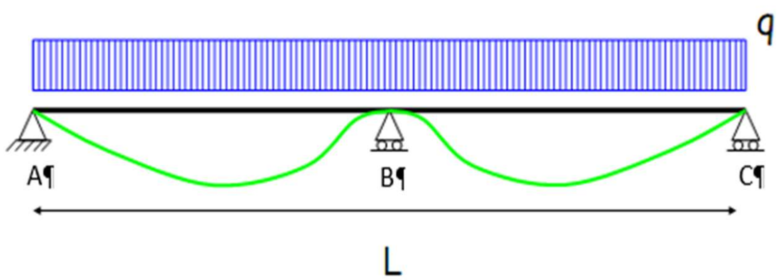


Fig. 10: Continuous beam subjected to a uniformly distributed load along its entire length

Using the force method:

Step 1: Static indeterminacy

The beam has three supports → three reactions. For a planar beam, the static equilibrium equations give only two independent equations for vertical reactions.

Degree of static indeterminacy is: ***3–2 = 1***.

We select the reaction at the support ***B*** (let's call it X_1) as the redundant.

Step 2: Primary structure

We remove the internal restraint at ***B*** (i.e., assume a hinge at ***B***), making the structure statically determinate.

Step 3: Compatibility equation

The vertical displacement at ***B*** due to:

The applied loads is denoted: Δ_{1P}.

The redundant at the support ***B*** is denoted: X_1.

The compatibility equation for this case is provided by:

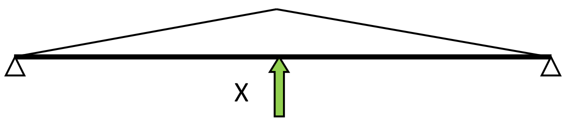
$$\delta_{11}X_1 + \Delta_{1P} = 0$$

After performing the integrations to obtain δ_{11} and Δ_{1P}, and solving the resulting equation, we can determine X_1 as follow:

$$\delta_{11}X_1 + \Delta_{1P} = 0$$

$$U = 2\times\left\{\frac{1}{2EI}\int_0^{L/2}\left(\frac{(X=1)}{2}\times x\right)^2 dx\right\} = \frac{L^3X^2}{96EI}$$

$$\delta_{11} = \frac{\partial U}{\partial X} = \frac{\partial}{\partial X}\left(\frac{L^3X^2}{96EI}\right) = \frac{L^3X}{48EI}$$



While:

$$X = 1 \rightarrow \delta_{11} = \frac{L^3}{48EI}$$

$$\Delta_{1P} = \frac{-1}{EI}\times\int_0^{L/2}\int_0^{qL/2-qx^2/2} dydx \times \left(\frac{\int_0^{L/2}\int_0^{qL/2-qx^2/2} xdydx}{\int_0^{L/2}\int_0^{qL/2-qx^2/2} dydx}\right) = \frac{-5qL^4}{384EI}$$

We can also get Δ_{1P}, in another way as follow:

Surface area resulting from the bending moment curve due to a uniformly distributed load applied over half the beam:

$$S = \int_0^{L/2} \int_0^{q*x*L/2-q*x*x/2} 1 dy dx = \frac{L^3 q}{24}$$

Centroid position of the previously defined area:

$$C_G = \frac{\int_0^{L/2} \int_0^{q*x*L/2-q*x*x/2} x dy dx}{\int_0^{L/2} \int_0^{q*x*L/2-q*x*x/2} 1 dy dx} = \frac{5L}{16}$$

The parameter Δ_{1P} is defined as follows:

$$\Delta_{1P} = -2 \times \frac{L^3 q}{24EI} \times \frac{5L}{16} \times \frac{1}{2} = \frac{-5qL^4}{384EI}.$$

Finally, the value of the redundant can be determined, allowing us to compute the reactions of the beam.

$$X_1 = \frac{5qL}{8} = R_B$$

$$R_A + R_C + \frac{5qL}{8} = qL$$

$$\sum M\Big|_A = 0 \Rightarrow qL\frac{L}{2} - \frac{5qL}{8}\frac{L}{2} - R_C L = 0 \quad \Rightarrow \quad R_C = \frac{3qL}{16}$$

$$R_A = \frac{3qL}{16}$$

At this stage, the structure becomes statically determinate, allowing the calculation of bending moments and shear forces along its entire length.

Using the finite element method

Step 1: Discretization

Divide the beam into two elements: ***AB*** and ***BC***.

Each beam element has 2 nodes and 2 DOFs per node (rotation and vertical displacement):

Total DOFs = 6

But since ***A*** and ***C*** are fixed, we reduce the system by applying boundary conditions.

$$[K_e] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Step 2: Local Stiffness Matrix

For each element we can write the stiffness matrix:

Step 3: Assembly

Assemble the global stiffness matrix ***[K]***, apply the boundary conditions (fixed at ***A*** and ***C***), and solve for unknown displacements.

Step 4: Compute Moments

Use the calculated displacements to compute internal efforts including moments and shear forces.

Numerical computation:

The complete numerical analysis of the two-span continuous beam, performed using both the Force Method and the Finite Element Method (FEM) with the data presented above, leads to the results summarized in the comparison table below:

Comparison of Results:

Support	Force Method (kNm)	FEM (kNm)
X=L/4	0.015625 q	0.015625 q
X=L/2	-0.03125 q	-0.03125 q
X=4L/5	0.17 q	0.175 q

Slight differences arise due to the discretization inherent in the Finite Element Method (FEM). Increasing the number of elements enhances the accuracy of the results

The bending moment distribution for the entire beam is illustrated graphically in figure 11.

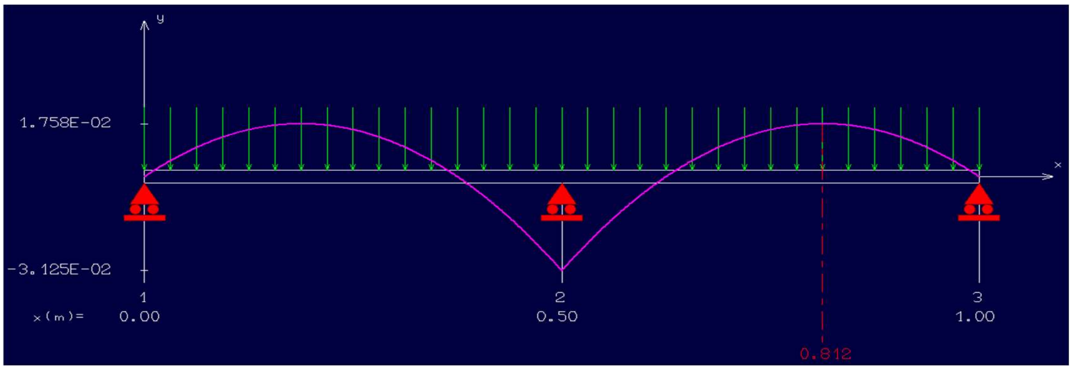


Fig. 11: Bending moment distribution for the entire beam

5. Continuous beams and frames

In structural engineering and mechanics, continuous beams and frames are fundamental elements widely used in buildings, bridges, and industrial structures. Unlike simple beams supported at only two points, continuous beams and frames are structural systems supported at three or more points or interconnected into multi-member assemblies. This continuous nature introduces additional internal redundancies, enhancing structural efficiency, load distribution, and serviceability but also requiring more advanced analysis techniques due to their static indeterminacy.

This section explores the definitions, behavior, advantages, limitations, and analysis approaches for continuous beams and frames, serving as a theoretical foundation for the stress analysis procedures applied later in this study.

5.1.Continuous beams

5.1.1. Definition

A continuous beam is a beam extending over more than two supports[8]. It can have uniform or varying cross-sections and be subjected to various load types (point loads, distributed loads, moments). The presence of multiple supports makes it statically indeterminate, necessitating additional compatibility conditions for internal force calculation, figure 12.

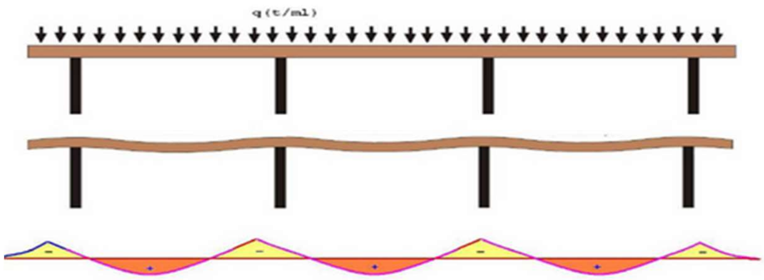


Fig. 12: Illustration of the continuous beam with its displacements and bending moments

5.1.2. Characteristics

- Statically indeterminate: Requires methods like the Force Method, Slope-Deflection Method, Moment Distribution Method, or Finite Element Method (FEM) for analysis;
- Improved load distribution: Bending moments are redistributed across spans, reducing peak moments compared to simply supported beams;
- Better deflection control due to continuous connectivity;
- Moment reversal may occur in some spans (positive to negative moment transitions).

5.1.3. Common Applications

- Multi-span bridges;
- Continuous floor girders in buildings;
- Pipe racks and conveyor support systems.

5.2. Frame Structures

5.2.1. Definition

A frame is an assembly of beams (horizontal members) and columns (vertical members) rigidly connected to form a stable structure[9]. Frames can be planar (2D) or space (3D) systems, often designed to resist vertical loads and lateral actions like wind or seismic forces, figure 13.

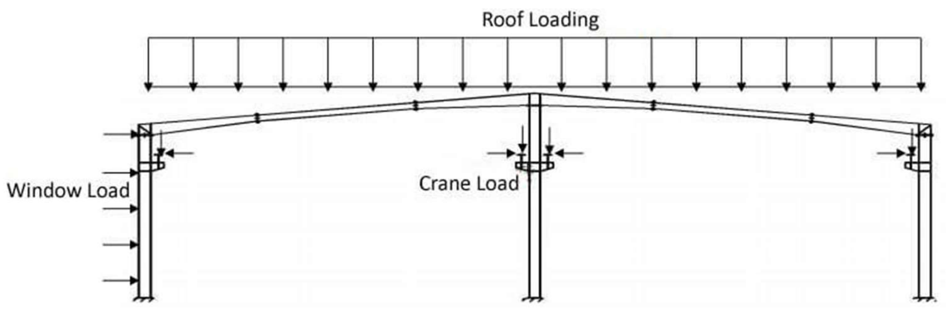


Fig. 13: Illustration of the frame subjected to different types of external loads

5.2.2. Characteristics

- Contains both bending and axial forces in members;
- May include inclined, horizontal, and vertical members;
- High static indeterminacy: Requires advanced analytical or numerical methods;
- Frames can be braced (with diagonal members) or unbraced (relying on member stiffness).

5.2.3. Types of frames

- Fixed-frame: Joints fixed against rotation;
- Hinged frame: Some or all joints allow rotation;
- Portal frame: Common in industrial sheds and warehouses:
- Braced frame: Includes diagonal members to improve lateral stability.

5.3. Stress behavior in continuous Beams and frames

Continuous beams and frames experience combined internal forces:

- Bending moments (positive and negative);
- Shear forces;
- Axial forces (notably in inclined and vertical members of frames).

At inclined sections or joints, internal force components must be resolved into local coordinate systems, affecting stress distribution.

Important considerations:

- Maximum positive moments typically occur at mid-span;

- Maximum negative moments occur at supports;
- Axial forces are significant in frames due to vertical loads and lateral actions.

5.4.Analytical challenges and methods

Due to static indeterminacy and complex loading scenarios, continuous beams and frames cannot be analyzed by equilibrium equations alone. Common analysis approaches include:

- Force Method (Flexibility Method);
- Displacement Method (Stiffness Method);
- Moment Distribution Method (for multi-span beams and portal frames);
- Finite Element Method (for complex or irregular geometries).

5.5.Advantages of continuous beams and frames

- Greater load-carrying capacity due to redundancy;
- Reduction in deflections and internal moments;
- More economical material usage;
- Better structural stability against lateral forces.

5.6.Limitations

- More complex analysis and design process;
- Sensitivity to support settlements and temperature effects;
- Requires careful detailing at joints to prevent cracking or failure;
- May develop negative moments at supports, requiring reinforcement.

5.7.Diagram descriptions

The following illustrations should be done:

- A continuous beam over three supports with load and moment diagrams;
- A planar rectangular frame under vertical and lateral loads, showing internal forces;
- Moment distribution sketches for continuous beams and frames.

5.8.Importance in modern structural engineering

Continuous beams and frames are prevalent in modern construction:

- Building skeletons (multi-story steel or concrete frames);
- Long-span bridges;
- Industrial buildings;
- Piping systems and supporting structures.

Their widespread use demands thorough understanding of their stress behavior and efficient analysis methods precisely the focus of your thesis comparing classical and numerical techniques.

6. Bibliographic research

The study presented in this work is supported by numerous bibliographic references. Focusing specifically on those closely related to our topic, we highlight the following relevant studies.

In the study conducted by G. E. Okolnikova et al. [10], a comparative analysis was performed between two structural analysis methods: the Finite Element Method (FEM) and the Force Method. The primary objective was to evaluate and compare the accuracy and computational efficiency of both approaches when applied to the analysis of a continuous steel beam commonly used in frame structures. The results showed minimal differences in the calculated internal moments, forces, and deformations, indicating a strong correlation between the two methods. Consequently, both FEM and the Force Method can be considered reliable and comparable tools for structural analysis in similar engineering applications.

In the study [11], the Finite Element Method (FEM) is employed to analyze stress distributions in beams subjected to non-uniform torsion. The analysis involves the computation of primary and secondary warping functions, torsional constants, and the warping moment of inertia, enabling an accurate representation of stress distribution across the entire cross-section. The numerical results are validated through comparison with analytical solutions, demonstrating a high level of accuracy and reliability. Additionally, the study proposes the use of chamfers as a design modification to reduce stress concentrations. Future work aims to incorporate nonlinear effects to simulate the gradual plastic deformation of the cross-sections.

The study [12] investigates the finite element analysis of thin-walled beams subjected to combined loading, with particular emphasis on constrained torsion. By employing an advanced 3D stiffness matrix that includes an additional degree of freedom for warping, the researchers significantly enhance the accuracy of deformation and stress analysis. This approach is especially effective for thin-walled open sections, where characteristic torsional parameters tend to be low. The findings indicate that incorporating these factors can lead to safer and more optimized structural designs. The study also considers trigonometric and approximate methods for modeling non-uniform torsion, offering valuable insights and practical solutions for the design of complex beam elements.

The study conducted by Pishro A.A. et al. [13] examines the structural performance of reinforced concrete beams strengthened with fiber-reinforced polymers (FRP) when subjected to combined torsion and bending. By integrating experimental investigations, finite element simulations using ABAQUS, and predictive modeling through artificial neural networks (ANN), the study provides a comprehensive evaluation of FRP effectiveness. The findings reveal a notable improvement in the load-carrying capacity and torsional resistance of the strengthened beams. Furthermore, the ANN models yield accurate predictions of the structural response, highlighting their potential as a complementary tool for design verification in advanced structural analysis frameworks.

The study [14] presents a comprehensive theoretical and numerical model for analyzing the behavior of thin-walled open-section beams subjected to significant torsion. The model accounts for torsional effects, axial shortening, pre-buckling deformations, and the coupling between bending and torsion. A three-dimensional framework is employed, featuring two nodes with seven degrees of freedom per node. The resulting equilibrium equations are inherently nonlinear and strongly coupled, although simplifications are possible for linear analysis. The numerical solution is obtained using a Newton-Raphson iterative method in conjunction with a custom-developed finite element code. To validate the model's robustness, both linear and nonlinear examples are analyzed, demonstrating high accuracy—particularly in scenarios involving bifurcation phenomena.

The study [15] introduces a numerical approach based on the Finite Element Method (FEM) for analyzing planar structural frames with no lateral displacement. In this formulation, the frames are modeled using bar elements and equivalent springs to represent bending stiffness. An iterative solution process is employed to determine the internal force distribution within the structure. The methodology is validated through two practical examples, demonstrating both the effectiveness and computational efficiency of the approach for analyzing complex frame configurations.

The study [16] proposes a finite element model for the static analysis of thin-walled curved beams, employing the concept of the Equivalent Laminated Composite Section (ELCS). This approach simplifies the formulation while maintaining a high level of accuracy in predicting structural behavior. The model effectively captures the mechanical response of curved beams and offers a substantial reduction in computational cost compared to conventional three-dimensional finite element methods.

The reference [17] provides a foundational theoretical framework for analyzing composite structural elements, including thin-walled and curved beams. The authors present closed-form and numerical solutions to bending, torsion, and buckling problems in composite structures. The comprehensive treatment of anisotropic material behavior and stress distribution under complex loading conditions makes it particularly valuable for the accurate modeling of curved beam elements using FEM.

The study [18] develops a nonlinear finite element model for analyzing space frames composed of curved beam elements. The model accounts for large displacements and rotations, and it utilizes a co-rotational formulation to improve accuracy in geometrically nonlinear regimes. Numerical examples demonstrate the model's capability to simulate the nonlinear behavior of curved beams and frame structures under various loading conditions.

This review paper [19] provides a detailed examination of curved steel member behavior, with a focus on stability under axial, bending, and torsional loads. The authors summarize various analytical, experimental, and numerical methods used in the modeling of curved beams. The paper underscores the importance of including warping and torsional effects for accurate analysis, and it positions FEM as a reliable tool for handling such complexities.

The study [20] presents a dynamic finite element model to analyze the response of curved beams under moving loads, which is critical in structural applications such as bridges and crane arms. The formulation includes geometric curvature, inertia effects, and damping. Time-history analyses reveal the influence of beam curvature on displacement and stress distribution. The results validate the importance of advanced FEM formulations for dynamic load scenarios.

The classic text provides in [21] a detailed treatment of the Force Method (also known as the Method of Consistent Deformations), covering statically indeterminate structures such as continuous beams and frames. It includes worked examples and theoretical foundations that remain essential for understanding classical structural analysis techniques.

The book [22] is widely used in academic institutions, especially in engineering programs. It covers both the Force Method and the Displacement Method in detail, with applications to continuous beams, rigid frames, and trusses. The step-by-step approach is helpful for teaching and reference purposes.

The work brought by Zienkiewicz in [23] is considered the foundational text and standard reference for FEM. It offers comprehensive coverage of theory, numerical implementation, and applications across structural mechanics, including beam and frame elements. The book also discusses stiffness matrix assembly and element formulations.

Renowned for its clarity and practical approach, the book [24] focuses on engineering applications of FEM. It includes beam, frame, and plate analysis, with emphasis on how to derive and use the stiffness matrix in structural contexts. Examples include both 1D and 2D elements.

Perfect for both students and researchers, the book [25] offers an accessible introduction to FEM. It develops the stiffness matrix method step by step and includes applications to axial members, beams, trusses, and frames. It's especially useful for learning the basic theory and coding simple FEM solvers.

The widely used textbook [26] includes in-depth chapters on the analysis of beams, frames, and trusses using both classical and matrix methods. It covers influence lines, moment distribution, and the Force and Displacement Methods with real-world examples. Ideal for both academic and professional use.

A highly respected text for graduate-level structural analysis is provided in [27], this book focuses on the matrix formulation of beam and frame structures, with detailed discussions of both the Force and Stiffness Methods. It provides the mathematical foundation for structural modeling in FEM software.

The bibliographic research presented in this section highlights a selection of key studies that directly support and inform the present work. These references illustrate the depth and diversity of current research on the Force Method, the Finite Element Method, and their application to beam and frame structures under complex loading conditions. From classical theoretical foundations to modern numerical simulations incorporating advanced material behavior and dynamic effects, the selected works demonstrate the rich landscape of structural analysis methodologies.

It is important to note, however, that this selection represents only a small point in a vast sea of available literature. The references cited here were carefully chosen for their direct relevance and comprehensive coverage of the specific topics addressed in our study. Many other valuable contributions exist, but the scope of this section has been intentionally focused to ensure clarity and alignment with the objectives of this work.

7. Conclusion

This chapter has laid the theoretical foundation for the analysis of internal forces and stresses in continuous beams and frame structures, focusing particularly on the complexities introduced by inclined sections. These structural configurations are frequently encountered in real-world engineering applications, such as bridge decks, sloped roof systems, mechanical trusses, and off-axis load scenarios, where stress analysis becomes notably more intricate due to the combined effects of normal and shear forces.

The Force Method, as a classical analytical technique, was presented in detail for its capacity to resolve statically indeterminate structures through a systematic approach grounded in equilibrium and compatibility principles. Its ability to provide exact solutions for internal forces by introducing redundant reactions and employing flexibility relationships makes it an indispensable tool in structural mechanics, especially for structures where conventional equilibrium equations alone are insufficient.

In parallel, the chapter introduced the Finite Element Method **(FEM)** as a powerful numerical alternative, offering significant advantages in terms of versatility and computational efficiency. Unlike purely analytical methods, FEM can accommodate complex geometries, heterogeneous material properties, and irregular loading conditions, making it particularly valuable for modern structural analysis.

By establishing the theoretical principles and operational frameworks of both the Force Method and the Finite Element Method, this chapter prepares the groundwork for the analyses conducted in the subsequent sections. The forthcoming chapters will apply these methodologies to practical case studies involving continuous beams and frame systems, allowing for a comparative assessment of their accuracy, applicability, and computational demands. This comparison will not only illustrate the theoretical insights gained from the Force Method but also demonstrate the practical advantages offered by numerical simulation through the Finite Element Method.

Chapter 2

Analysis of a Continuous Beam Under External Load Using the Force Method: Comparison with the Finite Element Method (FEM)

Chapter 2: *Analysis of a Continuous Beam Under External Load Using the Force Method: Comparison with the Finite Element Method (FEM)*

1. Introduction

In this chapter, we explore the application of the Force Method for the analysis of continuous beams subjected to external loads. The Force Method, also known as the Method of Consistent Deformations, is a classical approach used to solve statically indeterminate structures by introducing a set of redundant forces. Through this process, the original indeterminate system is transformed into a statically determinate basic system, allowing for a systematic and structured solution.

We begin by reviewing the fundamental principles underlying the Force Method and the procedural steps involved in its application. Particular attention is given to the identification of redundant forces, the construction of the basic system, and the formulation of compatibility conditions based on the deformations of the structure. The theoretical framework is then reinforced through detailed worked examples, illustrating the calculation of internal forces, bending moments, and shear forces for a continuous beam under various loading conditions.

