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Numerical Processing of Nonlinear Fredholm Integro-Differential Equations by Applying Projection Techniques

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إلى والدي الكريمين سبب وجودي في الحياة بهما أسمو العلا، وعليهما أرتكز

إلى من هم سند لي في الحياة إخواني وأخواتي

إلى أساتذتي الكرام، إلى أصدقائي الأعزاء، إلى زملائي الفضلاء...

إلى كل من ساهم و ساعدني في حياتي الدراسية...

إلى أولائك جميعا أهدي رسالتي الدكتوراه هذه ، التي أسال الله تعالى أن يتقبلها خالصة منا وأن ينفع بها....

يوسف حنكــه

Abstract

The main focus of this thesis is to present a numerical study of Fredholm integral equations of the nonlinear integro-differential type. This includes examining both the regular and weakly singular cases, as well as the fractional case. To obtain numerical solutions for these equations, we use popular projection methods like the Galerkin method and the collocation method, along with classical orthogonal polynomials. The primary benefit of this approach is that it allows us to transform the main equations for each case into a nonlinear algebraic system. We can then use iterative methods to solve these systems efficiently. To show the accuracy and effectiveness of our approach, we present several numerical examples throughout the thesis. These examples demonstrate how our numerical process accurately solve the given equations, which further confirms the effectiveness of our proposed method.

Keywords: Orthogonal polynomials. Fredholm integro-differential equations. Galerkim method. Collocation technique. Nonlinear equation

Resumé

L'objectif principal de cette thèse est de présenter une étude numérique des équations intégrales de Fredholm de type intégro-différentiel non linéaire. Cela comprend l'examen à la fois des cas réguliers et faiblement singuliers, ainsi que du cas fractionné. Pour obtenir des solutions numériques pour ces équations, nous utilisons des méthodes de projection populaires comme la méthode de Galerkin et la méthode de collocation, ainsi que des polynômes orthogonaux classiques. Le principal avantage de cette approche est qu'elle nous permet de transformer les équations principales de chaque cas en un système algébrique non linéaire. Nous pouvons ensuite utiliser des méthodes itératives pour résoudre efficacement ces systèmes. Pour démontrer la justesse et l'efficacité de notre approche, nous présentons plusieurs exemples numériques tout au long de la thèse. Ces exemples démontrent comment nos solutions numériques résolvent avec précision les équations données, ce qui confirme davantage l'efficacité de la méthode que nous proposons.

Mots-clés: Polynômes orthogonaux. Équations intégro-différentielles de Fredholm. Méthode de Galerkin. Technique de colocation. Équation non linéaire.

الملخــص

في هذه الأطروحة نقدم دراسة عددية لمعادلات فريدهولم التفاضلية التكاملية ذات نواة غير خطية. وهذا يشمل دراسة كل من حالات الأنوية المعتدلة وكذا الأنوية ضعيفة الإنفراد، بالإضافة إلى الحالة الكسرية؛ للحصول على حلول عددية لهذا النوع من المعادلات، نستخدم طرق الإسقاط الشهيرة مثل طريقة قالوركين وطريقة التجميع، إلى جانب كثيرات الحدود المتعامدة الكلاسيكية؛ الفائدة الأساسية من هذه الطريقة باعتبارها تسمح لنا بتحويل المعادلات الرئيسية لكل حالة إلى نظام جبري غير خطي مما يمكننا بعد ذلك استخدام الطرق التكرارية لحل هذه الأنظمة ؛ وأخيرا لإظهار دقة وفعالية تقريبنا المقترح نقدم العديد من الأمثلة العددية لتوضيح دقة و فعالية الطريقة.

الكلمات المفتاحية:

كثيرات الحدود المتعامدة، معادلات التفاضلية التكاملية لفريدهولم، طريقة قالوركين، طريقة التجميع، المعادلات غير الخطية.

Notations

$\mathcal{V}$	Vector space.
$\mathcal{B}$	Banach space.
$\langle ., . \rangle$	Scalar product.
$\mathcal{H}$	Hilbert space.
$\langle ., . \rangle$	Scalar product.
$\|x\|$	Norm of x.
$\mathbb{R}_n[X]$	Polynomials of degree less than or equal n.
∂	Partial differential.
$L^2([a,b])$	Lebesgue space.
$H^1([a,b])$	Sobolev space.
$\mathbb{C}$	Set of complex numbers.
$\mathbb{R}$	Set of real numbers.
$\mathbb{Z}$	Set of integer numbers.
$\mathbb{N}$	Set of positive integer numbers.
$\mathbb{N}^*$	Set of strict positive integer numbers.

Contents

Introduction

An integro-differential equation is a type of differential equation that involves both derivatives and integrals of the unknown function. Such equations are often used in physics, engineering, and applied mathematics to model a wide range of phenomena, including heat transfer, fluid mechanics, and signal processing, ... (see [6, 9, 23, 36]).

Solving integro-differential equations can be challenging, as they often do not have explicit solutions. Instead, numerical methods are used to approximate the solutions. Among these, we can enumerate finite difference methods or spectral methods: Laplace decomposition method [5], Legendre-Galerkin method [18], Bernoulli polynomials [8] , Pseudospectral methods, Piecewise linear approximation, Polynomial approximation, Rational approximation [21], B-spline method [16, 28], Euler matrix method [25], Exponential spline method [22], CAS wavelet [35], Differential transformation [13], Schauder bases [7], Homotopy perturbation method [39], collocation method [31], Haar wavelet bases [17].

On the other hand, there are many publications that give the numerical solution by using orthogonal polynomials; we find for instance: Chebyshev polynomials [14, 27, 33], Legender polynomials [37, 38], Hermite polynomials [24], Laguerre polynomials [30, 32], Jacobi polynomials [29], new class of orthogonal polynomials [3], Comparison of the orthogonals polynomial [12] . However, applying orthogonal polynomials to solve nonlinear integro-differential equations can be more complex than solving linear equations. In fact, there are still techniques that can be used to approximate the solution using orthogonal polynomials. One approach is to use a Galerkin method, which involves approximating the solution as a linear combination of a finite number of orthogonal polynomials, and then projecting the original nonlinear integro-differential equation onto the space of these polynomials. This yields a system of nonlinear algebraic equations for the coefficients of the polynomial expansion, which can then be solved using numerical techniques such as Newton's method or fixed-point iteration.

Another approach is to use a collocation method, which involves evaluating the integro-differential equation at a set of collocation points, and approximating the solution as a linear combination of orthogonal polynomials that satisfy the equation at these points. This yields a system of nonlinear algebraic equations that can also be solved numerically.

In this thesis, we undertake a numerical research for several forms of nonlinear integro-differential equations in which the unknown function and its derivatives are in the nonlinear kernel. The current study relies on the use of several kinds of orthogonal polynomials to get the required numerical solution.

The outline of the thesis is as follows: The first chapter aims to introduce some fundamental definitions and theorems in functional analysis, as well as the orthogonal polynomials, including their properties that are necessary for the subsequent chapters.

In the second chapter, we examine regular integro-differential equations with nonlinear kernels. In the first section, using Legender polynomials, we give the numerical solution for the following equation

$$\begin{cases} u(z) = f(z) + \displaystyle\int_0^1 K\left(z, y, u(y), u'(y)\right) dy, \quad \forall z \in [0,1], \\ u(0) = \rho. \end{cases}$$

In the second section, we develop the previous method presented in the first section by utilizing Chebyshev polynomials to solve the following equation [19]

$$\begin{cases} \psi(z) = f(z) + \displaystyle\int_0^1 K\left(z, y, \psi(y), \psi'(y), \psi''(y)\right) dy, \quad \forall z \in [0,1], \\ \psi(0) = \alpha, \quad \psi'(0) = \beta. \end{cases}$$

Through the third chapter, we investigate nonlinear integro-differentials equations with weakly singular kernels which takes the following form

$$\begin{cases} v(z) = f(z) + \displaystyle\int_a^b p(|z-t|) F\left(z, t, v(t), v'(t)\right) dt, z \in [a,b], \\ v(a) = v_0, \end{cases}$$

where

$$\lim_{x \to 0} p(x) = +\infty.$$

In the first section, we apply the Galerkin method along with Chebyshev polynomials of the second kind to approximate the solution of the nonlinear integro-differential equation. In the second section, we use the collocation approach based on Laguerre polynomials to solve the equation.

Finally, in the fourth chapter, we utilize Hermite polynomials to obtain an approximation for the solution of the fractional integro-differential equation of the form

$$z \in [0,1], u(z) = g(z) + \int_0^1 K\left(z, t, u(t), \mathcal{D}^\alpha u(t)\right) dt, \quad u(0) = 0,$$

where $\mathcal{D}^\alpha$ denotes the Caputo- Fabrizio derivative of order α [20].

During the period of the thesis study, we were able to publish the following articles:

1. Henka, Y., Lemita, S., Aissaoui, M. Z. (2023). Hermite wavelets collocation method for solving a Fredholm integro-differential equation with fractional Caputo-Fabrizio derivative. Proyecciones (Antofagasta), 42(4), 917-930.

2. Henka, Y., Lemita, S., Aissaoui, M. Z. (2022). Numerical study for a second order Fredholm integro-differential equation by applying Galerkin-Chebyshev-wavelets method. Journal of Applied Mathematics and Computational Mechanics, 21(4), 28-39.

Chapter 1

Preliminaries

1.1 General notions

In order to provide a comprehensive understanding of the material presented in this thesis, it is important to establish a solid background in the relevant mathematical concepts. This section introduces the reader to some fundamental definitions and theorems that will be used throughout the rest of the work.

Definition 1.1. *A norm on a vector space $\mathcal{V}$ is a mapping $\|\cdot\| : \mathcal{V} \to [0, \infty)$ such that for all $u, v \in \mathcal{V}$ and for all $\alpha \in \mathbb{C}$:*

(i) $\|u + v\| \leq \|u\| + \|v\|$ *(Triangle Inequality),*

(ii) $\|\alpha u\| = |\alpha|\|u\|$ *(Scalar Property),*

(iii) $\|u\| = 0$ *if and only if* $u = 0$.

The pair $(\mathcal{V}, \|\cdot\|)$ is then called a normed space.

Definition 1.2. *A space $\mathcal{B}$ is said to be a Banach space if $(\mathcal{B}, \|\cdot\|)$ is a complete normed space.*

Definition 1.3. *Let $\mathcal{V}$ be a vector space over $\mathbb{C}$. A scalar (or inner) product is a mapping $\langle .,. \rangle from \mathcal{V} \times \mathcal{V}$ into $\mathbb{C}$ such that*

(i) $\langle u, v \rangle = \overline{\langle v, u \rangle} \quad \forall u, v \in \mathcal{V}$,

(ii) $\langle u + w, v \rangle = \langle u, v \rangle + \langle w, v \rangle$ *and* $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle \quad \forall u, v \in \mathcal{V}, \lambda \in \mathbb{C}$,

(iii) $\forall u \in \mathcal{V} \quad \langle u, u \rangle \geq 0$ *and* $\langle u, u \rangle = 0$ *if and only if* $u = 0$.

We call a linear space with an inner product an inner product space or pre-Hilbert space.

Proposition 1.1. *Any inner product $\langle ., . \rangle$ defines a norm on $\mathcal{V}$ by setting:*

$$\|u\| = \sqrt{\langle u, u \rangle} \quad \forall u \in \mathcal{V}.$$

Exemple 1.1. *A Hilbert space is a complete inner product space.*

Exemple 1.2. *The space $\mathcal{H} = \mathbb{C}^n$ is a Hilbert space with the inner product:*

$$\langle v, u \rangle = \sum_{i=1}^{n} v_i \overline{u_i} \quad \forall v, u \in \mathbb{C}^n,$$

where $v = (v_1, v_2, \ldots, v_n)$ and $u = (u_1, u_2, \ldots, u_n)$.

Exemple 1.3. *Let* $\mathcal{H} = L^2([a,b])$. *Then* $\mathcal{H}$ *is a Hilbert space equipped with the inner product:*

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx \quad \forall f, g \in L^2([a,b]).$$

Exemple 1.4. *Let*

$$\mathcal{H} = \ell^2 = \left\{ u = (u_n)_{n\in\mathbb{N}^*} \subset \mathbb{C}, \quad \sum_{i=1}^{\infty} |u_i|^2 < \infty \right\},$$

and the scalar product

$$\langle u, v\rangle = \sum_{i=1}^{\infty} u_i \overline{v_i} \quad \forall u, v \in \ell^2.$$

Then $\mathcal{H}$ *is a Hilbert space.*

Theorem 1.1. *(Cauchy-Schwarz inequality) Let* $\mathcal{H}$ *be a Hilbert space. Then*

$$|\langle v, u\rangle| \leqslant \|u\|\|v\| \quad \text{for all } v, u \in \mathcal{H}.$$

Definition 1.4. *Let* $M = \{u_i \in \mathcal{H}; i \in I \subset \mathbb{N}\}$ *and* $\langle u_i, u_j\rangle = 0$ *for all* $i, j \in I$ *with* $i \neq j$. *Then,* M *is said to be an orthogonal system. Additionally, if* M *is orthogonal and* $\|u_i\| = 1$ *for all* $i \in I$, *then* M *is called an orthonormal system.*

Lemma 1.1. *(Bessel's inequality) Let* $\mathcal{H}$ *be a Hilbert space. If* $\{e_i \in \mathcal{H}, i \in I \subset \mathbb{N}\}$ *is an orthonormal basis, then for all* $u \in \mathcal{H}$

$$\sum_{k\geq 0} |\langle e_k, u\rangle|^2 \leq \|u\|^2.$$

Definition 1.5. *A set* C *is convex if and only if*

$$\forall u, v \in C, \quad \forall \theta \in [0,1] \quad \theta u + (1-\theta)v \in C.$$

Exemple 1.5. *In a normed space* E, *any subspace is a convex set.*

Theorem 1.2. *Let* K *be a non-empty closed convex subset of a Hilbert space* $\mathcal{H}$. *For any element* $u \in \mathcal{H}$, *there exists a unique point* $v \in K$ *such that*

$$\|u - v\| = \inf_{w\in K} \|u - w\|.$$

Theorem 1.3. *Let* $(E, \|\cdot\|)$ *be a normed space. A mapping* $T : E \to E$ *is called a contraction on* E *if there exists a positive constant* $\rho < 1$ *such that*

$$\|T(u) - T(v)\| \leq \rho\|u - v\| \text{ for all } u, v \in E.$$

Theorem 1.4. *[4] (**Banach's Fixed Point Theorem**). Let $(E, \|\cdot\|)$ be a Banach space and let $T : E \to E$ be a contraction on E. Then T has a unique fixed point $x \in E$, i.e. $T(x) = x$.*

Definition 1.6. *[10] (**Caputo-Fabrizio derivative**) Let α be a real number from the open interval $(0, 1)$. The fractional Caputo-Fabrizio derivative of order α for a function u belonging to the space $H^1[0, 1]$ is as follows:*

$$\mathcal{D}^\alpha u(z) = \frac{1}{1-\alpha}\int_0^z \exp\left[-\frac{\alpha}{1-\alpha}(z-s)\right] u'(s)ds.$$

Definition 1.7. *[26] (**Caputo-Fabrizio integral**) Let α be a real number from the open interval $(0, 1)$. We define the fractional integral of order α using the Caputo-Fabrizio operator for a function u belonging to the Sobolev space $H^1[0, 1]$ as follows:*

$$\mathcal{I}^\alpha u(z) = (1-\alpha)u(z) + \alpha\int_0^z u(s)ds. \tag{1.1}$$

Lemma 1.2. *Given a real number α such that $0 < \alpha < 1$ and a function u in the Sobolev space $H^1([a, b])$, the following identities hold:*

$$\begin{aligned}
&\mathcal{I}^\alpha\left(\mathcal{D}^\alpha u(z)\right) = u(z) - u(a),\\
&\mathcal{D}^\alpha\left(\mathcal{I}^\alpha u(z)\right) = u(z) - \exp\left[-\frac{\alpha}{1-\alpha}(z-a)\right]\cdot u(a).
\end{aligned}$$

Proof. See [26] □

1.2 Orthogonal polynomials

In this section, we address the topic of orthogonal polynomials, which are a special type of polynomials with the property that they are orthogonal with respect to a particular inner product. We explore some of the basic properties of orthogonal polynomials, including their recurrence relations and differential equations.

