الجمـــهورية الجزائريـــة الديمقراطيـــة الشعبيـــة
REPUBLIQUE ALGERIENNE DEMOCRATIQUE ET POPULAIRE

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وزارة التعليـم العالــي و البحث العلمـي
جامعـة 8 مـــاي 1945 قالمـــة
كليـــة الرياضيـات و الإعـلام الآلـي و علــوم المـادة
قسم الرياضيات

Lecture Notes

Mathematical Analysis III

Summary of lessons, examples and solved exercises

To students of the Second Year Engineering in Computer Science by

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Academic year
2025

Mathematical analysis III
Lecture Notes

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Preface

This course packet is designed as a comprehensive guide with solved exercises covering the official curriculum of the Mathematical Analysis III module. It is primarily aimed at second-year engineering students in computer science but may also be beneficial for second-year mathematics and technical science students.

At the end of each chapter, there is a selection of typical exercises with detailed solutions, crafted progressively to help students familiarize themselves with new concepts and ensure a solid understanding of key points.

The first chapter introduces parameterized integrals.

The second chapter covers the Laplace and Fourier transforms. The following chapter is dedicated to elements of topology.

Chapter 4 focuses on functions of several variables and vector-valued functions, emphasizing limits and continuity.

Chapter 5 addresses differential calculus and the Jacobian matrix, while the final chapter deals with optimization, both with and without constraints.

Chapter

1 Parameterized Integrals

Often, solving a differential equation involves computing an integral of the form:

$$\int_a^b f(x,t)\,dt.$$

In many instances, we do not have an explicit expression for this integral, which necessitates analyzing the function $F(x)$ as it is presented—specifically, as an integral dependent on the parameter x. This chapter outlines the conditions under which $F(x)$ is continuous and differentiable. We will apply these methods to the Laplace and Fourier transforms.

1.1 Integrals Depending on a Parameter

1.1.1 Function Defined by an Integral

Let $f:(x,t)\longrightarrow f(x,t)$ be a function of two variables, x and t. We consider x as a parameter and $t\in[a,b]$ as an integration variable. This allows us to define: $F(x)=\int_a^b f(x,t)\,dt$.
For a fixed x, for $F(x)$ to exist, it is enough that the partial application $t\longrightarrow f(x,t)$ be continuous on $[a,b]$. But this does not guarantee the continuity of the function F. We provide sufficient conditions for F to be continuous, and then differentiable.

Theorem 1.1.1. *Let I be an interval of $\mathbb{R}$ and $J=[a,b]$ a bounded closed interval. Let f be a continuous function on $I\times J$ with values in $\mathbb{R}$ (or $\mathbb{C}$).*
Then the function F defined for all $x\in I$ by $F(x)=\int_a^b f(x,t)\,dt$ is continuous on I.

Example 1.1.2. *Let $F(x)=\int_0^\pi \sin(x+t)\cdot e^{xt^2}\,dt$, defined for $x\in I=\mathbb{R}$.*
The function $(x,t)\mapsto f(x,t)=\sin(x+t)\cdot e^{xt^2}$ is continuous on $\mathbb{R}\times[0,\pi]$, so the function $x\mapsto F(x)$ is continuous on $\mathbb{R}$. We have $F(0)=\int_0^\pi \sin(t)\cdot 1\,dt=-\cos(t)\Big|_0^\pi=2$. Even though we do not have a formula for $F(x)$ in general, we can deduce from the continuity that $F(x)\longrightarrow F(0)=2$ as $x\longrightarrow 0$.

Theorem 1.1.3. *Let I be an interval of $\mathbb{R}$ and $J=[a,b]$ a bounded closed interval. Assume that:*

- *$(x,t) \mapsto f(x,t)$ is a continuous function on $I \times J$ (with values in $\mathbb{R}$ or $\mathbb{C}$),*
- *the partial derivative $(x,t) \mapsto \frac{\partial f}{\partial x}(x,t)$ exists and is continuous on $I \times J$.*

Then the function F defined for all $x \in I$ by $F(x) = \int_a^b f(x,t)\,dt$ is of class C^1 on I and:

$$F'(x) = \int_a^b \frac{\partial f}{\partial x}(x,t)\,dt.$$

We can remember the mnemonic abbreviation for the interchange of derivative and integral:

$$\frac{d}{dx}\int_a^b = \int_a^b \frac{\partial}{\partial x}.$$

Example 1.1.4. *Let us study $F(x) = \int_0^1 \frac{dt}{x^2+t^2}$ for $x \in (0,+\infty)$. Let $f(x,t) = \frac{1}{x^2+t^2}$. Then:*

- *f is continuous on $(0,+\infty) \times [0,1]$,*
- *$\frac{\partial f}{\partial x}(x,t) = -\frac{2x}{(x^2+t^2)^2}$ is continuous on $(0,+\infty) \times [0,1]$.*

We will have $F'(x) = \int_0^1 -\frac{2x}{(x^2+t^2)^2}\,dt$. For this example, we can explicitly calculate $F(x)$:

$$F(x) = \frac{1}{x}\arctan\left(\frac{1}{x}\right) \implies F'(x) = -\frac{1}{x^2}\arctan\left(\frac{1}{x}\right) - \frac{1}{x(1+x^2)}.$$

This proves $F'(x) = \int_0^1 -\frac{2x}{(x^2+t^2)^2}\,dt = -\frac{1}{x^2}\arctan\left(\frac{1}{x}\right) - \frac{1}{x(1+x^2)}$.

Theorem 1.1.5 (Fubini's Theorem). *Let $I = [\alpha,\beta]$ and $J = [a,b]$ be two bounded closed intervals. Let f be a continuous function on $I \times J$, with values in $\mathbb{R}$ (or $\mathbb{C}$). Then the function F defined for all $x \in I$ by $F(x) = \int_a^b f(x,t)\,dt$ is integrable on I and*

$$\int_\alpha^\beta F(x)\,dx = \int_\alpha^\beta \left(\int_a^b f(x,t)\,dt\right)dx = \int_a^b \left(\int_\alpha^\beta f(x,t)\,dx\right)dt.$$

The order of integration can be interchanged: $\int_\alpha^\beta \int_a^b f(x,t)\,dt\,dx = \int_a^b \int_\alpha^\beta f(x,t)\,dx\,dt$.

Example 1.1.6. *Let's calculate:*

$$I = \int_0^\pi \int_0^1 (t\sin x + 2x)\,dt\,dx.$$

First method: *We first integrate with respect to t, then with respect to x:*

$$I = \int_{x=0}^\pi \left(\int_{t=0}^1 (t\sin x + 2x)\,dt\right)dx = \int_0^\pi \left[\frac{t^2}{2}\sin x + 2xt\right]_{t=0}^{t=1} dx = \int_0^\pi \left(\frac{1}{2}\sin x + 2x\right)dx = \left[-\frac{1}{2}\cos x + x^2\right]_{x=0}^{x=\pi} = \pi^2 + 1.$$

Second method: *We use Fubini's theorem, which states that we can first integrate with respect to x, then with respect to t:*

$$I = \int_{t=0}^{1} (\int_{x=0}^{\pi} (t\sin x + 2x)\,dx)\,dt = \int_{t=0}^{1} \left[-t\cos x + x^2\right]_{x=0}^{x=\pi} dt = \int_{t=0}^{1} (2t+\pi^2)\,dt = [t^2+\pi^2 t]_{t=0}^{t=1} = \pi^2+1.$$

1.1.2 Varying Bounds

Another category of integrals is when the bounds are the parameters of the function: $G(x) = \int_{u(x)}^{v(x)} f(t)\,dt$, where u and v are functions of x.

Theorem 1.1.7. *Let f be a continuous function on a closed interval $[a,b]$ with values in $\mathbb{R}$ (or $\mathbb{C}$). Let I be an interval in $\mathbb{R}$ and $u, v : I \to [a,b]$ be functions of class C^1. Then the function G defined on the interval I by $G(x) = \int_{u(x)}^{v(x)} f(t)\,dt$ is of class C^1 and $G'(x) = v'(x)f(v(x)) - u'(x)f(u(x))$.*

Example 1.1.8. *Let's calculate the derivative of $G(x) = \int_x^{x^2} \frac{1}{\ln(t)}\,dt$ for $x > 1$.*
To apply Theorem 4, we restrict to an interval $[a,b]$ such that, for a fixed x, $x \in [a,b] \subset (1,+\infty)$. With $f(t) = \frac{1}{\ln t}$, $u(x) = x$, and $v(x) = x^2$, we have:
$G'(x) = v'(x)\cdot f(v(x)) - u'(x)\cdot f(u(x)) = 2x \cdot \frac{1}{\ln(x^2)} - 1 \cdot \frac{1}{\ln(x)} = \frac{x-1}{\ln x}$.

1.2 Improper Integrals Depending on a Parameter

Let $f : I \times [0,+\infty) \to \mathbb{R}$ be a function, where I is an interval in $\mathbb{R}$. Suppose that the integral of the partial application $t \mapsto f(x,t)$ is convergent on $[0,+\infty)$. We want to study the function that associates to $x \in I$ the value $F(x) = \int_0^{+\infty} f(x,t)\,dt$.
As you know, a convergent integral is defined as the limit of integrals on bounded intervals. Let's set $F_A(x) = \int_0^A f(x,t)\,dt$, so that $F(x) = \lim_{A\to+\infty} F_A(x)$. The results of the previous sections give conditions under which $F_A(x)$ is continuous and differentiable for fixed A. To pass to the limit as A tends to infinity, we will add a condition called the dominated convergence.

Definition 1.2.1 (Dominated Convergence)**.** *Let $f : I\times[0,+\infty) \to \mathbb{R}$ (or $\mathbb{C}$) be a continuous function. We say that $(x,t) \mapsto f(x,t)$ satisfies the hypothesis of dominated convergence if there exists $g : [0,+\infty) \to \mathbb{R}$ such that:*

1. *the integral $\int_0^{+\infty} g(t)\,dt$ is convergent,*
2. *and such that $\forall t \in [0,+\infty), \quad \forall x \in I, \quad |f(x,t)| \leqslant g(t)$.*

Remark 1.2.2. 1. *Note that g must necessarily be non-negative, so $\int_0^{+\infty} g(t)\,dt$ is actually absolutely convergent.*

2. *In the case of dominated convergence, for each* $x \in I$, $F(x) = \int_0^{+\infty} f(x,t)\,dt$ *is absolutely convergent.*

Example 1.2.3. *Let* $f(x,t) = \frac{\sin x + \sin t}{1+x^2+t^2}$. *Then* f *satisfies the hypothesis of dominated convergence on* $I = \mathbb{R}$ *because* $|f(x,t)| = \left|\frac{\sin x+\sin t}{1+x^2+t^2}\right| \leqslant \frac{2}{1+t^2} = g(t)$, *with* $\int_0^{+\infty} g(t)\,dt$ *converging.*

Theorem 1.2.4. *Let* I *be an interval of* $\mathbb{R}$ *and* $J = [0,+\infty)$. *Let* f *be a continuous function on* $I \times J$ *with values in* $\mathbb{R}$ *(or* $\mathbb{C}$*) and which satisfies the hypothesis of dominated convergence. Then the function* F *defined for all* $x \in I$ *by* $F(x) = \int_0^{+\infty} f(x,t)\,dt$ *is continuous on* I.

Theorem 1.2.5. *Let* I *be an interval of* $\mathbb{R}$ *and* $J = [0,+\infty)$. *Assume that:*

- $(x,t) \mapsto f(x,t)$ *is a continuous function on* $I \times J$ *(with values in* $\mathbb{R}$ *or* $\mathbb{C}$*).*
- *the partial derivative* $(x,t) \mapsto \frac{\partial f}{\partial x}(x,t)$ *exists, is continuous on* $I \times J$, *and satisfies the hypothesis of dominated convergence.*

Then F, *defined for all* $x \in I$ *by* $F(x) = \int_0^{+\infty} f(x,t)\,dt$, *is of class* C^1 *on* I *and:* $F'(x) = \int_0^{+\infty} \frac{\partial f}{\partial x}(x,t)\,dt$.

Example 1.2.6. *Let* $f(x,t) = \frac{x}{1+(xt)^2}$. *Then* f *is continuous on* $\mathbb{R} \times [0,+\infty)$ *and*

$$F_A(x) = \int_0^A f(x,t)\,dt = \int_0^A \frac{x}{1+(xt)^2}\,dt = \int_0^{xA} \frac{x}{1+u^2}\,du = [\arctan(u)]_0^{xA} = \arctan(xA) \text{ with } u = xt.$$

Therefore: $F(x) = \int_0^{+\infty} f(x,t)\,dt = \lim_{A\to+\infty} F_A(x) = \lim_{A\to+\infty} \arctan(xA) = \begin{cases} +\frac{\pi}{2} & \text{if } x > 0, \\ -\frac{\pi}{2} & \text{if } x < 0, \\ 0 & \text{if } x = 0. \end{cases}$

Thus, F *is discontinuous.*

Theorem 1.2.7 (Fubini's Theorem)**.** *Let* $I = [\alpha,\beta]$ *be a bounded closed interval and* $J = [0,+\infty)$. *Let* f *be a continuous function on* $I \times J$, *with values in* $\mathbb{R}$ *(or* $\mathbb{C}$*), and which satisfies the hypothesis of dominated convergence. Then the function* F *is integrable on* I *and*

$$\int_\alpha^\beta F(x)\,dx = \int_\alpha^\beta \left(\int_0^{+\infty} f(x,t)\,dt\right)dx = \int_0^{+\infty} \left(\int_\alpha^\beta f(x,t)\,dx\right)dt.$$

1.3 Chapter 1 Exercises

Exercise 1

Consider the following functions:

$$F(x) = \int_1^2 \frac{e^{-x^2 t^3}}{1+t^2} dt \quad \text{and} \quad G(x) = \int_0^1 \frac{e^{x(t+1)}}{t+1} dt.$$

1. Study the continuity and differentiability of F and G on $\mathbb{R}$.
2. Calculate G'.

Correction ▼ [1]

Exercise 2

Consider the function

$$F(x) = \int_0^1 \frac{1}{(t^2+x^2)(t^2+1)} dt.$$

1. Show that F is continuous on $\mathbb{R}^*$.
2. Find the function $F(x)$ (compute the integral).
3. Deduce the value of $\int_0^1 \frac{1}{(t^2+1)^2} dt$.

Correction ▼ [2]

Exercise 3

Consider the following functions:

$$F(x) = \left(\int_0^x e^{-t^2} dt\right)^2, \quad G(x) = \int_0^1 \frac{e^{-(1+t^2)x^2}}{t^2+1} dt.$$

1. Show that F and G are differentiable on $[0, +\infty)$ and compute F' and G'.
2. Show that for all $x \in [0, +\infty)$, $F(x) + G(x) = \frac{\pi}{4}$.
3. Deduce the value of $\int_0^{+\infty} e^{-t^2} dt$.

Correction ▼ [3]

Exercise 4

Let

$$F(x) = \int_1^{+\infty} \frac{e^{-xt}}{(1+t)\sqrt{t}}.$$

1. Show that F is continuous on $[0, +\infty[$.
2. Show that F is differentiable on $]0, +\infty[$.

Correction ▼ [4]

Chapter

2 Laplace Transform and Fourier Transform

2.1 Laplace Transform

This section serves as an introduction to the Laplace transform, a mathematical operation extensively utilized in electronics. It effectively converts a time-dependent function t into a frequency-dependent function s. Additionally, the Laplace transform is valuable for solving differential equations, as it transforms analytical operations such as differentiation and integration into algebraic operations like multiplication and division.

Definition 2.1.1. *Let f be a continuous function on the interval $[0,+\infty[$, with values in $\mathbb{R}$ (or $\mathbb{C}$). The Laplace transform of f is the function F defined by:*

$$F(s) = \int_0^{+\infty} f(t)e^{-st}dt \tag{2.1}$$

If one wants to emphasize the dependence on the function f (rather than the parameter s), then one can write this same integral as:

$$\mathcal{L}(f) = \int_0^{+\infty} f(t)e^{-st}dt \tag{2.2}$$

Remark 2.1.2.

- *We will assume that s is a real parameter, and the functions to be continuous.*
- *When we write $F(s)$, this will conventionally mean that the integral converges.*

Example 2.1.3. *1. Let $f(t) = 1$, the constant function equal to 1. Then, for $s > 0$,*

$$F(s) = \int_0^{+\infty} 1 \cdot e^{-st}dt = \left[-\frac{e^{-st}}{s}\right]_0^{+\infty} = \lim_{t\to+\infty}\left(\frac{-e^{-st}}{s}\right) - \left(\frac{-e^{0}}{s}\right) = \frac{1}{s} \tag{2.3}$$

2. Let $f(t) = e^t$. Then, for $s > 1$,

$$F(s) = \int_0^{+\infty} e^t e^{-st}dt = \int_0^{+\infty} e^{(1-s)t}dt = \left[\frac{e^{(1-s)t}}{1-s}\right]_0^{+\infty} = \frac{1}{s-1} \tag{2.4}$$

3. Let $f(t) = t$. We perform integration by parts with $u(t) = t$, $v'(t) = e^{-st}$. Then, for $s > 0$:

$$F(s) = \int_0^{+\infty} t \cdot e^{-st}dt = \left[t \cdot \frac{-e^{-st}}{s}\right]_0^{+\infty} - \int_0^{+\infty} 1.\frac{-e^{-st}}{s}dt = 0 + \frac{1}{s}\left[\frac{-e^{-st}}{s}\right]_0^{+\infty} = \frac{1}{s^2} \tag{2.5}$$

2.1.1 Properties

Proposition 2.1.4.

1. ***Linearity:*** $\mathcal{L}(\lambda f + \mu g) = \lambda\mathcal{L}(f) + \mu\mathcal{L}(g)$.
2. ***Differentiation:*** $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$.
3. ***Integration:*** $\mathcal{L}\left(\int f\right) = \frac{1}{s}\mathcal{L}(f)$ *where* $\int f$ *is the primitive of* f *vanishing at* $t = 0$.
4. ***Delay theorem:*** $\mathcal{L}(f(t-\tau)) = e^{-s\tau}\mathcal{L}(f(t))$.
5. ***Initial value theorem:*** $\lim_{s\to+\infty} sF(s) = f(0)$, $\lim_{s\to+\infty} F(s) = 0$.
6. ***Final value theorem:*** *If the limit of* $f(t)$ *exists and is finite as* $t \to +\infty$, *then* $\lim_{s\to 0} sF(s) = \lim_{t\to+\infty} f(t)$.

2.1.2 Common Laplace transforms

Here are some classical Laplace transforms:

$f(t)$	$\mathcal{L}(f(t)) = F(s)$	Validity
c	$\frac{c}{s}$	$s > 0$
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
$e^{\alpha t}$	$\frac{1}{s-\alpha}$	$s > \alpha$
$t^n e^{\alpha t}$	$\frac{n!}{(s-\alpha)^{n+1}}$	$s > \alpha$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$s > 0$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$s > 0$
$\sqrt{t}$	$\frac{1}{2}\sqrt{\frac{\pi}{s^3}}$	$s > 0$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$	$s > 0$

2.1.3 Inverse Laplace Transform

Theorem 2.1.5. *Let* $f, g : [0, +\infty) \to \mathbb{R}$ *be two continuous functions, and let* F *and* G *be their Laplace transforms. If for all* $s > 0$, $F(s) = G(s)$, *then for all* $t > 0$, $f(t) = g(t)$.

This theorem allows us to talk about the inverse Laplace transform, i.e., going from $F(s)$ to $f(t)$. There is no explicit formula at our disposal to make this transition. This is why the tables of Laplace transforms are useful: knowing $F(s)$, we manually search for the corresponding $f(t)$.

Example 2.1.6. *Let's find the function* $f(t)$ *corresponding to the Laplace transform:*

$$F(s) = \frac{2s-1}{s^2+1} - \frac{3}{(s-2)^2}?$$

We can decompose $F(s)$ as: $F(s) = \frac{2s}{s^2+1} - \frac{1}{s^2+1} - \frac{3}{(s-2)^2}$.
Using the linearity of the Laplace transform and the tables of common transforms, the corresponding function $f(t)$ is: $f(t) = 2\cos t - \sin t - 3te^{2t}$.

2.1.4 Differential Equations

If $F(s)$ is the Laplace transform of a function $f(t)$, then $sF(s) - f(0)$ is the Laplace transform of $f'(t)$. The Laplace transform thus replaces the operation of differentiation on $f(t)$ by a multiplication by s on $F(s)$. Here is how we can solve differential equations:

Differential equation
↓ **Laplace transform**
Algebraic equation
↓
Algebraic solution
↓ **Inverse Laplace transform**
Differential solution

We transform a differential problem into an algebraic problem, solve the algebraic problem, and then transform the algebraic solution into a differential solution. To respect the conventions, in the following we will note the functions as $y(t)$ instead of $f(t)$.

Example 2.1.7. *What is the solution to the differential equation: $y'(t) + y(t) = t$ with $y(0) = 3$?*

1. ***Laplace transforms:*** *Let's calculate the Laplace transforms of the objects that appear:*
   - Let $F(s) = \mathcal{L}(y)$,
   - Then we know that $\mathcal{L}(y') = sF(s) - y(0)$,
   - Finally, $\mathcal{L}(t) = \frac{1}{s^2}$.

2. ***From the differential equation to the algebraic equation:*** *Since $y'(t) + y(t) = t$, then $\mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}(t)$. This gives $sF(s) - y(0) + F(s) = \frac{1}{s^2}$. And since $y(0) = 3$ by hypothesis, then: $(s+1)F(s) = 3 + \frac{1}{s^2}$.*

3. ***Solving the algebraic equation:*** *It is simply a matter of:* $F(s) = \frac{3}{s+1} + \frac{1}{s^2(s+1)}$.

   But we will need the decomposition into partial fractions: $F(s) = -\frac{1}{s} + \frac{1}{s^2} + \frac{4}{s+1}$.

4. ***Back to the differential solution:*** *We need to find the function* $y(t)$ *corresponding to our algebraic solution* $F(s)$*. This is where the tables are useful:*

   - For $F_1(s) = \frac{1}{s}$*, it is* $y_1(t) = 1$*,*
   - For $F_2(s) = \frac{1}{s^2}$*, it is* $y_2(t) = t$*,*
   - For $F_3(s) = \frac{1}{s+1}$*, it is* $y_3(t) = e^{-t}$*.*

   So by linearity, the solution is $y(t) = -y_1(t)+y_2(t)+4y_3(t)$*, and thus:* $y(t) = -1+t+4e^{-t}$.

   We can verify that this function satisfies $y'(t) + y(t) = t$ *and* $y(0) = 3$.

Example 2.1.8. *Let's solve the following differential equation:*

$$y''(t) - 4y(t) = 3e^{-t} \text{ with } y(0) = 0 \text{ and } y'(0) = 1.$$

1. *Let's denote* $F(s) = \mathcal{L}(y)$*. We have* $\mathcal{L}(y') = sF(s) - y(0)$*, and thus*

$$\mathcal{L}(y'') = s\mathcal{L}(y') - y'(0) = s^2F(s) - sy(0) - y'(0).$$

   Given our initial conditions, we have here $\mathcal{L}(y'') = s^2F(s) - 1$*. Finally,* $\mathcal{L}(e^{-t}) = \frac{1}{s+1}$.

2. *The equation* $y''(t) - 4y(t) = 3e^{-t}$ *becomes* $(s^2 - 4)F(s) = 1 + \frac{3}{s+1}$.

3. *Thus, after decomposition into partial fractions:*

$$F(s) = \frac{1}{s^2-4} + \frac{3}{(s+1)(s^2-4)} = \frac{\frac{1}{2}}{s+2} + \frac{\frac{1}{2}}{s-2} - \frac{1}{s+1}$$

4. *Using the tables, we recognize the solution:*

$$y(t) = \frac{1}{2}e^{-2t} + \frac{1}{2}e^{2t} - 3e^{-t}.$$

2.2 Fourier Transform

This section is an introduction to the Fourier transform. Like the Laplace transform, the Fourier transform changes a function that depends on time into a function that depends on frequency and is widely used in signal theory. The Fourier transform applies to non-periodic functions, in contrast to Fourier series.

2.2.1 Definition

Definition 2.2.1. *Let* f *be a piecewise continuous function on* $\mathbb{R}$*, with values in* $\mathbb{R}$ *(or*

$\mathbb{C}$). *The Fourier transform of f is the function F defined by:*

$$\mathcal{F}(s) = \int_{-\infty}^{+\infty} f(t)e^{-ist}dt$$

We also denote it as $\mathcal{F}(f)$.

Example 2.2.2. 1. *Let f be the function defined by $f(t) = 1$ if $t \in [-1, +1]$, and $f(t) = 0$ otherwise. Then*

$$\mathcal{F}(s) = \int_{-\infty}^{+\infty} f(t)e^{-ist}dt = \int_{-1}^{1} e^{-ist}dt = \left[\frac{e^{-ist}}{-is}\right]_{-1}^{1} = \frac{2\sin(s)}{s}.$$

2. *What is the Fourier transform $F(s)$ of the function defined by $f(t) = e^{-\alpha|t|}$, with $\alpha > 0$?*

$$\mathcal{F}(s) = \int_{-\infty}^{+\infty} f(t)e^{-ist}dt = \int_{-\infty}^{0} e^{\alpha t}e^{-ist}dt + \int_{0}^{+\infty} e^{-\alpha t}e^{-ist}dt$$

$$= \left[\frac{e^{(\alpha-is)t}}{\alpha - is}\right]_{-\infty}^{0} + \left[\frac{e^{(-\alpha-is)t}}{-\alpha - is}\right]_{0}^{+\infty}$$

$$= \frac{1}{\alpha - is} + 0 + 0 + \frac{1}{\alpha + is} = \frac{2\alpha}{\alpha^2 + s^2}.$$

Remark 2.2.3.
- *Different definitions of the Fourier transform may be found in the literature, with different constants. So one must be careful about the specific definition being used.*
- *The improper integral has two uncertain points at $-\infty$ and $+\infty$. By definition, an improper integral $\int_{-\infty}^{+\infty} g(t)dt$ converges if and only if the integral $\int_{-\infty}^{0} g(t)dt$ converges and the integral $\int_{0}^{+\infty} g(t)dt$ converges as well.*
- *Unlike the Laplace transform, the Fourier transform often takes values in $\mathbb{C}$, even if the original function is defined on $\mathbb{R}$.*

2.2.2 Properties

Proposition 2.2.4.
1. *Linearity: $\mathcal{F}(\lambda f + \mu g) = \lambda\mathcal{F}(f) + \mu\mathcal{F}(g)$.*
2. *Parity: If f is an even function, then $F(s) = 2\int_0^{+\infty} f(t)\cos(st)dt$. If f is an odd function, then $F(s) = -2i\int_0^{+\infty} f(t)\sin(st)dt$.*
3. *Differentiation: $\mathcal{F}(f') = is\mathcal{F}(f)$.*
4. *Time-delay Theorem: $\mathcal{F}[f(t-\tau)] = e^{-is\tau}\mathcal{F}[f(t)]$.*

2.2.3 Inverse Fourier Transform

The Fourier transform maps $f(t)$ to $F(s)$. There exists an inverse Fourier transform that allows us to go back from $F(s)$ to $f(t)$.

Theorem 2.2.5. *If $F(s)=\int_{-\infty}^{+\infty} f(t)e^{-ist}dt$ has an absolutely convergent integral $\int_{-\infty}^{+\infty}|F(s)|ds$, then*

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} F(s)e^{+ist}ds.$$

One must pay attention to the constant $\frac{1}{2\pi}$ and the positive sign in e^{+ist}. We accept this theorem. In other words, if we denote the inverse Fourier transform as $\mathcal{F}^{-1}$, we have:

$$\mathcal{F}^{-1}(f)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)e^{+ist}dt.$$

Then the inverse Fourier transform $\mathcal{F}^{-1}$ is the operation that allows us to go back from F to f:

$$\mathcal{F}^{-1}\{\mathcal{F}(f)\}=f \text{ and } \mathcal{F}\{\mathcal{F}^{-1}(f)\}=f.$$

It is remarkable that the inverse transform has a form very close to the direct transform.

Example 2.2.6. *What is the Fourier transform of $g(t)=\frac{1}{1+t^2}$?*
In example 13, we saw that the Fourier transform of $f(t)=e^{-|t|}$ is $F(s)=\frac{1}{1+s^2}$, which has an absolutely convergent integral. This means that, according to Theorem 9, the inverse Fourier transform of $g(t)=\frac{1}{1+t^2}$ is: $G(s)=\frac{e^{-|s|}}{2}$.
We just said that:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} g(t)e^{+ist}dt=G(s)$$

So, evaluating this expression at $-s$, we get: $\frac{1}{2\pi}\int_{-\infty}^{+\infty} g(t)e^{-ist}dt=G(-s)$. In other words:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{1+t^2}e^{-ist}dt=\frac{e^{-|s|}}{2}$$

This allows us to conclude that the Fourier transform of $g(t)=\frac{1}{1+t^2}$ is:

$$\mathcal{F}(g)=\int_{-\infty}^{+\infty}\frac{e^{-ist}}{1+t^2}dt=\pi e^{-|s|}$$

In particular, when we take the real part of this last equality: $\int_{-\infty}^{+\infty}\frac{\cos(st)}{1+t^2}dt=\pi e^{-|s|}$. which gives for $s=1$: $\int_{-\infty}^{+\infty}\frac{\cos t}{1+t^2}dt=\frac{\pi}{e}$. The correspondence is therefore:

$$e^{-|t|}\xrightarrow{\mathcal{F}}\frac{2}{1+s^2}$$
$$\frac{e^{-|t|}}{2}\xleftarrow{\mathcal{F}^{-1}}\frac{1}{1+s^2}$$

Relation with the Laplace transform

The Fourier and Laplace transforms, when they are well-defined, are related by the following relationship:

$$\mathcal{F}(f)(s) = \mathcal{L}(f_+)(+is) + \mathcal{L}(f_-)(-is)$$

where f_+ and f_- are defined on $[0,+\infty[$ by: $f_+(t) = f(t)$ and $f_-(t) = f(-t)$ for $t \geqslant 0$.

Example 2.2.7. *Let's calculate the Fourier transform of $f(t) = t^2 e^{-|t|}$.*
We denote $f_+(t)$ and $f_-(t)$ as above. Since the function f is even, then $f_+ = f_-$.
From the Laplace transform tables, we know that for $f_+(t) = t^2 e^{-t}$ (with $t \geqslant 0$), which is of the type $t^n e^{\alpha t}$, we have $\mathcal{L}(f_+) = \frac{2}{(s+1)^3}$. We can thus deduce that:

$$\mathcal{F}(f)(s) = \mathcal{L}(f_+)(+is) + \mathcal{L}(f_-)(-is) = \frac{2}{(is+1)^3} + \frac{2}{(-is+1)^3} = \frac{4-12s^2}{(1+s^2)^3}.$$

2.2.4 Convolution product

The convolution product, denoted by $*$, is a bilinear operator and a commutative product.

Definition 2.2.8. *Let (f,g) be two functions defined on $\mathbb{R}$ (or $\mathbb{C}$). If f and g are integrable on $\mathbb{R}$, we define:*

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)\,dy$$

Basic Properties:
Let f and g be two functions in $L^1(\mathbb{R})$.

1. **Commutativity:** The convolution product is commutative:

$$(f * g)(x) = (g * f)(x)$$

2. **Translation:** Let $\tau_h(\cdot)$ denote the translation of a function $f(x)$: $\tau_h(f(x)) = f(x-h)$. We then have:

$$\tau_h(f) * g = \tau_h(f * g)$$

3. **Derivative:** If f and g are $C^1(\mathbb{R})$, and if f' and g' are in $L^1(\mathbb{R})$, then we have:

$$(f * g)' = f' * g = f * g'$$

Fourier Transform and Convolution: A key property of the Fourier transform $\mathcal{F}$ is that it transforms the convolution product into multiplication.