To provide a broader perspective and validate the results, the solutions obtained using the Force Method will be systematically compared to those derived from the Finite Element Method (FEM). This comparative study highlights the strengths and limitations of each approach and offers deeper insight into their practical applications in structural analysis.

The objective of this chapter is to develop a comprehensive understanding of the Force Method, its practical implementation, and its relationship with modern numerical techniques such as FEM, preparing students and practitioners to confidently apply these methods in real-world engineering problems.

2. Methods of analysis

2.1. Algorithm for solving structures using the force method

The structural analysis [28] of statically indeterminate systems using the force method generally proceeds through the following steps:

1. Determine the degree of static indeterminacy;
2. Select the primary system;
3. Formulate the equivalent system;
4. Establish the system of canonical equations;
5. Construct unit and load diagrams of internal forces (shear forces, bending moments, etc.);
6. Calculate the coefficients and free terms in the canonical equations;
7. Build the cumulative unit diagram;
8. Verify the coefficients and free terms through a universal check;
9. Solve the canonical system to determine the redundant reactions;
10. Draw the final internal force diagrams for the original structure;
11. Perform static and kinematic verification.

It should be noted that steps 7, 8, and 11 are optional but useful for checking the accuracy of the analysis. For systems with only one redundant force, steps 7 and 8 are unnecessary, as the cumulative unit diagram coincides with the unit diagram.

2.2.Fundamental physical relationships of the force method

Structural Analysis [29] requires that the equations governing the following fundamental physical relationships be satisfied:

(i) Equilibrium of forces and moments;

(ii) Compatibility of deformations among members and at supports;

(iii) Material behavior relating stresses to strains;

(iv) Strain-displacement relations;

(v) Boundary conditions.

In this context, the application of the Force Method (also known as the Flexibility Method) to the static analysis of continuous beams follows these main steps:

- ✓ Transform the statically indeterminate structure into a statically determinate basic system by removing selected unknown forces or support reactions and replacing them with known or unit forces;
- ✓ Apply the principle of superposition to calculate the forces required to restore compatibility with the original structure;
- ✓ Identify the unknowns to be solved, typically the redundant forces introduced by the removal of constraints;
- ✓ Establish the system of equations in which the coefficients of the unknowns are the flexibility coefficients, representing the deformations due to unit forces.

3. Static analysis of a continuous beam with three spans under uniform and concentrated loads using the three-moment technique

We consider a continuous beam ***(ABCD)*** comprising three spans, with a constant flexural rigidity ***EI***. The beam is subjected to a uniformly distributed load ***q=5 kN/m*** over spans ***AB*** and ***CD***, and a concentrated load ***P=40 kN*** applied at the mid-span ***BC***, figure 1.

Using the Three-Moment Technique, we aim to determine:

- ✓ The reactions at the supports;
- ✓ The bending moment and shear force values and their diagrams.

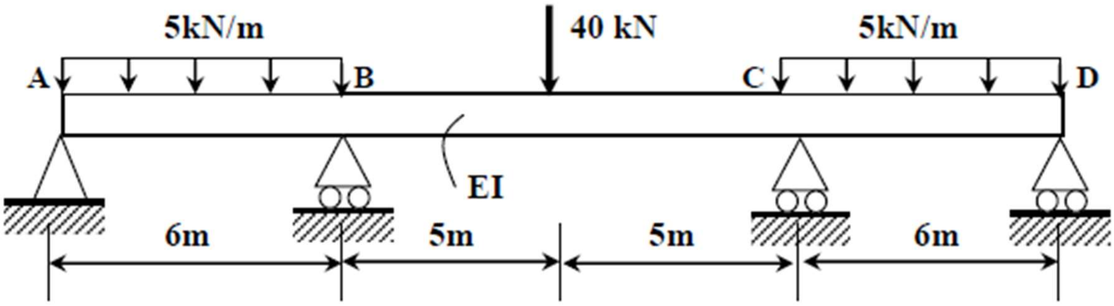


Fig. 1: Continuous beam subjected to various types of external loads

Degree of hyperstaticity:

d=5−3=2; the structure is twice hyperstatic.

The general formula of the Three-Moment Technique is given by the following expression [28,29,30]:

$$\ell_1 M_A + 2(\ell_1 + \ell_2) M_B + \ell_2 M_C = 6EI(\omega_{BR} - \omega_{BL}) \tag{1}$$

Point B

$$6M_A + 32M_B + 10M_C = 6EI(\omega_{BR} - \omega_{BL})$$

Since $M_A=0$, we can write:

$$32M_B + 10M_C = 6EI(\omega_{BR} - \omega_{BL}) \tag{2}$$

Point C

$$10M_B + 32M_C + 6M_D = 6EI(\omega_{CR} - \omega_{CL})$$

Since $M_D=0$, we can write:

$$10M_B + 32M_C = 6EI(\omega_{CR} - \omega_{CL}) \tag{3}$$

Referring to the diagram in Figure 2, which shows the formulas for the rotations at the simple supports for each loading case, we can express the following:

$$\omega_{AR} = -\frac{1}{24EI} P\ell^3 \text{ ; } \omega_{BL} = \frac{1}{24EI} q\ell_1^3 \text{ ; } \omega_{CR} = -\frac{1}{24EI} q\ell_3^3 \text{ ; } \omega_{CL} = \frac{1}{16EI} q\ell_2^2$$

Substituting these expressions into formulas (2) and (3), we obtain the following system:

$$32M_B + 10M_C = 6\left(-\frac{1}{16}P10^2 - \frac{1}{24}q6^3\right)$$
$$10M_B + 32M_C = 6\left(-\frac{1}{24}q6^3 - \frac{1}{16}q10^2\right) \quad (4)$$

The system of equations (4) takes the following matrix form:

$$\begin{bmatrix} 32 & 10 \\ 10 & 32 \end{bmatrix} \begin{Bmatrix} M_B \\ M_C \end{Bmatrix} = \begin{Bmatrix} -1770 \\ -1770 \end{Bmatrix} \quad (5)$$

Solving the matrix equation system (5) leads to the following solution, which represents the bending moment values at supports ***B*** and ***C***.

$$\begin{Bmatrix} M_B \\ M_C \end{Bmatrix} = \begin{Bmatrix} -42.1429\ kN.m \\ -42.1429\ kN.m \end{Bmatrix}$$

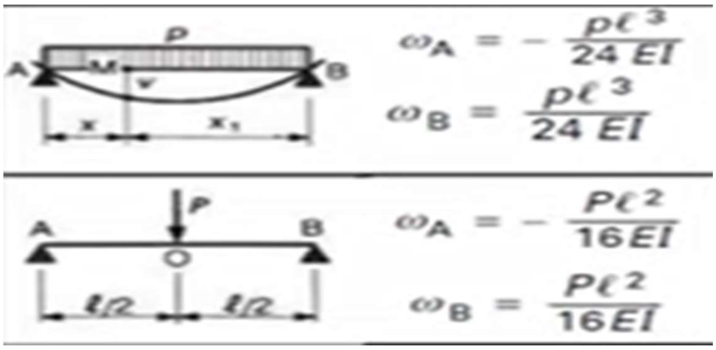


Fig. 2: Static diagram of the rotation for specific load geometries

Calculation of reactions: using the principle of decomposition.

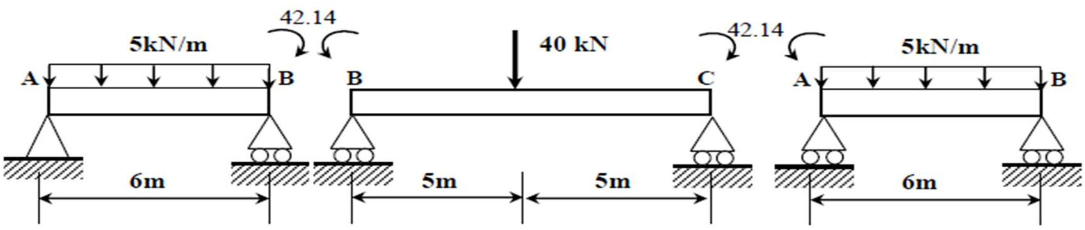


Fig. 3: Decomposed schematic of the continuous beam based on the principle of superposition

For span AB

$$\sum F = 0 \rightarrow R_A + R_B = 30$$

$$\sum M = 0 \rightarrow 6R_B - 3\times 30 - 42.14 = 0$$

Which gives:

$$R_A = 7.9762 \ \ kN$$

$$R_B = 22.0238\, kN$$

For span BC

$$\sum F = 0 \rightarrow R_B + R_C = 40$$

$$\sum M = 0 \rightarrow 10R_C - 5\times 40 - 42.14 + 42.14 = 0$$

Which gives:

$$R_B = 20 \ \ kN$$

$$R_C = 20\, kN$$

For span CD

$$\sum F = 0 \rightarrow R_C + R_D = 30$$

$$\sum M = 0 \rightarrow 6R_D - 3\times 30 + 42.14 = 0$$

Which gives:

$$R_C = 22.0238 \ \ kN$$

$$R_D = 7.9762\, kN$$

Superposition of results:

Finally, the reaction values of the continuous beam, after applying the superposition principle, are determined as follows, as shown in figure. 4:

$$R_A = 7.9762 \ \ kN$$

$$R_B = 22.0238 + 20 = 42.0238\, kN$$

$$R_C = 20 + 22.0238 = 42.0238\, kN$$

$$R_D = 7.9762 \ \ kN$$

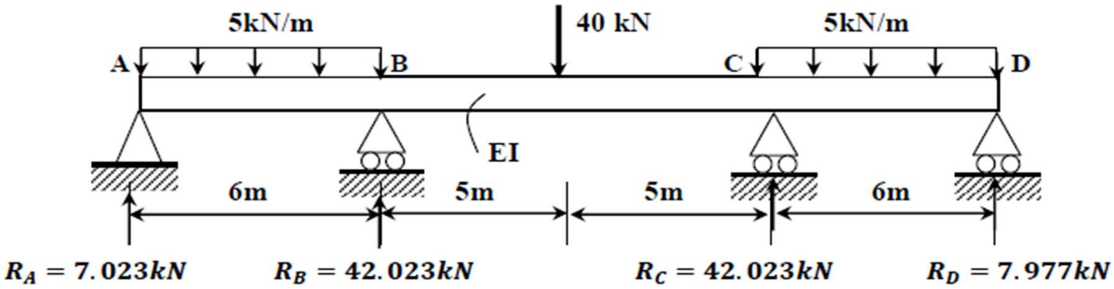


Fig. 4: Schematic representation of the reactions along the entire length of the continuous beam

Shear force diagrams of the continuous beam.

Once the reactions of the continuous beam have been determined, we can now derive the shear force diagram for the entire beam by applying the elasticity principles of statically determinate systems. The graphical results of the shear forces along the entire length of the continuous beam are shown in figure 5.

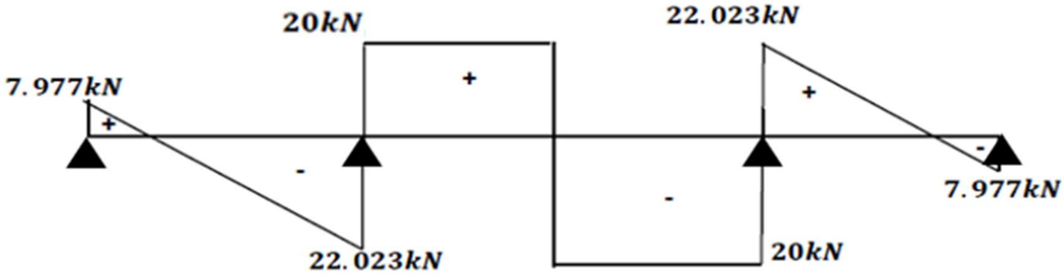


Fig. 5: Schematic representation of the shear force (kN) in the continuous beam

Bending moment diagram of the continuous beam.

In the same manner as for the shear force, we can determine the bending moment diagram for the entire beam. The graphical results of the bending moment along the entire length of the beam are shown in figure 6.

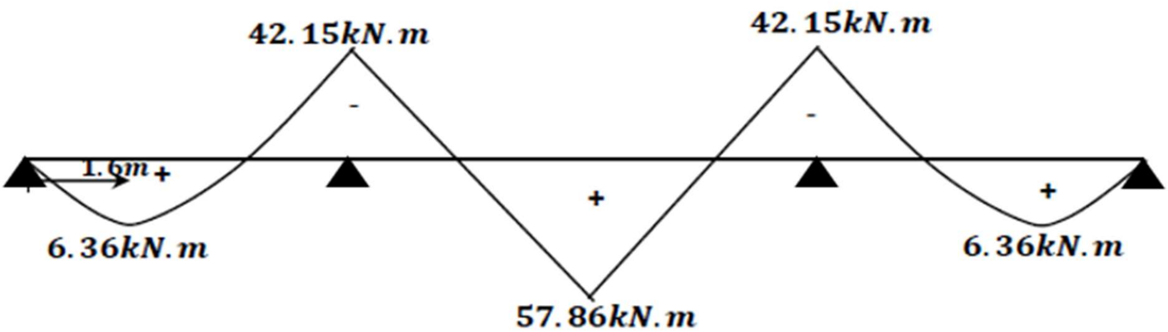


Fig. 6: Schematic representation of the bending moment (kNm) in the continuous beam

Determination of normal stresses in the cross-sections of the continuous beam

Referring to the diagram in figure 6, which shows that the maximum bending moment occurs at the midpoint of span ***BC*** with a value of $M_{\max} = 57.86\ kNm$, the maximum normal stress is expected at the same location and is calculated using the following formula:

$$\sigma_{\max} = \pm \frac{M_{\max}}{I} y_{\max}$$

I : is the moment of inertia of the section; $y_{\max}$: is half the height of the beam section.

The stress formula can be illustrated schematically as shown in figure. 7:

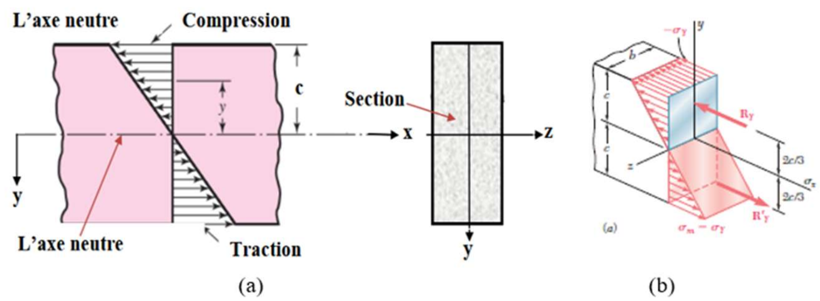


Fig. 7: Schematic representation of stress distribution across a structural section:
(a) 2D view; (b) 3D view

Comparison of results with the finite element method (FEM)

To check the accuracy of the results obtained using the Force Method in the static analysis of the continuous beam, we now perform a comparison with the Finite Element Method [31]. The idea is to repeat the analysis of the same structure using a different approach—this time, the Finite Element Method. For this purpose, we use the software RDM6, which is based on the principles of the finite element method. The software provides graphical outputs that show how the beam behaves under various external loads. These diagrams represent the internal forces and stresses in the beam's sections and allow us to evaluate the consistency of the results between the two methods.

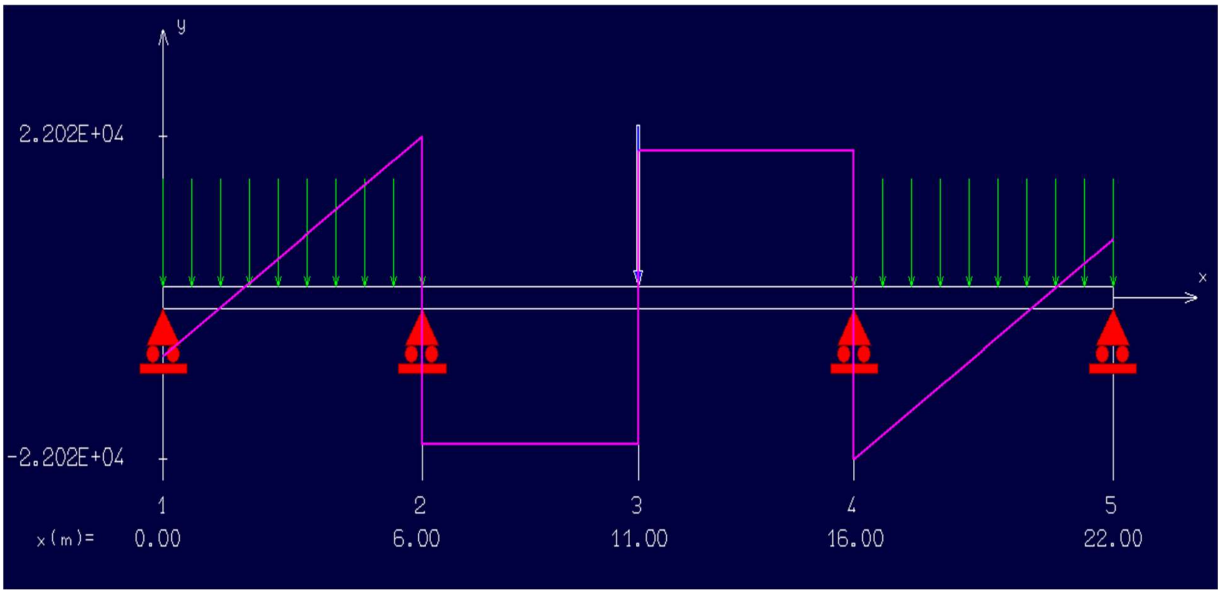


Fig. 8: Shear force (kN) distribution along the entire length of the beam

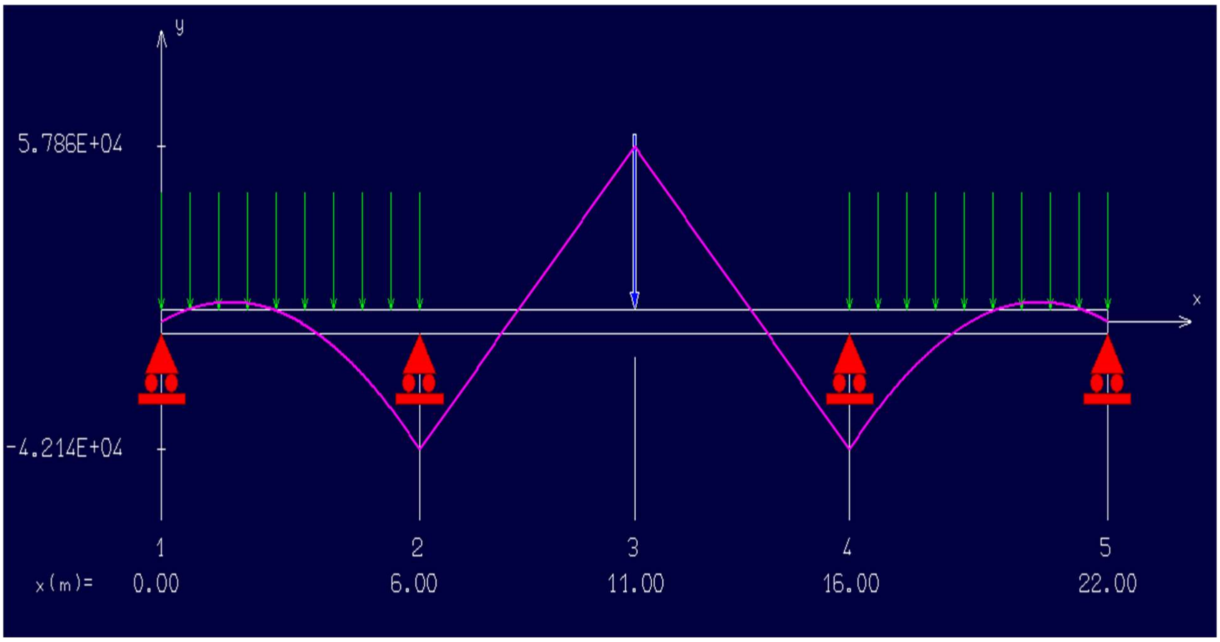


Fig. 9: Distribution of bending moments (kNm) along the full length of the beam

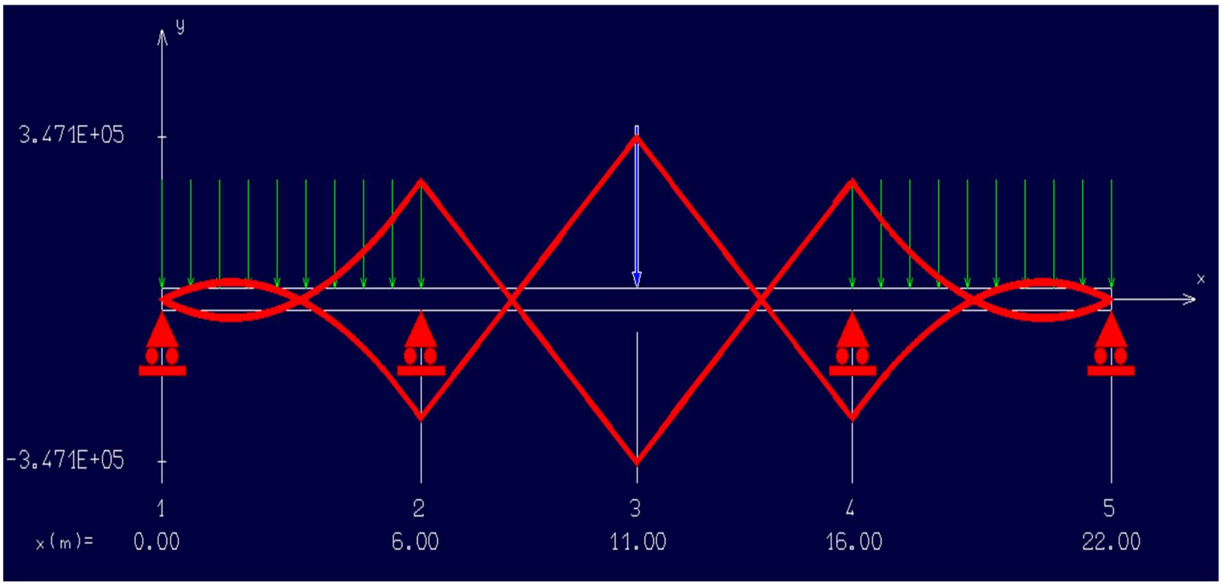


Fig. 10: Distribution of normal stresses (MPa) along the entire length of the beam

Observation

Both methods yield consistent results for the reactions at various supports of the beam, as well as for the shear forces, bending moments, and normal stresses along its entire length. In principle, the force method provides exact results, as it is based on analytical formulations. In contrast, the Finite element method (FEM) produces approximate results due to its reliance on discretizing the structure. However, the accuracy of the FEM can be remarkably high, with errors often approaching zero. One of the key advantages of the FEM, particularly through software like RDM6, lies in its ability to present results both graphically and numerically. This allows users to gain a clear understanding of the structural behavior, useful for identifying areas prone to stress concentration and enabling realistic and in-depth analysis. Consequently, this leads to optimal use of the structure and maximized performance benefits.