Definition 1.8. *The two functions u and v are said to be orthogonal with respect to the weight function $w(z)$ on $[a, b]$*

$$\langle u, v\rangle = \int_a^b w(z)u(z)v(z)dz = 0.$$

Proposition 1.2. *Let $(P_i)_{i\geq 0}$ be a family of orthogonal polynomials. Then, there is a recurrence relation between P_{n+1}, P_n and P_{n-1}*

$$\forall n \in \mathbb{N}^\star, \quad \exists a_n, b_n, c_n \in \mathbb{R}, \quad P_{n+1} = (a_n X + b_n) P_n + c_n P_{n-1}. \tag{1.2}$$

Proof. The family $(XP_n, P_n, P_{n-1}, \ldots, P_0)$ is a family of polynomials having different degrees and so, it is a free family of $\mathbb{R}_{n+1}[X]$ with $n+2$ vectors. We deduce that this family is a basis of $\mathbb{R}_{n+1}[X]$ and there exist reals a_n, b_n, c_n and α_i for $0 \leq i \leq n-2$, such that

$$P_{n+1} = a_n XP_n + b_n P_n + c_n P_{n-1} + \sum_{i=1}^{n-2} \alpha_i P_i.$$

We use the orthogonality of $(P_i)_{i\geq 0}$

$$\forall 0 \leq i \leq n-2, \quad < P_{n+1}, P_i >= a_n < XP_n, P_i > +\alpha_i \|P_i\|^2 = 0.$$

According to the expression of the scalar product $< XP_n, P_i\rangle = \langle P_n, XP_i\rangle$, since $XP_i \in \mathbb{R}_{n-1}[X]$ we have $\langle XP_n, P_i\rangle = 0$, then

$$\forall 0 \leq i \leq n-2, \quad \alpha_i = 0.$$

□

1.2.1 Legendre polynomials

The Legendre polynomials take the following form by the Rodrigués formula:

$$L_n(z) = \frac{1}{n!2^n}\frac{d^n}{dz^n}\left[\left(z^2-1\right)^n\right]. \tag{1.3}$$

Theorem 1.5. *The Legendre polynomials $L_n(z)$ are given by the following recurrence formula*

$$(n+1)L_{n+1}(z) = (2n+1)zL_n(z) - nL_{n-1}(z) \quad L_1(z) = z, \quad L_0(z) = 1.$$

Exemple 1.6. *With $n = 4$ we get the following six polynomials:*

$$\begin{aligned}
L_0(z) &= 1,\\
L_1(z) &= z,\\
L_2(z) &= \frac{1}{2}\left(3z^2-1\right),\\
L_3(z) &= \frac{1}{2}\left(5z^3-3z\right),\\
L_4(z) &= \frac{1}{8}\left(35z^4-30z^2+3\right),\\
L_5(z) &= \frac{1}{8}\left(63z^5-70z^3+15z\right).
\end{aligned}$$

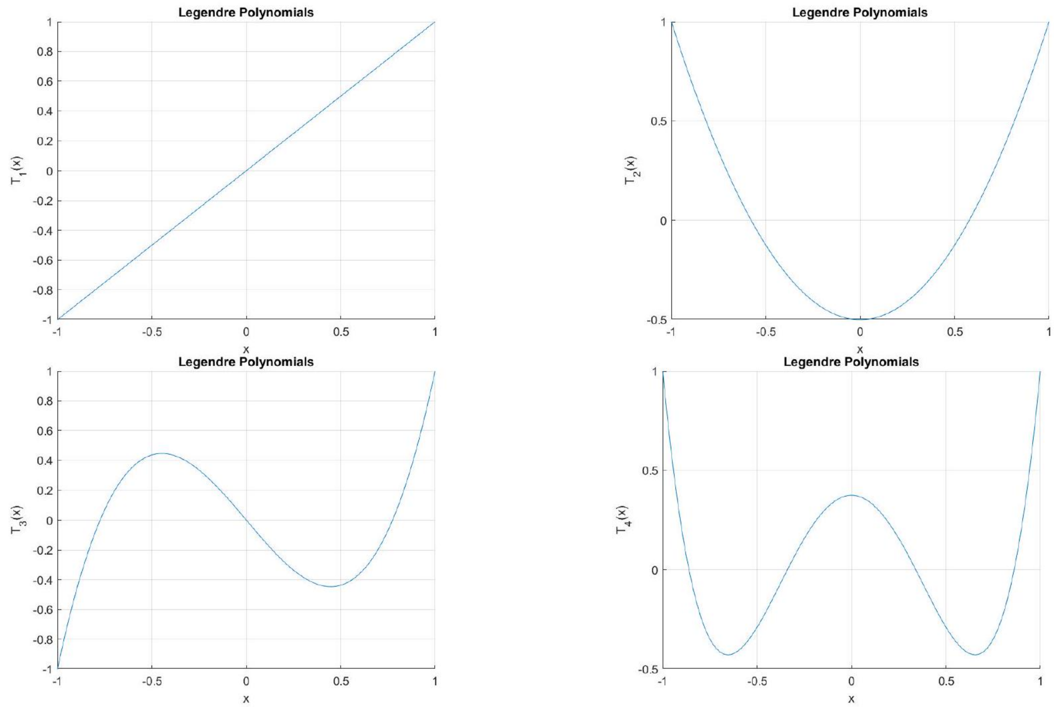

Figure 1.1: Legendre polynomials for $1 \leq n \leq 4$.

Proposition 1.3. *Legendre polynomials are orthogonal on* $[-1, 1]$ *and*

$$\int_{-1}^{1} L_i(z)L_j(z)dz = \begin{cases} \dfrac{2}{2i+1}, i = j, \\ 0, \quad i \neq j. \end{cases}$$

Proof. For $i < j$

$$\int_{-1}^{1} \frac{d^i}{dz^i}\left[\left(z^2-1\right)^i\right]\frac{d^j}{dz^j}\left[\left(z^2-1\right)^j\right]dz = -\int_{-1}^{1} \frac{d^{i+1}}{dz^{i+1}}\left[\left(z^2-1\right)^i\right]\frac{d^{j-1}}{dz^{j-1}}\left[\left(z^2-1\right)^j\right]dz + \left[\frac{d^i}{dz^i}\left[\left(z^2-1\right)^i\right]\frac{d^{j-1}}{dz^{j-1}}\left[\left(z^2-1\right)^j\right]\right]_{-1}^{1}. \tag{1.4}$$

Since

$$\left[\frac{d^i}{dz^i}\left[\left(z^2-1\right)^i\right]\frac{d^{j-1}}{dz^{j-1}}\left[\left(z^2-1\right)^j\right]\right]_{-1}^{1} = 0 \text{ because } \forall k \in (0, 1, \ldots, j-1), \frac{d^k}{dz^k}\left[\left(z^2-1\right)^j\right]_{-1}^{1} = 0,$$

then the relation(1.4) becomes

$$\int_{-1}^{1} \frac{d^i}{dz^i}\left[\left(z^2-1\right)^i\right]\frac{d^j}{dz^j}\left[\left(z^2-1\right)^j\right]dz = -\int_{-1}^{1} \frac{d^{i+1}}{dz^{i+1}}\left[\left(z^2-1\right)^i\right]\frac{d^{j-1}}{dz^{j-1}}\left[\left(z^2-1\right)^j\right]dz. \tag{1.5}$$

By integrating in the right-hand member j times, we get

$$\begin{aligned}\int_{-1}^{1}\frac{d^i}{dz^i}\left[\left(z^2-1\right)^i\right]\frac{d^j}{dz^j}\left[\left(z^2-1\right)^j\right]dz &= (-1)^j\int_{-1}^{1}\frac{d^{i+j}}{dz^{i+j}}\left[\left(z^2-1\right)^i\right]\frac{d^0}{dz^0}\left[\left(z^2-1\right)^j\right]dz\\ &\quad+\left[\frac{d^{i+j-1}}{d^{i+j-1}}\left[\left(z^2-1\right)^i\right]\frac{d^0}{dz^0}\left[\left(z^2-1\right)^j\right]\right]_{-1}^{1}\\ &=(-1)^j\int_{-1}^{1}\frac{d^{i+j}}{dz^{i+j}}\left[\left(z^2-1\right)^i\right]\left(z^2-1\right)^j dz.\end{aligned}$$

Since

$$\frac{d^{i+j}}{dz^{i+j}}\left[\left(z^2-1\right)^i\right]=0 \text{ if } i<j,$$

then,

$$\int_{-1}^{1}\frac{d^i}{dz^i}\left[\left(z^2-1\right)^i\right]\frac{d^j}{dz^j}\left[\left(z^2-1\right)^j\right]dz=0.$$

If $i=j$,

$$(n+1)\int_{-1}^{1}L_{n+1}(z)L_{n-1}(z)dz-(2n+1)\int_{-1}^{1}zL_n(z)L_{n-1}(z)dz+n\int_{-1}^{1}L_{n-1}^2(z)dz=0. \quad (1.6)$$

$$(n+1)\int_{-1}^{1}L_{n+1}(z)L_n(z)dz-(2n+1)\int_{-1}^{1}zL_n^2(z)dz+n\int_{-1}^{1}L_{n-1}(z)L_n(z)dz=0.$$

$$(n+1)\int_{-1}^{1}L_{n+1}^2(z)dz-(2n+1)\int_{-1}^{1}zL_n(z)L_{n+1}(z)dz+n\int_{-1}^{1}L_{n-1}(z)L_{n+1}(z)dz=0. \quad (1.7)$$

Since

$$\int_{-1}^{1}L_{n+1}(z)L_{n-1}(z)dz=0,\ \int_{-1}^{1}L_{n+1}(z)L_n(z)dz=0,$$
$$\int_{-1}^{1}L_{n-1}(z)L_n(z)dz=0 \text{ and } \int_{-1}^{1}L_{n-1}(z)L_{n+1}(z)dz=0,$$

then from (1.6) we deduce

$$-(2n+1)\int_{-1}^{1}zL_n(z)L_{n-1}(z)dz+n\int_{-1}^{1}L_{n-1}^2(z)dz=0.$$

Therefore,

$$\int_{-1}^{1}zL_n(z)L_{n-1}(z)dz=\frac{n}{(2n+1)}\int_{-1}^{1}L_{n-1}^2(z)dz=0,$$

and from (1.7), we deduce

$$n\int_{-1}^{1}L_n^2(z)dz-(2n-1)\int_{-1}^{1}zL_{n-1}(z)L_n(z)dz=0.$$

Hence,

$$\int_{-1}^{1}zL_{n-1}(z)L_n(z)dz=\frac{n}{(2n-1)}\int_{-1}^{1}L_n^2(z)dz.$$

Thus
$$\int_{-1}^{1} L_n^2(z)dz = \frac{(2n-1)}{(2n+1)} \int_{-1}^{1} L_{n-1}^2(z).$$
By recurrence, it comes
$$\int_{-1}^{1} L_n^2(z)dz = \frac{1}{(2n+1)} \int_{-1}^{1} L_0^2(z)dz = \frac{2}{(2n+1)}.$$
□

Corollary 1.1. *The Legendre polynomials can be defined by a differential equation for any integer n, such that $L_n(z)$ is a solution for the differential equation:*
$$\left(1-z^2\right) P'' - 2zP' + n(n+1)P = 0.$$

Proof. We have
$$L_n'(z) = 2nz\left(z^2-1\right)^{n-1},$$
after multiplication by $\left(z^2-1\right)$, we obtain
$$\left(z^2-1\right) L_n'(z) = 2nzL_n(z). \tag{1.8}$$
By differentiating the equality (1.8) $n+1$ times, with Leibniz's formula, we find
$$\begin{aligned}&\left(z^2-1\right)\frac{d^{n+2}}{dz^{n+2}}\left[L_n(z)\right] + (n+1)(2z)\frac{d^{n+1}}{dz^{n+1}}\left[L_n(z)\right] + \frac{n(n+1)}{2}(2)\frac{d^n}{dz^n}\left[L_n(z)\right]\\ =&2n\left[z\frac{d^{n+1}}{dz^{n+1}}\left[L_n(z)\right] + (n+1)\frac{d^n}{dz^n}\left[L_n(z)\right]\right],\end{aligned}$$
i.e.
$$\left(z^2-1\right)\left[\frac{d^n}{dz^n}\left[L_n(z)\right]\right]'' + 2z\left[\frac{d^n}{dz^n}\left[L_n(z)\right]\right]' - n(n+1)\frac{d^n}{dz^n}\left[L_n(z)\right] = 0,$$
that gives, after multiplication by $\dfrac{1}{n!2^n}$
$$\forall n \in \mathbb{N}, \left(1-z^2\right) L_n''(z) - 2zL_n'(z) + n(n+1)L_n(z) = 0.$$
□

1.2.2 The first kind of Chebyshev polynomials

The Chebyshev polynomials of the first kind are given by the Rodrigués representation
$$U_n(z) = \frac{(-1)^n\sqrt{\pi}\left(1-z^2\right)^{1/2}}{2^n\left(n-\frac{1}{2}\right)!}\frac{d^n}{dz^n}\left[\left(1-z^2\right)^{n-1/2}\right]. \tag{1.9}$$

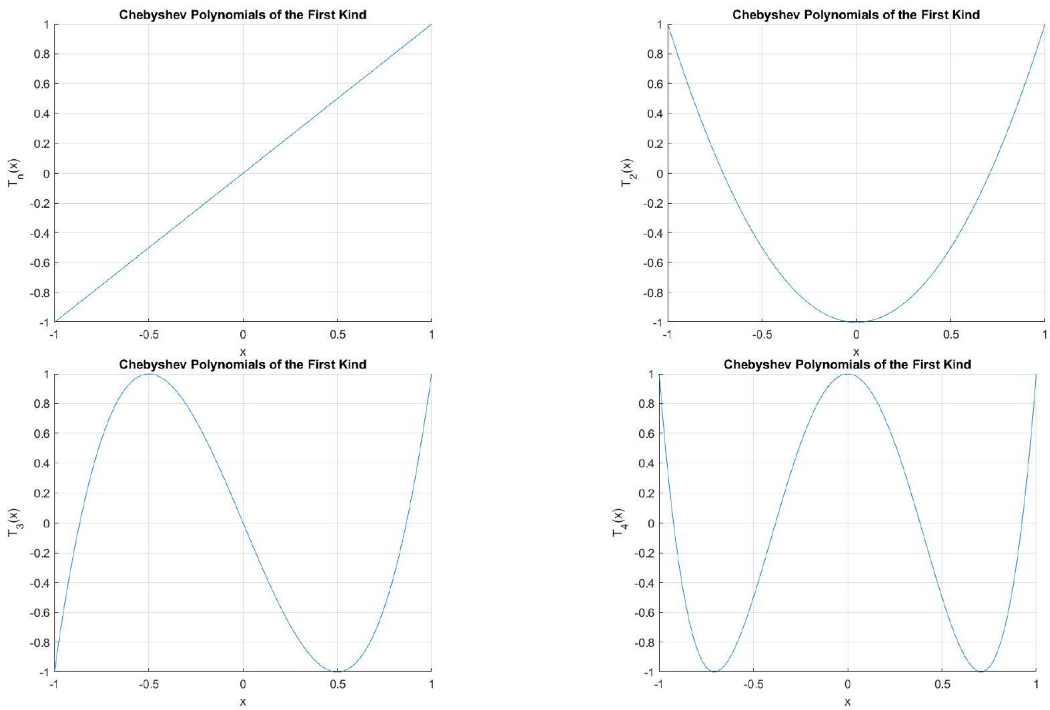

Figure 1.2: Chebyshev polynomials for $1 \leq n \leq 4$.

Definition 1.9. *The first kind of Chebyshev polynomials of order n is given by*

$$U_n(z) = \cos\left(n \cos^{-1}(z)\right), \quad \forall z \in [-1, 1].$$

Theorem 1.6. *The Chebyshev polynomilas of first kind can be defined by the following recurrence relation*

$$\begin{cases} U_{n+1}(z) = 2zU_n(z) - U_{n-1}(z), \\ U_0(z) = 1, \quad U_1(z) = z, \quad \textit{for} \quad n \geqslant 1. \end{cases}$$

Proposition 1.4. *These polynomials are orthogonal with respect to the weight function $w(z) = 1/\sqrt{1-z^2}$ on the interval $[-1, 1]$, and*

$$\int_{-1}^{1} w(z)U_n(z)U_m(z)dz = \begin{cases} \pi, & n = m = 0, \\ \pi/2, & n = m \neq 0, \\ 0, & n \neq m. \end{cases}$$

Proof. We have

$$\cos(n\theta + m\theta) + \cos(n\theta - m\theta) = 2\cos(n\theta)\cos(m\theta).$$

We put $z = \cos\theta$. Then

$$dz = -\sin\theta d\theta$$

$$-dz = \sin\theta d\theta = \sqrt{1-(\cos\theta)^2}d\theta \Longrightarrow d\theta = -\frac{dz}{\sqrt{1-z^2}}$$

$$\int_{-1}^{1} T_n(z)T_m(z)\frac{dz}{\sqrt{1-z^2}} = -\int_{1}^{-1} T_n(z)T_m(z)\frac{dz}{\sqrt{1-z^2}} = \int_0^\pi \cos(n\theta)\cos(m\theta)d\theta.$$

Then if $n \neq m$

$$\int_0^\pi \cos(n\theta)\cos(m\theta)d\theta = 0,$$

if $n = m \neq 0$

$$\int_0^\pi \cos(n\theta)^2 d\theta = \int_0^\pi \frac{1+\cos(2n\theta)}{2}d\theta = \frac{\pi}{2},$$

if $n = m = 0$

$$\int_0^\pi \cos(n\theta)^2 d\theta = \int_0^\pi d\theta = \pi.$$

□

Corollary 1.2. *The Chebyshev polynomial of degree n is a solution for the differential equation:*

$$\left(1-z^2\right)P''(z) - zP'(z) + n^2P(z) = 0, \quad \forall z \in [-1,1].$$

Proof. Let $n \in \mathbb{N}$. By deriving the equality $T_n(\cos\theta) = \cos(n\theta)$, we get

$$\forall\theta \in \mathbb{R}, \quad (-\sin\theta)T_n'(\cos\theta) = -n\sin(n\theta).$$

By deriving this equality a second time

$$\forall\theta \in \mathbb{R}, \quad (-\cos\theta)T_n'(\cos\theta) + \sin^2\theta T_n''(\cos\theta) = -n^2\cos(n\theta) = -n^2T_n(\cos\theta),$$
$$\forall z \in [-1,1], \quad (-z)T_n'(z) + \left(1-z^2\right)T_n''(z) = -n^2T_n(z).$$

Hence,

$$\forall n \in \mathbb{N}, \quad \forall z \in [0,1], \quad \left(1-z^2\right)T_n''(z) - zT_n'(z) + n^2T_n(z) = 0.$$

□

1.2.3 Chebyshev polynomials of the second kind

The Chebyshev polynomials of the second kind $U_n(z)$ are given by the following formula:

$$U_n(z) = \frac{(-1)^n(n+1)\sqrt{\pi}}{2^{n+1}\left(n+\frac{1}{2}\right)!\,(1-z^2)^{1/2}}\frac{d^n}{dz^n}\left[\left(1-z^2\right)^{n+1/2}\right].$$

Theorem 1.7. *The second kind of Chebyshev polynomials are defined by the following recurrence relation:*

$$\begin{cases} U_{n+1}(z) = 2zU_n(z) - U_{n-1}(z), \\ U_0(z) = 1, \quad U_1(z) = 2z, n \geq 1. \end{cases} \tag{1.10}$$

Exemple 1.7. *The first six Chebyshev polynomials of the second kind are*

$$\begin{aligned} U_0(z) &= 1, \\ U_1(z) &= 2z \\ U_2(z) &= 4z^2 - 1, \\ U_3(z) &= 8z^3 - 4z \\ U_4(z) &= 16z^4 - 12z^2 + 1 \\ U_5(z) &= 32z^5 - 32z^3 + 6z. \end{aligned}$$

Proposition 1.5. *The present polynomials are orthogonal with respect to the weight function* $w(z) = \sqrt{1-z^2}$ *on the interval* $[-1, 1]$. *Moreover, we have*

$$\int_{-1}^{1} w(t)U_i(t)U_j(t)dt = \begin{cases} \frac{\pi}{2}, & i = j, \\ 0, & i \neq j. \end{cases}$$

Corollary 1.3. *The Chebyshev polynomials of the second kind of the degree* n *is a solution for the differential equation:*

$$\left(1 - z^2\right) y'' - 3zy' + n(n+2)y = 0. \tag{1.11}$$

Proof. By setting $z = \cos(\theta)$, we have

$$U_n(\cos(\theta))\sin(\theta) = \sin((n+1)\theta),$$

and by differentiating this equality twice, we get

$$-\sin(\theta)U_n(\cos(\theta)) - 3\cos(\theta)\sin(\theta)U'(\cos(\theta)) + \sin(\theta)^3 U_n''(\cos(\theta)) = -(n+1)^2\sin((n+1)\theta).$$