Proposition 2.2.9. *For $f, g \in L^1(\mathbb{R})$, we have:*

$$\mathcal{F}((f * g)(x)) = \mathcal{F}(f(x)) \cdot \mathcal{F}(g(x)).$$

2.3 Chapter 2 Exercises

Exercise 5

Calculate the Laplace transform of the following functions:

$$a)\ t \longrightarrow e^{at}\ ,\quad b)\ t \longrightarrow t^n e^{at}\ ,\quad c)\ t \longrightarrow sin(\omega t)\ ,\quad d)\ t \longrightarrow e^{-4t} sin(5t)\ ,\quad e)\ t \longrightarrow t^2 cost\ .$$

Correction ▼ [5]

Exercise 6

Find the original functions of the following Laplace transforms:

1. $\frac{1}{(s+1)(s-2)}$	2. $\frac{-1}{(s-2)^2}$	**3.** $\frac{5s+10}{s^2+3s-4}$	**4.** $\frac{s-7}{s^2-14s+50}$
5. $\frac{s}{s^2-6s+13}$	**6.** $\frac{e^{-2s}}{s+3}$	**7.** $\frac{a}{s^2-a^2}$	**8.** $\frac{s^2}{(s+3)^3}$

Correction ▼ [6]

Exercise 7

Solve the following differential equations (using the Laplace transform):

$$a)\ x' + 3x = 0\ ,\quad b)\ x' + 3x = cos(3t)\ \ with\ \ x(0) = 0\ ,$$

$$c)\ x'' + x = t\ \ with\ \ x(0) = 1\ \ and\ \ x'(0) = 0\ .$$

Correction ▼ [7]

Exercise 8

Calculate the Fourier transform of:

1. $f(t) = e^{-\alpha|t|} \quad (\alpha > 0)$.
2. $g(t) = \begin{cases} 1 & \text{if } t \in]-3,3[\\ 0 & \text{otherwise} \end{cases}.$
3. $g(t) = \begin{cases} e^{-|t|} + 1 & \text{if } t \in]-3,3[\\ e^{-|t|} & \text{otherwise} \end{cases}.$

Correction ▼ [8]

Exercise 9

For $\alpha > 0$, we define $f(t) = e^{-\alpha|t|}$.

1. Calculate the Fourier transform of f.
2. Using the reciprocity formula, deduce the Fourier transform of $t \mapsto \frac{1}{1+t^2}$.
3. Calculate $f \star f$; deduce the Fourier transform of $t \mapsto \frac{1}{(1+t^2)^2}$.
4. Determine the Fourier transform of $t \mapsto \frac{t}{(1+t^2)^2}$.

Correction ▼ [9]

Chapter

3 Elements of Topology

In this chapter, we will examine the key concepts of topological, metric, and normed spaces, which are essential to modern analysis. These notions provide a framework for studying continuity and convergence, and are particularly important for the analysis of functions of several variables.

3.1 Metric Spaces

Definition 3.1.1 (Distance). *Let E be a non-empty set. A distance on E is a function $d : E \times E \longrightarrow \mathbb{R}^+$ that satisfies, for all $(x,y,z) \in E^3$:*

1. *$d(x,y) = 0$ if and only if $x = y$;*
2. *$d(x,y) = d(y,x)$;*
3. *$d(x,z) \leqslant d(x,y) + d(y,z)$.*

If d is a distance on E, the pair (E,d) is called a metric space.

Remark 3.1.2. *Property 3 is known as the "triangle inequality." It leads to the following inequality, called the "second triangle inequality":*

$$|d(x,y) - d(y,z)| \leqslant d(x,z).$$

Example 3.1.3. *Let E be a set. The function $d : E \times E \longrightarrow \mathbb{R}^+$ defined by*

$$d(x,x) = 0 \quad \text{and} \quad d(x,y) = 1 \quad \text{if} \quad x \neq y$$

*is called the **discrete metric**. We verify that d satisfies the properties of a distance (or metric):*

1. *For all $x, y \in E$, we have:*
   - *If $x = y$, then $d(x,y) = 0$, which is non-negative.*
   - *If $x \neq y$, then $d(x,y) = 1$, which is also non-negative.*

   Thus, $d(x,y) \geqslant 0$ for all $x, y \in E$.
2. *This property states that $d(x,y) = 0$ if and only if $x = y$. We can see that:*

- *If $d(x,y) = 0$, then by definition, x must equal y.*
- *Conversely, if $x = y$, then $d(x,y) = 0$.*

3. *This property states that $d(x,y) = d(y,x)$ for all $x,y \in E$. We have:*
   - *If $x = y$, then $d(x,y) = d(y,x) = 0$.*
   - *If $x \neq y$, then $d(x,y) = d(y,x) = 1$.*

4. *The triangle inequality states that for all $x,y,z \in E$:*

$$d(x,z) \leqslant d(x,y) + d(y,z).$$

We will check the possible cases:

(a) ***Case 1:*** $x = y = z$
   - *Here, $d(x,z) = d(x,y) + d(y,z) = 0 + 0 = 0$, satisfying the inequality.*

(b) ***Case 2:*** $x = y \neq z$
   - *Then $d(x,z) = 1$ and $d(x,y) + d(y,z) = 0 + 1 = 1$, satisfying the inequality.*

(c) ***Case 3:*** $x \neq y = z$
   - *Then $d(x,z) = 1$ and $d(x,y) + d(y,z) = 1 + 0 = 1$, satisfying the inequality.*

(d) ***Case 4:*** $x \neq y \neq z$
   - *In this case, we have $d(x,y) = 1$, $d(y,z) = 1$, and $d(x,z) = 1$. Thus:*

$$d(x,z) = 1 \leqslant 1 + 1 = 2,$$

which also satisfies the inequality.

Example 3.1.4. *On $\mathbb{R}$, the usual distance between two real numbers x and y is given by*

$$d_{|.|}(x,y) = |x - y|.$$

The properties of the metric $d_{|.|}$ can be summarized as follows:

1. *For all $x,y \in \mathbb{R}$, we have $d_{|.|}(x,y) \geqslant 0$ since the absolute value is always non-negative.*
2. *$d_{|.|}(x,y) = 0$ if and only if $x = y$. This directly follows from the definition of absolute value.*
3. *For all $x,y \in \mathbb{R}$, we have $d_{|.|}(x,y) = d_{|.|}(y,x)$, which is a property of absolute values: $|x - y| = |y - x|$.*
4. *For all $x,y,z \in \mathbb{R}$, we have*

$$d_{|.|}(x,z) \leqslant d_{|.|}(x,y) + d_{|.|}(y,z),$$

which follows from the property of absolute values:

$$|x - z| \leqslant |x - y| + |y - z|.$$

The metric space $(\mathbb{R}, |.|)$ is often denoted as $(\mathbb{R}, d_{|.|})$.

3.2 Normed Vector Spaces

Let $\mathbb{K}$ denote $\mathbb{R}$ or $\mathbb{C}$.

Definition 3.2.1 (Norm). *Let E be a vector space over $\mathbb{K}$. A mapping $\|.\|$ from E to $\mathbb{R}^+$ is a norm if it satisfies for all $(x, y) \in E$ and all $\lambda \in \mathbb{K}$:*

1. $\|x\| = 0$ *if and only if* $x = 0$.
2. $\|\lambda x\| = |\lambda|\|x\|$.
3. $\|x + y\| \leqslant \|x\| + \|y\|$.

   A vector space E equipped with a norm is called a normed vector space. The real number $\|x\|$ is called the norm of the vector x.

Example 3.2.2. *The mapping $\|\cdot\|_1 : \mathbb{K}^n \to \mathbb{R}$ defined by*

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

is a norm. We will verify the following properties:

1. *For all $x \in \mathbb{K}^n$,*
$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \geqslant 0.$$
   Since the absolute value $|x_i|$ is always non-negative, the sum of non-negative numbers is also non-negative.

2. *We have:*
$$\|x\|_1 = 0 \iff x = 0.$$
   If $\|x\|_1 = 0$, then $\sum_{i=1}^{n} |x_i| = 0$, which implies that each $|x_i| = 0$ and hence $x_i = 0$ for all i. Conversely, if $x = 0$, then clearly $\|x\|_1 = 0$.

3. *For any scalar $\alpha \in \mathbb{R}$ and $x \in \mathbb{K}^n$,*
$$\|\alpha x\|_1 = \sum_{i=1}^{n} |\alpha x_i| = \sum_{i=1}^{n} |\alpha||x_i| = |\alpha| \sum_{i=1}^{n} |x_i| = |\alpha|\|x\|_1.$$

4. *For all $x, y \in \mathbb{K}^n$,*
$$\|x + y\|_1 = \sum_{i=1}^{n} |x_i + y_i| \leqslant \sum_{i=1}^{n} (|x_i| + |y_i|) = \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i| = \|x\|_1 + \|y\|_1.$$
   This follows from the triangle inequality of absolute values, which states that $|a + b| \leqslant |a| + |b|$ for any real numbers a and b.

Example 3.2.3. *The mapping $\|\cdot\|_2 : \mathbb{K}^n \to \mathbb{R}$ defined by*

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}}$$

is a norm. We will verify the following properties:

1. *For all $x \in \mathbb{K}^n$,*

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}} \geqslant 0.$$

Since each $|x_i|^2$ is non-negative, their sum is also non-negative, and the square root of a non-negative number is non-negative.

2. *We have:*

$$\|x\|_2 = 0 \iff x = 0.$$

If $\|x\|_2 = 0$, then $\sum_{i=1}^{n} |x_i|^2 = 0$, which implies that each $|x_i| = 0$ and hence $x_i = 0$ for all i. Conversely, if $x = 0$, then clearly $\|x\|_2 = 0$.

3. *For any scalar $\alpha \in \mathbb{R}$ and $x \in \mathbb{K}^n$,*

$$\|\alpha x\|_2 = \left(\sum_{i=1}^{n} |\alpha x_i|^2 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^{n} |\alpha|^2 |x_i|^2 \right)^{\frac{1}{2}} = |\alpha| \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}} = |\alpha| \|x\|_2.$$

This shows that scaling a vector by a scalar scales its norm by the absolute value of that scalar.

4. *For all $x, y \in \mathbb{K}^n$,*

$$\|x + y\|_2 = \left(\sum_{i=1}^{n} |x_i + y_i|^2 \right)^{\frac{1}{2}}.$$

Using the Cauchy-Schwarz inequality, we have:

$$|x_i + y_i|^2 \leqslant (|x_i| + |y_i|)^2 = |x_i|^2 + 2|x_i||y_i| + |y_i|^2.$$

Thus,

$$\|x + y\|_2^2 \leqslant \sum_{i=1}^{n} (|x_i|^2 + 2|x_i||y_i| + |y_i|^2) = \sum_{i=1}^{n} |x_i|^2 + \sum_{i=1}^{n} |y_i|^2 + 2 \sum_{i=1}^{n} |x_i||y_i| \leqslant \|x\|_2^2 + \|y\|_2^2.$$

Taking the square root gives:

$$\|x + y\|_2 \leqslant \|x\|_2 + \|y\|_2.$$

Example 3.2.4. *The mapping* $\|\cdot\|_\infty : \mathbb{K}^n \to \mathbb{R}$ *defined by*

$$\|x\|_\infty = \max_{1\leqslant i\leqslant n} |x_i|$$

is a norm. We will verify the following properties:

1. *For all* $x \in \mathbb{K}^n$,
$$\|x\|_\infty = \max_{1\leqslant i\leqslant n} |x_i| \geqslant 0.$$
Since the absolute value $|x_i|$ *is always non-negative, the maximum of a set of non-negative numbers is also non-negative.*

2. *We have:*
$$\|x\|_\infty = 0 \iff x = 0.$$
If $\|x\|_\infty = 0$, *then* $\max_{1\leqslant i\leqslant n} |x_i| = 0$, *which implies that each* $|x_i| = 0$ *and hence* $x_i = 0$ *for all* i. *Conversely, if* $x = 0$, *then clearly* $\|x\|_\infty = 0$.

3. *For any scalar* $\alpha \in \mathbb{R}$ *and* $x \in \mathbb{K}^n$,
$$\|\alpha x\|_\infty = \max_{1\leqslant i\leqslant n} |\alpha x_i| = |\alpha| \max_{1\leqslant i\leqslant n} |x_i| = |\alpha|\|x\|_\infty.$$

4. *For all* $x, y \in \mathbb{K}^n$,
$$\|x+y\|_\infty = \max_{1\leqslant i\leqslant n} |x_i + y_i| \leqslant \max_{1\leqslant i\leqslant n} (|x_i| + |y_i|) \leqslant \max_{1\leqslant i\leqslant n} |x_i| + \max_{1\leqslant i\leqslant n} |y_i| = \|x\|_\infty + \|y\|_\infty.$$
The first inequality follows from the triangle inequality for absolute values, and the second inequality holds because the maximum of a sum is less than or equal to the sum of the maxima.

Every normed space is equipped with a distance.

Theorem 3.2.5. *Let* $(E, \|\cdot\|)$ *be a normed space over* $\mathbb{K}$. *The mapping* $d : E \times E \to \mathbb{R}$ *defined by*
$$d(x,y) = \|x-y\|$$
is a distance induced by the norm of E. *Thus, every normed vector space is a metric space.*

Proof. For all $x, y \in E$, we have
$$d(x,y) = 0 \iff \|x-y\| = 0 \iff x = y.$$
This shows that d satisfies the separation axiom. The symmetry follows from the fact that, for all $x, y \in E$,
$$\|x-y\| = \|-(y-x)\| = \|y-x\|.$$
To verify the triangle inequality, we take $x, y, z \in E$ and write
$$d(x,y) + d(y,z) = \|x-y\| + \|y-z\| \geqslant \|(x-y)+(y-z)\| = \|x-z\| = d(x,z).$$
□

Definition 3.2.6. 1. *Let E be a set, and let d_1 and d_2 be two distances on E. We say that d_1 and d_2 are equivalent if there exist constants $m, M > 0$ such that for all $x, y \in E$:*
$$md_1(x,y) \leqslant d_2(x,y) \leqslant Md_1(x,y).$$

2. *Let X be a vector space, and let N_1 and N_2 be two norms on X. We say that N_1 and N_2 are equivalent if there exist strictly positive constants m and M such that for all $x \in X$, the following inequalities hold:*

$$mN_1(x) \leqslant N_2(x) \leqslant MN_1(x).$$

Example 3.2.7. *Let (E, d_E) and (F, d_F) be two metric spaces. The distances d_1 and d_∞ defined earlier are equivalent.*
Indeed, for every $(x,y),(x',y') \in E \times F$, we have the following inequalities:

$$d_\infty((x,y),(x',y')) = \max(d_E(x,x'), d_F(y,y'))$$

By the triangle inequality, this can be bounded as follows:

$$d_\infty((x,y),(x',y')) \leqslant d_E(x,x') + d_F(y,y').$$

Thus, we find:
$$d_\infty((x,y),(x',y')) \leqslant d_1((x,y),(x',y')),$$
where
$$d_1((x,y),(x',y')) = d_E(x,x') + d_F(y,y').$$
On the other hand, we can also establish:

$$d_1((x,y),(x',y')) = d_E(x,x') + d_F(y,y') \leqslant 2\max(d_E(x,x'), d_F(y,y')) = 2d_\infty((x,y),(x',y')).$$

We have thus established the inequalities:

$$d_\infty \leqslant d_1 \leqslant 2d_\infty.$$

3.3 Balls, Open Sets, Closed Sets, Topology, Neighborhood

Definition 3.3.1. *Let $(E, \|.\|)$ be a normed space.*

1. *The closed ball centered at $a \in E$ with radius r for the norm $\|.\|$ is defined by*

$$B_f(a,r) = \{x \in E \mid d(a,x) \leqslant r\} = \{x \in E \mid \|x-a\| \leqslant r\}.$$

2. *The open ball centered at $a \in E$ with radius r for the norm $\|.\|$ is defined by*

$$B(a,r) = \{x \in E \mid d(a,x) < r\} = \{x \in E \mid \|x-a\| < r\}.$$

Example 3.3.2. *In $\mathbb{R}$ equipped with the usual distance $d(x,y) = |x-y|$, we define the open ball $B(x,r)$ and the closed ball $B_f(x,r)$ as follows:*

1. *The open ball centered at x with radius r is defined by:*

$$B(x,r) = \{y \in \mathbb{R} \mid |y-x| < r\} = (x-r, x+r).$$

2. *The closed ball centered at x with radius r is defined by:*

$$B_f(x,r) = \{y \in \mathbb{R} \mid |y-x| \leqslant r\} = [x-r, x+r].$$

Example 3.3.3. *In $\mathbb{R}^2$ equipped with the $\|\cdot\|_\infty$ norm, defined as*

$$\|(x,y)\|_\infty = \max\{|x|,|y|\},$$

we can describe the associated distance as

$$d((x_1,y_1),(x_2,y_2)) = \|(x_1-x_2, y_1-y_2)\|_\infty = \max\{|x_1-x_2|,|y_1-y_2|\}.$$

1. *The open ball centered at the origin $0 = (0,0)$ with radius 1 is defined by:*

$$B(0,1) = \{(x,y) \in \mathbb{R}^2 \mid \|(x,y)\|_\infty < 1\}.$$

This means that the open ball $B(0,1)$ consists of all points (x,y) such that the maximum of the absolute values of the coordinates is less than 1.

Characterization of the Open Ball: The condition $\|(x,y)\|_\infty < 1$ can be rewritten as:

$$\max\{|x|,|y|\} < 1.$$

This implies that both $|x| < 1$ and $|y| < 1$. Therefore, the open ball can be represented as:

$$B(0,1) = \{(x,y) \in \mathbb{R}^2 \mid -1 < x < 1 \text{ and } -1 < y < 1\} = (-1,1)^2.$$

2. *The closed ball centered at the origin with radius 1 is defined by:*

$$B_f(0,1) = \{(x,y) \in \mathbb{R}^2 \mid \|(x,y)\|_\infty \leqslant 1\} = [-1,1]^2.$$

This includes all points (x,y) such that $\max\{|x|,|y|\} \leqslant 1$, which corresponds to the square with vertices at $(-1,-1)$, $(-1,1)$, $(1,-1)$, and $(1,1)$.

Definition 3.3.4. *Let (E,d) be a metric space. A subset A of E is said to be open if for every point $a \in A$, there exists a radius $r > 0$ such that the open ball centered at a with radius r is contained in A:*

$$B(a,r) \subseteq A.$$

Example 3.3.5. *1. An open interval* $]a,b[$*, where* $a<b$ *are two real numbers, is open in* $(\mathbb{R},|\cdot|)$*. To see this, let us fix* $x\in]a,b[$*. Then we have* $x-a>0$ *and* $b-x>0$*. We set*

$$r=\min(x-a,b-x).$$

Now, let y *be such that* $|y-x|<r$*. Since* $|y-x|<r$*, it follows that both* $x-y$ *and* $y-x$ *are greater than* $-r$*. Therefore, we can show:*

$$y-a=(y-x)+(x-a)>(x-a)-r>0.$$

Similarly, we can show:

$$b-y=(b-x)+(x-y)>(b-x)-r>0.$$

Thus, we conclude that $y\in]a,b[$*, demonstrating that* $B(x,r)\subseteq]a,b[$*.*

2. An interval of the form $[a,b[$ *is not open in* $(\mathbb{R},|\cdot|)$*. Indeed, for any* $r>0$*, the ball* $B(a,r)$*, which corresponds to the open interval* $]a-r,a+r[$*, is not contained in* $]a,b[$*. Specifically, it includes points* $y<a$*, which are not in the interval* $[a,b[$*.*

Theorem 3.3.6. *Let* (E,d) *be a metric space,* $r\geqslant 0$*, and* $x_0\in E$*. The open ball* $B(x_0,r)$ *is an open set in* (E,d)*.*

Theorem 3.3.7. *Let* E *be a set, and let* d_1 *and* d_2 *be two distances on* E*. Suppose that* d_1 *and* d_2 *are equivalent. Then, the following equivalence holds for any subset* A *of* E*:*

$$(A \text{ is open in } (E,d_1)) \iff (A \text{ is open in } (E,d_2)).$$

Definition 3.3.8. *A subset* A *of a metric space* (E,d) *is said to be closed if its complement* $E\setminus A$ *is open.*

Example 3.3.9. *Let* (E,d) *be a metric space. In this context, we will demonstrate that both the entire set* E *and the empty set* $\emptyset$ *are considered to be open and closed sets.*

- *The empty set is open because for every point* $a\in\emptyset$ *(which does not exist), the condition holds vacuously.*
- *The entire set* E *is open because for any point* $x\in E$*, we can find a radius* $r>0$ *such that the open ball* $B(x,r)\subseteq E$*, since* E *contains all points in the space.*
- *The complement of the empty set is the entire space* E*, which is open by definition. Therefore,* $\emptyset$ *is also closed.*
- *The complement of* E *is the empty set* $\emptyset$*, which we established is open. Therefore,* E *is also closed.*

Theorem 3.3.10. *Let (E,d) be a metric space, $r \geqslant 0$, and $x_0 \in E$. The closed ball $B_f(x_0, r)$ is a closed set in (E,d).*

Theorem 3.3.11. *Let (E,d) be a metric space. The following properties are verified:*

1. *If $(O_i)_{i\in I}$ is a family of open sets, then*
$$\bigcup_{i\in I} O_i$$
is also an open set. (Any union of open sets is an open set.)
2. *If $O_1, O_2, \ldots, O_n$ are open sets, then*
$$O_1 \cap O_2 \cap \ldots \cap O_n$$
is also an open set. (Any finite intersection of open sets is an open set.)
3. *If $(F_i)_{i\in I}$ is a family of closed sets, then*
$$\bigcap_{i\in I} F_i$$
is also a closed set. (Any intersection of closed sets is a closed set.)
4. *If $F_1, F_2, \ldots, F_n$ are closed sets, then*
$$F_1 \cup F_2 \cup \ldots \cup F_n$$
is also a closed set. (Any finite union of closed sets is a closed set.)

Example 3.3.12. *In $\mathbb{R}$ equipped with the usual distance, we have:*

1. *An interval of the form*
$$[c,d] = \{x \in \mathbb{R} \mid c \leqslant x \leqslant d\}$$
is a closed set. This is because the complement of the closed interval $[c,d]$ is the set
$$(-\infty, c) \cup (d, +\infty).$$
This complement consists of the union of two open intervals. Thus, the complement is open.
2. *An interval of the form*
$$(-\infty, a] = \{x \in \mathbb{R} \mid x \leqslant a\}$$
is a closed set. The complement, $(a, +\infty)$, is open, which confirms that $(-\infty, a]$ is closed.

3. *For every* $n > 0$, *the set*

$$A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$$

is an open interval. However, the intersection

$$\bigcap_{n\geqslant 1} A_n = \{0\}$$

is not open, since the only point in the intersection is 0, *which does not have an open contained entirely in* $\{0\}$.

Similarly, for every $n \geqslant 1$, *the set*

$$B_n = \left[0, 1 - \frac{1}{n}\right]$$

is a closed interval. However, the union

$$\bigcup_{n\geqslant 1} B_n = [0, 1[$$

is not closed (see Example 3.3.5).

Definition 3.3.13. *Let* (E, d) *be a metric space. The topology of* (E, d) *is defined as*

$$\mathcal{T} = \{U \subseteq E \mid U \text{ is open set }\}.$$

Definition 3.3.14. *Let* A *and* B *be subsets of the metric space* E. *We say that* B *is a neighborhood of* A *if there exists an open ball* O *in* E *such that*

$$A \subseteq O \subseteq B,$$

If $A = \{x\}$, *we simply say that* B *is a neighborhood of* x. *We denote by* $V(x)$ *the set of neighborhoods of* x.

Remark 3.3.15. *We note the ambiguity of the terminology: the notion of a neighborhood is not necessarily related to the concept of proximity.*
Indeed, in $\mathbb{R}$, *the interval* $]0, 1]$ *is a neighborhood of every point in* $]0, 1[$ *but not of* 0. *In fact, a set is considered a neighborhood of a point if that point is contained within the set.*
In particular, the entire set E *is a neighborhood of each of its points.*

3.4 Interior, Closure, Boundary of a Set

Definition 3.4.1. *Given a metric space* (E, d) *and a subset* $A \subseteq E$, *the interior of* A, *denoted $\mathring{A}$, is the union of all open sets contained in A, that is,*

$$\mathring{A} = \bigcup_{O \text{ open}, O \subseteq A} O\,.$$

Remark 3.4.2.
- *By definition, $\mathring{A}$ is an open set, contained in A, and it contains all other open sets contained in A: it is the largest open set contained in A.*
- *A set A is open if and only if*

$$A = \mathring{A}.$$

Example 3.4.3. *In $(\mathbb{R}, |\cdot|)$, the interior of the interval $]0,1]$ is equal to $]0,1[$. Indeed, $]0,1[$ is an open set contained in $]0,1]$. Furthermore, for any $r > 0$, the open ball*

$$B(1,r) = (1-r, 1+r)$$

is not contained in $]0,1]$, since it includes points greater than 1. Therefore, 1 does not belong to the interior of $]0,1]$.

Definition 3.4.4. *Let (E,d) be a metric space, and let A be a subset of E. The closure of A, denoted $\overline{A}$, is defined as the intersection of all closed sets containing A, that is,*

$$\overline{A} = \bigcap_{F \text{ closed}, F \supseteq A} F.$$

We say that A is dense in E if

$$\overline{A} = E.$$

Remark 3.4.5.
- *$\overline{A}$ is a closed set, contains A, and is contained in all other closed sets that contain A: it is the smallest closed set containing A.*
- *A set A is closed if and only if*

$$A = \overline{A}.$$

Theorem 3.4.6. *Let (E,d) be a metric space, A a subset of E, and $x \in E$. The following properties are equivalent:*

1. *$x \in \overline{A}$.*
2. *For every open set U such that $x \in U$, we have $U \cap A \neq \emptyset$.*

Example 3.4.7. *In the metric space $(\mathbb{R}, |\cdot|)$, the closure of the interval $]0,1]$ is equal to $[0,1]$. To demonstrate this, we first observe that $[0,1]$ is a closed set that contains $]0,1]$. Therefore, we have:*

$$]0,1] \subseteq [0,1].$$

Next, we need to show that 0 *is in the closure* $\overline{]0,1]}$*. For any* $\epsilon > 0$*, consider the open ball* $B(0,\epsilon) = (-\epsilon,\epsilon)$*. We can choose* ϵ *to be any positive value, no matter how small. Since there are points in* $]0,1]$ *(for example, any* x *such that* $0 < x < \epsilon$*) that lie within this ball. Thus,*

$$0 \in \overline{]0,1]}.$$

hence,

$$\overline{]0,1]} = [0,1].$$

Remark 3.4.8. *If* (E,d) *is any metric space, and if* $a \in A$ *and* $r > 0$*, it is not necessarily true that* $\overline{B}(a,r) = B_f(a,r)$ *For example, consider the set* $\mathbb{N}$ *equipped with the discrete metric. The open ball* $B(0,1) = \{0\}$*, so*

$$\overline{B}(0,1) = \{0\}$$

However, by the definition of the discrete metric, we find that

$$B_f(0,1) = \mathbb{N}.$$

Example 3.4.9. *In* $\mathbb{R}$ *equipped with the usual distance, both* $\mathbb{Q}$ *and* $\mathbb{R} \setminus \mathbb{Q}$ *are dense.*
To show this, let $x \in \mathbb{R}$ *and* $r > 0$*. The open ball is given by*

$$B(x,r) =]x-r, x+r[.$$

It is known that between any two distinct real numbers, there always exists a rational number and an irrational number. This implies that

$$B(x,r) \cap \mathbb{Q} \neq \emptyset \quad \text{and} \quad B(x,r) \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \emptyset.$$

Thus, for any $x \in \mathbb{R}$*, we find that there exist points in both* $\mathbb{Q}$ *and* $\mathbb{R} \setminus \mathbb{Q}$ *within any open ball around* x*. Hence,* $\overline{\mathbb{Q}} = \mathbb{R}$ *and* $\overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}$*.*

Definition 3.4.10. *For a subset* A *of a metric space* E*, the boundary* ∂A *is defined as the set difference between its closure and its interior:*

$$\partial A \equiv \overline{A} \setminus \mathring{A} = \{x \in E \mid x \in \overline{A} \text{ and } x \notin \mathring{A}\}.$$

Example 3.4.11. *In* $\mathbb{R}$*,*

$$\partial(]0,1]) = \{1\}.$$

3.5 Case in the Space $\mathbb{R}^n$

This section highlights key concepts from topology in the vector space $\mathbb{R}^n$.
• The Euclidean norm, denoted $\|x\|$, is defined as

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

• The distance between points $A = (a_1, \ldots, a_n)$ and $M = (x_1, \ldots, x_n)$ is expressed as

$$\|M - A\| = \sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2}.$$

• The open ball centered at A with radius $r > 0$ is defined by

$$B(A, r) = B_r(A) = \{M \in \mathbb{R}^n \mid \|M - A\| < r\}.$$

• A subset U of $\mathbb{R}^n$ is a neighborhood of point $A \in U$ if it contains an open ball centered at A. Additionally, U is classified as open if every point $A \in U$ has an open ball contained within U.
• In $\mathbb{R}^2$, points are represented as (x, y), leading to:

$$\|(x, y)\| = \sqrt{x^2 + y^2},$$

$$B_r(x_0, y_0) = \{(x, y) \in \mathbb{R}^2 \mid (x - x_0)^2 + (y - y_0)^2 < r^2\}.$$

The norm of a vector $x = (x_1, \ldots, x_n)$ is defined as

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \in \mathbb{R}.$$

This establishes $\mathbb{R}^n$ as n-dimensional Euclidean space. The norm induces a metric given by

$$d(x, y) = \|x - y\| \quad \text{for } x, y \in \mathbb{R}^n.$$

Thus, $\mathbb{R}^n$ becomes a metric space and serves as an example of a topological space as follows:

Open Sets in $\mathbb{R}^n$

- For a point $x \in \mathbb{R}^n$ and $r > 0$, the ball

  $$B_r(x) = \{y \in \mathbb{R}^n : \|x - y\| < r\}$$

  is an open set.
- Open balls $B_r(x)$ are open sets in $\mathbb{R}^n$.
- A subset $O \subseteq \mathbb{R}^n$ is open if, for each point $x \in O$, there exists $\epsilon > 0$ such that

  $$B_\epsilon(x) \subseteq O.$$

- A subset $U \subseteq \mathbb{R}^n$ is closed if its complement $\mathbb{R}^n \setminus U$ is open.

Example 3.5.1. 1. *Examples of open sets in $\mathbb{R}$ include open intervals, such as* $(0, 1)$.

2. *The Cartesian product of n open intervals (an open rectangle) is open in* $\mathbb{R}^n$.
3. *Closed intervals serve as examples of closed sets in* $\mathbb{R}$.
4. *The Cartesian product of n closed intervals (a closed rectangle) is closed in* $\mathbb{R}^n$.
5. *The empty set $\emptyset$ and $\mathbb{R}^n$ itself are both open and closed sets.*
6. *However, not every subset of $\mathbb{R}^n$ is open or closed. Many subsets are neither open nor closed. For example, the interval* $(0, 1]$ *in $\mathbb{R}$ and the product of an open interval and a closed interval in* $\mathbb{R}^2$.

3.6 Chapter 3 Exercises

Exercise 10

Let $n \in \mathbb{N}^*$ and let $d_1, d_2, d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+$ be defined, for $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$, by:
$d_1(x,y) = \sum_{i=1}^n |x_i - y_i|$, $d_2(x,y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$, $d_\infty(x,y) = \max\{|x_i - y_i|\ ,\ i = \overline{1,n}\}$.

1. Verify that d_1, d_2, and d_∞ are distances on $\mathbb{R}^n$.
2. Assume $n = 2$. Draw the unit ball for each of these distances.
3. Show that d_1, d_2, and d_∞ are equivalent for $n = 2$.

Correction ▼ [10]

Exercise 11

We define the following three functions on $\mathbb{R}^2$:

$$N_1((x,y)) = |x| + |y|, \quad N_2((x,y)) = \sqrt{x^2 + y^2}, \quad N_\infty((x,y)) = \max(|x|, |y|).$$

1. Prove that N_1, N_2, N_∞ define three norms on $\mathbb{R}^2$.
2. Prove that: $\forall \alpha \in \mathbb{R}^2, \quad N_\infty(\alpha) \leqslant N_2(\alpha) \leqslant N_1(\alpha) \leqslant 2N_\infty(\alpha)$.
3. Are N_1, N_2, N_∞ equivalent?

Correction ▼ [11]

Exercise 12

Show that the following sets are open:

1. $A = \{(x,y) \in \mathbb{R}^2 : y > 0\}$
2. $B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 4x \text{ and } y > 0\}$
3. $C = \{x \in \mathbb{R} : x^3 > x\}$
4. $D = \{x \in \mathbb{R} : 0 < x < 1 \text{ and } \frac{1}{x} \notin \mathbb{Z}\}$

Correction ▼ [12]

Exercise 13

Indicate whether each of the following intervals is or is not a neighborhood of 0 for $\mathbb{R}$ equipped with the usual distance: $a)$ $]-\frac{1}{2}, \frac{1}{2}]$, $b)$ $]-1, 0]$, $c)$ $[0, \frac{1}{2}[$, $d)$ $]0, 1]$.