4. Static analysis of a continuous beam with two spans under uniform and concentrated loads using the three-moment technique

We consider a continuous beam ***(ABC)*** comprising two spans, with a constant flexural rigidity *EI*. The beam is subjected to a uniformly distributed load of ***q=6 kN/m*** over spans ***AB,*** and a concentrated load of ***P=40 kN*** applied at the midpoint of span ***BC***, figure 11.

Using the ***Three-Moment Technique***, we aim to determine:

- ✓ The reactions at the supports;
- ✓ The bending moment and shear force diagrams;
- ✓ Normal stress in the sections of the beam.

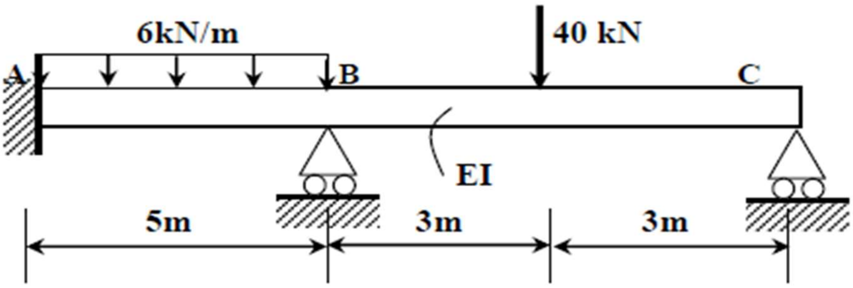


Fig. 11: Continuous beam subjected to various types of loads

For this type of structure, specific techniques must be employed to address the complexity of the task, notably by introducing the principle of fictitious modifications. This approach allows us to analyze the structure (a continuous beam) within the same framework used in the previous example. To achieve this, a fictitious support is introduced at point ***A***, along with an associated fictitious length L_0

Degree of hyperstaticity:

d=5−3=2; the structure is twice hyperstatic.

Based on the principle expressed in formula (1), we can write:

Point A:

$$0M_{A'} + 10M_A + 5M_B = 6EI\left(\omega_{AR} - \omega_{AL}\right) \quad (6)$$

Since $M_{A'}=0$, we can write:

$$10M_A + 5M_B = 6EI\left(\omega_{AR} - \omega_{AL}\right) \quad (7)$$

Point B:

$$5M_A + 22M_B + 6M_C = 6EI\left(\omega_{BR} - \omega_{BL}\right) \quad (8)$$

Since $M_C=0$, we can write:

$$5M_A + 22M_B = 6EI\left(\omega_{BR} - \omega_{BL}\right) \quad (9)$$

Referring to the figure 2, which shows the formulas for the rotations at the simple supports for each loading case, we can express the following expressions:

$$\omega_{AR} = -\frac{1}{24EI} q\ell^3$$

$$\omega_{AL} = 0$$

$$\omega_{BR} = -\frac{1}{16EI}P\ell^2$$

$$\omega_{BL} = \frac{1}{24EI}q\ell^3$$

Substituting these expressions into formulas (7) and (9), we obtain the following system:

$$10M_A + 5M_B = 6\left(-\frac{1}{24}q5^3 - 0\right)$$

$$5M_A + 22M_B = 6\left(-\frac{1}{16}P6^2 - \frac{1}{24}q5^3\right) \quad (10)$$

The system of equations (10) takes the following matrix form:

$$\begin{bmatrix} 10 & 5 \\ 5 & 22 \end{bmatrix}\begin{Bmatrix} M_A \\ M_B \end{Bmatrix} = \begin{Bmatrix} -187.5 \\ -727.5 \end{Bmatrix} \quad (11)$$

Solving the matrix equation system (11) leads to the following solution, which represents the bending moment values at supports ***A*** and ***B***.

$$\begin{Bmatrix} M_A \\ M_B \end{Bmatrix} = \begin{Bmatrix} -2.5\ kN.m \\ -32.5\ kN.m \end{Bmatrix}$$

Calculation of support reactions using the principle of decomposition

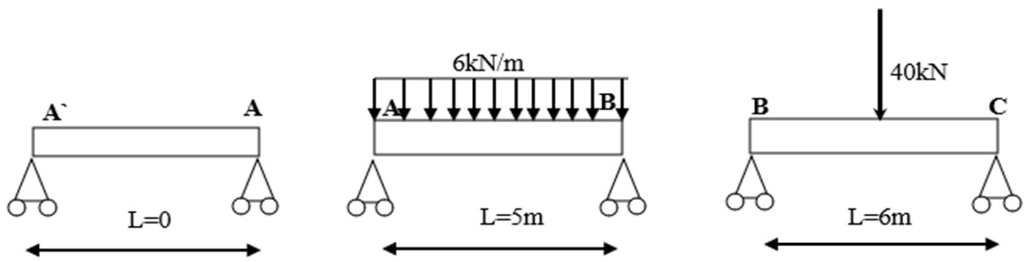


Fig. 12: Structural representation of a continuous beam divided according to the fictitious modification principle

For span AB

$$\sum F = 0 \rightarrow R_A + R_B = 30$$

$$\sum M = 0 \rightarrow 5R_B - 32.5 + 2.5 - 30 \times 2.5 = 0$$

Which gives:

$$R_A = 9kN$$

$$R_B = 21kN$$

For span BC

$$\sum F = 0 \rightarrow R_B + R_C = 40$$

$$\sum M = 0 \rightarrow 6R_C - 40 \times 3 + 32.5 = 0$$

Which gives:

$$R_B = 25.4kN$$

$$R_C = 14.6kN$$

Superposition of results:

Finally, the reaction values of the continuous beam, after applying the superposition principle, are determined and shown in figure 13.

$$R_A = 9kN$$

$$R_B = 21 + 25.4 = 46.4kN$$

$$R_C = 14.6kN$$

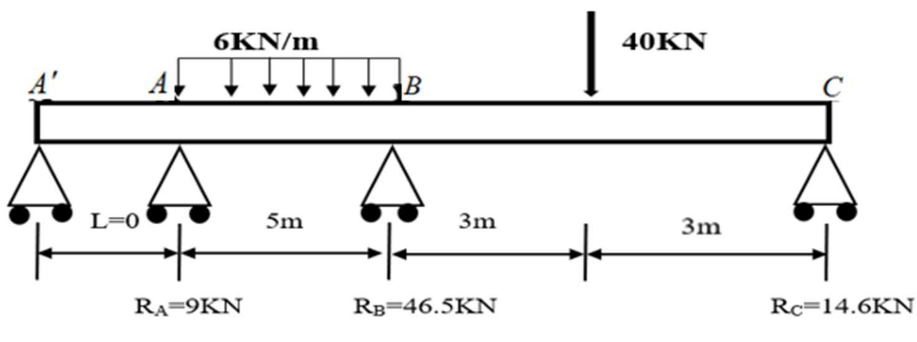


Fig. 13: Schematic illustrating the support reaction values (kN) along the entire length of the continuous beam

Shear force diagrams of the continuous beam

Once the reactions of the continuous beam have been determined using the force method, the shear force diagram along the entire length of the beam can be established by applying the principles of elasticity applicable to statically determinate systems. The graphical results are presented in figure 14.

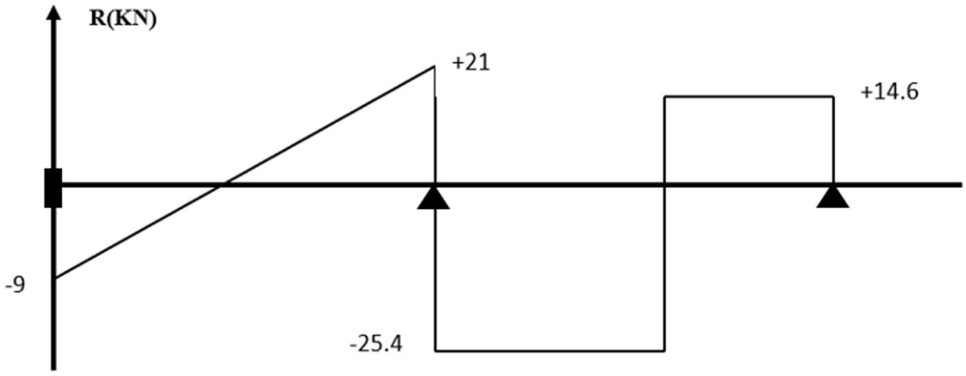


Fig. 14: Shear force (kN) distribution along the entire length of the beam

Bending moment diagram of the continuous beam.

After determining the reactions at the supports of the continuous beam, the next step—following the same approach used for the shear force—is to compute the bending moment values at various points along the structure. This leads to the construction of the corresponding bending moment diagram for the entire length of the beam. The graphical representation of the bending moment distribution is shown in figure 15.

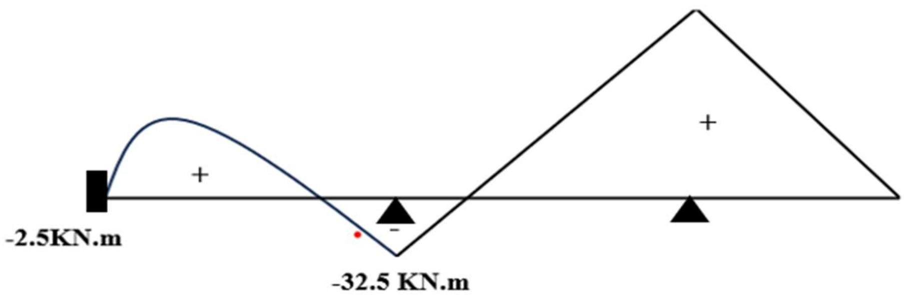


Fig. 15: Distribution of bending moments (kNm) along the entire length of the beam

Determination of stresses in the cross sections of the continuous beam

Referring to the diagram in figure 15, which shows that the maximum bending moment occurs at the midpoint of span ***BC*** with a value of $M_{\max} = 43.7\ kNm$, the maximum stress is expected at the same location and is calculated using the following formula:

$$\sigma_{\max} = \pm \frac{M_{\max}}{I} y_{\max}$$

I : is the moment of inertia of the section;

$y_{\max}$: is half the height of the beam's cross-section (fig. 7).

Comparison of results with the finite element method (FEM)

As previously mentioned, the results obtained using the force method need to be verified for accuracy. To this end, we now perform a comparison with the Finite Element Method. Following the same approach as in the previous example, we use the RDM6 software—based on the principles of the finite element method—to obtain both numerical and graphical outputs that illustrate the internal forces and stresses within the beam’s sections. This not only provides an alternative means of determining these quantities but also enables us to assess the consistency between the two methods.

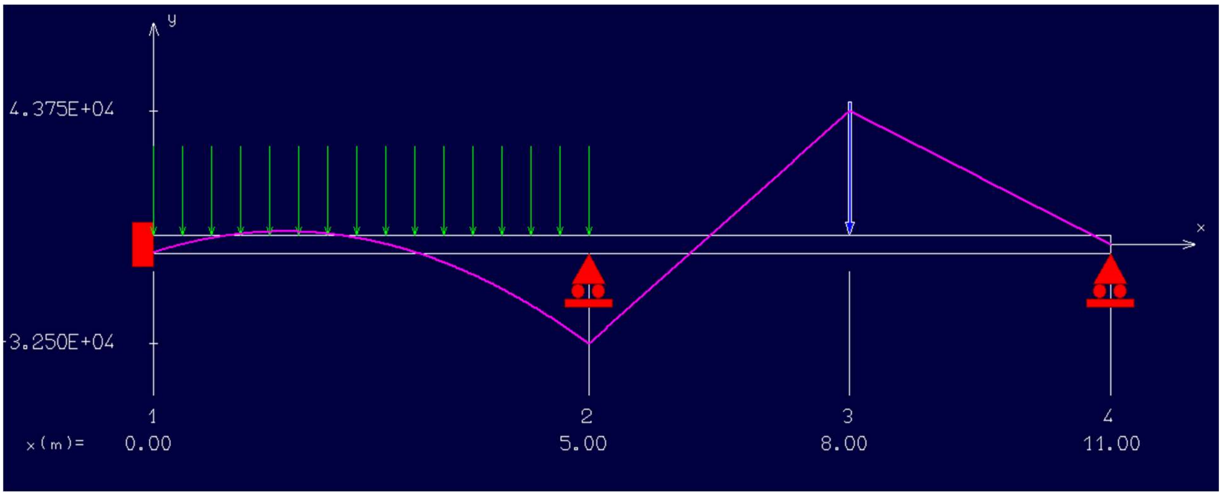


Fig. 16: Distribution of shear forces (kN) along the entire length of the continuous beam

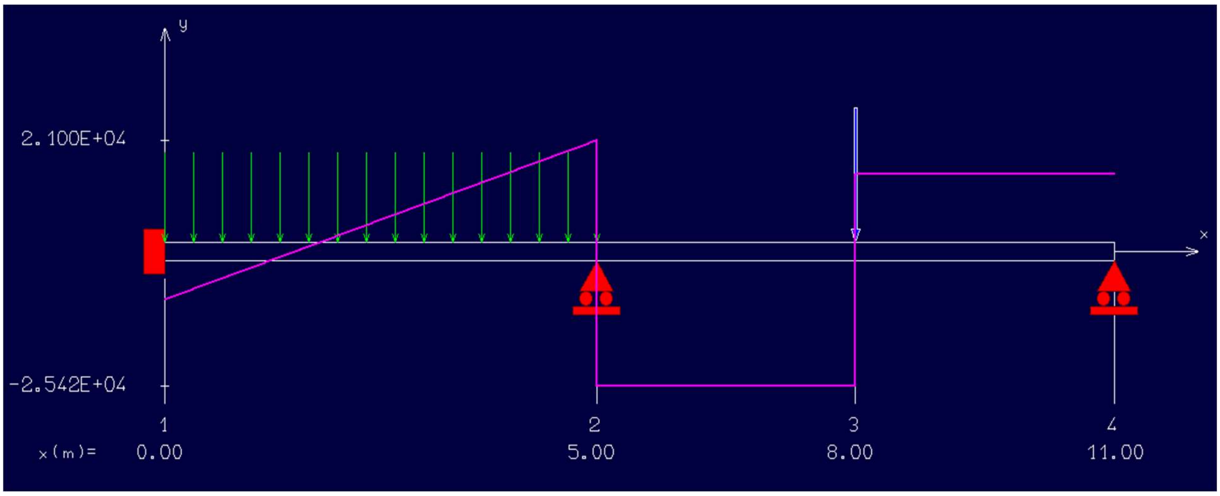


Fig. 17: Distribution of bending moments (kNm) along the entire length of the continuous beam

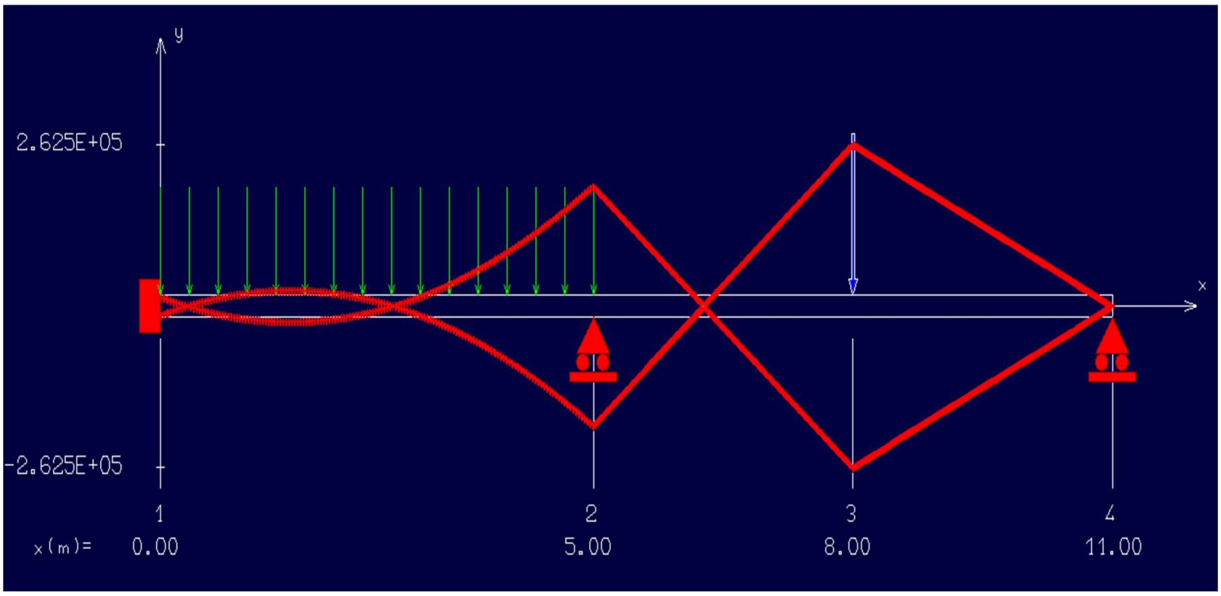


Fig. 18: Distribution of normal stress (MPa) along the entire length of the continuous beam (m)

Observation

Both methods made it possible to obtain results for the reactions at various supports of the continuous beam, as well as for the shear forces, bending moments, and normal stresses along its entire length. As mentioned earlier, the Force Method provides an exact solution, as it is based on analytical formulations. In contrast, the Finite Element Method (FEM) yields approximate results due to the discretization of the structure. However, the accuracy of the FEM can be remarkably high, with errors often approaching zero. One of the key advantages of the FEM—particularly when implemented in software such as RDM6—lies in its ability to present results both graphically and numerically. Moreover, it enables the analysis of structures with complex geometries, which represents a significant advantage that traditional analytical methods cannot offer.

5. Static analysis of a continuous beam with three spans under different types of external loads using the force method and verification with FEM

Beam Analysis

In the final example examined in this chapter, we analyze a continuous beam featuring various types of supports and subjected to multiple external loads. This structurally more complex system is illustrated in figure 19. The analysis begins with the determination of the unknown support reactions. Once these have been evaluated, the beam can be regarded as statically determinate. Subsequently, we compute the shear forces and bending moments along the entire length of the beam. These internal force distributions make it possible to determine the normal and shear stresses within the cross-sections of the structure.

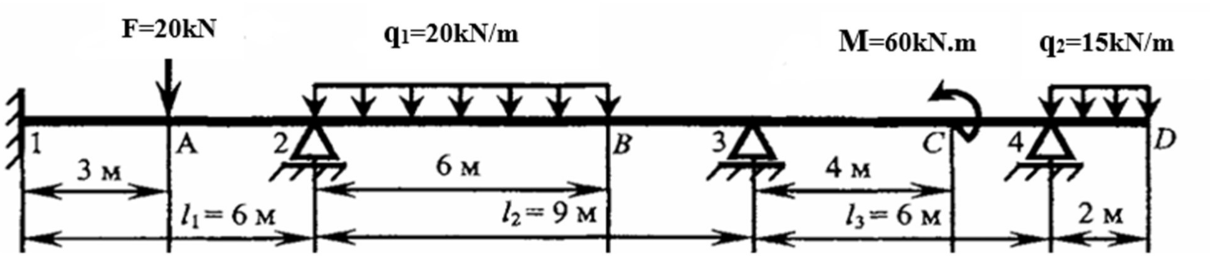


Fig. 19: Configuration of a continuous beam with various supports and external loads

All geometrical and physical data of the structure, including the values of the external loads, are presented in figure 19.

Structural analysis using the force method

Firstly, we should determine the degree of static indeterminacy of the beam illustrated in figure 19 using the formula:

$\boldsymbol{L = C_0 + 2H - 3D}$

Here, C_0 is the number of support constraints**;** H is the number of intermediate hinges**;** D is the number of disks.

For continuous beams (when $H = 0$ and $D = 1$), it simplifies to: $L = C_0 - 3$

In our case, $C_0 = 6$. Then $L = 6 - 3 = 3$, which means the beam is statically indeterminate to the third degree.

After determining the degree of static indeterminacy, the next step is to select the primary system. This is done by introducing hinges at the sections above the intermediate supports and at the fixed end, as illustrated in figure 20. The unknowns in this analysis are the support bending moments, denoted X_1, X_2, X_3.