Thus,

$$-3\cos(\theta)\sin(\theta)U'(\cos(\theta)) + \sin(\theta)^3 U_n''(\cos(\theta)) = -n(n+2)\sin((n+1)\theta),$$

and by dividing by $\sin(\theta)$, we find

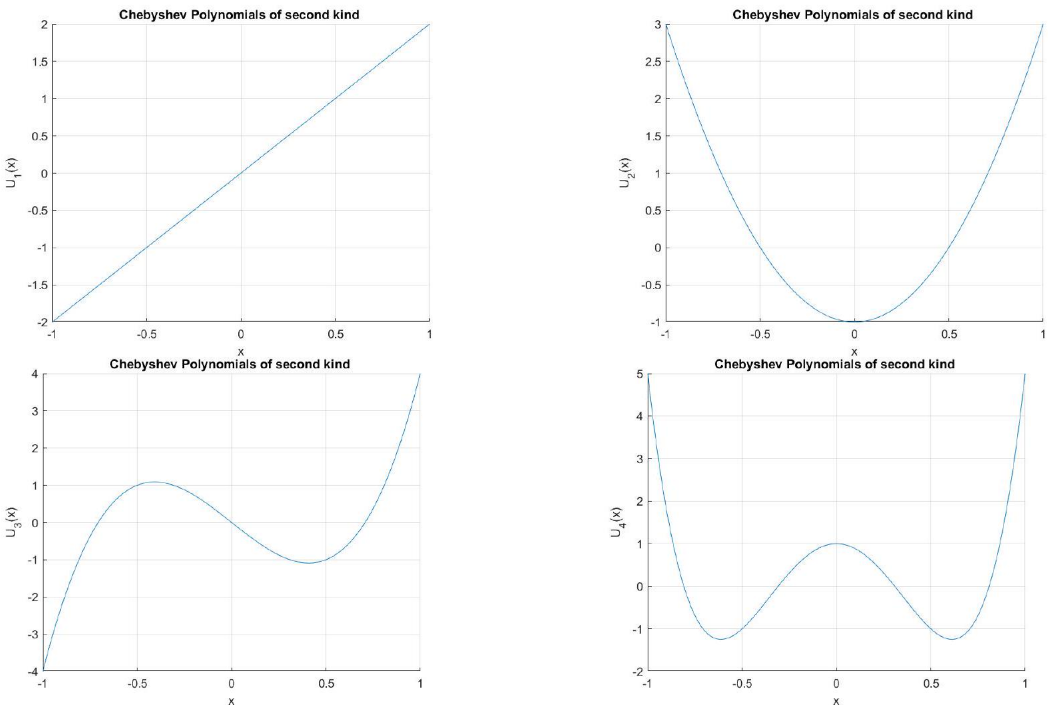

Figure 1.3: Chebyshev polynomials of the second kind for $1 \leq n \leq 4$.

$$\forall \theta \in \mathbb{R} \quad -3\cos(\theta)U'(\cos(\theta)) + \sin(\theta)^2 U_n''(\cos(\theta)) = -n(n+2)\frac{\sin((n+1)\theta)}{\sin(\theta)}.$$

Then,

$$\forall z \in [-1, 1] \quad \left(1 - z^2\right) U_n''(z) - 3zU_n'(z) + n(n+2)U_n(z) = 0.$$

□

1.2.4 Hermite polynomials

The Hermite polynomials can be defined by the Rodrigués formula

$$H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2}. \tag{1.12}$$

Theorem 1.8. *The Hermite polynomials $H_n(z)$ satisfy the following relation:*

$$2zH_{n+1}(z) = H_{n+2}(z) + 2(n+1)H_n(z). \tag{1.13}$$

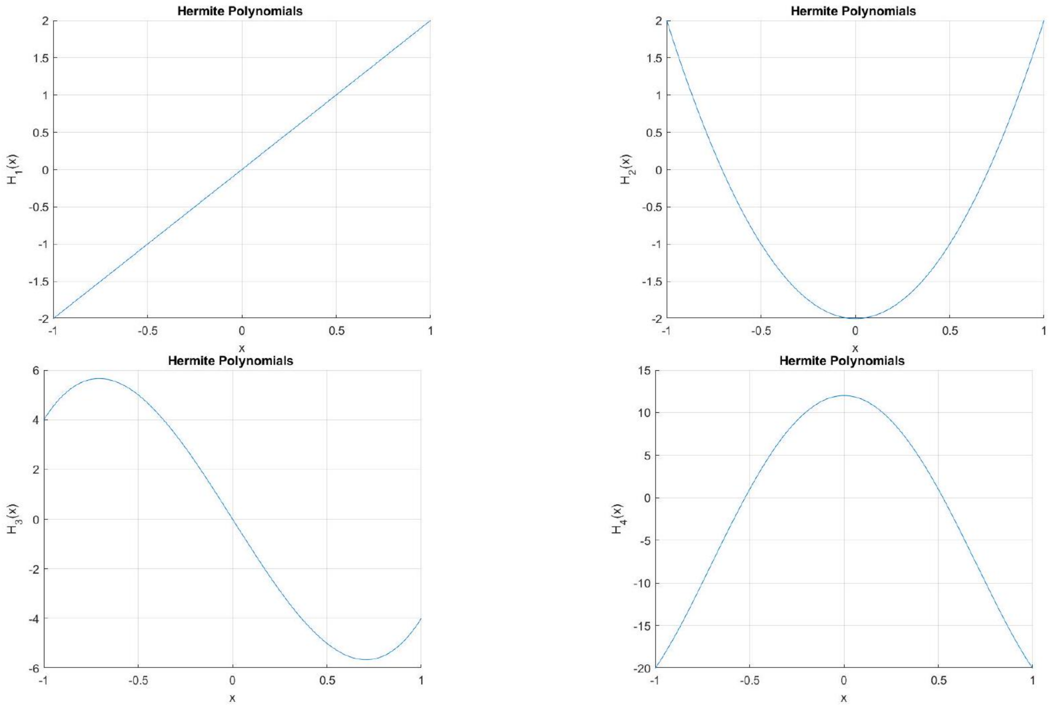

Figure 1.4: Hermite polynomials for $1 \leq n \leq 4$.

Exemple 1.8. *The first six Hermite polynomials $H_n(z)$ are*

$$
\begin{aligned}
H_0(z) &= 1, \\
H_1(z) &= 2z, \\
H_2(z) &= 4z^2 - 2, \\
H_3(z) &= 8z^3 - 12z, \\
H_4(z) &= 16z^4 - 48z^2 + 12, \\
H_5(z) &= 32z^5 - 160z^3 + 120z.
\end{aligned}
$$

Proposition 1.6. *These polynomials are orthogonal on the interval $\mathbb{R}$ with respect to the weight function $w(z) = e^{-z^2}$. Moreover we have*

$$\int_{-1}^{1} w(z) H_n(z) H_m(z) dz = \begin{cases} n! 2^n \sqrt{\pi}, & n = m, \\ 0, & n \neq m. \end{cases}$$

Lemma 1.3. *We have*

$$\int_{-\infty}^{+\infty} \exp\left(-u^2\right) du = \sqrt{\pi}.$$

Proof. The exponential generating function is given by

$$G(z,w) = \exp\left(2zw - w^2\right) = \sum_{n=0}^{\infty} \frac{H_n(z)w^n}{n!}.$$

We have

$$\int_{-\infty}^{+\infty} G(z,w)G(z,t)\exp\left(-z^2\right)dz = \sqrt{\pi}\exp(2wt),$$

and then,

$$\int_{-\infty}^{+\infty} \sum_{m\geq 0} \frac{t^m}{m!} H_m(z) \sum_{n\geq 0} \frac{w^n}{n!} H_n(z)\exp\left(-z^2\right)dz = \sqrt{\pi}\exp(2wt).$$

Thus,

$$\begin{aligned}\sum_{n\geq 0}\left(\sum_{m\geq 0}\frac{t^m}{n!m!}\int_{-\infty}^{+\infty} H_n(z)H_m(z)\exp\left(-z^2\right)dz\right)w^n &= \sqrt{\pi}\exp(2wt)dz \\ &= \sqrt{\pi}\sum_{n\geq 0}\left(\frac{2^n}{n!}t^n\right)w^n,\end{aligned}$$

and therefore,

$$\sum_{m\geq 0}\frac{t^m}{m!}\int_{-\infty}^{+\infty} H_n(z)H_m(z)\exp\left(-z^2\right)dz = \sqrt{\pi}2^n t^n.$$

Hence,

$$\int_{-\infty}^{+\infty} H_n(z)H_m(z)\exp\left(-z^2\right)dz = 0, \text{ if } n \neq m.$$

On the other hand, if $n = m$ by multiplying the equality (1.13) respectively by $H_n(z)$ and $H_{n-1}(z)$ we deduce

$$H_n^2(z) - 2zH_{n-1}(z)H_n(z) + 2(n-1)H_{n-2}(z)H_n(z) = 0, n \geq 2,$$

and

$$H_{n+1}(z)H_{n-1}(z) - 2zH_n(z)H_{n-1}(z) + 2nH_{n-1}^2(z) = 0, n \geq 2.$$

Therefore,

$$H_n^2(z) + 2(n-1)H_{n-2}(z)H_n(z) = H_{n+1}(z)H_{n-1}(z) + 2nH_{n-1}^2(z). \tag{1.14}$$

Multiplying (1.14) by $\exp\left(-z^2\right)$, we find

$$\begin{aligned}&H_n^2(z)\exp\left(-z^2\right) + 2(n-1)\exp\left(-z^2\right)H_{n-2}(z)H_n(z) \\ &\quad - \exp\left(-z^2\right)H_{n+1}(z)H_{n-1}(z) - 2n\exp\left(-z^2\right)H_{n-1}^2(z) \\ &= 0.\end{aligned}$$

The integration of the latter gives

$$\int_{-\infty}^{+\infty} \exp\left(-z^2\right) H_n^2(z)dz + 2(n-1)\int_{-\infty}^{+\infty} \exp\left(-z^2\right) H_{n-2}(z)H_n(z)dz$$
$$- \int_{-\infty}^{+\infty} \exp\left(-z^2\right) H_{n+1}(z)H_{n-1}(z)dz - 2n\int_{-\infty}^{+\infty} \exp\left(-z^2\right) H_{n-1}^2(z)dz$$
$$=0.$$

Due to orthogonality, we have

$$\int_{-\infty}^{+\infty} \exp\left(-z^2\right) H_{n-2}(z)H_n(z)dz = 0 \text{ et } \int_{-\infty}^{+\infty} \exp\left(-z^2\right) H_{n+1}(z)H_{n-1}(z)dz = 0.$$

Therefore,

$$\int_{-\infty}^{+\infty} \exp\left(-z^2\right) H_n^2(z)dz = 2n\int_{-\infty}^{+\infty} \exp\left(-z^2\right) H_{n-1}^2(z)dz.$$

By applying this formula $n-1$ times, we get

$$\int_{-\infty}^{+\infty} H_n^2(z)\exp\left(-z^2\right) = 2^{n-1}n!\int_{-\infty}^{+\infty} \exp\left(-z^2\right) H_1^2(z)dz = 2^{n-1}n!\int_{-\infty}^{+\infty} \exp\left(-z^2\right)(2z)^2dz. \tag{1.15}$$

But,

$$4\int_{-\infty}^{+\infty} \exp\left(-z^2\right)(z)^2dz = 4\int_{-\infty}^{+\infty} z\exp\left(-z^2\right)xdz = 4\left[-\frac{1}{2}z\exp\left(-z^2\right)\right]_{-\infty}^{+\infty} + 2\int_{-\infty}^{+\infty} \exp\left(-z^2\right)dz$$
$$= 2\sqrt{\pi}.$$

By substituting this result in (1.15) , we obtain

$$\int_{-\infty}^{+\infty} H_n^2(z)\exp\left(-z^2\right)dz = \sqrt{\pi}2^n n!.$$

□

Corollary 1.4. *The Hermite polynomial of degree* $nH_n(z)$ *is a solution for the differential equation:*

$$y'' - 2zy' + 2ny = 0.$$

Proof. We have

$$H_n(z) = (-1)^n e^{z^2}\frac{d^n}{dz^n}e^{-z^2},$$

and then,

$$H_n'(z) = (-1)^n 2ze^{z^2}\frac{d^n}{dz^n}e^{-z^2} + (-1)^n e^{z^2}\frac{d^{n+1}}{dz^{n+1}}e^{-z^2}.$$

Multiplying by e^{-z^2} gives

$$H_n'(z)e^{-z^2} = (-1)^n 2z\frac{d^n}{dz^n}e^{-z^2} + (-1)^n\frac{d^{n+1}}{dz^{n+1}}e^{-z^2}. \tag{1.16}$$

Differentiating (1.16) leads to

$$-2zH_n'(z)e^{-z^2} + H_n''(z)e^{-z^2} = (-1)^n 2\frac{d^n}{dz^n}e^{-z^2} + (-1)^n 2z\frac{d^{n+1}}{dz^{n+1}}e^{-z^2} + (-1)^n\frac{d^{n+2}}{dz^{n+2}}e^{-z^2},$$

and using the Rodrigués formula (1.12), we find

$$-2zH_n'(z) + H_n''(z) = 2H_n(z) - 2zH_{n+1} + H_{n+2}.$$

Therefore, by the recurrence relation (1.13),

$$H_n''(z) - 2zH_n'(z) + 2nH_n(z) = 0.$$

□

1.2.5 Laguerre polynomials

The Rodrigués representation for Laguerre polynomials is

$$L_n(z) = \frac{e^x}{n!}\frac{d^n}{dz^n}\left(z^n e^{-z}\right).$$

Theorem 1.9. *The Laguerre polynomials satisfy the recurrence relation:*

$$(n+2)L_{n+2}(z) = (2n+3-z)L_{n+1}(z) - n + 1L_n(z).$$

Exemple 1.9. *The first six Laguerre polynomials are*

$$\begin{aligned}
L_0(z) &= 1,\\
L_1(z) &= -z + 1,\\
L_2(z) &= \frac{1}{2}\left(z^2 - 4z + 2\right),\\
L_3(z) &= \frac{1}{6}\left(-z^3 + 9z^2 - 18z + 6\right),\\
L_4(z) &= \frac{1}{24}\left(z^4 - 16z^3 + 72z^2 - 96z + 24\right),\\
L_5(z) &= \frac{1}{120}\left(-z^5 + 25z^4 - 200z^3 + 600z^2 - 600z + 120\right).
\end{aligned}$$

Proposition 1.7. *The Laguerre polynomials are orthogonal with repect to the weight function $w(z) = \exp(-z)$ on the interval $[0, +\infty[$. Moreover, we have*

$$\int_0^{+\infty} w(z)L_n(z)L_m(z)dz = \begin{cases} 1, & n = m,\\ 0, & n \neq m.\end{cases}$$

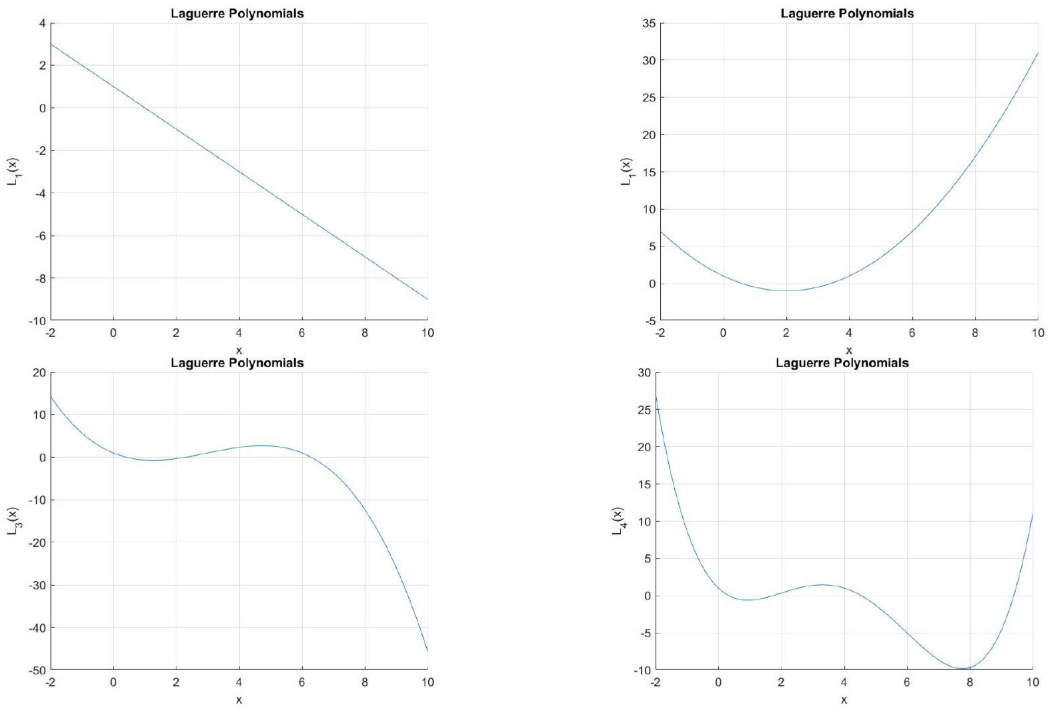

Figure 1.5: Laguerre polynomials for $1 \leq n \leq 4$.

Corollary 1.5. *The Laguerre polynomial $L_n(z)$ satisfies the following differential equation:*

$$zy'' + (1-z)y' + ny = 0.$$

Proof. The generating function is

$$g(z,t) = \exp\left[\frac{-zt}{(1-t)}\right] = \sum_{n\geq 0} L_n(z)z^n,$$

and we have

$$(1-t)^2\frac{\partial g}{\partial t}(z,t) = (1-z-t)g(z,t), \quad (t-1)\frac{\partial g}{\partial z}(z,t) = \mathrm{tg}(z,t),$$

$$\frac{\partial^2 g}{\partial z^2}(z,t) = \frac{t}{t-1}\frac{\partial g}{\partial z}(z,t).$$

Then,

$$z\frac{\partial^2 g}{\partial z^2}(z,t) + (1-z)\frac{\partial g}{\partial z}(z,t) + t\frac{\partial g}{\partial t}(z,t) = 0.$$

Therefore,

$$\sum_{n\geq 0}\left(zL_n''(z) + (1-z)L_n'(z)\right)t^n + t\sum_{n\geq 1} nL_n(z)t^{n-1} = 0.$$

Hence,

$$\sum_{n\geq 0} \left(zL_n''(z) + (1-z)L_n'(z) + nL_n(z)\right) t^n = 0,$$

$$zL_n''(z) + (1-z)L_n'(z) + nL_n(z) = 0, \qquad \text{for all } n \geq 0.$$

□

Chapter 2

Regular Nonlinear Fredholm Integro Differential Equations

2.1 Fredholm integro-differential equations of the first order

In this section, we present both analytical and numerical investigations of Fredholm integro-differential equations of this particular type:

$$\begin{cases} u(z) = f(z) + \displaystyle\int_0^1 K\left(z, y, u(y), u'(y)\right) dy, \\ u(0) = \rho, \end{cases} \tag{2.1}$$

where $K, \dfrac{\partial K}{\partial z} \in C\left([0,1]^2 \times \mathbb{R}^2\right)$, $f(z) \in H^1([0,1])$ and $u(z) \in H^1[0,1]$. This section aims to investigate the existence and uniqueness of the solution of problem (2.1) and propose a numerical process for its approximation. To achieve this goal, we analyze the necessary conditions for the existence and uniqueness of the solution. Then, we apply our numerical method that can approximate the solution with a high level of accuracy.

