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Sobolev Spaces and Variational Method

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Contents

0.1 Introduction

Sobolev spaces are function spaces that incorporate both differentiability and integrability conditions. They are widely used in partial differential equations, functional analysis, and mathematical physics to study weak solutions of differential equations. The key idea is to control the regularity of a function not only locally (as in traditional differentiability) but also globally in terms of its distributional derivatives.

The concept of Sobolev spaces was introduced by Sergei Sobolev in the 1930s. Sobolev was a Soviet mathematician who worked in the field of partial differential equations. He was the first to systematically study weakly differentiable functions and develop a theory of these spaces based on their integrability and differentiability properties. Sobolev spaces have since become a central tool in the study of partial differential equations and have applications in various areas of mathematics and physics.

Laurent Schwartz is known for his work in the field of mathematics. He developed a theory of generalized functions, now known as distributions, which extends the notion of functions to include objects that are not necessarily continuous but still have a well-defined behavior. He applied his theory of distributions to the study of partial differential equations, leading to the development of new methods for solving these equations. He made contributions to the fields of real analysis and complex analysis, including the study of Fourier series, he used his mathematical skills to study a wide range of problems in mathematical physics, including the study of wave propagation and electromagnetic fields. Overall, Laurent Schwartz made significant contributions to the fields of mathematics and mathematical physics, and his work has had a lasting impact on these areas of study.

Sobolev space is a vector space of functions equipped with a norm that is a combination of Lp-norms of the function together with its derivatives up to a given order. The derivatives are understood in a suitable weak sense to make the space complete, i.e. a Banach space. Their importance comes from the fact that weak solutions of some important partial differential equations exist in appropriate Sobolev spaces, even when there are no strong solutions in spaces of continuous functions with the derivatives understood in the classical sense.

This course is intended for first-year master's students and aims to provide definitions and fundamental properties of Sobolev space and variational method, as well as their applications to the solution of differentiel partial equations. However, this course may also interest students from other mathematical disciplines, as there are several applications of this theory not only in the theory of PDEs or stochastic processes, but also in the study of existence and uniqueness of differential equations and functional differential equations arising from quantum mechanics and control theory. The methods using sobolev space and variational methods are also applied today in solving concrete equations that arise in population dynamics or transport theory.

During the preparation of this manuscript, I mainly relied on references [1, 2, 3, 4, 6, 8, 9, 18]. Students can also consult reference [5, 7, 12, 14].

This lecture notes is divided into four parts.

In the first part, classification of seconde ordre linear PDE are presented. After that, we describes some tools for studying partial differential equations (PDEs): measure theory and distributions.

The second part is dedicated to the theoretical study of Sobolev space (definitions and properties). It is first introduced the space $H^1(\Omega)$, then $H^m(\Omega)$, with $m \geq 1,$ and then we generalized the study in the space $W^{m,p}(\Omega)$, where $m \geq 1,\ p \in [0, +\infty[$. They are widely used in partial differential equations, functional analysis, and mathematical physics to study weak solutions of differential equations.

In the third part, we study the existence and uniqueness of elliptic boundary value problems, by using a variational formulation and the Lax-Milgram theorem. Here we replace the equation by an equivalent variational formulation, obtained by integrating the equation multiplied by a test function.

Finally, Application of Variational Formulation and Lax Milgram Theorem is presented for different various examples with different boundary conditions or very general elliptic operators, not necessarily symmetric.

Each part concludes with a chapter of exercises, with detailed solutions (or hints for solutions, depending on the difficulty) provided for most of them.

0.2 Notations and abbreviations

a.e	Almost everywhere
$\mathbb{R}$	Real field.
$\mathbb{C}$	Complex field.
$\|.\|_H$	Norm of the linear space X.
$(.,.)_X$	Scalar product in X.
1_K	The indicator function of K.
Ω	Open set of $\mathbb{R}^n$.
$\triangle$	The laplacian.
$\triangledown$	The gradient.
div	The divergence.
$d\Gamma$	Lebesgue measure on boundary.
PDE	Partial differential equations.
$L(E,F)$	The space of linear bounded operators from E to F.
Φ	Empty set.
$\overline{X}$	Closure.
$B.C$	Boundary Conditions.
$[f]$	The equivalence classes of f.
$\mathcal{C}(\Omega)$	Space of continuous functions.
$\mathcal{D}(\Omega)$	The space of functions in C^∞ with a compact support in Ω.
$\mathcal{D}'(\Omega)$	The space of all distributions on Ω.
$H^m(\Omega)$	Sobolev space of order m greater than 1.
$H^{-m}(\Omega)$	The dual of the sobolev space $H_0^m(\Omega)$.
$W^{m,p}(\Omega)$	The Sobolev space with $p \in [1,+\infty]$ and m is a nonnegative integer.

Chapter 1

Distributions

$This$ chapter describes some tools for studying partial differential equations (PDEs): classification of 2nd-order linear PDEs, measure theory and distributions. They are used to obtain existence (and often uniqueness) results for a few examples of PDE problems of various natures (elliptic, parabolic or hyperbolic).

1.1 Classification of second-order linear PDEs

Partial differential equations (PDEs) play a key role in many fields of science and engineering, such as fluid mechanics, thermodynamics, electromagnetism, finance and biology... In particular, second-order PDEs can be used to describe a number of basic physical phenomena, such as diffusion and propagation.

1.1.1 Definitions and properties

Definition 1.1 *: A partial differential equation (PDE) is an equation containing an unknown function of several variables and some of its partial derivatives with respect to these variables. The order of the largest derivative in the equation is called the order of a PDE.*

Definition 1.2 *:*
1- A linear partial differential equation (PDE) is a type of PDE where the unknown function and its derivatives appear linearly.
2- There are three primary categories of nonlinear partial differential equations (PDEs):
• Semi-linear PDEs are the closest to linear PDEs, as they only involve linear terms in the highest order derivatives. The coefficients of these linear terms depend on the independent variables.
• In a quasilinear PDE the highest order derivatives likewise appear only as linear terms, but with

coefficients possibly functions of the unknown and lower-order derivatives.

• *A PDE without any linearity properties is called fully nonlinear, and possesses nonlinearities on one or more of the highest-order derivatives.*

Example 1.1 *:*

1) The heat equation is a 2nd order linear PDE.

2) Navier-Stokes equations is a quasilinear PDE.

The majority of differential equations that appear in physics involve partial derivatives with respect to spatial and temporal variables, and are therefore partial differential equations. Here, we'll focus to second-order partial differential equations it means equations of the form

$$\sum_{i,j=1}^{n} \alpha_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \varphi(x) + \sum_{i=1}^{n} \beta_i \frac{\partial}{\partial x_i} \varphi(x) + \gamma \varphi(x) = g(x), \tag{1.1}$$

where α_{ij}, β_i and γ are fixed functions and g is a given function, often called the source term of the equation. If $g(x)$ is zero, the equation is said to be homogeneous, otherwise it is inhomogeneous.

1.1.2 PDE classification

Second-order PDEs can be classified into three main families: elliptic, hyperbolic and parabolic. Consider a 2nd-order partial differential equation involving two variables (x, t) :

$$\begin{aligned} A\frac{\partial^2}{\partial t^2}\varphi(x,t) + B\frac{\partial^2}{\partial x \partial t}\varphi(x,t) + C\frac{\partial^2}{\partial x^2}\varphi(x,t) + D\frac{\partial}{\partial t}\varphi(x,t) \\ + E\frac{\partial}{\partial x}\varphi(x,t) + \gamma\varphi(x,t) = g(x,t). \end{aligned} \tag{1.2}$$

The solutions of these different partial differential equations behave in different ways, corresponding to very different physical situations. Classification is based on the following considerations:

1- If $B^2 - 4AC > 0$: the PDE is said to be **hyperbolic**, the classic example is the wave equation

$$\frac{\partial^2}{\partial x^2}\varphi(x,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi(x,t) = g(x,t), \tag{1.3}$$

2- If $B^2 - 4AC < 0$: the PDE is said to be **elliptic**, the classic example is the Laplace equation

$$\frac{\partial^2}{\partial x^2}\varphi(x,y) + \frac{\partial^2}{\partial y^2}\varphi(x,y) = g(x,y), \tag{1.4}$$

3- If $B^2 - 4AC = 0$: the PDE is parabolic, the classic example is the heat equation

$$\frac{\partial^2}{\partial x^2}\varphi(x,t) - \alpha\frac{\partial}{\partial t}\varphi(x,t) = g(x,t). \tag{1.5}$$

Remark 1.1 : *In general, the parameters A, B, C, D and E can be functions of x and t, so the sign of the discriminant $(B^2 - 4AC)$ can vary from one point to another and the equation changes regime.*

If there are n independent variables $x_1, x_2, ..., x_n$, a general linear partial differential equation of second order has the form

$$Lu = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} \quad + \text{lower-order terms} = 0.$$

The classification depends upon the signature of the eigenvalues of the coefficient matrix

$$M(x) = (\alpha_{ij})_{1\leq i,j\leq n}.$$

The terminology elliptic, hyperbolic and parabolic comes from the fact that when the matrix $M(x)$ is constant, the curves

$$x^T M x = cte,$$

are respectively ellipsoids, hyperboloids and paraboloids respectively, then we have:

Definition 1.3 :
• *A second-order linear PDE is said to be elliptic if the matrix $M(x)$ has only non-zero eigenvalues, all of which have the same sign (all positive or all negative).*
• *A PDE is said to be hyperbolic if $M(x)$ has only non-zero eigenvalues, all of which have the same sign except for one with the opposite sign.*
• *A P.D.E. is said to be parabolic if $M(x)$ has $(N - 1)$ non-zero eigenvalues of the same sign and one zero eigenvalue (all positive or all negative, except one that is zero).*

1.1.3 Boundary conditions

In addition to initial conditions (which define the value of the solution at a given initial time), PDEs problem can have also boundary conditions (B.C.) which are imposed on a partial differential equation to determine a unique solution. These conditions are specified at the boundaries of the domain on which the PDE is defined.
Boundary conditions can be classified into different types, depending on the nature of the problem and the physical or mathematical requirements. Here are some commonly encountered types of boundary conditions:

- **Dirichlet boundary condition** : These specify the values of the unknown function at the boundaries.
- **Neumann boundary condition**, which can be defined as the derivative of the solution with respect to the normal on a part of the boundary.
- **Mixed condition**: There are several ways of obtaining mixed conditions:
The first is to impose Dirichlet conditions in some directions and Neumann conditions in others at the same point on the boundary.
The second way is a linear combination of a Dirichlet condition and a Neumann condition which called Robin conditions.
The third way is to impose a weighted average of both conditions on part of the boundary.
Example
The temperature of a rod of length L is a function $u(x,t)$ of position x and time t. It is assumed that the ends are maintained at zero temperature

$$u(0,t) = 0,\ u(L,t) = 0.$$

These boundary conditions are said to be homogeneous (equal to zero).
Well-posed boundary problems

Definition 1.4 : *A boundary problem is a partial differential equation with boundary conditions on the all boundary of the domain on which it is posed.*
This problem is well-posed in the Hadamard sense if it has a unique solution that depends continuously on the data (second member, domain, boundary data, etc.). If one of these criteria is not satisfied, the problem is said to be ill-posed (the problem may not have a unique solution, or the solution may not be stable, making it more difficult to solve).

Remark 1.2 :
• *The classification of second-order linear PDEs in $\mathbb{R}^2$ is interesting because it reflects a set of common characteristics both qualitatively through the physical phenomena represented and in terms of mathematical (Boundary condition, well-posed problem, ...).*
• *The choice and formulation of appropriate boundary conditions depend on the specific problem being studied and the physical or mathematical requirements. By combining the PDE with the specified boundary conditions, a well-posed problem can be formulated, allowing for the determination of a unique solution.*

1.2 Lebesgue Integral

The Lebesgue integral is a more general extension of the Riemann integral, and is often used to study functions that are not continuous, but can be measured rigorously. Lebesgue spaces play an important role in functional analysis, partial differential equation theory and probability.

Definition 1.5 : *Let (X, Σ, μ) be a measured space. The function $f : X \to \overline{\mathbb{R}}$ is said to be integrable or summable over X if*

$$\int_X |f| \, d\mu < \infty.$$

The vector space of all integrable functions on X is denoted by $\mathcal{L}^1(X, \Sigma, \mu)$ (or simply $\mathcal{L}^1(X)$).

Definition 1.6 : *If Ω is an open set of $\mathbb{R}^n$ and $f : \Omega \to \overline{\mathbb{R}}$, then f is said to be locally integrable on Ω if it is measurable and*

$$\int_K |f| \, d\mu < \infty \text{ for any compact } K \subset \Omega.$$

The space of all locally integrable functions on Ω is denoted by $\mathcal{L}^1_{loc}(\Omega)$.

Definition 1.7 : *The vector space of numerical power functions p $(1 \leq p < +\infty)$ integrable on X is the space:*

$$\mathcal{L}^P(X, \Sigma, \mu) = \left\{ f : X \longrightarrow \overline{\mathbb{R}} \text{ mesurable} : \int_X |f|^p \, d\mu < \infty \right\}.$$

It will be equipped with the semi-norm

$$\|f\|_{\mathcal{L}^P} = \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}}.$$

If $p = +\infty$, we introduce the space of functions essentially bounded on the space X

$$\begin{aligned} \mathcal{L}^\infty(X, \Sigma, \mu) &= \{ f : X \longrightarrow \overline{\mathbb{R}} \text{ mesurable, } \exists C > 0 \text{ such as } |f(x)| \leq C \text{ a.e} \} \\ &= \{ f : X \longrightarrow \overline{\mathbb{R}} \text{ mesurable, } f \text{ bounded a.e} \}. \end{aligned}$$

It will be equipped with the semi-norm

$$\begin{aligned} \|f\|_{\mathcal{L}^\infty} &= \inf \{ C > 0; \ |f(x)| \leq C \text{ a.e} \} \\ &= \inf \left\{ \sup_{x \notin A} |f(x)|, \ A \in \Sigma \text{ and } \mu(A) = 0 \right\}. \end{aligned}$$

Definition 1.8 : *We define the space $L^p(X,\Sigma,\mu)$ ($1 \le p < +\infty$) by:*

$$L^p(X,\Sigma,\mu) = \mathcal{L}^P(X,\Sigma,\mu) \diagup \mathcal{R},$$

where $\mathcal{R}$ is the equivalence relation defined by:

$$\forall f,\ g \in \mathcal{L}^P(X,\Sigma,\mu),\ f\mathcal{R}g \Leftrightarrow f = g \quad \text{a.e.}$$

$L^p(X,\Sigma,\mu)$ is the space of equivalence classes of $\mathcal{L}^p(X,\Sigma,\mu)$ by the equivalence relation $\mathcal{R}$.

Remark 1.3 :
1- Using the equivalence relation $\mathcal{R}$ we have

$$h \in [f] \in L^p(X,\Sigma,\mu) \to h = f \qquad a.e,$$

with $[f]$ is the equivalence classes of f.
Note that despite the fact that it is technically incorrect to say L^p is the space of Lebesgue integrable functions even though it is really the space of equivalence classes of these functions modulo equality almost everywhere.
2- That is, we 'identify' two elements of L^p if and only if their difference is null, which is to say they are equal off a set of measure zero. Note that the set which is ignored here is not fixed, but can depend on the functions.
3- The triangle inequality makes the space $L^p,\ 1 \le p \le \infty$ into a metric space with distance

$$d(f,g) = \|f-g\|_{L^p}.$$

Theorem 1.1 *The space $L^p(X,\Sigma,\mu)$ for $1 \le p \le \infty$ is a Banach space.*

Proof. The proof of this theorem typically involves showing that the space $L^p(X,\Sigma,\mu)$ is a normed vector space and then demonstrating that it is complete, which means that Cauchy sequences in this space converge to a limit within the space. ■

Lebegue space

Definition 1.9 : *Let p be an element of $[1,+\infty]$ and Ω be an open of $\mathbb{R}^n$, we call Lebesgue space[1], and we denote $L^p(\Omega)$, the vector space of f functions (classes of functions) of Ω in $\mathbb{C}$, Lebesgue functions that are measurable, verify*

- $\displaystyle\int_\Omega |f(x)|^p\,dx < \infty,\quad if\ 1 \le p < +\infty,$
- $\displaystyle\underset{x\in\Omega}{supess}\,|f(x)| = \inf\{C > 0;\ |f(x)| \le C\ \text{a.e}\} < \infty,\ if\ p = +\infty.$

[1] Lebesgue spaces are named after the French mathematician Henri Lebesgue (1875-1941).

We provide $L^p(\Omega)$ *of the norm*

$$f \to \begin{cases} \|f\|_p = \left(\int_\Omega |f(x)|^p \, dx\right)^{\frac{1}{p}}, & 1 \leq p < +\infty, \\ \|f\|_\infty = \underset{x\in\Omega}{supess} |f(x)|, & p = +\infty. \end{cases}$$

Theorem 1.2 *(Riesz-Fisher Theorem)*
(i) For $1 \leq p \leq +\infty$*, the space* $(L_p, \|.\|_p)$ *is complete, so it's a Banach space.*
(ii) For $p = 2$ *: the space* $L^2(\Omega)$ *is a Hilbert space for the inner product*

$$(f, g)_{L^2(\Omega)} = \int_\Omega f(x).g(x)dx.$$

(iii) For $1 \leq p < +\infty$*, the space* $(L_p, \|.\|_p)$ *is separable.*

Minkowski's inequality: Let $1 \leq p \leq +\infty$, $f, g \in L^p(\Omega)$, then $(f+g) \in L^p$ and

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p \,. \tag{1.6}$$

Holder inequality:
Let $1 \leq p, q \leq +\infty$, with p, q conjugate ($\frac{1}{p} + \frac{1}{q} = 1$). If $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then the application

$$\begin{aligned} L^p(\Omega) \times L^q(\Omega) &\to \mathbb{R}, \\ (f, g) &\to \int_\Omega f(x).g(x)dx \end{aligned} \tag{1.7}$$

is continuous bilinear with values in $L^1(\Omega)$ and

$$\|fg\|_1 \leq \|f\|_p \|g\|_q \Rightarrow \int_\Omega |f.g|dx \leq \left(\int_\Omega |f|^p dx\right)^{1/p} \left(\int_\Omega |g|^q dx\right)^{1/q}.$$

The case $p = q = 2$ gives the Cauchy-Schwartz inequality

$$\int_\Omega |f.g| \, dx \leqslant \left(\int_\Omega |f|^2 dx\right)^{1/2} \left(\int_\Omega |g|^2 dx\right)^{1/2}. \tag{1.8}$$

Remark 1.4 : *For* $1 \leq p \leq +\infty$*, with* p, q *conjugate then the dual of* $L^p(\Omega)$ *is* $L^q(\Omega)$*.*
The Lebesgue integral is linear, additive, monotone, and satisfies the triangle inequality.

Fatou's lemma:

This lemma is used to study the properties of integrals of non-negative measurable functions, and deals with the limits of sequences of measurable functions. It's named after the French-Senegalese mathematician Fatou. In simpler terms, it tells us that when we have a sequence of functions that are increasing pointwise, the integral of the limit of the sequence is bounded below by the limit of the integrals of the individual functions. This result has applications in various areas of mathematics, including probability theory and analysis.

Fatou's lemma is formulated as follows:

Lemma 1.1 *[11] Let $(f_n)_{n\in\mathbb{N}}$ a sequence of integrable functions from Ω into $\mathbb{R}^+$, verify $|f_n| \leq g$ where g is a positive integrable function. Then,*

$$\int_\Omega (\lim_{n\to+\infty} f_n) dx = \lim_{n\to+\infty} \int_\Omega f_n \, dx.$$

The same result requires, for the Riemann integral, uniform convergence. An almost immediate consequence of Fatou's lemma is the dominated convergence theorem of Lebesgue, which is by far the most commonly used in practice.

Dominated convergence theorem "DCT "

Theorem 1.3 *[11] Let (X, Σ, μ) be a measured space and $1 \leq p \leq +\infty$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions of L^p such that:*

1- $f_n(x) \to f(x)$ as $n \to +\infty$ a.e in X.

2- There exists a function g in X integrable with values in $\mathbb{R}^+$, such that

$$|f_n(x)| \leq g(x) \quad \text{a.e in } X.$$

Then f is integrable and we have.

$$f_n \to f \text{ in } L^p(X) \Longrightarrow \int_X |f_n - f|^p d\mu \to 0 \text{ as } n \to +\infty \, .$$

Remark 1.5 *:*

The dominated convergence theorem is an important tool for studying the convergence of function sequences. It can be used to prove convergence properties for integrals of functions, and is used in many applications in mathematical analysis, such as functional analysis, measure theory and integration theory.

1.3 Distributions

In mathematics, a distribution is an extension of the concept of a function, allow us to represent quantities such as probability densities, electric potentials, force fields, etc., that cannot be described by classical functions. Here, we provide only a few elements of this theory.

1.3.1 The Space $C^k(\Omega)$

Definition 1.10 : *Let Ω be a non-empty open set in $\mathbb{R}^n$. The support of a function $f : \Omega \longrightarrow \mathbb{R}$, denoted as supp$(f)$, is the closed subset defined by:*

$$\text{supp}(f) = \overline{\{x \in \Omega : f(x) \neq 0\}},$$

which means that

$$x_0 \notin \text{supp}(f) \Leftrightarrow \exists V \in \mathcal{V}(x_0) : f(x) = 0, \ \forall x \in V.$$

Definition 1.11 : *Let Ω be a non-empty open set in $\mathbb{R}^n$. For any $k \in \overline{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}$, we define the space $C^k(\Omega)$ as follows:*

$$C^k(\Omega) = \{f : \Omega \longrightarrow \mathbb{R} \text{ or } \mathbb{C} : D^\alpha f \in C(\Omega), \forall \alpha \in \mathbb{N}^n; \ |\alpha| \leq k\},$$

In other words, a function $f : \Omega \longrightarrow \mathbb{R}$ is said to be of class C^k on Ω if all its partial derivatives up to order k exist and are continuous, with

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdot \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}, \ |\alpha| = \sum_{i=1}^{n} \alpha_i.$$

Remark 1.6 : *For any $k \in \overline{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}$, the space $C^k(\overline{\Omega})$ will be the function space $f \in C^k(\Omega)$ such that all partial derivatives $D^\alpha f$ extend continuously to $\overline{\Omega}$ for $|\alpha| \leq k$.*
In other words, $C^k(\overline{\Omega})$ is the set of restrictions of functions f from $C^k(\mathbb{R}^n)$ to $\overline{\Omega}$ that satisfy

$$\lim_{x \in \Omega, \, |x| \longrightarrow +\infty} |D^\alpha f(x)| = 0, \ \forall \alpha \in \mathbb{N}^n; \ |\alpha| \leq k. \tag{1.9}$$

We put

$$C^\infty(\Omega) = \bigcap_{k \geq 0} C^k(\Omega), \tag{1.10}$$

and

$$C^\infty(\overline{\Omega}) = \bigcap_{k \geq 0} C^k(\overline{\Omega}).$$

1.3.2 Space of test function $\mathcal{D}(\Omega)$

Test functions are an important tool in mathematical analysis for defining distributions. They allow us to test the value of a distribution for simple functions and are used to better understand the properties of the distribution.

Definition 1.12 : *Let Ω a non-empty open set of $\mathbb{R}^n$, we denote by*

$$\mathcal{D}(\Omega) = \bigcup_{K \text{ compact},\ K \subset \Omega} \mathcal{D}_K(\Omega),$$

the space of indefinitely differentiable functions on Ω $(C^\infty(\Omega))$ test function with

$$\mathcal{D}_K(\Omega) = \{u \in C^\infty(\Omega),\ supp(u) \subset K\}.$$

Example 1.2 : *Let the function φ given by*

$$\begin{aligned} \varphi :\ & \mathbb{R}^n \longrightarrow \mathbb{R} \\ & x \longmapsto \varphi(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases} . \end{aligned}$$

Then $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and its support is exactly the closed unit ball

$$\overline{B(0,1)} = \{x \in \mathbb{R}^n : |x| \leq 1\}.$$

Remark 1.7 :
1- Let Ω be a non-empty open set in $\mathbb{R}^n$. $\mathcal{D}(\mathbb{R}^n)$ is a vector space over $\mathbb{C}$, not reduced to zero, and an algebra for the multiplication of functions.
2- For all $p \in [1, \infty)$, the space $\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$.

1.3.3 Convergence in $\mathcal{D}(\Omega)$

The topology that can be defined on the space $\mathcal{D}(\Omega)$ is not simple. It cannot be defined by a metric (distance) or a norm.
We equip $\mathcal{D}(\Omega)$ with a pseudo-topology, meaning that we define convergent sequences in $\mathcal{D}(\Omega)$. In other words, we can equip $\mathcal{D}(\Omega)$ with a topology, called the inductive limit of the topologies of $\mathcal{D}_K(\Omega)$, where K ranges over the compact subsets of Ω, in the following way if $A \subset \mathcal{D}(\Omega)$ is convex, it is said to be a neighborhood of 0 in $\mathcal{D}(\Omega)$. It can be shown that this forms (basis of 0-neighborhoods for a topology on $\mathcal{D}(\Omega)$) making it a locally convex topological vector space (but not metrizable).

Definition 1.13 : *We say that a sequence of functions $(\varphi_k)_{k\geq 0} \in \mathcal{D}(\Omega)$ converges to a test function $\varphi \in \mathcal{D}(\Omega)$ for the topology of $\mathcal{D}(\Omega)$ if and only if there exists a fixed compact $K \subset \Omega$ such that:*
1) *supp$\varphi_k \subset K$, and supp$\varphi \subset K$, $\forall k \geq 0$.*
2) $\forall \alpha \in \mathbb{N}^n$, *on a $D^\alpha \varphi_k \to D^\alpha \varphi$ uniformelly in K as $k \to \infty$, then*

$$\forall \alpha \in \mathbb{N}^n : \sup_{x \in K} |D^\alpha(\varphi_k - \varphi)(x)| \underset{k\to+\infty}{\to} 0 .$$

Remark 1.8 :
1- Condition (1) is important because if the $(\varphi_k)_{k\geq 0}$ supports are compact, their number is infinite and their union isn't necessary compact.
2- Condition (2) shows how demanding this notion of convergence (it's the price to pay for $\mathcal{D}(\Omega)$ to be closed).
3- $\mathcal{D}(\Omega)$ is a dense subspace of $L^2(\Omega)$.

1.3.4 Distributions Space

The space $\mathcal{D}(\Omega)$ will allow us to define, through "duality," the concept of distribution.

Definition 1.14 : *A distribution or generalized function on Ω of $\mathbb{R}^n$ is any linear and continuous map with respect to the topology of $\mathcal{D}(\Omega)$. In other words, a distribution on Ω is a mapping*

$$\boldsymbol{T} : \mathcal{D}(\Omega) \to \mathbb{K}\ (\mathbb{R} \text{ or } \mathbb{C})$$

satisfies the following:
1) $\boldsymbol{T}$ is a linear map.
2) $\boldsymbol{T}$ is continuous, i.e. if $\varphi_k \to 0$ in $\mathcal{D}(\Omega)$, then $(\boldsymbol{T}, \varphi_k) \to 0$ in $\mathbb{K}$, when $k \to +\infty$.
The space of all distributions on Ω is denoted $\mathcal{D}'(\Omega)$ (it is the topological dual of $\mathcal{D}(\Omega)$).

Proposition 1.1 : *Let Ω be an open set in $\mathbb{R}^n$. Then $T \in \mathcal{D}'(\Omega)$ if and only if for every compact K in Ω,*

$$\exists c > 0,\ \exists m \in \mathbb{N}, \forall \varphi \in \mathcal{D}_K(\Omega),\ |\langle T, \varphi\rangle| \leq c \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\leq m}} \sup_{x\in K} |D^\alpha \varphi(x)|, \tag{1.11}$$

The distribution is said to be of order m if the preceding inequality is verified with a number m independent of the compact K.

Remark 1.9 :

1) The integer m involved in (1.11) may depend on the compact K. However, if the same m holds for all compacts, we say that T is of order m.

2) Let T be in $\mathcal{D}'(\Omega)$. We say that T is positive if, for any $\varphi \in \mathcal{D}(\Omega)$ such that $\varphi \geq 0$, we have $(T, \varphi) \geq 0$. Then every positive distribution is of order zero.

3) For $T \in \mathcal{D}'(\Omega)$, the support of T, which we denote $suppT$, is the complementary of the largest open where T is zero.

Locally integrable functions : $L^1_{loc}(\Omega)$

Let $L^1_{loc}(\Omega)$ be the space of (classes of) functions $f : \Omega \to \mathbb{C}$ which are locally integrable on Ω of $\mathbb{R}^n$(for the Lebesgue measure) is given by:

$$L^1_{loc}(\Omega) = \left\{ f : \Omega \to \overline{\mathbb{R}},\ \forall K \text{ compact } \subset \Omega,\ \int\limits_K |f|\, dx < \infty \right\},$$

can be identified as a subspace of $\mathcal{D}'(\Omega)$, en effet $\forall\ f \in L^1_{loc}(\Omega)$, we have

$$\begin{aligned} T_f :\ & \mathcal{D}(\Omega) \to \mathbb{R} \\ & \varphi \longmapsto \langle T_f, \varphi \rangle = \int\limits_\Omega f(x)\varphi(x)dx. \end{aligned}$$

If $\varphi \in \mathcal{D}(\Omega)$ and K its support (compact), we get

$$|\langle T_f, \varphi \rangle| \leq \int\limits_\Omega |f(x)| \,.\, |\varphi(x)|\, dx \leq \|f\|_{L^1(K)} \cdot \|\varphi\|_{L^\infty(K)} \,.$$

Example 1.3 : ***The Dirac mass***

In the resolution of certain problems in physics, it is sometimes useful to consider objects, called "functions" in an abusive language, but are not well-defined as pointwise representations. The most famous example of this is the Dirac measure, which, if considered as a function, is "zero outside of 0 and infinite at 0". If $a \in \Omega \subset \mathbb{R}^n$ is a fixed point in Ω, then we call the Dirac delta in a (or Dirac mass) the distribution

$$\begin{aligned} \delta_a :\ & \mathcal{D}(\Omega) \to \mathbb{R} \\ & \varphi \longmapsto \delta_a(\varphi) = \langle \delta_a, \varphi \rangle = \varphi(a). \end{aligned}$$

The Dirac distribution at the point a is not a function (since a function that is zero everywhere except at one point has an integral of zero, not equal to 1. It can be interpreted, in the context of electricity, as a point charge.