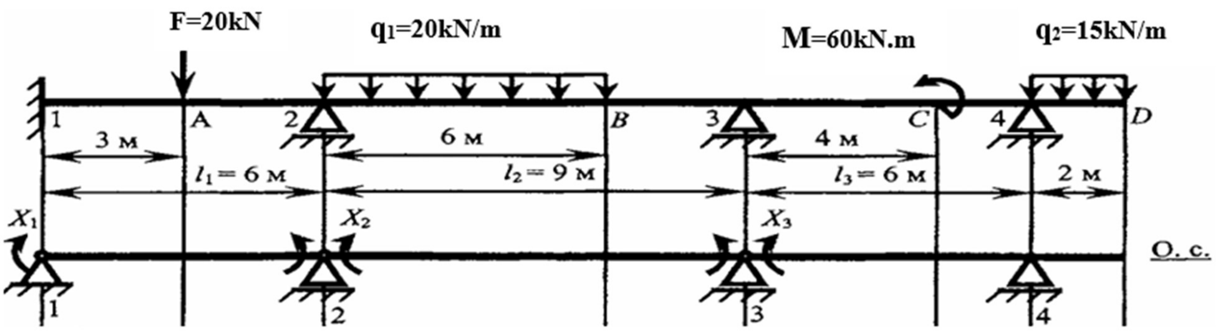


Fig. 20: Structural representation illustrating the applied loads and support conditions of the continuous beam

The canonical system of equations based on the force method principle for this case is given by:

$$\begin{cases} \boldsymbol{\delta}_{11}\mathbf{X}_1 + \boldsymbol{\delta}_{12}\mathbf{X}_2 + \boldsymbol{\delta}_{13}\mathbf{X}_3 + \boldsymbol{\Delta}_{1F} = \mathbf{0} \\ \boldsymbol{\delta}_{21}\mathbf{X}_1 + \boldsymbol{\delta}_{22}X_2 + \boldsymbol{\delta}_{23}\mathbf{X}_3 + \boldsymbol{\Delta}_{2F} = \mathbf{0} \\ \boldsymbol{\delta}_{31}\mathbf{X}_1 + \boldsymbol{\delta}_{32}\mathbf{X}_2 + \boldsymbol{\delta}_{33}\mathbf{X}_3 + \boldsymbol{\Delta}_{3F} = \mathbf{0} \end{cases} \qquad (12)$$

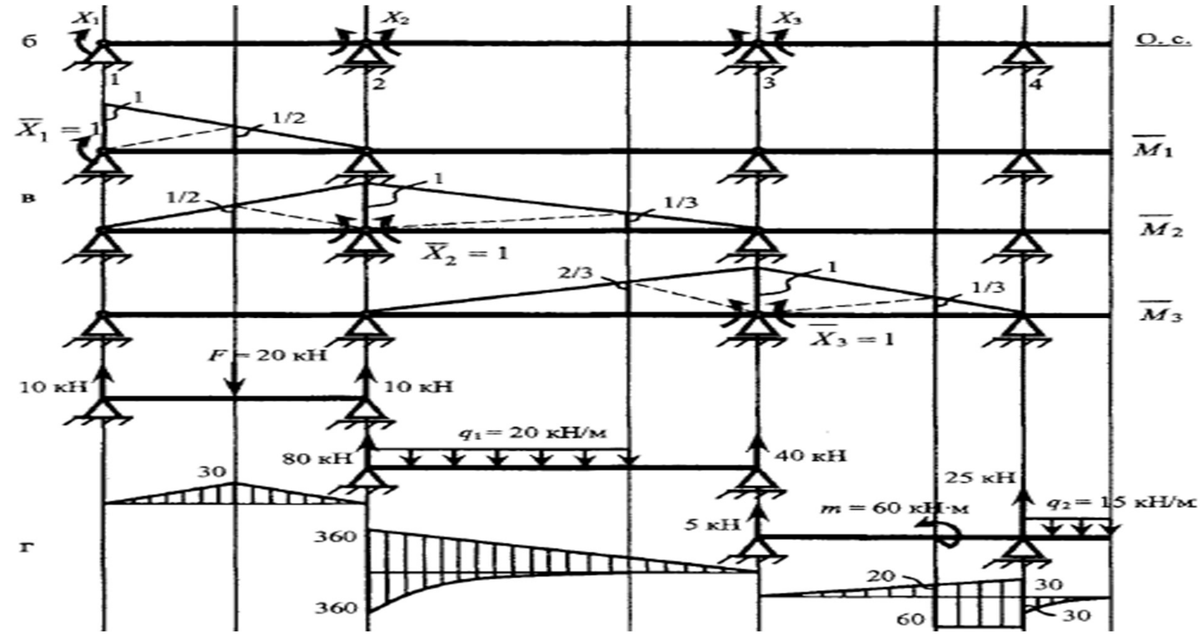


Fig. 21: Presentation of the unit moment diagrams corresponding to the three redundant unknowns

The construction of the unit moment diagrams corresponding to the three redundant unknowns for the structure under study leads to the different schemes shown in figure 21. Each span of the continuous beam is treated as an independent simply supported beam.

The various coefficients of the canonical system of equations (12) are calculated using Maxwell–Mohr's integral formula, which gives:

$$\delta_{11} = \sum \int \left(\bar{M}^T \bar{M} \right) dz = \frac{1}{2EJ} \times 6 \times 1 \times \frac{2}{3} \times 1 = \frac{2}{EJ};$$

$$\delta_{22} = \frac{5}{EJ}; \ \delta_{33} = \frac{5}{EJ};$$

$$\delta_{12} = \delta_{21} = \frac{1}{EJ}; \ \delta_{13} = \delta_{31} = 0; \ \delta_{23} = \frac{1.5}{EJ};$$

$$\Delta_{1F} = \frac{45}{EJ}; \ \Delta_{2F} = \frac{525}{EJ}; \ \Delta_{3F} = \frac{430}{EJ}.$$

The canonical system of equations (12) can be expressed in the following matrix form:

$$\begin{bmatrix} \boldsymbol{\delta}_{11} & \boldsymbol{\delta}_{12} & \boldsymbol{\delta}_{13} \\ \boldsymbol{\delta}_{21} & \boldsymbol{\delta}_{22} & \boldsymbol{\delta}_{23} \\ \boldsymbol{\delta}_{31} & \boldsymbol{\delta}_{32} & \boldsymbol{\delta}_{33} \end{bmatrix} \begin{Bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \mathbf{X}_3 \end{Bmatrix} = - \begin{Bmatrix} \boldsymbol{\Delta}_{1F} \\ \boldsymbol{\Delta}_{2F} \\ \boldsymbol{\Delta}_{3F} \end{Bmatrix} \tag{13}$$

After substituting the expressions for each parameter, equation (13) becomes:

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 5 & 1.5 \\ 0 & 1.5 & 5 \end{bmatrix} \begin{Bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \mathbf{X}_3 \end{Bmatrix} = - \begin{Bmatrix} 45 \\ 525 \\ 430 \end{Bmatrix} \tag{14}$$

Solving the system of equations (14) yields the following values for the unknown redundants:

$X_1 = +23.61\, kN.m$

$X_2 = -92.22\, kN.m$

$X_3 = -58.33\, kN.m$

Determination of support reactions in each span of the continuous beam based on the principle of decomposition

Span AB

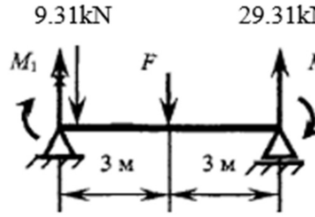


Fig. 22. Schematic representation of the span ***AB*** based on the decomposition principle

$$\begin{cases} R_A + R_B = 20 \\ \sum M\big|_A = 0 \Rightarrow R_B \times 6 - 20 \times 3 - 23.61 - 92.22 = 0 \end{cases} \rightarrow \begin{cases} R_A = -9.31 kN \\ R_B = 29.31 kN \end{cases}$$

Span BC

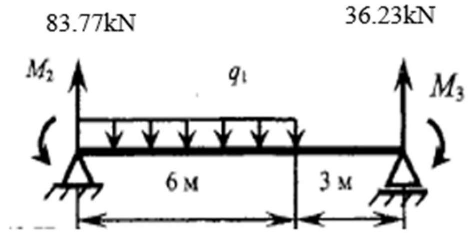


Fig. 23. Schematic representation of the span ***BC*** based on the decomposition principle

$$\begin{cases} R_B + R_C = 120 \\ \sum M\Big|_B = 0 \Rightarrow R_C \times 9 - 20 \times 6 \times 3 + 92.22 - 58.33 = 0 \end{cases} \rightarrow \begin{cases} R_B = 83.77kN \\ R_C = 36.23kN \end{cases}$$

Span CD

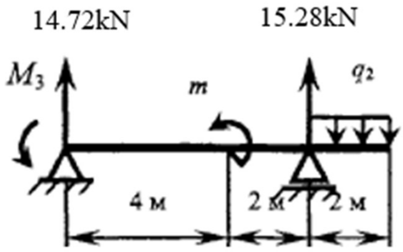


Fig. 24. Schematic representation of the span ***CD*** based on the decomposition principle

$$\begin{cases} R_C + R_D = 30 \\ \sum M\Big|_C = 0 \Rightarrow R_D \times 6 - 15 \times 2 \times 7 + 60 + 58.33 = 0 \end{cases} \rightarrow \begin{cases} R_C = 14.72kN \\ R_D = 15.28kN \end{cases}$$

Superposition of reactions

Finally, the reaction values of the continuous beam, after applying the superposition principle, are determined as follows, as shown graphically in figure 25.

$$\begin{aligned} R_A &= -9.31kN \\ R_B &= 83.77 + 29.31 = 113.08kN \\ R_C &= 36.23 + 14.72 = 50.95kN \\ R_D &= 15.28kN \end{aligned} \qquad (15)$$

Bending moment diagrams of the continuous beam

The classical method commonly used in the field of strength of materials is applied to determine the bending moment values at each support of the continuous beam, taking into account both the values of the redundants and the external loads. This procedure yields the following numerical results and also enables the construction of the bending moment diagram along the length of each span of the continuous beam, as illustrated graphically in figure 25.

$$M_1 = 1\times 23.61 = 23.61\ \text{kN}\cdot\text{m}$$

$$M_A = \frac{1}{2}\times 23.61 + \frac{1}{2}\times(-92.22) + 30 = -4.31\ \text{kN}\cdot\text{m};$$

$$M_2 = 1\times(-92.22) = -92.22\ \text{kN}\cdot\text{m};$$

$$M_B = \frac{1}{3}\times(-92.22) + \frac{2}{3}\times(-58.33) + 120 = 50.37\ \text{kN}\cdot\text{m};$$

$$M_3 = 1\times(-58.33) = -58.33\ \text{kN}\cdot\text{m};$$

$$M_4 = -30\ \text{kN}\cdot\text{m};$$

$$M_D = 0;$$

$$M_C^L = \frac{1}{3}\times(-58.33) + 20 = 0.56\ \text{kN}\cdot\text{m};$$

$$M_C^R = 0.56 - 60 = -59.44\ \text{kN}\cdot\text{m}.$$

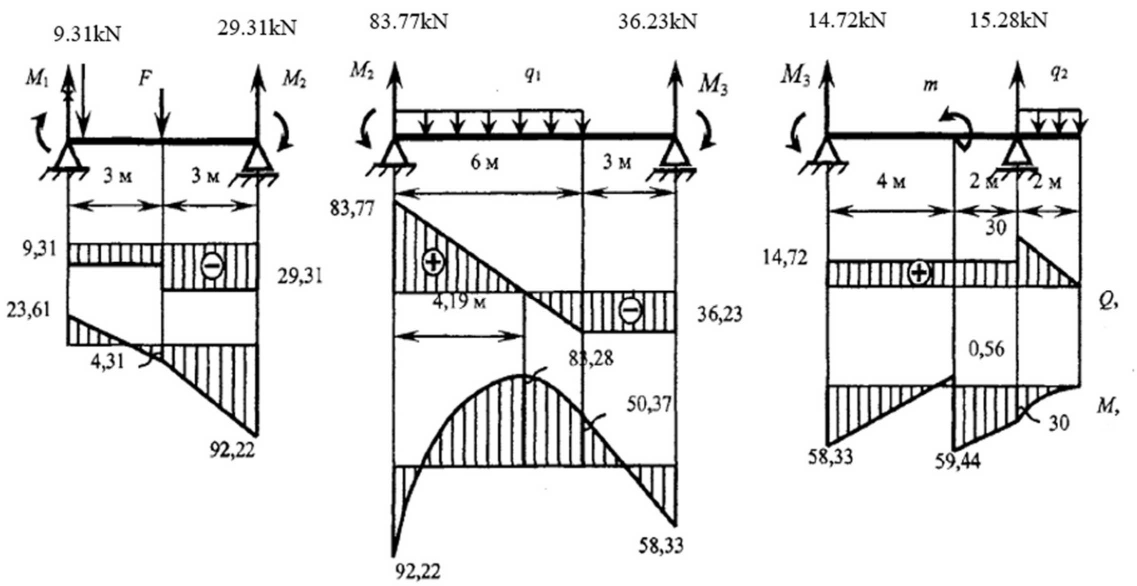


Fig. 25: Bending moment and shear forces diagrams along each span of the continuous beam, resulting from the external loads and support reactions to which the beam is subjected

After determining the bending moments along each span of the continuous beam separately, we can combine them to obtain the final bending moment diagram along the entire length of the beam, as shown in figure 26.

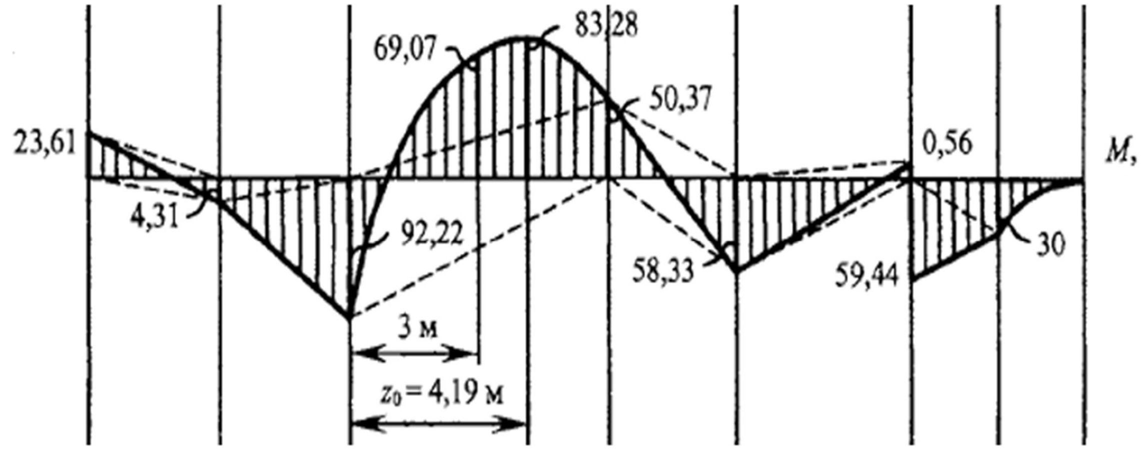


Fig. 26: Bending moment (kNm) diagram along the entire length of the continuous beam, resulting from the external loads and support reactions to which the beam is subjected

Shear force diagrams of the continuous beam

In the same manner as for the bending moment, the classical method commonly used in the field of strength of materials is also applied to determine the shear force values at each support of the continuous beam, taking into account both the values of the redundants and the external loads. This procedure yields the numerical results, expressed in terms of the support reaction values given above by expressions (15), and also enables the construction of the shear force diagram along each span of the continuous beam, as illustrated graphically in figures 25 and 27.

After determining the shear forces along each span of the continuous beam separately, we can combine them to obtain the final shear forces diagram along the entire length of the beam, as shown in figure 27.

Fig. 27: Shear force (kN) diagram along the entire length of the continuous beam, resulting from the external loads and support reactions to which the beam is subjected

Comparison of results with the finite element method (FEM)

To ensure the certainty of the results obtained using the force method, it is essential to compare them against those derived from an alternative, well-established technique. In this context, a comparative analysis was conducted using the finite element method (FEM), wherein the structure was modeled and analyzed following the same procedure as in the preceding examples, utilizing the RDM6 software. This software generates both numerical data and graphical representations, including internal force distributions and stress diagrams for various sections of the beam. Adopting this dual approach not only offers an independent means of evaluating internal forces and moments, but also facilitates a direct comparison between the two methods, thereby enabling a rigorous assessment of the consistency and reliability of the results produced by the force method.

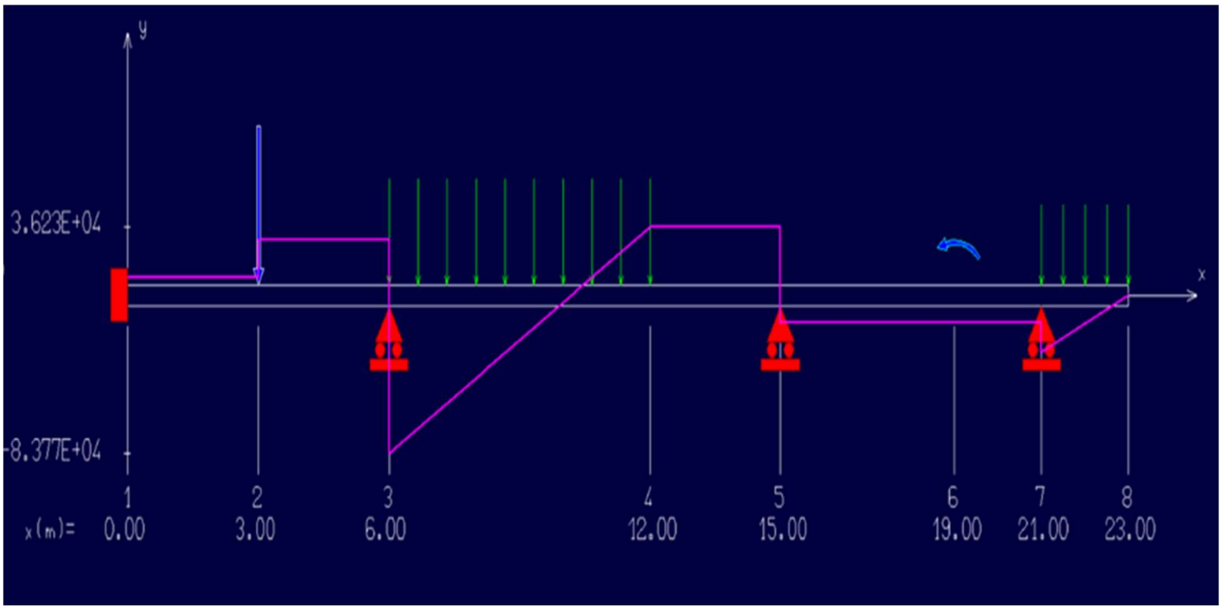


Fig. 28: Shear force (kN) distribution along the entire length of the beam

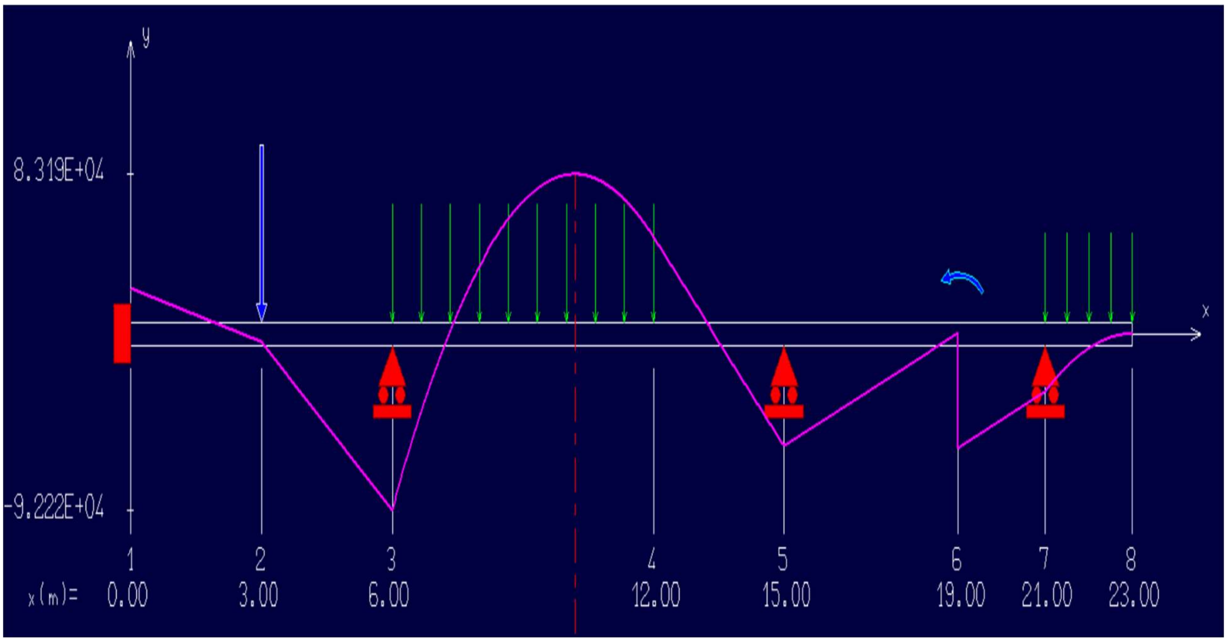


Fig. 29: Distribution of bending moments (kNm) along the full length of the beam

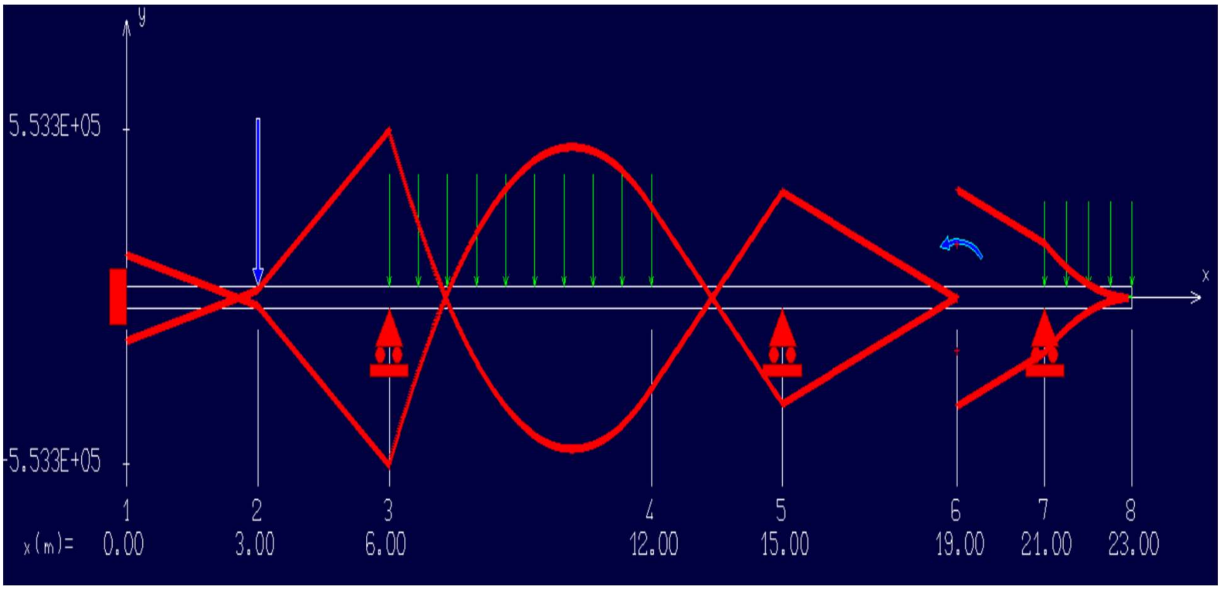


Fig. 30: Distribution of normal stresses (MPa) along the entire length of the beam

Observation

To validate the results obtained through the force method, a comparison with those produced by an established numerical technique is necessary. In this study, the finite element method (FEM) was employed for this purpose, using the RDM6 software to analyze the structure following the same modeling approach as in the earlier examples. The software provides comprehensive output in both graphical and numerical formats, including internal force distributions and stress diagrams across the beam sections. This complementary methodology not only offers an independent framework for evaluating internal actions, but also enables a clear and effective comparison between the analytical and numerical solutions, thereby reinforcing the reliability and coherence of the force method results.