2.1.1 Existence and uniqueness

Consider the Sobolev space $\mathcal{H} = H^1([a,b],\mathbb{R})$ equipped with the following norm

$$\forall u \in \mathcal{H} \quad \|u\|_{\mathcal{H}} = \|u\|_{L^2[a,b]} + \|u'\|_{L^2[a,b]}\,.$$

As a starting point, the following assumptions are made:

$$(\mathcal{S}) \left\| \begin{array}{l} \exists \alpha_r, \beta_r > 0, \text{such that } r = \overline{0,1} \forall z, x \in [0,1], \text{for all } u, v, \bar{u}, \bar{v} \in \mathbb{R}, \\ |K(z,x,u,v) - K(z,x,\bar{u},\bar{v})| \leqslant \alpha_0 |u - \bar{u}| + \beta_0 |v - \bar{v}|, \\ |\partial_z K(z,x,u,v) - \partial_z K(z,x,\bar{u},\bar{v})| \leqslant \alpha_1 |u - \bar{u}| + \beta_1 |v - \bar{v}|, \\ 0 < \gamma = \max\{\alpha_0 + \alpha_1, \beta_0 + \beta_1\} < 1. \end{array} \right.$$

Let $f \in \mathcal{H}$ and define the operator:

$$\begin{aligned} \forall z \in [0,1], \quad & L : \mathcal{H} \longrightarrow \mathcal{H} \\ & u \longmapsto L(u)(z) = f(z) + \int_0^1 K\left(z, y, u(y), u'(y)\right) \mathrm{d}y, \end{aligned}$$

with $[L(u)]'(z) = f'(z) + \displaystyle\int_0^1 \partial_z K\left(z, y, u(y), u'(y)\right) \mathrm{d}y.$

Theorem 2.1. *The equation (2.1) admits a unique solution in $\mathcal{H}$ based on the assumption $\mathcal{S}$.*

Proof. Let $\varphi, \psi \in \mathcal{H}$. Then, for all $z, x \in [0,1]$

$$|K(z,x,\varphi(x),\varphi'(x)) - K(z,x,\psi(x),\psi'(x))| \leqslant \alpha_0 |\varphi(x) - \psi(x)| + \beta_0 |\varphi'(x) - \psi'(x)|\,.$$

It follows that for all $z \in [0,1]$ and by Cauchy-Schwarz inequality

$$\begin{aligned} ll|L(\varphi)(z) - L(\psi)(z)| &\leqslant \alpha_0 |\varphi(y) - \psi(y)| + \beta_0 |\varphi'(y) - \psi'(y)| \\ &\leqslant \alpha_0 \|\varphi - \psi\|_{L^2} + \beta_0 \|\varphi' - \psi'\|_{L^2}\,. \end{aligned}$$

$$|L(\varphi)(z) - L(\psi)(z)|^2 \leqslant \alpha_0^2 \|\varphi - \psi\|_{L^2}^2 + \beta_0^2 \|\varphi' - \psi'\|_{L^2}^2 + 2\alpha_0\beta_0 \|\varphi - \psi\|_{L^2} \|\varphi' - \psi'\|_{L^2}$$

$$\|L(\varphi) - L(\psi)\|_{L^2}^2 \leqslant \left(\alpha_0 \|\varphi - \psi\|_{L^2} + \beta_0 \|\varphi' - \psi'\|_{L^2}\right)^2.$$

Hence,

$$\|L(\varphi) - L(\psi)\|_{L^2} \leqslant \alpha_0 \|\varphi - \psi\|_{L^2} + \beta_0 \|\varphi' - \psi'\|_{L^2}\,.$$

Similarly, we can find

$$\|L(\varphi)' - L(\psi)'\|_{L^2} \leqslant \alpha_1 \|\varphi - \psi\|_{L^2} + \beta_1 \|\varphi' - \psi'\|_{L^2}\,.$$

and then,

$$\|L(\varphi) - L(\psi)\|_{\mathcal{H}} \leqslant \gamma \|\varphi - \psi\|_{\mathcal{H}}.$$

Since $\gamma < 1$, the equation (2.1) has a unique solution. □

2.1.2 Numerical study

We obtain the numerical solution using Legendre wavelets and the operational matrix of integration. The Galerkin method is applied to obtain a nonlinear algebraic system, which is subsequently solved using the iterative method. Additionally, we provide several illustrative examples.

The Legendre wavelets can be defined as:

$$\ell_{i,j}(z) = \begin{cases} 2^{\frac{k-1}{2}} \sqrt{2j+1} L_j\left(2^k z - 2i + 1\right), & \dfrac{i-1}{2^{k-1}} \leqslant z \leqslant \dfrac{i}{2^{k-1}}, \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

where $k \in \mathbb{N}^*, j \in \mathbb{N}, 0 \leq j \leq m-1, i = 1, 2, \ldots, 2^{k-1}$, and $L_j(z)$ is the Legendre polynomial with degree j. Furthermore, this family of Legendre wavelets $\{\ell_{i,j}\}$ also defines an orthonormal basis for $L^2([0,1])$.

Function approximation

Using the Legendre wavelets basis, each function $u(z)$ in $L^2([0,1])$ can be expressed by the following formula:

$$u(z) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} c_{i,j} \ell_{i,j}(z), \tag{2.3}$$

where $c_{i,j} = \langle u, \ell_{i,j} \rangle$, and $\langle ., . \rangle$ denotes the inner product in $L^2([0,1])$.

By truncating the infinite series (2.3), we can write:

$$u_m(z) \approx \sum_{i=1}^{2^{k-1}} \sum_{j=0}^{m-1} c_{i,j} \ell_{i,j}(z) = C^T P_m(z), \tag{2.4}$$

where C^T and $P_m(z)$ are $2^{k-1}m \times 1$ matrices given by:

$$C^T = \left[c_{1,0}, c_{1,1}, \ldots, c_{1,m-1}, c_{1,0}, c_{1,1}, \ldots, c_{1,m-1}, \ldots, c_{2^{k-1},0}, \ldots, c_{2^{k-1},m-1}\right],$$

$$P_m(z) = \left[\ell_{1,0}, \ell_{1,1}, \ldots, \ell_{1,m-1}, \ell_{1,0}, \ell_{1,1}, \ldots, \ell_{1,m-1}, \ldots, \ell_{2^{k-1},0}, \ell_{2^{k-1},1} \ldots, \ell_{2^k-1,m-1}\right]^T.$$

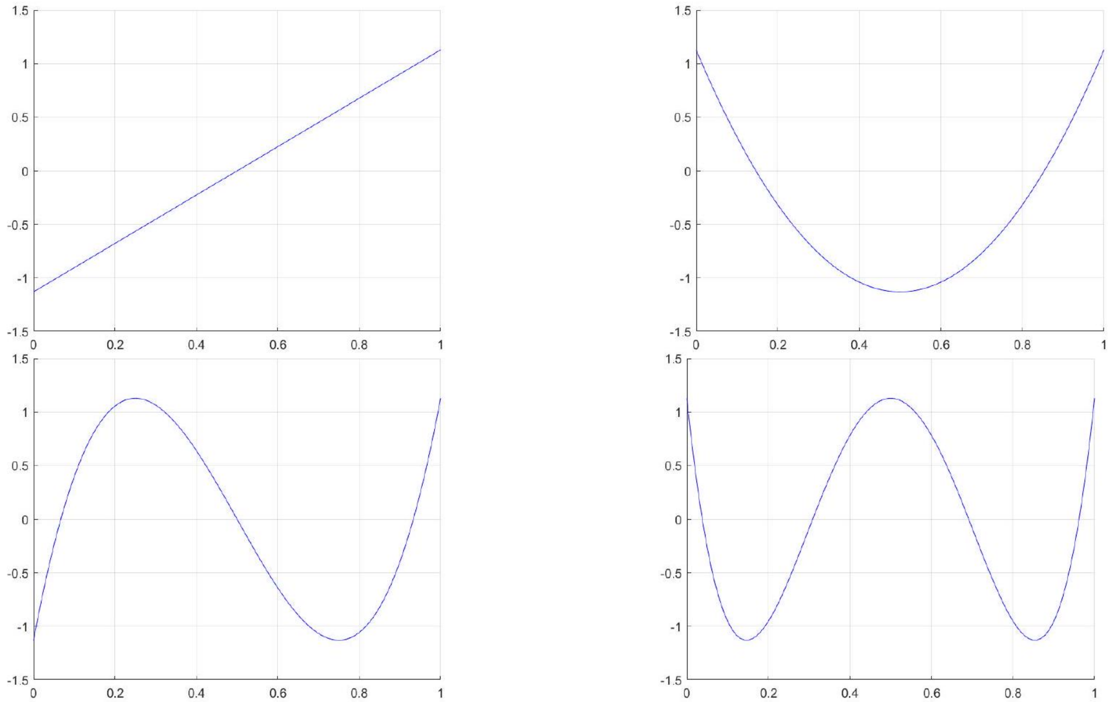

Figure 2.1: Legendre wavelets for k=1 and $1 \leq n \leq 4$

Theorem 2.2. *[40] Let $u \in L^2[0,1]$ be with a bounded second derivative, (e.g., $u''(z) \leq M$). The truncated series $u_m(z)$ given in (2.4) converges uniformly to the function $u(z)$. Moreover,*

$$|c_{i,j}| \leq \frac{24^{\frac{1}{2}} M}{2^{5k+1}(2j+1)(2j-1)(2j-3)^{\frac{1}{2}}}.$$

Operational matrix of integration

As a particular case, we take $k = 1$ in (2.4), to get:

$$C^T = [c_0, c_1, \ldots, c_{m-1}],$$

$$P_m(z) = [\ell_0(z), \ell_1(z), \ldots, \ell_{m-1}(z)].$$

The matrix M_m is defined as a matrix consisting of the coefficients of Legendre wavelets.

$$M_m = \begin{pmatrix} 1 & -\sqrt{3} & \sqrt{5} & \cdots & (-1)^{m-1}\sqrt{2m-1} \\ 0 & 2\sqrt{3} & -6\sqrt{5} & \cdots & \vdots \\ 0 & 0 & 6\sqrt{5} & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & \frac{(2m-2)!}{((m-1)!)^2}\sqrt{2m-1} \end{pmatrix}.$$

In the canonical polynomial basis $X_m(z) = \left(1, z, z^2, \ldots, z^{m-1}\right)$, the vector $P_m(z)$ of Legendre wavelets can be rewritten as follows:

$$P_m(z) = X_m(z)M_m.$$

Consider the matrix N which contains the coefficients of the integral of canonical polynomial basis:

$$N = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{m} \end{pmatrix}.$$

Then, we have the following integration matrix :

$$\int_0^z C^T P_m(s) = C^T \tilde{N} P_{m+1}(z) = C^T Q_m(z), \tag{2.5}$$

where $Q_m(z) = \tilde{N} P_{m+1}(z)$ and $\tilde{N}$ is given by:

$$\tilde{N} = M_m^{-1} N M_{m+1}.$$

Method description

Let be given the integro-differential equation:

$$u(z) = f(z) + \int_0^1 K\left(z, y, u(y), u'(y)\right) dy, \quad u(0) = \varrho.$$

First, let us differentiate the above equation with respect to the variable z to get:

$$u'(z) = f'(z) + \int_0^1 \partial_z K\left(z, y, u(y), u'(y)\right) dy. \tag{2.6}$$

Now, we approximate the unknown function $u'(z)$ of equation (2.6) by using Legendre wavelets (2.4) :

$$u'_m(t) = C^T P_m(t). \tag{2.7}$$

Next, we integrate the equation (2.7) with respect to the variable t from 0 to z :

$$u_m(z) = \varrho + C^T Q_m(z), \tag{2.8}$$

where $Q_m(z)$ is described in (2.5). By substituting relations (2.7) and (2.8) into the equation (2.6) we obtain the following non linear equation corresponding to the m unknown coefficients $(c_i, i = 0, 1, \dots, m-1)$:

$$C^T P_m(z) = f'(z) + \int_0^1 \partial_z K\left(z, y, \varrho + C^T Q_m(y), C^T P_m(y)\right) dy.$$

To obtain the coefficients $C^T = [c_0, c_1, \dots, c_{m-1}]$, we apply the Galerkin projection method corresponding to the Legendre's wavelet basis, i.e. by multiplying the above equation by $\ell_j(z), j = 0, 1, \dots, m-1$ and integrating with respect to the variable z from 0 to 1 , we get the following non linear algebraic system:

$$\begin{aligned}
c_0 &= \int_0^1 f'(z)\ell_0(z)dz + \int_0^1\int_0^1 \ell_0(z)\partial_z K\left(z, y, \varrho + C^T Q_m(y), C^T P_m(y)\right) dydz,\\
c_1 &= \int_0^1 f'(z)\ell_1(z)dz + \int_0^1\int_0^1 \ell_1(z)\partial_z K\left(z, y, \varrho + C^T Q_m(y), C^T P_m(y)\right) dydz,\\
\vdots &= \vdots\\
c_{m-1} &= \int_0^1 f'(z)\ell_{m-1}(z)dz + \int_0^1\int_0^1 \ell_{m-1}(z)\partial_z K\left(z, y, \varrho + C^T Q_m(y), C^T P_m(y)\right) dydz.
\end{aligned} \tag{2.9}$$

The exact solution C^T can be difficult or even impossible to obtain in several cases. Therefore, approximative methods are widely used, generally iterative methods. In our study, we choose the Picard successive approximations method. We start with the initial vector C_0^T and consider the sequence of vectors $\left(C_k^T\right)_{k\in\mathbb{N}}$. We then consider the following system:

$$\begin{aligned}
c_0^{k+1} &= \int_0^1 f'(z)\ell_0(z)dz + \int_0^1\int_0^1 \ell_0(z)\partial_z K\left(z, y, \varrho + C_k^T Q_m(y), C_k^T P_m(y)\right) dydz,\\
c_1^{k+1} &= \int_0^1 f'(z)\ell_1(z)dz + \int_0^1\int_0^1 \ell_1(z)\partial_z K\left(z, y, \varrho + C_k^T Q_m(y), C_k^T P_m(y)\right) dydz,\\
\vdots &= \vdots\\
c_{m-1}^{k+1} &= \int_0^1 f'(z)\ell_{m-1}(z)dz + \int_0^1\int_0^1 \ell_{m-1}(z)\partial_z K\left(z, y, \varrho + C_k^T Q_m(y), C_k^T P_m(y)\right) dydz.
\end{aligned} \tag{2.10}$$

One can solve the algebraic system (2.10) by using the Picard successive method to get the coefficients $(c_i, i = 0, 1, \ldots, m-1)$. Then we substitute them into the formula (2.8), which represents an approximate solution $u_m^k(z)$ to the main equation (2.3).

Convergence analysis

Theorem 2.3. *The numerical solution u_m converges to the true solution u in the Sobolev space $\mathcal{H}$.*

Proof. We will discuss the demonstration of this theorem in a similar way through the next section.

In this section we only demonstrate the convergence of the system (2.10).

Consider the Banach space $\mathbb{R}^m$ equipped the norm

$$\forall x \in \mathbb{R}^m \quad \|x\|_2 = \sqrt{\sum_{i=0}^{m-1} |x_i|^2}.$$

Let's suppose that:

$$\left\| \begin{array}{l} \forall z, x \in [0,1], \text{ for all } a, \bar{a}, b, \bar{b} \in \mathbb{R}, \\ \left|\partial_z K(z,x,a,b) - \partial_z K(z,x,\bar{a},\bar{b})\right| \leqslant \alpha|a-\bar{a}| + \beta|b-\bar{b}|, \\ \delta = \alpha + \beta < 1. \end{array} \right.$$

□

Theorem 2.4. *Under the assumption $(\mathcal{A})$, the nonlinear system (2.9) has a unique solution.*

Proof. Let us define the following operator:

$T : (T_1, T_2, \cdots, T_3) : \mathbb{R}^m \longrightarrow \mathbb{R}^m$

$$C^T \longmapsto T_i\left(C^T\right) = \int_0^1 \ell_i(z) f'(z) dz + \int_0^1 \int_0^1 \ell_i(z) \partial_z K\left(z, y, MC^T P(y), C^T P(y)\right).$$

Let C^T and G^T be two vectors from $\mathbb{R}^m$. We have

$$\begin{aligned} \left\|T\left(C^T\right) - T\left(G^T\right)\right\|_2 &\leq \left\|L\left(C^T P(t)\right)' - L\left(G^T P(t)\right)'\right\|_{L^2} \quad \text{Bessel's inequality} \\ &\leq \alpha \left\|MC^T P(t) - MG^T P(t)\right\|_{L^2} + \beta \left\|C^T P(t) - G^T P(t)\right\|_{L^2} \\ &\leq \alpha\|M\| \left\|C^T P(t) - G^T P(t)\right\|_{L^2} + \beta \left\|C^T P(t) - G^T P(t)\right\|_{L^2} \\ &\leq \alpha\|M\| \left\|C^T - G^T\right\|_2 + \beta \left\|C^T - G^T\right\|_2 \\ &\leq (\alpha+\beta) \left\|C^T - G^T\right\|_2 \\ &\leq \delta \left\|C^T - G^T\right\|_2, \end{aligned}$$

where the operator L is defined above. Since $\delta < 1$, by Banach's theorem 1.4, the system (2.9) has a unique solution. □

Theorem 2.5. *Under the assumption $\mathcal{A}$, for any initial vector C_0^T, the sequence $\left(C_k^T\right)_{k\in\mathbb{N}}$ converges to the vector C^T.*

Proof. We have

$$\left\|C_{k+1}^T - C^T\right\| \leq \delta \left\|C_k^T - C^T\right\|.$$

Therefore, by recurrence on k, we get

$$\left\|C_{k+1}^T - C^T\right\| \leq \delta^{k+1} \left\|C_0^T - C^T\right\|.$$

Since $\delta < 1$, $\left\|C_{k+1}^T - C^T\right\| \to 0$ when $k \to +\infty$. □

Corollary 2.1. $\left\|u_m^k - u\right\|_{\mathcal{H}} \leq \left\|u_m^k - u_m\right\|_{\mathcal{H}} + \left\|u_m - u\right\|_{\mathcal{H}} \to 0$ *when* $k, m \to +\infty$.