Correction ▼ [13]

Exercise 14

Determine the Interior and Closure of $\mathbb{Q}$, $[0,1] \cap \mathbb{Q}$, and $]0,1[\cap \mathbb{Q}$ (in $\mathbb{R}$ with its usual distance)

Correction ▼ [14]

Chapter

4 Functions of several variables : Limit and Continuity

In this chapter, we will study functions of multiple variables. These functions will therefore be of the form

$$f : E \subset \mathbb{R}^n \to \mathbb{R},$$

where $n \geqslant 1$ is a natural integer. In other words, the elements of the domain set E will be n-tuples of the type $(x_1,\ldots,x_n)$, which can be considered as vectors, and the elements of the codomain set will be real numbers. We will examine the properties of these functions, focusing on their limits and continuity, which are essential for understanding their behavior in higher dimensions. Additionally, we will investigate how these concepts extend to vector-valued functions, enriching our analysis of multivariable phenomena.

4.1 Domain of definition

Definition 4.1.1. *Let E be a subset of $\mathbb{R}^n$. A function $f : E \to \mathbb{R}$ associates with each $(x_1,\ldots,x_n) \in E$ a single real number $f(x_1,\ldots,x_n)$.*

Definition 4.1.2. *If we are first given an expression for $f(x_1,\ldots,x_n)$, then the domain of definition of f is the largest subset $D_f \subset \mathbb{R}^n$ such that, for each $(x_1,\ldots,x_n) \in D_f$, $f(x_1,\ldots,x_n)$ is well-defined. The function is then $f : D_f \to \mathbb{R}$.*

Example 4.1.3.

1. *Let $f(x,y) = \ln(1+x+y)$. It is necessary for $1+x+y$ to be strictly positive in order to calculate its logarithm. Thus, we have:*

$$D_f = \{(x,y) \in \mathbb{R}^2 \mid 1+x+y > 0\}.$$

To plot this set, we first draw the line defined by the equation $1+x+y=0$. We then determine which side of the line corresponds to the set $1+x+y>0$. Here, it is above the line.

2. *Let $f(x,y) = \exp\left(\frac{x+y}{x^2-y}\right)$. The denominator must not be zero:*

$$D_f = \{(x,y) \in \mathbb{R}^2 \mid x^2 - y \neq 0\}.$$

The points in the domain of definition are all the points in the plane that are not on the parabola defined by the equation $y = x^2$.

3. *Let* $f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2-2}}$. *The expression under the square root must be positive (to allow for taking the square root) and must not be zero (to allow for taking the inverse). Therefore, we have:*

$$D_f = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 > 2\}.$$

In other words, these are all the points outside the closed ball centered at $(0,0,0)$ *with radius* $\sqrt{2}$.

4.2 Limit of Functions from $\mathbb{R}^n$ to $\mathbb{R}$

The concepts of limit and continuity for functions of a single variable generalize to multiple variables without additional complexity: it suffices to replace the absolute value with the Euclidean norm.

4.2.1 Definition

Let f be a function $f : E \subset \mathbb{R}^n \to \mathbb{R}$ defined in the neighborhood of $x_0 \in \mathbb{R}^n$, except possibly at x_0.

Definition 4.2.1. *The function f has the limit l as x approaches x_0 if:*

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in E, (0 < \|x - x_0\| < \delta \Rightarrow |f(x) - l| < \epsilon).$$

We then write

$$\lim_{x \to x_0} f = l, \text{ or } \lim_{x \to x_0} f(x) = l, \text{ or } f(x) \to l.$$

We would similarly define $\lim_{x \to x_0} f(x) = +\infty$ *by:*

$$\forall A > 0, \exists \delta > 0, \forall x \in E, (0 < \|x - x_0\| < \delta \Rightarrow |f(x)| > A).$$

Remark 4.2.2. • *The concept of limit does not depend on the norms used here.*
• *If it exists, the limit is unique.*

4.2.2 Operations on Limits

To calculate limits, we rarely resort to this definition. Instead, we use general theorems: operations on limits and bounding. These are the same statements as for functions of a single variable: there is no difficulty or novelty.

Proposition 4.2.3. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be defined in the neighborhood of $x_0 \in \mathbb{R}^n$ and such that f and g have limits at x_0. Then:*

$$\lim_{x \to x_0} (f + g) = \lim_{x \to x_0} f + \lim_{x \to x_0} g,$$

$$\lim_{x \to x_0} (f \cdot g) = \lim_{x \to x_0} f \times \lim_{x \to x_0} g.$$

And if g does not vanish in a neighborhood of x_0:

$$\lim_{x\to x_0} \frac{f}{g} = \frac{\lim_{x\to x_0} f}{\lim_{x\to x_0} g}.$$

Remark 4.2.4.
- *The results above also hold for infinite limits with the usual conventions:*

$$\ell + \infty = +\infty, \quad \ell - \infty = -\infty, \quad \frac{1}{0} = +\infty, \quad \frac{1}{\infty} = 0, \quad \ell \times \infty = \infty \quad (\ell \neq 0), \quad \infty \times \infty = \infty.$$

- *The indeterminate forms are:* $+\infty - \infty$, $\frac{0}{0}$, $\frac{\infty}{0}$, $0 \times \infty$, ∞^0, 1^∞, *and* 0^0.
- *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a function of several variables such that* $\lim_{x\to x_0} f(x) = \ell$,
  Let $g : \mathbb{R} \to \mathbb{R}$ *be a function of a single variable such that* $\lim_{t\to\ell} g(t) = \ell'$.
  Then the function of several variables $g \circ f : \mathbb{R}^n \to \mathbb{R}$ *defined by* $(g \circ f)(x) = g(f(x))$ *satisfies*

$$\lim_{x\to x_0} (g \circ f)(x) = \ell'.$$

Theorem 4.2.5. *(Squeeze Theorem)*
Let $f, g, h : \mathbb{R}^n \to \mathbb{R}$ *be three functions defined in a neighborhood* U *of* $x_0 \in \mathbb{R}^n$.

- *If, for all* $x \in U$, *we have* $f(x) \leqslant h(x) \leqslant g(x)$,
- *and if* $\lim_{x\to x_0} f = \lim_{x\to x_0} g = \ell$, *then* h *has a limit at the point* x_0 *and* $\lim_{x\to x_0} h = \ell$.

4.2.3 Limit along a Path

The uniqueness of the limit implies that, regardless of the way we approach the point x_0, the limit value is always the same.

Proposition 4.2.6. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a function defined in the neighborhood of* $x_0 \in \mathbb{R}^n$, *except possibly at* x_0.

1. *If* f *has a limit* ℓ *at the point* x_0, *then the restriction of* f *to any curve passing through* x_0 *has a limit at* x_0, *and this limit is* ℓ.
2. *By contraposition, if the restrictions of* f *to two curves passing through* x_0 *have different limits at the point* x_0, *then* f *does not have a limit at the point* x_0.

Let's elaborate on the case of functions of 2 variables:

- A curve passing through the point $(x_0, y_0) \in \mathbb{R}^2$ is a continuous function $\gamma : \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (x(t), y(t))$, such that $\gamma(0) = (x_0, y_0)$.
- The restriction of f along γ is the function of one variable $f \circ \gamma : t \mapsto f(x(t), y(t))$.
- If f has limit ℓ at (x_0, y_0), then the proposition asserts that $f(x(t), y(t)) \to \ell$.

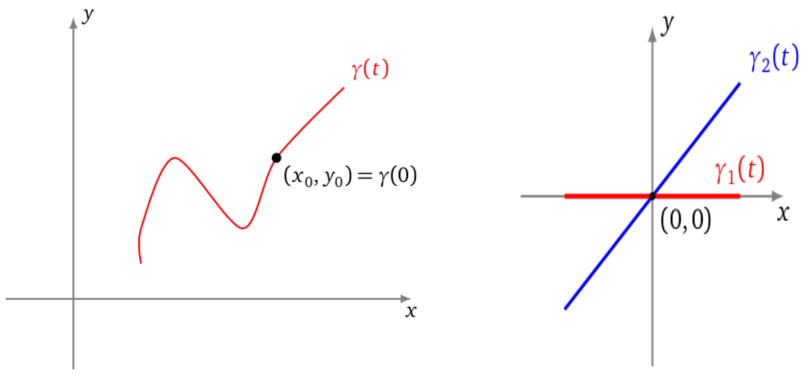
Example 4.2.7. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

- *If we take the path $\gamma_1(t) = (t,0)$, then $(f \circ \gamma_1)(t) = f(t,0) = 0$. Therefore, as $t \to 0$, $(f \circ \gamma_1)(t) \to 0$. If we take the path $\gamma_2(t) = (t,t)$, then $(f \circ \gamma_2)(t) = f(t,t) = \frac{t^2}{2t^2} = \frac{1}{2}$. Thus, as $t \to 0$, $(f \circ \gamma_2)(t) \to \frac{1}{2}$.*

  Below, in the figure on the left, are the two paths in the plane; in the two figures on the right, two different views of the values taken by f along these paths.

- *If f had a limit ℓ, then for any path $\gamma(t)$ such that $\gamma(t) \to (0,0)$ as $t \to 0$, we would have $(f \circ \gamma)(t) \to \ell$. We would then obtain $\ell = 0$ and $\ell = \frac{1}{2}$, which would contradict the uniqueness of the limit. Thus, f does not have a limit at $(0,0)$.*



4.3 Continuity of Functions from $\mathbb{R}^n$ to $\mathbb{R}$

Definition 4.3.1. *1. $f : E \subset \mathbb{R}^n \to \mathbb{R}$ is continuous at $x_0 \in E$ if $\lim_{x \to x_0} f(x) = f(x_0)$.*
2. A function f is continuous on E if it is continuous at every point of E.

By the properties of limits, if f and g are two functions continuous at x_0, then:

- The function $f + g$ is continuous at x_0,
- Similarly, $f \cdot g$ and $\frac{f}{g}$ (with $g(x) \neq 0$ in a neighborhood of x_0) are continuous at x_0,
- If $h : \mathbb{R} \to \mathbb{R}$ is continuous, then $h \circ f$ is continuous at x_0.

Example 4.3.2. *1. The functions defined by $(x,y) \mapsto x + y$, $(x,y) \mapsto xy$, and all polynomial functions in two variables x and y are continuous on $\mathbb{R}^2$ (for example, $(x,y) \mapsto x^2 + 3xy$). Likewise, all rational functions in two variables are continuous where they are defined.*

2. Since the exponential function is continuous, $(x,y) \mapsto e^{xy}$ is continuous on $\mathbb{R}^2$.

3. *The function defined by* $f(x,y) = \frac{2}{\sqrt{x^2+y^2}}$ *is continuous on* $\mathbb{R}^2 \setminus \{(0,0)\}$.

Definition 4.3.3. *(Extension by Continuity).*
Let $f : E \subset \mathbb{R}^n \to \mathbb{R}$. *Let* x_0 *be a limit point of* E *not belonging to* E. *If* $f(x)$ *has a limit* ℓ *as* $x \to x_0$, *we can extend the domain of definition of* f *to* $E \cup \{x_0\}$ *by setting* $f(x_0) = \ell$. *The extended function is continuous at* x_0. *We say that we have obtained an extension of* f *by continuity at the point* x_0.

Example 4.3.4. *Let* $f : \mathbb{R}^2 \setminus \{(0,0)\}$ *be defined by*

$$f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}.$$

- *Limit at the origin.*

  We use the inequalities $|x| \leqslant \sqrt{x^2+y^2}$ *and* $|y| \leqslant \sqrt{x^2+y^2}$. *Thus,*

$$|f(x,y)| = \frac{|x| \cdot |y|}{\sqrt{x^2+y^2}} \leqslant \sqrt{x^2+y^2} \to 0 \quad \text{as } (x,y) \to (0,0).$$

- *Extension.*

  To extend f *at* $(0,0)$, *we choose the limit obtained as the value. We set* $f(0,0) = 0$. *(We denote the extended function still by* $f : \mathbb{R}^2 \to \mathbb{R}$.*)*

- *Continuity.*

  By our choice of $f(0,0)$, f *is continuous at* $(0,0)$. *Outside the origin,* f *is continuous as a sum, product, composition, and inverse of continuous functions.*

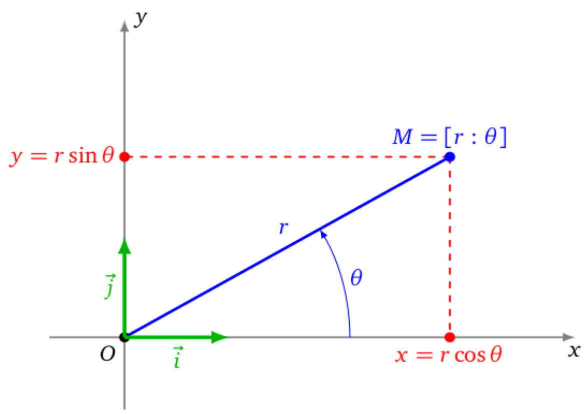
Conclusion: *The extended function is continuous over the entire* $\mathbb{R}^2$.

4.4 Polar Coordinates

Instead of locating a point in the plane $\mathbb{R}^2$ by its Cartesian coordinates (x,y), we can do so using its distance from the origin and the angle formed with the horizontal: these are the polar coordinates.

4.4.1 Definition

Let M be a point in the plane $\mathbb{R}^2$. Let $O = (0,0)$ be the origin. Let $(O,\vec{i},\vec{j})$ be a direct orthonormal coordinate system.
• We denote $r = \|OM\|$, the distance from M to the origin.
• We denote θ as the angle between $\vec{i}$ and OM.



We denote $[r : \theta]$ as the polar coordinates of the point M. In this course, r will always be positive. The angle is not uniquely determined; several choices are possible. To ensure uniqueness, we can restrict θ to the interval $[0, 2\pi[$ or $-\pi < \theta < \pi$. Generally, polar coordinates are not assigned to the origin (the angle would not have meaning).

4.4.2 From Polar Coordinates to Cartesian Coordinates

We recover the Cartesian coordinates (x, y) from the polar coordinates $[r : \theta]$ using the formulas:

$$x = r\cos\theta, \quad y = r\sin\theta.$$

In other words, we have defined a function:

$$(0, +\infty) \times [0, 2\pi[\to \mathbb{R}^2, \quad (r, \theta) \mapsto (r\cos\theta, r\sin\theta).$$

4.4.3 From Cartesian Coordinates to Polar Coordinates

We recover r and θ from (x, y) using the following formulas:

$$r = \sqrt{x^2 + y^2},$$

and, in the case where $x > 0$ and $y \geqslant 0$, $\theta = \arctan\left(\frac{y}{x}\right)$.
For points in other quadrants, we reduce to the principal quadrant where $x > 0$ and $y \geqslant 0$.

4.4.4 Limit and Continuity

When considering functions $f : E \subset \mathbb{R}^2 \to \mathbb{R}$, it is sometimes easier to prove results about limits, continuity, etc., using polar coordinates.

Proposition 4.4.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined in the neighborhood of $(0, 0) \in \mathbb{R}^2$, except possibly at $(0, 0)$. If*

$$\lim_{r \to 0} f(r\cos\theta, r\sin\theta) = \ell \in \mathbb{R}$$

exists independently of θ, meaning there exists a function $\epsilon(r) \to 0$ such that, for all $r > 0$

and all θ, we have:

$$|f(r\cos\theta, r\sin\theta) - \ell| \leqslant \epsilon(r),$$

then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \ell.$$

To clarify this proposition and explain the different practical cases of the limit, here's how to proceed. We express $f(x,y)$ in polar coordinates by calculating $f(r\cos\theta, r\sin\theta)$.

1. If $\lim_{r\to 0} f(r\cos\theta, r\sin\theta)$ exists and does not depend on the variable θ, then this limit is the limit of f at the point $(0,0)$.

2. If $\lim_{r\to 0} f(r\cos\theta, r\sin\theta)$ does not exist (or the limit is not finite), then f does not have a finite limit at the point $(0,0)$.

3. If $\lim_{r\to 0} f(r\cos\theta, r\sin\theta) = \ell(\theta)$ depends on θ, then f does not have a limit at the point $(0,0)$. To justify this, we provide two values θ_1 and θ_2 such that $\ell(\theta_1) \neq \ell(\theta_2)$.

Example 4.4.2. *1. Let*

$$f(x,y) = \frac{x^3}{x^2+y^2}$$

$$f(r\cos\theta, r\sin\theta) = \frac{r^3\cos^3\theta}{r^2} = r\cos^3\theta.$$

As $|\cos^3\theta| \leqslant 1$, we have $r|\cos^3\theta| \leqslant r$ (for all r and also for all θ), with $\epsilon(r) := r \to 0$. This implies that

$$f(r\cos\theta, r\sin\theta) \to 0 \quad \text{as } r \to 0.$$

The limit exists (independently of the values taken by θ), so the function f does indeed have a limit at $(0,0)$:

$$f(x,y) \to 0 \quad \text{as } (x,y) \to (0,0).$$

For those who want to do everything by hand with more details, we can also write $|f(r\cos\theta, r\sin\theta)| \leqslant r$, in other words $|f(x,y)| \leqslant \sqrt{x^2+y^2}$. Thus,

$$f(x,y) \to 0 \quad \text{as } (x,y) \to (0,0).$$

2. Let

$$f(x,y) = \frac{y}{x^2+y^3}.$$

$$f(r\cos\theta, r\sin\theta) = \frac{1}{r}\frac{\sin\theta}{\cos^2\theta + r\sin^3\theta}.$$

Fix θ such that $\sin\theta \neq 0$ (i.e., $\theta = 0(\mathrm{mod}\ \pi)$). Then, as $r \to 0$, $f(r\cos\theta, r\sin\theta)$ does not have a finite limit. In particular, the function $(x,y) \mapsto f(x,y)$ does not have a finite limit at $(0,0)$.

3. *Let*

$$f(x,y) = \frac{xy}{x^2+y^2}.$$

We have $f(r\cos\theta, r\sin\theta) = \frac{r^2\cos\theta\sin\theta}{r^2} = \cos\theta\sin\theta = \frac{\sin(2\theta)}{2}$.
For fixed θ*, the function* $r \mapsto f(r\cos\theta, r\sin\theta)$ *does have a limit* $\ell(\theta) = \frac{1}{2}\sin(2\theta)$ *as* $r \to 0$. *However, this limit depends on the angle* θ*: if* $\theta = 0$*, then* $\ell(\theta) = 0$*; on the other hand, if* $\theta = \frac{\pi}{4}$*, then* $\ell(\theta) = \frac{1}{2}$*. Since the limit depends on the angle, the function of two variables* $(x,y) \mapsto f(x,y)$ *does not have a limit at* $(0,0)$.

4.5 Vector Functions

Definition 4.5.1. *A function is said to be a vector function or vector-valued function when the codomain is* $\mathbb{R}^p$*, with* $p \geqslant 2$*:*

$$F : \mathbb{R}^n \to \mathbb{R}^p, \quad x = (x_1,\dots,x_n) \mapsto (f_1(x),\dots,f_p(x))$$

Each component f_j*, for* $j = 1,\dots,p$*, is a function of several variables with real values:* $f_j : \mathbb{R}^n \to \mathbb{R}$*. We denote* $x \mapsto F(x)$ *or* $(x_1,\dots,x_n) \mapsto F(x_1,\dots,x_n)$.

Example 4.5.2. *From* $\mathbb{R}$ *to* $\mathbb{R}^2$*:* $F(t) = (t^2, t)$.
From $\mathbb{R}^2$ *to* $\mathbb{R}^2$*:* $F(x,y) = (e^x\cos y, e^x\sin y)$.

4.5.1 Limit of Functions from $\mathbb{R}^n$ to $\mathbb{R}^p$

Definition 4.5.3. *Let* $F : E \subseteq \mathbb{R}^n \to \mathbb{R}^p$ *defined by* $x \mapsto F(x) = (f_1(x),\dots,f_p(x))$, $a = (a_1,a_2,\dots,a_n) \in E$ *and* $l = (l_1,l_2,\dots,l_p) \in \mathbb{R}^p$*, then:*

$$\lim_{x\to a} F(x) = l \iff \lim_{x\to a} f_i(x) = l_i \quad \text{for } i = 1,\dots,p.$$

4.5.2 Continuity of Functions from $\mathbb{R}^n$ to $\mathbb{R}^p$

Definition 4.5.4. *The function* f *is continuous at the point* $a \in E$ *(respectively on* E*) if and only if all the components* f_i *(for* $i = 1,\dots,p$*) of* f *are continuous at the point* a *(respectively on* E*).*

• All algebraic operations on continuous functions remain valid for continuous vector-valued functions.

4.6 Chapter 4 Exercises

Exercise 15

In each case, determine and represent the domain of definition of the following functions:

1. $f_1(x,y) = \frac{\sqrt{-y+x^2}}{\sqrt{y}}$.
2. $f_2(x,y) = \frac{\ln(y)}{\sqrt{x-y}}$.
3. $f_3(x,y) = \frac{\sqrt{4-x^2-y^2}}{\sqrt{x^2+y^2-1}}$.
4. $f_4(x,y) = \ln(x-y^2)$.

Correction ▼ [15]

Exercise 16

Find the limit of the following functions at (0,0) (without using polar coordinates).

1. $f_1(x,y) = \frac{x^2+y^2+1}{y}\sin(y)$.
2. $f_2(x,y) = \frac{x^4+y^4}{x^2+y^2}$.
3. $f_3(x,y) = \frac{x^2y^2}{x^2+y^2}$.
4. $f_4(x,y) = \frac{\sin(x^2)+\sin(y^2)}{\sqrt{x^2+y^2}}$.
5. $f_5(x,y) = \frac{1-\cos(xy)}{xy^2}$.
6. $f_6(x,y) = \frac{e^{xy}-1}{e^x-1}$.

Correction ▼ [16]

Exercise 17

Do the following functions have a limit at the origin? (Use the path method)

1. $f_1(x,y) = \frac{xy}{x^2+y^2}$.
2. $f_2(x,y) = \frac{xy^4}{x^4+y^6}$.
3. $f_3(x,y) = \frac{x+y}{x^2+y^2}$.

Correction ▼ [17]

Exercise 18

Calculate the limit, if it exists, or show that it does not exist, for the following functions using polar coordinates.

1. $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$.
2. $\lim_{(x,y)\to(1,0)} \frac{y^3}{(x-1)^2+y^2}$.
3. $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+xy+y^2}$.

Correction ▼ [18]

Exercise 19

Let

$$f(x,y) = \begin{cases} xy\left(\frac{x^2-y^2}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Show that f is continuous on $\mathbb{R}^2$.

Correction ▼ [19]

Chapter 5

Differential Calculus and Jacobian Matrix

5.1 Differential Calculus

For a function of several variables, there is a derivative for each of the variables, called the partial derivative. The set of partial derivatives allows us to reconstruct a linear approximation of the function: this is the differential.

5.1.1 Partial Derivatives

Definition 5.1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function of a single variable. The derivative of f at $x_0 \in \mathbb{R}$, if it exists, is given by:*

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Example 5.1.2. *The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is differentiable, with derivative $f'(x_0) = 2x_0$. Indeed, as h approaches 0, we have: $\frac{(x_0+h)^2 - x_0^2}{h} = 2x_0 + h \to_{h \to 0} 2x_0$.*

Definition 5.1.3. *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$, where U is an open set in $\mathbb{R}^n$. We say that f has a partial derivative with respect to the variable x_i at the point $x_0 = (a_1, \dots, a_n) \in \mathbb{R}^n$ if the function of one variable*

$$x_i \mapsto f(a_1, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_n)$$

is differentiable at the point a_i. In other words, we define the partial derivative of f with respect to x_i at the point $x_0 = (a_1, \dots, a_n)$ by

$$\lim_{h \to 0} \frac{f(a_1, \dots, a_{i-1}, a_i + h, a_{i+1}, \dots, a_n) - f(a_1, \dots, a_n)}{h}$$

if this limit exists.

Notation. This limit is denoted by $\frac{\partial f}{\partial x_i}(x_0)$.
This is the partial derivative of f with respect to x_i at the point x_0. The symbol ∂ is read as "d round." Another notation is $\partial_{x_i} f(x_0)$ or $f'_{x_i}(x_0)$.

Thus, there are n partial derivatives at the point x_0: $\frac{\partial f}{\partial x_1}(x_0)$, $\frac{\partial f}{\partial x_2}(x_0)$, $\dots$, $\frac{\partial f}{\partial x_n}(x_0)$.
In the case of a function of two variables $(x,y) \mapsto f(x,y)$, we have:

$$\frac{\partial f}{\partial x}(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$$

and

$$\frac{\partial f}{\partial y}(x_0,y_0) = \lim_{k\to 0} \frac{f(x_0,y_0+k)-f(x_0,y_0)}{k}.$$

Example 5.1.4. *For $f:\mathbb{R}^3 \to \mathbb{R}$ defined by $f(x,y,z) = \cos(x+y^2)e^{-z}$, we have:*
$\frac{\partial f}{\partial x}(x,y,z) = -\sin(x+y^2)e^{-z}$, $\frac{\partial f}{\partial y}(x,y,z) = -2y\sin(x+y^2)e^{-z}$, $\frac{\partial f}{\partial z}(x,y,z) = -\cos(x+y^2)e^{-z}$.

Example 5.1.5. *The function $f:\mathbb{R}^2 \to \mathbb{R}$ defined as*

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{at } (0,0) \end{cases}$$

has partial derivatives at every point but is not continuous at $(0,0)$:

1. *Discontinuity at the Origin: Along the path $\gamma(t) = (t,t)$ for $t \neq 0$, we have*

$$f(\gamma(t)) = \frac{t^2}{2t^2} = \frac{1}{2} \quad \text{which does not tend to } f(0,0) = 0.$$

   Thus, f is not continuous at $(0,0)$.

2. *Partial Derivatives Away from the Origin: Consider the point $(x_0,y_0) \neq (0,0)$. In a neighborhood of this point, f is defined by $f(x,y) = \frac{xy}{x^2+y^2}$. The function $x \mapsto f(x,y_0)$ is therefore continuous and differentiable in the neighborhood of x_0. The partial derivative is obtained by differentiating the function of one variable $x \mapsto f(x,y_0)$. Thus, we have*

$$\frac{\partial f}{\partial x}(x_0,y_0) = \frac{y_0^3 - x_0^2 y_0}{(x_0^2+y_0^2)^2}.$$

   Similarly, by differentiating the function $y \mapsto f(x_0,y)$, we find $\frac{\partial f}{\partial y}(x_0,y_0) = \frac{x_0^3 - x_0 y_0^2}{(x_0^2+y_0^2)^2}$.

3. *Partial Derivatives at the Origin:*

   Since the function f is defined at $(0,0)$ by a special formula, we must return to the definition of what partial derivatives are using limits. To calculate $\frac{\partial f}{\partial x}(0,0)$, we evaluate at $(x_0,y_0) = (0,0)$:

$$\frac{f(0+h,0)-f(0,0)}{h} = \frac{0}{h} = 0 \quad \to \quad 0 \quad \text{as } h \to 0.$$

   Therefore, $\frac{\partial f}{\partial x}(0,0) = 0$.

   Similarly: $\frac{f(0,0+k)-f(0,0)}{k} = \frac{0}{k} = 0 \quad \to \quad 0$ as $k \to 0$. Thus, $\frac{\partial f}{\partial y}(0,0) = 0$.

Conclusion: For any point $(x_0,y_0) \in \mathbb{R}^2$, the partial derivatives $\frac{\partial f}{\partial x}(x_0,y_0)$ and $\frac{\partial f}{\partial y}(x_0,y_0)$ exist.

5.1.2 Directional Derivative

It is possible to generalize the concept of partial derivative.

Definition 5.1.6. *Let $f : \mathbb{R}^n \to \mathbb{R}$. Let $v \in \mathbb{R}^n$ be a non-zero vector. The directional derivative of f at $x_0 \in \mathbb{R}^n$ in the direction of the vector v is defined, if it exists, by*

$$D_v f(x_0) = \lim_{t\to 0} \frac{f(x_0 + tv) - f(x_0)}{t}.$$

For a function $f : \mathbb{R}^2 \to \mathbb{R}$, the directional derivative at the point (x_0, y_0) in the direction of the vector $v = (h,k)$ is given by

$$D_v f(x_0, y_0) = \lim_{t\to 0} \frac{f(x_0 + th, y_0 + tk) - f(x_0, y_0)}{t}.$$

Example 5.1.7. *Let f be the function defined on $\mathbb{R}^2$ by*

$$f(x,y) = \begin{cases} \frac{x^3+y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

For any non-zero vector $v = (h,k)$, we have:

$$\lim_{t\to 0} \frac{f(0+th, 0+tk) - f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{(th)^3+(tk)^3}{(th)^2+(tk)^2} - 0}{t} = \frac{h^3+k^3}{h^2+k^2}.$$

Thus, f has a directional derivative in the direction of any non-zero vector at the point $(0,0)$, and when $v = (h,k)$: $D_v f(0,0) = \frac{h^3+k^3}{h^2+k^2}$. In general, if the vector v is a vector from the canonical basis, we recover a partial derivative. Let $f : \mathbb{R}^2 \to \mathbb{R}$.
1. If $v = (1,0)$, we have $D_v f(x,y) = \frac{\partial f}{\partial x}(x,y)$. 2. If $v = (0,1)$, we have $D_v f(x,y) = \frac{\partial f}{\partial y}(x,y)$.

5.1.3 Differentiability

For a function $f : \mathbb{R} \to \mathbb{R}$ of a single variable, another way to express that it is differentiable at x_0 is to verify that there exists $\ell \in \mathbb{R}$ such that

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0) - \ell \cdot h}{h} = 0.$$

We denote this ℓ by $f'(x_0)$, so that we have $f(x_0+h) \approx f(x_0) + f'(x_0)\cdot h$ (for small real h). In other words, we approximate the mapping $h \mapsto f(x_0+h) - f(x_0)$ by a linear function $h \mapsto f'(x_0)\cdot h$.
We will perform the same work in higher dimensions.

Definition 5.1.8. *Let $f : \mathbb{R}^n \to \mathbb{R}$. The function f is differentiable at $x_0 \in \mathbb{R}^n$ if there* *exists a linear mapping* $\ell : \mathbb{R}^n \to \mathbb{R}$ *such that:*

$$\lim_{\|h\|\to 0} \frac{f(x_0+h) - f(x_0) - \ell(h)}{\|h\|} = 0.$$

The mapping ℓ *is the differential of* f *at* x_0 *and is denoted by* $df(x_0)$.

In the case of functions of one variable, we have $df(x_0) = f'(x_0)$ (and $df(x_0)(h) = f'(x_0)\cdot h$). For functions of several variables, we will see shortly how to express the differential using partial derivatives. Note that $df(x_0)$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}$ (like f), and thus $df(x_0)(h)$ is a real number (for each $h \in \mathbb{R}^n$).
Just as for functions of one variable, if a function is differentiable, then it is continuous.

Proposition 5.1.9. *If* f *is differentiable at* $x_0 \in \mathbb{R}^n$, *then* f *is continuous at* x_0.

5.1.4 Differential

Proposition 5.1.10. *If* $f : \mathbb{R}^n \to \mathbb{R}$ *is differentiable at* $x_0 \in \mathbb{R}^n$, *then its partial derivatives exist, and we have:*

$$df(x_0)(h) = h_1 \frac{\partial f}{\partial x_1}(x_0) + \cdots + h_n \frac{\partial f}{\partial x_n}(x_0)$$

where $h = (h_1, \ldots, h_n)$.

In particular, when it exists, the differential is unique. For $f : \mathbb{R}^2 \to \mathbb{R}$ differentiable at (x_0, y_0), the formula is:

$$df(x_0,y_0)(h,k) = h\frac{\partial f}{\partial x}(x_0,y_0) + k\frac{\partial f}{\partial y}(x_0,y_0).$$

To show that a function is differentiable, one can use the fact that the sum, product, inverse (of a non-zero function), and composition of differentiable functions are also differentiable. Otherwise, it is necessary to revert to the definition. For example, for $f : \mathbb{R}^2 \to \mathbb{R}$:

1. First, compute the partial derivatives $\frac{\partial f}{\partial x}(x_0,y_0)$ and $\frac{\partial f}{\partial y}(x_0,y_0)$.

2. Write the candidate for the differential as $\ell(h,k) = h\frac{\partial f}{\partial x}(x_0,y_0) + k\frac{\partial f}{\partial y}(x_0,y_0)$.