6. Conclusion

In this chapter, we studied the force method, an analytical tool for determining the internal forces of continuous beams under different loads. First, the basic concepts and principles of the force method are clearly explained. Emphasis is placed on identifying redundant forces and establishing compatibility conditions, which are essential for analyzing complex structures using this traditional method.

To illustrate the practical application of the force method, several detailed examples are given. These examples include various loading scenarios and support arrangements commonly seen in civil and mechanical engineering. By performing these calculations step by step, it is shown how to determine internal forces such as bending moment and shear force. This not only deepens the theory introduced previously, but also provides simple guidance for students and engineers who encounter similar problems.

An important part of this chapter is the comparison of the results of the force method with those of the finite element method (FEM). This comparison highlights the advantages and disadvantages of both methods. The force method is well suited for hand calculations and helps to understand the force distribution in a structure. However, it can become more difficult and time consuming as complexity increases. In contrast, the FEM, supported by advanced computational tools, offers greater flexibility and efficiency when analyzing complex structures, unusual shapes, and varying material properties.

The comparison showed that both methods produced similar results for the cases analyzed, confirming the reliability of the Force Method for moderate structural systems. At the same time, it demonstrated how useful FEM is for handling more complex situations where traditional methods may struggle.

By combining both analytical and numerical approaches in this chapter, we offered a well-rounded method for analyzing continuous beams. This dual approach gives students, researchers, and engineers essential tools for solving a wide range of structural problems. It also emphasizes the importance of mastering classic techniques like the Force Method, not just for verifying numerical results but also for gaining a deeper understanding of how structures behave.

In summary, this chapter successfully explained the theoretical basis of the Force Method, applied it through practical examples, and compared its results with those from FEM. The knowledge gained here will serve as a strong foundation for the following chapters, where more advanced topics in structural analysis and design will be explored. It also highlights the value of combining traditional and modern methods in engineering practice to ensure accuracy, efficiency, and a thorough understanding of structural performance.

Chapter 3

Analysis of a Frames Under External Load Using the Force Method: Comparison with the Finite Element Method (FEM)

Chapter 3: *Analysis of a Frames Under External Load Using the Force Method: Comparison with the Finite Element Method (FEM)*

1. Introduction

This chapter focuses on the stress analysis of frame structures subjected to external loads, using both the Force Method and the Finite Element Method (FEM). It continues the comparative investigation initiated in Chapter 2, which was limited to continuous beams, by extending the study to frames—structural systems composed of several interconnected members arranged in various orientations (typically horizontal, vertical, and inclined) and joined by rigid connections. These joints are capable of transmitting not only axial and shear forces but also bending moments, making the internal force distribution more complex.

Compared to continuous beams, frame structures typically exhibit a higher degree of static indeterminacy, especially when composed of multiple loops or redundant members. This complexity increases the mathematical and computational effort required for their analysis. Nevertheless, understanding how these structures behave under external loads is essential in many fields of engineering, such as civil, mechanical, and aerospace applications.

The main objective of this chapter is to perform a detailed analysis of a statically indeterminate frame using the Force Method and to compare the results with those obtained through the Finite Element Method. This comparison serves two purposes:

- ✓ To demonstrate the application and implementation of both analytical and numerical methods on frame structures.
- ✓ To evaluate and interpret the similarities and differences in the outcomes, highlighting each method's strengths, limitations, and suitability depending on the engineering context.

The content of this chapter is organized as follows:

- Section 3.2 introduces the theoretical background required for analyzing frame structures with the Force Method and the Finite Element Method.
- Section 3.3 presents a complete worked example in which a representative frame under external load is analyzed using both approaches.
- Section 3.4 compares and discusses the results, including internal force distributions, nodal displacements, and reaction forces.
- Section 3.5 provides a summary of the main findings and highlights the key conclusions drawn from this comparative study.

2. Theoretical framework

The accurate analysis of frame structures under external loads requires a thorough understanding of the underlying theoretical principles that govern their mechanical behavior. This section provides this essential background by presenting the analytical and numerical foundations of the Force Method and the Finite Element Method (FEM) as applied specifically to plane frame structures. Frame structures are inherently more complex than simple beams due to their configuration: they consist of multiple interconnected members—often in horizontal, vertical, or inclined positions—joined by rigid connections that transmit not only axial and shear forces but also bending moments. This structural arrangement introduces additional degrees of static indeterminacy, making frames an ideal subject for comparing the capabilities of the Force Method and FEM.

The Force Method, a classical analytical technique, addresses static indeterminacy by reducing the structure to a statically determinate form and applying compatibility conditions to solve for the unknown redundant forces. This method is particularly insightful for engineers because it highlights the internal force flow and the role of each redundant element. However, its manual application becomes cumbersome as the degree of indeterminacy increases, especially in the case of complex frame geometries.

In contrast, the Finite Element Method is a modern computational technique that systematically breaks down the structure into smaller elements and builds a global system of equations based on stiffness relations. It is especially well-suited for frames due to its ability to handle large numbers of elements, varying member properties, and complex boundary conditions with high accuracy and efficiency.

This section is divided into three parts:

- Subsection 2.1 defines the structural characteristics of frame systems and the typical types of internal forces and displacements encountered in their analysis.
- Subsection 2.2 explains the step-by-step application of the Force Method to frame structures, including the selection of redundants, formulation of compatibility conditions, and determination of internal forces.
- Subsection 2.3 presents the finite element formulation for frames, covering the derivation and assembly of stiffness matrices, treatment of boundary conditions, and calculation of internal responses.

By establishing a solid theoretical foundation, this section prepares the reader for the practical implementation and comparative evaluation of both methods in the worked example that follows in Section 3.

2.1.Theoretical framework

A frame is a structural system made up of straight members interconnected at nodes (joints), forming either open or closed configurations. Each member may be subjected to:

- Axial forces (tension or compression),
- Shear forces, and
- Bending moments.

Frame joints are typically modeled as rigid, meaning that the angle between connected members remains unchanged under loading. This rigidity allows for the transmission of moments between members, which distinguishes frames from pin-jointed trusses.

The analysis of frames must account for:

- ✓ Multiple degrees of freedom at each node: two translations and one rotation in planar frames,

- ✓ Redundancy: frames are often statically indeterminate, meaning equilibrium equations alone are insufficient to determine all internal forces,
- ✓ Compatibility of deformations: displacements and rotations at joints must be consistent with the continuity of connected members.

These characteristics make frame analysis a challenging but essential component of structural mechanics.

2.2. Force method applied to frame structures

The Force Method, or method of consistent deformations, provides a classical approach to solving statically indeterminate frames. The general procedure involves the following steps:

Determine the degree of static indeterminacy

Calculate the number of unknown reactions and internal forces that cannot be resolved using equilibrium alone.

Select redundant forces

Choose a set of redundant forces (e.g., moments or reactions) whose removal reduces the structure to a statically determinate primary system.

Formulate compatibility conditions

Impose compatibility equations to ensure that displacements or rotations at the locations of the removed redundants are consistent with the original structure.

Compute flexibility coefficients

For each compatibility equation, determine how much displacement is caused by a unit load acting in the direction of each redundant. These values form the flexibility matrix.

Solve for redundant forces

Use the compatibility equations and flexibility matrix to solve for the unknown redundants.

Determine internal forces and reactions

Superimpose the effects of the external loads and the redundants on the basic determinate structure to compute internal forces and reactions.

This method provides clear insight into the structural behavior and is useful for hand calculations, but it becomes increasingly complex for systems with high indeterminacy or non-uniform properties.

2.3.Finite element formulation for frame analysis

The Finite Element Method (FEM) offers a powerful and general framework for analyzing frames, especially when computational tools are available. The procedure involves the following key steps:

Discretization

Divide the frame into a series of beam elements, each connected at nodes. In a planar frame, each node has three degrees of freedom: two translations (horizontal and vertical) and one rotation.

Element stiffness matrix

For each element, construct a local stiffness matrix that relates the nodal forces to nodal displacements based on beam theory. The matrix includes contributions from both axial and bending behavior.

Transformation to global coordinates

If elements are not aligned with the global coordinate axes, apply coordinate transformations to express their stiffness matrices in global terms.

Assembly of the global stiffness matrix

Combine all element matrices into a single global stiffness matrix for the entire structure, respecting connectivity at nodes.

Application of boundary conditions

Modify the global system to account for supports and constraints by eliminating or adjusting the corresponding degrees of freedom.

Solution of the system

Solve the resulting system of linear equations to find nodal displacements.

Post-processing

Use the computed displacements to evaluate internal axial forces, shear forces, and bending moments within each element, as well as reaction forces at the supports.

The FEM is highly adaptable and precise, especially for frames with irregular geometry, variable cross-sections, or non-uniform material properties. It is widely used in engineering practice due to its robustness and automation in commercial software.

3. Worked example: comparative analysis of a frame using the force method and FEM

This section presents a detailed comparative study between the Force Method and the Finite Element Method through the analysis of a representative plane frame structure subjected to external horizontal loads. The objective is to demonstrate how each method can be applied in practice and to compare the resulting internal forces, displacements, and support reactions.

3.1. Description of the frame and loading

We consider a rigid planar frame composed of two perpendicular members:

A vertical column ***AB*** of height ***h=3 m*** and a horizontal beam ***BC*** of length ***L=5 m*** (fig. 1).

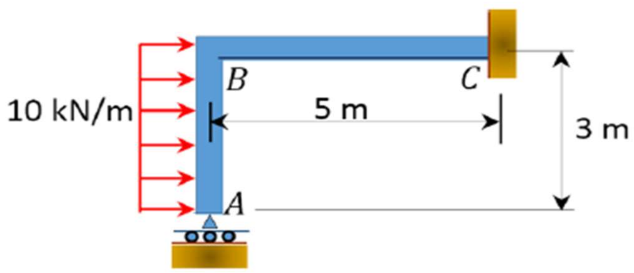


Fig. 1: Planar frame composed of two perpendicular members

3.2. Structural analysis of the frame using the force method

The base of the column at node ***A*** is simply supported (considered only the vertical reaction V_A).

The right end at node ***C*** is fixed (vertical reaction V_C, horizontal reaction H_C, moment M_C).

A uniform distributed load ***q=10 kN/m*** is applied horizontally along member ***AB***.

Objective is to compute the internal forces and displacements using the Force Method, then compare with the Finite Element Method (FEM).

The horizontal reaction H_A at ***A*** is ignored since ***A*** only provides vertical support.

The redundant chosen is the vertical reaction at support ***A***, denoted V_A. therefore, the structure is statically indeterminate to the first degree:

The compatibility equation for this indeterminate frame is:

$$\Delta_{AP} + V_A \delta_{AA} = 0 \quad (1)$$

The flexibility or compatibility coefficients ***Δ*** and ***δ*** are computed by graph multiplication method, as shown in fig. 2 and fig. 3:

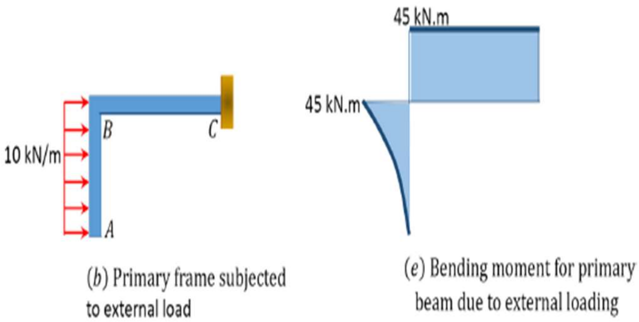


Fig. 2: Bending moment diagram due to the external horizontal load

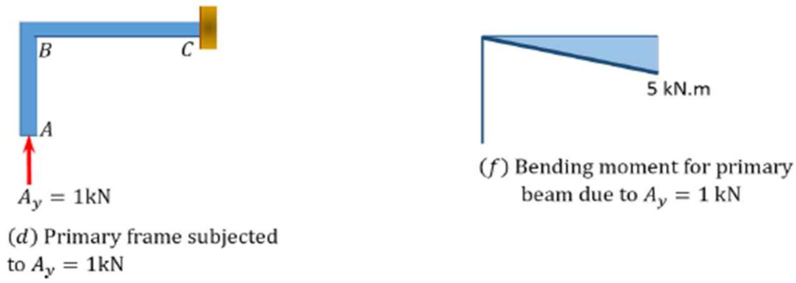


Fig. 3: Bending moment diagram due to the redundant force

$$\delta_{AA} = \frac{1}{2EI}(5\times5)\left(\frac{2\times5}{3}\right) = \frac{41.67}{EI}$$

$$\Delta_{AP} = -\frac{1}{EI}45\times\frac{1}{2}\times5\times5 = -\frac{562.5}{EI}$$

Substituting the flexibility coefficients into the compatibility equation (1), we found:

$$V_A = 13.5\ kN$$

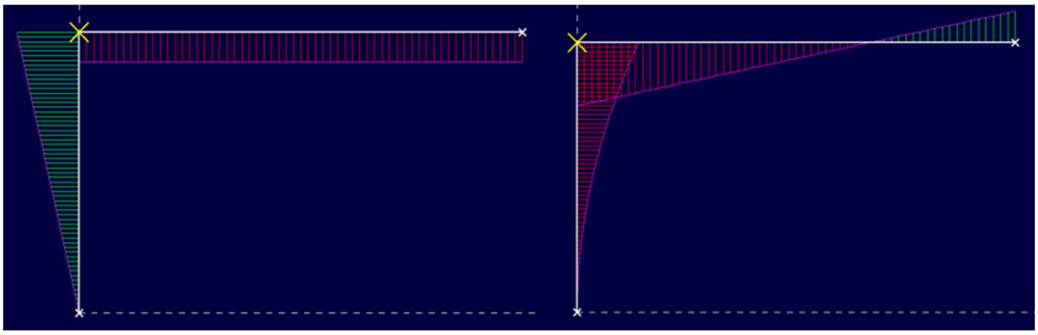
Computing the reactions at the fixed support C

$$\sum M\Big|_c = 0 \rightarrow -13.5\times5+10\times3\times1.5+M_c = 0 \rightarrow M_c = 22.56\ kNm$$

$$\sum F_y = 0 \rightarrow -V_C + 13.5 = 0 \rightarrow V_C = 13.5\ kN$$

$$\sum F_x = 0 \rightarrow -H_c + 10\times3 = 0 \rightarrow H_c = 30\ kN$$

Finally, the shear forces and bending moments diagrams over the entire frame are presented in the figure 4.



(a) (b)

Fig. 4: Shear forces (a) and bending moments (b) diagrams over the entire frame

The Force Method effectively determines the unknown vertical reaction V_A at the pinned support located at node A, along with all remaining reactions at the fixed support at node C. Once these support reactions are known, the shear force and bending moment diagrams can be constructed for the entire frame, thereby enabling the determination of both normal and shear stresses at any section. This confirms that the frame has been fully analyzed under the applied horizontal uniformly distributed load.

3.3. Structural analysis of the frame using the finite element method FEM

Let's consider the same frame descripted above in the previous analysis using the FEM principle [31] :

Discretization of the structure according to the FEM

The structure is discretized into two elements as provided in this table:

Element	Node 1	Node 2	Length of the element	Orientation
1	A	B	3 m	Vertical
2	B	C	5 m	Horizontal

Degrees of freedom

Each node of the structure possesses three degrees of freedom:

• Horizontal displacement ***u***

• Vertical displacement ***v***

• Rotation ***θ***

Node	DOFs
A	***u1, v1, θ1***
B	***u2, v2, θ2***
C	***u3, v3, θ3***

Boundary conditions

• Node ***A: u1 = 0, v1 = 0***

• Node ***C: u3 = 0, v3 = 0, θ3 = 0***

The remain unknowns of the frame under study are: ***u2, v2, θ1, θ2***

Local stiffness matrices

Each element uses the standard 2D beam element stiffness matrix in local coordinates.

$$[K^e]=\begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

Equivalent nodal forces UDL

For the vertical member ***AB*** subjected to a horizontal uniformly distributed load ***q***, the equivalent nodal loads are given by the following force vector:

$$f^{(e)} = \frac{qL}{2}\begin{bmatrix} 1 \\ 0 \\ \frac{L}{6} \\ 1 \\ 0 \\ -\frac{L}{6} \end{bmatrix}$$

Final procedure

(1) Assemble the global stiffness matrix ***K***;
(2) Assemble the global load vector ***F***;
(3) Apply the boundary conditions to reduce the size of the matrix system;
(4) Solve the reduced system to determine the unknown displacements;
(5) Compute the support reactions, and consequently determine the internal forces and stresses.

Results

The final solution leading to the reactions at the supports of the frame is given in this table:

Reactions	Value
VA	***+13.5 kN***
HC	***-30 kN***
VC	***-13.5 kN***
MC	***+22.5 kNm***

The Finite Element Method provides the same support reaction values for the frame, thus confirming the reliability of the results obtained via the force method. In addition, the FEM also computes the nodal displacements, offering a more comprehensive view of the structural response.

Remark

The Finite Element Method validates and complements the solution obtained by the Force Method. It yields identical support reactions and further provides a complete displacement field as well as the internal force distribution throughout the frame. The consistency between the classical and numerical approaches highlights the effectiveness of the FEM in analyzing statically indeterminate structures subjected to complex loading conditions and geometries.

4. Analysis of a frame with two vertical parallel members and a horizontal member under various external loads

This study focuses on the application of the Force Method and the Finite Element Method (FEM) to determine the support reactions of the structure and, consequently, to compute the internal forces and normal stresses within the frame's sections. The objective is to verify that the structure maintains static equilibrium—a fundamental principle in mechanics and structural analysis—and is therefore capable of withstanding the applied external loads in its intended application.

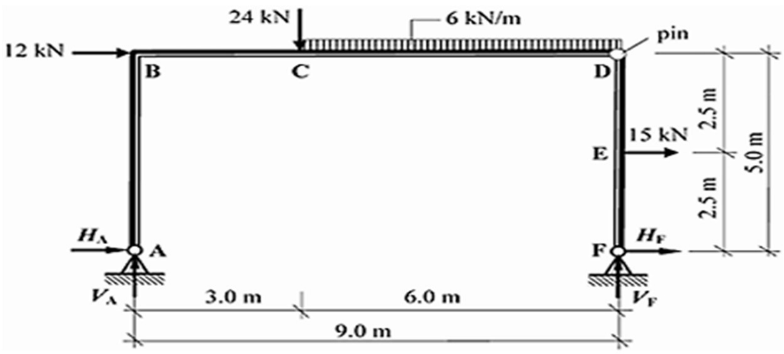


Fig. 5: Frame under multiple external loads and different support conditions

4.1. Analysis of the frame using the force method

Based on the three equations of equilibrium and the principle that the moments are zero at a pinned connection, the following expressions can be written:

$$\sum F_y = 0 \rightarrow V_A - 24.0 - 6.0 \times 6.0 + V_f = 0$$

$$\sum F_x = 0 \rightarrow H_A + 12.0 + 15.0 + H_f = 0$$

$$\sum M\big|_A = 12.0 \times 5.0 + 24.0 \times 3.0 + 6.0 \times 6.0 \times 6.0 + 15.0 \times 2.5 - V_f \times 9.0 = 0$$

$$\sum M_{pin} = 0 \rightarrow -15.0 \times 2.5 - H_f \times 5.0 = 0 \rightarrow H_f = 7.5\ kN$$

After determining $H_f = 7.5\ kN$, we can determine the others reactions using the above equations of equilibrium:

$$\sum F_x = 0 \rightarrow H_A = -19.5\ kN$$

$$\sum M\big|_A = 0 \rightarrow V_f = 42.83\ kN$$

$$\sum F_y = 0 \rightarrow V_A = 17.17\ kN$$

The graphical illustration of the reactions values is presented in the figure 6.

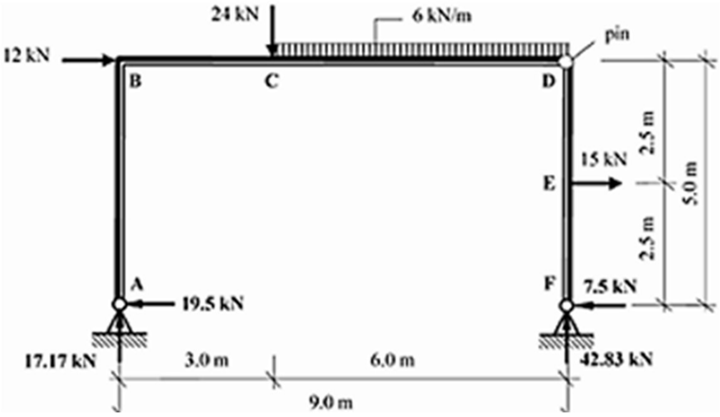


Fig. 6: Schematic representation of the frame reactions

The values of the moments at the nodes of the frame are determined under the assumption that positive bending moments induce tension on the inner side of the frame, as follows:

$$M_B = (19.5 \times 5.0) = 97.50\ kNm$$

$$M_C = (17.17 \times 3.0) + (19.5 \times 5.0) = 149.0\ kNm$$

$$M_D = 0(pin)$$

$$M_E = -(7.5 \times 2.5) = -18.75 kNm$$

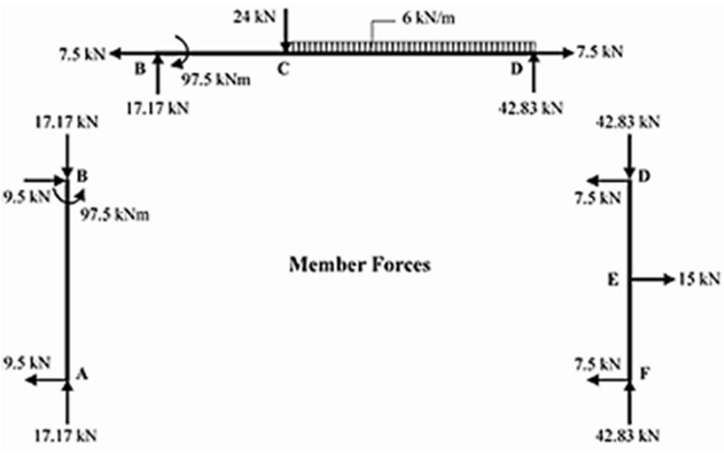


Fig. 7: Schematic representation of the frame members showing the applied forces and moments

Once the support reactions are known, the frame becomes a statically determinate structure, allowing for the construction of the bending moment, shear force, and axial force diagrams, as shown in figures 8 and 9.