2.1.3 Numerical examples

Here, several numerical examples are given to demonstrate the efficiency of our proposed method. We mention that the numerical results are computed by using the following error function:

$$E_m = \|u_m(z) - u(z)\|_{H^1([0,1])} = \sqrt{\|u_m(z) - u(z)\|_{L^2([0,1])}^2 + \|u_m'(z) - u'(z)\|_{L^2([0,1])}^2},$$

where $u(z)$ is the exact solution and $u_m(z)$ the approximate solution given by our proposed method. The results in the tables below are obtained when $\left\|C_{k+1}^T - C_k^T\right\| \leq \varepsilon$, for different values of ε.

First example

Let us consider the Fredholm integro-differential equation:

$$\begin{cases} \forall z \in [0,1], u(z) = f(z) + \displaystyle\int_0^1 \frac{1}{5} \sin\left[2(y + z + u(y)) + (1-y)e^y - u'(y)\right] dy, \\ u(0) = 0, \end{cases}$$

where

$$f(z) = ze^z - \frac{1}{5}\left[\sin^2(1+z) - \sin^2(z)\right].$$

The exact solution is $u(z) = ze^z$.

m	$E_m, \varepsilon = 10^{-15}$	$E_m, \varepsilon = 10^{-10}$	$E_m, \varepsilon = 10^{-7}$	$E_m, \varepsilon = 10^{-5}$
2	2.4042e − 02	2.4042e − 02	2.4042e − 02	2.4042e − 02
3	1.8363e − 03	1.8363e − 03	1.8363e − 03	1.8363e − 03
4	1.0855e − 04	1.0855e − 04	1.0855e − 04	1.0855e − 04
5	5.2218e − 06	5.2218e − 06	5.2218e − 06	5.2457e − 06
6	2.1146e − 07	2.1146e − 07	2.1193e − 07	5.4274e − 07
7	7.3891e − 09	7.3891e − 09	1.5907e − 08	4.9991e − 07
8	2.3076e − 10	2.3117e − 10	1.4089e − 08	4.9985e − 07
9	4.4605e − 11	4.6703e − 11	1.4087e − 08	4.9985e − 07

Table 2.1: Numerical results for the first example .

Second example

Consider the following integro-differential equation:

$$\begin{cases} u(z) = \frac{1}{4}[\cos(z+1) + 3\cos(z)] + \int_0^1 \frac{\sin(z+y)}{2\,(1 + u(y)^2 + u'(y)^2)} dy, \quad z \in [0,1], \\ u(0) = 0. \end{cases}$$

The exact solution is $u(z) = \sin(z)$.

Third example

Consider the following equation:

$$\begin{cases} u(z) = f(z) - \int_0^1 \frac{\sin(z)}{4} \sqrt{3 + u^2(y) + 0.25\pi^{-2}\,(u'(y))^2} dy, x \in [0,1], \\ u(0) = 0, \end{cases}$$

where

$$f(z) = \frac{1}{2}\sin(z) + |\sin(2\pi z - \pi)|,$$

and the exact solution is $u(z) = |\sin(2\pi z - \pi)|$. In this example, to obtain an approximate solution $u_m(z)$, we must take $k = 2$ in the Legendre wavelets formula (2.2), because $u'(z)$ has a discontinuous point at $z = 0.5$.

m	$E_m, \varepsilon = 10^{-15}$	$E_m, \varepsilon = 10^{-10}$	$E_m, \varepsilon = 10^{-7}$	$E_m, \varepsilon = 10^{-5}$
2	1.5073e − 03	1.5073e − 03	1.5073e − 03	1.5073e − 03
3	1.7277e − 04	1.7277e − 04	1.7277e − 04	1.7277e − 04
4	4.7584e − 06	4.7584e − 06	4.7584e − 06	4.7588e − 06
5	3.6343e − 07	3.6343e − 07	3.6344e − 07	3.6913e − 07
6	7.1237e − 09	7.1237e − 09	7.3462e − 09	6.4994e − 08
7	4.0789e − 10	4.0789e − 10	1.8400e − 09	6.4604e − 08
8	8.4572e − 12	8.5713e − 12	1.7943e − 09	6.4602e − 08
9	5.8232e − 12	5.9877e − 12	1.7943e − 09	6.4602e − 08

Table 2.2: Numerical results for the second example .

m	$E_m, \varepsilon = 10^{-10}$	$E_m, \varepsilon = 10^{-5}$	$E_m, \varepsilon = 10^{-3}$
3	1.7436e − 02	1.7436e − 02	1.7436e − 02
4	1.7436e − 02	1.7436e − 02	1.7436e − 02
5	2.6345e − 04	2.6345e − 04	2.6347e − 04
6	2.6345e − 04	2.6345e − 04	2.6347e − 04
7	2.2965e − 06	2.2973e − 06	3.9486e − 06
8	2.2965e − 06	2.2973e − 06	3.9486e − 06
9	2.8432e − 07	2.9053e − 07	3.2246e − 06

Table 2.3: Numerical results for the third example .

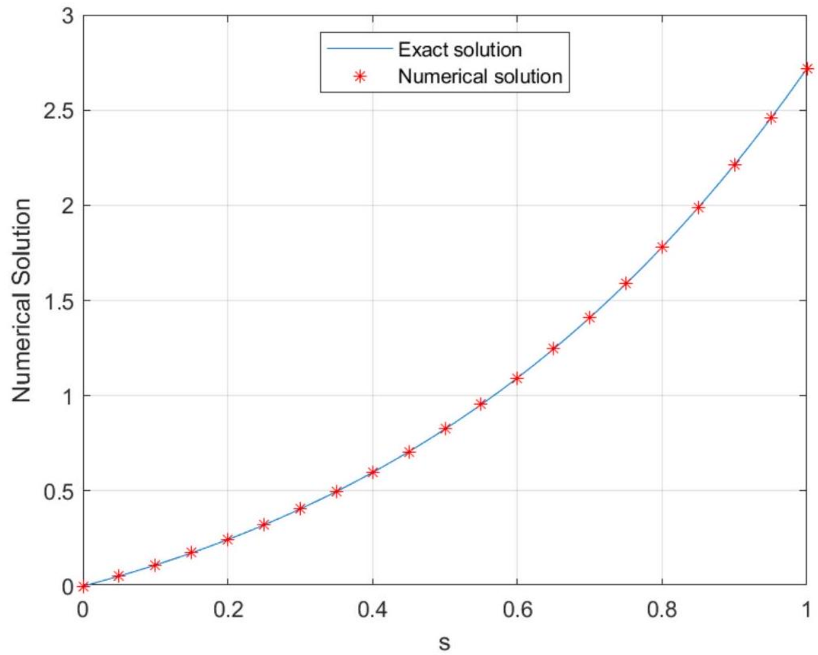

Figure 2.2: $u(y)$ vs $u_m(y)$ with $m = 9$ for the first example .

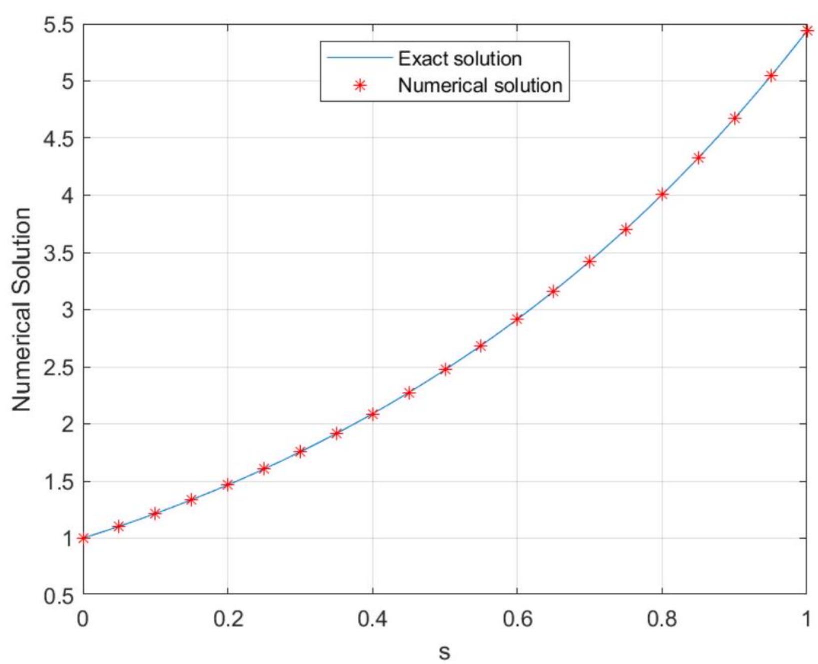

Figure 2.3: $u'(y)$ vs $u'_m(y)$ with $m = 9$ for the first example.

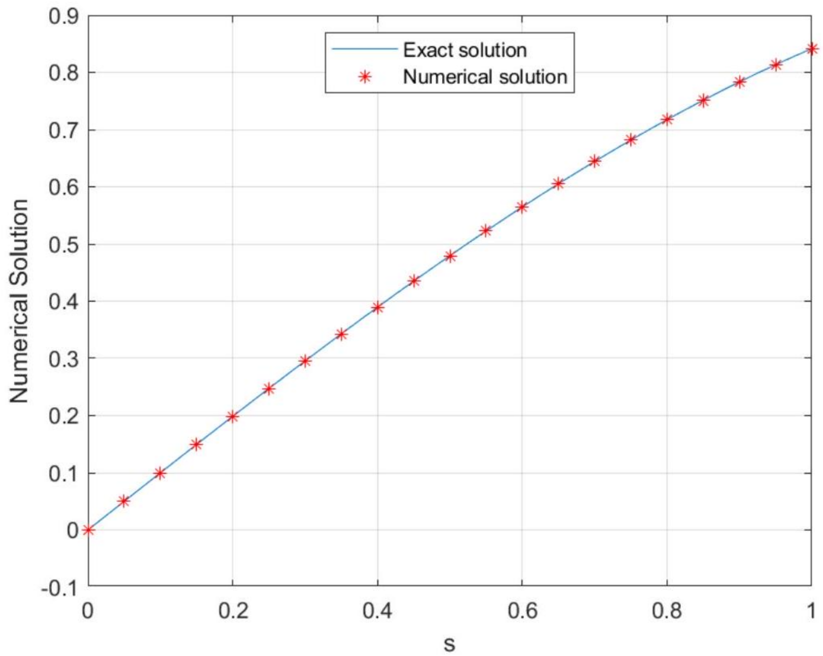

Figure 2.4: $u(y)$ vs $u_m(y)$ with $m = 9$ for the second example .

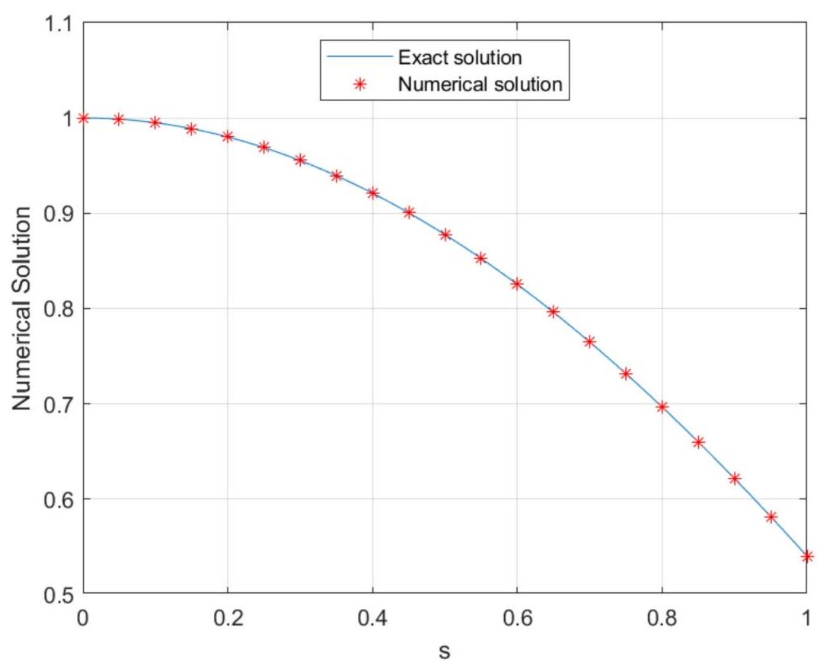

Figure 2.5: $u'(y)$ vs $u'_m(y)$ with $m = 9$ for the second example .

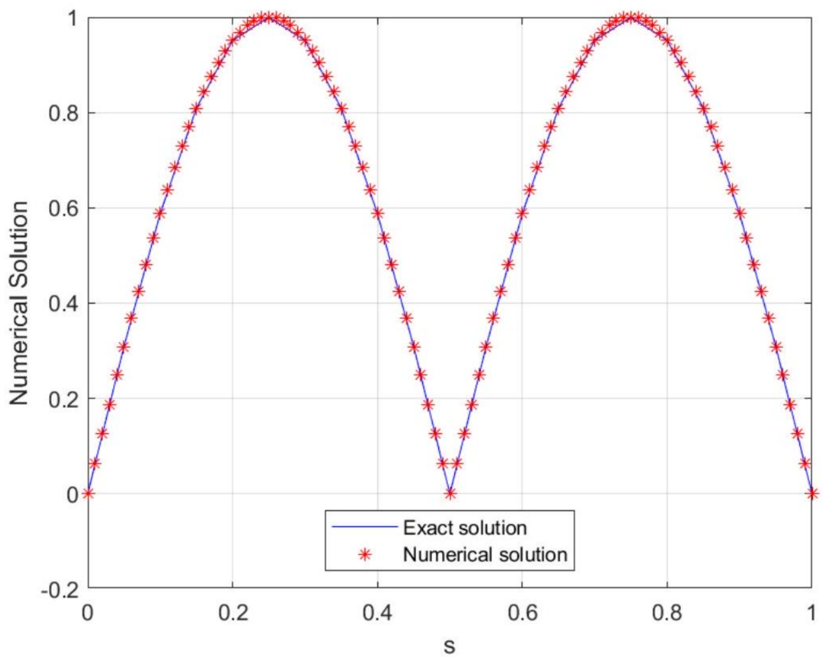

Figure 2.6: $u(y)$ vs $u_m(y)$ with $m = 7$ for the third example .

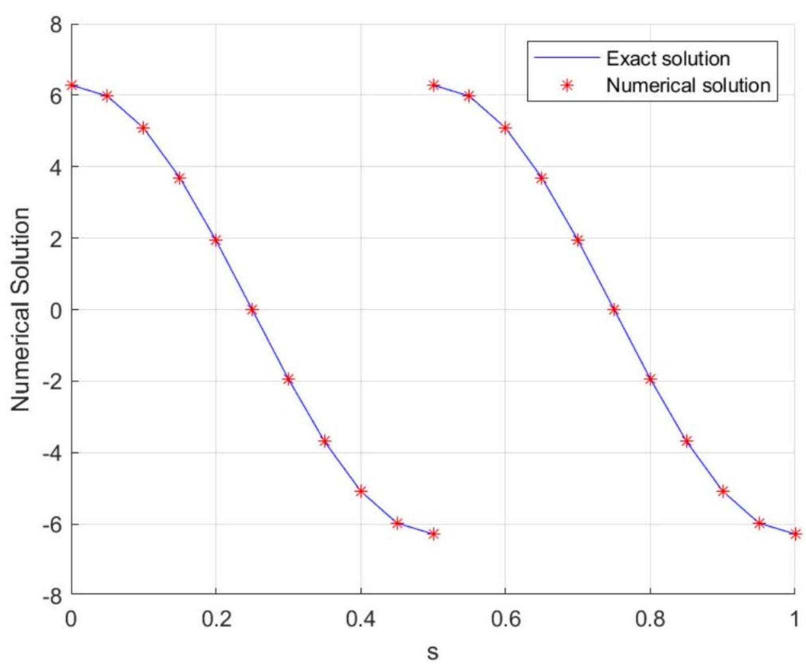

Figure 2.7: $u'(y)$ vs $u'_m(y)$ with $m = 7$ for the third example .

Interpretation of results:

Tables 2.1, 2.2 and 2.3, show the error function E_m for different values of m for the previous examples. In all cases, it seems that accuracy increases as m increased. On the other hand, Figures 4.2, $\cdots$, 2.7 show the comparison between the exact solution $u(s)$ (with its derivative $u'(s)$) and the approximate solution $u_m(s)$ (with its derivative $u'_m(s)$) for the three examples. It also appears that the exact and approximate solutions are almost identical. So, we confirm from these results the efficiency and validity of our proposed method.

2.2 Second order Fredholm integro-differential equations

Our aim through this section is to introduce a numerical study for the following type of integro-differential equation of a second order:

$$\begin{cases} \psi(z) = f(z) + \int_0^1 K\left(z, y, \psi(y), \psi'(y), \psi''(y)\right) dy, \\ \psi(0) = \alpha, \quad \psi'(0) = \beta, \end{cases} \tag{2.11}$$

where $K, \partial_z K, \partial_z^2 K \in \mathcal{C}^0\left([0,1]^2 \times \mathbb{R}^3\right), \alpha, \beta \in \mathbb{R}, \psi(z), f(z) \in H^2([0,1])$.

We utilize Chebyshev wavelets to obtain an approximate solution, and to improve accuracy, we extend our analysis to wavelets defined on subintervals. We also construct a double operational matrix of integration over different subintervals. Finally, we provide several examples to demonstrate the efficiency of our approach.