3. Finally, prove the limit as $\|(h,k)\| \to 0$:

$$\frac{f(x_0+h, y_0+k) - f(x_0,y_0) - \ell(h,k)}{\|(h,k)\|} \to 0.$$

Example 5.1.11. *Study the differentiability at every point of the function* f *defined by*

$$f(x,y) = \begin{cases} x - 3y + \frac{x^4}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

- *Away from* $(0,0)$*, the function* f *is differentiable because* f *is a sum, product, and inverse of differentiable functions (since* x^2+y^2 *is only zero at the origin).*
- *At* $(0,0)$*, we need to study differentiability manually.*
- *Partial derivative with respect to x:*

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{h+h^2}{h} = 1.$$

- *Partial derivative with respect to y:*

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{-3k}{k} = -3.$$

- *The candidate for the differential is therefore:*

$$\ell(h,k) = h\frac{\partial f}{\partial x}(0,0) + k\frac{\partial f}{\partial y}(0,0) = h-3k.$$

- *We calculate:*

$$0 \leqslant \frac{f(0+h,0+k)-f(0,0)-\ell(h,k)}{\sqrt{h^2+k^2}} = \frac{h^4}{(h^2+k^2)^{\frac{3}{2}}} \leqslant \frac{h^4}{|h|^3} = |h| \to 0 \quad \text{as } (h,k)\to(0,0).$$

Therefore, f *is differentiable at the point* $(0,0)$ *and* $df(0,0)(h,k) = h-3k$*.*

5.1.5 Connection with Partial Derivatives

Partial Derivatives. We saw in Proposition 5.1.10 that if $f:\mathbb{R}^2\to\mathbb{R}$ is differentiable at (x_0,y_0), then

$$df(x_0,y_0)(1,0) = \frac{\partial f}{\partial x}(x_0,y_0) \quad \text{and} \quad df(x_0,y_0)(0,1) = \frac{\partial f}{\partial y}(x_0,y_0).$$

In any dimension, for $f:\mathbb{R}^n\to\mathbb{R}$ differentiable at $x_0\in\mathbb{R}^n$, and e_i the i-th vector of the canonical basis:

$$df(x_0)(e_i) = \frac{\partial f}{\partial x_i}(x_0).$$

Directional Derivative. More generally, if $f:\mathbb{R}^n\to\mathbb{R}$ is differentiable at $x_0\in\mathbb{R}^n$, then $df(x_0)(v) = D_v f(x_0)$. For $f:\mathbb{R}^2\to\mathbb{R}$, this means that if $v=(h,k)$, then:

$$D_{(h,k)}f(x_0,y_0) = h\frac{\partial f}{\partial x}(x_0,y_0) + k\frac{\partial f}{\partial y}(x_0,y_0).$$

If f is not differentiable, this formula may be false.
Gradient.

The gradient is another way to represent the differential. The gradient of f at x_0 is the vector

$$\operatorname{grad} f(x_0) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x_0) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x_0) \end{pmatrix}.$$

If f is differentiable at x_0, then

$$df(x_0)(v) = \langle \operatorname{grad} f(x_0) | v \rangle,$$

where $\langle u|v \rangle$ is the inner product of u and v.

Summary.

When f is differentiable, the differential, the directional derivative, and the gradient encode the same information and are related by the formulas:

$$D_v f(x_0) = df(x_0)(v) = \langle \operatorname{grad} f(x_0) | v \rangle.$$

Example 5.1.12. *Let f be the function defined by $f(x,y) = \ln(1 + x + y^2)$.*

1. *Determine the domain of definition U of f.*
2. *Calculate the partial derivatives of f.*
3. *Show that f is differentiable on U.*
4. *Calculate the gradient of f at $(0,1)$ and express the differential at that point.*
5. *Calculate the directional derivative of f at $(0,1)$ in the direction of the vector $(2,1)$.*

Solution.

1. *The domain is $U = \{(x,y) \in \mathbb{R}^2 \mid 1 + x + y^2 > 0\}$.*
2. *The partial derivatives are:*
$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{1+x+y^2}, \quad \frac{\partial f}{\partial y}(x,y) = \frac{2y}{1+x+y^2}.$$
3. *f is differentiable on U as it is the sum, product, and composition of differentiable functions.*
4. *The gradient is obtained directly from the partial derivatives:*
$$\textit{grad } f(0,1) = \begin{pmatrix} \frac{\partial f}{\partial x}(0,1) \\ \frac{\partial f}{\partial y}(0,1) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}.$$
The differential at this point $df(0,1) : \mathbb{R}^2 \to \mathbb{R}$ is the linear mapping defined by
$$df(0,1)(h,k) = \langle \textit{grad } f(0,1) | (h,k) \rangle = h + \frac{1}{2}k.$$
5. *Since f is differentiable, the directional derivative is simply the linear combination of the partial derivatives: $D_{(2,1)} f(0,1) = 2 \cdot \frac{\partial f}{\partial x}(0,1) + 1 \cdot \frac{\partial f}{\partial y}(0,1) = 2$.*
We could also perform the calculation using the formula $D_{(2,1)} f(0,1) = df(0,1)(2,1)$.

5.1.6 Functions of Class C^1

In practice, functions are often of class C^1, which implies differentiability and is easier to verify.

Definition 5.1.13. *Let $f : \mathbb{R}^n \to \mathbb{R}$. We say that f is of class C^1 if the partial derivatives $\frac{\partial f}{\partial x_i}$ exist and are continuous (for $i = 1, \dots, n$).*

Of course, one can limit the definition to an open set. For example, if U is an open set in $\mathbb{R}^2$, $f : U \to \mathbb{R}$ will be of class C^1 on U if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous on U.

Theorem 5.1.14. *If f is of class C^1, then f is differentiable.*

5.1.7 Taylor's Formula of Order 1

Another way to say that f is differentiable is to state that f admits a first-order Taylor expansion. For $f : \mathbb{R}^2 \to \mathbb{R}$, at the point (x_0, y_0), if f is differentiable, then

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + h\frac{\partial f}{\partial x}(x_0, y_0) + k\frac{\partial f}{\partial y}(x_0, y_0) + o(\|(h,k)\|).$$

Knowing the values of f, $\frac{\partial f}{\partial x}$, and $\frac{\partial f}{\partial y}$ only at (x_0, y_0), we obtain an approximation of f at any (x, y) close to (x_0, y_0).

Notation: Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a function defined in the neighborhood of $(0,0)$. We say that g is negligible with respect to $\|(x,y)\|^n$ in the neighborhood of $(0,0)$ and we denote $g = o(\|(x,y)\|^n)$ if

$$\lim_{(x,y)\to(0,0)} \frac{g(x,y)}{\|(x,y)\|^n} = 0.$$

Example 5.1.15. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = \sin x \cdot e^{2y}$.*

$$\frac{\partial f}{\partial x}(x,y) = \cos x \cdot e^{2y} \quad \textit{and} \quad \frac{\partial f}{\partial y}(x,y) = 2\sin x \cdot e^{2y}.$$

Both partial derivatives exist and are continuous, hence f is of class C^1 on all $\mathbb{R}^2$.
In particular, f is differentiable at every point $(x_0, y_0) \in \mathbb{R}^2$ and

$$df(x_0,y_0)(h,k) = h\cos x_0 e^{2y_0} + 2k\sin x_0 e^{2y_0}.$$

For example, for $(x_0, y_0) = \left(\frac{\pi}{6}, 1\right)$, we have the Taylor expansion:

$$f\left(\frac{\pi}{6} + h, 1 + k\right) = f\left(\frac{\pi}{6}, 1\right) + h\frac{\partial f}{\partial x}\left(\frac{\pi}{6}, 1\right) + k\frac{\partial f}{\partial y}\left(\frac{\pi}{6}, 1\right) + o(\|(h,k)\|).$$

Thus:

$$f\left(\frac{\pi}{6} + h, 1 + k\right) = \frac{1}{2}e^2 + \frac{\sqrt{3}}{2}e^2 h + e^2 k + \epsilon(h,k)\sqrt{h^2 + k^2},$$

where $\epsilon(h,k) \to 0$ as $(h,k) \to (0,0)$.

Abstract

$$
\begin{array}{ccccc}
 & & f \text{ is of class } C^1 & \xleftrightarrow{\text{definition}} & f \text{ has continuous partial derivatives} \\
 & & \downarrow & & \\
f \text{ is continuous} & \longleftarrow & f \text{ is differentiable} & \xleftrightarrow{\text{definition}} & f \text{ has a first-order Taylor expansion} \\
 & & \downarrow & & \\
 & & f \text{ has partial derivatives (and directional derivative)} & &
\end{array}
$$

Counterexamples

If f is differentiable, then f admits partial derivatives and directional derivatives in all directions. The converse is false, by the following example.

Example 5.1.16. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by*

$$f(x,y) = \begin{cases} \frac{y^3}{\sqrt{x^2+y^4}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Show that f has a directional derivative in every non-zero vector at the point $(0,0)$, but it is not differentiable there.

Solution.

1. Let $v = (h,k) \neq (0,0)$.

- If $h = 0$, we have $\frac{f(t\cdot v)-f(0,0)}{t} = \frac{f(0,tk)}{t} = k$.

- If $h \neq 0$, we have

$$\left|\frac{f(t\cdot v)-f(0,0)}{t}\right| = \left|\frac{k^3t^2}{\sqrt{h^2t^2+h^4t^4}}\right| \leqslant \left|\frac{k^3}{h}\right||t| \to_{t\to 0} 0.$$

Thus, $D_v f(0,0) = k$ if $h = 0$, and 0 if $h \neq 0$.

2. With $v = (1,0)$, we have $\frac{\partial f}{\partial x}(0,0) = 0$, and with $v = (0,1)$, we have $\frac{\partial f}{\partial y}(0,0) = 1$. The candidate for the differential at $(0,0)$ is therefore $\ell(h,k) = k$. However, the expression

$$\epsilon(h,k) = \frac{f(h,k)-f(0,0)-\ell(h,k)}{\sqrt{h^2+k^2}} = \frac{k^3 - k\sqrt{h^2+k^4}}{\sqrt{h^2+k^2}\sqrt{h^2+k^4}}$$

does not tend to 0 as $(h,k) \to 0$, since $\lim_{t\to 0^+} \epsilon(t,t) = -\frac{1}{\sqrt{2}}$. Thus, f is not differentiable at the point $(0,0)$.

If f is of class C^1, then it is differentiable. The converse is false, as demonstrated by the following example.

Example 5.1.17. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *such that* $f(x,y) = \begin{cases} y^2 \sin(\frac{1}{x^2+y^2}) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ *Show that* f *is differentiable at every point in* $\mathbb{R}^2$ *but not of class* C^1 *at the origin.*
Solution.

- ***Outside the origin:***

  The partial derivatives are given by

  $$\frac{\partial f}{\partial x}(x,y) = -\frac{2xy^2}{(x^2+y^2)^2}\cos(\frac{1}{x^2+y^2}), \quad \frac{\partial f}{\partial y}(x,y) = 2y\sin(\frac{1}{x^2+y^2}) - \frac{2y^3}{(x^2+y^2)^2}\cos(\frac{1}{x^2+y^2}).$$

  These exist and are continuous on $\mathbb{R} \setminus \{(0,0)\}$*. Thus,* f *is of class* C^1 *on* $\mathbb{R}^2 \setminus \{(0,0)\}$ *and, by Theorem 5.1.14, is therefore differentiable on* $\mathbb{R}^2 \setminus \{(0,0)\}$*.*

- ***Differentiability at the point*** $(0,0)$**.**

  We will calculate the partial derivatives of f *at the point* $(0,0)$*. Since* $f(x,0) = 0$*, we have*

  $$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0.$$

  And since $f(0,y) = y^2 \sin\left(\frac{1}{y^2}\right)$*, we find*

  $$\frac{\partial f}{\partial y}(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k \to 0} k \sin\left(\frac{1}{k^2}\right) = 0.$$

  Thus, the candidate for the differential at $(0,0)$ *is* $\ell(h,k) = 0$*.*

  Moreover,

  $$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - \ell(h,k)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{k^2}{\sqrt{h^2+k^2}} \sin\left(\frac{1}{h^2+k^2}\right) \leqslant |k| \to 0,$$

  therefore f *is differentiable at the point* $(0,0)$*.*

Conclusion.
The function f *is differentiable on* $\mathbb{R}^2$*. Furthermore,* $\frac{\partial f}{\partial x}(t,t) = -\frac{\cos\left(\frac{1}{2t^2}\right)}{2t}$ *does not have a limit as* $t \to 0$*. Thus, the partial derivative* $\frac{\partial f}{\partial x}$ *is not continuous at* $(0,0)$*. Hence,* f *is not of class* C^1 *at the origin.*

5.1.8 Second Order Partial Derivatives

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function. The two partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are also functions from $\mathbb{R}^2$ to $\mathbb{R}$; suppose that these are also differentiable functions. Then we can compute the partial derivatives of $\frac{\partial f}{\partial x}$: $\frac{\partial}{\partial x}(\frac{\partial f}{\partial x})$, $\frac{\partial}{\partial y}(\frac{\partial f}{\partial x})$.

We can also compute the partial derivatives of $\frac{\partial f}{\partial y}$: $\frac{\partial}{\partial x}(\frac{\partial f}{\partial y})$, $\frac{\partial}{\partial y}(\frac{\partial f}{\partial y})$.

We denote these partial derivatives as: $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y \partial x}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y^2}$.
These are functions from $\mathbb{R}^2$ to $\mathbb{R}$.
More generally, for $f : \mathbb{R}^n \to \mathbb{R}$, we denote the first order partial derivatives by $\frac{\partial f}{\partial x_i} : \mathbb{R}^n \to \mathbb{R}$ (for $1 \leqslant i \leqslant n$) and the second order partial derivatives by $\frac{\partial^2 f}{\partial x_j \partial x_i}$ (for $1 \leqslant i, j \leqslant n$).

5.1.9 Schwarz's Theorem

For $f : \mathbb{R}^2 \to \mathbb{R}$, there are four second order partial derivatives to compute, but generally, two of them are equal.
Example 1.
Let $f : U \to \mathbb{R}$ be defined by $f(x,y) = x^2 \cos(y) + \ln(x - y^2)$ on $U = \{(x,y) \in \mathbb{R}^2 \mid x - y^2 > 0\}$.
Then:

$$\frac{\partial f}{\partial x}(x,y) = 2x\cos(y) + \frac{1}{x-y^2}, \quad \frac{\partial f}{\partial y}(x,y) = -x^2\sin(y) - \frac{2y}{x-y^2}.$$

We can now differentiate again to obtain the second order partial derivatives:

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 2\cos(y) - \frac{1}{(x-y^2)^2}, \quad \frac{\partial^2 f}{\partial y \partial x}(x,y) = -2x\sin(y) + \frac{2y}{(x-y^2)^2}.$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y) = -2x\sin(y) + \frac{2y}{(x-y^2)^2}, \quad \frac{\partial^2 f}{\partial y^2}(x,y) = -x^2\cos(y) - \frac{2x+2y^2}{(x-y^2)^2}.$$

We note from the previous example that $\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y)$. This is a general phenomenon that we will detail.

> **Definition 5.1.18.** *A function $f : \mathbb{R}^n \to \mathbb{R}$ is of class C^2 if f is of class C^1 (that is, its partial derivatives exist and are continuous) and if its partial derivatives are also of class C^1.*

The Schwarz theorem states that the result does not depend on the order in which the derivatives are taken.

> **Theorem 5.1.19.** *(Schwarz's Theorem) Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be a function of class C^2. For all $i, j \in \{1, \ldots, n\}$, we have:*
>
> $$\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right).$$
>
> *Thus, for $f : \mathbb{R}^2 \to \mathbb{R}$ of class C^2, we have: $\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y)$.*

For $f : \mathbb{R}^3 \to \mathbb{R}$ of class C^2, there are 9 second-order partial derivatives, but only 6 calculations to perform: $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial z^2}$, $\frac{\partial^2 f}{\partial y \partial x}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial z \partial x}$, $\frac{\partial^2 f}{\partial x \partial z}$, $\frac{\partial^2 f}{\partial z \partial y}$, $\frac{\partial^2 f}{\partial y \partial z}$.

Example 5.1.20. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by*

$$f(x,y) = \begin{cases} \frac{xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

We verify that f is of class C^1 on $\mathbb{R}^2$ and that

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \frac{y^5-x^2y^3}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}, \quad \frac{\partial f}{\partial y}(x,y) = \begin{cases} \frac{3x^3y^2+xy^4}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

The increment ratio $\frac{\frac{\partial f}{\partial x}(0,y)-\frac{\partial f}{\partial x}(0,0)}{y-0} = 1 \to 1$ *as* $y \to 0$
shows that $\frac{\partial^2 f}{\partial y \partial x}(0,0) = 1$.
Similarly, the increment ratio $\frac{\frac{\partial f}{\partial y}(x,0)-\frac{\partial f}{\partial y}(0,0)}{x-0} = 0 \to 0$ *as* $x \to 0$
This shows that $\frac{\partial^2 f}{\partial x \partial y}(0,0) = 0$. *The mixed partial derivatives are not equal at* $(0,0)$. *We conclude that at least one of the second partial derivatives* $\frac{\partial^2 f}{\partial x \partial y}$ *or* $\frac{\partial^2 f}{\partial y \partial x}$ *is not continuous at* $(0,0)$. *In other words, the function f is not of class C^2 at $(0,0)$, and the Schwarz theorem does not apply.*

5.1.10 Implicit Function

It involves replacing the study of a function of two variables with that of a function of a single variable.
Let $F : \mathbb{R}^2 \to \mathbb{R}$. We consider the level curve $\mathcal{C} : F(x,y) = 0$. We say that the function $y = \phi(x)$ is implicitly defined by $F(x,y) = 0$ if $F(x,\phi(x)) = 0$, meaning that $(x,\phi(x)) \in \mathcal{C}$. We then state that $y = \phi(x)$ is an implicit function of $F(x,y) = 0$.

Theorem 5.1.21. *(Implicit Function Theorem) Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function and let (x_0,y_0) be a point such that $F(x_0,y_0) = 0$. If $\frac{\partial F}{\partial y}(x_0,y_0) \neq 0$, then:*

1. *There exists a C^1 function $\phi : I \to J$ defining an implicit function $y = \phi(x)$, where I is an open interval containing x_0, J is an open interval containing y_0, and $y_0 = \phi(x_0)$. More precisely, for all $(x,y) \in I \times J$, we have:*

$$F(x,y) = 0 \iff y = \phi(x)$$

   In particular, for all $x \in I$, $F(x,\phi(x)) = 0$.

2. *Furthermore, for all $x \in I$, we have $\frac{\partial F}{\partial y}(x,\phi(x)) \neq 0$, and the derivative of ϕ is given by:*

$$\phi'(x) = -\frac{\frac{\partial F}{\partial x}(x,\phi(x))}{\frac{\partial F}{\partial y}(x,\phi(x))}$$

We have a symmetric statement when exchanging x and y if the other partial derivative does not vanish. If $\frac{\partial F}{\partial x}(x_0, y_0) \neq 0$, there exists a function $\tilde{\phi} : J \to I$ of the variable y, defined around y_0, which defines the implicit function $\tilde{\phi}(y)$, such that in a neighborhood of (x_0, y_0):

$$F(x,y) = 0 \iff x = \tilde{\phi}(y)$$

(this implies $F(\tilde{\phi}(y), y) = 0$).
If both partial derivatives $\frac{\partial F}{\partial x}(x_0, y_0)$ and $\frac{\partial F}{\partial y}(x_0, y_0)$ simultaneously vanish, the point (x_0, y_0) is called a singular point, and the Implicit Function Theorem does not apply.

Example 5.1.22. *Let's examine what the Implicit Function Theorem signifies for the circle defined by the equation $x^2 + y^2 - 1 = 0$, where $\frac{\partial F}{\partial y} = 2y$.*

- *At the point $(0,1)$:*

$$\frac{\partial F}{\partial y}(0,1) = 2 \neq 0.$$

  Thus, we can express y as a function of x around the point $(0,1)$. We can give the formula: $y = \sqrt{1-x^2}$.

- *At the point $(1,0)$:*

$$\frac{\partial F}{\partial y}(1,0) = 0.$$

  We can no longer apply the Implicit Function Theorem, and indeed, y is no longer a function of x around $(1,0)$. This is because for points (x,y) close to $(1,0)$ with x fixed, the equation $x^2 + y^2 - 1 = 0$ admits two possible solutions for y. However, around $(1,0)$, we can express x as a function of y.

5.2 Differential Forms and Exterior Differential

5.2.1 Differential Forms

Differential Forms of Degree 0

Let $D \subset \mathbb{R}^n$, $n = 2,3$ be a bounded domain.

Definition 5.2.1. *A differential form of degree 0 on D is simply a continuous function $f : D \to \mathbb{R}$.*

Differential Forms of Degree 1

Definition 5.2.2. *A differential form of degree 1 on D is an expression of the form:*

$$\alpha = \sum_{j=1}^{n} f_j(x)\, dx_j,$$

where the f_j are continuous functions from D to $\mathbb{R}$.

It will often be assumed that the functions f_j are regular (infinitely differentiable on D).

Example 5.2.3. 1. $\alpha = \cos(x_1 x_2^2)\,dx_1 + (x_1^2 x_3 - x_2)\,dx_2 + \tan(x_2 x_3)\,dx_3$ *is a differential form of degree* 1 *on* $\mathbb{R}^3$.

2. $\alpha = (x_1^2 + \sin x_2)\,dx_1 + e^{x_1 - x_2}\,dx_2$ *is a differential form of degree* 1 *on* $\mathbb{R}^2$.

For now, dx_j can be seen as "integration elements," similar to their use in $\int_a^b f(x)\,dx$.

Differential Forms of Degree 2

Definition 5.2.4. *A differential form of degree* 2 *on D is an expression of the form:*

$$\alpha = \sum_{j=1}^{n} \sum_{k=1}^{n} f_{jk}(x)\,dx_j \wedge dx_k,$$

where the f_{jk} are continuous functions from D to $\mathbb{R}$.

It will often be assumed that the functions f_{jk} are regular (infinitely differentiable on D).

Example 5.2.5. $\alpha = (x_1^2 \cos x_2)\,dx_1 \wedge dx_2 + (x_3 - 2x_1 x_2)\,dx_2 \wedge dx_3$ *is a differential form of degree* 2 *on* $\mathbb{R}^3$.

Calculation Rules

The following calculation rules apply to the symbols dx_i and $\wedge$, analogous to those of the vector product of two vectors:

$$dx_i \wedge dx_i = 0, \quad 1 \leqslant i \leqslant n,$$

$$dx_i \wedge dx_j = -dx_j \wedge dx_i, \quad 1 \leqslant i, j \leqslant n.$$

Example 5.2.6.

$$dx_2 \wedge dx_2 = 0, \quad dx_2 \wedge dx_1 = -dx_1 \wedge dx_2, \text{ etc.}$$

By applying these rules, one can simplify the expressions of a form. For instance, if $n = 2$ and α is a degree 2 form, we have:

$$\begin{aligned}\alpha &= f_{11}(x)dx_1 \wedge dx_1 + f_{12}(x)dx_1 \wedge dx_2 + f_{21}(x)dx_2 \wedge dx_1 + f_{22}(x)dx_2 \wedge dx_2 \\ &= (f_{12}(x) - f_{21}(x))dx_1 \wedge dx_2 = (f_{21}(x) - f_{12}(x))dx_2 \wedge dx_1 \\ &= g_{12}(x)dx_1 \wedge dx_2,\end{aligned}$$

where $g_{12} = f_{12} - f_{21}$.

Similarly, if $n = 3$ and α is a degree 2 form, we have:

$$\alpha = f_{11}(x)dx_1 \wedge dx_1 + f_{12}(x)dx_1 \wedge dx_2 + f_{13}(x)dx_1 \wedge dx_3 + f_{21}(x)dx_2 \wedge dx_1 + f_{22}(x)dx_2 \wedge dx_2 +$$
$$f_{23}(x)dx_2 \wedge dx_3 + f_{31}(x)dx_3 \wedge dx_1 + f_{32}(x)dx_3 \wedge dx_2 + f_{33}(x)dx_3 \wedge dx_3$$
$$= (f_{12}(x) - f_{21}(x))dx_1 \wedge dx_2 + (f_{23}(x) - f_{32}(x))dx_2 \wedge dx_3 + (f_{31}(x) - f_{13}(x))dx_3 \wedge dx_1$$
$$= g_{12}(x)dx_1 \wedge dx_2 + g_{23}(x)dx_2 \wedge dx_3 + g_{31}(x)dx_1 \wedge dx_3,$$

by applying the rules:

$$dx_1 \wedge dx_1 = dx_2 \wedge dx_2 = dx_3 \wedge dx_3 = 0,$$

$$dx_2 \wedge dx_1 = -dx_1 \wedge dx_2, \quad dx_3 \wedge dx_1 = -dx_1 \wedge dx_3, \quad dx_3 \wedge dx_2 = -dx_2 \wedge dx_3.$$

Remark 5.2.7. *It is useful to adopt the practice of writing degree 2 forms using only the symbols*

$$dx_1 \wedge dx_2,\ dx_2 \wedge dx_3,\ \text{and}\ \ dx_3 \wedge dx_1\ .$$

Understanding the Notion of Degree

The degree of a differential form corresponds to the power of the "d" symbols present in its expression: a degree 0 form is a function that has no "d" in its expression. A degree 1 form contains dx_j, while a degree 2 form contains $dx_i \wedge dx_j$.
Another way to understand degree is through the dimensionality of spaces: a curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ is a 1-dimensional space (one parameter is sufficient to describe points on the curve), and a surface in $\mathbb{R}^3$ (a domain in $\mathbb{R}^2$) is a 2-dimensional space (two parameters are needed to describe points on a surface). Following this perspective, a point in $\mathbb{R}^2$ or $\mathbb{R}^3$ is a 0-dimensional space.
Now, if I have a function f (a degree 0 form), the simplest thing I can do is evaluate its value at a point, which can be understood as "integrating the function f over a 0-dimensional space." Degree 1 forms are likewise objects intended to be integrated over curves, while degree 2 forms are meant to be integrated over surfaces. In other words, a differential form of degree d will be integrated over a space of dimension d. A differential form of degree d is often simply referred to as a d-form.

5.2.2 Integral of a 1-Form Along an Oriented Arc

Definition 5.2.8. *Let γ be an oriented arc in $\mathbb{R}^2$ (i.e., a curve in $\mathbb{R}^2$ with a specified direction of traversal), and let $\alpha = f_1(x)\,dx_1 + f_2(x)\,dx_2$ be a 1-form in $\mathbb{R}^2$. We define the integral of α along γ as follows:*
We choose a parameterization of γ:

$$[a,b] \ni t \mapsto x(t) = (x_1(t), x_2(t))$$

compatible with the orientation of γ. We define:

$$\int_\gamma \alpha := \int_a^b (f_1(x_1(t), x_2(t))x_1'(t) + f_2(x_1(t), x_2(t))x_2'(t))\, dt.$$

We can remember the following rules:

1. We write
$$\int_\gamma \alpha = \int_\gamma f_1(x)\, dx_1 + f_2(x)\, dx_2,$$

2. Along γ, we replace $x = (x_1, x_2)$ with $x(t) = (x_1(t), x_2(t))$, and dx_i with $\frac{dx_i}{dt}dt$, that is, $x_i'(t)dt$.

3. We integrate the resulting expression from a to b.

Definition 5.2.9. *Let γ be an oriented arc in $\mathbb{R}^3$, and let*

$$\alpha = f_1(x)dx_1 + f_2(x)dx_2 + f_3(x)dx_3$$

be a 1-form in $\mathbb{R}^3$. We define the integral of α along γ as follows:
We choose a parameterization of γ:

$$[a, b] \ni t \mapsto x(t) = (x_1(t), x_2(t), x_3(t))$$

that is compatible with the orientation of γ. We then set:

$$\int_\gamma \alpha := \int_a^b (f_1(x(t))x_1'(t) + f_2(x(t))x_2'(t) + f_3(x(t))x_3'(t))\, dt.$$

Again, the integral is independent of the parameterization as long as it remains compatible with the orientation.

5.2.3 Exterior Differential

We now define an operation denoted by d that transforms a form of degree k into a form of degree $k+1$. This operation, which serves as a type of differentiation, is called the exterior differential.
A differential form α is said to be of class C^k if the functions appearing in its expression are of class C^k. We begin with the case of degree 0 forms, which are simply functions.

Definition 5.2.10. *Let $f : D \to \mathbb{R}$ be a function of class C^1. We define the exterior differential df as:*

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x)\, dx_i.$$

Thus, the form df is a degree 1 form.

Example 5.2.11. *In dimension 2, if $f(x_1,x_2) = \sin(x_1 x_2^2)$, then we have:*

$$df = x_2^2\cos(x_1x_2^2)\,dx_1 + 2x_1x_2\cos(x_1x_2^2)\,dx_2.$$

Definition 5.2.12. *Let*

$$\alpha = \sum_{i=1}^{n} f_i(x)\,dx_i$$

be a degree 1 form assumed to be of class C^1 (meaning that the functions f_i are of class C^1). We then define the exterior differential of α as:

$$d\alpha = \sum_{i=1}^{n} df_i \wedge dx_i,$$

which is a degree 2 form.

Example 5.2.13. 1. *If $n = 2$ and*

$$\alpha = f_1(x)\,dx_1 + f_2(x)\,dx_2,$$

then we have:

$$\begin{aligned} d\alpha &= \frac{\partial f_1}{\partial x_1}(x)\,dx_1\wedge dx_1 + \frac{\partial f_1}{\partial x_2}(x)\,dx_2\wedge dx_1 + \frac{\partial f_2}{\partial x_1}(x)\,dx_1\wedge dx_2 + \frac{\partial f_2}{\partial x_2}(x)\,dx_2\wedge dx_2 \\ &= \left(\frac{\partial f_2}{\partial x_1}(x) - \frac{\partial f_1}{\partial x_2}(x)\right)dx_1\wedge dx_2, \end{aligned}$$

after applying the calculation rules.

2. *If $n = 3$ and*

$$\alpha = f_1(x)\,dx_1 + f_2(x)\,dx_2 + f_3(x)\,dx_3,$$

then we have:

$$\begin{aligned} d\alpha = {} & \frac{\partial f_1}{\partial x_1}(x)\,dx_1\wedge dx_1 + \frac{\partial f_1}{\partial x_2}(x)\,dx_2\wedge dx_1 + \frac{\partial f_1}{\partial x_3}(x)\,dx_3\wedge dx_1 + \frac{\partial f_2}{\partial x_1}(x)\,dx_1\wedge dx_2 + \frac{\partial f_2}{\partial x_2}(x)\,dx_2\wedge dx_2 + \\ & \frac{\partial f_2}{\partial x_3}(x)\,dx_3\wedge dx_2 + \frac{\partial f_3}{\partial x_1}(x)\,dx_1\wedge dx_3 + \frac{\partial f_3}{\partial x_2}(x)\,dx_2\wedge dx_3 + \frac{\partial f_3}{\partial x_3}(x)\,dx_3\wedge dx_3 \\ = {} & \left(\frac{\partial f_2}{\partial x_1}(x) - \frac{\partial f_1}{\partial x_2}(x)\right)dx_1\wedge dx_2 + \left(\frac{\partial f_3}{\partial x_2}(x) - \frac{\partial f_2}{\partial x_3}(x)\right)dx_2\wedge dx_3 + \left(\frac{\partial f_1}{\partial x_3}(x) - \frac{\partial f_3}{\partial x_1}(x)\right)dx_3\wedge dx_1. \end{aligned}$$

5.3 Jacobian Matrix

For a function of several variables, there is not just one derivative but several: one for each variable. If the function is also vector-valued, then for each component and for each variable, there is a derivative. All these derivatives are grouped in the Jacobian matrix.