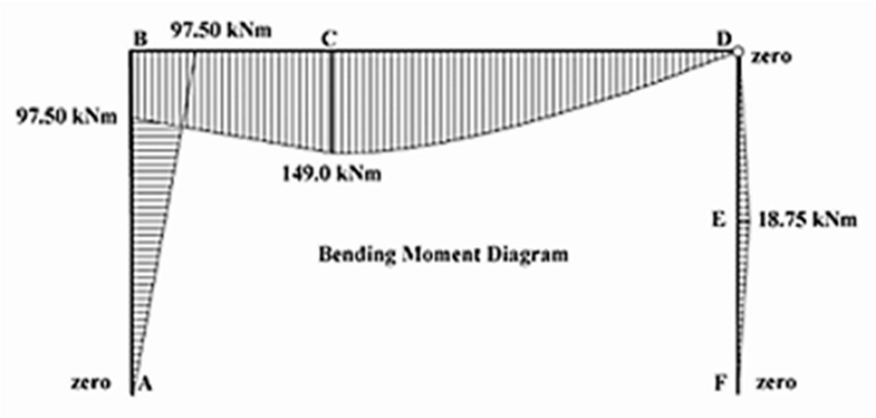
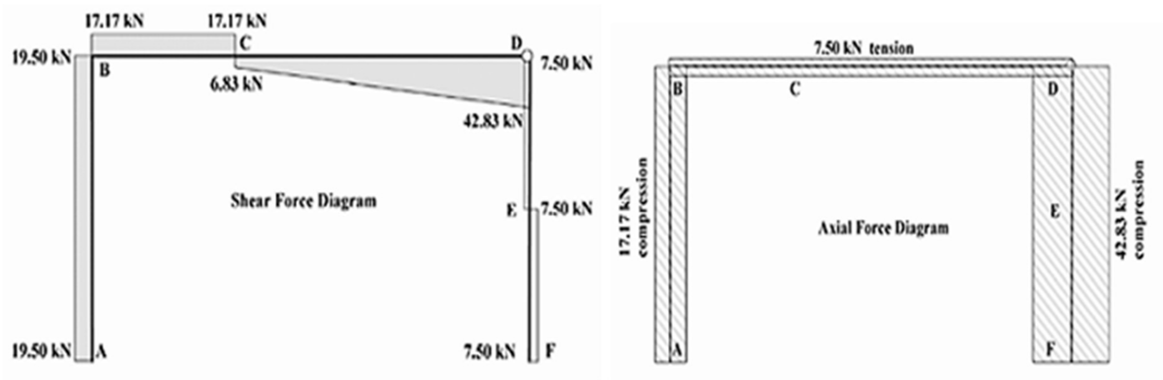


Fig. 8: Schematic representation of the bending moment distribution along the frame



(a) (b)

Fig. 9: Schematic representation of the shear force (a) and axial force (b) distributions along the frame

4.2.Analysis of the frame using the finite element method FEM

A comparison of the obtained results is essential regardless of the analysis method used, in order to verify the accuracy and reliability of the outcomes. To this end, a comparative study is carried out using the finite element method (FEM) through the RDM6 software, which provides both numerical and graphical outputs illustrating the internal forces and stresses within the members of the frame under investigation. This approach not only offers an alternative means of determining these quantities but also allows for the evaluation of consistency between the two methods.

By entering the geometric and physical properties of the frame into the RDM6 software (which is based on the principles of the finite element method), the following graphical results are obtained.

Figures 10, 11 and 12 respectively illustrate the distributions of shear force, bending moment and axial force.

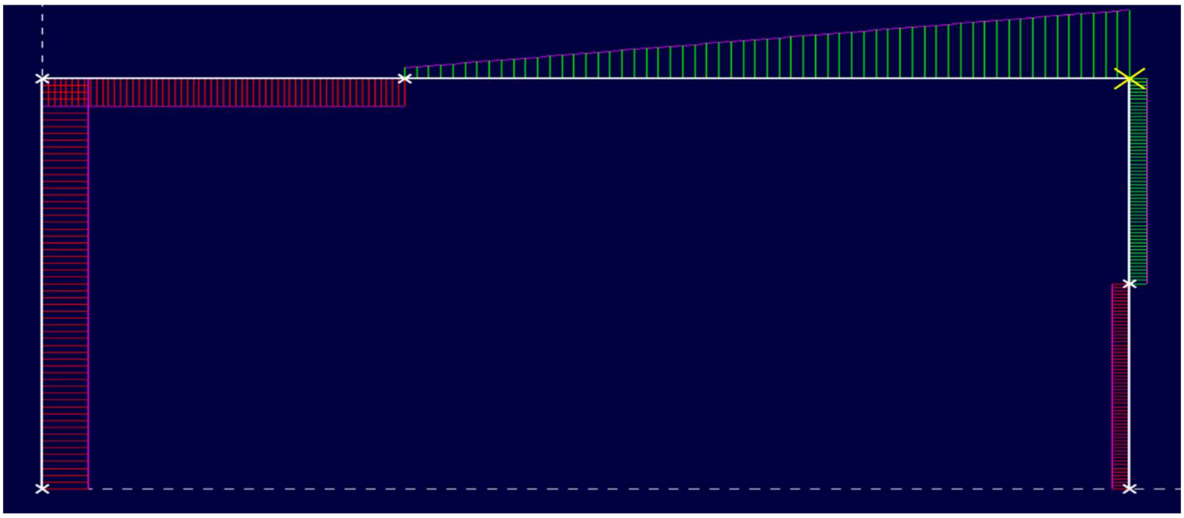


Fig. 10: Shear force distribution along the frame

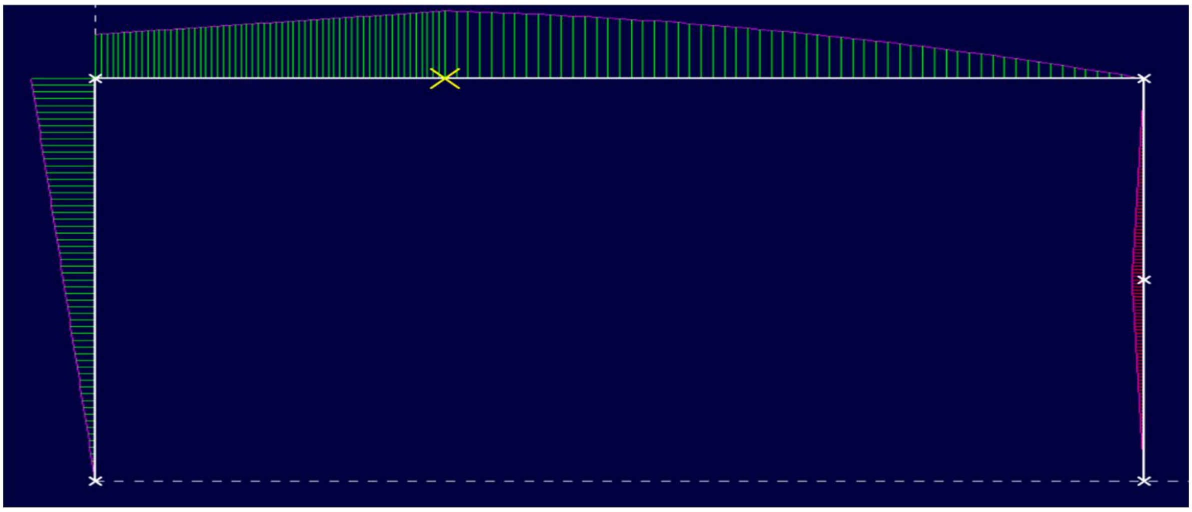


Fig. 11: Bending moments distribution along the frame

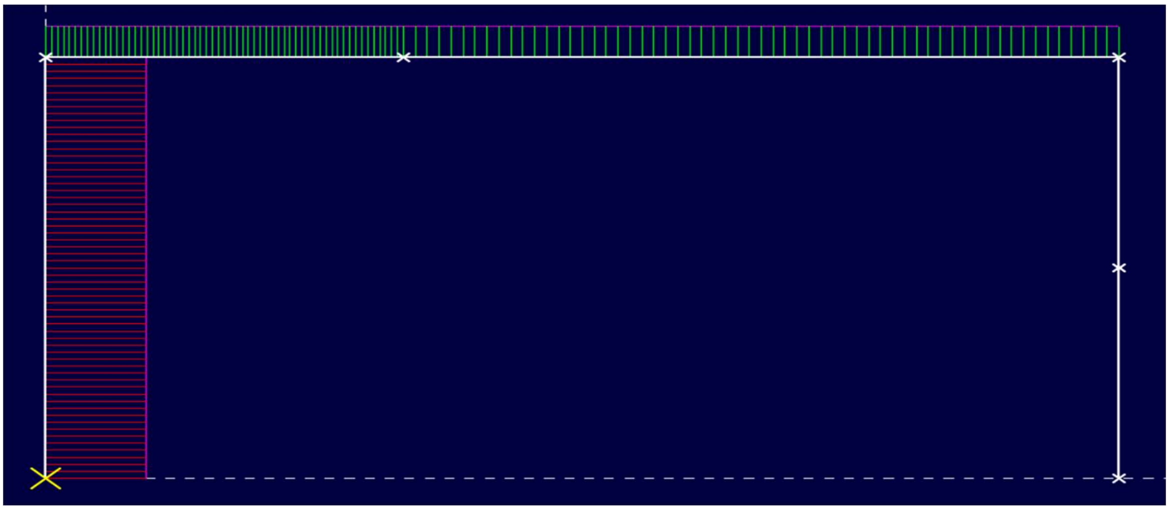


Fig. 12: Axial force distribution along the frame

Observation

As previously discussed in Chapter 2, the force method yields exact analytical solutions derived from classical structural analysis principles. In contrast, the finite element method (FEM) produces approximate results due to the discretization of the structure into a finite number of elements. A key strength of FEM—especially when implemented via software like RDM6—lies in its ability to deliver both precise numerical data and clear graphical representations of internal force distributions. Moreover, FEM is particularly well-suited for analyzing structures with complex geometries and multiple loading scenarios, offering flexibility and modeling capabilities that go beyond those of traditional analytical approaches.

5. **Analysis of a frame with three vertical members and two horizontal members under various external loads**

The main objective of this study is to apply the force method and the finite element method (FEM) to determine the support reactions of the structure illustrated in fig. 13. The analysis also provides the internal force values and the corresponding normal stresses in the various sections of the frame.

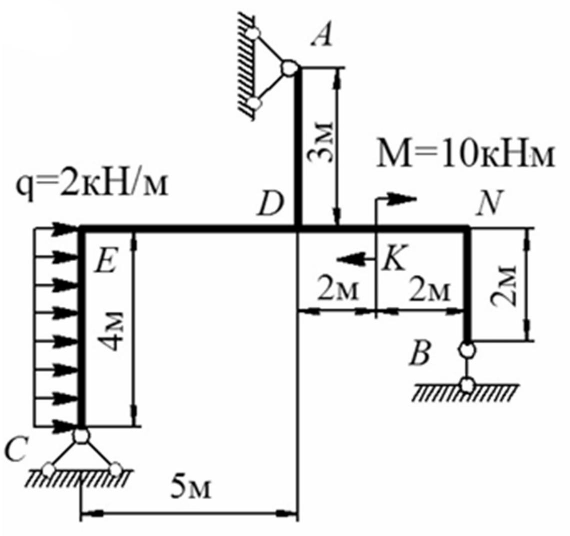


Fig. 13: Frame under study subjected to various external load

Analysis of the frame using the force method

Degree of static indeterminacy

$$D = 5 - 3 = 2$$

The primary system is selected as shown in fig. 14, after which the corresponding system of canonical equations can be formulated.

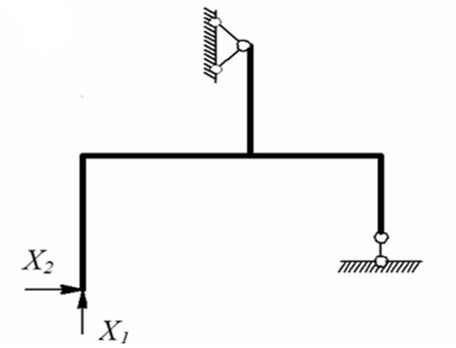


Fig. 14: Primary system of the frame

The system of canonical equations corresponding to this frame is given by the following expressions:

$$\begin{cases} X_1 \cdot \delta_{11} + X_2 \cdot \delta_{12} + \Delta_{1P} = 0 \\ X_1 \cdot \delta_{21} + X_2 \cdot \delta_{22} + \Delta_{2P} = 0 \end{cases} \quad (2)$$

Moments diagrams due to the redundants forces

We begin by determining the moment diagrams caused by the redundant forces, as shown in Fig. 15. It is noted that the reaction values indicated in figure 15 are readily obtained by directly applying the equations of equilibrium.

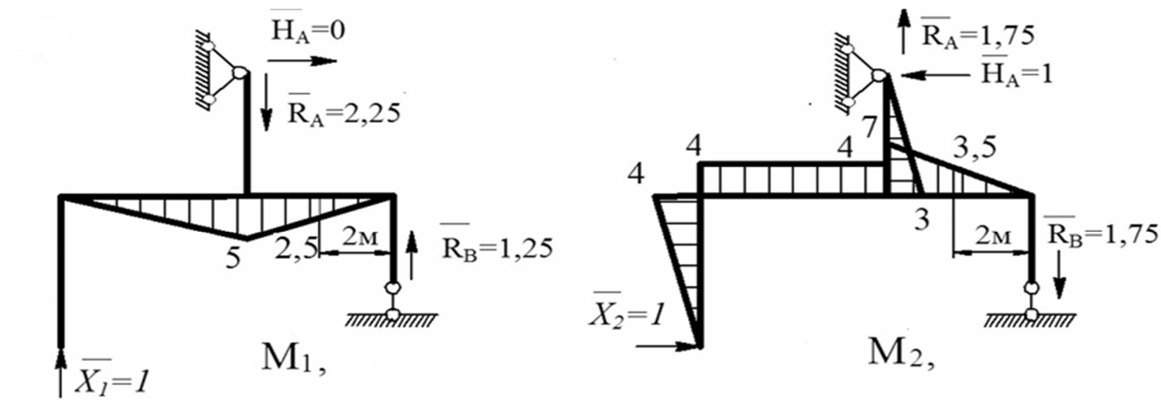


Fig. 15: Schematic representation of moment diagrams due to the redundant forces

Moments diagrams due to the external loads

By applying the same approach to the external loads—using the equations of equilibrium along with the fundamental laws of strength of materials—the corresponding moment diagrams can be determined,

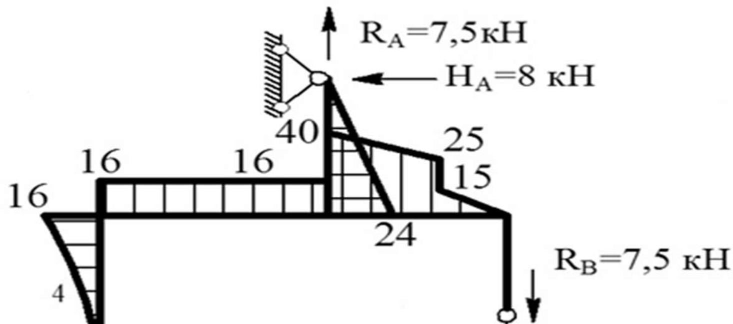


Fig. 16: Schematic representation of the bending moment distribution along the frame due to the external loads

To determine the coefficients of the canonical equations, the classical method of moment integration is applied based on the diagrams shown in Figs. 15 and 16. This yields the following results:

$$\delta_{11} = \frac{1}{EI}\cdot\frac{5}{6}\left(-2\cdot 0+2\cdot 5\cdot 0.5+0.5\cdot 5\cdot 0\right)+\frac{4}{6}\left(2\cdot 5\cdot 5+2\cdot 0\cdot 0.5+0\cdot 5\cdot 0.5\right)=\frac{75}{EI};$$

$$\delta_{22} = \frac{1}{EI}\cdot\frac{4}{6}\left(-2\cdot0+2\cdot4\cdot4+0\cdot4\cdot4\right)+\frac{5}{6}\left(2\cdot4\cdot4+2\cdot4\cdot4+4\cdot4\cdot4\right)+\frac{4}{6}\left(2\cdot7\cdot7+2\cdot0\cdot0+0\cdot7\cdot0.7\right)$$

$$+\frac{3}{6}\left(2\cdot3\cdot3+2\cdot0\cdot0+0\cdot3\cdot0.3\right)=\frac{175.6}{EI};$$

$$\delta_{12} = \frac{-1}{EI}\cdot\left(\frac{5}{6}\cdot(2\cdot0.4-2\cdot5.4-5\cdot4+0\cdot5)+\frac{4}{6}\cdot(2\cdot0.0-2\cdot5.7+7\cdot0+5\cdot0)\right)=-\frac{96.7}{EI};$$

$$\Delta_{1P} = \frac{-1}{EJ}\left(\frac{-5}{6}\cdot(2\cdot16\cdot0-2\cdot5\cdot16-16\cdot5+0\cdot16)+\frac{2}{6}\cdot(-2\cdot40\cdot5-2\cdot2.5\cdot25-5\cdot25-2.5\cdot40)+\right.$$

$$\left.+\frac{2}{6}\cdot(-2\cdot15\cdot2.5+2\cdot0\cdot0+15\cdot0+0\cdot2.5)=\right)-\frac{475}{EI}$$

$$\Delta_{2P} = \frac{1}{EI}\left(\frac{4}{6}\cdot(2\cdot0\cdot0+2\cdot4\cdot16+4\cdot0+16\cdot0)-\frac{2}{3}\cdot4\cdot4\cdot2+\frac{5}{6}\cdot(2\cdot4\cdot16+2\cdot4\cdot16+4\cdot16+16\cdot4)+\right.$$

$$+\frac{3}{6}\cdot(2\cdot0\cdot0+2\cdot3\cdot24+0\cdot24+0\cdot3)+\frac{2}{6}\cdot(2\cdot7\cdot40+2\cdot3.5\cdot25+40\cdot3.5+25\cdot7)+$$

$$\left.+\frac{2}{6}\cdot(2\cdot3.5\cdot15+2\cdot0\cdot0+0\cdot3.5+0\cdot15)\right)=\frac{841}{EI}$$

The system of canonical equations (2) takes the following form:

$$\begin{cases} 75X_1-96.7X_2=475 \\ -96.7X_1+175.6X_2=-841 \end{cases} \tag{3}$$

The solution to system of equations (3) yields the following values for X_1 and X_2:

$$X_1 = 0.3\ \text{kN}$$

$$X_2 = -4.6\ \text{kN}$$

Bending moments diagrams of the frame

At this stage of the structural analysis, each unit moment diagram corresponding to a redundant force is multiplied by its respective calculated value. Accordingly, the unit moment diagram is multiplied by the value of the redundant force to obtain the resulting moment diagram.

$$M_1^* = M_1 \times X_1$$

$$M_2^* = M_2 \times X_2$$

This step is of significant importance, as it enables the individual assessment of each redundant force's influence on the overall bending moment distribution along the structure, as shown in fig. 17. After performing the necessary computations, the resulting moment diagrams take the following form, figure 17:

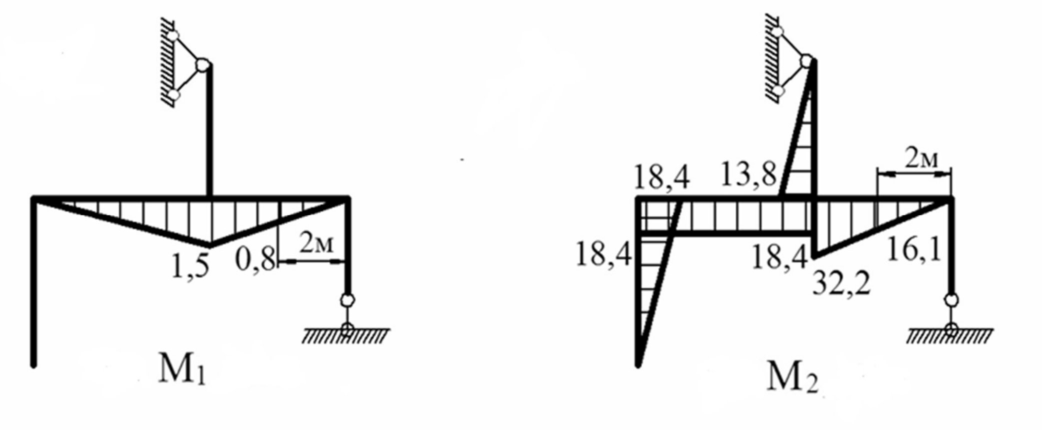


Fig. 17: Bending moment diagrams obtained after multiplying by the corresponding redundant force values

By superimposing—i.e., algebraically summing—all the bending moment diagrams resulting from the redundant forces and external loads (shown in figures 16 and 17), the final bending moment distribution along the entire frame is obtained, as shown in fig. 18.

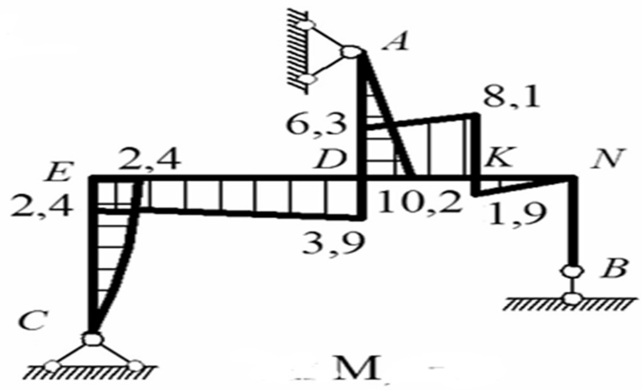


Fig. 18: Final bending moment distribution along the entire frame

Shear force and axial force diagrams of the frame

Using the same approach as for the bending moments, the shear and axial forces at the nodes of the frame are determined, followed by the construction of their respective diagrams. This is accomplished using classical methods commonly employed in structural mechanics. The procedure yields the results illustrated graphically in figure. 19.