Example 1.4 *: **Principale value***

Given the function $x \to \frac{1}{x}$, which does not belong to the space $L^1_{loc}(\mathbb{R})$, we can define a distribution called the Cauchy principal value denoted $vp\frac{1}{x}$ as follows:

$$\begin{aligned}\langle vp\frac{1}{x}, \varphi\rangle &= \lim_{\varepsilon\to 0} \int\limits_{|x|\geq\varepsilon} \frac{\varphi(x)}{x}dx \\ &= \lim_{\varepsilon\to 0}\left[\int_{-\infty}^{\varepsilon} \frac{\varphi(x)}{x}dx + \int_{\varepsilon}^{+\infty} \frac{\varphi(x)}{x}dx\right], \ \forall x \in \mathcal{D}(\mathbb{R}).\end{aligned}$$

We note that the existence of the limit is due to the compensation of the two divergent integrals.

Remark 1.10 *:*

1) All functions $f \in L^1_{loc}(\Omega)$ define a distribution T_f on Ω, which is of order 0.

2) We have $T_f = T_g$ for $g \in L^1_{loc}(\Omega)$ if and only if $f = g$ almost everywhere.

3) A distribution associated with a function $f \in L^1_{loc}(\Omega)$ can be interpreted, in the context of electricity, as a distributed charge with density f.

Definition 1.15 *All distribution that can be identified with a locally integrable function is called a regular distribution, and any other distribution is referred to as a singular distribution (the Dirac distribution δ_0 is not a regular distribution).*

Remark 1.11 *:*

1) *For Ω a open set of $\mathbb{R}^n$, we get $L^p(\Omega) \subset L^p_{loc}(\Omega)$, $\forall p \in [1, +\infty[$.*

2) *Distribution support T (denoted $suppT$) is the set of x in Ω such that: For any neighborhood V of x, there exists $\varphi \in \mathcal{D}(V)$ such that $\langle T, \varphi\rangle \neq 0$.*

Derivation in the sense of distributions

A fundamental property of distributions is that it is possible to derive them, but in a (weak) sense that we will now define.

Let Ω be an open set of $\mathbb{R}^n$ and $f \in C^m(\Omega)$ then $D^\alpha f \in L^1_{loc}(\Omega)$, $\forall |\alpha| \leq m$ and

$$\forall\varphi \in \mathcal{D}(\Omega), \ \langle T_{D^\alpha f}, \varphi\rangle = \int\limits_{\Omega} (D^\alpha f)\,\varphi dx$$

$$\overset{\text{par parties}}{=} (-1)^{|\alpha|} \int\limits_{\Omega} f\,(D^\alpha \varphi)\,dx = (-1)^{|\alpha|}\,\langle T_f, D^\alpha\varphi\rangle.$$

Definition 1.16 : *Let $T \in \mathcal{D}'(\Omega)$ and $\alpha \in \mathbb{N}^n$, we define the derivative of index α, denoted $D^\alpha T$ by*

$$\langle D^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle, \ \forall \varphi \in \mathcal{D}(\Omega),$$

with

$$supp(D^\alpha T) \subset supp(T).$$

Therefore, we define $DT = \frac{\partial T}{\partial x_i}$, $\forall\ i = 1, ...n$ by

$$\forall \varphi \in \mathcal{D}(\Omega), \left\langle \frac{\partial T}{\partial x_i}, \varphi \right\rangle = -\left\langle T, \frac{\partial \varphi}{\partial x_i} \right\rangle.$$

Remark 1.12

1) *Since $\varphi \to \left\langle T, \frac{\partial \varphi}{\partial x_i} \right\rangle$ is a linear continuous form in $\mathcal{D}(\Omega)$ because*

$$\varphi_n \to \varphi \Rightarrow \varphi_n' \overset{\mathcal{D}}{\to} \varphi',$$

this defines $\left(\frac{\partial T}{\partial x_i}\right)$ as a distribution.

2) *$D^\alpha : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ is continuous operator.*

3) *We have an equality between $D^\alpha f$(in the classical sense) and $D^\alpha f$ in the distribution sense if $f \in C^m(\Omega)$.*

4) *Any distribution is infinitely differentiable and its derivatives are distributions.*

5) $\forall f \in C^\infty(\Omega), \ \forall T \in \mathcal{D}'(\Omega),$

$$\frac{\partial}{\partial x_i}(fT) = \frac{\partial f}{\partial x_i} T + f.\frac{\partial T}{\partial x_i}, \forall i = 1, ..., n.$$

6) For all distribution $T \in \mathcal{D}'(\Omega)$, and $\forall \alpha, \beta \in \mathbb{N}^n$, we have

$$D^\alpha D^\beta T = D^\beta D^\alpha T = D^{\alpha+\beta} T.$$

Example 1.5 : *The Heaviside function H is defined on $\mathbb{R}$ by:*

$$\begin{aligned} H: \ & \mathbb{R} \to \mathbb{R} \\ & x \longmapsto H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} . \end{aligned}$$

The function $H \in L^1_{loc}(\mathbb{R})$, so we can define the distribution:

$$\begin{aligned} T_H: \ & \mathcal{D}(\mathbb{R}) \to \mathbb{R} \\ & \langle T_H, \varphi \rangle = \int_{\mathbb{R}} H(x)\varphi(x)dx \end{aligned} .$$

If $\varphi \in \mathcal{D}(\Omega)$, *then*

$$\langle DT_H, \varphi\rangle = -\langle T_H, \varphi'\rangle = \int_{\mathbb{R}} H(x)\varphi'(x)dx = -\int_0^{+\infty} \varphi'(x)dx = \varphi(0),$$

Therefore $DT_H = \delta_0$ *is the Dirac mass at the origin. Deriving again* H' *we have:*

$$\langle D^2T_H, \varphi\rangle = \langle \delta', \varphi\rangle = -\langle \delta, \varphi'\rangle = -\varphi'(0),\ \forall\varphi \in \mathcal{D}(\Omega).$$

More generally,

$$\langle \delta_0^m, \varphi\rangle = (-1)^m \langle \delta_0, \varphi^{(m)}\rangle = (-1)^m\varphi^m(0), \forall m \geq 1,\ \forall\varphi \in \mathcal{D}(\mathbb{R}).$$

Convergence in the space $\mathcal{D}'(\Omega)$

Definition 1.17 *: Let* $(T_n)_{n\in\mathbb{N}} \subset \mathcal{D}'(\Omega)$.
1) We say that T_n *converges to* 0 *in* $\mathcal{D}'(\Omega)$ *denoted* $T_n \to 0$ *in* $\mathcal{D}'(\Omega)$ *if and only if*

$$\langle T_n, \varphi\rangle \longrightarrow 0, \forall\varphi \in \mathcal{D}(\Omega).$$

2) We say that $(T_n)_{n\in\mathbb{N}}$ *converges to* $T \subset \mathcal{D}'(\Omega)$ *if and only if*

$$(T_n - T) \to 0 \ in\ \mathcal{D}'(\Omega).$$

3) Let $T \in \mathcal{D}'(\Omega)$, *we have*

$$T_n \to 0 \ in\ \mathcal{D}'(\Omega) \Rightarrow D^\alpha T_n \to 0 \ in\ \mathcal{D}'(\Omega), \forall\alpha \in \mathbb{N}^n.$$

4) Let $(T_n)_{n\in\mathbb{N}} \subset \mathcal{D}'(\Omega)$, *such that*

$$T_n \to T \ in\ \mathcal{D}'(\Omega) \Rightarrow D^\alpha T_n \to D^\alpha T \ in\ \ \mathcal{D}'(\Omega),$$

this expresses the continuity of the derivation operator in $\mathcal{D}'(\Omega)$.

Example 1.6 *Let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence of* $L^2(\Omega)$ *such that* $f_n \to f$ *in* $L^2(\Omega)$, *we have:*

$$\|f_n - f\|_{L^2} \to 0 \Rightarrow T_{f_n} \to T_f \ in\ \mathcal{D}'(\Omega).$$

In fact, for $\varphi \in \mathcal{D}(\Omega)$,

$$|\langle T_{f_n - f}, \varphi\rangle| = \left|\int_{\Omega} (f_n - f)(x)\varphi(x)dx\right| = \left|\langle f_n - f, \varphi\rangle_{L^2(\Omega)}\right|.$$

According to Cauchy-Schwartz, we obtain

$$|\langle T_{f_n-f}, \varphi\rangle| \leq \|f_n - f\|_{L^2}.\|\varphi\|_{L^2} \to 0.$$

Remark 1.13 : *Based on the example above, we can deduce the following result :*

$$L^2(\Omega) \hookrightarrow \mathcal{D}'(\Omega) \text{ continuous imbbeding}.$$

Regularizing sequences

Definition 1.18 : *A regularizing sequence is any sequence $(\varphi_k)_{k\in\mathbb{N}^*}$ of function such that*

1) $\varphi_k \in \mathcal{D}(\mathbb{R}^n)$ *and supp*$\varphi_k \subset B\left(0, \frac{1}{k}\right), \forall k \in \mathbb{N}^*$.

2) $\int_{\mathbb{R}^n} \varphi_k(x)dx = 1, \forall k \in \mathbb{N}^*$ *and* $\varphi_k \geq 0$ *on* $\mathbb{R}^n$.

Remark 1.14 :
1- Regularizing sequences are mainly used in distribution theory, in order to move from a problem involving generalized functions to a restriction to regular functions, which are simpler to study.
2- The convolution of a distribution T by a test function φ is a function of class C^∞, whose support is included in the Minkowski sum of the support of φ and the support of the distribution T.
3- Let T be a distribution and $(\varphi_k)_{k\in\mathbb{N}^*}$ a regularizing sequence. Then the sequence of regular distributions associated with the functions

$$(T * \varphi_k) \ converges \ to \ T \ in \ \mathcal{D}'.$$

Green's Formula

Definition 1.19 : *Let $\eta(x)$ be the outgoing normal unit vector (i.e. directed away from Ω) at the point $x \in \Gamma = \partial\Omega$. If u is a fairly regular function defined on $\overline{\Omega}$, we note*

$$\frac{\partial u}{\partial \eta}(x) = \nabla u(x).\eta(x), \ x \in \Gamma.$$

the normal derivative of u on Γ.

Theorem 1.4 : *(Ostrogradsky Formula): Let Ω be an open set of $\mathbb{R}^n$ of class $\mathcal{C}^1$ with the boundary $\partial\Omega = \Gamma$ and let F a function in $\mathcal{C}^1\left(\overline{\Omega}\right)$ with values in $\mathbb{R}^n$ (a vector field). Then*

$$\int_\Omega div\left(F(x)\right)dx = \int_\Gamma F(x).\eta(x)d\Gamma.$$

Corollary 1.1 : *(Green's Formula)*
Let Ω a bounded open of $\mathbb{R}^n$ of class $\mathcal{C}^1$. So for all functions $u \in \mathcal{C}^2\left(\overline{\Omega}\right)$ and $v \in \mathcal{C}^1\left(\overline{\Omega}\right)$ we have

$$\int_\Omega \triangle u(x)v(x)dx = \int_\Gamma \frac{\partial u}{\partial \eta}(x).v(x)d\Gamma - \int_\Omega \nabla u(x).\nabla v(x)dx.$$

Remark 1.15 :
To say that a regular function u has bounded support in the closed $\overline{\Omega}$ means that it vanishes at infinity if the closure is unbounded.
We also say that the function u has compact support in $\overline{\Omega}$ this does not imply that u vanishes on the boundary. In particular, this assumption about the bounded support of the function u in $\overline{\Omega}$ is unnecessary if the open set Ω is bounded. However, if it is unbounded, this assumption guarantees that the integrals involving u are finite.

1.4 Exercises

Exercise 1.1: Prove that the application T definite by:

$$\begin{aligned} T \quad &: \quad \mathcal{D}(\mathbb{R}^n) \to \mathbb{C} \\ u \quad &\to \quad \langle T, u\rangle = \int_{\mathbb{R}^n} \frac{\partial u}{\partial x_1}(t)dt \end{aligned}$$

is a distribution. Prove that it is of finite order, and specify the order.

Exercise 1.2:

Prove that

1- Any regulier distribution is of order zero.

2- The Dirac distribution is of order zero.

3- The $Vp(\frac{1}{x})$ distribution is of order 1 and $\frac{d}{dt} sign\ x = 2\delta$.

Exercise 1.3:

Let p a reel such that $1 < p \leq +\infty$, (f_k) a sequence of functions of $L^p(\Omega)$ verify:

1- $\exists C > 0, \forall k \in \mathbb{N}, \|f_k\|_p \leq C.$

2- $\exists T \in \mathcal{D}'(\Omega), T = \lim\limits_{k\to+\infty} [f_k]\,.$

Show that T is the regular distribution associated with a element of $L^p(\Omega)$.

Exercise 1.4:

A-

Let $f \in L^1(\mathbb{R})$ and let T be the linear form defined by

$$< T, \varphi >= \int_{\mathbb{R}} f(x)\varphi'(x)dx + \varphi'(0),\ \forall\, \varphi \in \mathcal{D}.$$

1. Write T in reduced form and deduce that it is a distribution on $\mathbb{R}$. What is its order?

2. Assume that f is derivable in $\mathbb{R}^*$ and that $f(0^+)$ and $f(0^-)$ exist. Write T in terms of T'_f, δ_0 and its derivatives.

B-

1- Let $\varphi \in \mathcal{D}(\mathbb{R})$ such that:

$$supp\varphi \subset]1, 2[,\ 0 \leq \varphi(x) \leq 1\ and\ \varphi(x) = 1\ for\ a \leq x \leq b,\ 1 < a < b < 2.$$

For $n \in \mathbb{N}^*$ let's set: $\varphi_n(x) = e^{-n}\varphi(nx)$. Show that $(\varphi_n)_{n\in\mathbb{N}^*}$ converges to zero in $\mathcal{D}(\mathbb{R})$.

2- Prove that there is no distribution $T \in \mathcal{D}'(\Omega)$ such that

$$\langle T, \varphi\rangle = \int_{\mathbb{R}} \exp\left(\frac{1}{x^2}\right)\varphi(x)dx.$$

Exercise 1.5:

For $x \in \mathbb{R}$ and $\varepsilon > 0$, let's set that

$$f_\varepsilon(x) = \ln(x + i\varepsilon) = \ln|x + i\varepsilon| + i\arg(x + i\varepsilon).$$

a) Prove that if $\varepsilon \to 0$, f_ε converge in $\mathcal{D}'(\Omega)$ to the distribution f_0 given by:

$$f_0 = \begin{cases} \ln(x), & x > 0 \\ \ln|x|, & x < 0 \end{cases}$$

b) Calculate $\dfrac{df_0}{dx}$ in the sense of distributions

c) Deduce that in $\mathcal{D}'(\Omega)$, we have $\dfrac{1}{x + i\,0} = \lim\limits_{\varepsilon \to 0} \dfrac{1}{x + i\,\varepsilon} = VP\left(\dfrac{1}{x}\right) - i\pi x.$

d) Prove that $\dfrac{1}{x - i\,0} = \lim\limits_{\varepsilon \to 0} \dfrac{1}{x + i\,\varepsilon} = VP\left(\dfrac{1}{x}\right) + i\pi x.$

e) Deduce that

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{\pi\left(x^2 + \varepsilon^2\right)} = \delta.$$

Exercise 1.6:

Let $f \in L^1_{loc}(\mathbb{R})$. Prove that the function

$$u(x,t) = f(t - x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}$$

in the space $L^1_{loc}(\mathbb{R} \times \mathbb{R})$. Verify in the distribution sens: $\dfrac{du}{dt} + \dfrac{du}{dx} = 0.$

Exercise 1.7:

Consider the sequence of functions $(f_n)_{n \geq 1}$ defined on $\mathbb{R}$ by:

$$f_n(x) = \begin{cases} 0, & x \leq -\frac{1}{n} \\ n^2\left(\frac{1}{n} + x\right), & -\frac{1}{n} \leq x \leq 0 \\ n^2\left(\frac{1}{n} + x\right), & 0 \leq x \leq \frac{1}{n} \\ 0, & x \geq \frac{1}{n} \end{cases}$$

1) Study the convergence of the sequence $(f_n)_{n \geq 1}$ in $L^2(\mathbb{R})$, $L^1(\mathbb{R})$ and $\mathcal{D}'(\mathbb{R})$.

Let us consider the Dirac distribution δ defined by:

$$\langle \delta, \varphi \rangle = \varphi(0), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}).$$

2) Prove that the sequence $(f_n)_{n \geq 1}$ converges in $\mathcal{D}'(\mathbb{R})$.

3) Prove that there is no function $f_\delta \in L^1_{loc}(\mathbb{R})$ such that

$$\langle \delta, \varphi \rangle = \int_{\mathbb{R}^n} f_\delta \varphi(x)dx.$$

Exercise 1.8:

Consider the function f defined for all $x \in \mathbb{R}^*$ by: $f(x) = H(x)xe^{\lambda x}$ where $H(x)$ is the Heaviside function defined on $\mathbb{R}$ by

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

i) Justify why f defines a distribution.

ii) Calculate in the sense of distributions $\frac{d^2}{dx^2}\left(\frac{df}{dx} - \lambda f\right)$.

Exercise 1.9:

Let be the function of two real variables

$$u(t,x) = \frac{H(x)}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}, \text{ with } u(0,x) = 0;$$

1) Check that u is of class C^1 on $\mathbb{R}^2$.

2) Calculate in the sense of the distributions Du where D is the differential operator

$$D = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}.$$

Exercise 1.10:

Let $\Omega =]a,b[$ where $(a < b)$. We want to show that if a distribution $u \in \mathcal{D}'(]a,b[)$ has a zero derivative (in the sense of distributions), then u is a constant function.

Exercice 1.11

We consider the function k defined on $\mathbb{R}^2$ as follows:

$$k(x,t) = 0, \text{ if } t < 0, \; k(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{1}{4t}\right), \text{ if } t > 0.$$

1) Prove that

$$\lim_{t \to 0^+} k(x,t) = 0, \; x \neq 0, \; \lim_{t \to 0^+} k(0,t) = +\infty,$$

$$\int_{-\infty}^{+\infty} k(x,t)dx = 1, \forall t > 0.$$

2) Calculate in the sense of distributions

$$\frac{\partial^2 k}{\partial t \partial x}, \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) k.$$

Chapter 2

Sobolev Space

Introduction

Sobolev spaces are spaces of functions that combine both differentiability and integrability conditions. They are widely used in partial differential equations, functional analysis, and mathematical physics to study weak solutions of differential equations.

Let's consider the following partial differential equation defined on an open set $\Omega \in \mathbb{R}^n$:

$$\begin{cases} -\triangle u(x) + c(x)u(x) = f(x) & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \end{cases}, \tag{2.1}$$

with $c \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$ being two given functions, and u is an unknown function. Solving the partial differential equation (2.1) means finding a function $u \in C^2(\Omega)$ that satisfies (2.1). In the theory of partial differential equations, it is often challenging to prove the existence of a solution.

However, if it has been established that a solution u exists, assuming the open set Ω is sufficiently regular, it can be shown that u satisfies, for any function $\varphi \in D(\Omega)$:

$$\int_\Omega \nabla u(x)\nabla\varphi(x)dx + \int_\Omega c(x)u(x)\varphi(x)dx = \int_\Omega f(x)\varphi(x)dx. \tag{2.2}$$

The idea behind Sobolev spaces is to introduce function spaces u such that $u \in L^2(\Omega)$ and $\nabla u \in (L^2(\Omega))^n$, which is a Hilbert space, and to seek not the solutions to the initial problem, but the functions u from this space that satisfy (2.2).

Such functions are called weak solutions of the partial differential equation. The benefit of this approach lies in the ability to utilize the complete machinery of Hilbert spaces, including projection theorems and the Lax-Milgram theorem, to demonstrate the existence of weak solutions. Consequently, we are not directly solving the original problem but instead tackling a variation of it using these powerful mathematical tools.

2.1 Sobolev Space $\mathbf{H}^1(\Omega)$

2.1.1 Definitions and Basic properties

Definition 2.1 : *Let Ω be a non-empty open set in $\mathbb{R}^n$. We define the Sobolev space of order 1 on Ω as:*

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) : \underbrace{\frac{\partial u}{\partial x_i} \in L^2(\Omega)}_{\text{calculated in the sense of distributions}} \quad , \ 1 \leq i \leq n \right\}, \tag{2.3}$$

equipped with the inner product

$$\begin{aligned} (u,v)_{H^1(\Omega)} &= (u,v)_{1,\Omega} = (u,v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{(L^2(\Omega))^n} \\ &= \int_\Omega u.v \, dx + \int_\Omega \nabla u.\nabla v \, dx \\ &= \int_\Omega \left(uv + \sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} \right) dx, \end{aligned}$$

and of the norm

$$\|u\|_{1,\Omega} = (u,u)_{1,\Omega}^{1/2}.$$

Therefore,

$$\begin{aligned} \|u\|_{1,\Omega} &= \left(\langle u,u \rangle_{H^1(\Omega)} \right)^{\frac{1}{2}} = \left(\langle u,u \rangle_{L^2(\Omega)} + \langle \nabla u, \nabla u \rangle_{(L^2(\Omega))^n} \right)^{\frac{1}{2}} \\ &= \left(\int_\Omega |u(x)|^2 \, dx + \int_\Omega |\nabla u(x)|^2 \, dx \right)^{\frac{1}{2}} \\ &= \left(\int_\Omega |u(x)|^2 \, dx + \sum_{i=1}^n \int_\Omega \left| \frac{\partial u(x)}{\partial x_i} \right|^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

Remark 2.1 : *According to the definition of the space $H^1(\Omega)$, we conclude that*

$$H^1(\Omega) \subset L^2(\Omega).$$

Example 2.1 :

1) *Let the function*

$$\begin{aligned} H : \ &]-1,1[\to \mathbb{R} \\ & x \longmapsto H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}. \end{aligned}$$

We have

$$\int_{-1}^1 |H(x)|^2 \, dx = 1 < \infty \Longrightarrow H \in L^2(]-1,1[).$$

So it defines a distribution T_H *on* $]-1,1[$ *and for all* $\varphi \in \mathcal{D}(]-1,1[)$ *we have*

$$\langle (T_H)', \varphi\rangle = -\langle T_H, \varphi'\rangle = \varphi'(0) = \langle \delta_0, \varphi'\rangle \Longrightarrow H' = \delta_0.$$

As $\delta_0 \notin L^2(]-1,1[)$, *then* $H \notin H^1(]-1,1[)$.
2) *Let* f *be a function defined by*

$$\begin{aligned} f:\ &]-1,1[\to \mathbb{R} \\ & x \longmapsto f(x) = |x| \end{aligned}.$$

It is clear that $f \in L^2(]-1,1[)$, *it remains to verify that* $f' \in L^2(]-1,1[)$.
We have $f \in L^2(]-1,1[)$, *so it defines a distribution* T_f *on* $]-1,1[$ *and for all* $\varphi \in \mathcal{D}(]-1,1[)$ *we have*

$$\begin{aligned} \langle (T_f)', \varphi\rangle &= -\langle T_f, \varphi'\rangle = -\int_{-1}^{1} f(x)\varphi'(x)dx \\ &= \int_{-1}^{0} x\varphi'(x)dx - \int_{0}^{1} x\varphi'(x)dx. \end{aligned}$$

We integrate by parts, we get

$$\langle (T_f)', \varphi\rangle = -\int_{-1}^{0} \varphi(x)dx + \int_{0}^{1} \varphi(x)dx.$$

Let the function

$$\begin{aligned} H:\ &]-1,1[\to \mathbb{R} \\ & x \longmapsto H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}, \end{aligned}$$

Then

$$(2H-1)(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases},$$

and

$$\int_{-1}^{1} |(2H-1)(x)|^2 dx = 2 < \infty,$$

Therefore

$$(2H-1) \in L^2(]-1,1[).$$

So it defines a distribution on $]-1,1[$ *noted* T_{2H-1}, *it means*

$$\begin{aligned} \langle T_{2H-1}, \varphi\rangle &= \int_{-1}^{1} (2H-1)(x)\varphi(x)dx \\ &= -\int_{-1}^{0} \varphi(x)dx + \int_{0}^{1} \varphi(x)dx \\ &= \langle T_{f'}, \varphi\rangle, \quad \forall \varphi \in \mathcal{D}(]-1,1[). \end{aligned}$$

Then

$$(T_f)' = T_{2H-1} \Rightarrow f' \in L^2(]-1,1[).$$

As a result,

$$f \in H^1(]-1,1[).$$

Theorem 2.1 *: The space $H^1(\Omega)$ is a Hilbert space with respect to the norm $\|.\|_{H^1(\Omega)}$ associated with the inner product $\langle\cdot,\cdot\rangle_{H^1(\Omega)}$.*

Proof. $H^1(\Omega)$ is a pre-Hilbert space with respect to the inner product $\langle\cdot,\cdot\rangle_{H^1(\Omega)}$. To show that $H^1(\Omega)$ is complete with respect to the norm $\|.\|_{H^1(\Omega)}$, we need to verify that it is a Banach space. Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $H^1(\Omega)$. Then,

$$\forall\varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall p, q \in \mathbb{N},\ p \geq\ q \geq n_0 \Rightarrow \|u_p - u_q\|_{H^1(\Omega)} < \varepsilon,$$

where

$$\|u_p - u_q\|^2_{H^1(\Omega)} = \|u_p - u_q\|^2_{L^2(\Omega)} + \sum_{i=1}^{n} \left\| \frac{\partial u_p}{\partial x_i} - \frac{\partial u_q}{\partial x_i} \right\|^2_{L^2(\Omega)} < \varepsilon.$$

So $(u_n)_{n\in\mathbb{N}}$ and $(\frac{\partial u_n}{\partial x_i})_{n\in\mathbb{N}}$ are Cauchy sequences in $L^2(\Omega)$ for every $i = 1,\ldots,n$. By the completeness of $L^2(\Omega)$, we have

$$\begin{aligned} \exists u &\in L^2(\Omega) \text{ such that } u_n \underset{n\to+\infty}{\to} u \text{ in } L^2(\Omega). \\ \exists u_i &\in L^2(\Omega), \text{ such that } \frac{\partial u_n}{\partial x_i} \underset{n\to+\infty}{\to} u_i \text{ in } L^2(\Omega),\ \forall i = 1,...,n,, \end{aligned}$$

We still need to verify that

$$u_i = \frac{\partial u}{\partial x_i}\ for\ every\ i = 1,\ldots,n.$$

Since the canonical injection of $L^2(\Omega)$ into $\mathcal{D}'(\Omega)$ is continuous ($L^2(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$), we have the following:

$$u_n \underset{n\to+\infty}{\to} u, \text{ in } \mathcal{D}'(\Omega) \text{ and } \frac{\partial u_n}{\partial x_i} \underset{n\to+\infty}{\to} u_i, \text{ in } \mathcal{D}'(\Omega).$$

Using the continuity of the derivative operator in $\mathcal{D}'(\Omega)$ (i.e., in the sense of distributions), we obtain

$$\frac{\partial u_n}{\partial x_i} \underset{n\to+\infty}{\to} \frac{\partial u}{\partial x_i}\ in\ \mathcal{D}'(\Omega), \forall i = 1,...,n.$$

By the uniqueness of the limit in $\mathcal{D}'(\Omega)$, we conclude that

$$u_i = \frac{\partial u}{\partial x_i}, \forall i = 1,\ldots,n.$$

We can conclude that

$$\frac{\partial u}{\partial x_i} \in L^2(\Omega)\ for\ every\ i = 1,\ldots,n.$$

therefore $u \in H^1(\Omega)$. This shows that $H^1(\Omega)$ is a complete space.

Remark 2.2 :
If Ω is bounded, then $C^1(\overline{\Omega}) \subset H^1(\Omega)$. The space $\mathcal{D}(\Omega)$ is a subspace of $H^1(\Omega)$, but it is not generally dense in $H^1(\Omega)$.

Theorem 2.2 : *The space $H^1(\Omega)$ is a separable space.*

Proof. $H^1(\Omega)$ is a separable space if it contains a countable dense subset in $H^1(\Omega)$. We have the space $L^2(\Omega)$ is separable, and the space $(L^2(\Omega))^{n+1}$ is also separable due to the property that the Cartesian product of a finite number of separable spaces remains separable. Now, let's consider the application

$$\begin{aligned} f \quad &: \quad H^1(\Omega) \to \left(L^2(\Omega)\right)^{n+1} \\ u \quad &\to \quad f(u) = \left(u, \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}\right) \end{aligned}$$

with

$$\|f(u)\|_{(L^2(\Omega))^{n+1}} = \left(\|u\|^2_{L^2(\Omega)} + \sum_{i=1}^{n} \left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^2(\Omega)}\right)^{1/2} = \|u\|_{H^1(\Omega)},$$

so f is an isometry. From the given expression:

$$f : H^1(\Omega) \to f\left(H^1(\Omega)\right),$$

we observe that f is bijective, implying that $H^1(\Omega)$ and $f(H^1(\Omega))$ are identifiable.
By using the property that every complete space is closed then

$$f\left(H^1(\Omega)\right) \text{ is a closed subspace of } (L^2(\Omega))^{n+1}.$$

Applying the property that a closed subspace of a separable Hilbert space is separable, we can deduce that $f(H^1(\Omega))$ is also separable. Therefore, we conclude that

$$H^1(\Omega) \text{ is a separable space.}$$

■

2.1.2 Space $H_0^1(\Omega)$

Definition 2.2 : *We call the space $H_0^1(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$ and we write*

$$H_0^1(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^1(\Omega)}.$$

$$H_0^1(\Omega) = \{u \in H^1(\Omega),\ \exists(\varphi_n)_n \in \mathbb{N} \in \mathcal{D}(\Omega),\ \textit{such that} \lim_{n\to+\infty} \|\varphi_n - v\|_{H^1(\Omega)} = 0\}. \qquad (2.4)$$

Remark 2.3 :

1- The Sobolev space $H_0^1(\Omega)$ is a closed subspace of $H^1(\Omega)$ where the functions and their first derivatives vanish on the boundary of the domain Ω.

2- If $\Omega = \mathbb{R}^n$, the space $\mathcal{D}(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$ i.e.

$$H_0^1(\mathbb{R}^n) = H^1(\mathbb{R}^n) \text{ with continuous and dense injection.}$$

3- $H_0^1(\Omega)$ is a Hilbert space with respect to the inner product $(.,.)_{H^1(\Omega)}$.