5.3.1 Vector-Valued Functions

A function is said to be a vector-valued function when the target space is not $\mathbb{R}$ but $\mathbb{R}^p$, with $p \geqslant 2$:

$$F : \mathbb{R}^n \to \mathbb{R}^p, \quad x = (x_1, \ldots, x_n) \mapsto (f_1(x), \ldots, f_p(x))$$

Each component f_j, for $j = 1, \ldots, p$, is a function of several variables with real values: $f_j : \mathbb{R}^n \to \mathbb{R}$. We denote

$$x \mapsto F(x) \text{ or } (x_1, \ldots, x_n) \mapsto F(x_1, \ldots, x_n).$$

5.3.2 Jacobian Matrix

Definition 5.3.1. *Let $F : \mathbb{R}^n \to \mathbb{R}^p$ be a function, whose components are $F = (f_1, \ldots, f_p)$. Let $x \in \mathbb{R}^n$. We assume that the partial derivatives $\frac{\partial f_i}{\partial x_j}$ exist at x (for all $i = 1, \ldots, p$ and $j = 1, \ldots, n$).*
The Jacobian matrix of F at $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is

$$J_F(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_p}{\partial x_1}(x) & \cdots & \frac{\partial f_p}{\partial x_n}(x) \end{pmatrix}$$

It is a matrix with p rows and n columns. The first row corresponds to the partial derivatives of f_1, the second row to the partial derivatives of f_2, and so on.
For $F : \mathbb{R}^2 \to \mathbb{R}^2$ with $F = (f_1, f_2)$ at $(x, y) \in \mathbb{R}^2$, it looks like this:

$$J_F(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \end{pmatrix}$$

Example 5.3.2. 1. *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x,y) = (x^2 + y^2, e^{x-y})$. At the point (x,y), we have:*

$$J_F(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} 2x & 2y \\ e^{x-y} & -e^{x-y} \end{pmatrix}$$

For example, at the point $(x_0, y_0) = (2,1)$, the Jacobian matrix is:

$$J_F(2,1) = \begin{pmatrix} 4 & 2 \\ e & -e \end{pmatrix}$$

2. *For $F(x,y,z) = (e^{xy}, z\sin x)$, we have:*

$$J_F(x,y,z) = \begin{pmatrix} ye^{xy} & xe^{xy} & 0 \\ z\cos x & 0 & \sin x \end{pmatrix}$$

5.3.3 Gradient

For a scalar-valued function $f : \mathbb{R}^n \to \mathbb{R}$ whose partial derivatives exist, the gradient vector is given by:

$$\operatorname{grad} f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{pmatrix}$$

This is a column vector that is the transpose of the Jacobian matrix (which, in this case, is a row vector):

$$\operatorname{grad} f(x) = J_f(x)^T.$$

We will return to the gradient in detail in the chapter "Gradient – Mean Value Theorem."

Physicists denote the gradient as $\nabla f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{pmatrix}$, where ∇ (read as "nabla") corresponds to the operator:

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}.$$

5.3.4 Differential

The theoretical counterpart of the Jacobian matrix is the differential associated with $F : \mathbb{R}^n \to \mathbb{R}^p$ at a point x. This section is more theoretical: for a first reading, one can simply remember that the differential $dF(x)$ is a linear map whose matrix (in the canonical basis) is the Jacobian matrix $J_F(x)$.

In other words:

$$dF(x)(h) = J_F(x) \times h$$

where $x \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$, while the result $dF(x)(h)$ is an element of $\mathbb{R}^p$.

We will examine what this means for the differential of a vector-valued function. Let $F : \mathbb{R}^n \to \mathbb{R}^p$ whose components are $F = (f_1, \dots, f_p)$ with each $f_j : \mathbb{R}^n \to \mathbb{R}$.

Definition 5.3.3.

- *$F : \mathbb{R}^n \to \mathbb{R}^p$ is differentiable at $x \in \mathbb{R}^n$ if each component $f_j : \mathbb{R}^n \to \mathbb{R}$ (for $j = 1, \dots, p$) is differentiable at x. We denote the differential of f_j at x as $df_j(x) : \mathbb{R}^n \to \mathbb{R}$.*
- *The differential of a differentiable vector-valued function $F : \mathbb{R}^n \to \mathbb{R}^p$ at $x \in \mathbb{R}^n$ is*

the linear map $dF(x) : \mathbb{R}^n \to \mathbb{R}^p$ defined by:

$$dF(x) = (df_1(x), \dots, df_p(x)).$$

Attention! The differential $dF(x)$ of F at $x \in \mathbb{R}^n$ is a linear map, so it is indeed a function (and not a vector). The evaluation of this function yields an expression involving vectors:

$$\forall h \in \mathbb{R}^n, \quad dF(x)(h) = (df_1(x)(h), \dots, df_p(x)(h)).$$

Proposition 5.3.4. *Let $F : \mathbb{R}^n \to \mathbb{R}^p$ be differentiable at $x \in \mathbb{R}^n$. Then*

$$dF(x)(h) = J_F(x) \times h$$

where $J_F(x)$ is the Jacobian matrix of F at x, for any $h \in \mathbb{R}^n$.

In other words, finding the differential at x amounts to calculating the Jacobian matrix at x. This proposition follows from the expression of each differential $df_j(x)$ using the partial derivatives $\frac{\partial f}{\partial x_j}$ (for $j = 1, \dots, n$).

Example 5.3.5. *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x,y) = (ye^{x^2}, x^2 - y)$. We will calculate $dF(x,y)(h,k)$ for any $(x,y),(h,k) \in \mathbb{R}^2$.*
- The Jacobian matrix of F is:

$$J_F(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} 2xye^{x^2} & e^{x^2} \\ 2x & -1 \end{pmatrix}$$

- At (x,y) and for $(h,k) \in \mathbb{R}^2$, we have:

$$dF(x,y)(h,k) = J_F(x,y) \times \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} (2xyh + k)e^{x^2} \\ 2xh - k \end{pmatrix}$$

- For example, at the point $(x_0, y_0) = (1,1)$, we have: $dF(1,1)(h,k) = ((2h+k)e, \quad 2h-k)$.

Remarks:
- If F has components of class C^1 (i.e., all partial derivatives exist and are continuous), then they are differentiable, and F is also differentiable. - If F is differentiable at x, then F is continuous at x. - If $L : \mathbb{R}^n \to \mathbb{R}^p$ is a linear map, then its differential is the map itself at every point: in other words, $dL(x) = L$ for all $x \in \mathbb{R}^n$.
Remarks:
There is another equivalent definition of the two concepts encountered:
- $F : \mathbb{R}^n \to \mathbb{R}^p$ is differentiable at $x \in \mathbb{R}^n$ if there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^p$ such that:

$$\lim_{\|h\| \to 0} \frac{F(x+h) - F(x) - L(h)}{\|h\|} = 0.$$

In this case, L is the differential of F at x and we denote it as $dF(x)$.

5.3.5 Jacobian Matrix of a Composition

Proposition 5.3.6. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable functions. Then $g \circ f$ is differentiable and*

$$(g \circ f)'(x) = g'(f(x)) \times f'(x).$$

Remark:
It may be interesting to denote x as the variable of the function f and y as the variable of the function g. The formula can then also be written as:

$$\frac{d(g \circ f)}{dx} = \frac{dg}{dy}(f(x)) \times \frac{df}{dx}(x).$$

By letting $y = f(x)$, we can consider g as a function of the variable y, but also (through composition) as a function of the variable x. We can then write, as physicists do:

$$\frac{dg}{dx} = \frac{dg}{dy} \times \frac{dy}{dx}.$$

This is a formula that is easily memorized by saying that we simplify the fraction by eliminating dy in the numerator and the denominator.
Now let's move on to the case of $F : \mathbb{R}^n \to \mathbb{R}^p$ and $G : \mathbb{R}^p \to \mathbb{R}^q$. The composition is then $G \circ F : \mathbb{R}^n \to \mathbb{R}^q$, and is defined by $(G \circ F)(x) = G(F(x))$.

Theorem 5.3.7. *If F and G are differentiable, then $G \circ F$ is differentiable, and the Jacobian matrices are related by the following formula:*

$$J_{G \circ F}(x) = J_G(F(x)) \times J_F(x)$$

Here, "×" denotes the product of the two Jacobian matrices.
It is particularly noted that if the components of F and G are of class C^1 (i.e., the partial derivatives exist and are continuous), then the functions are differentiable and the formula is valid. Moreover, $G \circ F$ is also of class C^1.

Example 5.3.8. *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x, y) = (x + y, e^{2x-y})$ and $G : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $G(x, y) = (xy, y \sin x, x^2)$. The Jacobian matrices of F and G are:*

$$J_F(x, y) = \begin{pmatrix} 1 & 1 \\ 2e^{2x-y} & -e^{2x-y} \end{pmatrix}, \quad J_G(x, y) = \begin{pmatrix} y & x \\ y \cos x & \sin x \\ 2x - y & 0 \end{pmatrix}.$$

Note that we need $J_G(F(x, y))$. Thus, in $J_G(x, y)$, we replace x with the first component of F (which is $x + y$) and y with the second component of F (which is e^{2x-y}). Therefore,

$$J_G(F(x, y)) = \begin{pmatrix} e^{2x-y} & x + y \\ e^{2x-y} \cos(x + y) & \sin(x + y) \\ 2(x + y) & 0 \end{pmatrix}.$$

To obtain the Jacobian matrix of the composition $G \circ F : \mathbb{R}^2 \to \mathbb{R}^3$, we apply the formula given by the product of matrices:

$$J_{G\circ F}(x,y) = J_G(F(x,y)) \times J_F(x,y).$$

We find:

$$J_{G\circ F}(x,y) = \begin{pmatrix} (1+2x+2y)e^{2x-y} & (1-x-y)e^{2x-y} \\ (\cos(x+y)+2\sin(x+y))e^{2x-y} & (\cos(x+y)-\sin(x+y))e^{2x-y} \\ 2x+2y & 2x+2y \end{pmatrix}.$$

Theorem 5.3.9. *If $F : \mathbb{R}^n \to \mathbb{R}^p$ is differentiable at x, and if $G : \mathbb{R}^p \to \mathbb{R}^q$ is differentiable at $F(x)$, then $G \circ F : \mathbb{R}^n \to \mathbb{R}^q$ is differentiable at x and we have:*

$$d(G\circ F)(x) = dG(F(x)) \circ dF(x).$$

We will apply the formula for the Jacobian matrix of a composition to calculate partial derivatives. The only two things to remember are, first, the formula $J_{G\circ F}(x) = J_G(F(x)) \times J_F(x)$, and second, how to apply it. Therefore, it is unnecessary to memorize the following formulas.

Case: $F : \mathbb{R} \to \mathbb{R}^2$, $G : \mathbb{R}^2 \to \mathbb{R}$

Proposition 5.3.10. *Let $F : \mathbb{R} \to \mathbb{R}^2$ be defined by $t \mapsto F(t) = (x(t), y(t))$, where $t \mapsto x(t)$ and $t \mapsto y(t)$ are differentiable functions, and let $G : \mathbb{R}^2 \to \mathbb{R}$ be defined by $(x,y) \mapsto G(x,y)$ as a differentiable function. Then $h = G \circ F : \mathbb{R} \to \mathbb{R}$ is defined by $t \mapsto h(t) = G(x(t), y(t))$. is differentiable and*

$$h'(t) = \frac{\partial G}{\partial x}(x(t),y(t)) \cdot x'(t) + \frac{\partial G}{\partial y}(x(t),y(t)) \cdot y'(t).$$

This is a direct application of the formula $J_h(t) = J_G(F(t)) \times J_F(t)$, with:

$$J_h(t) = \frac{dh}{dt} = h'(t), \quad J_G(x,y) = \begin{pmatrix} \frac{\partial G}{\partial x}(x,y) & \frac{\partial G}{\partial y}(x,y) \end{pmatrix}, \quad J_F(t) = \begin{pmatrix} \frac{dx}{dt}(t) \\ \frac{dy}{dt}(t) \end{pmatrix} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}.$$

Example 5.3.11. *Let $G(x,y) = \cos(y)e^x$. Calculate the derivative of the function $h : t \mapsto G(t^2, \sin t)$.*

Solution: *One method would be to write $h(t) = \cos(\sin t)e^{t^2}$ and then differentiate h... But let's use the formula $J_h(t) = J_G(F(t)) \times J_F(t)$, where we define $F(t) = (t^2, \sin t)$, so that $h = G \circ F$.*

Knowing that: $J_h(t) = h'(t)$, $\quad J_G(x,y) = \begin{pmatrix} \cos(y)e^x & -\sin(y)e^x \end{pmatrix}$, $\quad J_F(t) = \begin{pmatrix} 2t \\ \cos t \end{pmatrix}$,

we calculate $J_G(F(t))$ and obtain

$$h'(t) = 2t\cos(\sin t)e^{t^2} + \cos(t)\cdot(-\sin(\sin t)e^{t^2}) = 2t\cos(\sin t)e^{t^2} - \cos(t)\sin(\sin t)e^{t^2}.$$

Case $F : \mathbb{R} \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{R}$: More generally, we have the following result.

Proposition 5.3.12. *Let $F : \mathbb{R} \to \mathbb{R}^n$ be a function whose components are all differentiable, and let $G : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Then the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(t) = G(F(t))$ is differentiable and:*

$$h'(t) = \langle grad G(F(t)) \mid F'(t) \rangle.$$

Case: $F : \mathbb{R}^2 \to \mathbb{R}^2$, $G : \mathbb{R}^2 \to \mathbb{R}$

Proposition 5.3.13. *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $(x,y) \mapsto (f_1(x,y), f_2(x,y))$ and $G : \mathbb{R}^2 \to \mathbb{R}$ defined by $(u,v) \mapsto G(u,v)$ be differentiable functions. The function $H = G \circ F : \mathbb{R}^2 \to \mathbb{R}$ defined by $(x,y) \mapsto G(F(x,y))$ is differentiable, and we have:*

$$\frac{\partial H}{\partial x}(x,y) = \frac{\partial G}{\partial u}(F(x,y)) \cdot \frac{\partial f_1}{\partial x}(x,y) + \frac{\partial G}{\partial v}(F(x,y)) \cdot \frac{\partial f_2}{\partial x}(x,y),$$

$$\frac{\partial H}{\partial y}(x,y) = \frac{\partial G}{\partial u}(F(x,y)) \cdot \frac{\partial f_1}{\partial y}(x,y) + \frac{\partial G}{\partial v}(F(x,y)) \cdot \frac{\partial f_2}{\partial y}(x,y).$$

It is once again the formula $J_H(x,y) = J_G(F(x,y)) \times J_F(x,y)$, with:

$$J_H(x,y) = \begin{pmatrix} \frac{\partial H}{\partial x}(x,y) & \frac{\partial H}{\partial y}(x,y) \end{pmatrix}, \quad J_G(u,v) = \begin{pmatrix} \frac{\partial G}{\partial u}(u,v) & \frac{\partial G}{\partial v}(u,v) \end{pmatrix},$$

and

$$J_F(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \end{pmatrix}$$

Example 5.3.14. *Calculate the partial derivatives of the function $(x,y) \mapsto G(x-y, x+y)$ where $G : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function.*

Solution:

Let $F(x,y) = (x-y, x+y)$, and denote (u,v) as the variables of the function G and $H(x,y) = (G \circ F)(x,y) = G(x-y, x+y)$.

Thus, we have:

$$J_H(x,y) = \begin{pmatrix} \frac{\partial H}{\partial x}(x,y) & \frac{\partial H}{\partial y}(x,y) \end{pmatrix}, \quad J_G(u,v) = \begin{pmatrix} \frac{\partial G}{\partial u}(u,v) & \frac{\partial G}{\partial v}(u,v) \end{pmatrix},$$

and

$$J_F(x,y) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Therefore:

$$\frac{\partial H}{\partial x}(x,y) = \frac{\partial G}{\partial u}(x-y, x+y) \cdot 1 + \frac{\partial G}{\partial v}(x-y, x+y) \cdot 1,$$

$$\frac{\partial H}{\partial y}(x,y) = \frac{\partial G}{\partial u}(x-y, x+y) \cdot (-1) + \frac{\partial G}{\partial v}(x-y, x+y) \cdot 1.$$

5.4 Chapter 5 Exercises

Exercise 20

We define $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ by

$$f(x,y) = \frac{x^2}{(x^2+y^2)^{\frac{3}{4}}}.$$

Justify that we can extend f to a continuous function on $\mathbb{R}^2$. Study the existence of partial derivatives at $(0,0)$ for this extension.

Correction ▼ [20]

Exercise 21

Verify, using the definition, that the following functions are differentiable at the points (x_0, y_0).

1. $f_1(x,y) = xy - 3x^2, \quad (x_0,y_0) = (1,2).$
2. $f_2(x,y) = yx, \quad (x_0,y_0) = (4,1).$

Correction ▼ [21]

Exercise 22

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \frac{y^3}{\sqrt{x^2+y^4}} \quad \text{if } (x,y) \neq (0,0) \quad \text{and} \quad f(0,0) = 0.$$

Show that f has a directional derivative in every non-zero direction at the point $(0,0)$, but that it is not differentiable at that point.

Correction ▼ [22]

Exercise 23

Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{x^3-y^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Show that the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at every point in $\mathbb{R}^2$ and that f is continuous, but it is not differentiable at $(0,0)$.

Correction ▼ [23]

Exercise 24

Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} (x^2+y^2)^3 \cos\left(\frac{1}{x^2+y^2}\right), & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

1. Study the continuity of f on $\mathbb{R}^2$.
2. Calculate $\nabla f(x,y)$.
3. Is the function f of class C^1 on $\mathbb{R}^2$?
4. What can we conclude about the differentiability of f on $\mathbb{R}^2$?

Correction ▼ [24]

Exercise 25

Calculate the integral of the differential form α along γ in the following cases:

1. Let
$$\alpha = \frac{x}{x^2+y^2}\,dx + \frac{y}{x^2+y^2}\,dy$$
and let γ be the arc of the parabola given by the equation $y^2 = 2x+1$, joining the points $(0,-1)$ and $(0,1)$, traversed once in the direction of increasing y.
2. Let
$$\alpha = (x-y^3)\,dx + x^3\,dy$$
and let γ be the circle centered at the origin O with radius 1, traversed once in the positive direction.
3. Let
$$\alpha = xyz\,dx$$
and let γ be the arc defined by $x = \cos t, y = \sin t, z = \cos t \sin t$, where t varies from 0 to $\frac{\pi}{2}$.
4. Let
$$\alpha = y^2\,dx + x^2\,dy.$$
and let γ be any circle in the plane, traversed once in the counterclockwise direction.

Correction ▼ [25]

Exercise 26

Let F be the function defined by

$$F(x,y) = (x^2 + x\cos y,\ e^{x-y},\ y^3 x).$$

1. Justify the differentiability of F on $\mathbb{R}^2$.
2. Write the Jacobian matrix J_F at every point in $\mathbb{R}^2$.
3. Determine $dF(a)$, the differential of F at the point $a \in \mathbb{R}^2$.

Correction ▼ [26]

Exercise 27

Let f be the real function defined on $\mathbb{R}^2$ by

$$f(x,y) = \sin(x^2 - y^2)$$

and let g be a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by

$$g(x,y) = (x+y, x-y).$$

1. Justify that f and g are differentiable at every point $(x,y) \in \mathbb{R}^2$.
2. Calculate the partial derivatives of $f \circ g$ and the differential of $f \circ g$ at the point (x,y).
3. Calculate the Jacobian matrices of f and g at the point (x,y).
4. By applying the theorem on the composition of two differentiable functions, find the Jacobian matrix of $f \circ g$ at the point (x,y) and the differential of $f \circ g$ at the point (x,y).

Correction ▼ [27]

Chapter

6 Optimization with and without constraints

This chapter focuses on identifying the maximum and minimum values of functions. We will explore the conditions for the existence of local extrema and conclude with the concept of constrained extrema, where the search for extrema is subject to specific constraints. To facilitate understanding of these concepts in multiple variables, we will first review the case of a single-variable function.

6.1 Optimization without constraints

6.1.1 Case of a Single Variable

Let $f : \mathbb{R} \to \mathbb{R}$ be a function of one variable.

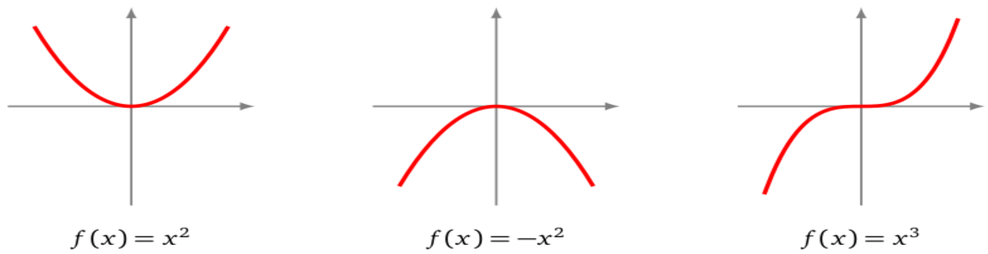
- f has a local maximum at $x_0 \in \mathbb{R}$ if there exists an open interval I containing x_0 such that:
$$\forall x \in I, \quad f(x) \leqslant f(x_0).$$
- f has a local minimum at $x_0 \in \mathbb{R}$ if there exists an open interval I containing x_0 such that:
$$\forall x \in I, \quad f(x) \geqslant f(x_0).$$
- f has a local extremum at $x_0 \in \mathbb{R}$ if f has either a local maximum or a local minimum at that point.
- f has a critical point at $x_0 \in \mathbb{R}$ if $f'(x_0) = 0$. Geometrically, this is a point of horizontal tangent.

Proposition 6.1.1. *If f is differentiable and has a local minimum or a local maximum at x_0, then $f'(x_0) = 0$. In other words, if x_0 is a local extremum, then it is a critical point.*

The converse is not always true. For example, for $f : x \mapsto x^3$, the point $x_0 = 0$ is a critical point, but it is neither a local maximum nor a local minimum (it is an inflection point).

Example 6.1.2.
1. *$f : x \mapsto x^2$, local minimum at 0, with $f'(0) = 0$ and $f''(0) > 0$.*
2. *$f : x \mapsto -x^2$, local maximum at 0, with $f'(0) = 0$ and $f''(0) < 0$.*

3. *$f : x \mapsto x^3$, neither a local minimum nor a local maximum at 0, with $f'(0) = 0$ and $f''(0) = 0$.*



Taylor's Formula of Order 2

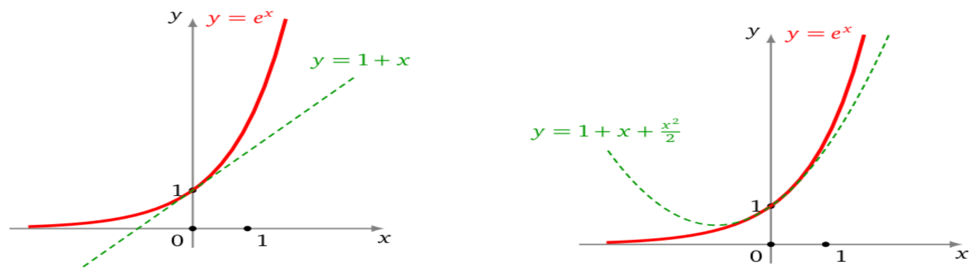
Let $f : \mathbb{R} \to \mathbb{R}$ be a function of one variable of class C^2.

Theorem 6.1.3. *(Taylor's Formula of Order 2) For any $x_0 \in \mathbb{R}$, we have*

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2} f''(x_0) + h^2 \varepsilon(h)$$

where $\varepsilon(h) \to 0$ as $h \to 0$.

The first-order Taylor expansion, $f(x_0 + h) \approx f(x_0) + hf'(x_0)$, corresponds to the approximation of the graph of f by its tangent at x_0 (left figure below). The second-order Taylor expansion, $f(x_0 + h) \approx f(x_0) + hf'(x_0) + \frac{h^2}{2} f''(x_0)$, corresponds to an approximation by a parabola (right figure).



Let us choose a value for x_0 such that $f'(x_0) = 0$ and $f''(x_0) \neq 0$. Then, for sufficiently small h, the term $\frac{h^2}{2} f''(x_0) + h^2 \varepsilon(h)$ has the same sign as $f''(x_0)$. If, for example, $f(x_0) > 0$, we can deduce that $f(x_0 + h) \geqslant f(x_0)$ (for h close to 0), and thus f has a local minimum at x_0.

Characterization of Minima and Maxima

The practical search for local extrema of a function of one variable proceeds as follows:

1. Identify the critical points given by the equation $f'(x) = 0$.

2. For each critical point x_0, calculate the second derivative:

- If $f''(x_0) > 0$, then f has a local minimum at x_0.
- If $f''(x_0) < 0$, then f has a local maximum at x_0.
- If $f''(x_0) = 0$, further investigation is needed.

6.1.2 Case of Two Variables

Let $f : U \to \mathbb{R}$ be a function of two variables, where U is an open subset of $\mathbb{R}^2$.

Definition 6.1.4. *We say that f has a local maximum (resp. local minimum) at $(x_0, y_0) \in U$ if there exists an open disk $D \subset U$, centered at (x_0, y_0), such that:*

$$\forall (x,y) \in D, \quad f(x,y) \leqslant f(x_0,y_0) \quad \text{(resp. } f(x,y) \geqslant f(x_0,y_0)\text{)}.$$

We say that f has a local extremum at (x_0, y_0) if it has a local maximum or a local minimum there.

Critical Point

Let us assume that f is of class C^2 on an open set U, meaning that its partial derivatives up to order 2 exist and are continuous.

Proposition 6.1.5. *If f has a local extremum at (x_0, y_0) in an open set U, then*

$$\frac{\partial f}{\partial x}(x_0,y_0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(x_0,y_0) = 0.$$

In other words, if f has a local minimum or maximum at a point, then the gradient of f is the zero vector at that point. The points in U where the gradient of f vanishes are called critical points of f. The previous result states that the extrema of a function on an open set can only occur at a critical point. The converse is false.
By definition, a critical point that is neither a local maximum nor a local minimum is called a saddle point.

Hessian matrix

The Jacobian matrix is the matrix of partial derivatives, while the Hessian matrix is the matrix of second-order partial derivatives.

Definition 6.1.6. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of n variables. The Hessian matrix of f at $x = (x_1, \ldots, x_n)$ is the $n \times n$ matrix:*

$$H_f(x) = \left(\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \right)_{1 \leqslant i,j \leqslant n}.$$

For a function of class C^2, according to Schwarz's theorem, this matrix is symmetric.

In the case of a function of two variables:

$$H_f(x,y) = \begin{pmatrix} \frac{\partial^2 f(x,y)}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \frac{\partial^2 f}{\partial y \partial x}(x,y) & \frac{\partial^2 f(x,y)}{\partial y^2} \end{pmatrix}.$$

For three variables, the Hessian matrix (evaluated at (x,y,z)) is given by:

$$H_f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix}.$$

Example 6.1.7. *Let's calculate the Hessian matrix of $f(x,y) = xy^2 + x^4 - y^4$.*
First, we compute the partial derivatives:

$$\frac{\partial f}{\partial x}(x,y) = 4x^3 + y^2, \quad \frac{\partial f}{\partial y}(x,y) = 2xy - 4y^3.$$

Thus, we have:

$$H_f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \frac{\partial^2 f}{\partial y \partial x}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix} = \begin{pmatrix} 12x^2 & 2y \\ 2y & 2x - 12y^2 \end{pmatrix}.$$

Characterization of Minima and Maxima

For a function $f : \mathbb{R}^2 \to \mathbb{R}$, we will use Monge's notation, which provides a simple criterion for detecting a local minimum or maximum.

Theorem 6.1.8. *(Monge's Criterion).*
Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of class C^2, and let (x_0, y_0) be a critical point of f. Define

$$r = \frac{\partial^2 f}{\partial x^2}(x_0,y_0), \quad s = \frac{\partial^2 f}{\partial x \partial y}(x_0,y_0), \quad t = \frac{\partial^2 f}{\partial y^2}(x_0,y_0).$$

Then:

- *If $rt - s^2 > 0$ and $r > 0$, then (x_0,y_0) is a local minimum of f.*
- *If $rt - s^2 > 0$ and $r < 0$, then (x_0,y_0) is a local maximum of f.*
- *If $rt - s^2 < 0$, then (x_0,y_0) is neither a local minimum nor a local maximum: it is a saddle point.*
- *If $rt - s^2 = 0$, no direct conclusion can be drawn (further analysis is needed).*

Remark: $rt - s^2$ is the determinant of the Hessian matrix at (x_0,y_0):

$$H_f(x_0,y_0) = \begin{pmatrix} r & s \\ s & t \end{pmatrix}.$$

Example 6.1.9.

1. *Let $f(x,y) = x^2 + y^2$. The point $(0,0)$ is the unique critical point of f. We calculate*

$$r = \frac{\partial^2 f}{\partial x^2}(0,0) = 2, \quad s = \frac{\partial^2 f}{\partial x \partial y}(0,0) = 0, \quad t = \frac{\partial^2 f}{\partial y^2}(0,0) = 2.$$

Thus, $rt - s^2 = 4$ with $r > 0$, so $(0,0)$ is indeed a local minimum of f.

2. *Let $f(x,y) = x^2 - y^2$. We find a single critical point: $(0,0)$. We calculate*

$$H_f(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

This time, $r = 2$, $s = 0$, and $t = -2$. Thus, $rt - s^2 = -4 < 0$, and therefore $(0,0)$ corresponds to a saddle point.

3. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = x^3 + y^3 - 3xy$.*

Partial Derivatives:

$$\frac{\partial f}{\partial y}(x,y) = 3y^2 - 3x, \quad \frac{\partial f}{\partial x}(x,y) = 3x^2 - 3y$$

Critical Points: The critical points are where $\frac{\partial f}{\partial x}(x,y) = 0$ and $\frac{\partial f}{\partial y}(x,y) = 0$ simultaneously. This gives us the equations:

$$x^2 = y \quad \text{and} \quad y^2 = x$$

which implies $x, y \geqslant 0$. From this, we have: $x^4 = y^2 = x$.

The positive solutions are $x = 0$ (and then $y = 0$) and $x = 1$ (and then $y = 1$). Thus, the critical points are $(0,0)$ and $(1,1)$.

Second Partial Derivatives:

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 6x, \quad \frac{\partial^2 f}{\partial x \partial y}(x,y) = -3, \quad \frac{\partial^2 f}{\partial y^2}(x,y) = 6y$$

- *Analysis at $(0,0)$: $H_f(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$*

  This means $r = 0$, $s = -3$, $t = 0$, so $rt - s^2 = -9 < 0$, thus $(0,0)$ is a saddle point.

- *Analysis at $(1,1)$: $H_f(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$*

  This gives $r = 6$, $s = -3$, $t = 6$, so $rt - s^2 = 27 > 0$ with $r > 0$, hence $(1,1)$ is a minimum point of f (it is a local minimum and not a global minimum).

This is an example where the criterion does not allow us to conclude definitively. Further analysis is required to complete the study.

Example 6.1.10. *Let $f(x,y) = 2x^3 - y^4 - 3x^2$. We find two critical points: $(0,0)$ and $(1,0)$. Moreover:*

$$H_f(0,0) = \begin{pmatrix} -6 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad H_f(1,0) = \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix}.$$

We cannot conclude because the determinant $rt - s^2$ is zero. We will analyze each case manually.

- *At $(0,0)$: Let's write $f(x,y) = x^2(2x-3) - y^4$. For $|x| \leqslant 1$, we have $2x - 3 \leqslant 0$, hence:*

  $$f(x,y) = x^2(2x-3) - y^4 \leqslant 0.$$

  Since $f(0,0) = 0$, we conclude that f has a local maximum at the point $(0,0)$.

- *At $(1,0)$: First, we restrict ourselves to points of the form $(1,y)$ (around $y_0 = 0$):*

  $$f(1,y) = -1 - y^4 \leqslant -1 = f(1,0).$$

  Next, we consider points of the form $(x,0)$ (around $x_0 = 1$, for example, for $|x-1| \leqslant 1$):

  $$f(x,0) = (x-1)^2(2x+1) - 1 \geqslant -1 = f(1,0).$$

  Thus, at $(1,0)$, it is neither a minimum nor a maximum: it is a saddle point.

6.2 Optimization with constraints

Theorem 6.2.1. *Let $f, g : U \to \mathbb{R}$ be C^1 functions on an open set $U \subset \mathbb{R}^2$. Let $(x_0, y_0) \in U$ such that f subject to the constraint $g(x,y) = 0$ has an extremum at the point (x_0, y_0) and $\nabla g(x_0, y_0) \neq (0,0)$. Then there exists a real number λ such that $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$. In other words, we have:*

$$\begin{aligned} g(x_0,y_0) &= 0 \\ \frac{\partial f}{\partial g}(x_0,y_0) &= \lambda \frac{\partial g}{\partial x}(x_0,y_0) \\ \frac{\partial f}{\partial g}(x_0,y_0) &= \lambda \frac{\partial g}{\partial y}(x_0,y_0) \end{aligned} \tag{6.1}$$

Note that we can also consider the constraint $g(x,y) = c$ (where c is a constant), which reduces to the case stated in the theorem by considering $g(x,y) - c = 0$.