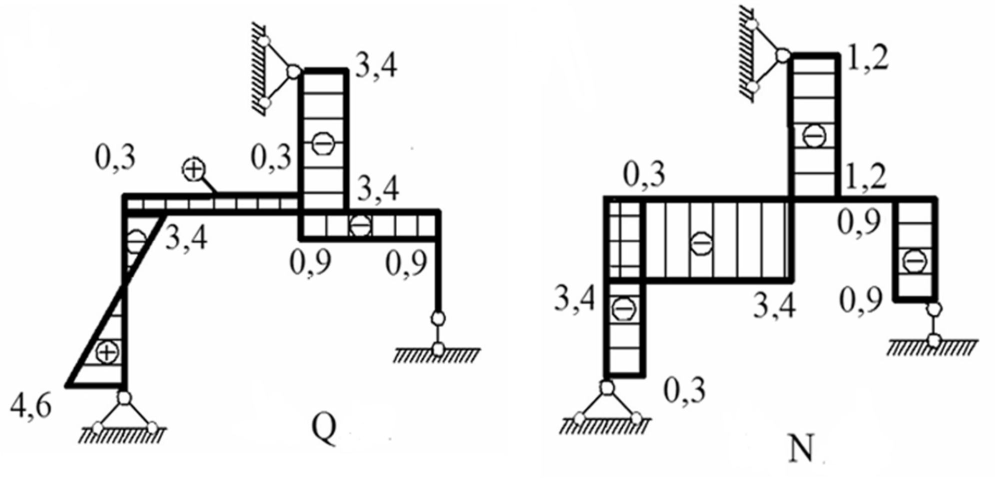


Fig. 19: Shear force ***Q*** and axial force ***N*** diagrams of frame

Comparison of results with the finite element method (FEM)

In every analysis, verifying the reliability of the results remains essential. Accordingly, a comparative study was conducted using the finite element method. Following the same procedure as in the previous example, the RDM6 software—based on FEM principles—was employed to generate both numerical values and graphical representations of internal forces and stresses within the frame members. This approach facilitates a direct comparison between the two methods, allowing for a thorough assessment of the consistency and accuracy of the results.

Entering the geometric and physical properties of the frame into the RDM6 software yields the following results, which represent the bending moment, shear force, axial force, and normal stress diagrams along the entire frame. These results are illustrated in figures 20, 21, 22, and 23.

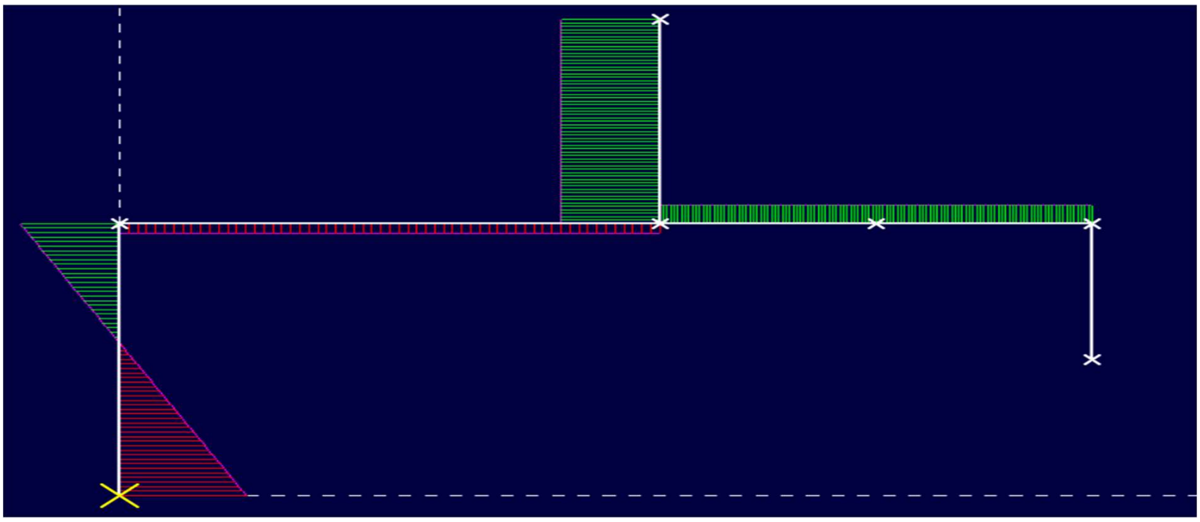


Fig. 20: Shear force distribution along the entire frame

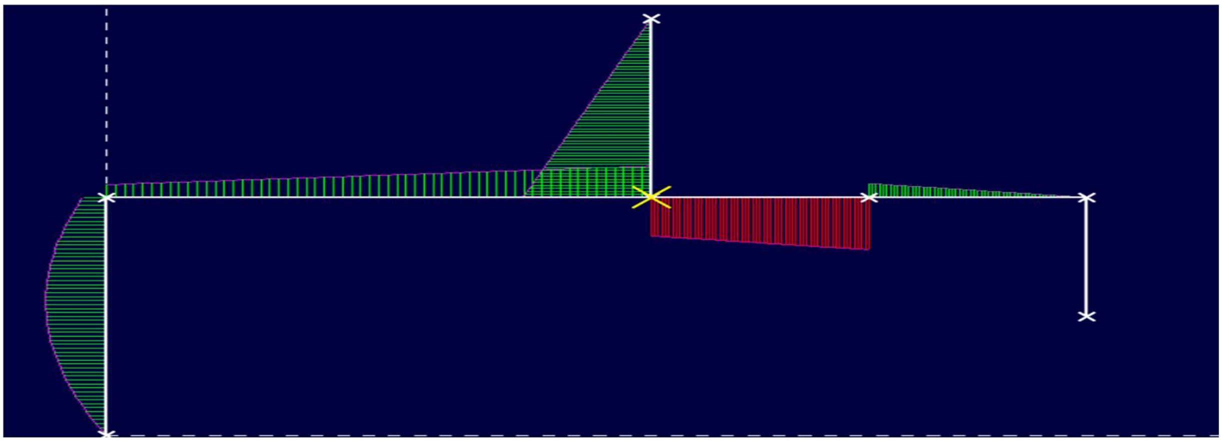


Fig. 21: Bending moment distribution along the entire frame

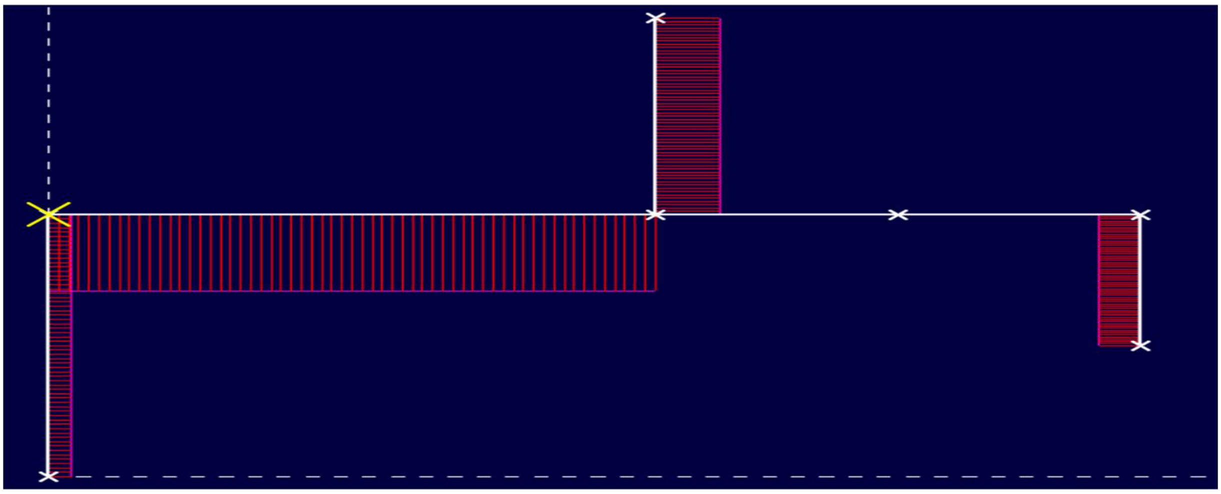


Fig. 22: Axial force distribution along the entire frame

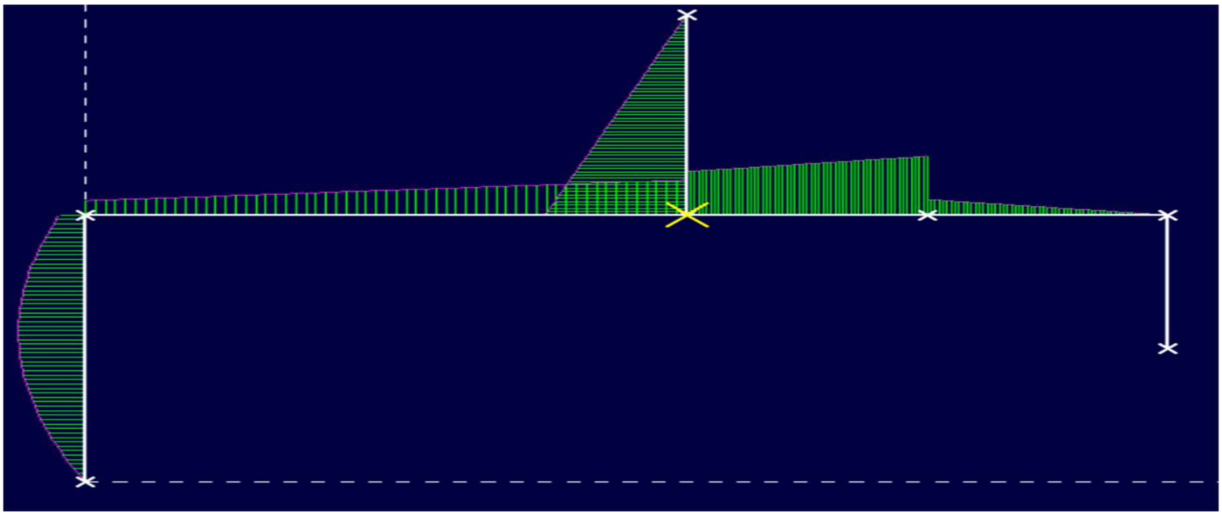


Fig. 23: Normal stress distribution along the entire frame

Remark

For all the previously analyzed frames using both the force method and the finite element method (FEM), there is a high degree of concordance between the results. This highlights the potential for coupling classical analytical methods with numerical approaches, allowing them to be applied together for certain types of structures and reinforcing the strengths of both methods. A key advantage of FEM—particularly when implemented through software such as RDM6—is its suitability for analyzing structures with complex geometries and loading configurations, offering capabilities that extend beyond those of traditional analytical techniques.

6. **Analysis of a frame with two identical vertical members and two horizontal members, having different flexural rigidities in the vertical and horizontal directions, subjected to various external loads**

This example involves the structural analysis of a planar frame subjected to both distributed and concentrated loads. The frame consists of two identical vertical columns with identical flexural rigidity and two horizontal beams that also share the same flexural rigidity. The structure is subjected to various external loads and features different geometric dimensions, as illustrated in figure 24.

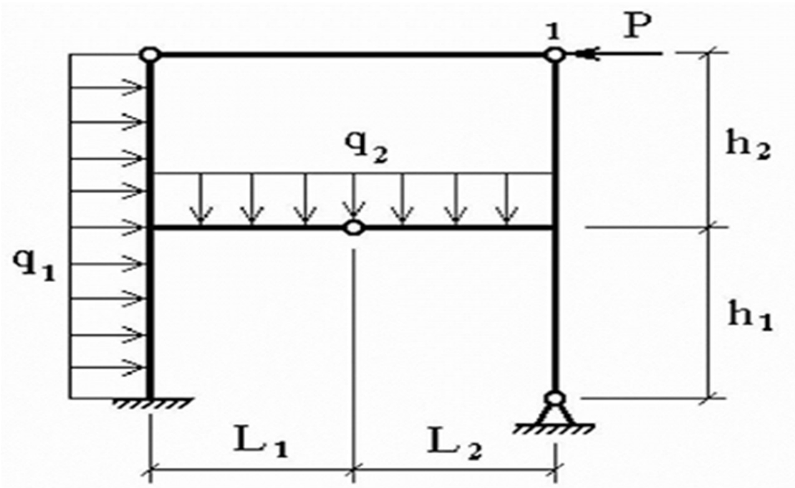


Fig. 24: Planar frame subjected to various external load

The values of the structural parameters are as follows:

L_1=L_2=2.2 m; h_1=h_2=2.5 m; q_1=12 kN/m; q_2=15 kN/m; P=30 kN; EI_p/EI_{ct}=4/3EI

Analysis of the structure using the force method

Determination of the degree of static indeterminacy

$$DSI = -(3B - 2H - C0) = -(3\times3 - 2\times3 - 5) = 2$$

Selection of the primary system

The primary system is obtained by removing the two redundant restraints and replacing them with the unknown forces X_1 and X_2, as illustrated in figure 25.

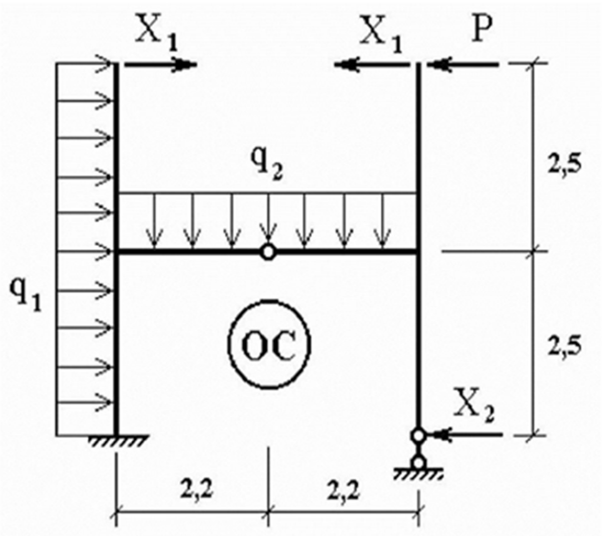


Fig. 25: Schematic illustration of the primary system

The system of canonical equations, with X_1 and X_2 as the unknowns, is given by:

$$\begin{cases} \delta_{11} \cdot X_1 + \delta_{12} \cdot X_2 + \Delta_{1P} = 0 \\ \delta_{21} \cdot X_1 + \delta_{22} \cdot X_2 + \Delta_{2P} = 0 \end{cases} \tag{4}$$

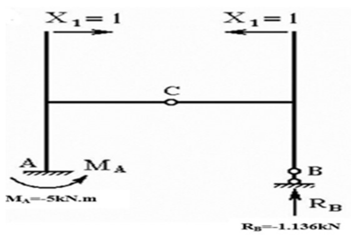
Determination of the parameters of the system (4)

Determination of the reactions due to the unknown redundant X_1 and X_2

The reaction for the redundant force $X_1 = 1$

$$\sum M|_C = 1 \times 2.5 + R_B \times 2.2 = 0 \rightarrow R_B = -1.136kN$$

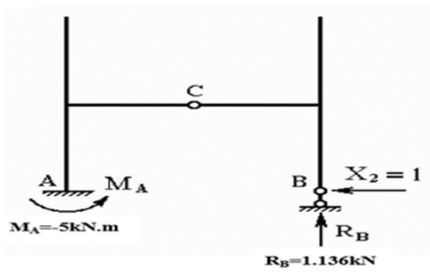
$$\sum M|_A = M_A + R_B \times 4.4 = 0 \rightarrow M_A = -5kN.m$$



The reaction for the redundant force $X_2 = 1$

$$\sum M|_C = -1 \times 2.5 + R_B \times 2.2 = 0 \rightarrow R_B = 1.136kN$$

$$\sum M|_A = M_A + R_B \times 4.4 = 0 \rightarrow M_A = -5kN.m$$



Determination of the bending moments

Following the same procedure explained in the previous examples, the bending moment diagrams due to the redundant forces are obtained, as shown in figure 26.

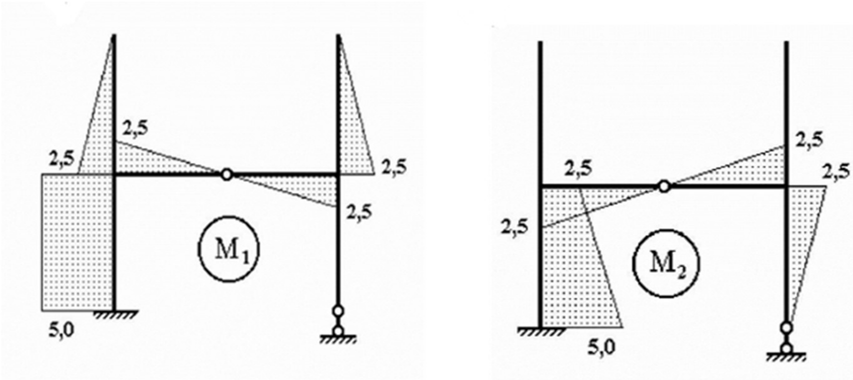


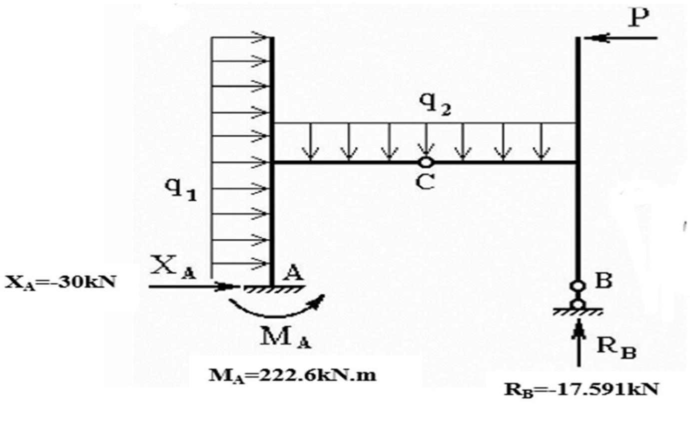
Fig. 26: Bending moment diagrams along the frame due to the redundants forces

Determination of the reactions due to the external loads

$$\sum M\big|_A = M_A - \frac{q_1 \times 5^2}{2} - \frac{q_2 \times 4.4^2}{2} + P \times 5 + R_B \times 4.4 = 0 \rightarrow M_A = 222.6 kN.m$$

$$\sum X = X_A + q_1 \times 5 - P = 0 \rightarrow X_A = -30 kN$$

$$\sum M\big|_C = -q_2 \times 2.2 \times 1.1 + P \times 2.5 + \text{R}_B \times 2.2 = 0 \rightarrow R_B = -17.591 kN$$



Determination of the bending moments

Following the same procedure explained in the previous examples, the bending moment diagrams due to the external loads are obtained, as shown in figure 27.

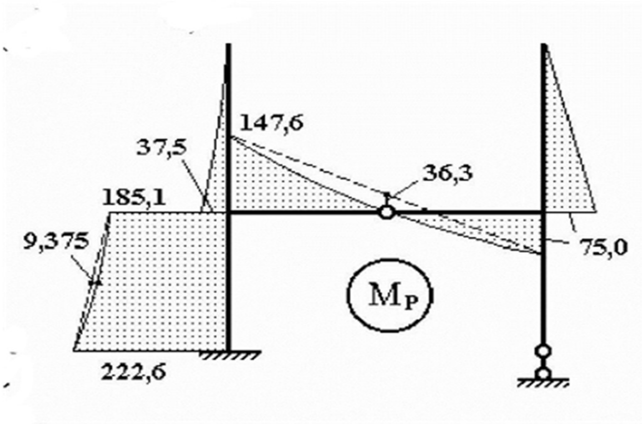


Fig. 27: Bending moment diagrams along the frame due to the external loads

The diagrams presented in figures 26 and 27 are employed to determine the values of the parameters in the system of canonical equations, following the procedure outlined in the previous examples.

$$\delta_{11} = \frac{1}{3EI}\times 2\times\left(\frac{1}{2}\times 2.5\times 2.5\times 2.5\right)+\frac{1}{4\cdot EI}\times 2\times\left(\frac{1}{2}\times 2.5\times 2.2\times 2.5\right)+\frac{1}{3\cdot EI}\times\left(5\times 2.5\times 5\right)=\frac{26,5972}{EI}$$

$$\delta_{22} = \frac{1}{4EI}\times 2\times\left(\frac{1}{3}\times 2.5\times 2.5\times 2.2\right)+\frac{1}{3EI}\times\left(\frac{1}{3}\times 2.5\times 2.5\times 2.5\right)+\frac{1}{3EI}\times\left(\frac{2.5}{6}\times(2\times 2.5\times 2.5+2.5\times 5+2.5\times 2.5+2.5\times 5)\right)=\frac{16,1806}{EI}$$

$$\delta_{12} = -\frac{1}{4EI}\times 2\times\left(\frac{1}{3}\times 2.5\times 2.5\times 2.2\right)-\frac{1}{3EI}\times\left(2.5\times 2.5\times 2.5\right)+\frac{1}{2EI}\times 5\times 2.5\times(2.5+5)=-\frac{17,9167}{EI}$$

$$\Delta_{1P} = \frac{1}{4EI}\times 75\times 2.5\times 2.2+\frac{1}{4EI}\times 147.6\times 2.5\times 2.2+\frac{1}{3EI}\times 2.5\times 75\times 2.5+\frac{1}{4EI}\times 37.5\times 2.5\times 2.5+$$
$$+\frac{1}{2\,EI}\times 5\times 2.5\times(185.1+222.6)=\frac{997.508}{EI}$$

$$\Delta_{2P} = -\frac{1}{4EI}\times 75\times 2.5\times 2.2-\frac{1}{4EI}\times 147.6\times 2.5\times 2.2-\frac{1}{3EI}\times 2.5\times 75\times 2.5+\frac{1}{4EI}\times 37.5\times 2.5\times 2.5+$$
$$+\frac{1}{2EI}\times 2.5\times(2.5\times 185.1+2.5\times 222.6+5\times 185.1+2.5\times 222.6)=-\frac{745.567}{EI}$$

The system of canonical equations (4) takes the following form:

$$\begin{cases} 26.5972X_1-17.9167X_2=-997.508 \\ -17.9167X_1+16.1806X_2=745.567 \end{cases} \quad (5)$$