Proposition 2.1 : *Let Ω be a bounded open subset of $\mathbb{R}^n$, then the space $\mathcal{D}(\Omega)$ is not dense in $H^1(\Omega)$ i.e. $H_0^1(\Omega) \subsetneq H^1(\Omega)$.*

Proof. Exercise. ■

Proposition 2.2 : *Let Ω be a bounded open of $\mathbb{R}^n$. If u in the space $H_0^1(\Omega)$ then the extension of u by 0 outside Ω is an element of $H^1(\mathbb{R}^n)$ i.e.*

$$\widetilde{u} = \begin{cases} u & \text{on } \Omega \\ 0 & \text{otherwise} \end{cases} \in H^1(\mathbb{R}^n).$$

Proof. Let $u \in H_0^1(\Omega)$, we have $H_0^1(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^1(\Omega)}$ then

$$\exists \varphi_k \in \mathcal{D}(\Omega) \text{ such that } \varphi_k \to u \text{ in } H^1(\Omega) \ \ as\ k \to +\infty.$$

Then

$$\forall |q| \leq 1, D^q(\varphi_k - u) \to 0 \ in\ L^2(\Omega), \ \ as\ k \to +\infty.$$

We have $\varphi_k \in \mathcal{D}(\Omega)$, so it extends by 0 into $(\mathbb{R}^n \backslash \Omega)$, then

$$\varphi_k \in \mathcal{D}(\mathbb{R}^n) \text{ and } \|\varphi_k\|_{H^1(\mathbb{R}^n)} = \|\varphi_k\|_{H^1(\Omega)},$$

since (φ_k) is a convergent sequence in $H^1(\Omega)$, then (φ_k) is convergent in $H^1(\mathbb{R}^n)$. This implies that (φ_k) is a Cauchy sequence in $H^1(\mathbb{R}^n)$,

$$\exists v \in H^1(\mathbb{R}^n) \text{ such that } \varphi_k \to v \ \ \text{in } H^1(\mathbb{R}^n) \ \ as\ k \to +\infty.$$

We still have to show that

$$\widetilde{u} = v = \begin{cases} u & \text{on } \Omega \\ 0 & \text{on } \mathbb{R}^n \backslash \Omega \end{cases}$$

$$\begin{aligned}
\forall |q| &\leq 1, \int_{\mathbb{R}^n} |D^q(\varphi_k - v)|^2 \, dx = \int_\Omega |D^q(\varphi_k - v)|^2 \, dx + \int_{\mathbb{R}^n \setminus \Omega} |D^q(\varphi_k - v)|^2 \, dx \\
&= \int_\Omega |D^q(\varphi_k - v)|^2 \, dx + \int_{\mathbb{R}^n \setminus \Omega} |D^q v|^2 \, dx \to 0 \; as \; k \to +\infty \\
&\Rightarrow \int_\Omega |D^q(\varphi_k - v)|^2 \, dx \to 0, \quad \text{and} \quad \int_{\mathbb{R}^n \setminus \Omega} |D^q v|^2 \, dx \to 0 \\
&\Rightarrow \int_{\mathbb{R}^n \setminus \Omega} |\nabla v|^2 \, dx + \int_{\mathbb{R}^n \setminus \Omega} |v|^2 \, dx = \|v\|^2_{H^1(\mathbb{R}^n \setminus \Omega)} = 0 \\
&\Rightarrow v = 0 \quad a.e \; on \; \mathbb{R}^n \setminus \Omega.
\end{aligned}$$

On the one hand,

$$\int_\Omega |D^q(\varphi_k - v)|^2 \, dx \underset{k \to +\infty}{\to} 0 \Rightarrow |D^q(\varphi_k - v)|^2 \underset{k \to +\infty}{\to} 0 \text{ in } L^2(\Omega).$$

On the other hand, we have

$$\int_\Omega |D^q(\varphi_k - u)|^2 \, dx \underset{k \to +\infty}{\to} 0 \Rightarrow D^q(\varphi_k - u) \underset{k \to +\infty}{\to} 0 \text{ in } L^2(\Omega),$$

and the uniqueness of the limit implies that $v = u$ a.e on Ω. ■

2.1.3 Poincare's Inequality

Poincare's inequality (named with the French mathematician Henri Poincary) allows bounding a function based on estimates of its derivatives and the geometry of its domain of definition. These estimates are of great importance in the calculus of variations.

Definition 2.3 *Let Ω be an open set in $\mathbb{R}^n$. We say that Ω is bounded in a direction if there exists a unit vector $\eta \in \mathbb{R}^n$ (i.e., $|\eta| = 1$) such that:*

$$\Omega \subset \widetilde{\Omega} = \{x \in \mathbb{R}^n : a < x.\eta < b\},$$

with

$$x.\eta = x_1\eta_1 + ... + x_n\eta_n.$$

We also say that $\widetilde{\Omega}$ is a strip in $\mathbb{R}^n$ with thickness $d = b - a$ in the direction η.

Theorem 2.3 *Let Ω be an open set in $\mathbb{R}^n$, bounded in a direction. Then, for all $v \in H_0^1(\Omega)$, we have*

$$\|v\|_{L^2(\Omega)} \leq C(\Omega) \|\nabla v\|_{L^2(\Omega)} = C(\Omega) \left(\sum_{i=1}^n \left\| \frac{\partial v}{\partial x_i} \right\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}},$$

where $C(\Omega) > 0$ is a universal constant depending on Ω.

Proof. :
Without loss of generality, we can assume that Ω is bounded in the x_n direction. Let $\Omega \subset \Omega' \times]a, b[$, where Ω' is an open set in $\mathbb{R}^{n-1}$ and we denote

$$x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}.$$

Since $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$, it is sufficient to prove the inequality in $\mathcal{D}(\Omega)$.
Let $v \in \mathcal{D}(\Omega)$, there exist $\widetilde{v} \in \mathcal{D}(\mathbb{R}^n)$, such that

$$\widetilde{v}(x) = \begin{cases} v(x) & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^n \backslash \Omega \end{cases},$$

Let $x = (x', x_n) \in \mathbb{R}^n$, we can write

$$\widetilde{v}(x) = \int_a^{x_n} \frac{\partial \widetilde{v}}{\partial x_n}(x', t)\, dt = -\int_{x_n}^b \frac{\partial \widetilde{v}}{\partial x_n}(x', t)\, dt,$$

then

$$2\widetilde{v}(x) = \int_a^{x_n} \frac{\partial \widetilde{v}}{\partial x_n}(x', t)\, dt - \int_{x_n}^b \frac{\partial \widetilde{v}}{\partial x_n}(x', t)\, dt.$$

We find that

$$\begin{aligned} 2\left|\widetilde{v}(x)\right| &= \left|\int_a^{x_n} \frac{\partial \widetilde{v}}{\partial x_n}(x', t)\, dt - \int_{x_n}^b \frac{\partial \widetilde{v}}{\partial x_n}(x', t)\, dt\right| \\ &\leq \left|\int_a^{x_n} \frac{\partial \widetilde{v}}{\partial x_n}(x', t)\, dt\right| + \left|\int_{x_n}^b \frac{\partial \widetilde{v}}{\partial x_n}(x', t)\, dt\right| \\ &\leq \int_a^{x_n} \left|\frac{\partial \widetilde{v}}{\partial x_n}(x', t)\right| dt + \int_{x_n}^b \left|\frac{\partial \widetilde{v}}{\partial x_n}(x', t)\right| dt \\ &= \int_a^b \left|\frac{\partial \widetilde{v}}{\partial x_n}(x', t)\right| dt. \end{aligned}$$

Then

$$\begin{aligned} \left|\widetilde{v}(x)\right|^2 &\leq \frac{1}{2^2}\left(\int_a^b \left|\frac{\partial \widetilde{v}}{\partial x_n}(x', t)\right| dt\right)^2 \\ &\overset{Cauchy-Schwarz}{\leq} \frac{1}{2^2}\left(\int_a^b 1^2 dt\right)\left(\int_a^b \left|\frac{\partial \widetilde{v}}{\partial x_n}(x', t)\right|^2 dt\right) \\ &= \frac{b-a}{2^2}\int_a^b \left|\frac{\partial \widetilde{v}(x', t)}{\partial x_n}\right|^2 dt. \end{aligned}$$

Integrating with respect to the variable x_n from a to b, we arrive at

$$\begin{aligned} \int_a^b \left|\widetilde{v}(x)\right|^2 dx_n &\leq \frac{b-a}{2^2}\int_a^b \left(\int_a^b \left|\frac{\partial \widetilde{v}}{\partial x_n}(x', t)\right|^2 dt\right) dx_n \\ &\leq \left(\frac{b-a}{2}\right)^2 \int_a^b \left|\frac{\partial \widetilde{v}}{\partial x_n}(x', t)\right|^2 dt. \end{aligned}$$

As a result,

$$\int_{\widetilde{\Omega}} |\widetilde{v}(x)|^2 \, dx \leq \left(\frac{b-a}{2}\right)^2 \int_{\widetilde{\Omega}} \left|\frac{\partial \widetilde{v}}{\partial x_n}(x)\right|^2 dx,$$

where

$$\begin{aligned}\int_{\Omega} |v(x)|^2 \, dx &\leq \left(\frac{b-a}{2}\right)^2 \int_{\Omega} \left|\frac{\partial v}{\partial x_n}(x)\right|^2 dx \\ &\leq \left(\frac{b-a}{2}\right)^2 \int_{\Omega} \sum_{i=1}^{n} \left|\frac{\partial v}{\partial x_i}(x)\right|^2 dx,\end{aligned}$$

then

$$\|v\|_{L^2(\Omega)} \leq \frac{b-a}{2} \left(\int_{\Omega} \sum_{i=1}^{n} \left|\frac{\partial v}{\partial x_i}(x)\right|^2 dx\right)^{\frac{1}{2}} = \frac{1}{2} d \|\nabla v\|_{L^2(\Omega)}.$$

Using the density of $\mathcal{D}(\Omega)$ in $H_0^1(\Omega)$, we obtain

$$\exists C(\Omega) = \frac{d}{2}, \ \|v\|_{L^2(\Omega)} \leq C(\Omega) \|\nabla v\|_{L^2(\Omega)}, \ \forall v \in H_0^1(\Omega).$$

This finishes the proof. ■

Remark 2.4 :
1- The optimal constant C in Poincare's inequality is sometimes called the Poincare constant of the Ω domain.
In general, determining the Poincare constant is a very difficult task that depends on the geometry of the domain Ω. In some cases, bounds can be given.
2- In general, Poincare's inequality is not true in $H^1(\Omega)$, for example, let the function

$$v(x) = 1, with\ x \in \Omega =]-1, 1[.$$

We have $v \in H^1(]-1,1[)$ with

$$\|v\|_{L^2(]-1,1[)} = \sqrt{2},$$

and

$$\left\|\frac{\partial v}{\partial x}\right\|_{L^2(]-1,1[)} = 0,$$

then $\nexists c > 0$, such that

$$\|v\|_{L^2(]-1,1[)} \leq c \left\|\frac{\partial v}{\partial x}\right\|_{L^2(]-1,1[)}.$$

Corollary 2.1 *If Ω is bounded set of $\mathbb{R}^n$, the semi-norm*

$$|v|_{1,\Omega} = \|\nabla v\|_{L^2(\Omega)} = \left(\sum_{i=1}^{n} \left\|\frac{\partial v}{\partial x_i}\right\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}},$$

is a norm on $H_0^1(\Omega)$ *that is equivalent to the norm induced by* $H^1(\Omega)$*. We denote this norm as*

$$\|v\|_{H_0^1(\Omega)} = |v|_{1,\Omega} = \|\nabla v\|_{L^2(\Omega)} = \left(\sum_{i=1}^{n} \left\|\frac{\partial v}{\partial xi}\right\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}, \ \forall v \in H_0^1(\Omega).$$

Proof. We need to prove that for any $v \in H_0^1(\Omega)$, if $\|v\|_{H_0^1(\Omega)} = 0$, then $v = 0$. We have

$$\|\nabla v\|_{L^2(\Omega)} = \left(\sum_{i=1}^{n} \left\|\frac{\partial v}{\partial x_i}\right\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} = 0.$$

Using the Poincare inequality,

$$\|v\|_{L^2(\Omega)} \leq C(\Omega)\|\nabla v\|_{L^2(\Omega)} = C\ d\left(\sum_{i=1}^{n} \left\|\frac{\partial v}{\partial x_i}\right\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}},$$

we conclude that

$$\|v\|_{L^2(\Omega)} = 0 \Rightarrow v = 0.$$

2) Let's show that $\|.\|_{H_0^1(\Omega)}$ is equivalent to $\|.\|_{H^1(\Omega)}$.
$i)$ We have

$$\|v\|_{H^1(\Omega)}^2 = \|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2 \geq \|\nabla v\|_{L^2(\Omega)}^2.$$

Then

$$\|v\|_{H^1(\Omega)} \geq \|\nabla v\|_{L^2(\Omega)} = \|v\|_{H_0^1(\Omega)}.$$

$ii)$ On the other hand, we have

$$\begin{aligned} \|v\|_{H^1(\Omega)} &= \left(\|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} \\ &\overset{Ineg.Poincaré}{\leq} \left((C(\Omega))^2\|\nabla v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} \\ &= \left(1 + (C(\Omega))^2\right)^{\frac{1}{2}}\|\nabla v\|_{L^2(\Omega)} \\ &= K(\Omega)\|v\|_{H_0^1(\Omega)}. \end{aligned}$$

From these two inequalities, we get

$$|v|_{1,\Omega} \leq \|v\|_{H^1(\Omega)} \leq K(\Omega)|v|_{1,\Omega}, \tag{2.5}$$

we can conclude that the norms $|.|_{1,\Omega}$ and $\|.\|_{H^1(\Omega)}$ are equivalent. ■

2.1.4 Trace Theorem

The functions of $H^1(\Omega)$ (have derivatives in the space $L^2(\Omega)$ in the sense of distributions) are in general not continuous. However, they do not have too severe discontinuities, and admit "traces" on sub-varieties of dimension $(n-1)$.
Let Ω be an open set of $\mathbb{R}^n$ with boundary $\Gamma = \partial\Omega$. If $v \in H^1(\Omega)$, how can we define the value of v at the boundary of Ω, i.e. $v_{\backslash\Gamma} = ?$
1 • **If Ω is an open set of $\mathbb{R}$,** we have

Theorem 2.4 *[10] If $v \in H^1(]a,b[)$, then $v \in \mathcal{C}([a,b])$, thus there exists a positive constant c such that*

$$\sup_{x\in[a,b]} |v(x)| \leq c \|v\|_{H^1(]a,b[)}. \tag{2.6}$$

Remark 2.5 *:*
1- As $\sup_{x\in[a,b]} |v(x)|$ is the norm of uniform convergence of the space $\mathcal{C}([a,b])$, the theorem (2.4) means that $H^1(]a,b[)$ is contained with continuous injection in $\mathcal{C}([a,b])$,

$$H^1(]a,b[) \hookrightarrow \mathcal{C}([a,b]).$$

So there's no difficulty in defining $v(a)$ and $v(b)$ for all $v \in H^1(]a,b[)$. (In general, we say that the normed vector space E is continuously injected into the normed vector space F, and we'll write $E \hookrightarrow F$, if E is a vector subspace of F and if the canonical injection

$$\begin{aligned} j: \quad & E \longrightarrow F \\ & x \longmapsto j(x) = x, \end{aligned}$$

is continuous, that is

$$\exists c > 0 : \|x\|_F \leq c\|x\|_E, \; \forall x \in E).$$

2- This result (2.6) is false in higher dimensions.

• **In the case where Ω is an open set in $\mathbb{R}^n$, $n \geq 2$**
Functions in $H^1(\Omega)$ are not generally continuous. We begin with the case of the half-space $H^1(\mathbb{R}^n_+)$ and then the general case.
First Case: Let $\Omega = \mathbb{R}^n_+$.
For $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$, we can fix the coordinate $x_n = x_n^0$ and obtain a function

$$\begin{aligned} v : \mathbb{R}^{n-1} \quad & \longrightarrow \quad \mathbb{R} \\ x' = (x_1, ..., x_{n-1}) \quad & \longrightarrow \quad v(x') = u(x', x_n^0) \end{aligned}$$

which is continuous on $\mathbb{R}^{n-1}$. We then say that the function v is the trace of u on the hyperplane $x_n = x_n^0$.

We will show that we can define, for a function in $H^1(\mathbb{R}^n)$, an extension of the trace concept to hyperplanes $x_n = x_n^0$. We can assume $x_n^0 = 0$, and we will demonstrate that we can define a notion of trace for functions in $H^1(\mathbb{R}_+^n)$ on the boundary Γ, where

$$\mathbb{R}_+^n = \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}.$$

In this case, we have

$$\partial\mathbb{R}_+^n = \Gamma = \{x = (x', 0) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}\} \simeq \mathbb{R}^{n-1}.$$

Proposition 2.3 : *The space of smooth, compactly supported functions defined on the closure of the positive orthant is dense in the Sobolev space $H^1(\mathbb{R}_+^n)$:*

$$\mathcal{D}\left(\overline{\mathbb{R}_+^n}\right) \ \textit{is dense in } H^1\left(\mathbb{R}_+^n\right).$$

Proof.

Step 1: Approximation by functions in $\mathcal{D}(\mathbb{R}_+^n)$.

Since $\mathcal{D}(\mathbb{R}_+^n)$ consists of smooth, compactly supported functions, it is dense in $L^2(\mathbb{R}_+^n)$. In other words, for any $f \in L^2(\mathbb{R}_+^n)$, there exists a sequence f_k in $\mathcal{D}(\mathbb{R}_+^n)$ such that

$$f_k \to f \ \in L^2(\mathbb{R}_+^n) \ as \ k \to \infty.$$

Step 2: Approximation of weak Derivatives

Now, for any function $f \in H^1(\mathbb{R}_+^n)$, its weak derivatives up to order 1 are well-defined in the distributional sense. By Step 1, we can find a sequence f_k in $\mathcal{D}(\mathbb{R}_+^n)$ such that $f_k \underset{k\to+\infty}{\to} f$ in $L^2(\mathbb{R}_+^n)$.

Step 3: Approximation in $H^1(\mathbb{R}_+^n)$

We need to show that the weak derivatives of f_k also converge to the weak derivatives of f in $L^2(\mathbb{R}_+^n)$. This follows from the fact that $\mathcal{D}(\mathbb{R}_+^n)$ is dense in $L^2(\mathbb{R}_+^n)$ and the weak derivatives are linear operators, which means that

$$f_k \underset{k\to+\infty}{\to} f \ in \ L^2(\mathbb{R}_+^n),$$

implies the convergence of their weak derivatives in $L^2(\mathbb{R}_+^n)$. Hence, we can conclude that $\mathcal{D}(\overline{\mathbb{R}_+^n})$ is dense in $H^1(\mathbb{R}_+^n)$. ■

Remark 2.6 *: If $v \in \mathcal{D}\left(\overline{\mathbb{R}_+^n}\right)$, we can define the restriction $v|_\Gamma$ (by the definition of $\mathcal{D}\left(\overline{\mathbb{R}_+^n}\right)$).*

We will show that $v_{|\Gamma} \in L^2(\Gamma)$ and that it depends continuously on the norm of v in $H^1(\mathbb{R}_+^n)$. Thus, we can define a trace for any function in $H^1(\mathbb{R}_+^n)$ by density.

Proposition 2.4 *: For any function $v \in \mathcal{D}(\overline{\mathbb{R}_+^n})$, we have $v(x',0) \in L^2(\mathbb{R}^{n-1})$ and*

$$\|v(.,0)\|^2_{L^2(\mathbb{R}^{n-1})} \leq \|v\|^2_{H^1(\mathbb{R}_+^n)} .$$

Proof. Let $v \in \mathcal{D}\left(\overline{\mathbb{R}_+^n}\right)$ be a function that is very smooth and has a limit of zero at infinity. We have

$$\begin{aligned} v^2(x',0) &= -\int_0^{+\infty} \frac{\partial}{\partial x_n}\left(v^2(x',x_n)\right)dx_n \\ &= -2\int_0^{+\infty} v(x',x_n).\frac{\partial}{\partial x_n}v(x',x_n)dx_n \\ &\leq \int_0^{+\infty} v^2(x',x_n)dx_n + \int_0^{+\infty}\frac{\partial v^2}{\partial x_n}(x',x_n)dx_n. \end{aligned}$$

then

$$\int_{\mathbb{R}^{n-1}} v^2(x',0)dx' \leq \int_{\mathbb{R}^{n-1}}\int_0^{+\infty} v^2(x',x_n)dx_ndx' + \int_{\mathbb{R}^{n-1}}\int_0^{+\infty}\frac{\partial v^2}{\partial x_n}(x',x_n)dx_ndx',$$

therefore

$$\|v\|^2_{L^2(\mathbb{R}^{n-1})} \leq \|v\|^2_{H^1\left(\mathbb{R}_+^n\right)}, \ \forall v \in \mathcal{D}\left(\overline{\mathbb{R}_+^n}\right).$$

The proof is complete. ■

Remarks: Consider the application:

$$\begin{aligned} \gamma_0 : \mathcal{D}\left(\overline{\mathbb{R}_+^n}\right) &\to L^2\left(\mathbb{R}^{n-1}\right) \\ v &\to \gamma_0(v) = v\left(x',0\right) \end{aligned} \tag{2.7}$$

which is a continuous linear application (according to the previous proposition). As $\mathcal{D}\left(\overline{\mathbb{R}_+^n}\right)$ is dense in $H^1\left(\mathbb{R}_+^n\right)$, the application γ_0 extends into a continuous linear application, again denoted by γ_0 de $H^1\left(\mathbb{R}_+^n\right)$ in $L^2\left(\mathbb{R}^{n-1}\right)$. Then,

Lemma 2.1 *There is an application*

$$\begin{aligned} \gamma_0 : \ H^1\left(\mathbb{R}_+^n\right) &\to L^2\left(\mathbb{R}^{n-1}\right) \\ v &\to \gamma_0(v) \end{aligned}$$

where

$$\gamma_0(v) = v\left(x',0\right) = v_{\backslash\Gamma}, \forall v \in \mathcal{D}\left(\overline{\mathbb{R}_+^n}\right).$$

In other words, if $\Omega = \mathbb{R}^n_+$, *we can define the boundary value* $v_{\backslash\Gamma}$ *of an function* $v \in H^1\left(\mathbb{R}^n_+\right)$ *as a function of* $L^2(\Gamma)$.

Theorem 2.5 *: Let* Ω *be a regular open set, there exists an operator* $P \in L(H^1(\Omega), H^1(\mathbb{R}^n))$ *called* P*-prolongation such that*

$$\forall v \in H^1(\Omega), \quad Pv = v \quad a.e \ \ \Omega.$$

Definition 2.4 *An open set* Ω *of* $\mathbb{R}^n$ *is called 1-regular if* Ω *is bounded and if its boundary* Γ *is a variety of class* C^1 *of dimension* $(n-1)$, Ω *being locally on one side of* Γ.

Second Case: In the general case, we will assume that the result is still true for sufficiently regular open sets.
Let Ω be an open subset of $\mathbb{R}^n$. $\mathcal{D}(\overline{\Omega})$ denotes the space of restrictions to Ω of functions in $\mathcal{D}(\mathbb{R}^n)$.

Proposition 2.5 *If* Ω *is sufficiently regular, the space of test functions* $\mathcal{D}(\overline{\Omega})$ *is dense in* $H^1(\Omega)$.

Theorem 2.6 *Let* Ω *be an open set in* $\mathbb{R}^n$ *with a smooth boundary* $\Gamma = \partial\Omega$. *The application*

$$\begin{aligned} \gamma_0 : \mathcal{D}\left(\overline{\Omega}\right) &\rightarrow L^2(\Gamma) \\ v &\rightarrow \gamma_0(v) = v_{\backslash\Gamma} \end{aligned}$$

extends continuously to a linear and continuous application γ_0, *called the trace map, defined by*

$$\begin{aligned} \gamma_0 : H^1(\Omega) &\rightarrow L^2(\Gamma) \\ v &\rightarrow \gamma_0(v) = v_{\backslash\Gamma} \end{aligned}$$

Remark 2.7 *:*
1) $\ker\gamma_0 = H^1_0(\Omega) = \left\{v \in H^1(\Omega) : v_{\backslash\Gamma} = 0\right\}$.
2) The application γ_0 *is not surjective in general, but its image is a dense subspace in* $L^2(\Gamma)$.

Trace theorem application: Green's formula

Theorem 2.7 *Assume that* Ω *is a bounded open boundary* $\mathcal{C}^1$ *by pieces, then, if* u *and* v *are functions of* $H^1(\Omega)$, *we have*

$$\int_\Omega \frac{\partial u}{\partial x_i} v dx = -\int_\Omega u \frac{\partial v}{\partial x_i} dx + \int_\Gamma u v \eta_i d\Gamma, \quad 1 \leq i \leq n.$$

where $\eta = (\eta_1, ..., \eta_n)$ *designates the vector normal to* Γ *oriented outwards from* Γ.

2.1.5 Compactness theorem

Definition 2.5 : *Let E, F be two normed vector spaces. We say that E is injected into F with compact imbbeding and we write $E \hookrightarrow F$, if E is a vector subspace of F and if the canonical injection*

$$\begin{aligned} j: \quad & E \longrightarrow F \\ & x \longmapsto j(x) = x, \end{aligned}$$

is compact, i.e. all bounded in E is relatively compact in F.

Theorem 2.8 *(Rellich Kondrachov):*
Let Ω be a bounded and regular open set in $\mathbb{R}^n$. The canonical injection from $H^1(\Omega)$ to $L^2(\Omega)$ is compact, which means that every bounded set in $H^1(\Omega)$ is relatively compact in $L^2(\Omega)$. In other words, from any bounded sequence in $H^1(\Omega)$, we can extract a convergent subsequence in $L^2(\Omega)$.

To prove that the canonical injection is compact, we use the following lemma:

Lemma 2.2 *For a Banach space E to be reflexive it is necessary and sufficient that its unit ball be weakly compact, i.e. that from any bounded sequence $(v_n)_{n\in\mathbf{N}}$ one can extract a weakly convergent sub-sequence (v_μ). i.e. for $v \in E$*

$$\langle \tau, v_\mu \rangle \rightharpoonup \langle \tau, v \rangle \quad \forall \tau \in E'.$$

Proof. (Rellich Kondrachov Theorem)
To show that the canonical injection is compact, it is necessary and sufficient to show that the unit ball of $H^1(\Omega)$ is compact in the sense of $L^2(\Omega)$, i.e. show that from any bounded sequence in $H^1(\Omega)$ we can extract a convergent sub-sequence in $L^2(\Omega)$. We will show the elements of the demonstration for $\Omega = \mathbb{R}^n$.
Let $(v_n)_{n\in\mathbb{N}}$ be a bounded sequence of $H^1(\mathbb{R}^n)$ and therefore bounded in $L^2(\mathbb{R}^n)$. Now $L^2(\mathbb{R}^n)$ being a reflexive Hilbert. We'll use the previous lemma to characterize reflexive spaces. We can then extract a sub-sequence (v_μ) such that

$$v_\mu \underset{\mu\to+\infty}{\rightharpoonup} v \text{ in } L^2(\mathbb{R}^n) \text{ (weak convergence,)}$$

modulo a translation, we can assume $v = 0$ (without restricting generality). It then remains to show that (v_μ) converges strongly to 0 in $L^2(\mathbb{R}^n)$, i.e. that

$$\int_{\mathbb{R}^n} |v_\mu|^2 \, dx \longrightarrow 0.$$

Let $\widehat{v}_\mu$ be the Fourier transform of v_μ. According to Plancherel's theorem, the application

$$\begin{aligned} L^2(\mathbb{R}^n) &\rightarrow L^2(\mathbb{R}^n) \\ v_\mu &\rightarrow \widehat{v}_\mu \end{aligned}$$

is an isometry, we get.

$$\|\widehat{v}_\mu\|_{L^2(\mathbb{R}^n)} = \|v_\mu\|_{L^2(\mathbb{R}^n)} \Rightarrow \int_{\mathbb{R}^n} |\widehat{v}_\mu(x)|^2 \, dx = \int_{\mathbb{R}^n} |v_\mu(x)|^2 \, dx,$$

Let $M > 0$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} |\widehat{v}_\mu(\xi)|^2 \, dx &= \int_{|\xi|<M} |\widehat{v}_\mu(\xi)|^2 \, d\xi + \int_{|\xi|>M} \frac{1+|\xi|^2}{1+|\xi|^2} |\widehat{v}_\mu(\xi)|^2 \, d\xi \\ &\leq \int_{|\xi|<M} |\widehat{v}_\mu(\xi)|^2 \, d\xi + \frac{1}{1+M^2} \int_{|\xi|\geq M} \left(1+|\xi|^2\right) |\widehat{v}_\mu(\xi)|^2 \, d\xi. \end{aligned}$$

However, it is easy to see that

$$\|v\|_{H^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{1/2} |\widehat{v}(\xi)|^2 \, d\xi,$$

because

$$H^1(\mathbb{R}^n) = \left\{v \in L^2(\mathbb{R}^n), \quad \left(1+|\xi|^2\right) \widehat{v} \in L^2(\mathbb{R}^n)\right\},$$

and therefore

$$\int_{\mathbb{R}^n} |\widehat{v}_\mu|^2 \, d\xi \leq \int_{|\xi|<M} |\widehat{v}_\mu(\xi)|^2 \, d\xi + \frac{1}{1+M^2} \|v_\mu\|_{1,\mathbb{R}^n}.$$

Or,

$$\widehat{v}_\mu(\xi) = \int_{\mathbb{R}^n} e^{-2i\pi x\xi} v_\mu(x) dx,$$

because Ω is bounded, $\Omega \subset K$ (K compact)

$$\begin{aligned} \widehat{v}_\mu(\xi) &= \int_{\mathbb{R}^n} 1_K(x).e^{-2i\pi x\xi} v_\mu(x) dx \\ &= \int_K e^{-2i\pi x\xi} v_\mu(x) dx, \end{aligned}$$

where the 1_K is the indicator function of K i.e

$$1_K(x) = \begin{cases} 1 & \text{if } x \in K \\ 0 & \text{if not} \end{cases}$$

but the application

$$x \rightarrow 1_K e^{-2i\pi x\xi} \in L^2(\mathbb{R}^n) \ and \ v_\mu \longrightarrow 0 \ in \ L^2(\mathbb{R}^n),$$

where

$$\lim_{\mu \to +\infty} \widehat{v}_\mu(\xi) = 0.$$

Furthermore

$$\begin{aligned} |\widehat{v}_\mu(\xi)| &\leq \left(\int \left| 1_K \, e^{-2i\pi x \xi} \right|^2 dx \right)^{1/2} \|v_\mu\|_{0,\mathbb{R}^n} \\ &\leq c \, \|v_\mu\|_{0,\mathbb{R}^n} \leq c \, \|v_\mu\|_{1,\mathbb{R}^n} \\ &\leq C. \end{aligned}$$

We can apply the "DCT Theorem" to show that

$$\int_{|\xi|<M} |\widehat{v}_\mu(\xi)|^2 \, d\xi \longrightarrow 0 \text{ when } \mu \to +\infty.$$

For all M, we show that: $\lim\limits_{\mu \to +\infty} \|\widehat{v}_\mu\|^2_{0,\mathbb{R}^n} \leq \frac{1}{1+M^2}.c,$ therefore

$$\lim_{\mu \to +\infty} \|\widehat{v}_\mu\|^2_{0,\mathbb{R}^n} = 0 \quad \text{and} \quad v_\mu \longrightarrow 0 \text{ in } L^2 \text{ strongly}$$

which completes the demonstration of Rellich's Theorem. ■

2.1.6 The space $H^{-1}(\Omega)$

Definition 2.6 *The space $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$ and we write*

$$\left(H_0^1(\Omega)\right)' = H^{-1}(\Omega).$$

Corollary 2.2 *Let L a continuous linear form on $H_0^1(\Omega)$, then L can be identified with a distribution in space $H^{-1}(\Omega)$.*

Proof. From Riez's representation theorem, we have

$$\begin{aligned} \exists h &\in H_0^1(\Omega) \text{ such that } L(v) = \langle h, v \rangle_{H^1(\Omega)}, \quad \forall v \in H_0^1(\Omega), \\ \langle h, v \rangle_{H^1(\Omega)} &= \langle h, v \rangle_{L^2(\Omega)} + \sum_{i=1}^n \left\langle \frac{\partial h}{\partial x_i}, \frac{\partial v}{\partial x_i} \right\rangle_{L^2(\Omega)}. \end{aligned}$$

If $v \in \mathcal{D}(\Omega)$, we'll have

$$L(v) = \left\langle h - \sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2}, v \right\rangle,$$

if we set $g_i = -\frac{\partial h}{\partial x_i}$, we'll get

$$L(v) = \left\langle h + \sum_{i=1}^n \frac{\partial g_i}{\partial x_i}, v \right\rangle.$$

Hence L can be identified as a distribution

$$T = h + \sum_{i=1}^{n} \frac{\partial g_i}{\partial x_i}, \; h, g_i \in L^2(\Omega), \; T \in \mathcal{D}'(\Omega).$$

Then $T \in H^{-1}(\Omega)$ because

$$\langle T, \varphi \rangle = \left\langle h + \sum_{i=1}^{n} \frac{\partial g_i}{\partial x_i}, \varphi \right\rangle = \langle h, \varphi \rangle - \sum_{i=1}^{n} \left\langle g_i, \frac{\partial \varphi}{\partial x_i} \right\rangle, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Then

$$\begin{aligned} |\langle T, \varphi \rangle| &\leq \|h\| \,.\, \|\varphi\| + \sum_{i=1}^{n} \|g_i\| \,.\, \left\| \frac{\partial \varphi}{\partial x_i} \right\| \\ &\leq \left(c \, \|h\|_{L^2(\Omega)} + \sum_{i=1}^{n} \|g_i\|_{L^2(\Omega)} \right) \|\varphi\|_{H_0^1(\Omega)} \,. \end{aligned}$$

The proof is complete. ■

Theorem 2.9 *Let Ω be a non-empty open of $\mathbb{R}^n$, the map*

$$\begin{aligned} H^1(\Omega) &\rightarrow H^{-1}(\Omega) \\ u &\rightarrow \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + u, \end{aligned}$$

define an isometry from $H_0^1(\Omega)$ on $H^{-1}(\Omega)$.