6.2.1 Lagrangian Method

Let us form the Lagrangian $L(x,y,\lambda) = f(x,y) - \lambda g(x,y)$, where λ (the Lagrange multiplier) is an unknown. For this function to have an extremum, the gradient of L must be zero;

in other words, we seek the triplets (x,y,λ) such that

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = \lambda \frac{\partial g}{\partial x}(x,y), \\ \frac{\partial f}{\partial y}(x,y) = \lambda \frac{\partial g}{\partial y}(x,y), \\ 0 = g(x,y). \end{cases}$$

Let (x_0,y_0,λ_0) be a solution of this system. If $\nabla g(x_0,y_0) \neq 0$, then (x_0,y_0) is a critical point of the function f under the constraint g. These critical points satisfy the constraint, but now we need to classify these candidates.
As in the case of unconstrained extrema, we can formulate second-order conditions related to the critical points, but these only apply in certain cases. Let

$$\Delta(x_0,y_0,\lambda_0) \equiv \frac{\partial^2 L}{\partial x^2}(x_0,y_0,\lambda_0)\frac{\partial^2 L}{\partial y^2}(x_0,y_0,\lambda_0) - \left(\frac{\partial^2 L}{\partial x \partial y}(x_0,y_0,\lambda_0)\right)^2,$$

which is the determinant of the submatrix obtained from the Hessian of L by eliminating the last row and the last column.

- If $\Delta(x_0,y_0,\lambda_0) > 0$, $\frac{\partial^2 L}{\partial x^2}(x_0,y_0,\lambda_0) < 0$, and $\frac{\partial^2 L}{\partial y^2}(x_0,y_0,\lambda_0) < 0$, then there is a local maximum at (x_0,y_0).
- If $\Delta(x_0,y_0,\lambda_0) > 0$, $\frac{\partial^2 L}{\partial x^2}(x_0,y_0,\lambda_0) > 0$, and $\frac{\partial^2 L}{\partial y^2}(x_0,y_0,\lambda_0) > 0$, then there is a local minimum at (x_0,y_0).
- If $\Delta(x_0,y_0,\lambda_0) \leqslant 0$, we cannot conclude directly. We then study the sign of the difference
$$d(h,k) \equiv f(x_0+h,y_0+k) - f(x_0,y_0),$$
with h and k being related by the equation $g(x_0+h,y_0+k) = 0$. If this difference has a constant sign for (h,k) close to $(0,0)$, then it is a local extremum (a maximum if $d < 0$, a minimum if $d > 0$). Otherwise, f does not present a local extremum at (x_0,y_0).

Example 6.2.2. *To study the existence of extrema of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$f(x,y) = xy$$

under the constraint

$$g(x,y) = x + y - 6 = 0$$

we use the method of Lagrange multipliers.
Let us introduce the Lagrangian $L : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$:

$$L(x,y,\lambda) = f(x,y) - \lambda g(x,y) = xy - \lambda(6 - x - y)$$

By applying the necessary condition of Lagrange multipliers, we must calculate the triplets (x,y,λ) that are solutions to the following system:

$$\begin{cases} \frac{\partial L}{\partial x} = y + \lambda = 0 \\ \frac{\partial L}{\partial y} = x + \lambda = 0 \\ g(x,y) = x + y - 6 = 0 \end{cases}$$

By solving this system of equations, we find the critical point

$$(x, y, \lambda) = (3, 3, -3).$$

The only extremum candidate for f under the constraint g is therefore the point $(3, 3)$, and $f(3, 3) = 9$. The Hessian submatrix of L is

$$H_L(x, y, \lambda) = \begin{pmatrix} \frac{\partial^2 L}{\partial x^2}(x, y, \lambda) & \frac{\partial^2 L}{\partial x \partial y}(x, y, \lambda) \\ \frac{\partial^2 L}{\partial y \partial x}(x, y, \lambda) & \frac{\partial^2 L}{\partial y^2}(x, y, \lambda) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

For all x, y,

$$|H_L(x, y, \lambda)| = -1 \leqslant 0.$$

Thus, further investigation is needed. To determine the nature of this critical point, we must examine the sign of the difference

$$d(x, y) = f(x, y) - f(3, 3) = xy - 9$$

where x and y are related by the equation $0 = g(x, y) = x + y - 6$, which means $y = 6 - x$. We have

$$d(x, 6 - x) = -(x - 3)^2 < 0 \text{ for all } x \in \mathbb{R}.$$

This indicates that the difference d is negative for all x, which means that the point $(3, 3)$ is a local maximum of f under the constraint g.

6.3 Chapter 6 Exercises

Exercise 28

Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{xy^2}{x+y} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Is f of class C^2 at $(0,0)$?

Correction ▼ [28]

Exercise 29

Find the critical points of the following functions and determine their nature.

1. $$f(x,y) = (x^2 - y)e^{x-y}.$$
2. $$g(x,y) = x^3 + 6x^2 + 3y^2 - 12xy + 9x.$$
3. $$h(x,y) = 3x^3 + 3y^3 - x - y.$$

Correction ▼ [29]

Exercise 30

Find the critical points of the following functions and determine their nature.

1. $$f(x,y) = x^4 + y^4 - 4(x-y)^2.$$
2. $$g(x,y) = x^4 + y^3 - 3y - 2.$$

Correction ▼ [30]

Exercise 31

Determine the minima and maxima in the following cases:

1. $f_1(x,y) = x + y$ under the constraint $x^2 + y^2 = 1$.
2. $f_2(x,y) = 4x^2 + y^2$ under the constraint $x^2 + y^2 = 4$.

Correction ▼ [31]

Chapter

7 Solutions

7.1 Solutions to Chapter 1 Exercises

Correction of the exercise 1 ▲

1. • Let $f(t,x) = \frac{e^{-x^2t^3}}{1+t^2}$. The function $F(x) = \int_1^2 \frac{e^{-x^2t^3}}{1+t^2}dt$ is a proper integral.
   We have: f is continuous as the ratio of continuous functions on $[1,2]\times\mathbb{R}$ with a non-zero denominator and $\frac{\partial f}{\partial x}(t,x) = \frac{-2xt^3e^{-x^2t^3}}{1+t^2}$ is continuous on $[1,2]\times\mathbb{R}$.
   Conclusion: We use the theorem of conservation of differentiability under the integral for the case of a proper parameterized integral. We deduce that F is differentiable on $\mathbb{R}$, which also implies its continuity on $\mathbb{R}$.
   • Let $g(t,x) = \frac{e^{x(t+1)}}{t+1}$. The function $G(x) = \int_0^1 \frac{e^{x(t+1)}}{t+1}dt$ is a proper integral.
   We have: g is continuous as the ratio of continuous functions on $[0,1]\times\mathbb{R}$ with a non-zero denominator and $\frac{\partial g}{\partial x}(t,x) = e^{x(t+1)}$ is continuous on $[0,1]\times\mathbb{R}$.
   Conclusion: We use the theorem of conservation of differentiability under the integral for the case of a parameterized Riemann integral. We deduce that G is differentiable on $\mathbb{R}$, which also implies its continuity on $\mathbb{R}$.
2. We have
$$G'(x) = \int_0^1 g'(t,x)dt = \int_0^1 e^{x(t+1)}dt = \begin{cases} \frac{e^{2x}-e^x}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

Correction of the exercise 2 ▲

Let $f(t,x) = \frac{1}{(t^2+x^2)(t^2+1)}$. The function $F(x) = \int_0^1 \frac{1}{(t^2+x^2)(t^2+1)}dt$ is a proper parameterized integral.

1. Study the continuity of F on $\mathbb{R}_+^*$: Since f is continuous as the ratio of continuous functions on $[0,1]\times\mathbb{R}_+^*$ with a non-zero denominator, by using the theorem of conservation of continuity under the integral for the case of proper parameterized integrals, we deduce that F is continuous on $\mathbb{R}_+^*$.

   Moreover, the function F is even, which implies that F is continuous on all of $\mathbb{R}^*$.
2. From the previous question, we have in particular
$$\lim_{x\to 1} F(x) = F(1) = \int_0^1 \frac{1}{(t^2+1)^2}dt. \quad (1)$$

Furthermore,

$$f(t,x) = \frac{1}{(t^2+x^2)(t^2+1)} = \frac{1}{x^2-1}\left(\frac{1}{t^2+1} - \frac{1}{t^2+x^2}\right) \quad \text{for } x \neq 1 \text{ and } x \neq -1.$$

Thus,

$$\begin{aligned}\lim_{x\to 1} F(x) &= \lim_{x\to 1} \frac{1}{x^2-1}\left(\int_0^1 \frac{1}{t^2+1}dt - \int_0^1 \frac{1}{t^2+x^2}dt\right)\\ &= \lim_{x\to 1} \frac{1}{x^2-1}\left([\arctan(t)]_0^1 - \frac{1}{x}\left[\arctan\left(\frac{t}{x}\right)\right]_0^1\right)\\ &= \lim_{x\to 1} \frac{1}{x^2-1}\left(\frac{\pi}{4} - \frac{1}{x}\arctan\left(\frac{1}{x}\right)\right) = \lim_{x\to 1} \frac{\frac{\pi}{4} - \frac{1}{x}\arctan\left(\frac{1}{x}\right)}{x^2-1}.\end{aligned}$$

Using L'Hôpital's rule, we get

$$\lim_{x\to 1} F(x) = \lim_{x\to 1} \frac{-\left(-\frac{1}{x^2}\arctan\left(\frac{1}{x}\right) + \frac{1}{x}\cdot\frac{-\frac{1}{x^2}}{\left(\frac{1}{x}\right)^2+1}\right)}{2x} = \frac{\pi}{8} + \frac{1}{4}. \quad (2)$$

3. From (1) and (2), we deduce that

$$\int_0^1 \frac{1}{(t^2+1)^2}dt = \frac{\pi}{8} + \frac{1}{4}.$$

Correction of the exercise 3 ▲

1. Let us show that F is differentiable. Let $H(x) = \int_0^x e^{-t^2}dt$ and $h(t,x) = e^{-t^2}$. We apply the theorem of conservation of differentiability for proper parameterized integrals with variable limits, where $u(x) = 0$ and $v(x) = x$. The function h is C^1 on $[0,+\infty[\times[0,+\infty[$ (since it is the composition of C^1 functions), so H is differentiable and:

$$H'(x) = v'(x)h(v(x)) - u'(x)h(u(x)) = e^{-x^2}.$$

(We can obtain the same result by noting that the function H is just an antiderivative of the function $t \mapsto e^{-t^2}$).

Thus, F is differentiable since it is the product of differentiable functions, and we have $F'(x) = 2H'(x)H(x) = 2e^{-x^2}\int_0^x e^{-t^2}dt$.

Let us show that G is differentiable. Let $g(t,x) = \frac{e^{-(1+t^2)x^2}}{t^2+1}$. G is also a proper parameterized integral. The function g is C^1 on $[0,1]\times[0,+\infty[$ (since it is the composition and the ratio of C^1 functions). Therefore, G is differentiable and:

$$G'(x) = \int_0^1 \frac{\partial g}{\partial x}(t,x)dt = -2\int_0^1 \frac{x(t^2+1)e^{-(1+t^2)x^2}}{t^2+1}dt = -2\int_0^1 xe^{-(1+t^2)x^2}dt.$$

2. To show that $\forall x \in [0,+\infty[, F(x)+G(x) = \frac{\pi}{4}$, we start by showing that $\forall x \in [0,+\infty[, F'(x)+G'(x) = 0$. We have:

$$G'(x) = -2\int_0^1 xe^{-(1+t^2)x^2}dt = -2e^{-x^2}\int_0^1 xe^{-(tx)^2}dt.$$

Let us make the change of variables $u = tx$, $du = xdt$. We obtain:

$$G'(x) = -2e^{-x^2}\int_0^x e^{-u^2}du = -F'(x).$$

Thus, for all $x \in [0,+\infty[, F(x)+G(x) = C$. To find the constant C, we can take $x = 0$ and get $C = F(0)+G(0) = 0+\frac{\pi}{4} = \frac{\pi}{4}$ (easy to verify).

We conclude that $\forall x \in [0,+\infty[, F(x)+G(x) = \frac{\pi}{4}$.

3. To deduce the value of $\int_0^{+\infty} e^{-t^2}dt$, we first recall that it is a convergent improper integral (this can be shown, for example, using the order rule; $\lim_{t\to+\infty} t^\alpha \cdot e^{-t^2} = 0$ for $\alpha > 1$). Therefore, $\lim_{x\to+\infty}\int_0^x e^{-t^2}dt$ exists and is finite. We can then compute $\lim_{x\to+\infty}\left(\int_0^x e^{-t^2}dt\right)^2$, and then take its square root.

Thus, we need to calculate $\lim_{x\to+\infty} F(x)$, and for that we calculate $\lim_{x\to+\infty} G(x)$:

We have $G(x) \leqslant e^{-x^2}\int_0^1 \frac{1}{t^2+1}dt$ because $e^{-t^2x^2} \leqslant 1$ (we kept e^{-x^2} to ensure the limit is zero).

Since $\lim_{x\to+\infty} e^{-x^2} = 0$ and $\int_0^1 \frac{1}{t^2+1}dt = [\arctan(t)]_0^1 = \frac{\pi}{4}$, we have $\lim_{x\to+\infty} G(x) = 0$. This gives us $\lim_{x\to+\infty} F(x) = \frac{\pi}{4}$. Thus,

$$\int_0^{+\infty} e^{-t^2}dt = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}.$$

Correction of the exercise 4 ▲

$$F(x) = \int_0^\infty \frac{e^{-xt}}{(1+t)\sqrt{t}}dt.$$

Thus,

$$f(x,t) = \frac{e^{-xt}}{(1+t)\sqrt{t}}.$$

1. Study of the continuity of F:

Dominated Convergence:

$$\text{For all } t \geqslant 1 \text{ and } \forall x \in \mathbb{R}^+, \frac{e^{-xt}}{(1+t)\sqrt{t}} \leqslant \frac{e^{-xt}}{t^{3/2}} \leqslant \frac{1}{t^{3/2}} = g(t).$$

$$\text{Moreover, } \int_1^\infty \frac{1}{t^{3/2}}dt \text{ converges (Riemann, } \alpha = \frac{3}{2} > 1).$$

Thus, $\int_0^\infty f(t,x)dt$ satisfies the dominated convergence criterion on $\mathbb{R}^+$.

Let us apply the continuity theorem:

- f is continuous on $\mathbb{R}^+ \times [1, +\infty[$, as it is a ratio and product of continuous functions.
- $\int_0^\infty f(x,t)\,dt$ satisfies the dominated convergence criterion on $\mathbb{R}^+$. Therefore, F is continuous on $\mathbb{R}^+$.

2. Study of differentiability: we have

$$\forall t \geqslant 1 \text{ and } \forall x \in \mathbb{R}^+, \frac{\partial f}{\partial x}(x,t) = -\sqrt{t}\frac{e^{-xt}}{1+t}.$$

Dominated Convergence:

For all $t \in [1, +\infty[$ and $\forall x \in [\alpha, +\infty[, \alpha > 0$, $\dfrac{-\sqrt{t}e^{-xt}}{1+t} \leqslant \dfrac{\sqrt{t}e^{-xt}}{t} \leqslant \dfrac{e^{-xt}}{\sqrt{t}} \leqslant e^{-xt} \leqslant e^{-\alpha t} = g(t)$.

Now, $\int_0^\infty e^{-\alpha t}\,dt$ converges. Thus, $\int_1^\infty \frac{\partial f}{\partial x}(x,t)\,dt$ satisfies the dominated convergence criterion on any interval of the form $[\alpha, +\infty[$ with $\alpha > 0$.

Let us apply the differentiability theorem:

- f is of class C^1 on $\mathbb{R}^+ \times [1, +\infty[$, as it is a sum, ratio, and composition of C^1 functions.
- $\int_\alpha^\infty \frac{\partial f}{\partial x}(t,x)\,dt$ satisfies the dominated convergence criterion on $[\alpha, +\infty[, \alpha > 0$.

Thus, F is differentiable on any interval of the form $[\alpha, +\infty[\subset [0, +\infty[\Rightarrow F$ is differentiable on $]0, +\infty[$.

7.2 Solutions to Chapter 2 Exercises

Correction of the exercise 5 ▲

a) If $F(s)$ is the Laplace transform of $f(t)$, then we have:

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

In our case, $f(t) = e^{at}$, $s > a$, so we can rewrite the integral as follows:

$$\int_0^\infty e^{(a-s)t}\, dt = \left[\frac{e^{(a-s)t}}{a-s}\right]_0^{+\infty} = \frac{1}{s-a}.$$

b) For $f(t) = t^n e^{at}$. We perform integration by parts to find a recurrence formula for $n > 1$:

$$F_n(s) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-st} t^n e^{at}\, dt = \int_0^\infty t^n e^{(a-s)t}\, dt = \left[t^n \frac{-e^{(a-s)t}}{a-s}\right]_0^\infty + \frac{n}{a-s}\int_0^\infty t^{n-1} e^{(a-s)t}\, dt.$$

This simplifies to:

$$= \frac{n}{a-s} F_{n-1}(s).$$

We calculate $F_0(s) = \frac{1}{s}$, leading to the recurrence $F_n(s) = \frac{n!}{(s-a)^{n+1}}$, $s > a$.

c) The Laplace transform of $\sin(\omega t)$ can be calculated using Euler's formula. We know that $\sin(\omega t) = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$. Thus, using this exponential form, we can calculate the Laplace transform:

$$\begin{aligned}
\mathcal{L}\{\sin(\omega t)\} &= \mathcal{L}\left\{\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right\} \\
&= \frac{1}{2i}\left(\mathcal{L}\{e^{i\omega t}\} - \mathcal{L}\{e^{-i\omega t}\}\right) \\
&= \frac{1}{2i}\left(\frac{1}{s-i\omega} - \frac{1}{s+i\omega}\right) \\
&= \frac{1}{2i}\left(\frac{s+i\omega-(s-i\omega)}{(s-i\omega)(s+i\omega)}\right) \\
&= \frac{1}{2i}\left(\frac{2i\omega}{s^2+\omega^2}\right) \\
&= \frac{\omega}{s^2+\omega^2}.
\end{aligned}$$

Therefore, the Laplace transform of $\sin(\omega t)$ is $\frac{\omega}{s^2+\omega^2}$.
Here, $\mathcal{L}\{e^{i\omega t}\} = \int_0^\infty e^{i\omega t} e^{-st}\, dt = \frac{1}{s-i\omega}$ according to part a).
$\mathcal{L}\{e^{i\omega t}\}$ is the transform $F(s)$ of $f(t)$ such that $f(t) = e^{i\omega t}$.

d) For $f(t) = e^{-4t}\sin(5t)$:

$$F(s) = \int_0^\infty e^{-4t}\sin(5t)e^{-st}\, dt = \int_0^\infty \sin(5t)e^{-(s+4)t}\, dt = \frac{5}{(s+4)^2+5^2} = \frac{5}{(s+4)^2+25}$$

according to the previous question.

e) For $f(t) = t^2 \cos(t)$:

$$F(s) = \int_0^{\infty} t^2 \cos(t) e^{-st}\, dt = \frac{2s^3 - 6s}{(s^2+1)^3}.$$

Correction of the exercise 6 ▲

1) We decompose the fraction into partial fractions, that is, we look for a and b such that

$$\frac{1}{(s+1)(s-2)} = \frac{a}{s+1} + \frac{b}{s-2}.$$

We find $a = -\frac{1}{3}$ and $b = \frac{1}{3}$. Thus,

$$F(s) = -\frac{1}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2}.$$

It follows that the original function is

$$f(t) = -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t}.$$

2) Let $G(s) = \frac{1}{s-2}$ and $F(s) = -\frac{1}{(s-2)^2}$. Then $G'(s) = F(s)$. Now, the original function of G is $g(t) = e^{2t}$. Therefore,

$$F(s) = G'(s) = \int_0^{+\infty} \frac{\partial}{\partial s}(g(t)e^{-st})\, dt = \int_0^{+\infty} -te^{2t}e^{-st}\, dt,$$

which gives $f(t) = -te^{2t}$.

3) The denominator factors as $(s+4)(s-1)$. We decompose the fraction into partial fractions by writing it in the form

$$\frac{a}{s+4} + \frac{b}{s-1} = \frac{(a+b)s + (4b-a)s^2 - 3s + 4}{(s+4)(s-1)}.$$

By identification, we get the system $\{a+b=5, -a+4b=10\}$, which gives $a=2$ and $b=3$. Thus, the function is

$$\frac{2}{s+4} + \frac{3}{s-1}.$$

The original function is $2e^{-4t} + 3e^t$.

4) The discriminant of the quadratic in the denominator is negative, so it has no roots. We rewrite it in canonical form as

$$\frac{s-7}{s^2 - 14s + 50} = \frac{s-7}{(s-7)^2 + 1}.$$

The original function of $\frac{s}{s^2+1}$ is $\cos(t)$, so

$$\frac{s}{s^2+1} = \mathcal{L}(\cos(t))(s),$$

which leads to

$$\frac{s-7}{(s-7)^2+1} = \mathcal{L}(\cos(t))(s-7) = \mathcal{L}(e^{7t}\cos(t))(s).$$

Thus, the original function is $f(t) = e^{7t}\cos(t)$.
5) The discriminant of the quadratic in the denominator is negative, so it has no roots. We rewrite it in canonical form as

$$\frac{s}{s^2-6s+13} = \frac{s}{(s-3)^2+4} = \frac{s-3}{(s-3)^2+2^2} + \frac{3}{(s-3)^2+2^2} = \frac{s-3}{(s-3)^2+2^2} + \frac{3}{2}\frac{2}{(s-3)^2+2^2}.$$

By reasoning similarly as in the previous question, we find that the original function is

$$\cos(2t)e^{3t} + \frac{3}{2}\sin(2t)e^{3t}.$$

6) The original of $\frac{1}{s+3}$ is $f(t) = e^{-3t}$. To find the original of the function $e^{-2s/(s+3)}$, we use the shifting theorem:

$$\mathcal{L}(f(t-\tau)) = e^{-s\tau}\mathcal{L}(f(t)).$$

Thus,

$$e^{-2s}\frac{1}{s+3} = e^{-2s}\mathcal{L}(e^{-3t}) = \mathcal{L}(f(t-2)),$$

leading to the original $e^{-3(t-2)}$.
7) The original of $\frac{a}{s^2-a^2}$ is

$$\frac{a}{s^2-a^2} = \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right),$$

thus the original function is

$$f(t) = \frac{1}{2}\left(e^{at} - e^{-at}\right).$$

8)

$$\frac{s^2}{(s+3)^2} = \frac{9}{(s+3)^3} - \frac{6}{(s+3)^2} + \frac{1}{s+3} = \frac{1}{s+3} + 6\frac{-1}{(s+3)^2} + \frac{9}{2}\frac{2}{(s+3)^3}.$$

We have $\frac{1}{s+3} = \mathcal{L}(e^{-3t})$ and $\frac{-1}{(s+3)^2} = \left(\frac{1}{s+3}\right)' = \mathcal{L}'(e^{-3t}) = \int_0^\infty \frac{\partial}{\partial s}(e^{-3t}e^{-st})\,dt = \int_0^\infty -te^{-3t}e^{-st}\,dt$. Using the derivative property, we have $-\frac{1}{(s+3)^2} = \mathcal{L}(-te^{-3t})$ and $\frac{2}{(s+3)^3} = \mathcal{L}(t^2e^{-3t})$. Therefore,

$$\frac{s^2}{(s+3)^2} = \mathcal{L}\left(e^{-3t}(1-6t+\frac{9}{2}t^2)\right).$$

Thus, the original of $s \mapsto \frac{s^2}{(s+3)^2}$ is $t \mapsto e^{-3t}(1-6t+\frac{9}{2}t^2)$.

Correction of the exercise 7 ▲

Using the Laplace transform:
According to the property of the derivative transform,

$$\mathcal{L}(x(t)) = F(s),\ \mathcal{L}(x'(t)) = s\mathcal{L}(x(t)) - x(0) = sF(s) - x(0)$$

We have:
a) $sF(s) - x(0) + 3F(s) = 0$, or $(s+3)F(s) = x(0)$, and thus $F(s) = \frac{x(0)}{s+3}$, which gives, reverting to the original function, $x(t) = x(0)e^{-3t}$.

b) Here, we have $x(0) = 0$ but $\mathcal{L}(\cos(3t)) = \frac{s}{s^2+9}$, thus $(s+3)F(s) = \frac{s}{s^2+9}$ and

$$F(s) = \frac{1}{(s+3)(s^2+9)} = \frac{\alpha}{s+3} + \frac{\beta}{s+3i} + \frac{\gamma}{s-3i}.$$

By solving this equation, we obtain $\alpha = -\frac{1}{6}$, $\beta = \frac{1+i}{12}$, and $\gamma = \frac{1-i}{12}$. Thus,

$$F(s) = \mathcal{L}\Big(-\frac{1}{6}e^{-3t} + \frac{1+i}{12}e^{-3it} + \frac{1-i}{12}e^{3it}\Big) = \mathcal{L}\Big(-\frac{1}{6}e^{-3t} + \frac{1}{6}\cos(3t) + \frac{1}{6}\sin(3t)\Big).$$

Therefore, $x(t) = \frac{1}{6}\Big(\cos(3t) + \sin(3t) - e^{-3t}\Big)$.

c) Using the Laplace transform:
We have $\mathcal{L}(t) = \frac{1}{s^2}$ and $\mathcal{L}(x'') = s\mathcal{L}(x') - x'(0) = s(s\mathcal{L}(x) - x(0)) - x'(0) = s^2F(s) - s$.
The equation becomes:

$$s^2F(s) - s + F(s) = \frac{1}{s^2} \implies F(s) = \frac{1}{s^2+1}\cdot\frac{1+s^3}{s^2} = \frac{1}{s^2(s^2+1)} + \frac{s}{s^2+1} = \frac{1}{s^2} - \frac{1}{s^2+1} + \frac{s}{s^2+1}.$$

The inverse Laplace transform is: $x(t) = t - \sin(t) + \cos(t)$.

Correction of the exercise 8 ▲

1.

$$\begin{aligned}
\mathcal{F}(f_\alpha)(s) &= \int_{-\infty}^{\infty} e^{-\alpha|t|}e^{-ist}dt \\
&= \int_{-\infty}^{0} e^{-\alpha|t|}e^{-ist}dt + \int_{0}^{\infty} e^{-\alpha|t|}e^{-ist}dt \\
&= \int_{-\infty}^{0} e^{(\alpha-is)t}dt + \int_{0}^{\infty} e^{(-\alpha-is)t}dt \\
&= \left[\frac{1}{\alpha-is}e^{(\alpha-is)t}\right]_0^{-\infty} + \left[\frac{1}{-\alpha-is}e^{(-\alpha-is)t}\right]_0^{\infty} \\
&= \frac{1}{\alpha-is} + \frac{1}{-\alpha-is} = \frac{2\alpha}{\alpha^2+s^2}, \quad \forall s \in \mathbb{R}
\end{aligned}$$

The second method: Since f_α is even, we have

$$\mathcal{F}(f_\alpha)(s) = \int_{-\infty}^{\infty} f_\alpha(t)e^{-ist}dt = 2\int_0^{\infty} e^{-\alpha t}\cos(st)dt$$

Using the Laplace transform, we have

$$\mathcal{L}(\cos(at)) = \int_0^{\infty} e^{-st}\cos(at)dt = \frac{s}{s^2+a^2}, \quad \forall s > 0, a \in \mathbb{R}.$$

Replacing s with α and a with s in $\mathcal{L}(\cos(at))$, we obtain

$$\int_0^{\infty} e^{-\alpha t}\cos(st)dt = \frac{\alpha}{\alpha^2+s^2}, \quad s \in \mathbb{R}.$$

Finally,

$$\mathcal{F}(f_\alpha)(s) = \frac{2\alpha}{\alpha^2+s^2}, \quad \forall s \in \mathbb{R}.$$

2. Since g is even,

$$\begin{aligned}\mathcal{F}(g)(s) &= \int_{-\infty}^{+\infty} g(t)e^{-ist}dt \\ &= 2\int_0^3 \cos(st)dt \\ &= 2\left[\frac{\sin(st)}{s}\right]_0^3 \\ &= 2\frac{\sin(3s)}{s}, \quad \forall s \in \mathbb{R}^*\end{aligned}$$

$$\text{Since} \quad \mathcal{F}(g)(0) = \lim_{s\to 0}\frac{2\sin(3s)}{s} = 6$$

We deduce that

$$\mathcal{F}(g)(s) = \begin{cases} \frac{2\sin(3s)}{s} & \text{if } s \neq 0, \\ 6 & \text{if } s = 0. \end{cases}$$

3. Since h is even,

$$\begin{aligned}\int_{-\infty}^{\infty} h(t)e^{-ist}dt &= 2\int_0^{\infty}(e^{-t}+1)\cos(st)dt + 2\int_3^{\infty} e^{-t}\cos(st)dt \\ &= 2\int_0^3 e^{-t}\cos(st)dt + 2\int_0^3 \cos(st)dt + 2\int_3^{\infty} e^{-t}\cos(st)dt \\ &= 2\int_0^{\infty} e^{-t}\cos(st)dt + 2\int_0^3 \cos(st)dt\end{aligned}$$

Calculating $\int_3^{\infty} e^{-t}\cos(st)dt$ using the Laplace transform: We have

$$\mathcal{L}(\cos(at)) = \int_0^{\infty} e^{-st}\cos(at)dt = \frac{s}{s^2+a^2}, \quad \forall s > 0, a \in \mathbb{R}.$$

Replacing s with 1 and a with s in $\mathcal{L}(\cos(at))$, we obtain

$$\int_0^{\infty} e^{-t}\cos(st)dt = \frac{1}{1+s^2}, \quad \forall s \in \mathbb{R}.$$

$$\int_0^3 \cos(st)dt = \left[\frac{\sin(st)}{s}\right]_0^3 = \frac{\sin(3s)}{s}, \quad \forall s \in \mathbb{R}^*.$$

Finally,

$$\begin{aligned}\int_{-\infty}^{+\infty} h(t)e^{-ist}dt &= 2\int_0^{+\infty} e^{-t}\cos(st)dt + 2\int_0^3 \cos(st)dt \\ &= \frac{2}{1+s^2} + \frac{2\sin(3s)}{s}, \quad \forall s \in \mathbb{R}^*.\end{aligned}$$

$$\mathcal{F}(h)(s) = \begin{cases} \frac{2\sin(3s)}{s} + \frac{2}{1+s^2} & \text{if } s \neq 0, \\ 8 & \text{if } s = 0. \end{cases}$$

Correction of the exercise 9 ▲

1. Since f_α is even, we have

$$\mathcal{F}(f_\alpha)(s) = \int_{-\infty}^{\infty} f_\alpha(t)e^{-ist}dt = 2\int_0^{\infty} e^{-\alpha t}\cos(st)dt = \mathcal{L}(\cos(st))(\alpha) = \frac{2\alpha}{\alpha^2+s^2}, \quad \forall s \in \mathbb{R}.$$

2. We have $\mathcal{F}(e^{-\alpha|t|})(s) = \frac{2\alpha}{\alpha^2+s^2}$, so $\frac{1}{1+s^2} = \mathcal{F}\left(\frac{1}{2}e^{-|t|}\right)$ and $\mathcal{F}^{-1}\left(\frac{1}{1+s^2}\right) = \frac{1}{2}e^{-|t|}$. On the other hand,

$$\mathcal{F}^{-1}\left(\frac{1}{1+s^2}\right) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{1+s^2}e^{+ist}ds = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{1+S^2}e^{-iSt}dS = \frac{1}{2\pi}\mathcal{F}\left(\frac{1}{1+S^2}\right)(t).$$

(by changing variables $S = -s$). Thus,

$$\mathcal{F}\left(\frac{1}{1+S^2}\right)(t) = 2\pi \cdot \frac{1}{2}e^{-|t|} = \pi e^{-|t|}.$$

Therefore, the Fourier transform of the function $t \mapsto \frac{1}{1+t^2}$ is $s \mapsto \pi \cdot e^{-|s|}$.