2.2.1 Chebyshev wavelets

The Chebyshev wavelets are defined as follows:

$$\theta_{i,j}(z) = \begin{cases} 2^{\frac{k}{2}} \widetilde{\Theta}_j\left(2^k z - 2i + 1\right), & \frac{i-1}{2^{k-1}} \leqslant z \leqslant \frac{i}{2^{k-1}}, \\ 0, \quad \text{otherwise} \end{cases}$$

where

$$\widetilde{\Theta}_j(z) = \begin{cases} \frac{1}{\sqrt{\pi}}, & j = 0, \\ \sqrt{\frac{2}{\pi}} \Theta_j(z), & j \neq 0, \end{cases}$$

for $k \in \mathbb{N}^*, j \in \mathbb{N}, 0 \leq j \leq n-1, i = 2^p, p = 0, \cdots, k-1$. Θ_j is the Chebyshev polynomial of degree j. The Chebyshev wavelets denoted by $\theta_{i,j}$ form an orthonormal basis in the Hilbert space $L^2_{w_k}([0,1])$ with:

$$w_k(z) = \begin{cases} w_{1,k}(z), & 0 \leq z < \frac{1}{2^{k-1}}, \\ w_{2,k}(z), & \frac{1}{2^{k-1}} \leq z < \frac{2}{2^{k-1}}, \\ \vdots & \vdots \\ w_{2^{k-1},k}(z), & \frac{2^{k-1}-1}{2^{k-1}} \leq z < 1, \end{cases}$$

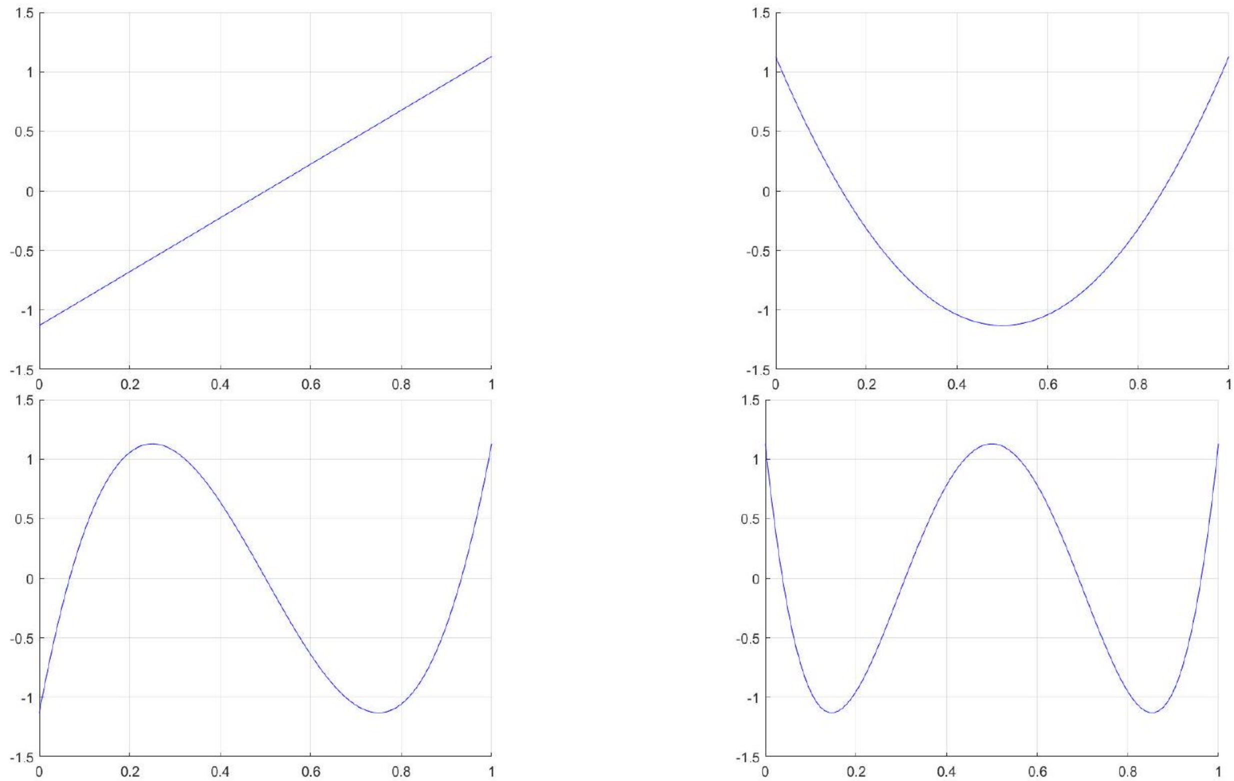

Figure 2.8: Chebyshev wavelets for $k = 1$ and $1 \leq n \leq 4$.

where $w_{i,k}(z) = w\left(2^k z - 2i + 1\right)$. The graphical representations of the Chebyshev wavelet charts are shown in Figure 2.8 for $k = 1$ and $1 \leq n \leq 4$.

Any function $\psi(z)$ in $L^2_{w_k}([0,1])$ can be written as follows:

$$\psi(z) = \sum_{i \geq 1} \sum_{j \geq 0} c_{i,j} \theta_{i,j}(z), \tag{2.12}$$

where $c_{i,j} = \langle \psi, \theta_{i,j} \rangle$, and $\langle ., . \rangle$ is the scalar product in $L^2_{w_k}([0,1])$.

By truncating the infinite series (2.12), we approximate the function $\psi(z)$ as follows:

$$\psi_n(z) = \sum_{i=1}^{2^{k-1}} \sum_{j=0}^{n-1} c_{i,j} \theta_{i,j}(z) = C^T P(z), \tag{2.13}$$

where $P(z)$ and C^T are $2^{k-1}n \times 1$ matrices:

$$C^T = \left[c_{1,0}, c_{1,1}, \ldots, c_{1,n-1}, c_{1,0}, c_{2,1}, \ldots, c_{2,n-1}, \ldots, c_{2^{k-1},0}, \ldots, c_{2^{k-1},n-1}\right],$$

and

$$P(z) = \left[\theta_{1,0}, \theta_{1,1}, \ldots, \theta_{1,n-1}, \theta_{2,0}, \theta_{2,1}, \ldots, \theta_{2,n-1}, \ldots, \theta_{2^{k-1},0}, \theta_{2^{k-1},1} \ldots, \theta_{2^{k-1},n-1}\right]^T.$$

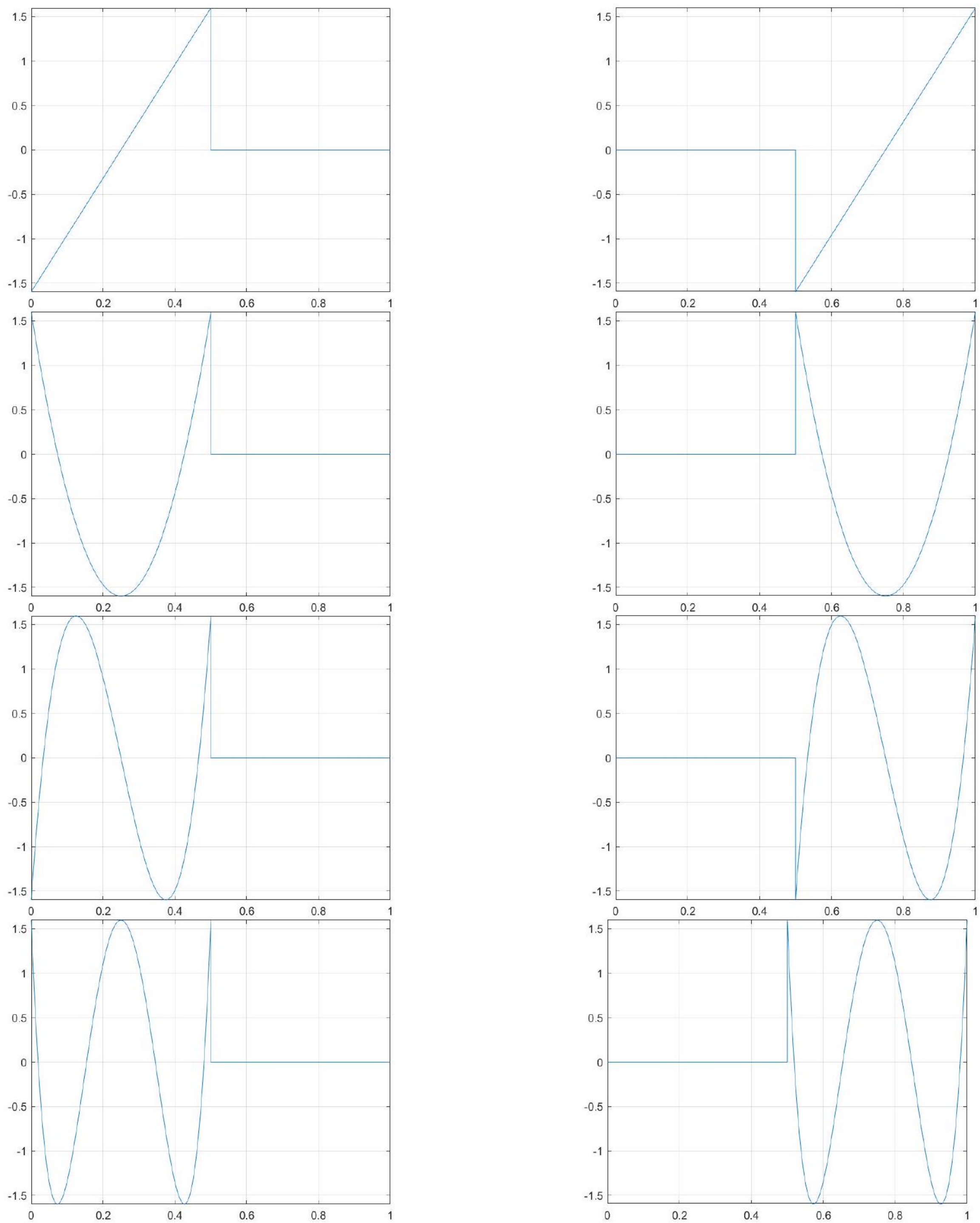

Figure 2.9: Chebyshev wavelets for $k = 2$ and $1 \leq n \leq 4$.

2.2.2 Operational integration matrix

We take $k = 2$. Then, $P(z)$ and C^T are simplified as follows:

$$C^T = [c_{1,0}, c_{1,1}, \ldots, c_{1,n-1}, c_{2,0}, c_{2,1}, \ldots, c_{2,n-1}],$$
$$P(z) = [\theta_{1,0}(z), \theta_{1,1}(z), \ldots, \theta_{1,n-1}(z), \theta_{2,0}(z), \theta_{2,1}(z), \ldots, \theta_{2,n-1}(z)].$$

Let W_n be a matrix of Chebyshev wavelet coefficients:

$$W_n = \begin{pmatrix} F_n & O_n \\ O_n & \tilde{F}_n \end{pmatrix},$$

$$\tilde{F}_n = \frac{2}{\sqrt{\pi}} \begin{pmatrix} 1 & -3\sqrt{2} & \ldots & T_{n-1}(-3)\sqrt{2} \\ 0 & 4\sqrt{2} & \ldots & \vdots \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \ldots & 4^{n-1}\sqrt{2} \end{pmatrix},$$

$$F_n = \frac{2}{\sqrt{\pi}} \begin{pmatrix} 1 & -\sqrt{2} & \ldots & (-1)^{n-1}\sqrt{2} \\ 0 & 4\sqrt{2} & \ldots & \vdots \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 4^{n-1}\sqrt{2} \end{pmatrix},$$

and

$$Z_n(z) = \left(1, z, z^2, \ldots, z^{n-1}, 1, z, z^2, \ldots, z^{n-1}\right),$$
$$P_n(z) = [\theta_{1,0}(z), \theta_{1,1}(z), \ldots, \theta_{1,n-1}(z), \theta_{2,0}(z), \theta_{2,1}(z), \ldots, \theta_{2,n-1}(z)].$$

So, we can write:

$$P_n(z) = Z_n(z) W_n.$$

Consider the matrix N_n that represents the integral matrix in the canonical basis:

$$N_n = \begin{pmatrix} G_n & O_n \\ O_n & \tilde{G}_n \end{pmatrix}$$

where

$$\tilde{G}_n = \begin{pmatrix} -\frac{1}{2} & 1 & 0 & 0 & \cdots & 0 \\ -\frac{1}{2^2} & 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2^n} & 0 & 0 & 0 & \cdots & \frac{1}{n} \end{pmatrix},$$

and

$$G_n = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{n} \end{pmatrix}.$$

Consequently, we find the initial operational matrix of integration:

$$\int_0^z C^T P_n(\xi) d\xi = C^T M_n P_{n+1}(z) = C^T Q_1(z).$$

By the same way we get the double operational matrix of integration:

$$\int_0^z \int_0^x C^T P_n(\xi) d\xi dx = C^T M_n M_{n+1} P_{n+2}(z) = C^T Q_2(z),$$

where

$$M_n = W_n^{-1} N_n W_{n+1}.$$

2.2.3 Method description

Consider the following problem:

$$\begin{cases} \psi(z) = f(z) + \int_0^1 K\left(z, \xi, \psi(\xi), \psi'(\xi), \psi''(\xi)\right) d\xi, \\ \psi(0) = \alpha, \quad \psi'(0) = \beta. \end{cases} \tag{2.14}$$

First, we differentiate the equation in (2.14) twice to get the following equation:

$$\psi''(z) = f''(z) + \int_0^1 \partial_z^2 K\left(z, \xi, \psi(\xi), \psi'(\xi), \psi''(\xi)\right) d\xi. \tag{2.15}$$

Next, we approximate the unknown function $\psi''(z)$ using Chebyshev wavelets:

$$\psi_n''(z) = C^T P_n(z). \tag{2.16}$$

By integrating the equation(2.16) from 0 to z, we find:

$$\psi_n'(z) = \beta + C^T Q_1(z). \tag{2.17}$$

Integrating again (2.17) from 0 to z produces

$$\psi_n(z) = \alpha + \beta z + C^T Q_2(z). \tag{2.18}$$

Then, by substituting (2.16), (2.17) and (2.18) into (2.15), we obtain:

$$C^T P_n(z) = f''(z) + \int_0^1 \partial_z^2 K\left(z, \xi, \alpha + \beta\xi + C^T Q_2(\xi), \beta + C^T Q_1(\xi), C^T P_n(\xi)\right) d\xi. \tag{2.19}$$

Let us now multiply the equation (2.19) by $\theta_{i,j}(z)w_{i,2}(z)$ for $j = 0, 1, \cdots, n-1$ and $i = 1, 2$, and then we integrate with respect to the variable z from 0 to 1. This gives the following nonlinear algebraic system:

$$\begin{aligned}
c_{i,0} &= y_{i,0} + \int_0^1 \int_0^1 \theta_{i,0}(z)w_{i,2}(z)\partial_z^2 K\left(z, \xi, \alpha + \beta\xi + C^T Q_2(\xi), \beta + C^T Q_1(\xi), C^T P_n(\xi)\right) d\xi dz \\
c_{i,1} &= y_{i,1} + \int_0^1 \int_0^1 \theta_{i,1}(z)w_{i,2}(z)\partial_z^2 K\left(z, \xi, \alpha + \beta\xi + C^T Q_2(\xi), \beta + C^T Q_1(\xi), C^T P_n(\xi)\right) d\xi dz \\
&\vdots \\
c_{i,n-1} &= y_{i,n-1} + \int_0^1 \int_0^1 \theta_{i,n-1}(z)w_{i,2}(z)\partial_z^2 K\left(z, \xi, \alpha + \beta\xi + C^T Q_2(\xi), \beta + C^T Q_1(\xi), C^T P_n(\xi)\right) d\xi dz
\end{aligned} \tag{2.20}$$

where $y_{i,j} = \langle f'', \theta_{i,j} \rangle$. By applying the Picard successive approximations method, we obtain the vector solution C^T for the system (2.20) outlined above. We can subsequently substitute this solution into (2.18) to get the numerical solution for the main equation (2.11).

2.2.4 Convergence analysis

In order to establish the convergence analysis for the numerical process outlined previously, we begin by considering the Sobolev space $\mathcal{H} = H^2([0,1], \mathbb{R})$ equipped with the norm

$$\forall \psi \in \mathcal{H}, \quad \|\psi\|_{\mathcal{H}} = \|\psi\|_{L^2[0,1]} + \|\psi'\|_{L^2[0,1]} + \|\psi''\|_{L^2[0,1]}.$$

Furthermore, let us consider the following additional assumptions:

$$(\mathcal{S}) \begin{cases} \exists A_r, B_r, C_r > 0, \text{ where } r = 0, 1, 2. \forall z, t \in [0,1], \forall x, \bar{x}, y, \bar{y}, s, \bar{s} \in \mathbb{R}, \\ |K(z,t,x,y,s) - K(z,t,\bar{x},\bar{y},\bar{s})| \leqslant A_0|x - \bar{x}| + B_0|y - \bar{y}| + C_0|s - \bar{s}|, \\ |\partial_z K(z,t,x,y,s) - \partial_z K(z,t,\bar{x},\bar{y},\bar{s})| \leqslant A_1|u - \bar{u}| + B_1|y - \bar{y}| + C_1|s - \bar{s}|, \\ \left|\partial_z^2 K(z,t,u,y,s) - \partial_z^2 K(z,t,\bar{u},\bar{y},\bar{s})\right| \leqslant A_2|u - \bar{u}| + B_2|y - \bar{y}| + C_2|s - \bar{s}|, \\ 0 < \gamma = \max\left\{\sum_{r=0}^{2} A_r, \sum_{r=0}^{2} B_r, \sum_{r=0}^{2} C_r\right\} < 1. \end{cases}$$

Theorem 2.6. *Under assumptions $(\mathcal{S})$, the numerical solution ψ_n converges to the exact solution ψ in the Hilbert space $\mathcal{H}$.*

Proof. Let T be an operator which is defined as a mapping from $\mathcal{H}$ to itself as follows:

$$\begin{aligned} T : \mathcal{H} &\longrightarrow \mathcal{H} \\ \psi &\longmapsto T(\psi)(z) = \int_0^1 K\left(z, \xi, \psi(\xi), \psi'(\xi), \psi''(\xi)\right) d\xi + f(z). \end{aligned}$$

Consequently, we can express the exact solution ψ to equation (2.11), along with its first and second derivatives ψ' and ψ'', through the following system of equations:

$$\begin{cases} \psi(z) = T(\psi)(z), \\ \psi'(z) = T'(\psi)(z), \\ \psi''(z) = T''(\psi)(z). \end{cases}$$

Applying the Galerkin projection method by using Chebyshev wavelets, defined in (2.13), we can approximate the preceding system as follows:

$$\begin{cases} \psi_n(z) = T_n\left(\psi_n\right)(z), \\ \psi_n'(z) = T_n'\left(\psi_n\right)(z), \\ \psi_n''(z) = T_n''\left(\psi_n\right)(z). \end{cases}$$

It is clear that

$$\begin{aligned} \left|\psi_n(z) - \psi(z)\right| = \left|T_n\left(\psi_n\right) - T(\psi)\right| &= \left|T_n\left(\psi_n\right) - T\left(\psi_n\right) + T\left(\psi_n\right) - T(\psi)\right| \\ &\leqslant \left|T_n\left(\psi_n\right) - T\left(\psi_n\right)\right| + \left|T\left(\psi_n\right) - T(\psi)\right|. \end{aligned}$$