Proof. To show that the map is an isometry from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$, we need to demonstrate that it preserves the norm of functions in $H_0^1(\Omega)$.

Let $u \in H_0^1(\Omega)$, then

$$\frac{\partial u}{\partial x_i} \in L^2(\Omega), \; for \; all \; 1 \leq i \leq n,$$

since $u \in H^1(\Omega)$. Also, since u has compact support in Ω, its second partial derivatives $(\frac{\partial^2 u}{\partial x_i^2})$ are also in $L^2(\Omega)$ for all $1 \leq i \leq n$.

Now, consider the map

$$\begin{aligned} T : H_0^1(\Omega) &\rightarrow H^{-1}(\Omega) \\ u &\rightarrow \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + u. \end{aligned}$$

We need to show that $\|Tu\|_{H^{-1}(\Omega)} = \|u\|_{H^1(\Omega)}$.

First, let's show that $\|Tu\|_{H^{-1}(\Omega)}$ is bounded by $\|u\|_{H^1(\Omega)}$. Using the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
|\langle Tu, v\rangle| &= \left|\sum_{i=1}^{n}\int_\Omega \frac{\partial^2 u}{\partial x_i^2} v dx + \int_\Omega uv, dx\right| \\
&\leq \sum_{i=1}^{n}\left|\int_\Omega \frac{\partial^2 u}{\partial x_i^2} v dx\right| + \left|\int_\Omega uv dx\right| \\
&\leq \sum_{i=1}^{n}\left\|\frac{\partial^2 u}{\partial x_i^2}\right\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\
&= \left(\sum_{i=1}^{n}\left\|\frac{\partial^2 u}{\partial x_i^2}\right\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right)\|v\|_{L^2(\Omega)} \\
&\leq (1 + \|u\|_{H^1(\Omega)})\|v\|_{H^1(\Omega)}.
\end{aligned}
$$

Thus,

$$\|Tu\|_{H^{-1}(\Omega)} \leq (1 + \|u\|_{H^1(\Omega)})\|u\|_{H^1(\Omega)}.$$

Now, let's show that $\|Tu\|_{H^{-1}(\Omega)}$ is at least $\|u\|_{H^1(\Omega)}$.
Let $v = u$ in the above inequality, then we have

$$
\begin{aligned}
|\langle Tu, u\rangle| &\leq (1 + \|u\|_{H^1(\Omega)})\|u\|_{H^1(\Omega)}\|u\|_{H^1(\Omega)} \\
&\leq (1 + \|u\|_{H^1(\Omega)})\|u\|_{H^1(\Omega)}^2.
\end{aligned}
$$

Since $u \in H_0^1(\Omega)$, we know that $\|u\|_{H^1(\Omega)} < \infty$. Therefore, dividing both sides by $(\|u\|_{H^1(\Omega)} + 1)$, we get:

$$\frac{|\langle Tu, u\rangle|}{\|u\|_{H^1(\Omega)} + 1} \leq \|u\|_{H^1(\Omega)}^2. \tag{2.8}$$

Thus,

$$\|Tu\|_{H^{-1}(\Omega)} \geq \|u\|_{H^1(\Omega)}. \tag{2.9}$$

By using (2.8) and (2.9), we obtain that

$$\|Tu\|_{H^{-1}(\Omega)} = \|u\|_{H^1(\Omega)},$$

and thus the map T is an isometry from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$. ■

Corollary 2.3 *The space $\mathcal{D}(\Omega)$ is dense in $H^{-1}(\Omega)$.*

Proof. The space $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$. Then, the image of $\mathcal{D}(\Omega)$ under the map

$$
\begin{aligned}
H^1(\Omega) &\to H^{-1}(\Omega) \\
u &\to \sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2} + u,
\end{aligned}
$$

is valued in the space $\mathcal{D}(\Omega)$, hence the density of $\mathcal{D}(\Omega)$ in $H^{-1}(\Omega)$. ■

2.2 Sobolev Space H$^m(\Omega)$

The study of certain partial differential equations, in particular for the bi-Laplacian, may require the use of Sobolev spaces of order m greater than 1, so we generalize the definition of the Sobolev space $H^1(\Omega)$.

2.2.1 Definitions and properties

Let Ω be a non-empty open of $\mathbb{R}^n$ and $m \geq 1$ be an integer. Let call a Sobolev space of order m on Ω the space

$$H^m(\Omega) = \left\{ u \in L^2(\Omega) : \underbrace{D^\alpha u \in L^2(\Omega)}_{\text{in the sense of distributions}}, \ \forall \alpha \in \mathbb{N}^n; \ |\alpha| \leq m \right\},$$

equipped with the inner product

$$\langle u, v \rangle_{H^m(\Omega)} = \sum_{|\alpha|\leq m} \langle D^\alpha u, D^\alpha v \rangle_{L^2(\Omega)} = \sum_{|\alpha|\leq m} \int_\Omega D^\alpha u(x) \overline{D^\alpha v(x)} dx,$$

where $\langle .,. \rangle_{L^2(\Omega)}$ is the scalar product of $L^2(\Omega)$, and

$$\begin{aligned} \|u\|_{H^m(\Omega)} &= \left(\langle u, u \rangle_{H^m(\Omega)} \right)^{\frac{1}{2}} = \left(\sum_{|\alpha|\leq m} \langle D^\alpha u, D^\alpha u \rangle_{L^2(\Omega)} \right)^{\frac{1}{2}} \\ &= \left(\sum_{|\alpha|\leq m} \int_\Omega D^\alpha u(x) . \overline{D^\alpha u(x)} dx \right)^{\frac{1}{2}} \\ &= \left(\sum_{|\alpha|\leq m} \int_\Omega |D^\alpha u(x)|^2 \, dx \right)^{\frac{1}{2}} \\ &= \left(\sum_{|\alpha|\leq m} \|D^\alpha u(x)\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}. \end{aligned}$$

Theorem 2.10 : *$H^m(\Omega)$ is a Hilbert space for the norm $\|.\|_{H^m(\Omega)}$ associated with the scalar product $\langle .,. \rangle_{H^m(\Omega)}$.*

Proof. The space $H^m(\Omega)$ provided with the scalar product $\langle .,. \rangle_{H^m(\Omega)}$ is a pre-Hilbertian space, it remains to show that it is complete for the norm $\|.\|_{H^m(\Omega)}$. Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $H^m(\Omega)$, then

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall p, q \in \mathbb{N}, \ (p \geq n_0 \text{ and } q \geq n_0) \Rightarrow \left(\|u_p - u_q\|_{H^m(\Omega)} < \varepsilon \right).$$

or

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall p, q \in \mathbb{N}, (p \geq n_0 \ \text{ and } q \geq n_0) \Rightarrow \left(\sum_{|\alpha| \leq m} \|D^\alpha (u_p - u_q)\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}} < \varepsilon.$$

Then

$$(p \geq n_0 \text{ and } q \geq n_0) \Rightarrow \left(\|D^\alpha (u_p - u_q)\|_{L^2(\Omega)} < \varepsilon \right), \ \forall \alpha \in \mathbb{N}^n, |\alpha| \leq m.$$

From this we deduce that $(D^\alpha u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega)$, for all $\alpha \in \mathbb{N}^n$, $|\alpha| \leq m$. Since $L^2(\Omega)$ is complete, we have

$$\exists v_\alpha \in L^2(\Omega) \ \text{ such that } D^\alpha u_n \to v_\alpha, \forall \alpha \in \mathbb{N}^n, \ \ |\alpha| \leq m.$$

More specifically,

$$u_n \to v_0 \text{ in } L^2(\Omega).$$

It remains to show that $v_\alpha = D^\alpha v_0, \forall \alpha \in \mathbb{N}^n$, $|\alpha| \leq m$.
The canonical injection of $L^2(\Omega)$ in $D'(\Omega)$ is continuous ($L^2(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$,) then

$$(D^\alpha u_n \to v_\alpha \ in \ L^2(\Omega)) \Rightarrow (T_{D^\alpha u_n} \to T_{v_\alpha} \ in \ \mathcal{D}'(\Omega)), \ \forall \alpha \in \mathbb{N}^n, \ \ |\alpha| \leq m. \tag{2.10}$$

In particular,

$$T_{u_n} \to T_{v_0} \ in \ \mathcal{D}'(\Omega).$$

From the continuity of the derivation operator in $\mathcal{D}'(\Omega)$, we obtain

$$D^\alpha T_{u_n} \to D^\alpha T_{v_0} \text{in } \mathcal{D}'(\Omega), \forall \alpha \in \mathbb{N}^n; |\alpha| \leq m. \tag{2.11}$$

Since $D^\alpha T_{u_n} = T_{D^\alpha u_n}$, by virtue of the uniqueness of the limit in $\mathcal{D}'(\Omega)$, we conclude from (2.10) and (2.11) that

$$T_{v_\alpha} = D^\alpha T_{v_0} = T_{D^\alpha v_0}.$$

This shows that

$$D^\alpha v_0 \in L^2(\Omega), \ \forall \alpha \in \mathbb{N}^n, \ |\alpha| \leq m,$$

then $v_\alpha \in H^m(\Omega)$, where $H^m(\Omega)$ is a complete space. ■

Remark 2.8 :
In the case of Sobolev spaces, a function's smoothness is quantified in relation to its derivatives. Functions within this space are distinguished by the number of derivatives of order m, where m is a positive integer. The higher this number, the greater the regularity of the function, signifying that it is "smoother".
Functions that have an infinite number of derivatives of order m for all m are considered "infinitely smooth" or "analytic" because they are regular to all orders of differentiation.

Lemma 2.3 : *Let Ω be a non-empty open set in $\mathbb{R}^n$. If $\ell, m \in \mathbb{N}$ with $\ell \geq m$. Then*

$$H^{\ell}(\Omega) \subset H^{m}(\Omega) \quad \textit{with continuous imbedding}.$$

Proof. Let $u \in H^{\ell}(\Omega)$, we have

$$D^{\alpha}u \in L^2(\Omega), \ \forall \alpha \in \mathbb{N}^n, \ |\alpha| \leq \ell.$$

As $\ell \geq m$, therefore

$$D^{\alpha}u \in L^2(\Omega), \ \forall \alpha \in \mathbb{N}^n, \ |\alpha| \leq m.$$

The result is $H^{\ell}(\Omega) \subset H^{m}(\Omega)$. Moreover, we have

$$\|u\|_{H^m} = \left(\sum_{|\alpha| \leq m} \int_{\Omega} |D^{\alpha}u(x)|^2 \, dx \right)^{\frac{1}{2}} \leq \|u\|_{H^{\ell}} = \left(\sum_{|\alpha| \leq \ell} \int_{\Omega} |D^{\alpha}u(x)|^2 \, dx \right)^{\frac{1}{2}}.$$

Then $H^{\ell}(\Omega) \hookrightarrow H^{m}(\Omega)$. ■

Theorem 2.11 : *$H^m(\Omega)$ is a separable space.*

Proof.
Let $L^2_N(\Omega)$ be the product space $(L^2(\Omega) \times L^2(\Omega) \times ... \times L^2(\Omega))$ N times, then the application

$$\begin{aligned} \|.\|_{L^2_N} : \ & L^2_N(\Omega) \to \mathbb{R} \\ & u = (u_1, ..., u_N) \longmapsto \|u\|_{L^2_N} = \left(\sum_{i=1}^{N} \|u_i\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}. \end{aligned}$$

is a norm on $L^2_N(\Omega)$. Let $u \in H^m(\Omega)$, then

$$D^{\alpha}u \in L^2(\Omega), \forall \alpha \in \mathbb{N}^n, |\alpha| \leq m \Rightarrow D^{\alpha^{(i)}}u \in L^2(\Omega), \ \forall i = \overline{1, N}.$$

For $u \in H^m(\Omega)$, we get

$$\left\| \left(D^{\alpha^{(1)}}u, D^{\alpha^{(2)}}u, ..., D^{\alpha^{(N)}}u \right) \right\|_{L^2_N} = \left(\sum_{i=1}^{N} \left\| D^{\alpha^{(i)}}u \right\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}} = \|u\|_{H^m(\Omega)}.$$

We define the application J by

$$J: \quad H^m(\Omega) \to L^2_N(\Omega)$$
$$u \longmapsto J(u) = \left(D^{\alpha^{(1)}}u, D^{\alpha^{(2)}}u, ..., D^{\alpha^{(N)}}u\right),$$

then J is linear and

$$\|J(u)\|_{L^2_N(\Omega)} = \|u\|_{H^m(\Omega)},$$

therefore J is an isometry of $H^m(\Omega)$ (which is complete) in $L^2_N(\Omega)$, hence the subspace

$$W = J(H^m(\Omega)) \; is \; closed \; in \; L^2_N(\Omega).$$

In addition, $L^2(\Omega)$ is separable, then $L^2_N(\Omega)$ is separable too. Then, $J(H^m(\Omega))$ is separable. Therefore,

$$H^m(\Omega) \cong J\left(H^m(\Omega)\right),$$

we deduce that $H^m(\Omega)$ is separable. ■

2.2.2 Space $H_0^m(\Omega)$

We note that $H_0^m(\Omega)$ the closure of $\mathcal{D}(\Omega)$ for the norm of $H^m(\Omega)$ i.e,

$$H_0^m(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^m(\Omega)}.$$

In other words,

$$H_0^m(\Omega) = \left\{u \in H^m(\Omega), \; \exists (u_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\Omega) : \|u_n - u\|_{H^m} \longrightarrow 0 \; as \; n \to +\infty\right\}.$$

Proposition 2.6 *Let $u \in H_0^m(\Omega)$ and let $\widetilde{u}$ the extension of u by zero on $\mathbb{R}^n \backslash \Omega$, i.e*

$$\widetilde{u}(x) = \begin{cases} u(x) & x \in \Omega \\ 0 & x \in \mathbb{R}^n \backslash \Omega \end{cases}.$$

If $\alpha \in \mathbb{N}^n$, $|\alpha| \leq m$, then $D^\alpha \widetilde{u} = \widetilde{D^\alpha u}$ in the sense of distributions. In other words $\widetilde{u} \in H^m(\mathbb{R}^n)$ and

$$\|u\|_{H^m(\Omega)} = \|\widetilde{u}\|_{H^m(\mathbb{R}^n)}.$$

Proof. Let $(\varphi_k)_{k\geq 1}$ a sequence of $\mathcal{D}(\Omega)$ which converges to u and $\alpha \in \mathbb{N}^n$, $|\alpha| \leq m$. If $\psi \in \mathcal{D}(\Omega)$, then

$$\begin{aligned}
(-1)^{|\alpha|} \int_{\mathbb{R}^n} \widetilde{u}(x) D^\alpha \psi(x)\, dx &= (-1)^{|\alpha|} \int_\Omega u(x) D^\alpha \psi(x)\, dx \\
&= \lim_{k\longrightarrow\infty} (-1)^{|\alpha|} \int_\Omega \varphi_k(x) D^\alpha \psi(x)\, dx \\
&= \lim_{k\longrightarrow\infty} \int_\Omega (D^\alpha \varphi_k(x))\, \psi(x)\, dx \\
&\overset{\text{TCD}}{=} \int_\Omega (D^\alpha \varphi_k(x))\, \psi(x)\, dx \\
&= \int_{\mathbb{R}^n} \widetilde{(D^\alpha \varphi_k(x))} \psi(x)\, dx.
\end{aligned}$$

Where $\mathcal{D}^\alpha \widetilde{u} = \widetilde{\mathcal{D}^\alpha u}$ in the sense of distributions. As $\mathcal{D}^\alpha u \in L^2(\Omega)$, then

$$\widetilde{\mathcal{D}^\alpha u} \in L^2(\mathbb{R}^n) \Rightarrow \mathcal{D}^\alpha \widetilde{u} \in L^2(\mathbb{R}^n).$$

So $\widetilde{u} \in H^m(\mathbb{R}^n)$ and $\|u\|_{H^m(\Omega)} = \|\widetilde{u}\|_{H^m(\mathbb{R}^n)}$. ■

Remark 2.9 :
1- As the operator $\sim$ is evidently linear, it is continuous from $H_0^m(\Omega)$ to $H^m(\mathbb{R}^n)$. More precisely, it is an isometry.
2- The space $\mathcal{D}(\mathbb{R}^n)$ is dense in $H^m(\mathbb{R}^n)$; i.e.

$$H_0^m(\mathbb{R}^n) = H^m(\mathbb{R}^n).$$

2.2.3 Poincare's inequality

It is sometimes necessary to show that a semi-norm on a closed subspace of $H^m(\Omega)$ is, in fact, equivalent to the usual norm. To establish this equivalence, we will use the following general result.

Theorem 2.12 *[19] Let Ω be a bounded open set in $\mathbb{R}^n$ (or bounded in one direction,) then*

$$\forall v \in H_0^m(\Omega) : \|v\|_{L^2(\Omega)} \leq C_m(\Omega) \sum_{|\alpha|=m} \|D^\alpha u\|_{L^2(\Omega)},$$

where $C_m(\Omega)$ is a constant depending on Ω.

Proposition 2.7 *: Let Ω be a bounded open set in $\mathbb{R}^n$ (or is bounded in one direction), the semi-norm*

$$|v|_{m,\Omega} = \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}},$$

is a norm on $H_0^m(\Omega)$ is equivalent to the norm induced by $\|v\|_{H^m(\Omega)}$, and we denote

$$\|v\|_{H_0^m(\Omega)} = |v|_{m,\Omega}.$$

2.2.4 The trace theorem for $H^m(\Omega)$

If $v \in H^2(\Omega)$, we can define its trace $\gamma_0 v \in L^2(\Gamma)$, but we can also do the same for each of the first derivatives $\dfrac{\partial v}{\partial x_i}$ for $i = \overline{1,n}$. We can also define

$$\gamma_1(v) = \sum_{i=1}^{n} \eta_i . \gamma_0 \left(\frac{\partial v}{\partial x_i} \right),$$

where $\eta = (\eta_1, ..., \eta_n)$ designates the vector normal to Γ oriented outwards from Γ. It can be shown that the application

$$\begin{aligned} (\gamma_0, \gamma_1): \quad H^2(\Omega) &\longrightarrow L^2(\Gamma) \times L^2(\Gamma) \\ v &\longmapsto (\gamma_0 v, \gamma_1 v) \end{aligned},$$

is continuous linear and its kernel is $H_0^2(\Omega)$.

Theorem 2.13 *Let Ω be a bounded open with a lipschitzian boundary, then*

$$H_0^2(\Omega) = \{u \in H^2(\Omega), \gamma_0 u = \gamma_0 \frac{\partial u}{\partial \eta} = 0 \text{ on } \Gamma\}.$$

More generally, we have the following theorem.

Theorem 2.14 *Let Ω be an open set in $\mathbb{R}^n$ with a sufficiently regular boundary, and let m be an integer greater than or equal to 1. For each multi-index $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq m-1$, there exists a linear and continuous mapping from the space $H^m(\Omega)$ to the space $(L^2(\Gamma))^m$, and we have the following*

$$\begin{aligned} H^m(\Omega) &\rightarrow \left(L^2(\Gamma)\right)^m \\ v &\rightarrow \gamma_0(D^\alpha v) \end{aligned}$$

$$H_0^m(\Omega) = \{v \in H^m(\Omega) : \forall \alpha \in \mathbb{N}^n, \ |\alpha| \leq m-1, \ \gamma_0(D^\alpha v) = 0\}.$$

2.2.5 Imbedding Theorems

Extension operators

Is there an extension operator P of $H^m(\Omega)$ in $H^m(\mathbb{R}^n)$ such that $(Pu)_{\backslash\Omega} = u$?
The answer is negative if the domain Ω is not regular, but if Ω is convex or has a lipschitzian boundary, such an operator exists for all $m \in \mathbb{N}$.

Theorem 2.15 : *Let Ω be a bounded open set in $\mathbb{R}^n$ with a Lipschitz boundary. Then there exists a continuous extension operator $P : H^m(\Omega) \longrightarrow H^m(\mathbb{R}^n)$ such that $(Pu)_{\backslash\Omega} = u$ (a.e).*

Remark 2.10 : *For $(0 \leq m' \leq m)$, we have*

$$\left(H^m(\Omega) \hookrightarrow H^{m'}(\Omega)\right) \Leftrightarrow \left(H^m(\Omega) \subset H^{m'}(\Omega)\right).$$

Corollary 2.4 : *If Ω is a open set of class $\mathcal{C}^\ell$ ($\ell \in \mathbb{N}$, $\ell \geq 1$) with Γ bounded, then*
1) If $m > \frac{n}{2} + k$, $k \in \mathbb{N}$, $\ell \geq k$, so $H^m(\Omega) \hookrightarrow \mathcal{C}^k\left(\overline{\Omega}\right)$.
2) If $m = \frac{n}{2}$, then $H^m(\Omega) \hookrightarrow L^p(\Omega)$, for all $p \in [2, \infty[$.
3) If $0 \leq m < \frac{n}{2}$, then $H^m(\Omega) \hookrightarrow L^p(\Omega)$, for all $p \in \left[2, \dfrac{2n}{n-2m}\right[$.

2.2.6 Compact Embeddings of Rellich

The Rellich-Kondrachov theorem deals with the compact Embeddings of Sobolev spaces.

Theorem 2.16 : *(Rellich-Kondrachov Theorem)*
Suppose Ω is a bounded domain with a sufficiently smooth boundary in $\mathbb{R}^n$. The Rellich-Kondrachov theorem states that the imbedding

$$H^m(\Omega) \hookrightarrow L^p(\Omega), \ \forall m \in \mathbb{N},$$

is compact for $1 \leq p < \frac{n}{n-m}$.
And for Ω be a bounded domain of class $\mathcal{C}^1$, then

$$H_0^{m+1}(\Omega) \hookrightarrow H_0^m(\Omega), \ \forall m \in \mathbb{N}.$$

Remark 2.11 *This theorem is an important result in functional analysis and plays a significant role in the study of partial differential equations (PDEs).*

2.2.7 The space $H^{-m}(\Omega)$

As $\mathcal{D}(\Omega) \hookrightarrow H_0^m(\Omega)$ (continuous injection) and

$$\overline{\mathcal{D}(\Omega)} = H_0^m(\Omega),$$

then the space $(H_0^m(\Omega))'$ (the dual of $H_0^m(\Omega)$) can be identified with a subspace of $\mathcal{D}'(\Omega)$ that we'll note $H^{-m}(\Omega)$.

Definition 2.7 *For $m \in \mathbb{N}$, we denote $H^{-m}(\Omega)$ as the dual space of $H_0^m(\Omega)$. Let T be an element of $H^m(\Omega)$.*

$$H^{-m}(\Omega) = \{T \in \mathcal{D}'(\Omega),\ T = \sum_{|\alpha|\leq m} D^\alpha v_\alpha,\ v_\alpha \in L^2(\Omega)\}, \tag{2.12}$$

Then, T defines a continuous linear functional on $\mathcal{D}(\Omega)$ equipped with the norm H^m.

Lemma 2.4 *Let $L \in H^{-m}(\Omega)$, there is $u = (u_\alpha)_{|\alpha|\leq m} \in (L^2(\Omega))^N$ such that*

$$\langle L, v\rangle = \sum_{|\alpha|\leq m} \langle D^\alpha v, u_\alpha\rangle,\ \forall v \in H_0^m(\Omega),$$

furthermore, $\|.\|_{H^{-m}(\Omega)}$ is given by

$$\|L\|_{H^{-m}(\Omega)} = \inf\left\{\|w\|_{L_N^2(\Omega)},\ w \in L_N^2(\Omega) : L(w) = \sum_{|\alpha|\leq m} \langle D^\alpha w, u_\alpha\rangle, \forall w \in H_0^m(\Omega)\right\}.$$

Remark 2.12

1) *$H^{-m}(\Omega)$ is a separable, reflexive, and complete space.*

2) *If $m > m'$, $H^{-m'}(\Omega) \subset H^{-m}(\Omega)$ with continuous injection*

$$H^{-m'}(\Omega) \hookrightarrow H^{-m}(\Omega).$$

3) *We have $\|u\|_{H^{-m}(\Omega)} \leq \|u\|_{L^2(\Omega)}$, $\forall u \in L^2(\Omega)$. In fact, for any $u \in L^2(\Omega)$ et $v \in H_0^m(\Omega)$ we get*

$$\begin{aligned} |\langle u, v\rangle| &\leq \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\leq \|u\|_{L^2(\Omega)} \|v\|_{H^m(\Omega)}. \end{aligned}$$

Therefore

$$\frac{|\langle u, v\rangle|}{\|v\|_{H^m(\Omega)}} \leq \|u\|_{L^2(\Omega)},\ \forall v \in H_0^m(\Omega),\ v \neq 0.$$

Hence

$$\|u\|_{H^{-m}(\Omega)} \leq \|u\|_{L^2(\Omega)}.$$

4) *We have*

$$|\langle u,v\rangle| \leq \|u\|_{H^m(\Omega)} \|v\|_{H^{-m}(\Omega)}\,,\ \forall u\in H_0^m(\Omega)\,,\ \forall v\in L^2(\Omega)\,.$$

In fact, we get

$$\begin{aligned} |\langle u,v\rangle| &= \|u\|_{H^m(\Omega)}\left|\left\langle \frac{u}{\|u\|_{H^m(\Omega)}},v\right\rangle\right| \\ &\leq \|u\|_{H^m(\Omega)}\|v\|_{H^{-m}(\Omega)}\,,\ \forall u\in H_0^m(\Omega),\forall v\in L^2(\Omega). \end{aligned}$$

2.3 The spaces $W^{m,p}(\Omega)$

In this section, we introduce the definition of the Sobolev space $W^{m,p}(\Omega)$ and establish some of their basic properties.