3. Let's start by calculating the convolution product. For $x > 0$:

$$f*f(x) = \int_{\mathbb{R}} e^{-\alpha(|x-y|+|y|)}\,dy = \int_{-\infty}^{0} e^{-\alpha(x-2y)}\,dy + \int_0^x e^{-\alpha x}\,dy + \int_x^{+\infty} e^{-\alpha(2y-x)}\,dy = e^{-\alpha x}\left(x + \frac{1}{\alpha}\right).$$

Since f is even, $f * f$ is also even, and we have

$$f * f(x) = e^{-\alpha|x|}\left(|x| + \frac{1}{\alpha}\right).$$

Now, the Fourier transform of $t \mapsto e^{-\alpha|t|}\left(|t| + \frac{1}{\alpha}\right)$, for $\alpha = 1$, is $s \mapsto \frac{4}{(1+s^2)^2}$, since the Fourier transform transforms the convolution product of two functions into a regular product:

$$\mathcal{F}(f \star f) = \mathcal{F}(f) \times \mathcal{F}(f) = \frac{2}{1+s^2} \times \frac{2}{1+s^2} = \frac{4}{(1+s^2)^2}.$$

Thus, $\mathcal{F}^{-1}\left(\frac{1}{(1+s^2)^2}\right) = \frac{1}{4}e^{-|t|}(|t|+1)$. We apply the Fourier inversion formula once again. The Fourier transform of $t \mapsto \frac{1}{(1+t^2)^2}$ is the function $s \mapsto \frac{\pi}{2}e^{-|s|}(|s|+1)$.

4. Note that the derivative of the function $t \mapsto \frac{1}{1+t^2}$ is $t \mapsto -\frac{2t}{(1+t^2)^2}$. Using the differentiation formulas, we deduce:

$$\mathcal{F}\left(\frac{t}{(1+t^2)^2}\right) = -\frac{1}{2}\mathcal{F}\left(\frac{d}{dt}\left(\frac{1}{1+t^2}\right)\right) = -\frac{1}{2} \cdot is\mathcal{F}\left(\frac{1}{1+t^2}\right) = -i\frac{\pi}{2}se^{-|s|}.$$

7.3 Solutions to Chapter 3 Exercises

Correction of the exercise 10 ▲

1. For d_1:

   - Non-negativity: $d_1(x,y) = \sum_{i=1}^n |x_i - y_i| \geqslant 0$, and $d_1(x,y) = 0 \iff |x_i - y_i| = 0$ for all i, which implies $x = y$.
   - Symmetry: $d_1(x,y) = \sum_{i=1}^n |x_i - y_i| = \sum_{i=1}^n |y_i - x_i| = d_1(y,x)$.
   - Triangle inequality:

   $$d_1(x,z) = \sum_{i=1}^n |x_i - z_i| \leqslant \sum_{i=1}^n |x_i - y_i| + \sum_{i=1}^n |y_i - z_i| = d_1(x,y) + d_1(y,z).$$

   For d_2:

   - $d_2(x,y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} \geqslant 0$, and $d_2(x,y) = 0 \iff x = y$.
   - $d_2(x,y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} = d_2(y,x)$.
   - By Cauchy-Schwarz,

   $$d_2(x,z) = \sqrt{\sum_{i=1}^n (x_i - z_i)^2} \leqslant \sqrt{\sum_{i=1}^n (x_i - y_i)^2} + \sqrt{\sum_{i=1}^n (y_i - z_i)^2} = d_2(x,y) + d_2(y,z).$$

   For d_∞:

   - $d_\infty(x,y) = \max\{|x_i - y_i|\} \geqslant 0$, and $d_\infty(x,y) = 0 \iff x = y$.
   - $d_\infty(x,y) = \max\{|x_i - y_i|\} = d_\infty(y,x)$.
   - $$d_\infty(x,z) \leqslant \max\{|x_i - y_i| + |y_i - z_i|\} = d_\infty(x,y) + d_\infty(y,z).$$

   Thus, d_1, d_2, and d_∞ are distances on $\mathbb{R}^n$.

2. The unit ball for each distance in $\mathbb{R}^2$ is defined as follows:

   - For d_1:

   $$B_1((0,0),1) = \{(x_1,x_2) \in \mathbb{R}^2 : d_1((0,0),(x_1,x_2)) < 1\} = \{(x_1,x_2) : |x_1| + |x_2| < 1\}.$$

   This describes a diamond shape centered at the origin.

   - For d_2:

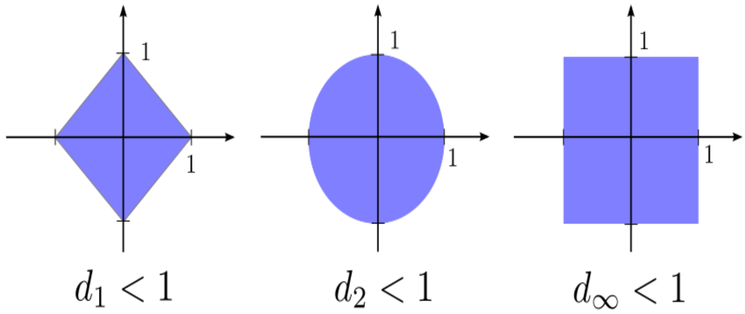
   $$B_2((0,0),1) = \{(x_1,x_2) \in \mathbb{R}^2 : d_2((0,0),(x_1,x_2)) < 1\} = \{(x_1,x_2) : x_1^2 + x_2^2 < 1\}.$$

   This describes a circular disk centered at the origin.

   - For d_∞:

   $$B_\infty((0,0),1) = \{(x_1,x_2) \in \mathbb{R}^2 : d_\infty((0,0),(x_1,x_2)) < 1\} = \{(x_1,x_2) : \max\{|x_1|,|x_2|\} < 1\}.$$

   This describes a square centered at the origin with vertices at $(1,1),(-1,1),(-1,-1),(1,-1)$.



3. To show that d_1, d_2, and d_∞ are equivalent for $n = 2$, we need to demonstrate that there exist constants $C_1, C_2 > 0$ such that for all $x, y \in \mathbb{R}^2$:

$$C_1 d_2(x,y) \leqslant d_1(x,y) \leqslant C_2 d_2(x,y)$$

and

$$C_1 d_\infty(x,y) \leqslant d_1(x,y) \leqslant C_2 d_\infty(x,y).$$

From d_2 to d_1: Using the Cauchy-Schwarz inequality:

$$d_1(x,y) = |x_1 - y_1| + |x_2 - y_2| \leqslant \sqrt{2}\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2} = \sqrt{2} d_2(x,y).$$

From d_1 to d_2: Using the fact that $\sqrt{a+b} \leqslant \sqrt{2}\max\{\sqrt{a}, \sqrt{b}\}$:

$$d_2(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2} \leqslant \sqrt{(|x_1-y_2| + |y_1-y_2|)^2} \leqslant d_1(x,y).$$

From d_∞ to d_1:

$$d_1(x,y) \leqslant 2 d_\infty(x,y).$$

From d_1 to d_∞:

$$d_\infty(x,y) \leqslant d_1(x,y).$$

Thus, we have shown that d_1, d_2, and d_∞ are equivalent distances for $n = 2$.

Correction of the exercise 11 ▲

1. For $N_1((x,y)) = |x| + |y|$:

   - Non-negativity: Clearly, $|x| \geqslant 0$ and $|y| \geqslant 0$, so $N_1((x,y)) \geqslant 0$. Moreover, $N_1((x,y)) = 0$ if and only if $|x| = 0$ and $|y| = 0$, which implies $(x,y) = (0,0)$.

   - Scalar multiplication: For $c \in \mathbb{R}$,

   $$N_1(c(x,y)) = N_1((cx,cy)) = |cx| + |cy| = |c|(|x| + |y|) = |c| N_1((x,y)).$$

   - Triangle inequality: For (x_0, y_0) and (x_1, y_1),

   $$N_1((x_0+x_1, y_0+y_1)) = |x_0+x_1| + |y_0+y_1| \leqslant |x_0| + |x_1| + |y_0| + |y_1| = N_1((x_0,y_0)) + N_1((x_1,y_1)).$$

Thus, N_1 is a norm.

For $N_2((x,y)) = \sqrt{x^2+y^2}$:

- Clearly, $N_2((x,y)) \geqslant 0$ and $N_2((x,y)) = 0$ if and only if $x = 0$ and $y = 0$.

- For $c \in \mathbb{R}$,

$$N_2(c(x,y)) = N_2((cx,cy)) = \sqrt{(cx)^2+(cy)^2} = |c|\sqrt{x^2+y^2} = |c|N_2((x,y)).$$

- Using the Cauchy-Schwarz inequality,

$$N_2((x_0+x_1,y_0+y_1))^2 = (x_0+x_1)^2 + (y_0+y_1)^2.$$

Expanding this gives:

$$= x_0^2 + x_1^2 + y_0^2 + y_1^2 + 2(x_0x_1 + y_0y_1).$$

By Cauchy-Schwarz,

$$2(x_0x_1+y_0y_1) \leqslant 2\sqrt{x_0^2+y_0^2}\sqrt{x_1^2+y_1^2} \leqslant N_2((x_0,y_0))^2 + N_2((x_1,y_1))^2.$$

Hence,

$$N_2((x_0+x_1,y_0+y_1)) \leqslant N_2((x_0,y_0)) + N_2((x_1,y_1)).$$

Thus, N_2 is a norm.

For $N_\infty((x,y)) = \max(|x|,|y|)$:

- Clearly, $N_\infty((x,y)) \geqslant 0$ and $N_\infty((x,y)) = 0$ if and only if $(x,y) = (0,0)$.

- For $c \in \mathbb{R}$,

$$N_\infty(c(x,y)) = N_\infty((cx,cy)) = \max(|cx|,|cy|) = |c|\max(|x|,|y|) = |c|N_\infty((x,y)).$$

- For (x_0,y_0) and (x_1,y_1),

$$N_\infty((x_0+x_1,y_0+y_1)) = \max(|x_0+x_1|,|y_0+y_1|) \leqslant \max(|x_0|+|x_1|,|y_0|+|y_1|) \leqslant N_\infty((x_0,y_0))+N_\infty((x_1,y_1)).$$

Thus, N_∞ is a norm.

2. We need to show:

$$\forall \alpha \in \mathbb{R}^2, \quad N_\infty(\alpha) \leqslant N_2(\alpha) \leqslant N_1(\alpha) \leqslant 2N_\infty(\alpha).$$

1. $N_\infty(\alpha) \leqslant N_2(\alpha)$: Since $N_\infty(\alpha) = \max(|x|,|y|)$, We have $|x| \leqslant \sqrt{x^2+y^2}$ and $|y| \leqslant \sqrt{x^2+y^2}$, hence $N_\infty(\alpha) \leqslant N_2(\alpha)$.

2. $N_2(\alpha) \leqslant N_1(\alpha)$: From the identity $(|x|+|y|)^2 \geqslant x^2+y^2$, we find that

$$N_2(\alpha) = \sqrt{x^2+y^2} \leqslant |x|+|y| = N_1(\alpha).$$

3. $N_1(\alpha) \leqslant 2N_\infty(\alpha)$: Since $|x| \leqslant N_\infty(\alpha)$ and $|y| \leqslant N_\infty(\alpha)$, we have

$$N_1(\alpha) = |x|+|y| \leqslant N_\infty(\alpha) + N_\infty(\alpha) = 2N_\infty(\alpha).$$

3. The norms N_1, N_2, N_∞ are equivalent if there exist constants $C_1, C_2 > 0$ such that:

$$C_1 N_1(x) \leqslant N_2(x) \leqslant C_2 N_1(x) \quad \text{and} \quad C_1 N_\infty(x) \leqslant N_2(x) \leqslant C_2 N_\infty(x).$$

From the inequalities established, we have:

$$N_\infty(\alpha) \leqslant N_2(\alpha) \leqslant N_1(\alpha) \leqslant 2N_\infty(\alpha).$$

Thus, the norms N_1, N_2, N_∞ are equivalent because they can be bounded relative to each other.

Correction of the exercise 12 ▲

1. We will use the definition of an open set to show that A is open in $\mathbb{R}^2$.

   Let $(x,y) \in A$ be given. Since $y > 0$, we will show that the open ball $B((x,y),y)$ is entirely contained within A. Specifically, we have:

$$(a,b) \in B((x,y),y) \implies (x-a)^2 + (y-b)^2 < y^2.$$

   From this inequality, we can derive:

$$(y-b)^2 < y^2 \implies |y-b| < y. \implies -y < b-y < y.$$

   From $b - y < y$, we get: $b < 2y$. From $-y < b - y$, $0 < b$. Combining these results, we have: $0 < b < 2y$. Thus, $(a,b) \in A$. This shows that $B((x,y),y) \subset A$, which demonstrates that the set A is open in $\mathbb{R}^2$.

2. The given set is the intersection $B = B_1 \cap B_2$, where

$$B_1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 4x\} \quad \text{and} \quad B_2 = \{(x,y) \in \mathbb{R}^2 : y > 0\}.$$

   Note that B_2 is open in $\mathbb{R}^2$ as shown in the previous example.

   The set B_1 can be rewritten as: $B_1 = \{(x,y) \in \mathbb{R}^2 : (x-2)^2 + y^2 < 4\} = B((2,0),2)$, which describes an open ball in $\mathbb{R}^2$ and is thus open.

   Since B is the intersection of two open sets, it follows that the set B is also open.

3. We have, $x^3 > x \iff x^3 - x > 0 \iff x(x^2-1) > 0. \iff x(x-1)(x+1) > 0.$

   The product $x(x-1)(x+1)$ is positive in the intervals $(-1,0)$ and $(1,\infty)$. Therefore, we conclude that:

$$C = (-1,0) \cup (1,\infty).$$

   Since both intervals are open, the set A is open in $\mathbb{R}$.

4. Consider the set $D = \{x \in \mathbb{R} : 0 < x < 1 \text{ and } \frac{1}{x} \notin \mathbb{Z}\}$.

   This set D can be interpreted as the interval $(0,1)$ from which the points $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ have been excluded. To express this more formally, we can write:

$$D = (0,1) \setminus \left\{ \frac{1}{n} : n \in \mathbb{Z}^+ \right\}.$$

The set D can be represented as :

$$D = \bigcup_{n=1}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n} \right).$$

Since D is the union of these open intervals, we conclude that D is open in $\mathbb{R}$.

Correction of the exercise 13 ▲

1. We have $0 \in]-\frac{1}{2}, \frac{1}{2}[\subseteq]-\frac{1}{2}, \frac{1}{2}]$, and the interval $]-\frac{1}{2}, \frac{1}{2}[$ is an open set. Thus, $]-\frac{1}{2}, \frac{1}{2}]$ is a neighborhood of 0.

   Explanation: A neighborhood of a point x is an open set that contains an interval around x. Here, since 0 is contained in the open interval $]-\frac{1}{2}, \frac{1}{2}[$, and this interval is part of the larger set $]-\frac{1}{2}, \frac{1}{2}]$, we conclude that $]-\frac{1}{2}, \frac{1}{2}]$ is a neighborhood of 0.

2. 2) and 3) Any open set containing 0 must include points both greater than and less than 0. Therefore, neither $]-1, 0]$ nor $[0, \frac{1}{2}[$ can be considered neighborhoods of 0.

   Explanation: The definition of an open set requires that for any point within it, there exists a surrounding interval that also lies within the set. The interval $]-1, 0]$ includes 0 but does not contain any points greater than 0, while $[0, \frac{1}{2}[$ contains 0 but does not include any points less than 0. Thus, neither of these intervals can serve as neighborhoods of 0.

3. The interval $]0, 1]$ does not even contain 0 and is therefore not a neighborhood of 0.

   Explanation: Since a neighborhood of 0 must include the point 0 itself, the interval $]0, 1]$ clearly cannot be a neighborhood of 0.

Correction of the exercise 14 ▲

1. The interior of $\mathbb{Q}$ is empty, that is, $\mathring{\mathbb{Q}} = \emptyset$. This means there are no open intervals in $\mathbb{R}$ that consist entirely of rational numbers. Since any open interval in $\mathbb{R}$ contains irrational numbers, it cannot be fully contained within $\mathbb{Q}$.

2. The closure of $\mathbb{Q}$ in $\mathbb{R}$ is $\overline{\mathbb{Q}} = \mathbb{R}$. This is because for any real number and any open interval around it, there will always be rational numbers present, ensuring that every point in $\mathbb{R}$ is included in the closure.

3. The interior of $[0, 1] \cap \mathbb{Q}$ is empty, that is, $\mathrm{Int}([0, 1] \cap \mathbb{Q}) = \emptyset$. Similar to $\mathbb{Q}$, there are no open sets within $[0, 1]$ that can be entirely made up of rational numbers. Any open interval in this range will also contain irrational numbers.

4. The closure of $[0, 1] \cap \mathbb{Q}$ is $\overline{[0, 1] \cap \mathbb{Q}} = [0, 1]$. This occurs because for any point in $[0, 1]$, there are rational numbers that can get arbitrarily close to it. Hence, all points in the interval from 0 to 1 are included in the closure.

5. The interior of $]0,1[\cap\mathbb{Q}$ is empty, that is, $\text{Int}(]0,1[\cap\mathbb{Q}) = \emptyset$. As with the previous cases, there are no open sets in the interval $]0,1[$ that consist solely of rational numbers. Any open interval will also contain irrationals.

6. The closure of $]0,1[\cap\mathbb{Q}$ is $\overline{]0,1[\cap\mathbb{Q}} = [0,1]$. Every point in $[0,1]$ can be reached by rational numbers that lie within the interval $]0,1[$. Therefore, the closure includes all points in the interval from 0 to 1.

7.4 Solutions to Chapter 4 Exercises

Correction of the exercise 15 ▲

Domains of Definition :

1. $f_1(x,y) = \frac{\sqrt{-y+x^2}}{\sqrt{y}}$

   Conditions:

   - $-y + x^2 \geqslant 0 \quad \Rightarrow \quad y \leqslant x^2$
   - $y > 0$

   Domain: $\{(x,y) \in \mathbb{R}^2 \mid 0 < y \leqslant x^2\}$

2. $f_2(x,y) = \frac{\ln(y)}{\sqrt{x-y}}$

   Conditions:

   - $y > 0$
   - $x - y > 0 \quad \Rightarrow \quad x > y$

   Domain: $\{(x,y) \in \mathbb{R}^2 \mid y > 0 \text{ and } x > y\}$

3. $f_3(x,y) = \frac{\sqrt{4-x^2-y^2}}{\sqrt{x^2+y^2-1}}$

   Conditions:

   - $4 - x^2 - y^2 \geqslant 0 \quad \Rightarrow \quad x^2 + y^2 \leqslant 4$
   - $x^2 + y^2 - 1 > 0 \quad \Rightarrow \quad x^2 + y^2 > 1$

   Domain: $\{(x,y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 \leqslant 4\}$

4. $f_4(x,y) = \ln(x - y^2)$

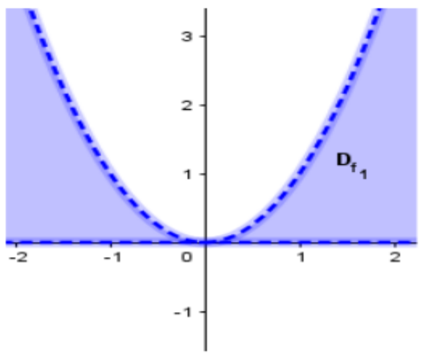
   Conditions:

   - $x - y^2 > 0 \quad \Rightarrow \quad x > y^2$

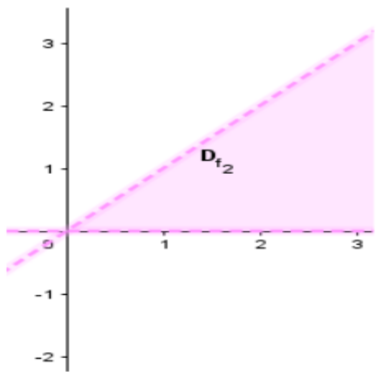
   Domain: $\{(x,y) \in \mathbb{R}^2 \mid x > y^2\}$

Graphical Representation :

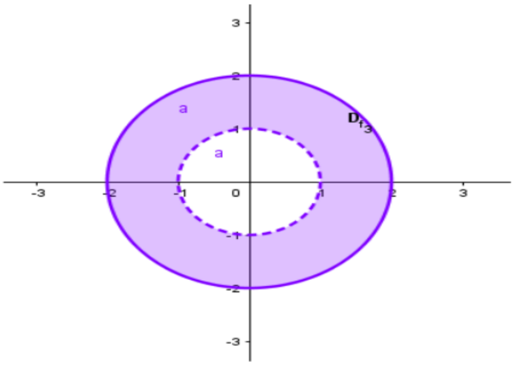
1. **For** f_1: Plot the parabola $y = x^2$ and shade the region above the x-axis (excluding the curve itself).



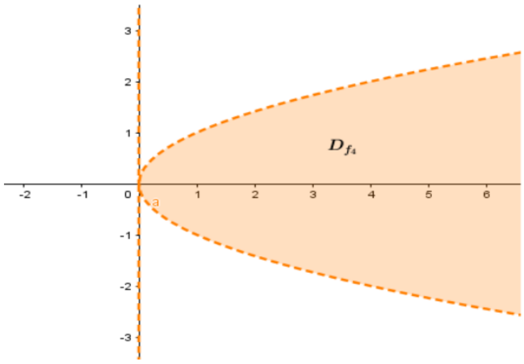
2. **For** f_2: Shade the area above the line $y = 0$ and to the right of the line $x = y$.



3. **For** f_3: Draw the circle with radius 2 (boundary for $x^2 + y^2 = 4$) and the circle with radius 1 (boundary for $x^2 + y^2 = 1$). Shade the annular region between these circles.



4. **For** f_4: Shade the area to the right of the parabola $x = y^2$.



Correction of the exercise 16 ▲

1. $x^2 + y^2 + 1$ approaches 1, and $\frac{\sin(x)}{x}$ approaches 1 as (x, y) approaches $(0, 0)$. Thus, f_1 approaches 1 as (x, y) approaches $(0, 0)$.

2. For $(x, y) \in \mathbb{R}^2$, $|x^4 + y^4| = (x^2 + y^2)^2 - 2x^2y^2 \leqslant (x^2 + y^2)^2 + 2 \times \left(\frac{1}{2}(x^2 + y^2)\right)^2 = \frac{3}{2}(x^2 + y^2)^2$, and thus for $(x, y) \neq (0, 0)$,
$$|f_2(x, y)| = \frac{|x^4 + y^4|}{x^2 + y^2} \leqslant \frac{3}{2}(x^2 + y^2).$$
Since $\lim_{(x,y)\to(0,0)} \frac{3}{2}(x^2 + y^2) = 0$, it follows that $\lim_{(x,y)\to(0,0)} f_2(x, y) = 0$ (by the squeeze theorem).

3. For all $(x, y) \in \mathbb{R}^2$, $x^2 - 2|xy| + y^2 = (|x| - |y|)^2 > 0$ and thus $|xy| \leqslant \frac{1}{2}(x^2 + y^2)$. Consequently, for $(x, y) \neq (0, 0)$,
$$|f_3(x, y)| = \frac{x^2y^2}{x^2 + y^2} \leqslant \frac{1}{4}(x^2 + y^2).$$
Since $\lim_{(x,y)\to(0,0)} \frac{1}{4}(x^2 + y^2) = 0$, it follows that $\lim_{(x,y)\to(0,0)} f_3(x, y) = 0$ (by the squeeze theorem).

4. We have
$$|f_4(x, y)| \leqslant \frac{|\sin(x^2)|}{x^2} \cdot \frac{x^2}{\sqrt{x^2 + y^2}} + \frac{|\sin(y^2)|}{y^2} \cdot \frac{y^2}{\sqrt{x^2 + y^2}}.$$
However, $\frac{\sin(x^2)}{x^2} \to 1$ as $(x, y) \to (0, 0)$, and $\frac{x^2}{\sqrt{x^2+y^2}} \leqslant \frac{(x^2+y^2)}{\sqrt{x^2+y^2}} \leqslant \sqrt{x^2 + y^2}$. Thus, $f_4(x, y)$ approaches 0 as (x, y) approaches $(0, 0)$.

5. We know that $\frac{1-\cos(t)}{t^2} \to \frac{1}{2}$ as $t \to 0$. Now, we can write $f_5(x, y) = x \cdot \frac{1-\cos(xy)}{(xy)^2}$. It follows that $\frac{1-\cos(xy)}{(xy)^2}$ approaches $\frac{1}{2}$ as (x, y) approaches $(0, 0)$. Thus, $f_5(x, y)$ approaches 0 as (x, y) approaches $(0, 0)$.

6. Using the fact that $\lim_{x\to 0}\frac{e^x-1}{x}=1$, we can simplify the expression:

$$\lim_{(x,y)\to(0,0)}\frac{e^{xy}-1}{e^x-1}=\lim_{(x,y)\to(0,0)}\frac{xy\frac{e^{xy}-1}{xy}}{x\frac{e^x-1}{x}}=0.$$

Therefore, $f_6(x,y)$ approaches 0 as (x,y) approaches $(0,0)$.

Correction of the exercise 17 ▲

1. - If we take the path $\gamma_1(t)=(t,t)$ (since $\gamma_1(0)=(x_0,y_0)=(0,0)$), then $(f\circ\gamma_1)(t)=f(t,t)=\frac{t^2}{2t^2}=\frac{1}{2}$. Thus, $\lim_{t\to 0}f(t,t)=\frac{1}{2}$.
   - If we take the path $\gamma_2(t)=(0,t)$ (since $\gamma_2(0)=(x_0,y_0)=(0,0)$), then $(f\circ\gamma_2)(t)=f(0,t)=0$. Thus, $\lim_{t\to 0}f(0,t)=0$.

   Since $\lim_{t\to 0}f(t,t)\neq\lim_{t\to 0}f(0,t)$, f does not have a limit at $(0,0)$.

2. - If we take the path $\gamma_1(t)=(t,t)$ (since $\gamma_1(0)=(x_0,y_0)=(0,0)$), then $(f\circ\gamma_1)(t)=f(t,t)=\frac{t}{1+t^2}$. Thus, $\lim_{t\to 0}f(t,t)=0$.
   - If we take the path $\gamma_2(t)=(t^2,t)$ (since $\gamma_2(0)=(x_0,y_0)=(0,0)$), then $(f\circ\gamma_2)(t)=f(t^2,t)=\frac{1}{t^2+1}$. Thus, $\lim_{t\to 0}f(t^2,t)=1$.

   Since $\lim_{t\to 0}f(t,t)\neq\lim_{t\to 0}f(t^2,t)$, f does not have a limit at $(0,0)$.

3. If we take the path $\gamma_1(t)=(t,t)$ (since $\gamma_1(0)=(x_0,y_0)=(0,0)$), then $(f\circ\gamma_1)(t)=f(t,t)=\frac{1}{t}$. Thus, $\lim_{t\to 0}f(t,t)=\infty$. Therefore, f does not have a limit at $(0,0)$.

Correction of the exercise 18 ▲

1. By switching to polar coordinates, we have:

$$\begin{cases}x=r\cos\theta\\ y=r\sin\theta\\ r>0,\theta\in[0,2\pi]\end{cases}$$

For the function $f(x,y)=\frac{xy}{\sqrt{x^2+y^2}}$, we get:

$$f(r\cos\theta,r\sin\theta)=\frac{r^2\cos\theta\sin\theta}{\sqrt{r^2(\cos^2\theta+\sin^2\theta)}}=\frac{r^2\cos\theta\sin\theta}{r}=r\cos\theta\sin\theta.$$

The absolute value of $f(r\cos\theta,r\sin\theta)$ is:

$$|f(r\cos\theta,r\sin\theta)|=|r\cos\theta\sin\theta|\leqslant r.$$

As r approaches 0, the limit of $f(r\cos\theta,r\sin\theta)$ is 0. Thus,

$$\lim_{(x,y)\to(0,0)}\frac{xy}{\sqrt{x^2+y^2}}=0.$$

2. By switching to polar coordinates, we have:

$$\begin{cases} x = 1 + r\cos\theta \\ y = r\sin\theta \\ r > 0, \theta \in [0, 2\pi] \end{cases}$$

For the function $f(x,y) = \frac{y^3}{(x-1)^2+y^2}$, we get:

$$f(1+r\cos\theta, r\sin\theta) = \frac{(r\sin\theta)^3}{((1+r\cos\theta)-1)^2 + (r\sin\theta)^2} = \frac{r^3\sin^3\theta}{(r\cos\theta)^2 + (r\sin\theta)^2} = r\sin^3\theta.$$

The absolute value of $f(1+r\cos\theta, r\sin\theta)$ is:

$$|f(1+r\cos\theta, r\sin\theta)| = |r\sin^3\theta| \leqslant r.$$

As r approaches 0, the limit of $f(1+r\cos\theta, r\sin\theta)$ is 0. Thus,

$$\lim_{(x,y)\to(1,0)} \frac{y^3}{(x-1)^2+y^2} = 0.$$

3. By using polar coordinates, we set $x = r\cos\theta$ and $y = r\sin\theta$ with $\theta \in [0, 2\pi]$ and $r > 0$. Then the function $f(r\cos\theta, r\sin\theta)$ can be expressed as:

$$f(r\cos\theta, r\sin\theta) = \frac{\cos\theta\sin\theta}{\cos\theta\sin\theta + 1}.$$

Thus,

$$\lim_{r\to 0} f(r\cos\theta, r\sin\theta) = \frac{\cos\theta\sin\theta}{\cos\theta\sin\theta + 1}.$$

The limit depends on θ, therefore it does not exist.

Correction of the exercise 19 ▲

The function f is continuous on $\mathbb{R}^2 - \{(0,0)\}$ because it is the quotient of continuous functions whose denominator does not vanish. It is also continuous at $(0,0)$ because by using polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ with $\theta \in [0, 2\pi]$ and $r > 0$, we have:

$$f(r\cos\theta, r\sin\theta) = \frac{r^2\cos\theta\sin\theta \cdot r^2(\cos^2\theta - \sin^2\theta)}{r^2(\cos^2\theta + \sin^2\theta)} = r^2\cos\theta\sin\theta(\cos^2\theta - \sin^2\theta).$$

And

$$|f(r\cos\theta, r\sin\theta)| = \left|r^2\cos\theta\sin\theta(\cos^2\theta - \sin^2\theta)\right| \leqslant 2r^2.$$

Thus, as $r \to 0$, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$, so $\lim_{(x,y)\to(0,0)} xy\frac{x^2-y^2}{x^2+y^2} = 0 = f(0,0)$. In conclusion, f is continuous on $\mathbb{R}^2$.

7.5 Solutions to Chapter 5 Exercises

Correction of the exercise 20 ▲

We define $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ by

$$f(x,y) = \frac{x^2}{(x^2+y^2)^{\frac{3}{4}}}.$$

To justify the extension of f to a continuous function on $\mathbb{R}^2$, we need to study the limit of $f(x,y)$ as (x,y) approaches $(0,0)$.
Using polar coordinates, we can rewrite $f(x,y)$ as follows:

$$f(x,y) = \frac{r^2\cos^2(\theta)}{r^{\frac{3}{2}}}$$

where $r = \sqrt{x^2+y^2}$ is the distance from (x,y) to the origin and θ is the angle that the vector (x,y) makes with the x-axis.
As (x,y) approaches $(0,0)$, we have $r \to 0$ and thus $\frac{r^2\cos^2(\theta)}{r^{\frac{3}{2}}} \to 0$. Therefore, we can extend f to a continuous function on $\mathbb{R}^2$ by defining $f(0,0) = 0$.
Now let us examine the existence of partial derivatives at $(0,0)$ for this extension.
The partial derivative with respect to x at $(0,0)$ is given by:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}.$$

Using the definition of f:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{\frac{h^2}{(h^2+0^2)^{\frac{3}{4}}} - 0}{h} = \lim_{h\to 0} \frac{h^2}{h(h^2)^{\frac{3}{4}}} = \lim_{h\to 0} \frac{h^2}{h^{\frac{11}{4}}} = \lim_{h\to 0} \frac{1}{h^{\frac{3}{4}}}.$$

This limit does not exist, so the partial derivative with respect to x does not exist at $(0,0)$.
Since $f(0,h) = 0$ for all real h, f has a partial derivative with respect to the second variable at $(0,0)$ which equals $\frac{\partial f}{\partial y} = 0$.
In conclusion, the extension of f to a continuous function on $\mathbb{R}^2$ is possible by defining $f(0,0) = 0$. However, the partial derivative with respect to x does not exist for this extension.