Based on the Cauchy-Schwarz inequality and hypotheses $(\mathcal{S})$, we deduce:

$$\begin{aligned} \left|T\left(\psi_n\right) - T(\psi)\right| &= \left|\int_0^1 \left(K\left(z, \xi, \psi_n(\xi), \psi_n'(\xi), \psi_n''(\xi)\right) - K\left(z, \xi, \psi(\xi), \psi'(\xi), \psi''(\xi)\right)\right) d\xi\right| \\ &\leqslant A_0 \int_0^1 \left|\psi_n(\xi) - \psi(\xi)\right| d\xi + B_0 \int_0^1 \left|\psi_n'(\xi) - \psi'(\xi)\right| d\xi + C_0 \int_0^1 \left|\psi_n''(\xi) - \psi''(\xi)\right| d\xi \\ &\leqslant A_0 \left\|\psi_n - \psi\right\|_{L^2[0,1]} + B_0 \left\|\psi_n' - \psi'\right\|_{L^2[0,1]} + C_0 \left\|\psi_n'' - \psi''\right\|_{L^2[0,1]}. \end{aligned} \tag{2.21}$$

Conversely, the research conducted in [2] assumes the convergence of the sequence $S_n = T_n\left(\psi_n\right)$ and establishes the following error convergence rate:

$$\left|T_n\left(\psi_n\right) - T\left(\psi_n\right)\right| \leqslant \mathcal{O}\left(n^{\mu_0}\right) \to 0. \tag{2.22}$$

So, from inequalities (2.21) and (2.22), we obtain:

$$\left\|\psi_n - \psi\right\|_{L^2[0,1]} \leqslant A_0 \left\|\psi_n - \psi\right\|_{L^2[0,1]} + B_0 \left\|\psi_n' - \psi'\right\|_{L^2[0,1]} + C_0 \left\|\psi_n'' - \psi''\right\|_{L^2[0,1]} + \mathcal{O}\left(n^{\mu_0}\right).$$

Similarly, we find:

$$\|\psi_n' - \psi'\|_{L^2[0,1]} \leqslant A_1 \|\psi_n - \psi\|_{L^2[0,1]} + B_1 \|\psi_n' - \psi'\|_{L^2[0,1]} + C_1 \|\psi_n'' - \psi''\|_{L^2[0,1]} + \mathcal{O}\left(n^{\mu_1}\right),$$
$$\|\psi_n'' - \psi''\|_{L^2[0,1]} \leqslant A_2 \|\psi_n - \psi\|_{L^2[0,1]} + B_2 \|\psi_n' - \psi'\|_{L^2[0,1]} + C_2 \|\psi_n'' - \psi''\|_{L^2[0,1]} + \mathcal{O}\left(n^{\mu_2}\right).$$

Therefore,

$$\mathcal{O}\left(n^{\mu}\right) = \mathcal{O}\left(n^{\mu_0}\right) + \mathcal{O}\left(n^{\mu_1}\right) + \mathcal{O}\left(n^{\mu_2}\right) \to 0.$$

Furthermore, when $0 < \gamma < 1$, we can deduce that:

$$\|\psi_n - \psi\|_{\mathcal{H}} \leqslant \frac{\mathcal{O}\left(n^{\mu}\right)}{1-\gamma} \to 0.$$

This indeed confirms the convergence of the approximate solution ψ_n to ψ in the Hilbert space $\mathcal{H}$. □

2.2.5 Numerical examples

First example

Let be given the integro-differential problem:

$$\begin{cases} \psi(z) = f(z) + \displaystyle\int_0^1 K\left(z, \xi, \psi(\xi), \psi'(\xi), \psi''(\xi)\right) d\xi, \\ \psi(0) = \psi'(0) = 0, \end{cases}$$

with $\quad f(z) = z\sin(z) + \dfrac{1}{2}\sin(z)(2\ln(2) - 1)$, and

$$K(u, v, x, y, z) = -\frac{1}{2}\sin(u)\ln\left[1 + \cos(v)v + \sin(v)x - \frac{1}{2}\sin(v)(z - \sin(v))\right]$$

and the exact solution is $\psi(z) = z\sin(z)$.

Second example

Let be given the Fredholm integro-differential problem:

$$\begin{cases} \psi(z) = f(z) - \displaystyle\int_0^1 \frac{3}{4}\cos(z)\sin\left[\frac{1}{2}\exp^{-y}\left(2\psi''(y) - \psi'(y) - \psi(y)\right)\right] dy, \quad \forall z \in [0,1], \\ \psi(0) = \dfrac{1}{4}, \quad \psi'(0) = -\dfrac{3}{4}, \end{cases}$$

with

$$f(z) = \begin{cases} \dfrac{1}{4}\left[(2z-1)^2\exp(z) + \cos(z)\left[\cos\left(\dfrac{1}{2}\right) - 2\cos(2) + \cos\left(\dfrac{7}{2}\right)\right]\right], \quad z \in \left[0, \dfrac{1}{2}\right], \\ \dfrac{1}{4}\left[-(2z-1)^2\exp(z) + \cos(z)\left[\cos\left(\dfrac{1}{2}\right) - 2\cos(2) + \cos\left(\dfrac{7}{2}\right)\right]\right], \quad z \in \left[\dfrac{1}{2}, 1\right]. \end{cases}$$

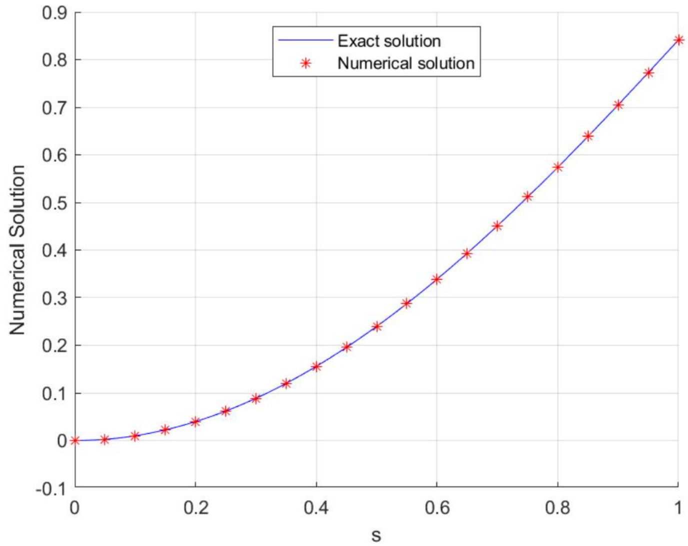

Figure 2.10: ψ vs ψ_n for the first example with $n = 7$.

n	3	4	5	6	7
E_n	$6.988\text{E}-05$	$1.433\text{E}-06$	$4.934\text{E}-08$	$6.723\text{E}-10$	$5.444\text{E}-10$
CPU time	0.109	0.123	0.154	0.194	0.310

Table 2.4: Numerical results of the first example.

and the exact solution is:

$$\psi(z) = \begin{cases} \frac{1}{4}(2z-1)^2 \exp(z), & 0 \leqslant z \leqslant \frac{1}{2}, \\ -\frac{1}{4}(2z-1)^2 \exp(z), & \frac{1}{2} \leqslant z \leqslant 1. \end{cases}$$

n	3	4	5	6	7
E_n	$7.949\text{E}-04$	$2.759\text{E}-05$	$7.617\text{E}-07$	$1.741\text{E}-08$	$8.503\text{E}-09$
CPU time	0.081	0.108	0.129	0.169	0.372

Table 2.5: Numerical results of the second example.

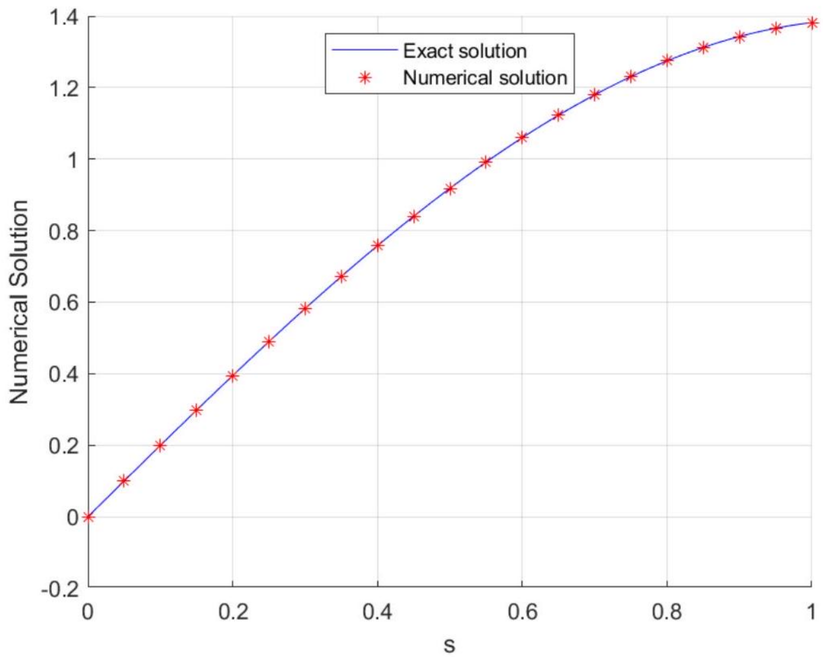

Figure 2.11: ψ' vs ψ_n' for the first example with $n = 7$.

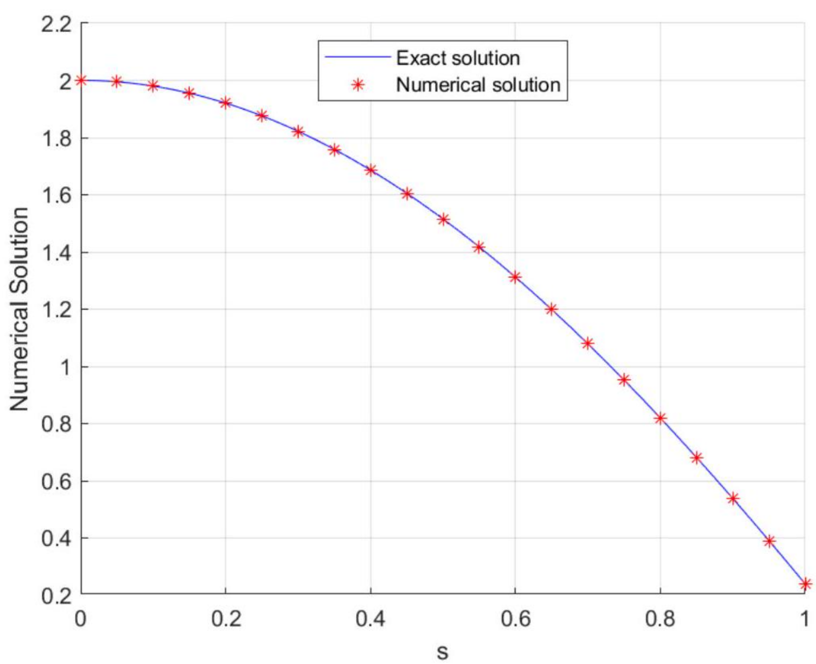

Figure 2.12: ψ'' vs ψ_n'' for the first example with $n = 7$.

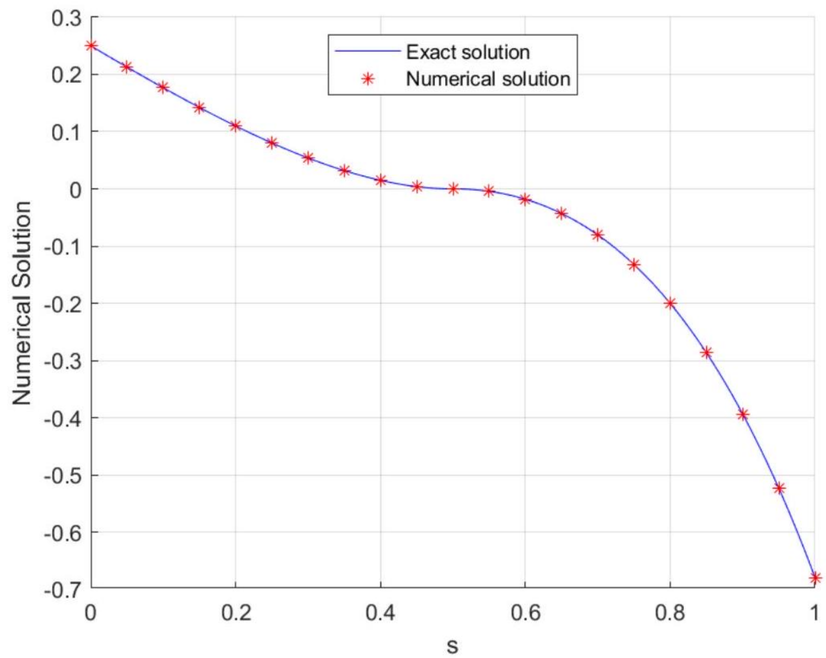

Figure 2.13: ψ vs ψ_n for the second example, with $n = 7$.

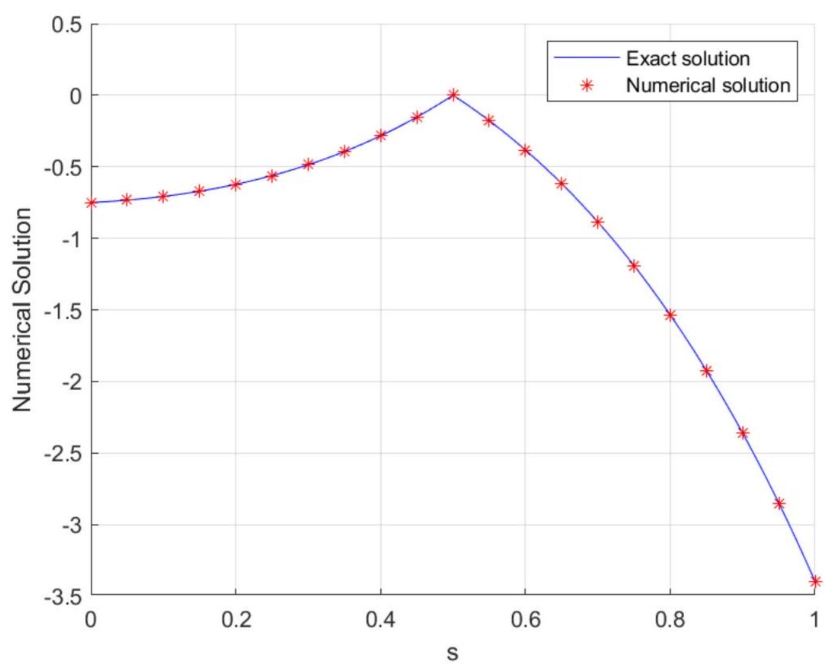

Figure 2.14: ψ' vs ψ'_n for the second example with $n = 7$.

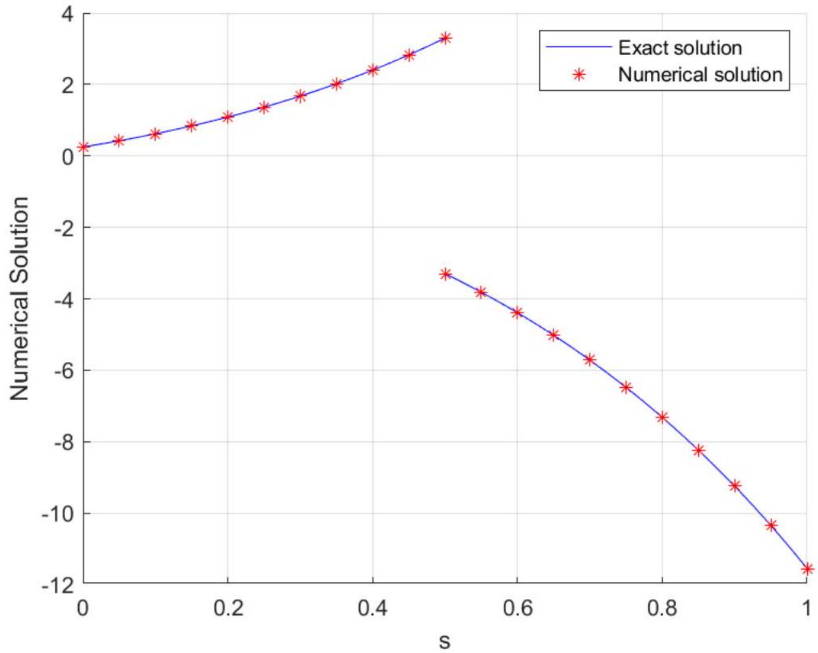

Figure 2.15: ψ'' vs ψ_n'' for the second example with $n = 7$.

Discussion

Based on both the tables and figures, it is evident that the error function diminishes significantly, particularly when the number n is large. This indicates that our suggested approach exhibits greater efficacy when dealing with higher degrees of the polynomial (approximate solution) n.

Chapter 3

Weakly Singular Nonlinear Fredholm Integro-Differential Equations

In this chapter, our primary objective is to delve into a numerical approximation method tailored for solving a class of nonlinear Fredholm integro-differential equations characterized by kernels that possess weak singularities:

$$\begin{cases} v(z) = f(z) + \int_a^b p(|z-t|)F\left(z,t,v(t),v'(t)\right)dt, z \in [a,b], \\ v(a) = v_0, \end{cases} \tag{3.1}$$

where

$$\left\| \begin{array}{l} f \in \mathcal{C}^1([a,b],\mathbb{R}), \quad F, \partial_z F \in C^0\left([a,b]^2 \times \mathbb{R}^2\right), \\ p(t) \in W^{1,1}(0,b-a), \\ \lim\limits_{t\to 0} p'(t) = +\infty. \end{array} \right.$$

3.1 Galerkin method

The Galerkin method and the operational matrix of integration are used to compute the approximate solution. Based on the features of the second kind of Chebyshev polynomials, we present a technique for removing the kernel singularity during the computing process, and then we find the desired numerical solution.