2.3.1 Definitions and properties

Let Ω be a non-empty open set in $\mathbb{R}^n$, $p\in[1,+\infty]$ and m is a nonnegative integer. The Sobolev space $W^{m,p}(\Omega)$ is defined by

$$W^{m,p}(\Omega)=\{u\in L^p(\Omega): D^\alpha u\in L^p(\Omega)\ ,\ \forall\alpha\in\mathbb{N}^n;\ |\alpha|\leq m\}\,, \tag{2.13}$$

where $D^\alpha u$ represents a weak partial derivative of u (in the sense of distributions). The space $W^{m,p}(\Omega)$ is equipped with the following norm

$$\|u\|_{W^{m,p}(\Omega)}=\begin{cases}\left(\sum\limits_{|\alpha|\leq m}\|D^\alpha u\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}} & if\quad 1\leq p<+\infty,\\ \max\limits_{|\alpha|\leq m}\|D^\alpha u\|_{L^\infty(\Omega)} & if\quad p=+\infty.\end{cases} \tag{2.14}$$

Theorem 2.17 : *$W^{m,p}(\Omega)$ is a Banach space.*

Proof. To show that the space $W^{m,p}(\Omega)$ is complete with respect to the norm $\|.\|_{W^{m,p}(\Omega)}$, we need to prove that every Cauchy sequence in $W^{m,p}(\Omega)$ converges to a limit in $W^{m,p}(\Omega)$. Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $W^{m,p}(\Omega)$, this means that

$$\forall\varepsilon>0,\exists n_0\in\mathbb{N},\forall s,q\in\mathbb{N},\ \ (s\geq n_0 \text{ and } q\geq n_0)\Rightarrow\|u_s-u_q\|_{W^{m,p}(\Omega)}<\varepsilon.$$

Therefore

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall s, q \in \mathbb{N}, (s \geq n_0 \text{ and } q \geq n_0) \Rightarrow \left(\sum_{|\alpha| \leq m} \| D^\alpha (u_s - u_q) \|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} < \varepsilon.$$

Then

$$(s \geq n_0 \text{ et } q \geq n_0) \Rightarrow \left(\| D^\alpha (u_s - u_q) \|_{L^p(\Omega)} < \varepsilon \right), \ \forall \alpha \in \mathbb{N}^n, |\alpha| \leq m.$$

From this we deduce that $(D^\alpha u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega)$, for any $\alpha \in \mathbb{N}^n$, $|\alpha| \leq m$. Since $L^p(\Omega)$ is complete, we get

$$\exists v_\alpha \in L^p(\Omega) \text{ such that } D^\alpha u_n \to v_\alpha, \forall \alpha \in \mathbb{N}^n, \ |\alpha| \leq m.$$

In particular,

$$u_n \to v_0 \text{ in } L^p(\Omega) \; as \; n \to +\infty.$$

To show that

$$v_\alpha = D^\alpha v_0, \ \forall \ \alpha \in \mathbb{N}^n, |\alpha| \leq m,$$

we can use the fact that the injection of $L^p(\Omega)$ into $L^1_{loc}(\Omega)$ implies that the sequence $(u_n)_{n \in \mathbb{N}}$ determines a distribution $T_{u_n} \in \mathcal{D}'(\Omega)$.
Using the Holder inequality, we obtain the following:

$$\begin{aligned} |T_{u_n}(\varphi) - T_u(\varphi)| &\leq \int_\Omega | u_n(x) - u(x) || \varphi(x) | \, dx \\ &\leq \|\varphi\|_q \|u_n - u\|_p, \ \forall \varphi \in \mathcal{D}(\Omega), \end{aligned} \tag{2.15}$$

where $q = \frac{p}{p-1}$. Hence $\forall \alpha \in \mathbb{N}^n, \ |\alpha| \leq m,$

$$T_{u_n}(\varphi) \to T_{u_\alpha}(\varphi), \ \ \forall \varphi \in \mathcal{D}(\Omega), \ as \; n \to +\infty. \tag{2.16}$$

and

$$T_{D^\alpha u_n}(\varphi) \to T_{u_\alpha}(\varphi), \ \ \forall \varphi \in \mathcal{D}(\Omega), \ as \; n \to +\infty. \tag{2.17}$$

Since

$$D^\alpha T_{u_n} = T_{D^\alpha u_n},$$

by the uniqueness of the limit in $\mathcal{D}'(\Omega)$, we can conclude that

$$T_{v_\alpha}(\varphi) = D^\alpha T_u = (-1)^{|\alpha|} T_u(D^\alpha \varphi).$$

This proves that

$$D^\alpha u \in L^p(\Omega), \ \forall \alpha \in \mathbb{N}^n, \ |\alpha| \leq m \Rightarrow v_\alpha \in W^{m,p}(\Omega).$$

Hence $W^{m,p}(\Omega)$ is a complete space. ■

Remark 2.13 :
1) In the case $p = 2$, Sobolev spaces become particularly due to their property of being Hilbert spaces, denoted as $H^m(\Omega)$.

$$W^{m,2}(\Omega) = H^m(\Omega).$$

2) Note that for $m = 0$, the space $W^{m,p}(\Omega)$ is the Lebesgue space $L^p(\Omega)$.
3) In the one-dimensional space $(n = 1)$, where a and b belong to the set of real numbers $\mathbb{R}$, each element of $W^{1,p}(]a,b[)$ can be associated with a continuous function. This means that there exists a representative function within the class that is continuous.
This can be attributed to the fact that in one dimension, every function in $W^{1,p}(]a,b[)$ can be expressed as the integral of its derivative.

$$\exists \widehat{u} \in C(]a,b[) \text{ and } v \in L^p(]a,b[);\; u = \widehat{u} \text{ a.e. and} \tag{2.18}$$

$$\widehat{u}(x) = \widehat{u}(a) + \int_a^x v(s)ds.$$

In dimensions higher than 1, this is not true.
4) The space $W^{1,\infty}(\Omega)$ is the space of Lipschitz functions.

Theorem 2.18 : *Given a non-empty open set Ω in $\mathbb{R}^n$, $p \in [1,+\infty]$, and $m \in \mathbb{N}^*$, equipped with the previous norm, we have the following:*

$$\begin{cases} \forall 1 < p < +\infty, & W^{m,p}(\Omega) \text{ is a reflexive space,} \\ \forall 1 \leq p < +\infty, & W^{m,p}(\Omega) \text{ is a separable space,} \end{cases}$$

Extension Theorems

Definition 2.8 *Let Ω be a domain in $\mathbb{R}^n$. We say that Ω verifies the m-extension property if there exists a continuous linear operator:*

$$P : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^n),$$

verifying:
1- $P(u)\chi_\Omega = u, \forall u \in W^{m,p}(\Omega)$,
2- For any $0 \leq k \leq m$, $\exists C_k > 0,\; \|Pu\|_{W^{k,p}(\mathbb{R}^n)} \leq C_k \|u\|_{W^{k,p}(\Omega)}$.

Theorem 2.19 : *Let Ω be a non-empty open of $\mathbb{R}^n$ possessing the m-extension property, $p \in [1,+\infty]$ and $m \in \mathbb{N}$. Then the space $\mathcal{D}(\mathbb{R}^n)$ is dense in $W^{m,p}(\Omega)$ i.e.*

$$\forall u \in W^{m,p}(\Omega),\; \exists (u_k)_{k\in\mathbb{N}} \in \mathcal{D}(\mathbb{R}^n),\; u = \lim_{k\to+\infty} (u_k\chi_\Omega).$$

Remark 2.14 *The existence of even m-extension operator for a domain* Ω *guarantees that* $W^{m,p}(\Omega)$ *inherits many properties possessed by* $W^{m,p}(\mathbb{R}^n)$. *For instance, if the embedding*

$$W^{m,p}(\mathbb{R}^n) \to L^q(\mathbb{R}^n),$$

in known to hold, then the embedding

$$W^{m,p}(\Omega) \to L^q(\Omega),$$

follows via the chain of inequalities: For any $0 \leq k \leq m$, $\exists N_k, C_k > 0$,

$$\begin{aligned} \|u\|_{W^{0,q}(\Omega)} &\leq \|Pu\|_{W^{0,q}(\mathbb{R}^n)} \\ &\leq N_k \|Pu\|_{W^{m,p}(\mathbb{R}^n)} \leq N_k C_k \|u\|_{W^{m,p}(\Omega)} . \end{aligned} \tag{2.19}$$

2.3.2 The Space $W_0^{m,p}(\Omega)$

Definition 2.9 *: Let* p *be a real number,* $1 \leq p < +\infty$, *and an integer* $m \geq 2$, *we call the Sobolev space* $W_0^{m,p}(\Omega)$ *the adherence of* $\mathcal{D}(\Omega)$ *in* $W^{m,p}(\Omega)$, *i.e.*

$$W_0^{m,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W^{m,p}(\Omega)}.$$

Remark 2.15 *For any* m, *we have*

$$W_0^{m,p}(\Omega) \subset W^{m,p}(\Omega) \subset L^p(\Omega).$$

Corollary 2.5 *: Let* p *be a real number,* $1 \leq p < +\infty$, *and an integer* $m \geq 2$, *we obtain*
1- For $\Omega = \mathbb{R}^n, W_0^{m,p}(\mathbb{R}^n) = W^{m,p}(\mathbb{R}^n)$.
2- For Ω *a ball or a bounded block of* $\mathbb{R}^n, W_0^{m,p}(\Omega) \neq W^{m,p}(\Omega)$.
3- For an open set Ω *in* $\mathbb{R}^n$ *that is contained in a strip or a band, we have the following result:*

$$\forall m \in \mathbb{N}^*,\ \exists C > 0,\ \forall u \in W_0^{m,p}(\Omega),\ \|u\|_{L^p(\Omega)} \leq C \sum_{|\alpha|=m} \|D^\alpha u\|^p_{L^p(\Omega)} ,$$

and

$$|||u||| = \left(\sum_{|\alpha|=m} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} ,$$

is a norm on $W_0^{m,p}(\Omega)$*equivalent to the norm of* $W^{m,p}(\Omega)$.
4- Note that

$$H_0^1(\Omega) = W_0^{1,2}(\Omega).$$

Lemma 2.5 *(Friedrichs)*
For $m \in \mathbb{N}$ and $p < \infty$, for any element $f \in W^{m,p}(\Omega)$ and $\theta \in \mathcal{D}(\Omega)$, there exists a sequence $(\varphi_n)_n \in \mathcal{D}(\Omega)$ that approximates $(f\theta)$ in $W^{m,p}(\Omega)$.

Proof. To show this, we can construct the sequence $(\varphi_n)_n$ as follows

$$\varphi_n \to \theta \textit{ in } \mathcal{D}(\Omega) \textit{ as } n \to +\infty.$$

This means that for any compact set $K \subset \Omega$,

$$\exists N \in \mathbb{N} \textit{ such that } \forall n \geq N, \varphi_n \textit{ and } \theta \textit{ coincide on} K.$$

Define

$$\psi_n = f\varphi_n, \ \forall n \in \mathbb{N}.$$

We can show that $(\psi_n)_n$ is a sequence in $W^{m,p}(\Omega)$ by using the properties of the Sobolev space $W^{m,p}(\Omega)$.
Since $f \in W^{m,p}(\Omega)$ and $\varphi_n \in \mathcal{D}(\Omega)$, which is a space of smooth functions with compact support, their product $(f\varphi_n)$ is a smooth function with compact support in Ω, satisfying the required regularity conditions for elements in $W^{m,p}(\Omega)$. By using suitable approximation arguments, we can show that

$$\psi_n \to f\theta \textit{ in } W^{m,p}(\Omega) \textit{ as } n \to +\infty.$$

This involves showing convergence of the derivatives of ψ_n and using properties of the Sobolev space norm. Therefore, we have established that for any

$$f \in W^{m,p}(\Omega) \textit{ and } \theta \in \mathcal{D}(\Omega),$$

there exists a sequence $(\varphi_n)_n \in \mathcal{D}(\Omega)$ that approximates

$$(f\theta) \in W^{m,p}(\Omega).$$

The proof is completed. ■

Remark 2.16 *:*
Similary, this means that we can approximate f in $W^{m,p}_{loc}(\Omega)$ by elements from $\mathcal{D}(\Omega)$. In other words, functions from $\mathcal{D}(\Omega)$ are dense in $W^{m,p}_{loc}(\Omega)$, allowing us to make approximations and apply operators defined on regular functions in the more general context of locally integrable functions.

2.3.3 Compact Embeddings of $W^{m,p}(\Omega)$

The Rellich Kandrachov Theorem

Theorem 2.20 *[1] Let Ω be a domain in $\mathbb{R}^n$, Ω_0 a bounded subdomain of Ω and Ω_0^k the intersection of Ω_0 with a k-dimensional plane in $\mathbb{R}^n$. Let j, m be integers, $m \geq 1$, and let $1 \leq p < +\infty$. Then the following Embeddings are compact:*

$$
\begin{aligned}
W_0^{j+m,p}(\Omega) &\rightarrow W^{j,q}(\Omega_0) \ \ if \ \ 0 < n - mp < k \leq n, 1 \leq q < kp/(n-mp), \\
W_0^{j+m,p}(\Omega) &\rightarrow W^{j,q}(\Omega_0) \ \ if \ \ n = mp, 1 \leq k \leq n, 1 \leq q < \infty, \\
W_0^{j+m,p}(\Omega) &\rightarrow C_B^j(\Omega_0), \\
W_0^{j+m,p}(\Omega) &\rightarrow W^{j,q}(\Omega_0^k) \ \ if \ \ 1 \leq q \leq \infty, \\
W_0^{j+m,p}(\Omega) &\rightarrow C^j(\overline{\Omega_0}) \ \ \ if \ \ mp > n, \\
W_0^{j+m,p}(\Omega) &\rightarrow C^{j,\lambda}(\overline{\Omega_0}) \ \ if \ \ mp > n \leq (m-1)p, \ 0 < \lambda < m - (n/p).
\end{aligned}
$$

2.3.4 The space $W^{-m,q}(\Omega)$

Definition 2.10 : *Let p and q be real numbers satisfying $1 \leq q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and let m be a positive integer. We define the Sobolev space and denote it as $W^{-m,q}(\Omega)$, which is the dual space of $W_0^{m,p}(\Omega)$. In other words, we have*

$$(W_0^{m,p}(\Omega))' = W^{-m,q}(\Omega).$$

Theorem 2.21 : *[1] Let $1 \leq p < +\infty$, for any $L \in (W^{m,p}(\Omega))'$ there exists an element $v = (v_\alpha)_{0 \leq |\alpha| \leq m}$ in the space $L_N^q = (L^q(\Omega) \times L^q(\Omega) .... \times L^q(\Omega))$ N times, such that*

$$\forall u \in W^{m,p}(\Omega), L(u) = \sum_{0 \leq |\alpha| \leq m} < D^\alpha u, v_\alpha > . \tag{2.20}$$

furthermore,

$$\|L\|_{(W^{m,p}(\Omega))'} = inf \|v\|_{L_N^q}. \tag{2.21}$$

The infimum is obtained by considering all $v \in L_N^q$ that satisfy equation (2.20) for every $u \in W^{m,p}(\Omega)$. Moreover, this infimum is attainable under these conditions.

Remark 2.17 :

1) For $1 \leq p < +\infty$, every element L of the space $W^{-m,q}(\Omega)$ is an extension to $W^{m,p}(\Omega)$ of a distribution $T \in \mathcal{D}'$. To see this suppose L is given by (2.20) for some $v \in L_N^q$ and define T_{v_n}, $T \in \mathcal{D}'$, by

$$T_{v_n}(\varphi) =< \varphi, v_n >, \varphi \in \mathcal{D}(\Omega), \ |\alpha| \leq m.$$

$$T = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha T_{v_n}. \tag{2.22}$$

2) For $1 \leq p < +\infty$, the dual space $(W_0^{m,p}(\Omega))'$ is isometrically isomorphic to a Banach space that comprises distributions $T \in \mathcal{D}'$ satisfying (2.22) for a certain $v \in L_N^q$.

3) If $1 \leq p < +\infty,\ m \leq 1$, then the dual space $(W^{-m,q}(\Omega))$ is a Banach space and evidently it is separable and reflexive if $1 < p < +\infty$.

2.4 Exercises

Exercise 2.1:

Let u be an application from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$x \to u(x) = \begin{cases} x & x \in [0,1], \\ -x+2 & x \in [1,4], \\ 0 & elsewhere \end{cases}$$

1- Does the application u belong to $H^1(\mathbb{R})$?

2- Are there any open intervals I of $\mathbb{R}$ for which the restriction of u to $I \in H^1(I)$.

Exercise 2.2:

I- Let u_α be the application of $\mathbb{R}$ in $\mathbb{R}$ defined by: $u_\alpha(x) = x^\alpha, \alpha \in \mathbb{R}$. For what values of α, u_α does it belong to $H^1(]0,2[)$?

II- The function $u :\to \mathbb{R}$ defined by

$$x \to u(x) = \begin{cases} a & x \in [0,2], \\ b & x \in [2,5], \end{cases}$$

belong to $H^1(]0,5[)$?

Exercise 2.3:

Let (r,θ) be the polar coordinates of $\mathbb{R}^2$ and

$$\Omega = \left\{(r,\theta), \;\; 0 < r < 1, \; 0 < \theta < \frac{\pi}{4}\right\}.$$

Let α be a real number in $\mathbb{R}$. We denote by f_α the mapping from Ω to $\mathbb{R}$, defined as $f_\alpha(r,\theta) = r^\alpha$.

1- For which values of α does f_α belong to $L^2(\Omega)$?

2- For which values of α does f_α belong to $H^1(\Omega)$?

Exercise 2.4:

Let $u \in L^1_{loc}(]0,1[)$ such that $Du = 0$. Show that

$$\exists a \in \mathbb{R}, u = a \;\; a.e.,$$

which means a function in $L^1_{loc}(]0,1[)$ with a derivative equal to zero is constant.

Exercise 2.5:

Let B be the open unit ball in $\mathbb{R}^2$. Show that the function

$$u(x) = \left|\log\left(|x/2|\right)\right|^\alpha,$$

belongs to $H^1(B)$ for $0 < \alpha < 1/2$, but is not bounded in the neighborhood of the origin.

Exercise 2.6:

Show that there cannot be a notion of trace for functions in $L^2(\Omega)$, meaning that there does not exist a constant $C > 0$ such that

$$\forall v \in L^2(\Omega), \| v|_{\partial\Omega}\|_{L^2(\partial\Omega)} \leq C \|v\|_{L^2(\Omega)} .$$

Exercise 2.7:

Consider the trace mapping

$$\begin{array}{rcl} \gamma_0 : \mathcal{D}(\mathbb{R}^+) & \to & \mathbb{R} \\ u & \to & \gamma_0 u \end{array}$$

1- By considering the sequence of functions $v_n = e^{-nx}$ for $x \geq 0$, and $n \geq 0$, show that γ_0 cannot be extended to a continuous linear mapping from $L^2(]-\infty, 0[)$ into $\mathbb{R}$.

2- Using the density of $\mathcal{D}(\mathbb{R}^+)$ in $H^1(]-\infty, 0[)$, show that γ_0 can be extended to a continuous linear mapping from $H^1(]-\infty, 0[)$ into $\mathbb{R}$, i.e.,

$$v(0) \leq \|v\|_{H^1(\mathbb{R}^*_+)} \text{ for } v \in \mathcal{D}(\mathbb{R}^+).$$

Exercise 2.8:

Let $v \in L^2(\mathbb{R}^n)$, and its Fourier transformation $\widehat{v}$ is defined by:

$$\widehat{v}(\zeta) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\zeta} v(x) dx, \quad x, \zeta \in \mathbb{R}^n.$$

Show that

$$H^1(\mathbb{R}^n) = \left\{ v \in L^2(\mathbb{R}^n), \quad \left(1 + |\xi|^2\right) \widehat{v} \in L^2(\mathbb{R}^n) \right\},$$

with the norm defined as

$$\|v\|_{H^1(\mathbb{R}^n)} = \int \mathbb{R}^n \left(1 + |\xi|^2\right)^{1/2} |\widehat{v}(\xi)|^2 d\xi.$$

Exercise 2.9:

Let $1 \leq p \leq \infty$ and $u \in W^{1,p}(]0;1[)$.

1. Prove that there exists $C \in \mathbb{R}$ such that

$$u(x) = C + \int_0^x Du(t)dt, \text{ a.e. } x \in]0,1[.$$

As a consequence, deduce that $u \in C([0;1], \mathbb{R})$ in the sense that there exists $v \in C([0,1], \mathbb{R})$ such that $u = v$ a.e. on $]0,1[$.

By identifying u and v, we can say that $W^{1,p}(]0,1[) \subset C([0,1],\mathbb{R})$.

2. Prove that

$$\|u\|_\infty \leq \|u\|_{W^{1,p}(]0,1[)}\,.$$

3. Let $u \in C([0,1],\mathbb{R})$. Suppose that there exists $w \in L^p(]0,1[)$ such that

$$u(x) = u(0) + \int_0^x w(t)dt, \text{ a.e. } x \in]0,1[.$$

Prove that

$$u \in W^{1,p}(]0,1[) \; et \;\; Du = w.$$

Exercise 2.10:

Let $1 \leq p \leq \infty$ and $u \in W^{1,p}(]0;1[)$.

1- For all u in $\mathcal{D}(\mathbb{R})$, show the inequality

$$\int_{I\times I} |u(x) - u(y)|^2 \, dxdy \leq mes(I)^3 \int_I |u'(t)| \, dt. \tag{1}$$

Deduce

$$\int_I |u(x)|^2 \, dx \leq \frac{mes(I)^2}{2} \int_I |u'(x)|^2 \, dx + \frac{1}{mes(I)} \left| \int_I u(x)dx \right|^2. \tag{2}$$

2- The inequality (2) is true for any element of $H^1(I)$.

Exercise 2.11:

Let I be an interval in $\mathbb{R}$, and let $p \in]1,+\infty[$. Prove that the norms $\|.\|_{W^{1,p}(I)}$, $|||.|||_{W^{1,p}(I)}$, define by:

$$\|u\|_{W^{1,p}(I)} = \|u\|_{L^p(I)} + \|u'\|_{L^p(I)}\,,$$

and

$$|||u|||_{W^{1,p}(I)} = \left(\|u\|^p_{L^p(I)} + \|u'\|^p_{L^p(I)}\right)^{1/p}.$$

are equivalent.

Exercise 2.12:

Let $n = 2$, $\frac{1}{2} < \alpha < 1$. Let

$$\Omega = \{(r,\theta), 0 < r < 1, 0 < \theta < \frac{\pi}{\alpha}\}.$$

and

$$u(r,\theta) = (r^\alpha - r)^{-\alpha} sin(\alpha\theta).$$

1) Prove that $\exists q^* > 1 \; such \; that \; \forall q \in [1, q^*[, u \in W^{1,q}(\Omega)$.

2) Calculate $-\Delta u$.

3) What's up.

Exercise 2.13:

Let $\Omega = (a, b)$, where $a < b$. Using the Ascoli's theorem, we will show that the canonical injection from $H^1(\Omega)$ to $C^0(\Omega)$ is compact.
Consequently, we can deduce that the canonical injection from $H^1(\Omega)$ to $L^2(\Omega)$ is also compact.

Exercise 2.14:

Let $\mathbb{R}^n \supset \Omega = \Omega_1 \cup \Omega_2$ be a domain and take $E = H_0^1(\Omega)$. Define

$$E_1 = \{u \in \Omega_1, u = 0 \ in \ \Omega/\Omega_1\}, \ E_2 = \{u \in \Omega_2, u = 0 \ in \ \Omega/\Omega_2\}.$$

Prove that E_1 and E_2 are closed subspaces in E, and

$$E_1^{\perp} \cap E_2^{\perp} = \{0\}.$$

Chapter 3

Variational Formulation of Boundary Problems

A classical solution (also known as a strong solution) of a PDE is a solution belongs the space $C^n(n = 1, 2, ...)$. Unfortunately, this classical formulation poses a number of problems for proving the existence of a solution. For this reason, we will replace the classical formulation with a variational one.

3.1 Variational Formulation

The principle of the variational approach to solving partial differential equations is to replace the equation by an equivalent variational formulation, obtained by integrating the equation multiplied by a test function.

3.1.1 Dirichlet Problem

Let Ω be a bounded open set in $\mathbb{R}^n$ with boundary $\Gamma = \partial\Omega$ of piecewise C^1 class. We consider the following homogeneous Dirichlet problem: Find a function $u : \overline{\Omega} \longrightarrow \mathbb{R}$ such tha

$$(P_c)\quad \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases}, \tag{3.1}$$

with

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

being the Laplacian of u and f is a given function in $L^2(\Omega)$. The boundary condition $u = 0$ on Γ is the homogeneous Dirichlet condition. The classical solution here is a function in $C^2(\Omega)$.

Definition 3.1 *A **classical** solution of (Pc) is a function $u \in \mathcal{C}^2(\overline{\Omega})$ that satisfies (Pc) in the usual sense.*

Variational formulation of the problem (P_c):
Let v be a test function. Multiplying both sides of the first equation of problem (P_c) by a test function $v \in H_0^1(\Omega)$ and then integrating the result over Ω (we verify that this integration is possible), we obtain

$$\int_\Omega -\triangle u.v dx = \int_\Omega f v dx, \quad \text{for any } v \in H_0^1(\Omega). \tag{3.2}$$

We now use Green's formula, we obtain

$$\int_\Omega \nabla u \, \nabla v \, dx - \int_\Gamma \frac{\partial u}{\partial \eta} v d\Gamma = \int_\Omega f v dx.$$

Then

$$(\mathrm{P_V}) \quad \forall v \in H_0^1(\Omega). \quad \int_\Omega \nabla u \, \nabla v \, dx = \int_\Omega f v dx \tag{3.3}$$

We say that $(\mathrm{P_V})$ is the variational formulation of the problem (P_c) i.e. we replace the problem (3.1) by the following one: Given a function $f \in L^2(\Omega)$, find a function $u \in H_0^1(\Omega)$ satisfying (3.3).

Reciprocally (Back to a classic solution):

Definition 3.2 *A **weak** solution of (Pv) is a function $u \in H_0^1(\Omega)$ verifying (Pv).*

Let $u \in H_0^1(\Omega)$, we get

$$\forall v \in H_0^1(\Omega), \quad \int_\Omega \nabla u \nabla v dx = \int_\Omega f v dx,$$

since $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$, we can write

$$\forall v \in \mathcal{D}(\Omega), \quad \int_\Omega \nabla u \, \nabla v \, dx = \int_\Omega f v dx.$$

Therefore, this equation is verified in the sense of distributions on Ω

$$(\nabla u, \ \nabla v \ )_{\mathcal{D}' \times \mathcal{D}} = (f, v)_{\mathcal{D}' \times \mathcal{D}} \, .$$

Then using the definition of the derivative in the sense of distributions, this equality becomes

$$(-\triangle u, v)_{\mathcal{D}' \times \mathcal{D}} = (f, v)_{\mathcal{D}' \times \mathcal{D}} \, .$$

As a result,

$$(-\triangle u - f, v)_{\mathcal{D}'\times\mathcal{D}} = 0, \quad \forall v \in \mathcal{D}(\Omega),$$

from which we deduce the following equality, in the sense of distributions

$$-\triangle u = f \quad \text{in } \mathcal{D}'(\Omega).$$

Therefore, equation (3.1) is satisfied in the sense of distributions over Ω (i.e., weakly) and since $f \in L^2(\Omega)$, we have

$$-\Delta u = f \;\text{ almost everywhere in } L^2(\Omega).$$

Moreover, since $u \in H_0^1(\Omega)$, we obtain $u|_\Gamma = 0$. Thus, the problem (3.3) becomes: find a function u such that

$$\begin{cases} -\triangle u = f & \text{in } \Omega \\ u_{\diagup\Gamma} = 0 & \Gamma = \partial\Omega \end{cases},$$

hence, **Variationnel Problem $\Longleftrightarrow$ Direct problem.**

Remark 3.1 *:*

1) Any sufficiently regular solution of the classical problem (P_c) is a solution of the variational problem (P_V).

2) The advantage of the variational formulation (3.3) is that it makes sense even if the solution u is only a function in $C^1(\Omega)$, unlike the classical formulation (3.1) which requires u to belong to $C^2(\Omega)$. Therefore, it is simpler to solve (3.3) than (3.1) because we are less demanding in terms of solution regularity.

3) The variational method has transformed a second-order problem into a first-order problem, but it has also transformed a linear problem into a quadratic problem.

3.1.2 Neumman's Problem

Let Ω be a bounded open $\mathbb{R}^n$ with regular $\Gamma = \partial\Omega$ boundary and consider the following homogeneous Neumman problem: Find $u : \overline{\Omega} \longrightarrow \mathbb{R}$ solution of the problem

$$(P_c) \quad \begin{cases} -\triangle u + u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } \Gamma \end{cases}. \tag{3.4}$$

Given f as a function in $L^2(\Omega)$, and

$$\frac{\partial u}{\partial \eta} = \nabla u.\overrightarrow{\eta},$$

where $\overrightarrow{\eta}$ represents the unit normal vector to the boundary Γ and pointing outward from Ω.

Remark 3.2 :
In terms of modeling, the Neumann condition $\frac{\partial u}{\partial \eta}$ represents a flux condition. For example, in the interpretation of thermal equilibrium, this condition corresponds to a prescribed heat flux across the boundary, as opposed to the Dirichlet condition that imposes a specified temperature on the boundary. The case $\frac{\partial u}{\partial \eta} = 0$ corresponds to perfect thermal insulation: no heat can enter or leave Ω.

Variational formulation of the problem (P_c):

By multiplying both sides of the first equation in the problem (P_c) by a test function $v \in H^1(\Omega)$ and integrating over Ω, we obtain

$$\int_\Omega -\triangle u.v dx + \int_\Omega u.v dx = \int_\Omega f v dx, \ \forall v \in H^1(\Omega). \tag{3.5}$$

Applying Green's formula, we deduce

$$\int_\Omega \nabla u \ \nabla v \ dx + \int_\Omega u.v dx - \int_\Gamma \frac{\partial u}{\partial \eta} v d\Gamma = \int_\Omega f v dx.$$

Using the boundary condition, we find

$$(\mathrm{P_V}) \quad \int_\Omega \nabla u \ \nabla v \ dx + \int_\Omega u.v dx = \int_\Omega f v dx \quad \forall v \in H^1(\Omega). \tag{3.6}$$

This gives us the following result:

Let u be a solution of (P_c), then u is a solution of the problem (P_V).