Correction of the exercise 21 ▲

1. We have

$$\frac{\partial f}{\partial x}(1,2) = -4, \quad \frac{\partial f}{\partial y}(1,2) = 1.$$

   Thus, f is differentiable at $(1,2)$ if and only if

$$I = \lim_{(h,k)\to(0,0)} \frac{f(1+h,2+k) - f(1,2) - h\frac{\partial f}{\partial x}(1,2) - k\frac{\partial f}{\partial y}(1,2)}{\sqrt{h^2+k^2}} = 0.$$

Therefore,

$$I = \lim_{(h,k)\to(0,0)} \frac{(1+h)(2+k) - 3(1+h)^2 + 1 + 4h - k}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{kh - 3h^2}{\sqrt{h^2+k^2}}.$$

Using polar coordinates, we set $h = r\cos\theta$ and $k = r\sin\theta$ with $\theta \in [0, 2\pi[$ and $r > 0$.

$$f(r\sin\theta, r\cos\theta) = r^2 \sin\theta\cos\theta - 3r^2\cos^2\theta = r(\sin\theta\cos\theta - 3\cos^2\theta),$$

$$|f(r\sin\theta, r\cos\theta)| = |r(\sin\theta\cos\theta - 3\cos^2\theta)| \leqslant 4r.$$

Thus, $\lim_{r\to 0} f(r\sin\theta, r\cos\theta) = 0$. Therefore,

$$\lim_{(h,k)\to(0,0)} \frac{f(1+h, 2+k) - f(1,2) - h\frac{\partial f}{\partial x}(1,2) - k\frac{\partial f}{\partial y}(1,2)}{\sqrt{h^2+k^2}} = 0.$$

The function f_1 is differentiable at $(1,2)$.

2. We have

$$\lim_{(h,k)\to(0,0)} \frac{f(4+h, 1+k) - f(4,1) - h\frac{\partial f}{\partial x}(4,1) - k\frac{\partial f}{\partial y}(4,1)}{\sqrt{h^2+k^2}} = \lim_{(0,0)} \frac{(4+h)(1+k) - 4 - h - 4k}{\sqrt{h^2+k^2}}$$

$$= \lim_{(0,0)} \frac{hk}{\sqrt{h^2+k^2}} = 0.$$

(Using polar coordinates). Thus, the function f_2 is differentiable at $(4,1)$.

Correction of the exercise 22 ▲

1. Let $v = (h,k) \neq (0,0)$.

$$\lim_{t\to 0} \frac{f(t\cdot v) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(0,tk)}{t} = k, \quad \text{if } h = 0$$

$$\text{and} \quad \lim_{t\to 0} \frac{f(t\cdot v) - f(0,0)}{t} = 0, \quad \text{if } h \neq 0.$$

Thus, $D_v f(0,0) = k$ if $h = 0$, and $D_v f(0,0) = 0$ if $h \neq 0$.

2. With $v = (1,0)$, we have $\frac{\partial f}{\partial x}(0,0) = 0$, and with $v = (0,1)$, we have $\frac{\partial f}{\partial y}(0,0) = 1$. The candidate for the differential at $(0,0)$ is thus $\ell(h,k) = k$. However, the expression

$$\varepsilon(h,k) = \frac{f(h,k) - f(0,0) - \ell(h,k)}{\sqrt{h^2+k^2}} = \frac{k^3 - k\sqrt{h^2+k^4}}{\sqrt{h^2+k^2}\sqrt{h^2+k^4}}$$

does not tend to 0 as $(h,k) \to (0,0)$, because $\lim_{t\to 0^+} \varepsilon(t,t) = -\frac{1}{\sqrt{2}}$. Therefore, f is not differentiable at the point $(0,0)$.

Correction of the exercise 23 ▲

1. The function f has partial derivatives on $\mathbb{R}^2 - \{(0,0)\}$ (since f is the quotient of functions that have partial derivatives where the denominator does not vanish for $(x,y) \in \mathbb{R}^2 - \{(0,0)\}$):

$$\frac{\partial f}{\partial x}(x,y) = \frac{x^4 + 3x^2y^2 + 2xy^3}{(x^2+y^2)^2}, \quad (x,y) \neq (0,0),$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{-y^4 - 3y^2x^2 - 2yx^3}{(x^2+y^2)^2}, \quad (x,y) \neq (0,0).$$

For $(x,y) = (0,0)$, we have

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 1, \quad \frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = -1.$$

Thus, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at every point in $\mathbb{R}^2$.

2. The function f is continuous on $\mathbb{R}^2 - \{(0,0)\}$ (since f is the quotient of continuous functions). For $(x,y) = (0,0)$, we use polar coordinates by setting $x = r\cos\theta$ and $y = r\sin\theta$ with $\theta \in [0, 2\pi[$ and $r > 0$. Then,

$$f(r\sin\theta, r\cos\theta) = \frac{r^3(\cos^3\theta - \sin^3\theta)}{r^2(\cos^2\theta + \sin^2\theta)^2} = \frac{r(\cos^3\theta - \sin^3\theta)}{(\cos^2\theta + \sin^2\theta)^2} = r(\cos^3\theta - \sin^3\theta).$$

We have $|f(r\sin\theta, r\cos\theta)| = |r(\cos^3\theta - \sin^3\theta)| \leqslant 2r$, so $\lim_{r\to 0} f(r\sin\theta, r\cos\theta) = 0$. Thus,

$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$, which shows that f is continuous at $(0,0)$.

3. The partial derivatives $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ exist, and f is continuous at $(0,0)$. The function f is differentiable at $(0,0)$ if and only if

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - h\frac{\partial f}{\partial x}(0,0) - k\frac{\partial f}{\partial y}(0,0)}{\sqrt{h^2+k^2}} = 0.$$

We have

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - h\frac{\partial f}{\partial x}(0,0) - k\frac{\partial f}{\partial y}(0,0)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{h^3 - k^3}{(h^2+k^2)^{\frac{3}{2}}} + \frac{k - k}{\sqrt{h^2+k^2}},$$

by switching to polar coordinates $h = r\cos\theta$ and $k = r\sin\theta$ where $\theta \in [0, 2\pi[$ and $r > 0$, we then have

$$f(r\sin\theta, r\cos\theta) = \cos^3\theta - \sin^3\theta - \cos\theta + \sin\theta.$$

The limit of f depends on θ, thus f is not differentiable at $(0,0)$.

Correction of the exercise 24 ▲

1. Study the continuity of f on $\mathbb{R}^2$: The function f is clearly continuous at every point other than $(0,0)$ because it is defined by a composition of continuous functions. To check the continuity at $(0,0)$, we need to calculate the limit

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} (x^2+y^2)^3 \cos\left(\frac{1}{x^2+y^2}\right)$$

Using the property $|\cos\theta| \leqslant 1$, we can establish the following upper bound:

$$|(x^2+y^2)^3 \cos\left(\frac{1}{x^2+y^2}\right)| \leqslant |(x^2+y^2)^3|$$

This bound approaches zero as (x,y) approaches $(0,0)$. Therefore,

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0).$$

Thus, f is continuous on $\mathbb{R}^2$.

2. For $(x,y) \neq (0,0)$, we have

$$\nabla f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} 6x(x^2+y^2)^2\cos\left(\frac{1}{x^2+y^2}\right) + 2x(x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right) \\ 6y(x^2+y^2)^2\cos\left(\frac{1}{x^2+y^2}\right) + 2y(x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right) \end{pmatrix}.$$

For $(x,y) = (0,0)$, we have

$$\nabla f(0,0) = \text{grad } f(0,0) = \begin{pmatrix} \frac{\partial f}{\partial x}(0,0) \\ \frac{\partial f}{\partial y}(0,0) \end{pmatrix} = \begin{pmatrix} \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} \\ \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

3. Let's check the continuity of the partial derivatives of f on $\mathbb{R}^2$. We have

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} 6x(x^2+y^2)^2\cos\left(\frac{1}{x^2+y^2}\right) + 2x(x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$$

So, $\lim_{(0,0)} \frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial x}(0,0)$ (Using polar coordinates), and

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} 6y(x^2+y^2)^2\cos\left(\frac{1}{x^2+y^2}\right) + 2y(x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Thus, $\lim_{(0,0)} \frac{\partial f}{\partial y}(x,y) = 0 = \frac{\partial f}{\partial y}(0,0)$. The function f is therefore of class $C^1(\mathbb{R}^2)$.

4. Since the function f is of class C^1 on $\mathbb{R}^2$, it is differentiable on $\mathbb{R}^2$.

Correction of the exercise 25 ▲

1. Let γ be the parametrized arc given by

$$t \mapsto \left(\frac{t^2-1}{2}, t\right), \quad t \text{ varying from } -1 \text{ to } 1.$$

Then, we compute the integral:

$$\int_\gamma \alpha = \int_{-1}^{1} \left(\frac{\frac{t^2-1}{2}}{\left(\frac{t^2-1}{2}\right)^2 + t^2} t + \frac{t}{\left(\frac{t^2-1}{2}\right)^2 + t^2} \right) dt = 0 \quad \text{(odd function)}.$$

Thus,

$$\int_\gamma \alpha = 2\ln 2.$$

2. For the second case, we have:

$$\int_\gamma \alpha = \int_0^{2\pi} \Big((\cos t - \sin^3 t)(-\sin t) + \cos^3 t(\cos t) \Big) dt.$$

This simplifies to:

$$\int_0^{2\pi} \Big(\cos^4 t + \sin^4 t - \cos t \sin t \Big) dt = \int_0^{2\pi} \Big((\cos^2 t + \sin^2 t)^2 - 2\cos^2 t \sin^2 t - \cos t \sin t \Big) dt.$$

Further simplifying, we find:

$$\int_0^{2\pi} \left(1 - \frac{1}{2}\sin(2t) - \frac{1}{4}(1 - \cos(4t)) \right) dt = 2\pi\left(1 - \frac{1}{4}\right) = \frac{3\pi}{2}.$$

Thus,

$$\int_\gamma \alpha = \frac{3\pi}{2}.$$

3. In the third case, we have:

$$\int_\gamma \alpha = \int_0^{\frac{\pi}{2}} (\cos t \sin t \cos t \sin t)(-\sin t) dt = -\int_0^{\frac{\pi}{2}} \cos^2 t \sin^3 t \, dt.$$

This can be rewritten as:

$$-\int_0^{\frac{\pi}{2}} (-\cos^2 t \sin t + \cos^4 t \sin t) dt = \left[\frac{\cos^3 t}{3} - \frac{\cos^5 t}{5} \right]_0^{\frac{\pi}{2}} = -\frac{1}{3} + \frac{1}{5} = -\frac{2}{15}.$$

Thus,

$$\int_\gamma \alpha = -\frac{2}{15}.$$

4. Consider the circle γ centered at (a,b) with radius $R > 0$, traversed once in the counterclockwise direction. Alternatively, we can consider the parametrized arc γ given by
$$t \mapsto (a + R\cos t, b + R\sin t), \quad t \text{ varying from } 0 \text{ to } 2\pi.$$
We compute the integral:
$$\int_\gamma \alpha = \int_0^{2\pi} \Big((b + R\sin t)^2(-R\sin t) + (a + R\cos t)^2(R\cos t)\Big)dt.$$
$$= R\int_0^{2\pi} \Big((b + R\sin t)^2(-\sin t) + (a + R\cos t)^2 \cos t\Big)dt.$$
$$= R\int_0^{2\pi} \Big(a\cos t - b\sin t + 2aR\cos^2 t - 2bR\sin^2 t + R^2(\cos^3 t - \sin^3 t)\Big)dt$$
$$= R^2\int_0^{2\pi} \Big(2a\cos^2 t - 2b\sin^2 t + R(\cos^3 t - \sin^3 t)\Big)dt.$$
$$= R^2\int_0^{2\pi} \Big(a(1 + \cos t) - b(1 - \cos t) + R(\cos t - \sin t)(\cos^2 t + \cos t\sin t + \sin^2 t)\Big)dt$$
$$= R^2\int_0^{2\pi} (a - b + R(\cos t - \sin t)(1 + \cos t\sin t))\,dt$$
$$= R^2\left(2\pi(b - a) + \int_0^{2\pi} R(\cos t - \sin t + \cos^2 t\sin t - \sin^2 t\cos t)dt\right)$$
$$= 2\pi R^2(b - a).$$

Correction of the exercise 26 ▲

1. Since the components $f_1(x,y) = x^2 + x\cos(y)$, $f_2(x,y) = e^{x-y}$, and $f_3 = y^3x$ are differentiable at every point (x,y) in $\mathbb{R}^2$, it follows that F is differentiable at every point (x,y) in $\mathbb{R}^2$.

2. Let's write the Jacobian matrix J_F at every point in $\mathbb{R}^2$:
$$J_F(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \\ \frac{\partial f_3}{\partial x}(x,y) & \frac{\partial f_3}{\partial y}(x,y) \end{bmatrix} = \begin{bmatrix} 2x + \cos(y) & -x\sin(y) \\ e^{x-y} & -e^{x-y} \\ y^3 & 3y^2x \end{bmatrix}$$

3. Since F is differentiable, the differential of F is given by:
$$DF(x,y)(h) = J_F(x,y) \times h = \begin{bmatrix} 2x + \cos(y) & -x\sin(y) \\ e^{x-y} & -e^{x-y} \\ y^3 & 3y^2x \end{bmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$
$$DF(x,y)(h) = (Df_1(x,y)(h), Df_2(x,y)(h), Df_3(x,y)(h))$$

where

$$Df_1(x,y)(h) = \frac{\partial f_1}{\partial x}(x,y)\cdot h_1 + \frac{\partial f_1}{\partial y}(x,y)\cdot h_2 = (2x+\cos(y))\cdot h_1 - x\sin(y)\cdot h_2$$

$$Df_2(x,y)(h) = \frac{\partial f_2}{\partial x}(x,y)\cdot h_1 + \frac{\partial f_2}{\partial y}(x,y)\cdot h_2 = e^{x-y}\cdot h_1 - e^{x-y}\cdot h_2$$

$$Df_3(x,y)(h) = \frac{\partial f_3}{\partial x}(x,y)\cdot h_1 + \frac{\partial f_3}{\partial y}(x,y)\cdot h_2 = y^3\cdot h_1 + 3y^2x\cdot h_2$$

Correction of the exercise 27 ▲

1. Since $f(x,y) = \sin(x^2-y^2)$ and the components $g_1(x,y) = x+y$ and $g_2(x,y) = x-y$ are C^1 functions at every point (x,y) in $\mathbb{R}^2$, it follows that f and g are differentiable at every point (x,y) in $\mathbb{R}^2$.

2. Let's compute the partial derivatives of $f\circ g$ and the differential of $f\circ g$ at the point (x,y). First, we find $f\circ g$, where $f(x,y) = \sin(x^2-y^2)$ and $g(x,y) = (x+y, x-y)$:

$$f\circ g(x,y) = f(g(x,y)) = f(x+y, x-y) = \sin((x+y)^2-(x-y)^2) = \sin(4xy).$$

It follows that the differential of $f\circ g$ at the point (x,y) is given by:

$$D(f\circ g)(x,y)(h_1,h_2) = \frac{\partial}{\partial x}(f\circ g)(x,y)\cdot h_1 + \frac{\partial}{\partial y}(f\circ g)(x,y)\cdot h_2 = 4y\cos(4xy)\cdot h_1 + 4x\cos(4xy)\cdot h_2.$$

3. Let's compute the Jacobian matrices of f and g at the point (x,y):

$$J_f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x}(x,y) & \frac{\partial f}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} 2x\cos(x^2-y^2) & -2y\cos(x^2-y^2) \end{pmatrix}.$$

Setting $g(x,y) = (g_1(x,y), g_2(x,y))$, we find:

$$J_g(x,y) = \begin{bmatrix} \frac{\partial g_1}{\partial x}(x,y) & \frac{\partial g_1}{\partial y}(x,y) \\ \frac{\partial g_2}{\partial x}(x,y) & \frac{\partial g_2}{\partial y}(x,y) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

4. Now let's find the Jacobian matrix of $f\circ g$ at the point (x,y) by applying the chain rule:

$$J_{f\circ g}(x,y) = J_f(g(x,y))\cdot J_g(x,y) = \begin{bmatrix} 2(x+y)\cos(4xy) & -2(x-y)\cos(4xy) \end{bmatrix}\cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

This simplifies to:

$$= \begin{bmatrix} 4y\cos(4xy) & 4x\cos(4xy) \end{bmatrix}.$$

Thus,

$$D(f\circ g)(x,y)(h_1,h_2) = J_{f\circ g}(x,y)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = 4y\cos(4xy)\cdot h_1 + 4x\cos(4xy)\cdot h_2.$$

7.6 Solutions to Chapter 6 Exercises

Correction of the exercise 28 ▲

We have

$$f(x,y) = \begin{cases} \frac{xy^2}{x+y} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

To show that f is of class C^2 at $(0,0)$, we need to demonstrate that f is of class C^1 at $(0,0)$ (its partial derivatives exist and are continuous) and that its partial derivatives are of class C^1 (its second derivatives exist and are continuous).

Existence of first partial derivatives at $(0,0)$:

$$\lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0 = \frac{\partial f}{\partial x}(0,0).$$

$$\lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0 = \frac{\partial f}{\partial y}(0,0).$$

For $(x,y) \neq (0,0)$:

$$\frac{\partial f}{\partial x}(x,y) = \frac{y^2(x+y)-xy^2}{(x+y)^2} = \frac{y^3}{(x+y)^2}.$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{y^2(x+y)-xy^2}{(x+y)^2} = \frac{2x^2y+xy^2}{(x+y)^2}.$$

Existence of second derivatives at $(0,0)$:

$$\lim_{h\to 0} \frac{\frac{\partial f}{\partial x}(0,h)-\frac{\partial f}{\partial x}(0,0)}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1 = \frac{\partial^2 f}{\partial y \partial x}(0,0).$$

$$\lim_{h\to 0} \frac{\frac{\partial f}{\partial y}(h,0)-\frac{\partial f}{\partial y}(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0 = \frac{\partial^2 f}{\partial x \partial y}(0,0).$$

Since $\frac{\partial^2 f}{\partial y\partial x}(0,0) = 1 \neq 0 = \frac{\partial^2 f}{\partial x\partial y}(0,0)$, f is not of class C^2, according to Schwarz's theorem.

Correction of the exercise 29 ▲

1. The critical points of f:

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{cases} \iff \begin{cases} (x^2+2x-y)e^{x-y} = 0 \\ (y-x^2-1)e^{x-y} = 0 \end{cases} \iff \begin{cases} x^2+2x-y = 0 \\ y-x^2-1 = 0. \end{cases}$$

Thus, the only critical point of f is

$$\begin{cases} x_0 = \frac{1}{2}, \\ y_0 = \frac{5}{4}. \end{cases}$$

The Hessian matrix of f is

$$H_f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x\partial y}(x,y) \\ \frac{\partial^2 f}{\partial y\partial x}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix} = \begin{pmatrix} (x^2+4x-y+2)e^{x-y} & (-x^2-2x+y-1)e^{x-y} \\ (-x^2-2x+y-1)e^{x-y} & (x^2-y+2)e^{x-y} \end{pmatrix}$$

$$H_f\left(\frac{1}{2},\frac{5}{4}\right)=\begin{pmatrix}3e^{-\frac{3}{4}} & -e^{-\frac{3}{4}}\\ -e^{-\frac{3}{4}} & e^{-\frac{3}{4}}\end{pmatrix}=\begin{pmatrix}r & s\\ s & t\end{pmatrix}=rt-s^2=2e^{-\frac{3}{2}}>0$$

Thus, $\left(\frac{1}{2},\frac{5}{4}\right)$ is a local extremum of f. Since $r=\frac{\partial^2 f}{\partial x^2}\left(\frac{1}{2},\frac{5}{4}\right)=3e^{-\frac{3}{4}}>0$, $\left(\frac{1}{2},\frac{5}{4}\right)$ is a local minimum for f.

2. The critical points of g:

$$\begin{cases}\frac{\partial g}{\partial x}(x,y)=0\\ \frac{\partial g}{\partial y}(x,y)=0\end{cases}\iff\begin{cases}3x^2+12x-12y+9=0\\ 6y-12x=0\end{cases}$$

$$\Rightarrow\quad\begin{cases}y=2x\\ x^2-4x+3=0\end{cases}\quad\Rightarrow(x,y)\in\{(1,2),(3,6)\}.$$

To study the nature of the critical points $(1,2)$ and $(3,6)$, we calculate the second derivatives:

$$\begin{cases}\frac{\partial^2 g}{\partial x^2}=6x+12\\ \frac{\partial^2 g}{\partial y^2}=6\\ \frac{\partial^2 g}{\partial x\partial y}=-12\end{cases}$$

Thus, for the point $(1,2)$, we have

$$\begin{cases}\frac{\partial^2 g}{\partial x^2}(1,2)=18\\ \frac{\partial^2 g}{\partial y^2}(1,2)=6\\ \frac{\partial^2 g}{\partial x\partial y}(1,2)=-12\end{cases}\Rightarrow H_f(1,2)=\begin{pmatrix}18 & -12\\ -12 & 6\end{pmatrix}=\begin{pmatrix}r & s\\ s & t\end{pmatrix}.$$

$$|H_f(1,2)|=rt-s^2=-36<0$$

we conclude that the point $(1,2)$ is a saddle point for f; it is neither a local maximum nor a local minimum.
For the critical point $(3,6)$, $|H_f(3,6)|=rt-s^2=216>0$ and $r=\frac{\partial^2 f}{\partial x^2}(3,6)=30>0$. Thus, $(3,6)$ is a local minimum for f.

3. h is of class C^2 on $\mathbb{R}^2$.
The critical points of h:

$$\begin{cases}\frac{\partial h}{\partial x}(x,y)=9x^2-1=0\\ \frac{\partial h}{\partial y}(x,y)=9y^2-1=0\end{cases}$$

The system thus has 4 solutions which are the critical points of f:

$$\begin{cases}a_1=\left(\frac{1}{3},\frac{1}{3}\right)\\ a_2=\left(\frac{1}{3},-\frac{1}{3}\right)\\ a_3=\left(-\frac{1}{3},\frac{1}{3}\right)\\ a_4=\left(-\frac{1}{3},-\frac{1}{3}\right)\end{cases}$$

To determine the nature of the critical points, we calculate the second derivatives:

$$\frac{\partial^2 f}{\partial x^2}(x,y)=18x,\quad\frac{\partial^2 f}{\partial y^2}(x,y)=18y,\quad\frac{\partial^2 f}{\partial x\partial y}(x,y)=0$$

- At $a_1 = \left(\frac{1}{3}, \frac{1}{3}\right)$, we have $rt - s^2 = 36 > 0$ and $r = 6 > 0$, so f has a local minimum at a_1.
- At a_2 and a_3, we have $rt - s^2 = -36 < 0$, so f has no extremum at either of these two points.
- At a_4, we have $rt - s^2 = 36 > 0$ and $r = -6 < 0$, so f has a local maximum at a_4.

Correction of the exercise 30 ▲

1. The first-order partial derivatives of f are $\frac{\partial f}{\partial x}(x,y) = 4x^3 - 8(x-y)$ and $\frac{\partial f}{\partial y}(x,y) = 4y^3 + 8(x-y)$. The critical points are solutions to the system

$$\begin{cases} 4x^3 = 8(x-y) \\ -4y^3 = 8(x-y) \end{cases}$$

From this, we deduce that $x^3 = -y^3 = (-y)^3$. Since the cube function is injective, this gives us $x = -y$. Substituting this back into the first equation, we find $4x^3 = 16x$, or $x^3 - 4x = 0 \iff x(x^2-4) = 0 \iff x(x-2)(x+2) = 0$. Thus, the critical points of f are $(0,0)$, $(2,-2)$, and $(-2,2)$.

Now, let's analyze the nature of these critical points. The second-order partial derivatives are $\frac{\partial^2 f}{\partial x^2}(x,y) = 12x^2 - 8$, $\frac{\partial^2 f}{\partial y^2}(x,y) = 12y^2 - 8$, and $\frac{\partial^2 f}{\partial x \partial y}(x,y) = 8$.

- At $(2,-2)$, using standard notation, we have $r = 40$, $t = 40$, and $s = -8$. This time, $rt - s^2 > 0$ and $r > 0$, so the point $(2,-2)$ is a local minimum for f. The conclusion is the same for $(-2,2)$.
- At $(0,0)$, we have $r = -8$, $t = -8$, and $s = 8$, which gives $rt - s^2 = 0$. Therefore, we cannot conclude directly. However, we notice that $f(x,0) = x^4 - 4x^2$ is negative when x is small, while $f(x,x) = 2x^4$ is always positive. Thus, $(0,0)$ is neither a maximum nor a minimum since, as close as we want to $(0,0)$, we have points (x,y) with $f(x,y) > f(0,0)$ and other points (x,y) with $f(x,y) < f(0,0)$.

2. Let's start by calculating the first and second-order partial derivatives of f:
$\frac{\partial f}{\partial x}(x,y) = 4x^3$, $\frac{\partial f}{\partial y}(x,y) = 3y^2 - 3$, $\frac{\partial^2 f}{\partial x^2}(x,y) = 12x^2$, $\frac{\partial^2 f}{\partial x \partial y}(x,y) = 0$, $\frac{\partial^2 f}{\partial y^2}(x,y) = 6y$.

A point (x,y) is critical if and only if $x = 0$ and $y^2 = 1$. The two critical points of f are therefore $(0,1)$ and $(0,-1)$. At these two points, using standard notation, $rt - s^2 = 0$. Thus, we cannot conclude directly. Instead of studying the sign of $f(x,y) - f(x_0,y_0)$ in the neighborhood of (x_0,y_0), we will study the sign of $f(x_0+h, y_0+k) - f(x_0,y_0)$ in the neighborhood of $(0,0)$.

- At $(0,1)$. We set $y = 1+h$, with h close to 0. We then have $f(x,1+h) = x^4 + (1+h)^3 - 3(1+h) - 2 = x^4 + h^3 + 3h^2 - 4$. For x and h close to 0, $x^4 \geqslant 0$ and $3h^2 + h^3 \geqslant 0$ (in the neighborhood of 0, the h^2 term predominates). Thus, if x and h are close to 0, we have $f(x,1+h) \geqslant -4 = f(0,1)$. Therefore, $(0,1)$ is a local minimum for f.

- At $(0,-1)$ is similar, but this time we set $y=-1+h$. We then have $f(x,-1+h)=x^4-3h^2+h^3$. Here, we notice that $f(x,-1)=x^4 \geqslant 0 = f(0,-1)$ while $f(0,-1+h)=-3h^2+h^3<0$ if h is small enough. Thus, $(0,-1)$ is a saddle point.

Correction of the exercise 31 ▲

1. Let us introduce the Lagrangian:

$$L(x,y,\lambda)=x+y-\lambda(x^2+y^2-1)$$

We seek the critical points of L:

$$\begin{cases} 1-2\lambda x=0 \\ 1-2\lambda y=0 \\ 1-x^2-y^2=0 \end{cases}$$

$$\nabla L(x,y,\lambda)=\begin{pmatrix}0\\0\\0\end{pmatrix} \iff (x,y,\lambda)\in\left\{\left(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}},\pm\frac{1}{2}\right)\right\}$$

Now, let's analyze the nature of these critical points:

$$\frac{\partial^2 L}{\partial x^2}(x,y,\lambda)=-2\lambda \implies \frac{\partial^2 L}{\partial x^2}\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},-\frac{1}{2}\right)=2>0$$

$$\frac{\partial^2 L}{\partial y^2}(x,y,\lambda)=-2\lambda \implies \frac{\partial^2 L}{\partial y^2}\left(-\frac{\partial^2 L}{\partial y^2}\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},-\frac{1}{2}\right)=2$$

$$\frac{\partial^2 L}{\partial x\partial y}(x,y,\lambda)=0 \implies \frac{\partial^2 L}{\partial x\partial y}\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},-\frac{1}{2}\right)=0$$

$$\Delta(x,y,\lambda)=4\lambda^2 \implies \Delta\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},-\frac{1}{2}\right)=2>0$$

Thus, $\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$ is a minimum and $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ is a maximum for f under the constraint $x^2+y^2=1$.

2. Let $g(x,y)=x^2+y^2-4$ and $L(x,y,\lambda)=f(x,y)-\lambda g(x,y)=4x^2+y^2-\lambda(x^2+y^2-4)$. Search for the critical points of L:

$$\nabla L(x,y,\lambda)=\begin{pmatrix}8x-2\lambda x\\ 2y-2\lambda y\\ 4-x^2-y^2\end{pmatrix}$$

Thus,

$$\begin{cases} 2x(4-\lambda)=0 \\ 2y(1-\lambda)=0 \\ x^2+y^2=4 \end{cases}$$

The critical points of L are: $(x_1, y_1, \lambda_1) = (2, 0, 4)$, $(x_2, y_2, \lambda_2) = (-2, 0, 4)$, $(x_3, y_3, \lambda_3) = (0, 2, 1)$, $(x_4, y_4, \lambda_4) = (0, -2, 1)$.

Study of the Nature of Critical Points:

We have:

$$\frac{\partial^2 L}{\partial x^2}(x, y, \lambda) = 8 - 2\lambda$$

$$\frac{\partial^2 L}{\partial y^2}(x, y, \lambda) = 2 - 2\lambda$$

$$\frac{\partial^2 L}{\partial x \partial y}(x, y, \lambda) = 0$$

Therefore,

$$\frac{\partial^2 L}{\partial x^2}(x_1, y_1, \lambda_1) \times \frac{\partial^2 L}{\partial y^2}(x_1, y_1, \lambda_1) - \left(\frac{\partial^2 L}{\partial x \partial y}(x_1, y_1, \lambda_1)\right)^2 = 0,$$

$$\frac{\partial^2 L}{\partial x^2}(x_2, y_2, \lambda_2) \times \frac{\partial^2 L}{\partial y^2}(x_2, y_2, \lambda_2) - \left(\frac{\partial^2 L}{\partial x \partial y}(x_2, y_2, \lambda_2)\right)^2 = 0,$$

$$\frac{\partial^2 L}{\partial x^2}(x_3, y_3, \lambda_3) \times \frac{\partial^2 L}{\partial y^2}(x_3, y_3, \lambda_3) - \left(\frac{\partial^2 L}{\partial x \partial y}(x_3, y_3, \lambda_3)\right)^2 = 0,$$

$$\frac{\partial^2 L}{\partial x^2}(x_4, y_4, \lambda_4) \times \frac{\partial^2 L}{\partial y^2}(x_4, y_4, \lambda_4) - \left(\frac{\partial^2 L}{\partial x \partial y}(x_4, y_4, \lambda_4)\right)^2 = 0.$$

Conclusion: The Lagrange method only allows us to find candidate extrema of f under the constraint g but does not allow us to conclude whether they are indeed extrema.

Direct Study of the Nature of Critical Points:

We are not required to use the Hessian submatrix of L to establish the nature of the critical points. In fact, it is sufficient to study the sign of the distance function:

$$d_i(h, k) \equiv f(x_i + h, y_i + k) - f(x_i, y_i),$$

for $i = 1, 2, 3, 4$ and $(h, k) \approx (0, 0)$ with $g(x_i + h, y_i + k) = 0$:

$$d_i(h, k) = 4(x_i + h)^2 + (y_i + k)^2 - 4x_i^2 - y_i^2,$$

$$g_i(x_i + h, y_i + k) = (x_i + h)^2 + (y_i + k)^2 - 4.$$

If $\partial_y g(x_i, y_i) \neq 0$, then the equation $g(x_i + h, y_i + k) = 0$ implicitly defines k as a function of h near $h = 0$: solving $g(x_i + h, y_i + k) = 0$ gives

$$(y_i + k)^2 = 4 - (x_i + h)^2.$$

Substituting this expression into $d_i(h, k)$, we obtain the function of a single variable:

$$\tilde{d}_i(h) = 3(x_i + h)^2 + 4 - 4x_i^2 - y_i^2.$$

If $\partial_x g(x_i, y_i) \neq 0$, then the equation $g(x_i + h, y_i + k) = 0$ implicitly defines h as a function of k near $k = 0$: solving $g(x_i + h, y_i + k) = 0$ gives

$$(x_i + h)^2 = 4 - (y_i + k)^2.$$

Substituting this expression into $d_i(h,k)$, we find the function of a single variable:

$$\tilde{d}_i(k) = 16 - 3(y_i + k)^2 - 4x_i^2 - y_i^2.$$

Conclusion:

- $\tilde{d}_1(k) = -3k^2 \leqslant 0$ therefore the point (x_1, y_1) is a local minimum of f under the constraint g.
- $\tilde{d}_2(k) = -3k^2 \leqslant 0$ therefore the point (x_2, y_2) is a local minimum of f under the constraint g.
- $\tilde{d}_3(h) = 3h^2 \geqslant 0$ therefore the point (x_3, y_3) is a local maximum of f under the constraint g.
- $\tilde{d}_4(h) = 3h^2 \geqslant 0$ therefore the point (x_4, y_4) is a local maximum of f under the constraint g.

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