The solution of system (5) yields:

$$X_1 = -28.566\ \text{kN}$$
$$X_2 = 13.239\ \text{kN}$$

Each unit moment diagram is scaled by the value of its corresponding redundant force. For example, the unit moment diagram M_1 is multiplied by the redundant force X_1 to produce the resulting moment diagram due to X_1. This procedure is systematically repeated for all redundant forces to construct the complete and final moment distribution resulting from the redundants, as illustrated in figure 28.

$$M_1^* = M_1 \times X_1$$
$$M_2^* = M_2 \times X_2$$

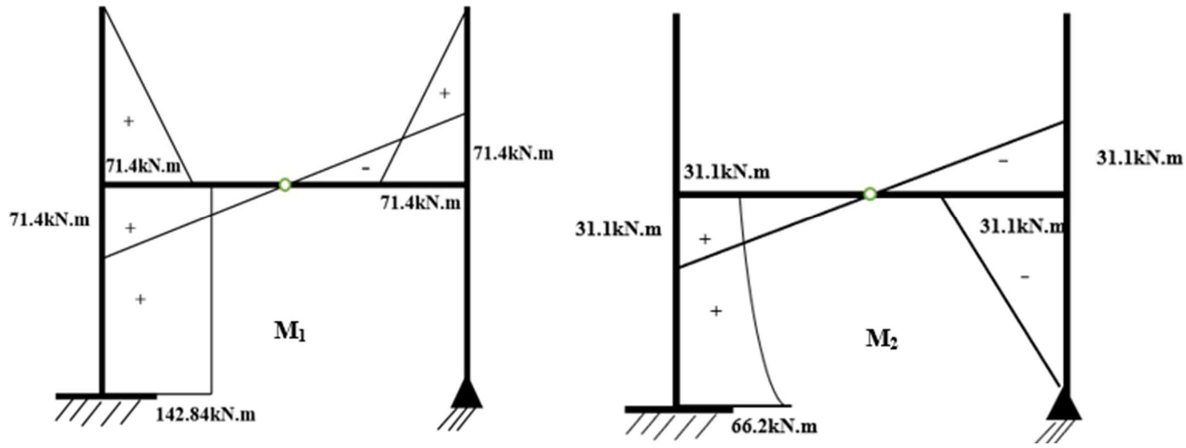


Fig. 28: Resulting bending moment diagrams M_1 and M_2 scaled by their respective redundant force values

The final bending moment diagrams for the entire structure are obtained by applying the principle of superposition, that is, by algebraically summing all individual diagrams resulting from the redundant forces and the external loads, as explained in the previous example. The corresponding graphical representation is shown in figure 29

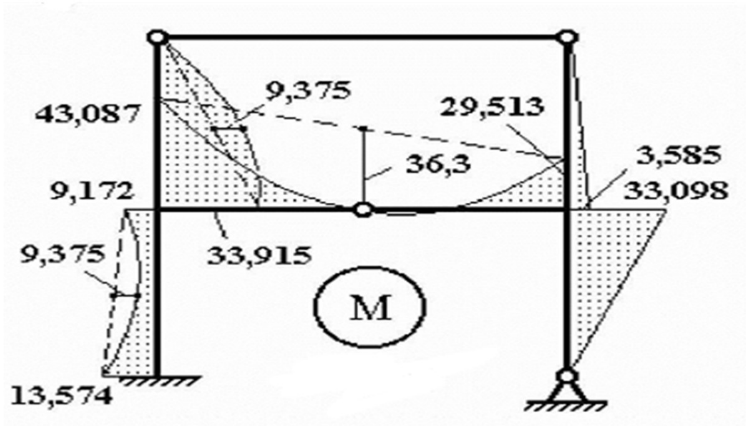


Fig. 29: Final bending moment diagram m_t showing the distribution throughout the entire structure

Determination of shear and axial force diagrams

Following the same manner as for the bending moment to determine the final shear and axial forces distribution along the entire frame as explained in the previous example. The procedure of calculation for this structure leads to the results graphically illustrated the figure 30.

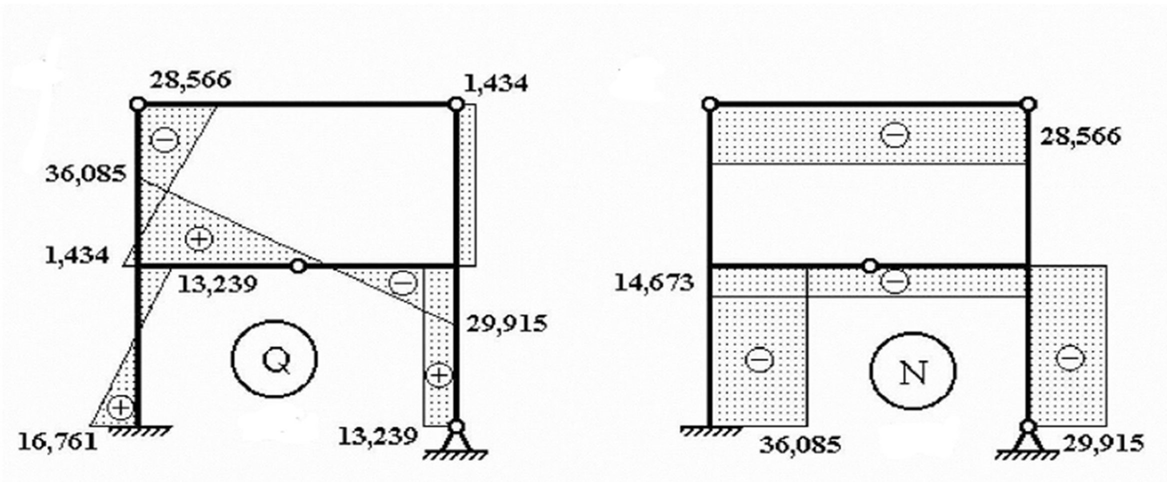


Fig. 30: Final shear force ***Q*** and axial force ***N*** diagrams showing their distribution along the entire structure

Comparison of the force method results with those obtained using the finite element method

As previously discussed, it is important to verify the accuracy of the results obtained using the force method. For this purpose, a comparison is carried out with the finite element method (FEM). Using the same procedure as in the previous example, the RDM6 software—which is based on the principles of the finite element method—is utilized to generate both numerical and graphical representations of the internal forces and stresses within the frame members. This serves not only as an alternative approach for determining these quantities but also as a means of evaluating the consistency and reliability of the two methods.

By inputting the geometric dimensions and mechanical properties of the frame into the RDM6 software, we obtain the resulting diagrams for bending moments, shear forces, axial forces, and normal stresses distributed throughout the entire structure. These outputs are presented in Figures 31, 32, 33, and 34.

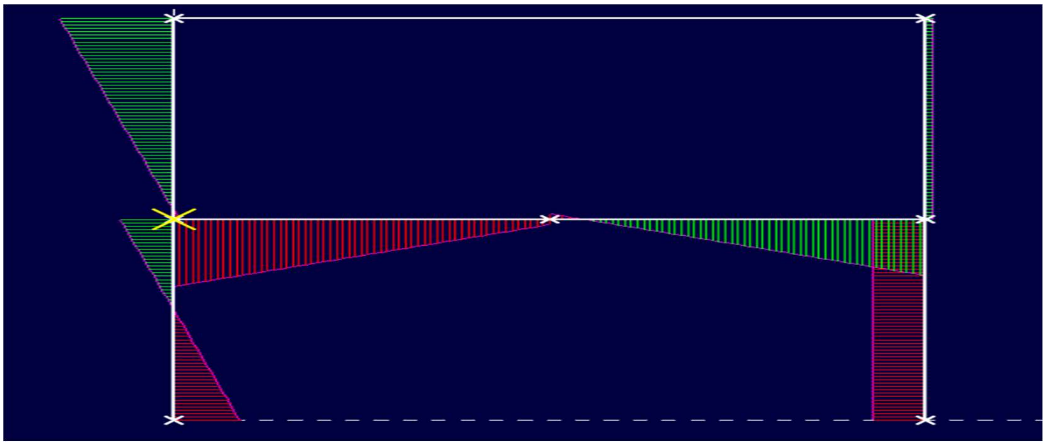


Fig. 31: Shear force distribution along the entire frame

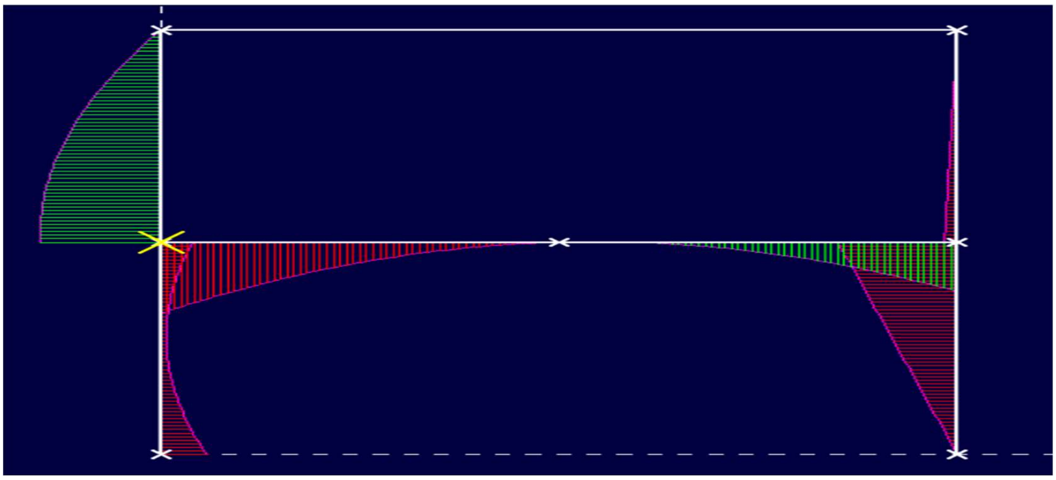


Fig. 32: Bending moment distribution along the entire frame

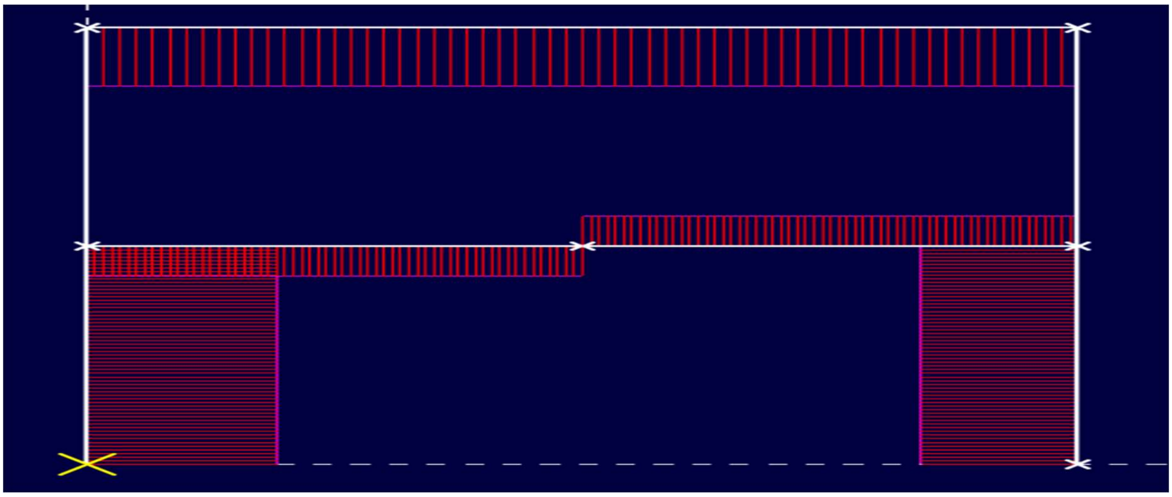


Fig. 33: Axial force distribution along the entire frame

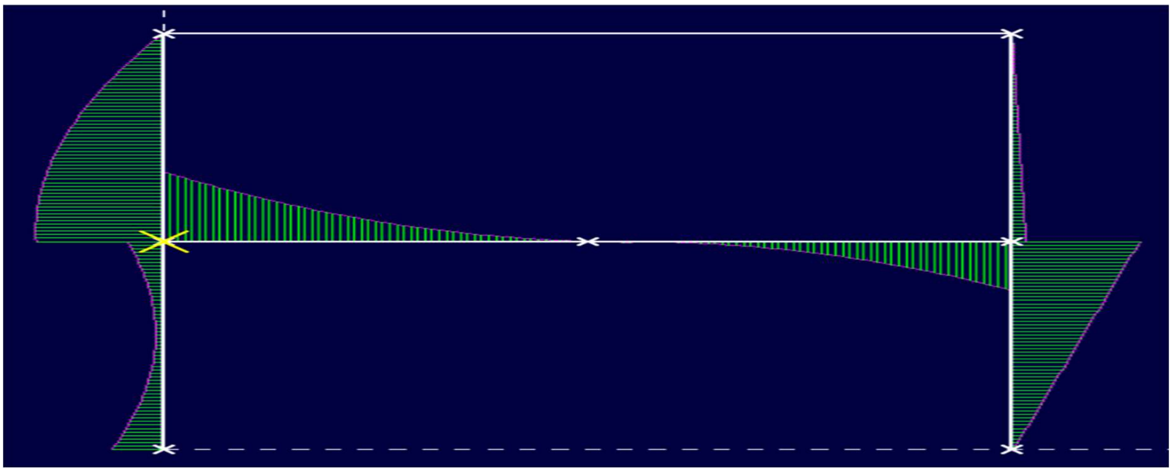


Fig. 34: Normal stress distribution along the entire frame

Observation

Both methods allow for the determination of support reactions as well as the internal force distributions—shear forces, bending moments, and axial forces—throughout the frame. While the Force Method provides exact analytical results grounded in classical structural theory, the Finite Element Method (FEM) delivers approximate solutions by discretizing the structure into smaller elements. Nevertheless, the accuracy of FEM is remarkably high, and the minor differences observed are generally negligible in practical engineering applications. A key strength of FEM, especially when implemented using tools like RDM6, lies in its ability to generate both comprehensive numerical data and clear graphical representations of internal force distributions. In addition, FEM is particularly well-suited for analyzing structures with complex geometries or loading conditions, where traditional analytical techniques may become impractical or overly complicated.

7. Conclusion

This chapter has explored the structural analysis of frame systems using a comparative approach between the Force Method and the Finite Element Method (FEM), with particular emphasis on statically indeterminate structures subjected to external loads. The Force Method, grounded in classical mechanics, enables the exact determination of support reactions and internal forces by converting the original indeterminate system into a determinate one and applying compatibility conditions. Its strength lies in the analytical clarity and deep physical insight it provides. However, its applicability becomes increasingly limited as the structural complexity grows—especially in cases involving irregular geometries or non-uniform boundary conditions—making the manual formulation and resolution of equations highly cumbersome.

On the other hand, the FEM offers remarkable flexibility and computational efficiency. By discretizing the structure into smaller, manageable finite elements, FEM facilitates the analysis of complex geometries, varied loading scenarios, and heterogeneous material properties. When applied through specialized software such as RDM6, FEM not only produces highly accurate numerical results but also offers intuitive graphical visualizations of internal forces, nodal displacements, and stress distributions. This graphical output enhances interpretability, aiding engineers in diagnosing and optimizing structural behavior.

Throughout the analyzed examples, both methods consistently produced concordant results in terms of support reactions and internal force distributions. This high degree of agreement underscores the reliability and robustness of both techniques. It also demonstrates the importance of cross-verification in structural engineering, as the convergence of results from fundamentally different approaches enhances confidence in the overall solution.

While the Force Method remains pedagogically valuable and effective for relatively simple configurations, FEM proves indispensable for modern engineering practice, especially in the context of large-scale or geometrically complex structures. The combined use of both methods—leveraging the analytical transparency of the Force Method with the computational power of FEM—offers a powerful toolkit for structural engineers.

In conclusion, this chapter not only highlights the individual merits of the Force Method and FEM but also emphasizes their complementary nature. Future work may extend this comparative framework to include dynamic analyses, nonlinear material behavior, and advanced hybrid modeling techniques, thereby further broadening the scope and applicability of structural analysis methodologies in engineering design and research.

General Conclusion

This work has provided a comprehensive study of structural analysis through three chapters, progressively introducing and applying both classical and modern methods to evaluate internal forces, reactions, and stress distributions in statically indeterminate beams and frames. Each chapter has contributed to developing a deeper understanding of analytical techniques and computational tools used in structural engineering.

Chapter 1 laid the theoretical foundation by presenting the **principles of the Force Method and the Finite Element Method (FEM)**. The comparison of these two methods highlighted their mathematical formulations, historical development, and respective areas of applicability. Special attention was given to the conceptual differences between exact analytical solutions provided by the Force Method and the numerical approximations obtained through FEM. This theoretical framework was essential to guide the practical analyses in the subsequent chapters.

Chapter 2 applied both methods to the analysis of **continuous beams**, offering a step-by-step examination of support reactions, internal forces, and stress distributions under various loading conditions. The Force Method enabled the derivation of precise solutions using equilibrium and compatibility conditions, while FEM, implemented through RDM6 software, delivered approximate but highly accurate results, complemented by graphical outputs. The agreement between both methods validated the reliability of each and illustrated the power of cross-verification.

Chapter 3 extended the comparative analysis to **frame structures**, including both straight and inclined members subjected to horizontal and vertical loads. The complexity of the geometry and boundary conditions illustrated the limitations of purely analytical approaches and the necessity of numerical methods for real-world applications. Once again, FEM proved particularly effective for dealing with multiple degrees of freedom and for providing detailed insights into structural behavior. The results confirmed the consistency of both methods, reinforcing the importance of using them in tandem for thorough and reliable analysis.

In summary, this study demonstrates the complementary strengths of classical and numerical techniques in structural analysis. The **Force Method** remains an essential tool for educational purposes and for simple to moderately complex structures, offering transparency and physical insight. Conversely, the **Finite Element Method** is indispensable for modern engineering applications, particularly when dealing with complex geometries and loading conditions. By integrating both approaches, engineers can achieve a more complete understanding of structural behavior, enhance the accuracy of results, and reinforce the integrity of their analyses.

The methodology and examples presented in this work serve not only as a technical reference but also as a pedagogical tool to train students and young engineers in the effective application of structural analysis methods. Future research may explore additional complexities such as **dynamic loading, material nonlinearity, and advanced hybrid techniques**, further enhancing the reliability and sophistication of structural design and assessment.

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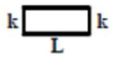
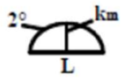
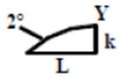
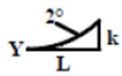
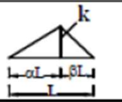
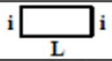
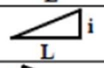
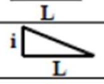
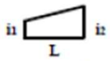
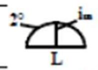
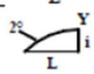
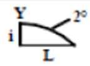
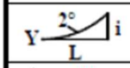
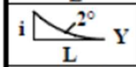
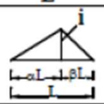
Appendix A:

Table showing the moment multiplication expressions used to compute the parameters of the canonical equations for determining the redundant forces in the Force Method

M_k / M_i							
	Lik	$\frac{1}{2}Lik$	$\frac{1}{2}Li(k_1+k_2)$	$\frac{2}{3}Lik_m$	$\frac{2}{3}Lik$	$\frac{1}{3}Lik$	$\frac{1}{2}Lik$
	$\frac{1}{2}Lik$	$\frac{1}{3}Lik$	$\frac{1}{6}Li(k_1+2k_2)$	$\frac{1}{3}Lik_m$	$\frac{5}{12}Lik$	$\frac{1}{4}Lik$	$\frac{1}{6}L(1+\alpha)ik$
	$\frac{1}{2}Lik$	$\frac{1}{6}Lik$	$\frac{1}{6}Li(2k_1+k_2)$	$\frac{1}{3}Lik_m$	$\frac{1}{4}Lik$	$\frac{1}{12}Lik$	$\frac{1}{6}L(1+\beta)ik$
	$\frac{1}{2}L(i_1+i_2)k$	$\frac{1}{6}L(i_1+2i_2)k$	$\frac{1}{6}L(2i_1k_1+i_1k_2+i_2k_1+2i_2k_2)$	$\frac{1}{3}L(i_1+i_2)k_m$	$\frac{1}{12}L(3i_1+5i_2)k$	$\frac{1}{12}L(i_1+3i_2)k$	$\frac{1}{6}Lk[(1+\beta)i_1+(1+\alpha)i_2]$
	$\frac{2}{3}Li_mk$	$\frac{1}{3}Li_mk$	$\frac{1}{3}Li_m(k_1+k_2)$	$\frac{8}{15}Li_mk_m$	$\frac{7}{15}Li_mk$	$\frac{1}{5}Li_mk$	$\frac{1}{3}L(1+\alpha\beta)i_mk$
	$\frac{2}{3}Lik$	$\frac{5}{12}Lik$	$\frac{1}{12}Li(3k_1+5k_2)$	$\frac{7}{15}Lik_m$	$\frac{8}{15}Lik$	$\frac{3}{10}Lik$	$\frac{1}{12}L(5-\beta-\beta^2)ik$
	$\frac{2}{3}Lik$	$\frac{1}{4}Lik$	$\frac{1}{12}Li(5k_1+3k_2)$	$\frac{7}{15}Lik_m$	$\frac{11}{30}Lik$	$\frac{2}{15}Lik$	$\frac{1}{12}L(5-\alpha-\alpha^2)ik$
	$\frac{1}{3}Lik$	$\frac{1}{4}Lik$	$\frac{1}{12}Li(k_1+3k_2)$	$\frac{1}{5}Lik_m$	$\frac{3}{10}Lik$	$\frac{1}{5}Lik$	$\frac{1}{12}L(1+\alpha+\alpha^2)ik$
	$\frac{1}{3}Lik$	$\frac{1}{12}Lik$	$\frac{1}{12}Li(3k_1+k_2)$	$\frac{1}{5}Lik_m$	$\frac{2}{15}Lik$	$\frac{1}{30}Lik$	$\frac{1}{12}L(1+\beta+\beta^2)ik$
	$\frac{1}{2}Lik$	$\frac{1}{6}L(1+\alpha)ik$	$\frac{1}{6}Li[(1+\beta)k_1+(1+\alpha)k_2]$	$\frac{1}{3}L(1+\alpha\beta)ik_m$	$\frac{1}{12}L(5-\beta-\beta^2)ik$	$\frac{1}{12}L(1+\alpha+\alpha^2)ik$	$\frac{1}{3}Lik$