We say that (P_V) is the variational formulation of problem (P_c) i.e. we replace problem (3.4) by the following one: Given a function $f \in L^2(\Omega)$, find a function $u \in H^1(\Omega)$ satisfying

$$\int_\Omega \nabla u \ \nabla v \ dx + \int_\Omega u.v dx = \int_\Omega f v dx \quad \forall v \in H^1(\Omega).$$

Reciprocally: (Back to the classic solution):
Let $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla u \nabla v dx + \int_\Omega u.v dx = \int_\Omega f v dx, \ \ \forall v \in H^1(\Omega).$$

Therefore

$$\int_\Omega \nabla u \ \nabla v \ dx + \int_\Omega u.v dx = \int_\Omega f v dx \ , \ \ \forall v \in \mathcal{D}(\Omega).$$

This equation is verified in the sense of distributions on Ω, we can write

$$(\nabla u, \ \nabla v \ )_{\mathcal{D}' \times \mathcal{D}} + (u, \ v \ )_{\mathcal{D}' \times \mathcal{D}} \quad = \quad (f, v)_{\mathcal{D}' \times \mathcal{D}} \, .$$

Using the definition of the derivative in the sense of distributions, this equality becomes

$$(-\triangle u+u,v)_{\mathcal{D}'\times\mathcal{D}}=(f,v)_{\mathcal{D}'\times\mathcal{D}}\,.$$

Therefore

$$(-\triangle u+u-f,v)_{\mathcal{D}'\times\mathcal{D}}=0,\quad \forall v\in\mathcal{D}(\Omega). \tag{3.8}$$

We conclude that

$$-\triangle u+u=f\quad \text{in } \mathcal{D}'(\Omega).$$

As $f\in L^2(\Omega)$, we have the equation

$$-\triangle u+u=f\ a.e\ in\ L^2(\Omega).$$

Multiplying this equation by v and integrating over Ω and then applying the Green's formula, we obtain

$$\frac{\partial u}{\partial\eta}=0\ on\ \Gamma=\partial\Omega.$$

Thus, the problem (3.6) becomes: to find u that satisfies

$$\begin{cases}-\triangle u+u=f & \text{in } \Omega\\ \frac{\partial u}{\partial \eta}=0 & \text{on } \Gamma\end{cases}.$$

Hence, there is an equivalence between the variational problem and the direct problem.

3.2 Variationnel Problem

To formulate a variational problem for a boundary value problem, we need:

1) A Hilbert space V with a norm $\|.\|_V$.

2) A continuous bilinear form $a(\cdot,\cdot)$ on $V\times V$.

3) A continuous linear form $l(\cdot)$ on V.

We consider a variational formulation of the following type: Find u in V such that it satisfies

$$\forall u\in V,\quad a(u,v)=l(v). \tag{3.7}$$

3.3 Lax-Milgram's Theorem

The Lax-Milgram theorem (*Peter Lax and Arthur Milgram* [1]) is used to establish the existence and uniqueness of solutions to partial differential equations formulated in variational form in a real Hilbert space.

Definition 3.3 *Let V be a Hilbert space. A bilinear form $a(\cdot,\cdot)$ defined on $V\times V$ is continuous if*

$$\exists c>0 : |a(u,v)| \leq c\,\|u\|_V\,\|v\|_V\,,\ \forall u,\ v\in V. \tag{3.9}$$

Definition 3.4 *For a Hilbert space V, a bilinear form $a(\cdot,\cdot)$ defined on $V\times V$ is said to be coercive (or V-elliptic) if there exists $\alpha>0$ such that:*

$$\forall v\in V : a(v,v) \geq \alpha\,\|v\|_V^2\,. \tag{3.10}$$

Remark 3.3 *:*

1- The simplest example is when $a(\cdot,\cdot)$ is the inner product on V. In this case, we indeed have all the previous properties with $c=\alpha=1$.

2- If for all $v\in V$, $a(v,v)<0$, we need to show coercivity with respect to $-a(\cdot,\cdot)$.

3- The assumption "a(.,.) is coercive" cannot be replaced by "a(.,.) is positive definite".

4- If there exists $v\neq 0$ in V such that $a(v,v)=0$, then $a(\cdot,\cdot)$ is not coercive.

Theorem 3.1 *(Lax-Milgram)*

Let $a(\cdot,\cdot)$ be a continuous and coercive bilinear form on a Hilbert space $V\times V$, and let $\ell(\cdot)$ be a continuous linear form on V. Then, the variational problem can be formulated as follows

$$(P_V)\begin{cases} \text{Find } u\in V \text{ such that:} \\ a(u,v)=\ell(v)\,,\ \forall v\in V \end{cases}, \tag{3.11}$$

has a unique solution in V.

Proof.

1) Uniqueness: Let u_1 and u_2 be two solutions of the problem (P_V) then

$$\begin{cases} a(u_1,v)=\ell(v) \\ a(u_2,v)=\ell(v) \end{cases} \Rightarrow a(u_1-u_2,v)=0,\ \forall v\in V.$$

[1]Peter Lax: (born in 1926 in Budapest) is a Hungarian-American mathematician who was awarded the Abel Prize in 2005.
Arthur Milgram (born on June 3, 1912, in Philadelphia and died on January 30, 1961, at the age of 48) was an American mathematician.

For $v = u_1 - u_2$, we obtain

$$a(u_1 - u_2, u_1 - u_2) = 0.$$

The coercivity of a implies the existence of $\alpha > 0$ such that

$$0 = a(u_1 - u_2, u_1 - u_2) \geq \alpha \|u_1 - u_2\|^2,$$

Hence

$$\|u_1 - u_2\| = 0 \Rightarrow u_1 = u_2.$$

2) Existence: For a fixed **v**, the application

$$w \to a(v, w),$$

is a continuous linear form of V in $\mathbb{R}$. The Riesz representation implies that there exists a unique element Au of V such that

$$a(u, w) = (Au, w) = \ell(w), \forall u, w \in V.$$

This defines a linear and continuous operator because

$$|(Au, w)| \quad = \quad |\, a(u, w)| \leq M \|u\| . \|w\| .$$

This implies that

$$\|Au\| = \sup_{\|w\|=1} |\, a(u, w)| \leq M \|u\| .$$

Similarly, the linear form l is continuous on V, and according to the Riesz representation theorem there exists a unique element $f \in V$ such that

$$\forall v \in V, \; l(v) = (f, v).$$

In conclusion, the variational problem (P_V) is equivalent to find $u \in V$ as a solution to the equation:

$$Au = f.$$

To solve this problem,

• We need to show that the operator A is a bijection from V to V, which implies the existence and uniqueness of u, and that its inverse is continuous, which proves the continuous dependence of u on l. or

• We can use the Banach fixed-point theorem, which involves showing that the equation $Au = f$, which is equivalent to

$$u = u - \rho(Au - f),\ \rho > 0,$$

has a fixed point. Let's consider the function $T : V \to V$ defined as

$$T(v) = v - \rho(Av - f).$$

Then, $, \forall v_1, v_2 \in V$, we have

$$\begin{aligned} \|T(v_1) - T(v_2)\|^2 &= \|v_1 - \rho(Av_1 - f) - v_2 + \rho(Av_2 - f)\|^2 \\ &= \|v_1 - v_2\|^2 - 2\rho a\,(v_2 - v_1, v_2 - v_1) + \rho^2 \|A(v_2 - v_1)\|^2 \\ &\leq \left(1 - 2\alpha\rho + \rho^2 \|A\|^2\right) \|v_2 - v_1\|^2 . \end{aligned}$$

If

$$\left(1 - 2\alpha\rho + \rho^2 \|A\|^2\right) < 1,$$

this implises that

$$0 < \rho < \frac{2\alpha}{\|A\|^2}.$$

The operator T is contractive, and therefore, it has a fixed point. We can conclude that the problem (P_V) has a solution in V. ■

Remark 3.4 :

1) In the case of complex Hilbert spaces and complex-valued variational problems, the Lax-Milgram theorem still holds for a bilinear or sesquilinear form.

2)This theorem forms the basis of finite element methods. In fact, it can be shown that if instead of find u in H, we seek $(u_n)_n$ in H_n, a finite-dimensional subspace of H with dimension n, then in the case where a is symmetric, u_n is the projection of u with respect to the inner product defined by $a(\cdot,\cdot)$

Minimization result

Let $a(.,.)$ be a symmetrical bilinear form, i.e.

$$\forall u, v \in V, a\,(u, v) = a\,(v, u)\,.$$

We introduce the quadratic functional defined for all $v \in V$ by

$$J(v) = \frac{1}{2}a(v, v) - \ell(v), \tag{3.12}$$

and consider the minimization problem: find $u \in V$ such that

$$J(u) = \min_{v\in V} J(v). \tag{3.13}$$

Then we have the following theorem

Theorem 3.2 :

Assuming that the bilinear form $a(\cdot,\cdot)$ is symmetric and coercive. Then, the problem (3.13) admits a unique solution $u \in V$, which is none other than the solution of the problem (P_V).

This theorem states that under the assumptions of symmetry and coercivity, the variational problem (3.13) has a unique solution in V and this solution coincides with the solution of the original problem (P_V).

Proof. If the bilinear form $a(.,.)$ is symmetrical, then for any w in V we have

$$J(u+w) = \frac{1}{2}\big[a(u,u) + 2a(u,w) + a(w,w)\big] - [L(u) + l(w)].$$

$$J(u+w) = J(u) + \Big(a(u,w) - L(w)\Big) + \frac{1}{2}a(w,w).$$

Since u is the only solution to the problem (P_V), this gives

$$J(u+w) = J(u) + \frac{1}{2}a(w,w).$$

And as $a(.,.)$ is coercive, we have :

$$J(u+w) \geq J(u) + \frac{\alpha}{2}\|w\|^2.$$

The conclusion is

$$J(u) \leq J(v), \forall v \in V,$$

hence the result. ■

Remark 3.5 :

1) When the bilinear form $a(.,.)$ is symmetric, the problem (P_V) corresponds to the minimization problem of a quadratic functional over a Hilbert space V. This is, in fact, the abstract formulation of several problems in the calculus of variations. This explains why the problem (P_V) is referred to as a variational problem.

2) The variational problem (P_V) corresponds to the Euler equation $(J'(u) = 0)$ associated with the minimization problem.

3) This equivalence can be exploited from a numerical perspective: to compute an approximation of the solution u to the variational problem, one can employ classical algorithms for minimizing quadratic functionals, such as the conjugate gradient method, for example.

3.4 Problem of regularity

There are essentially two types of regularity results
(1) A "local" regularity result, which depends only on the regularity of the coefficients of the elliptic operator considered and of f. For example, for the Dirichlet problem written in variational form which we know that it has a unique solution $u \in H_0^1(\Omega)$, we have the following regularity result for any integer $m \geq 0$:

$$if\ f \in H^m_{loc}(\Omega) \Rightarrow u \in H^{m+2}_{loc}(\Omega).$$

(2) A "global" regularity result, i.e., up to the boundary Γ of Ω. For this type of result, the regularity of the boundary Γ of the domain is essential, as well as the type of boundary condition. Regarding the Dirichlet problem, we have the following result:

Proposition 3.1 *Let u be the solution of the variational problem (P_V) with $f \in L^2(\Omega)$. If the boundary Γ of class C^1 then $u \in H^2(\Omega)$.*

Remark 3.6 *This result is also true for the Neumann problem with $g \in H^{\frac{1}{2}}(\Omega)$.*

Chapter 4

Applications in some problems

In this chapter, our aim is to apply the Lax-Milgram theorem to ensure the existence of solutions to various boundary value problems (different operators and boundary conditions are considered). These problems are first formulated in variational form.

4.1 Laplace equation with Dirichlet conditions

Beyond the physical problems governed by this equation (such as the steady-state heat equation, elastic membrane problem, electrostatic equilibrium, etc.), we choose this equation as a model problem due to its simplicity.
Let Ω be a bounded open set in $\mathbb{R}^n$ with a boundary $\Gamma = \partial\Omega$ that is piecewise C^1. We consider the following homogeneous Dirichlet problem:

$$(Pc)\begin{cases} \text{Find } u \in H_0^1(\Omega) & \text{such that} \\ -\Delta u = f & f \in L^2(\Omega) \\ u|_{\Gamma} = 0 \end{cases}$$

Prove that the problem (Pc) has a unique solution.
I) **Equivalence between the problem** (Pc) **and the problem** (P_V).
We've already shown (in chapter 3) that the (Pc) problem is equivalent to the following variational problem:

$$(P_V)\begin{cases} \text{Find } u \in H_0^1(\Omega) & \text{such that:} \\ a(u,v) = l(v) \end{cases}$$

with

$$\begin{aligned} a(u,v) &= \int_\Omega \nabla u \nabla v dx = \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dx, \\ l(v) &= \int_\Omega f v dx. \end{aligned}$$

II) **Prove that the problem** (P_V) **has a unique solution**

By verifying the conditions of Lax Milgram's theorem

1) The bilinear form $a(.,.)$ is continuous on $H_0^1(\Omega) \times H_0^1(\Omega)$, in fact using the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} |a(u,v)| &\leq \left| \int_\Omega \nabla u . \nabla v dx \right| \\ &\leq \left(\int_\Omega |\nabla u|^2 dx \right)^{1/2} \left(\int_\Omega |\nabla v|^2 dx \right)^{1/2} \\ &\leq \|u\|_{H_0^1(\Omega)} \times \|v\|_{H_0^1(\Omega)} . \end{aligned}$$

On the other hand, since Ω is a bounded open of $\mathbb{R}^n$, the bilinear form $a(.,.)$ is coersive. In fact

$$a(u,u) = \int_\Omega |\nabla u|^2 dx = \|u\|^2_{H_0^1(\Omega)} ,$$

it means that

$$a(u,u) \geq \alpha \|u\|^2_{H_0^1(\Omega)} , \quad \text{with } \alpha = 1.$$

2) The linear form l is continuous on $H_0^1(\Omega)$, in fact according to the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |l(v)| &= \left| \int_\Omega f.v \, dx \right| \leq \left(\int_\Omega |f|^2 dx \right)^{1/2} \left(\int_\Omega |v|^2 dx \right)^{1/2} \\ &\leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\leq c_\Omega \|f\|_{L^2(\Omega)} \|v\|_{H_0^1(\Omega)}, \\ &\leq C \|v\|_{H_0^1(\Omega)}. \end{aligned}$$

where

$$C = c_\Omega \times \|f\|_{L^2(\Omega)} \leq +\infty \; because \; f \in L^2(\Omega),$$

et c_Ω is the constant of the Poincare inequality.

According to Lax Milgram's theorem, the problem (Pc) has a unique solution in $H_0^1(\Omega)$.

4.2 Laplace equation with Neumann conditions

Let Ω be a bounded open $\mathbb{R}^n$ of boundary $\Gamma = \partial\Omega$ of class C^1. Let's show that the problem (Pc) has a unique solution

$$(Pc)\begin{cases} \text{Find } u \in H^1(\Omega) & \text{such that} \\ -\Delta u + u = f & f \in L^2(\Omega) \\ \frac{\partial u}{\partial \eta} = 0 & on \;\; \Gamma \end{cases}$$

I) **Equivalence between the** (Pc) **problem and the** (P_V) **problem**

The variational formulation of this problem is of the following form

$$(P_V)\begin{cases} \text{Find } u \in H^1(\Omega) & \text{such that} \\ a(u,v) = l(v) \end{cases}$$

with

$$\begin{aligned} a(u,v) &= \int_\Omega \nabla u \nabla v dx + \int_\Omega uv dx = \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dx + \int_\Omega uv dx, \\ l(v) &= \int_\Omega fv dx. \end{aligned}$$

We have shown (in section 3.1.2) that $\qquad (Pc) \Leftrightarrow (P_V).$

II) **Prove that the problem** (P_V) **has a unique solution**

Verify that the conditions of Lax Milgram's theorem.

1) First of all, the space $V = H^1(\Omega)$ is a Hilbert space for the usual norm $\|.\|_{H^1(\Omega)}$.

2) Since the bilinear form a is continuous on $H^1(\Omega) \times H^1(\Omega)$, we have, thanks to the Cauchy-Schwartz inequality :

$$\begin{aligned} |a(u,v)| &\leq \left| \int_\Omega \nabla u.\nabla v dx \; + \int_\Omega u \; v \; dx \right| \\ &\leq \left| \int_\Omega \nabla u.\nabla v dx \right| + \left| \int_\Omega u \; v \; dx \right| \\ &\leq \left(\int_\Omega |\nabla u|^2 dx \right)^{1/2} \left(\int_\Omega |\nabla v|^2 dx \right)^{1/2} + \left(\int_\Omega |u|^2 dx \right)^{1/2} \left(\int_\Omega |v|^2 dx \right)^{1/2} \\ &\leq \|\nabla u\|_{L^2(\Omega)} \times \|\nabla v\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \times \|v\|_{L^2(\Omega)} . \end{aligned}$$

We have

$$\|u\|^2_{H^1(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\,,$$

then

$$\|\nabla u\|^2_{L^2(\Omega)} \leq \|u\|^2_{H^1(\Omega)} \ \ and \ \ \|u\|^2_{L^2(\Omega)} \leq \|u\|^2_{H^1(\Omega)}\,.$$

This shows that

$$|a(u,v)| \ \ \leq \ \ 2\,\|u\|_{H^1(\Omega)} \times \|v\|_{H^1(\Omega)}\,.$$

Moreover, the bilinear form $a(.,.)$ is coersive. This is because

$$a(u,u) = \int_\Omega |\nabla u|^2\,dx + \int_\Omega |u|^2\,dx = \|u\|^2_{H^1(\Omega)}\,.$$

Therefore

$$a(u,u) \geq \alpha \|u\|^2_{H^1(\Omega)}, \quad (\alpha = 1).$$

3) Moreover, the bilinear form $a(.,.)$ is coersive. This is because

$$\begin{aligned} |l(v)| \ &= \ \left|\int_\Omega f.v\ dx\right| \leq \left(\int_\Omega |f|^2\,dx\right)^{1/2} \left(\int_\Omega |v|^2\,dx\right)^{1/2} \\ &\leq \ \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\ &\leq \ c\ \|v\|_{L^2(\Omega)} \\ &\leq \ c\ \|v\|_{H^1(\Omega)}\,. \end{aligned}$$

with

$$c = \|f\|_{L^2(\Omega)} < +\infty \ because\ f \in L^2(\Omega).$$

With all the assumptions of the Lax-Milgram theorem satisfied, we deduce that the variational problem (Pc) has a unique solution u in $H^1(\Omega)$.

Remark 4.1 :

According to Theorem (3.2) this solution is the unique element of $H^1(\Omega)$ which minimizes the functional

$$J(v) = \frac{1}{2}\int_\Omega \left(\sum_{i=1}^{n} |\frac{\partial v}{\partial x_i}|^2 + |v|^2 dx\right) - \int_\Omega fv dx. \tag{4.1}$$

in the space $H^1(\Omega)$.

4.3 Second-order elliptic equation in $\mathbb{R}$

Problem 01: Let $I =]0,1[$. Find $u : \overline{I} \longrightarrow \mathbb{R}$ verifying

$$(\text{Pc}) \quad \begin{cases} -\dfrac{d^2u}{dx^2} + u = f & \text{on } I \\ u(0) = u(1) = 0 \end{cases},$$

where f is a given function in $L^2(I)$.

Variational formulation of (Pc):

Let u be a regular element ($u \in \mathcal{C}^2(\overline{I})$) solution of (Pc) and let $v \in H_0^1(I)$ be a test function. Multiplying by v in (Pc) and integrating from 0 to 1, we get

$$\int_0^1 \left(-\frac{d^2u}{dx^2} v + uv \right) dx = \int_0^1 f v dx.$$

Integrating by parts, we obtain

$$-\int_0^1 \frac{d^2u}{dx^2} v dx = -\left[u' v \right]_0^1 + \int_0^1 u' v' dx.$$

We have $v \in H_0^1(I)$, then

$$v(0) = v(1) = 0,$$

(note that this property is satisfied by the u solution), it remains

$$\int_0^1 \left(u' v' + uv \right) dx = \int_0^1 f v dx.$$

we get $u \in V = H_0^1(I)$ and

$$(\text{PD}_\text{V}) \quad a(u,v) = \ell(v), \ \forall v \in V,$$

with

$$a(u,v) = \int_0^1 \left(u' v' + uv \right) dx,$$

and

$$\ell(v) = \int_0^1 f v dx.$$

The problem of finding $u \in H_0^1(I)$ satisfying (P_V) is called the variational formulation of the boundary value problem (Pc).

Existence and uniqueness of the solution of ($\mathrm{P_V}$)

1) The bilinear form $a(.,.)$ is continuous, in fact, for $v \in H_0^1(I)$, we obtain

$$\begin{aligned} |a(u,v)| &= \left| \int_0^1 \left(u^{'}v^{'} + uv \right) dx \right| \\ &\leq \int_0^1 \left| u^{'}v^{'} \right| dx + \int_0^1 |uv|\, dx \\ &\overset{\text{Ch-Sc}}{\leq} \left(\int_0^1 \left| u^{'} \right|^2 dx \right)^{\frac{1}{2}} \left(\int_0^1 \left| v^{'} \right|^2 dx \right)^{\frac{1}{2}} + \left(\int_0^1 |u|^2 dx \right)^{\frac{1}{2}} \left(\int_0^1 |v|^2 dx \right)^{\frac{1}{2}} \\ &\leq \left\| u^{'} \right\|_{L^2(I)} \left\| v^{'} \right\|_{L^2(I)} + \|u\|_{L^2(I)} \|v\|_{L^2(I)} . \end{aligned}$$

then

$$|a(u,v)| \leq \|u\|_{H_0^1(I)} \|v\|_{H_0^1(I)} .$$

Therefore $a(.,.)$ is continuous on $H_0^1(I) \times H_0^1(I)$.

2) The coercivity of $a(.,.)$: For $v \in H_0^1(I)$, we have

$$\begin{aligned} a(v,v) &= \int_0^1 \left(v^{'} \right)^2 dx + \int_0^1 v^2 dx \\ &= \left\| v^{'} \right\|_{L^2(I)}^2 + \|v\|_{L^2(I)}^2 = \|v\|_{H^1(I)}^2 . \end{aligned}$$

Therefore $a(.,.)$ is coersive ($\alpha = 1$).

3) Continuity of $\ell(.)$: For $v \in H_0^1(I)$, we have

$$\begin{aligned} |\ell(v)| &= \left| \int_0^1 fv dx \right| \leq \int_0^1 |f|\,|v|\, dx \\ &\overset{\text{Ch-Sc}}{\leq} \left(\int_0^1 |f|^2 dx \right)^{\frac{1}{2}} \left(\int_0^1 |v|^2 dx \right)^{\frac{1}{2}} \\ &= \|f\|_{L^2(I)} \cdot \|v\|_{L^2(I)} \\ &\leq c\, \|v\|_{H_0^1(I)} . \end{aligned}$$

Then $\ell(.)$ is continous in $H_0^1(I)$.

Using the Lax-Milgram theorem, we obtain the existence of a unique $u \in H_0^1(I)$ solution of ($\mathrm{P_V}$).

Then, the weak solution of (P_V) is the classical solution $u \in \mathcal{C}^2\left(\overline{I}\right)$ of (Pc) .

Regularity and return to the classic solution

Let $u \in H_0^1(I)$ be a solution of (P$_\text{V}$) .

We have $u \in H_0^1(I)$, so

$$u(0) = u(1) = 0,$$

the boundary conditions of (Pc) are verified.

As $\overline{\mathcal{D}(I)} = H_0^1(I)$, then

$$\int_0^1 \left(u'\varphi' + u\varphi\right) dx = \int_0^1 f\varphi dx,\ \forall\varphi \in \mathcal{D}(I),$$

therefore

$$\int_0^1 u'\varphi' dx = \int_0^1 (f-u)\varphi dx,\ \forall\varphi \in \mathcal{D}(I).$$

as a result

$$\left\langle T_{u'}, \varphi'\right\rangle_{\mathcal{D}'\times\mathcal{D}} = \left\langle -\left(T_{u'}\right)', \varphi\right\rangle_{\mathcal{D}'\times\mathcal{D}} = \langle -T_{u''}, \varphi\rangle_{\mathcal{D}'\times\mathcal{D}} = \left\langle T_{f-u}, \varphi'\right\rangle_{\mathcal{D}'\times\mathcal{D}}.$$

Hence: $-T_{u''} = T_{f-u}$, which shows that $u'' = u - f \in L^2(I)$ because

$$f \in L^2(I) \text{ and } u \in H_0^1(I) \Longrightarrow u \in L^2(I).$$

Since

$$u \in H_0^1(I) \Longrightarrow u' \in L^2(I),$$

it comes $u \in H^2(I)$ and

$$u'' = u - f \text{ in the sense of distributions.}$$

We have $u \in H_0^1(I) \subset H^1(I) \hookrightarrow \mathcal{C}\left(\overline{I}\right)$, according to the trace theorem, then $u - f \in \mathcal{C}\left(\overline{I}\right)$ hence $u \in \mathcal{C}^2\left(\overline{I}\right)$ and that

$$-\frac{d^2u}{dx^2} + u = f \text{ on } I.$$

Finally, the solution $u \in H_0^1(I)$ of (P$_\text{V}$) is therefore a classical solution of (Pc).

Remark 4.2 *If we only assume $f \in \mathcal{C}(\overline{I})$, the variational problem defined in (P_V) also has a unique solution $u \in H_0^1(I)$, which can be easily characterized as $u \in H^2(I)$ and*

$$\begin{cases} u'' = u - f & \text{a.e on } I \\ u(0) = u(1) = 0 \end{cases}.$$

Problem 02: Let $I =]0,1[$. Find $u : \overline{I} \longrightarrow \mathbb{R}$ such that

$$(\text{Pc}) \quad \begin{cases} -\dfrac{d^2u}{dx^2} + u = f & \text{on } I \\ u(0) = \alpha, \ u(1) = \beta \end{cases},$$

where f is a given function (in $\mathcal{C}(\overline{I})$ or in $L^2(I)$) and the boundary conditions

$$u(0) = \alpha \; and \; u(1) = \beta,$$

are called non-homogeneous Dirichlet conditions.
Let u_0 be a regular function such that $u_0(0) = \alpha$ and $u_0(1) = \beta$, and let $\widehat{u} = u - u_0$, then

$$\begin{cases} -\dfrac{d^2\widehat{u}}{dx^2} + \widetilde{u} = f + \dfrac{d^2u_0}{dx^2} - u_0 & \text{on } I \\ \widehat{u}(0) = \widehat{u}(1) = 0. \end{cases}$$

This brings us back to the homogeneous problem for $\widehat{u}$ (Problem 01).
Problem 03:
Let $I =]0,1[$. Find $u : \overline{I} \longrightarrow \mathbb{R}$ verifying

$$(\text{Pc}) \quad \begin{cases} -\dfrac{d^2u}{dx^2} + u = f & \text{on } I \\ u'(0) = u'(1) = 0 \end{cases},$$

where f is a given function (for example in $\mathcal{C}\left(\overline{I}\right)$, or in $L^2(I)$).
Variationnel Formulation of (Pv)
Let u solution of (Pc) and let $v \in H^1(I)$ test function. Multiplying by v in (Pc) and integrating from 0 to 1, we get

$$\int_0^1 \left(u^{'}v^{'} + uv\right) dx = \int_0^1 fv dx.$$

We put

$$a(u,v) = \langle u, v\rangle_{H^1(I)} = \int_0^1 \left(u^{'}v^{'} + uv\right) dx,$$

and

$$\ell(v) = \int_0^1 fv dx,$$

with $f \in \mathcal{C}\left(\overline{I}\right)$.
As in problem 1, we can apply the Lax-Milgram theorem, which gives the existence of a unique $u \in H^1(I)$ verifying

$$(\text{P}_\text{V}) \qquad a(v,v) = \ell(v), \ \forall v \in H^1(I)$$

Regularity: Let $u \in H^1(I)$ solution of $(\mathrm{P_V})$.
As $\mathcal{D}(I) \subset H^1(I)$, then

$$\int_0^1 \left(u^{'}\varphi^{'} + u\varphi\right) dx = \int_0^1 f\varphi dx, \ \forall \varphi \in \mathcal{D}(I),$$

therfore

$$\int_0^1 u^{'}\varphi^{'} dx = \int_0^1 (f-u)\,\varphi dx, \ \forall \varphi \in \mathcal{D}(I).$$

as a result

$$\left\langle T_{u'}, \varphi^{'} \right\rangle_{\mathcal{D}' \times \mathcal{D}} = \left\langle -\left(T_{u'}\right)^{'}, \varphi \right\rangle_{\mathcal{D}' \times \mathcal{D}} = \left\langle -T_{u''}, \varphi \right\rangle_{\mathcal{D}' \times \mathcal{D}} = \left\langle T_{f-u}, \varphi^{'} \right\rangle_{\mathcal{D}' \times \mathcal{D}}.$$

Hence $-T_{u''} = T_{f-u}$, which shows that

$$u^{''} = u - f \in L^2(I),$$

because

$$f \in \mathcal{C}\left(\overline{I}\right) \Longrightarrow f \in L^2(I) \text{ and } u \in H^1(I) \Longrightarrow u \in L^2(I)).$$

As we also have $u^{'} \in L^2(I)$, it comes $u \in H^2(I)$ and $u^{''} = u - f$ in the sense of distributions. As $u - f \in \mathcal{C}\left(\overline{I}\right)$, by using the trace theorem, we obtain

$$f \in \mathcal{C}\left(\overline{I}\right) \text{ and } u \in H^1(I) \hookrightarrow \mathcal{C}\left(\overline{I}\right),$$

and

$$u^{''} = u - f \in \mathcal{C}\left(\overline{I}\right), u^{''} = u - f \in L^2(I) \subset L^1(I) \Rightarrow u \in \mathcal{C}^2\left(\overline{I}\right),$$

then

$$-\frac{d^2u}{dx^2} + u = f, \text{ on } I.$$

Let's find the boundary conditions verified by u. Integrating by parts parts, we obtain

$$-\int_0^1 \frac{d^2u}{dx^2} v dx = -\left[u^{'}v\right]_0^1 + \int_0^1 u^{'}v^{'} dx,$$

therefore

$$\int_0^1 u^{'}v^{'} dx = \left[u^{'}v\right]_0^1 - \int_0^1 \frac{d^2u}{dx^2} v dx,$$

but

$$\begin{aligned} a(v,v) &= \ell(v) \Leftrightarrow \int_0^1 u^{'}v^{'} dx + \int_0^1 uv dx = \int_0^1 fv dx \\ &\Leftrightarrow \int_0^1 u^{'}v^{'} dx = \int_0^1 (f-u)\,v dx. \end{aligned}$$

Hence

$$\begin{aligned}\left[u'v\right]_0^1 - \int_0^1 \frac{d^2u}{dx^2} v dx &= \int_0^1 (f-u)\, v dx \\ &\Longrightarrow \left[u'v\right]_0^1 + \int_0^1 \left(-\frac{d^2u}{dx^2} + u - f\right) v dx = 0.\end{aligned}$$

$$-\frac{d^2u}{dx^2} + u = f \Rightarrow \int_0^1 \left(-\frac{d^2u}{dx^2} + u - f\right) v dx = 0,$$

then

$$\left[u'v\right]_0^1 = 0, \ \forall v \in H^1(I).$$

Let's choose $v(t) = t$, we have $v \in H^1(I)$ which gives

$$u'(1) = 0,$$

and for $v(t) = 1 - t$, we get $v \in H^1(I)$ which implies

$$u'(0) = 0.$$

The solution $u \in H^1(I)$ of $(\mathrm{P_V})$ is therefore the solution to the boundary problem

$$(\mathrm{Pc}) \quad \begin{cases} -\dfrac{d^2u}{dx^2} + u = f & \text{on } I \\ u'(0) = u'(1) = 0 \end{cases}.$$

Remark 4.3 : *Assuming only $f \in L^2(I)$, the variational problem defined in (P_V) also has a unique solution $u \in H^1(I)$ which we characterize by $u \in H^2(I)$ and*

$$(Pc) \quad \begin{cases} -\dfrac{d^2u}{dx^2} + u = f & \text{a.e on } I \\ u'(0) = u'(1) = 0 \end{cases}$$

4.4 Exercises

Exercise 4.1:

Let $I =]0,1[$ and $u : \overline{I} \longrightarrow \mathbb{R}$ such that

$$(Pc) \quad \begin{cases} -\dfrac{d^2u}{dx^2} + u = f & \text{on } I \\ u(0) = u(1) = 0 \end{cases},$$

where $f \in L^2(I)$.

1- Show that any solution of (Pc) is a solution of a problem (P_V) of the form

$$(P_V) \begin{cases} \text{Found } u \in H_0^1(I) \ \text{ such that} \\ a(u,v) = \ell(v)\,, \ \forall v \in H_0^1(I) \end{cases},$$

2- Prove that the problem (P_V) has a unique solution.

Exercise 4.2:

Let $I =]0,1[$, and a map $u : \overline{I} \longrightarrow \mathbb{R}$ verify

$$(Pc) \quad \begin{cases} -\dfrac{d^2u}{dx^2} + u = f & \text{on } I \\ u(0) = a, \ u(1) = b \end{cases},$$

whre f is a given function in $L^2(I)$ and $a, b \in \mathbb{R}^*$.

1- Prove that the problem (Pc) has a unique solution

2- Same question, using the following Neumann conditions

$u'(0) = \alpha$, $u'(1) = \beta, \forall(\alpha, \beta) \neq (0,0)$.

Exercise 4.3:

Let $f \in L^2(\Omega)$. Consider the problem

$$(1) \begin{cases} -\triangle u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \eta} = g & \text{on } \partial\Omega \end{cases},$$

The solution of this problem is defined up to a constant. To remove this indeterminacy, we will search for u in the quotient space

$$V = \{v \in H^1(\Omega), \int_\Omega v(x)dx = 0\},$$

if $u \in H^2(\Omega)$ is the solution of (1), then we necessarily have the following condition :

$$\int_\Omega f dx = 0.$$

1- Prove that V is a Hilbert space equipped with the norm of $H^1(\Omega)$.
2- One can show (by a proof by contradiction and using the fact that $H^1(\Omega)$ is compactly embedded into $L^2(\Omega)$) that there exists a strictly positive constant C such that:

$$\int_\Omega v^2 dx \leq C \int_\Omega |\nabla v|^2 dx, \forall v \in V. \qquad (2)$$

From this, deduce that V is a Hilbert space for the norm :

$$\|v\|_V = (\int_\Omega |\nabla v|^2 dx)^{1/2}.$$

3- Write the variational formulation P_V associated with the problem

$$\{Find\ u \in\ V\ solution\ of\ (1)\}.$$

Show that it has a unique solution.
4- Deduce the existence and uniqueness of a solution for the problem P_V.

Exercise 4.4:

Let Ω be a bounded, regular open set of class C^1, $f \in C(\Omega)$ and $g \in C(\Omega)$ be two given functions, and β a positive real number. We consider the problem: Find $u \in C^2(\overline{\Omega})$ such that

$$(1) \begin{cases} -\triangle u = f & \text{in } \Omega \\ \beta u + \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega \end{cases},$$

When $\beta > 0$, we say that the boundary condition $\beta u + \frac{\partial u}{\partial \nu} = g$ is of Robin type (or Fourier).
1- Prove that any solution of (1) is a solution of problem

$$(2) \begin{cases} \text{find } u \in V \text{ such that} \\ a(u,v) = \ell(v), \ \forall v \in V \end{cases},$$

where $V = C^1(\Omega)$ and $a(.,.)$, $L(.)$ are bilinear and linear forms that will be specified.
2- Show that if $\beta > 0$, problem (2) has at most one solution, and the same holds for problem (1).

Exercise 4.5:

Let Ω be a bounded, regular open set in $\mathbb{R}^n$. We consider the plate equation

$$(1) \begin{cases} \triangle(\triangle u) = f & \text{in } \Omega \\ u = 0 & \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega \end{cases}$$

Let X be the space of functions v in $C^2(\Omega)$ such that v and $\frac{\partial v}{\partial \eta}$ vanish on $\partial\Omega$. Suppose u is a function in $C^4(\Omega)$.

Show that u is a solution of the boundary value problem (1) if and only if u belongs to X and satisfies the equality

$$\int_\Omega (\triangle u(x))\,(\triangle v(x))\,dx = \int_\Omega f(x)v(x)dx, \quad \text{for any fonction } v \in X.$$

Exercise 4.6:

Using the variational approach, demonstrate the existence and uniqueness of the solution of

$$(1)\begin{cases} -\triangle u + u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases},$$

Where Ω is an arbitrary open set in the space $\mathbb{R}^n$ and $f \in L^2(\Omega)$.
In particular, show that adding a zero-order term to the Laplacian allows us not to require the assumption that Ω is bounded.

Exercise 4.7:

Demonstrate that the unique solution $u \in H^1(\Omega)$ of the variational formulation

$$\int_\Omega (\nabla u.\nabla v + uv)dx = \int_{\partial\Omega} gvds + \int_\Omega fvdx, \quad \forall v \in H1(\Omega)$$

satisfies the following energy estimate

$$\|u\|_{H^1(\Omega)} \leq C\left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right).$$

Where $C > 0$ is a constant that does not depend on u, f and g.

Exercise 4.8:

Assuming that Ω is a bounded open set in $\mathbb{R}^n$ and $f \in (L^2(\Omega))^n$, show the existence and uniqueness of a weak solution in $(H_0^1(\Omega))^n$ to the Lamé system

$$-\mu\triangle u - (\mu + \lambda)\nabla(divu) = f \quad in\ \Omega.$$

Verify that we can weaken the assumptions of positivity on the Lamé coefficients by assuming only that

$$\mu > 0, and\ \ 2\mu + \lambda > 0.$$

Exercise 4.9:

Let V be the space of velocity fields with zero divergence defined by

$$V = \left\{ v \in H_0^1(\Omega)^n : divv = \sum_{i=1}^{N} \frac{\partial v_i}{\partial x_i} = 0\ a.e.\ in\ \Omega. \right\}$$

Let $J(v)$ the energy defined for $v \in V$ by

$$J(v) = \frac{1}{2}\int_{\Omega} \mu \left|\nabla v\right|^2 dx - \int_{\Omega} f.vdx$$

Let $u \in V$ the unique solution of the variational formulation

$$\int_{\Omega} \mu \nabla u.\nabla v dx = \int_{\Omega} f.vdx, \quad \forall v \in V.$$

Prove that u is also the unique minimum point of the energy, that is

$$J(u) = \min_{v \in V} J(v).$$

Conversely, demonstrate that if $u \in V$ is a minimum point of the energy $J(v)$, then u is the unique solution of the variational formulation.

Exercise 4.10:

Let V be a Hilbert space. According to the conditions of the Lax-Milgram theorem, the variational problem

$$(P_V) \begin{cases} \text{Find } u \in V \text{ such that} \\ a(u,v) = \ell(v), \ \forall v \in V \end{cases},$$

is equivalent to problem :

$$\{Find\ u \in V\ such\ that: \quad Au = F\}.$$

Prove that the operator A is a bijection from V to V.

Exercise 4.11:

For a Lipschitz domain $\Omega \in \mathbb{R}^n$ and a function $A \in C^1(\Omega, \mathbb{R}^{n \times n})$, find the weak formulation for the problem

$$(1) \begin{cases} -div(A\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases},$$

Chapter 5

Correction of exercises

5.1 First chapter exercises

Exercice 1.2

Let Ω be an open set in $\mathbb{R}^n$. Then $T \in \mathcal{D}'(\Omega)$ if and only if for every compact $K \subset \Omega$,

$$\exists c > 0,\ \exists m \in \mathbb{N}, \forall \varphi \in \mathcal{D}_K(\Omega),\ |\langle T, \varphi\rangle| \leq c \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \leq m}} \sup_{x \in K} |D^\alpha \varphi(x)|, \tag{5.1}$$

1- Let K be a compact of Ω. If $\varphi \in \mathcal{D}(\Omega)$, we get

$$|\langle T_f, \varphi\rangle| \leq \int_\Omega |f(x)| \,.\, |\varphi(x)|\, dx \leq \|f\|_{L^1(K)} \cdot \|\varphi\|_{L^\infty(K)} \,.$$

This implies that the continuity criterion (5.1) is verified with

$$C = \|f\|_{L^1(K)} \ \textit{and}\ m = 0.$$

Then, all functions $f \in L^1_{loc}(\Omega)$ define a distribution T_f on Ω, which is of order 0.

2- Similarly, we have

$$|\langle \delta_{x_0}, \varphi\rangle| = |\varphi(x_0)| \leq \|\varphi\|_{L^\infty(K)},\ \varphi \in \mathcal{D}(\Omega),$$

which implies that the condition (5.1) is verified with

$$C = 1 \ \textit{and}\ m = 0.$$

The distribution δ_{x_0} is therefore of order 0.

3- Prove that the $vp(\frac{1}{x})$ distribution is of order 1. Let K be a compact of $\mathbb{R}$, there exists $a>0$ such that

$$\langle vp\frac{1}{x},\varphi\rangle = \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\frac{\varphi(x)}{x}dx = \lim_{\varepsilon\to 0}\left[\int_{-\infty}^{\varepsilon}\frac{\varphi(x)}{x}dx+\int_{\varepsilon}^{+\infty}\frac{\varphi(x)}{x}dx\right],\ \forall x\in\mathcal{D}\left(\mathbb{R}\right).$$

we can write

$$\begin{aligned}\langle vp\frac{1}{x},\varphi\rangle &= |\int_0^a \frac{\varphi(x)-\varphi(-x)}{x}dx| \\ &= |\int_0^a\int_{-1}^1 \varphi'(ux)dudx| \leq 2a\|\varphi'\|_{C(K)}.\end{aligned}$$

which implies that the condition (5.1) is verified with

$$C = 2a\ and\ m = 1.$$

The distribution $vp(\frac{1}{x})$ is therefore of order 1.

Exercice 1.3

Let p a reel such that $1<p\leq+\infty$, (f_k) a sequence of functions of $L^p(\Omega)$ verify:

$$\exists C>0,\ \forall k\in\mathbb{N},\ \|f_k\|_p\leq C.$$

$$\exists T\in\mathcal{D}'(\Omega),\ T=\lim_{k\to+\infty}[f_k].$$

Show that T is the regular distribution associated with a element of $L^p(\Omega)$. Let q such that $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\forall u\in\mathcal{D}(\Omega), |<T,u>=\lim_{k\to+\infty}|<[f_k],u>|\leq C\|u\|_{L^q(\Omega)}.$$

Let φ be the mapping:

$$\begin{aligned}\mathcal{D}(\Omega) &\to \mathbb{C}\\ u &\to \varphi(u)=(T,u),\end{aligned}$$

φ is linear and continuous on $\mathcal{D}(\Omega)$ equipped with the topology of $L^q(\Omega)$, where q satisfies $1\leq q<+\infty$, $\mathcal{D}(\Omega)$ is dense in $L^q(\Omega)$, φ admits a unique linear and continuous extension from $L^q(\Omega)$ to $\mathbb{C}$. In other words, there exists:

$$f\in L^p(\Omega),\ \varphi(u)=(T,u)=\int_\Omega f(x)u(x)dx.$$

Exercice 1.4:

A- We have

$$\begin{aligned} < T, \varphi > &= < T_f, \varphi' > + < \delta_0, \varphi' > = - < (T_f)', \varphi > -' < \delta_0', \varphi > \\ &= < -(T_f)' - \delta_0', \varphi >, \forall \varphi \in \mathcal{D}(\mathbb{R}). \end{aligned}$$

In other words,

$$T = -(T_f)' - \delta_0' \quad in \ \mathcal{D}'(\mathbb{R}).$$

Which implies that T is of order 1.

2. If f is derivable in $\mathbb{R}^*$ and $f(0+)$ and $f(0-)$ exist, then using the jump formula, we obtain

$$(T_f)' = T_{f'} + (f(0^+) - f(0^-))\delta_0.$$

Taking into account the result of the first question, it follows that

$$T = T_{f'} + (f(0^+) - f(0^-))\delta_0 - \delta_0' \quad in \ \mathcal{D}'(\mathbb{R}).$$

Exercise 1.6:

To prove that the function

$$u(x,t) = f(t-x), \quad (t,x) \in \mathbb{R} \times \mathbb{R},$$

in the space $L^1_{loc}(\mathbb{R}\times\mathbb{R})$, we need to show that it is locally integrable, which means that its integral over any compact K subset of $\mathbb{R} \times \mathbb{R}$ is finite.

Then, for any $\varepsilon > 0$, we can find a compact subset $K_1 \subset \mathbb{R}$ such that $K \subset K_1 \times \mathbb{R}$. Since f is locally integrable, we have:

$$\exists M_1 > 0, \int_{K_1} |f(x)|dx < M_1.$$

Then

$$\int_K |u(x,t)|dxdt = \int_{K_1} \int_{\mathbb{R}} |f(y)|dydx, y = t - x.$$

Therefore,

$$\int_K |u(x,t)|dxdt = M_1 Mes(K_1) < +\infty.$$

Since the integral of $|u(x,t)|$ over any compact subset K of $\mathbb{R} \times \mathbb{R}$ is finite, $u(x,t) = f(t-x)$ is locally integrable on $\mathbb{R} \times \mathbb{R}$, then

$$u(x,t) \in L^1_{loc}(\mathbb{R} \times \mathbb{R}).$$

Exercice 1.7

Consider the sequence of functions $(f_n)_{n\geq 1}$ defined on $\mathbb{R}$ by:

$$f_n(x)=\begin{cases} 0, & x\leq -\frac{1}{n} \\ n^2\left(\frac{1}{n}+x\right), & -\frac{1}{n}\leq x\leq 0 \\ n^2\left(\frac{1}{n}-x\right), & 0\leq x\leq \frac{1}{n} \\ 0, & x\geq \frac{1}{n}\end{cases}$$

1) Study the convergence of the sequence $(f_n)_{n\geq 1}$ in $L^2(\mathbb{R})$, $L^1(\mathbb{R})$ and $\mathcal{D}'(\mathbb{R})$.

(a) **Convergence in** $L^2(\mathbb{R})$**:**

For any $n\geq 1$, the function $f_n(x)$ is zero almost everywhere except in the interval $-\frac{1}{n}\leq x\leq \frac{1}{n}$. In this interval, $f_n(x)$ is bounded by n^2, and its integral over the entire real line is finite

$$\int_{-\infty}^{+\infty}|f_n(x)|^2dx=\int_{-\frac{1}{n}}^{0}n^4(\frac{1}{n}+x)^2dx+\int_{0}^{+\frac{1}{n}}n^4(\frac{1}{n}-x)^2dx=\frac{2n}{3}.$$

So

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}|f_n(x)|^2dx=+\infty.$$

Therefore, the sequence $(f_n)_{n\geq 1}$ isn't converges in $L^2(\mathbb{R})$.

(c) **Convergence in** $\mathcal{D}'(\mathbb{R})$**:**

We need to show that the sequence $(f_n)_{n\geq 1}$ converges in the space of distributions $\mathcal{D}'(\mathbb{R})$. Let's consider a test function $\varphi\in\mathcal{D}(\mathbb{R})$. We have $f_n\in L^1_{loc}(\mathbb{R})$, so it defines a regular distribution The distributional limit of f_n is given by

$$\lim_{n\to\infty}(f_n,\varphi)=\lim_{n\to\infty}\int_{-\infty}^{+\infty}f_n(x)\varphi(x)dx.$$

According to the mean value theorem, we have

$$\forall x\in\mathbb{R},\exists\theta(x)\ such\ that:\varphi(x)=\varphi(0)+x\varphi'(\theta(x)).$$

Now, we can calculate the integral

$$\begin{aligned}\int_{-\infty}^{+\infty}f_n(x)\varphi(x)dx &= \int_{-\frac{1}{n}}^{+\frac{1}{n}}f_n(x)(\varphi(0)+x\varphi'(\theta(x)))dx \\ &= \varphi(0)\int_{-1/n}^{+1/n}f_n(x)dx+\int_{-1/n}^{+1/n}xf_n(x)\varphi'(\theta(x))dx \\ &= \varphi(0)+\int_{-1/n}^{+1/n}xf_n(x)\varphi'(\theta(x))dx \\ &\leq \varphi(0)+\underset{t\in\mathbb{R}}{max}|\varphi'(t)|\int_{-1/n}^{+1/n}xf_n(x)dx \\ &\leq \varphi(0)+\frac{2}{n}\underset{t\in\mathbb{R}}{max}|\varphi'(t)\end{aligned}$$

Thus, we have:

$$\lim_{n\to\infty}(f_n,\varphi) = \lim_{n\to\infty}\int_{-\infty}^{+\infty} f_n(x)\varphi(x)dx = \varphi(0).$$

This means that the distributional limit of f_n is the Dirac distribution δ at $x = 0$. Hence, the sequence $(f_n)_{n\geq 1}$ converges in $\mathcal{D}'(\mathbb{R})$ to the Dirac distribution δ.
3- Suppose there exists a function $f_\delta \in L^1_{loc}(\mathbb{R})$ such that

$$\langle \delta, \varphi\rangle = \int_{\mathbb{R}^n} f_\delta \varphi(x)dx,$$

for all test functions $\varphi \in \mathcal{D}(\mathbb{R})$. Then, by the definition of the Dirac distribution, we have

$$\varphi(0) = \int_{\mathbb{R}} f_\delta(x)\varphi(x)dx, \forall \varphi \in \mathcal{D}(\mathbb{R}).$$

However, there is no locally integrable function f_δ that satisfies this condition, as the integral on the right-hand side will always be 0 for any test function φ whose support does not include the origin. But on the left-hand side, $\varphi(0)$ can take non-zero values. This contradiction shows that such a function f_δ cannot exist in $L^1_{loc}(\mathbb{R})$.
Therefore, there is no function $f_\delta \in L^1_{loc}(\mathbb{R})$ that satisfies

$$\langle \delta, \varphi\rangle = \int_{\mathbb{R}^n} f_\delta \varphi(x)dx.$$

Exercice 1.10:

Let $\Omega =]a, b[$ where $(a < b)$. We want to show that if a distribution $u \in \mathcal{D}'(]a, b[)$ has a zero derivative (in the sense of distributions), then u is a constant function.
To prove this, let's assume that u has a zero derivative, i.e., $u' = 0$ as a distribution. We want to show that this implies u is constant.
Suppose u is not constant; that means

$$\exists x_1, x_2 \in]a, b[,\ x_1 \neq x_2,\ u(x_1) \neq u(x_2).$$

Without loss of generality, assume $u(x_1) < u(x_2)$. Since u is a distribution, we can find two test functions

$$\varphi_1, \varphi_2 \in \mathcal{D}(]a, b[)\ such\ that\ \varphi_1(x_1) \neq \varphi_2(x_2).$$

Now consider the test function $\varphi = \varphi_2 - \varphi_1$. We get

$$\varphi \in \mathcal{D}(]a, b[),\ and\ \varphi \neq 0.$$

However, we have: $(u', \varphi) = 0($ since $u = 0$ as a distribution$)$. By integration by parts, we get:

$$(u, \varphi') = 0, \forall\ \varphi' \in \mathcal{D}(]a, b[) \Rightarrow u = 0,$$

which contradicts our assumption that $u(x_1) \neq u(x_2)$. Therefore, our initial assumption that u is not constant must be false. Thus, if u has a zero derivative (in the sense of distributions) on $]a, b[$, then u must be a constant function.

Exercice 1.11:

We consider the function k defined on $\mathbb{R}^2$ as follows:

$$\begin{aligned} k(x,t) &= 0, \text{ if } t < 0, \\ k(x,t) &= \frac{1}{\sqrt{4\pi t}} \exp(-\frac{1}{4t}), \text{ if } t > 0. \end{aligned}$$

1) To show the properties of k:

a) As t approaches 0, the term $\frac{1}{\sqrt{4\pi t}} \exp(-\frac{1}{4t})$ in the expression for k becomes infinitely large, thus,

$$\lim_{t \to 0^+} k(0,t) = +\infty.$$

When x tends to 0, both cases of the definition of k yield 0, hence

$$\lim_{x \to 0} k(x,t) = 0.$$

b) Prove that $\int_{-\infty}^{+\infty} k(x,t)dx = 1$ for all $t > 0$.

We can calculate the integral for $t > 0$ as follows:

$$\int_{-\infty}^{+\infty} k(x,t)dx = \int_{0}^{+\infty} \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{1}{4t}\right) dx = 1.$$

c) Prove that $\int_{-\infty}^{+\infty} k(x,t)dx = 1$ for all $t < 0$.

Since $t < 0$, the integral over the whole real line is equal to 0 because $k(x,t) = 0$ for $t < 0$. Hence,

$$\int_{-\infty}^{+\infty} k(x,t)dx = 0, \ \forall t < 0.$$

2) To compute $\frac{\partial^2 k}{\partial t \partial x}$ and $\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) k$ in the sense of distributions.

a) Calculate $\frac{\partial^2 k}{\partial t \partial x}$:

We differentiate $k(x,t)$ with respect to t and then with respect to x, we get

$$\frac{\partial k}{\partial t} = \begin{cases} 0, & \text{if } t < 0, \\ -\frac{1}{8\sqrt{\pi}} \frac{1}{t^{3/2}} \exp\left(-\frac{1}{4t}\right), & \text{if } t > 0, \end{cases}$$

$$\frac{\partial^2 k}{\partial t \partial x} = \begin{cases} 0, & \text{if } t < 0, \\ \frac{1}{32\sqrt{\pi}} \frac{1}{t^{5/2}} \exp\left(-\frac{1}{4t}\right), & \text{if } t > 0. \end{cases}$$

b) Compute $\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)k$, we obtain

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)k=\begin{cases}0, & \text{if } t<0,\\ \left(-\frac{1}{8\sqrt{\pi}}\frac{1}{t^{3/2}}-\frac{1}{32\sqrt{\pi}}\frac{1}{t^{5/2}}\right)\exp\left(-\frac{1}{4t}\right), & \text{if } t>0.\end{cases}$$

Again, it is important to note that these derivatives are calculated in the sense of distributions, as the function has some singularities at $t=0$.

5.2 Seconde chapter exercises

Exercise 2.1:——————————————————————————————-

Let u be an application from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$x\to u(x)=\begin{cases}x & x\in[0,1],\\ -x+2 & x\in[1,4],\\ 0 & elsewhere\end{cases}$$

1- Does the application u belong to $H^1(\mathbb{R})$?

To determine whether the function u belongs to the Sobolev space $H^1(\mathbb{R})$, we need to check if it is locally integrable on $\mathbb{R}$ and if its weak derivative is square-integrable on $\mathbb{R}$.

a- Let's check if u is locally integrable: Since u is piecewise defined, it is locally integrable on each subinterval.

On $[0,1]$, $u(x)=x$ is continuous, and thus locally integrable.

On $[1,4]$, $u(x)=-x+2$ is also continuous and thus locally integrable.

On the intervals outside $[0,4]$, $u(x)=0$, and it is trivially locally integrable.

Since u is locally integrable on each interval, it is locally integrable on $\mathbb{R}$, it means $u\in L^2(\mathbb{R})$. En effet,

$$\int_{\mathbb{R}}|u(x)|^2dx=\frac{8}{3}.$$

b- Next, let's find if the weak derivative $\partial_x u\in L^2(\mathbb{R})$ On a

$$x\to u'(x)=\begin{cases}1 & x\in[0,1],\\ -1 & x\in[1,4],\\ 0 & elsewhere\end{cases}$$

Since the function u belongs to $L^2(\Omega)$, thus it defines a distribution T_u sur $\mathbb{R}$, et pour $\varphi\in D(\mathbb{R}$, on a

$$<T_u',\varphi>=-<T_u,\varphi'>=-\int_0^1 x\varphi'(x)dx-\int_1^4(-x+2)\varphi'(x)dx.$$

On intègre par parties, on obtient

$$\begin{aligned} < T'_u, \varphi > &= -(x\varphi(x)|_0^1 - \int_0^1 \varphi(x)dx) - ((-x+2)\varphi(x)|_1^4 - \int_1^4 \varphi(x)dx) \\ &= \int_0^1 \varphi(x)dx + \int_1^4 \varphi(x)dx + 2\varphi(4) \\ &= \int_0^1 \varphi(x)dx + \int_1^4 \varphi(x)dx + 2(\delta_4, \varphi). \end{aligned}$$

Therefore, u is not belongs to $H^1(\mathbb{R})$ because $\delta_4 \notin L^2(\mathbb{R})$.

2- Now, let's find the open intervals I of $\mathbb{R}$ for which the restriction of u to I is in $H^1(I)$.

Any open interval I of $\mathbb{R}$ not containing 4 will satisfy $u \in H^1(I)$.

Exercice 2.3:—————————————————————————————-

Let (r, θ) be the polar coordinates of $\mathbb{R}^2$ and

$$\Omega = \left\{(r,\theta), \ \ 0 < r < 1, \ 0 < \theta < \frac{\pi}{4}\right\}.$$

Let α be a real number in $\mathbb{R}$. We denote by f_α the mapping from Ω to $\mathbb{R}$, defined as

$$f_\alpha(r,\theta) = r^\alpha.$$

1- For which values of α does f_α belong to $L^2(\Omega)$?

The function $f_\alpha(r,\theta)$ belongs to $L^2(\Omega)$ if the integral of its square over the domain Ω is finite. In other words, we need to check if the following integral is finite

$$\int_\Omega |f_\alpha(r,\theta)|^2 r dr d\theta = \int_0^1 \int_0^{\frac{\pi}{4}} |r^\alpha|^2 r d\theta dr$$

Let's compute this integral

$$\int_0^1 \int_0^{\frac{\pi}{4}} |r^\alpha|^2 r dr d\theta = \frac{\pi}{4} \int_0^1 r^{2\alpha+1} dr.$$

So, the function $f_\alpha(r,\theta) = r^\alpha$ belongs to $L^2(\Omega)$ for $2\alpha + 1 > 0$ i.e. $\alpha > -1/2$.

2- For which values of α, does f_α belong to $H^1(\Omega)$?

The function $f_\alpha(r,\theta)$ belongs to $H^1(\Omega)$ if it is in $L^2(\Omega)$ and its first-order partial derivatives with respect to r and θ are also in $L^2(\Omega)$. The partial derivatives of f_α are given by:

$$\frac{\partial f_\alpha}{\partial r} = \alpha r^{\alpha-1}, \quad \frac{\partial f_\alpha}{\partial \theta} = 0.$$

Now, let's check the square integrability of these derivatives:

$$\begin{aligned} \int_\Omega \left|\frac{\partial f_\alpha}{\partial r}\right|^2 r dr d\theta &= \int_0^1 \int_0^{\frac{\pi}{4}} \left|\alpha r^{\alpha-1}\right|^2 d\theta r dr \\ &= \int_0^1 \int_0^{\frac{\pi}{4}} \alpha^2 r^{2\alpha-1} dr d\theta \end{aligned}$$

This integral is finite if and only if $2\alpha - 1 \geq 0$.
Therefore, the function $f_\alpha(r,\theta) = r^\alpha$ belongs to $H^1(\Omega)$ for $\alpha \geq \frac{1}{2}$.
Exercise 2.6:——————————————————————————————

To show that there cannot be a notion of trace for functions in $L^2(\Omega)$, we'll assume the contrary, i.e., there exists a constant $C > 0$ such that for all $v \in L^2(\Omega)$, the following inequality holds:

$$\| v|_{\partial\Omega} \|_{L^2(\partial\Omega)} \leq C \, \|v\|_{L^2(\Omega)}$$

where $v|_{\partial\Omega}$ represents the restriction of v to the boundary $\partial\Omega$.

Method 1:

Let's construct a counterexample to disprove this assumption. Consider the following function $v_n \in L^2(\Omega)$ defined as follows:

$$v_n(r,\theta) = \{nr^{-\frac{1}{2}} \; for \; r \in (0, 1/n) \; and \; 0 \text{ otherwise}\}.$$

where n is a positive integer. The domain Ω is given as:

$$\Omega = \left\{ (r,\theta) \mid 0 < r < 1, \; 0 < \theta < \frac{\pi}{4} \right\}.$$

Now, let's calculate the L^2 norms of v_n and its trace on $\partial\Omega$.

$$\begin{aligned} \|v_n\|^2_{L^2(\Omega)} &= \int_0^1 \int_0^{\frac{\pi}{4}} |v_n(r,\theta)|^2 d\theta dr = \int_0^1 \int_0^{\frac{\pi}{4}} \left| nr^{\frac{-1}{2}} \right|^2 d\theta dr \\ &= n^2 \int_0^1 \int_0^{\frac{\pi}{4}} \frac{1}{r} d\theta dr \; = 0, \end{aligned}$$

and

$$\| v_n|_{\partial\Omega} \|^2_{L^2(\partial\Omega)} = \int_{\partial\Omega} |v_n(r,\theta)|^2 d\theta = \int_0^{\frac{\pi}{4}} |v_n|^2 d\theta = n^2 \cdot \frac{\pi}{4}. \tag{5.2}$$

Now, let's look at the assumed inequality:

$$n^2 \cdot \frac{\pi}{4} = \| v_n|_{\partial\Omega} \|_{L^2(\partial\Omega)} \leq C \, \|v_n\|_{L^2(\Omega)} = 0.$$

Therefore, there is no constant C that can satisfy the inequality for all v_n. Hence, we have a contradiction, and our initial assumption was false. Therefore, there cannot be a notion of trace for functions in $L^2(\Omega)$.

Method 2:

To simplify matters, let's choose the open set Ω to be the unit ball. Construct a sequence of smooth functions in $\overline{\Omega}$ equal to 1 on $\partial\Omega$, and in the norm $L^2(\Omega)$, it tends towards zero.

Let T be a reguler function of $[0,+\infty[$ in $\mathbb{R}^+$ such that $T(0) = 1$, $T(s) = 0$ for $s > 1$ and $0 \leq T(s) \leq 1$ for all s. The sequence u^n of functions from the ball Ω to $\mathbb{R}$ is defined as

$$u^n(x) = T(n(1-|x|)). \tag{5.3}$$

For every n and for every $x \in \partial\Omega$, $|u^n(x)| = 1$. Furthermore, the sequence $u^n(x)$ is bounded by 1 for every $x \in \Omega$. Moreover, $u^n(x) = 0$for every x belonging to the ball of radius $1 - 1/n$ and for any C for n large enough

$$\|u^n\|_{L^2(\partial\Omega)} = \|u^0\|_{L^2(\partial\Omega)} > C\|u^n\|_{L^2(\Omega)}.$$

The trace operator defined from $C(\overline{\Omega}) \cap L^2(\Omega)$ to $L^2(\partial\Omega)$ is not continuous. Consequently, it cannot be extended to a continuous mapping from $L^2(\Omega)$ in $L^2(\partial\Omega)$.

Exercise 2.11:

To show that the norms $\|.\|_{W^{1,p}(I)}$ and $|||.|||_{W^{1,p}(I)}$ are equivalent, we need to show that there exist positive constants C_1 and C_2 such that for any function u in the Sobolev space $W^{1,p}(I)$, the following inequalities hold:

$$C_1 \|u\|_{W^{1,p}(I)} \leq |||u|||_{W^{1,p}(I)} \leq C_2 \|u\|_{W^{1,p}(I)},$$

where $\|u\|_{W^{1,p}(I)}$ and $|||u|||_{W^{1,p}(I)}$ are the norms defined in the question.
Let's start by proving the first inequality:

$$C_1 \|u\|_{L^p(I)} + C_1 \|u'\|_{L^p(I)} \leq \|u\|_{W^{1,p}(I)},$$

where $C_1 = 1$. Using the Minkowski inequality for L^p norms, we have:

$$\begin{aligned}
\|u\|_{L^p(I)} + \|u'\|_{L^p(I)} &= \|u\|_{L^p(I)} + \|u'\|_{L^p(I)} \cdot 1 \\
&\leq \|u\|_{L^p(I)} + \|u'\|_{L^p(I)} \cdot \|1\|_{L^p(I)} \\
&\leq \|u\|_{L^p(I)} + \|u'\|_{L^p(I)} \cdot \|u\|_{L^p(I)} \\
&= \left(1 + \|u'\|_{L^p(I)}\right) \|u\|_{L^p(I)}
\end{aligned}$$

Since $1 + \|u'\|_{L^p(I)} > 1$ (as $p > 1$), we can take

$$C_1 = 1 + \|u'\|_{L^p(I)}$$

to get the desired inequality:

$$\|u\|_{L^p(I)} + \|u'\|_{L^p(I)} \leq \left(1 + \|u'\|_{L^p(I)}\right) \|u\|_{L^p(I)} = C_1 \|u\|_{L^p(I)}$$

Now, let's prove the second inequality

$$\|u\|_{W^{1,p}(I)} \leq C_2 \|u\|_{L^p(I)} + C_2 \|u'\|_{L^p(I)}$$

where

$$C_2 = 2^{1/p} + 1.$$

We start by applying the Minkowski inequality for L^p norms to the definition of the norm $|||u|||_{W^{1,p}(I)}$:

$$\begin{aligned} |||u|||_{W^{1,p}(I)} &= \left(\|u\|^p_{L^p(I)} + \|u'\|^p_{L^p(I)}\right)^{1/p} = \left(\|u\|^p_{L^p(I)} + \|u'\|^p_{L^p(I)} \cdot 1\right)^{1/p} \\ &\leq \left(\|u\|^p_{L^p(I)} + \|u'\|^p_{L^p(I)} \cdot \|1\|_{L^p(I)}\right)^{1/p} \\ &= \left(\|u\|^p_{L^p(I)} + \|u'\|^p_{L^p(I)}\right)^{1/p} \\ &= \|u\|_{W^{1,p}(I)} \end{aligned}$$

Now, since $p > 1$, we can apply Young's inequality to the sum inside the norm:

$$\begin{aligned} \|u\|_{W^{1,p}(I)} &= \left(\|u\|^p_{L^p(I)} + \|u'\|^p_{L^p(I)}\right)^{1/p} \\ &\leq \left(\|u\|^p_{L^p(I)} + \left(\|u'\|^p_{L^p(I)}\right)^{1/(p-1)}\right)^{1/p} \\ &= \left(\|u\|^p_{L^p(I)} + \|u'\|_{L^p(I)}\right)^{1/p} \\ &= \left(\|u\|^p_{L^p(I)} + \|u'\|_{L^p(I)} \cdot 1\right)^{1/p} \\ &\leq \left(\|u\|^p_{L^p(I)} + \|u'\|^p_{L^p(I)}\right)^{1/p} \\ &= |||u|||_{W^{1,p}(I)} \end{aligned}$$

Therefore, we have shown that the norms $\|.\|_{W^{1,p}(I)}$ and $|||.|||_{W^{1,p}(I)}$ are equivalent with

$$C_1 = 1 + \|u'\|_{L^p(I)}\,, \; C_2 = 2^{1/p} + 1.$$

Exercise 2.12:

To prove that

$$\exists q^* > 1 \text{ such that } \forall q \in [1, q^*[, u \in W^{1,q}(\Omega),$$

we need to show that the partial derivatives of u are in $L^q(\Omega)$ for q values within the given range.

1) Step 1 : We want to show that $u \in L^q(\Omega_\alpha),\; for\; q < 2\alpha$.

$$\int_{\Omega_\alpha} |u|^q dx = \int_{\Omega_\alpha} |(r^\alpha - r)^{-\alpha} sin(\alpha\theta)|^q dx = \int_0^{\pi/\alpha} |sin(\alpha\theta)|^q dx \int_0^1 \left(\frac{1}{r^\alpha} - r^\alpha\right)^r rdr.$$

On a

$$\left(\frac{1}{r^\alpha}-r^\alpha\right)^r \sim \frac{1}{r^{\alpha q-1}},$$

which is integrable on $]0,1[$ if and only if $\alpha q-1<1$, i.e $q<2\alpha$.
Hence $u\in L^q(\Omega_\alpha),\ if\ q<2\alpha.$
2) Step 2 : We want to show that $\frac{\partial u}{\partial x},\ \frac{\partial u}{\partial y}\in L^q(\Omega_\alpha)\ for\ q<2\alpha+1.$
let's calculate the partial derivatives of u:

$$\frac{\partial u}{\partial r}=-\alpha(r^\alpha-r)^{-\alpha-1}(r^{\alpha-1}-1)\sin(\alpha\theta).$$

$$\frac{\partial u}{\partial \theta}=\alpha(r^\alpha-r)^{-\alpha}\cos(\alpha\theta).$$

But

$$\frac{\partial u}{\partial r}=cos(\theta)\frac{\partial u}{\partial x}+sin(\theta)\frac{\partial u}{\partial y}.$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta}=-sin(\theta)\frac{\partial u}{\partial x}+cos(\theta)\frac{\partial u}{\partial y}.$$

Hence, if $\frac{\partial u}{\partial r}$ and $\frac{1}{r}\frac{\partial u}{\partial \theta}$ are in $L^q(\Omega)$ then $\frac{\partial u}{\partial x},\ \frac{\partial u}{\partial y}$ will be in $L^q(\Omega)$.

$$\int_{\Omega_\alpha}\left|\frac{\partial u}{\partial r}\right|^q dx=\int_0^{\pi/\alpha}|\alpha sin(\alpha\theta)|^q d(\theta)\int_0^1\left|\frac{1}{r^\alpha+1}-r^{\alpha-1}\right|^q rdr.$$

On a

$$\left|\frac{1}{r^\alpha+1}-r^{\alpha-1}\right|^q r\sim\frac{1}{r^{(\alpha+1)q-1}},$$

which is integrable on $]0,1[$ if and only if $(\alpha+1)q-1<1$, i.e $q<2\alpha+1$, and

$$\int_{\Omega_\alpha}\left|\frac{1}{r}\frac{\partial u}{\partial \theta}\right|^q=\int_0^{\pi/\alpha}|\alpha cos(\alpha\theta)|^q d(\theta)\int_0^1\frac{1}{r^q}\left|\frac{1}{r^\alpha}-r^\alpha\right|^q rdr.$$

On a

$$\left|\frac{1}{r^\alpha+1}-r^{\alpha-1}\right|^q r\sim\frac{1}{r^{(\alpha+1)q-1}},$$

which is integrable on $]0,1[$ if and only if $(\alpha+1)q-1<1$, i.e $q<2\alpha+1.$
It follows that

$$\frac{\partial u}{\partial x},\ \frac{\partial u}{\partial y}\in L^q(\Omega_\alpha)\ for\ q<\frac{2}{\alpha+1}.$$

2) We take $q^*=\frac{2}{\alpha+1}$. Since both partial derivatives are in $L^q(\Omega)$ for $q<q^*$, we can conclude that

$$u\in W^{1,q}(\Omega)\ for\ q\in[1,q^*[.$$

2) To calculate $-\Delta u$, we need to find the Laplacian of u, which is given by:

$$\Delta=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Let's calculate the second derivatives:

$$\frac{\partial^2 u}{\partial r^2} = sin(\alpha\theta)(\alpha(\alpha+1)r^{-\alpha-2} - \alpha(\alpha-1)r^{-\alpha-2}).$$

$$\frac{1}{r}\frac{\partial u}{\partial r} = sin(\alpha\theta)(-\alpha r^{-\alpha-2} - \alpha(\alpha-1)r^{-\alpha-2}).$$

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\alpha^2 sin(\alpha\theta)(r^{-\alpha-2} - r^{\alpha-2}).$$

While computing the laplacian, we obtains the following results: $-\Delta u = 0$.

3) It follows that the PDE

$$\begin{cases} -\Delta v = 0 \ in \ \Omega \\ v = 0 \quad on \ \partial\Omega \end{cases},$$

has two solutions in $W^{1,q}(\Omega_\alpha)$ and the zero function. The problem is that Ω_α is not of class C^1. Then we lose the uniqueness of the solution.

Exercise 2.13:

To show that the canonical injection maps $u \in H^1(\Omega)$ in $C^0(\Omega)$ is compact, we need to demonstrate that it maps bounded sets in $H^1(\Omega)$ to relatively compact sets in $C^0(\Omega)$.

First, let's consider a bounded set B in $H^1(\Omega)$. By definition of $H^1(\Omega)$, we know that for any $u \in B$, both u and its derivative u' are in $L^2(\Omega)$.

Now, let's show that the set $\{u(x) : u \in B\}$ is equicontinuous, i.e.,

$$\forall\varepsilon > 0, \exists\delta > 0 \ such \ that \ \forall x, y \in \Omega, |x-y| < \delta \Rightarrow |u(x) - u(y)| < \varepsilon, \ \forall u \in B.$$

Since B is bounded in $H^1(\Omega)$, it means that

$$\exists M > 0, \|u\|_{H^1(\Omega)} \leq M, \forall u \in B \Rightarrow \|u\|_{L^2(\Omega)} \leq M_1 \ and \ \|u'\|_{L^2(\Omega)} \leq M_2,$$

for some constants M_1 and M_2. This implies that $|u(x)| \leq M_1$ and $|u'(x)| \leq M_2$ for almost every $x \in \Omega$.

Now, for any $x, y \in \Omega$ with $|x - y| < \delta$, we can apply the mean value theorem to get: where c is some point between x and y. So, we can take

$$\delta = \varepsilon/M_2,$$

and the set $\{u(x) : u \in B\}$ is equicontinuous.

Since $\{u(x) : u \in B\}$ is bounded and equicontinuous, the Ascoli's theorem guarantees that it is relatively compact in $C^0(\Omega)$. Therefore, the canonical injection from $H^1(\Omega)$ to $C^0(\Omega)$ is compact. Since the injection from $H^1(\Omega)$ to $L^2(\Omega)$ is the same as the injection from $H^1(\Omega)$ to $C^0(\Omega)$ (as $L^2(\Omega)$ is a subspace of $C^0(\Omega)$), we can conclude that the canonical injection from $H^1(\Omega)$ to $L^2(\Omega)$ is also compact.

5.3 Third chapter exercises

Exercise 3.2:

Let $I=]0,1[$ and let a map $u:\overline{I}\longrightarrow\mathbb{R}$ which verify

$$(Pc)\quad\begin{cases}-\dfrac{d^2u}{dx^2}+u=f & \text{on } I\\ u'(0)=\alpha,\ u'(1)=\beta\end{cases},$$

where f is a given function in $C(I)$ and $\forall(\alpha,\beta)\neq(0,0)$.

Show that the problem (Pc) has a unique solution.

Variationnal Formulation (Pc)**:**

Let u be a regular element ($u\in\mathcal{C}^2(\overline{I})$) that is a solution of problem (Pc). Let $v\in H^1(I)$ be a test function. By multiplying both sides of (Pc) by v and integrating from 0 to 1, we obtain the following expression

$$\int_0^1\left(-\frac{d^2u}{dx^2}v+uv\right)dx=\int_0^1 fvdx.$$

By performing integration by parts, we obtain

$$-\int_0^1\frac{d^2u}{dx^2}vdx=-\left[u^{'}v\right]_0^1+\int_0^1 u^{'}v^{'}dx.$$

Therefore

$$-\left[u^{'}v\right]_0^1+\int_0^1 u^{'}v^{'}dx+\int_0^1 uvdx=\int_0^1 fvdx,$$

and

$$\int_0^1 u^{'}v^{'}dx+\int_0^1 uvdx=\beta v(1)-\alpha v(0)+\int_0^1 fvdx.$$

for $u,v\in V=H^1(I)$

$$(\mathrm{P_V})\quad a(u,v)=\ell(v),\ \forall v\in V,$$

with

$$a(u,v)=\int_0^1\left(u^{'}v^{'}+uv\right)dx,$$

and

$$\ell(v)=\ell_1(v)+\ell_2(v)$$

with

$$\ell_1(v)=\int_0^1 fvdx,\ \ell_2(v)=\beta v(1)-\alpha v(0).$$

Existence and uniqueness of the solution of $(\mathrm{P_V})$**:**

It is clear that $a(.,.)$ is continuous on $H^1(I)\times H^1(I)$. Moreover, since

$$a(v,v)=\|v\|^2_{H^1(I)},$$

and $a(.,.)$ is coercive on $H^1(I)$, it is evident that $\ell_1(.)$ is continuous on $H^1(I)$. The remaining part is to show that $\ell_2(.)$ is continuous on $H^1(I)$.
For every $v \in V = H^1(I)$, we have

$$\begin{aligned} |\ell_2(v)| &= |\beta v(1) - \alpha v(0)| \leq |\beta| |v(1)| + |\alpha| |v(0)| \\ &\leq |\beta| \underset{x\in\Omega}{supess} |v(x)| + |\alpha| \underset{x\in\Omega}{supess} |v(x)| \\ &\leq |\beta| \|v\|_{L^\infty} + |\alpha| \|v\|_{L^\infty} . \end{aligned}$$

Since $H^1(I) \hookrightarrow L^\infty(I)$ it means that

$$\exists C > 0, \|v\|_{L^\infty(I)} \leq C \|v\|_{H^1(I)} ,$$

then

$$|\ell_2(v)| \leq C(|\beta| + |\alpha|) \|v\|_{H^1(I)} ,$$

so $\ell_1(.)$ is continuous in $H^1(I)$.
Using the Lax-Milgram theorem, we obtain the existence of a unique $u \in H^1(I)$ solution of $(\mathrm{PDn_V})$. Then the solution $u \in \mathcal{C}^2(\overline{I})$ of (Pc) is therefore a solution of $(\mathrm{P_V})$.

Regularity and Return to the Classical Solution:

Let $u \in H^1(I)$ solution of $(\mathrm{P_V})$.
Since $\mathcal{D}(I) \subset H^1(I)$, then $\beta\varphi(1) - \alpha\varphi(0) = 0$ and

$$\int_0^1 u'\varphi' dx + \int_0^1 u\varphi dx = \int_0^1 f\varphi dx, \ \forall\varphi \in \mathcal{D}(I),$$

therefore

$$\int_0^1 u'\varphi' dx = \int_0^1 (f-u)\varphi dx, \ \forall\varphi \in \mathcal{D}(I).$$

hence

$$\left\langle T_{u'}, \varphi' \right\rangle_{\mathcal{D}'\times\mathcal{D}} = \left\langle -(T_{u'})', \varphi \right\rangle_{\mathcal{D}'\times\mathcal{D}} = \langle -T_{u''}, \varphi \rangle_{\mathcal{D}'\times\mathcal{D}} = \left\langle T_{f-u}, \varphi' \right\rangle_{\mathcal{D}'\times\mathcal{D}} .$$

then $-T_{u''} = T_{f-u}$, this show that $u'' = u - f \in L^2(I)$, because

$$f \in \mathcal{C}(\overline{I}) \Longrightarrow f \in L^2(I),$$

and

$$u \in H^1(I) \Longrightarrow u \in L^2(I).$$

Indeed, since $u \in H^1(I)$, we have $u' \in L^2(I)$, which implies that u has a weak second derivative.
Since $u - f \in \mathcal{C}(\overline{I})$ because $f \in \mathcal{C}(\overline{I})$ and

$$u \in H^2(I) \subset H^1(I) \hookrightarrow \mathcal{C}(\overline{I}),$$

therefore $u \in \mathcal{C}^2\left(\overline{I}\right)$, in fact

$$u^{''} = u - f \in \mathcal{C}\left(\overline{I}\right) \ et \ u^{''} = u - f \in L^2(I) \subset L^1(I),$$

then

$$-\frac{d^2u}{dx^2} + u = f, \text{ on } I.$$

For any $v \in H^1(I)$, we have

$$\begin{aligned} -\int_0^1 \frac{d^2u}{dx^2} v dx &= -\left[u^{'}v\right]_0^1 + \int_0^1 u^{'}v^{'}dx \\ &= -u^{'}(1)v(1) + u^{'}(0)v(0) + \int_0^1 u^{'}v^{'}dx. \end{aligned}$$

Then

$$\int_0^1 u^{'}v^{'}dx = u^{'}(1)v(1) - u^{'}(0)v(0) - \int_0^1 \frac{d^2u}{dx^2} v dx$$

Since $a(u,v) = \ell(v)$, then

$$\int_0^1 u^{'}v^{'}dx + \int_0^1 uvdx = \beta v(1) - \alpha v(0) + \int_0^1 fvdx,$$

therefore

$$\int_0^1 u^{'}v^{'}dx = -\int_0^1 uvdx + \beta v(1) - \alpha v(0) + \int_0^1 fvdx.$$

Consequently

$$u^{'}(1)v(1) - u^{'}(0)v(0) - \int_0^1 \frac{d^2u}{dx^2} v dx + \int_0^1 uvdx - \beta v(1) + \alpha v(0) - \int_0^1 fvdx = 0,$$

then

$$v(1)\left(u^{'}(1) - \beta\right) + v(0)\left(\alpha - u^{'}(0)\right) + \int_0^1 \left(-\frac{d^2u}{dx^2} + u - f\right) v dx = 0.$$

where

$$v(1)\left(u^{'}(1) - \beta\right) + v(0)\left(\alpha - u^{'}(0)\right) = 0,$$

because $-\frac{d^2u}{dx^2} + u = f$. By simple choices of v, we obtain

$$u^{'}(0) = \alpha \ and \ u^{'}(1) = \beta.$$

Exercise 3.3:———————————————————————————-

Let $f \in L^2(\Omega)$, we consider the problem

$$(1)\begin{cases} -\triangle u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega \end{cases},$$

By integrating equation (1) over Ω, using the Green's formula, and considering that $\frac{\partial u}{\partial \nu} = 0$ on Γ, we obtain the compatibility condition

$$\int_\Omega f dx = 0.$$

2- We have

$$V = \{v \in H^1(\Omega), \int_\Omega v(x)dx = 0\}.$$

Therefore,

$$V = \text{Ker } L \quad \text{with } L : v \in H^1(\Omega) \to \int_\Omega v(x)dx \in \mathbb{R}.$$

Since L is continuous, V is a closed vector subspace of the Hilbert space $(H^1(\Omega), \|.\|_{H^1(\Omega)})$, making it a Hilbert space itself with the norm $\|.\|_{H^1(\Omega)}$.

3- With inequality

$$\int_\Omega v^2 dx \leq C \int_\Omega |\nabla v|^2 dx, \forall v \in V. \quad (2)$$

we get that $\|\nabla v\|_{L^2(\Omega)}$ is a norm in V that is equivalent to the norm $\|.\|_{H^1(\Omega)}$. Thus, V is a Hilbert space with the norm:

$$\|v\|_V = \|\nabla v\|_{L^2(\Omega)}.$$

4- The variational formulation reads as follows: Find $u \in V$ such that

$$\int_\Omega \nabla u \cdot \nabla v, dx = \int_\Omega f v, dx \quad \forall v \in V. \tag{3}$$

According to the Lax-Milgram theorem, this problem has a unique solution. Indeed, $(V, \|\|_V)$ is a Hilbert space. The bilinear form

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v dx,$$

is continuous and coercive on $V \times V$, and the linear form $L(v) = \int_\Omega f v dx$, is continuous (According to inequality (2)).

5- Let's show that the unique solution to problem (3) is also a solution to equation (1). We proceed as follows: Let $v \in H^1(\Omega)$. Then we define

$$\overline{v} = v - \int_\Omega v, dx \in V.$$

According to the variational formulation (3), we have:

$$\int_\Omega \nabla u.\nabla \overline{v} dx \; = \int_\Omega f.\overline{v} dx,$$

which can be rewritten as:

$$\int_\Omega \nabla u.\nabla v dx = \int_\Omega f v dx - \Big(\int_\Omega f dx\Big) - \Big(\int_\Omega v dx\Big).$$

Now, using the compatibility condition, we obtain:

$$\int_\Omega \nabla u.\nabla v dx = \int_\Omega f v dx, \quad \forall v \in H^1(\Omega).$$

Assuming $u \in H^2(\Omega)$ and using the Green's formula, we deduce that:

$$\int_\Omega (-\triangle u - f) v dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v dx, \qquad \forall v \in H^1(\Omega).$$

Now, we proceed with the classical argument.
Take $v \in \mathcal{D}(\Omega) \subset H^1(\Omega)$. Since $\mathcal{D}(\Omega)$ is a dense subset of $L^2(\Omega)$, the boundary term disappears, and we deduce that

$$-\triangle u = f \;\; in \; L^2(\Omega),$$

meaning it holds almost everywhere in Ω. Then, we have:

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} v dx = 0 \qquad \forall v \in H^1(\Omega).$$

As the image of $H^1(\Omega)$ under the trace operator is dense in $L^2(\partial\Omega)$, we conclude that:

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{in } L^2(\partial\Omega),$$

which implies that

$$\frac{\partial u}{\partial \nu} = 0 \; almost \; everywhere \; on \; \partial\Omega.$$

Exercise 3.5:

Let u be a regular solution of the plate equation (1), for any $v \in X$,

$$\int_\Omega \triangle(\triangle u)\, v \; dx = \int_\Omega f v \; dx.$$

By integrating equation, we obtain

$$-\int_\Omega \nabla(\triangle u).\nabla v \; dx + \int_{\partial\Omega} \frac{\partial(\triangle u)}{\partial \nu} v dx = \int_\Omega f v \; dx.$$

we have $v = 0$ on $\partial\Omega$, we deduce that

$$-\int_\Omega \nabla(\triangle u).\nabla v \; dx = \int_\Omega f v \; dx,$$

Then, by a new integration by parts and using the fact that $\frac{\partial v}{\partial \nu} = 0$ on $\partial\Omega$, we obtain

$$\int_\Omega (\triangle u).(\triangle v) \; dx = \int_\Omega f v \; dx.$$

Reciprocal. Suppose that u is a solution of the variational problem. By performing two successive integrations by parts, we obtain,

$$\int_{\Omega} (\triangle (\triangle u) - f).v \, dx = 0,$$

for any $v \in X$. We deduce that

$$\triangle (\triangle u) - f = 0.$$

Exercise 3.6:———————————————————————————————-

Using the variational approach, we demonstrate the existence and uniqueness of the solution for the problem:

$$(1) \begin{cases} -\triangle u + u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases},$$

where Ω is any open set in the space $\mathbb{R}^n$, and $f \in L^2(\Omega)$.

In particular, shown that adding a zeroth-order term to the Laplacian allows us to avoid the assumption that Ω is bounded.

Step 1. The variational formulation:

We multiply the verified equation by a test function v that is zero on $\partial\Omega$. By performing integration by parts, we obtain that

$$\int_{\Omega} (\nabla u.\nabla v \ + uv) \, dx = \int_{\Omega} fv \, dx.$$

In order for this expression to make sense, we need to choose u and v in $H_0^1(\Omega)$. Thus, the variational problem associated with equation (1) consists of determining $u \in H_0^1(\Omega)$ such that:

$$a(u; v) = L(v) \ \ for\ any\ v \in H_0^1(\Omega),$$

where

$$a(u, v) = \int_{\Omega} (\nabla u.\nabla v \ + uv) \, dx,$$

$$L(v) = \int_{\Omega} fv \, dx.$$

Step 2. Solving of the variational problem

The continuity of $a(.,.)$ and $L(.)$ is evident, as well as the coercivity of the bilinear form $a(.,.)$. Indeed,

$$a(u, u) = \|u\|_{H^1(\Omega)}.$$

The assumptions of the Lax-Milgram theorem are satisfied. Therefore, there exists a unique solution to the variational problem.

Finally, by performing the same integrations by parts as in the first step, we verify that

$$-\triangle u + u = f,$$

as elements of $L^2(\Omega)$ and thus almost everywhere in Ω. Finally, since $u \in H_0^1(\Omega)$ and Ω is a regular open set, the trace of u is well-defined, and

$$u = 0 \text{ almost everywhere on } \partial\Omega.$$

Exercise 3.7:

Show that the unique solution $u \in H^1(\Omega)$ of the variational formulation

$$\int_\Omega (\nabla u.\nabla v + uv)dx = \int_{\partial\Omega} gvds + \int_\Omega fvdx, \quad \forall v \in H^1(\Omega),$$

Verifies the following energy estimate

$$\|u\|_{H^1(\Omega)} \leq C\left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right),$$

Where $C > 0$ is a constant that does not depend on u, f, and g. It suffices to apply the variational formulation to the test function $v = u$. This leads to the conclusion that:

$$\|u\|_{H^1(\Omega)}^2 = \int_\Omega (|\nabla u|^2 + |u|^2)dx = \int_{\partial\Omega} gu + \int_\Omega fudx.$$

By applying the Cauchy-Schwarz inequality to the second term

$$\|u\|_{H^1(\Omega)}^2 \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \|u\|_{L^2(\partial\Omega)}.$$

By the Trace Theorem (where the application γ_0 is a linear and continuous mapping from $H^1(\Omega)$ to $L^2(\partial\Omega)$), there exists a positive constant C (which depends only on Ω) such that

$$\|u\|_{L^2(\partial\Omega)} \leq C\|u\|_{H^1(\Omega)},$$

then

$$\|u\|_{H^1(\Omega)}^2 \leq C\left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right)\|u\|_{H^1(\Omega)}.$$

From this, we deduce the desired inequality.

Exercise 3.10:

Let V be a Hilbert space. According to the conditions of the Lax-Milgram theorem, the variational problem

$$(P_V)\begin{cases} \text{Find } u \in V \text{ such that} \\ a(u,v) = \ell(v), \ \forall v \in V \end{cases},$$

is equivalent to problem:

$$\{Find\ u \in V\ such\ that \quad Au = F\}.$$

Show that the operator A is a bijection from V to V.

For v fixed, the application $w \to a(v, w)$ is continuous linear from V to $\mathbb{R}$. The Riesz representation theorem implies that there exists an element Av in V such that

$$a(u, w) = (Au, w) = \ell(v), \forall u, w \in V.$$

This defines a linear and continuous operator because

$$|(Au, w)| \quad = \quad |\, a(u, w)| \leq M \|u\| . \|w\|$$

Then

$$\|Au\| = \sup_{\|w\|=1} |\, a(u, w)| \leq M \|u\| .$$

Similary, the linear form l is contiuous from V, the Riesz representation theorem implies that there exists an unique element $f \in V$ such that

$$\forall v \in V,\ l(v) = (f, v).$$

Thus, the problem (P_v) is equivalent to seeking $u \in V$ as a solution of

$$Au = f.$$

Then, the operator A is bijective.

The operator A is linear and continuous; as a result, A^{-1} is also continuous from V to V. In fact, we can show that A^{-1} is a bounded operator (and hence continuous) from V to V.

First, we deduce from the coercivity that:

$$\forall v \in V,\ \alpha \|v\|^2 \leq a(v, v) = (Av, v) \leq \|Av\| \|v\| ,$$

it means

$$\forall v \in V, \|Av\| \geq \alpha \|v\| . \tag{1}$$

This proves that A is injective ($Av = 0 \Rightarrow v = 0$).

To show that A is surjective, meaning $AV = V$, we can use the inequality (1) to demonstrate that the image $AV \subset V$ is a closed subspace. Indeed, if $(u_n)_{n\in\mathbb{N}}$ is a sequence in V such that $(Au_n)_{n\in\mathbb{N}}$ is a convergent sequence, then the bound holds for all $n, m \in \mathbb{N}$.

$$\|Av_m - Av_n\| \geq \alpha \|v_m - v_n\| ,$$

To show that the sequence $(u_n)_{n\in\mathbb{N}}$ is Cauchy, it converges to $u \in V$, and the sequence $(Au_n)_{n\in\mathbb{N}}$ converges to Au.

Finally, we demonstrate that the image $A(V)^{\perp}$ is dense. Suppose $v \in A(V)^{\perp}$. For any $u \in V$, we have

$$a(u, v) = (Au, v) = 0,$$

in particular,

$$a(v; v) = 0.$$

We deduce that $v = 0$, and thus $A(V)^{\perp} = 0$. This is one of the characterizations of the density of a subspace.
Since AV is both closed and dense, it follows that $AV = V$, meaning A is surjective. Consequently, we have demonstrated that $A : V \to V$ is a linear, bijective, and continuous mapping. The inequality

$$\|v\| \leq \frac{1}{\alpha} \|Av\|.$$

allow us to show that A^{-1} is continuous with a norm bounded by $\frac{1}{\alpha}$.

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