3.1.1 Function approximation

Consider the Hilbert space $\mathcal{H} = L^2_w[-1,1]$ that is equipped with the inner product $\langle u, v\rangle = \int_{-1}^{1} w(\xi)u(\xi)v(\xi)d\xi$ with $w(\xi) = \sqrt{1-\xi^2}$, and the orthonormal basis

$$\mathcal{B} = \left\{ \varphi_n = \sqrt{\frac{2}{\pi}}\phi_n, \quad n = 0, 1, \cdots \right\},$$

where ϕ_n is the Chebyshev polynomial of the second kind of degree n described in the first chapter. Define the following orthogonal projection:

$$\begin{aligned} \forall \xi \in [-1,1], \quad & \mathcal{P}_n : \mathcal{H} \longrightarrow \mathbb{P}_n \\ & v \longmapsto \mathcal{P}_n(v)(\xi) = \sum_{i=0}^{n} \langle v, \varphi_i\rangle \varphi_i(\xi) = C^T P_n(\xi), \end{aligned} \tag{3.2}$$

where

$$C^T = [c_0, c_1, \cdots, c_n], \quad c_i = \langle v, \varphi_i\rangle, i = 0, \cdots, n \tag{3.3}$$

and

$$P_n(\xi) = [\varphi_0(\xi), \varphi_1(\xi), \cdots, \varphi_n(\xi)].$$

$\mathbb{P}_n$ is the space of polynomials of degree less than or equals n. We approximate any function $v(\xi)$ of $\mathcal{H}$ as:

$$v(\xi) \approx (\mathcal{P}_n v)(\xi) = C^T P_n(\xi). \tag{3.4}$$

Theorem 3.1. *Let $f(\xi) \in L^2_\omega[-1,1]$, satisfying $|f''(\xi)| \leqslant L$. Then, the series (3.2) converges uniformly to $f(\xi)$. Furthermore, the coefficients in (3.3) satisfy the following inequality:*

$$|c_n| < \frac{4\sqrt{2\pi}L}{(n+1)^2}, \quad \forall n \geqslant 2.$$

Proof. We have

$$c_n = \int_{-1}^{1} f(\xi)\varphi_n(x)\omega(\xi)dx. \tag{3.5}$$

If we set $\xi = \cos\theta$ in (3.5), then we get

$$\begin{aligned} c_n &= \sqrt{\frac{2}{\pi}} \int_0^{\pi} f(\cos(\theta)) \sin(n+1)\theta \sin\theta d\theta \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{\pi} f(\cos(\theta))[\cos n\theta - \cos(n+2)\theta] d\theta, \end{aligned}$$

that, after integrating twice by parts, gives

$$c_n = \frac{1}{\sqrt{2\pi}} \int_0^{\pi} f''(\cos\theta) S_n(\theta) d\theta,$$

where

$$S_n(\theta) = \frac{\sin\theta}{n} \left(\frac{\sin(n-1)\theta}{n-1} - \frac{\sin(n+1)\theta}{n+1} \right) - \frac{\sin\theta}{n+2} \left(\frac{\sin(n+1)\theta}{n+1} - \frac{\sin(n+3)\theta}{n+3} \right).$$

Therefore,

$$\begin{aligned} |c_n| &= \frac{1}{\sqrt{2\pi}} \left| \int_0^{\pi} f''(\cos(\theta)) S_n(\theta) d\theta \right| \\ &= \frac{1}{\sqrt{2\pi}} \left| \int_0^{\pi} f''(\cos(\theta)) S_n(\theta) d\theta \right| \\ &\leqslant \frac{L}{\sqrt{2\pi}} \int_0^{\pi} |S_n(\theta)| \, d\theta \\ &\leqslant \pi \frac{L}{\sqrt{2\pi}} \left[\frac{1}{n} \left(\frac{1}{n-1} + \frac{1}{n+1} \right) + \frac{1}{n+2} \left(\frac{1}{n+1} + \frac{1}{n+3} \right) \right] \\ &= \frac{2\sqrt{2\pi} L}{(n^2 + 2n - 3)} \\ &< \frac{4\sqrt{2\pi} L}{(n+1)^2}. \end{aligned}$$

□

3.1.2 Operational integration matrix

The operatinal matrix of integration for $n = 4$ is obtained as follows:

$$\begin{aligned} \int_{-1}^{z} \varphi_0(t) dt &= \begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 & 0 & 0 \end{bmatrix} P(z), \\ \int_{-1}^{z} \varphi_1(t) dt &= \begin{bmatrix} \frac{-3}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 \end{bmatrix} P(z), \\ \int_{-1}^{z} \varphi_2(t) dt &= \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & 0 & \frac{-1}{6} & 0 & 0 \end{bmatrix} P(z), \\ \int_{-1}^{z} \varphi_3(t) dt &= \begin{bmatrix} \frac{-1}{4} & 0 & \frac{-1}{8} & 0 & \frac{1}{8} & 0 \end{bmatrix} P(z), \\ \int_{-1}^{z} \varphi_4(t) dt &= \begin{bmatrix} \frac{1}{5} & 0 & 0 & \frac{-1}{10} & 0 & \frac{1}{10} \end{bmatrix} P(z). \end{aligned}$$

Therefore,

$$\int_{-1}^{z} C^T P(t)dt = C^T M_{5\times 6} Q(z),$$

where

$$Q(z) = \left[\varphi_0(z), \varphi_1(z), \cdots, \varphi_{n+1}(z)\right],$$

and

$$M_{5\times 6} = \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{3}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{6} & 0 & -\frac{1}{6} & 0 & 0 \\ -\frac{1}{4} & 0 & -\frac{1}{8} & 0 & \frac{1}{8} & 0 \\ \frac{1}{5} & 0 & 0 & -\frac{1}{10} & 0 & \frac{1}{10} \end{pmatrix}.$$

Similarly, we can find the operational matrix of integration M for an arbitrary value n.

3.1.3 Method description

By taking the derivative of the equation (3.1), we get the following equation:

$$\begin{aligned} v'(z) = f'(z) &+ \int_{-1}^{1} \operatorname{sign}(z-t) p'(|z-t|) F\left(z, t, v(t), v'(t)\right) dt \\ &+ \int_{-1}^{1} p(|z-t|) \partial_z F\left(z, t, v(t), v'(t)\right) dt, \end{aligned} \tag{3.6}$$

where

$$\operatorname{sign}(z-t) = \begin{cases} +1, & z > t, \\ 0, & z = t, \\ -1, & z < t. \end{cases}$$

Now, we approximate the derivative of the unknown function $v'(z)$ by using (3.4). Thus,

$$v'(z) = C^T P(z). \tag{3.7}$$

Integrating (3.7) with respect to z from 0 to z gives

$$v(z) = v_0 + C^T M Q(z). \tag{3.8}$$

Now, by substituting (3.8) and (3.7) into (3.6), we get the following equation for the unknown vector C^T:

$$
\begin{aligned}
C^T P(z) = f'(z) &+ \int_{-1}^{1} \operatorname{sign}(z-t) p'(|z-t|) F\left(z, t, v_0 + C^T M Q(t), C^T P(t)\right) dt \\
&+ \int_{-1}^{1} p(|z-t|) \partial_z F\left(z, t, v_0 + C^T M Q(t), C^T P(t)\right) dt.
\end{aligned} \tag{3.9}
$$

By multiplying the equation (3.9) by $w(z)\varphi_i(z)$ for $i = 0, \cdots, n$, and then integrating with respect to the variable z, we find

$$
\begin{aligned}
c_0 = \langle f', \varphi_0 \rangle &+ \int_{-1}^{1}\int_{-1}^{1} w(z)\varphi_0(z) \operatorname{sign}(z-t) p'(|z-t|) F\left(z, t, v_0 + C^T M Q(t), C^T P(t)\right) dt \\
&+ \int_{-1}^{1}\int_{-1}^{1} w(z)\varphi_0(z) p(|z-s|) \partial_z F\left(z, t, v_0 + C^T M Q(t), C^T P(t)\right) dt, \\
c_1 = \langle f', \varphi_1 \rangle &+ \int_{-1}^{1}\int_{-1}^{1} w(z)\varphi_1(z) \operatorname{sign}(z-t) p'(|z-t|) F\left(z, t, v_0 + C^T M Q(t), C^T P(t)\right) dt \\
&+ \int_{-1}^{1}\int_{-1}^{1} w(z)\varphi_1(z) p(|z-s|) \partial_z F\left(z, t, v_0 + C^T M Q(t), C^T P(t)\right) dt, \\
&\vdots \vdots \\
c_n = \langle f', \varphi_n \rangle &+ \int_{-1}^{1}\int_{-1}^{1} w(z)\varphi_n(z) \operatorname{sign}(z-t) p'(|z-t|) F\left(z, t, v_0 + C^T M Q(t), C^T P(t)\right) dt \\
&+ \int_{-1}^{1}\int_{-1}^{1} w(z)\varphi_n(z) p(|z-s|) \partial_z F\left(z, t, v_0 + C^T M Q(t), C^T P(t)\right) dt.
\end{aligned} \tag{3.10}
$$

Analytically, the first integration for each equation in (3.10) exists because $p' \in L^1([-1, 1])$. However, the Matlab software cannot run. Hence, we use the integration by parts and the following property for the weight function $w(z)$:

$$
w(-1) = w(1) = 0.
$$

Then, the equations (3.10) are simplified as follows:

$$
\begin{aligned}
c_0 &= \langle f', \varphi_0 \rangle + \int_{-1}^{1}\int_{-1}^{1} \left(w(z)\varphi_0(z)\right)' p(|z-t|) F\left(z, t, v_0 + C^T M Q(t), C^T P(t)\right) dtdz \\
c_1 &= \langle f', \varphi_1 \rangle + \int_{-1}^{1}\int_{-1}^{1} \left(w(z)\varphi_1(z)\right)' p(|z-t|) F\left(z, t, v_0 + C^T M Q(t), C^T P(t)\right) dtdz \\
\vdots &= \vdots \\
c_n &= \langle f', \varphi_n \rangle + \int_{-1}^{1}\int_{-1}^{1} \left(w(z)\varphi_n(z)\right)' p(|z-t|) F\left(z, t, v_0 + C^T M Q(t), C^T P(t)\right) dtdz.
\end{aligned} \tag{3.11}
$$

The solution of the nonlinear equations system (3.11) can be obtained by using Picard successive approximations, which lead to the following system:

$$\begin{aligned} c_0^{k+1} &= \langle f', \varphi_0 \rangle + \int_{-1}^{1}\int_{-1}^{1} (w(z)\varphi_0(z))' p(|z-t|) F\left(z, t, v_0 + C_k^T M Q(t), C_k^T P(t)\right) dt, \\ c_1^{k+1} &= \langle f', \varphi_1 \rangle + \int_{-1}^{1}\int_{-1}^{1} (w(z)\varphi_1(z))' p(|z-t|) F\left(z, t, v_0 + C_k^T M Q(t), C_k^T P(t)\right) dt, \\ \vdots &= \vdots \\ c_n^{k+1} &= \langle f', \varphi_n \rangle + \int_{-1}^{1}\int_{-1}^{1} (w(z)\varphi_n(z))' p(|z-t|) F\left(z, t, v_0 + C_k^T M Q(t), C_k^T P(t)\right) dt, \end{aligned} \tag{3.12}$$

where $k \in \mathbb{N}, c_0^{(i)} = \langle f', \varphi_i \rangle$, for all $i = 0, \cdots, n$ and $C_k^T = \left[c_0^k, c_1^k, \cdots, c_n^k\right]$.

3.1.4 Convergence analysis

In this section, to examine the convergence analysis of our method, we need to recall the following useful theorem.

Theorem 3.2. *There exists a constant $\alpha > 0$ such that, for any function $u \in H1(I)$, the following estimate holds:*

$$\|\mathcal{P}_n u - u\|_{L^2(I)} \leq \alpha n^{-1} \|u\|_{H^1(I)}.$$

Proof. For details, see [34] . □

Suppose the following hypotheses:

$$(\mathcal{A}) \left\| \begin{array}{l} \exists A, A', B, B' \in \mathbb{R}_+, \quad \forall \zeta, \tau \in [-1,1], \forall u, u', v, v' \in \mathbb{R}, \\ |F(\zeta, \tau, u, v) - F(\zeta, \tau, u', v')| \leq A\,|u - u'| + B\,|v - v'|, \\ |\partial_\zeta F(\zeta, \tau, u, v) - \partial_\zeta F(\zeta, \tau, u', v')| \leq A'\,|u - u'| + B'\,|v - v'|, \\ 0 < \sqrt{2}\left(B\,\|p'\|_{L^1(I)} + B'\|p\|_{L^1(I)}\right) < 1. \end{array} \right.$$

Theorem 3.3. *Under the assumptions $(\mathcal{A})$, the numerical solution $v_n(\xi)$ converges to the exact solution $v(\xi)$. Furthermore,*

$$\|v_n - v\|_{H^1(I)} \leq \beta n^{-1} \|v\|_{H^1(I)},$$

where β is a constant.

Proof. Denote $\mathcal{P}_n v = v_n(\xi)$ and $\mathcal{P}_{n-1} v' = v'_n(\xi)$. Then, from (3.6) and (3.9), we can write

$$\begin{aligned}
&v'_n(\xi) - v'(\xi) = \\
&\quad \int_{-1}^{1} \operatorname{sign}(\xi-\zeta) p'(|\xi-\zeta|) F\left(\xi,\zeta,v_n(\zeta),v'_n(\zeta)\right) d\zeta + \int_{-1}^{1} p(|\xi-\zeta|) \partial_\xi F\left(\xi,\zeta,v_n(\zeta),v'_n(\zeta)\right) d\zeta \\
&\quad - \int_{-1}^{1} \operatorname{sign}(\xi-\zeta) p'(|\xi-\zeta|) F\left(\xi,\zeta,v(\zeta),v'(\zeta)\right) d\zeta - \int_{-1}^{1} p(|\xi-\zeta|) \partial_\xi F\left(\xi,\zeta,v(\zeta),v'(\zeta)\right) d\zeta \\
&\quad = \int_{-1}^{1} \operatorname{sign}(\xi-\zeta) p'(|\xi-\zeta|) \left[F\left(\xi,\zeta,v_n(\zeta),v'_n(\zeta)\right) d\zeta - F\left(\xi,\zeta,v(\zeta),v'(\zeta)\right)\right] d\zeta \\
&\quad + \int_{-1}^{1} p(|\xi-\zeta|) \left[\partial_\xi F\left(\xi,\zeta,v_n(\zeta),v'_n(\zeta)\right) d\zeta - \partial_\xi F\left(\xi,\zeta,v(\zeta),v'(\zeta)\right)\right] d\zeta.
\end{aligned}$$

Therefore, under the hypotheses $(\mathcal{A})$, we get

$$\begin{aligned}
|v'_n(\xi) - v'(\xi)| &\leq \int_{-1}^{1} |p'(|\xi-\zeta|)| \left[A\,|v_n(\zeta)-v(\zeta)| + B\,|v'_n(\zeta)-v'(\zeta)|\right] d\zeta \\
&+ \int_{-1}^{1} |p(|\xi-\zeta|)| \left[A'\,|v_n(\zeta)-v(\zeta)| + B'\,|v'_n(\zeta)-v'(\zeta)|\right] d\zeta \\
&\leq A\,\|p'\|_{L^1(I)}\,\|v_n - v\|_{L^\infty(I)} + B\,\|p'\|_{L^1(I)}\,\|v'_n - v'\|_{L^\infty(I)} \\
&+ A'\|p\|_{L^1(I)}\,\|v_n - v\|_{L^\infty(I)} + B'\|p\|_{L^1(I)}\,\|v'_n - v'\|_{L^\infty(I)}\,.
\end{aligned}$$

Since L^∞ is dense in L^2, we achieve

$$\begin{aligned}
\|v'_n - v'\|_{L^2(I)} \leq \sqrt{2}\left(A\,\|p'\|_{L^1(I)} + A'\|p\|_{L^1(I)}\right) \|v_n - v\|_{L^2(I)} \\
+ \sqrt{2}\left(B\,\|p'\|_{L^1(I)} + B'\|p\|_{L^1(I)}\right) \|v'_n - v'\|_{L^2(I)}\,.
\end{aligned}$$

Hence,

$$\|v'_n - v'\|_{L^2(I)} \leq \frac{\sqrt{2}\left(A\,\|p'\|_{L^1(I)} + A'\|p\|_{L^1(I)}\right)}{1-\sqrt{2}\left(B\,\|p'\|_{L^1(I)} - B'\|p\|_{L^1(I)}\right)}\,\|v_n - v\|_{L^2(I)}\,.$$

From the previous Theorem 3.2 , we conclude that

$$\|v_n - v\|_{H^1(I)} \leq \beta n^{-1} \|v\|_{H^1(I)},$$

where β is a constant given by:

$$\beta = \alpha\left(1 + \frac{\sqrt{2}\left(A\,\|p'\|_{L^1(I)} + A'\|p\|_{L^1(I)}\right)}{1-\sqrt{2}\left(B\,\|p'\|_{L^1(I)} - B'\|p\|_{L^1(I)}\right)}\right).$$

□

n	$E(n,2)$	$E(n,3)$	$E(n,4)$	$E(n,5)$
3	1.1723e − 01	1.1723e − 01	1.1723e − 01	1.1719e − 01
5	2.1376e − 03	2.0952e − 03	2.0917e − 03	2.0915e − 03
7	5.8607e − 04	5.0129e − 05	1.6428e − 05	1.6271e − 05
9	5.8485e − 04	4.6525e − 05	3.2965e − 06	8.5842e − 07

Table 3.1: Numerical results for Example 1.

3.1.5 Numerical examples

In order to illustrate the effectiveness of the proposed method, we consider two numerical examples for the integro-differential equation (3.1). Let us define the error function as follows:

$$E(n,k) = \max_{i=1:n} |v(z_i) - v_n(z_i)|,$$

where $v(z)$ is the exact solution, $v_n(z)$ the approximate solution of degree n, $z_i = -1 + \dfrac{2i}{n+1}$ for $n = 1 \cdots n$, and k is the order of the approximation in the algebraic system (3.12).

Example 1

Consider the following equation:

$$\begin{cases} v(z) = f(z) + \displaystyle\int_{-1}^{1} \sqrt{|z-t|} \cos\left[z + v(t) + e^t - v'(t)\right] dt, \quad \forall z \in [-1,1], \\ v(-1) = -e^{-1}, \end{cases}$$

with

$$f(z) = e^z - \frac{2}{3}\left(\sqrt{(1-z)^3} + \sqrt{(1+z)^3}\right)\cos(z),$$

and the exact solution is $v(z) = ze^z$.

Example 2

Consider the following equation:

$$\begin{cases} v(z) = \sin(z) + \dfrac{8}{7}\left((1+z)^{\frac{7}{4}} + (1-z)^{\frac{7}{4}}\right) + \displaystyle\int_{-1}^{1} |z-t|^{\frac{3}{4}} \sqrt{3 + v(t)^2 + v'(t)^2}dt, \\ v(-1) = \sin(-1), \quad \forall z \in [-1,1]. \end{cases}$$

such that the exact solution is $v(z) = \sin(z)$.

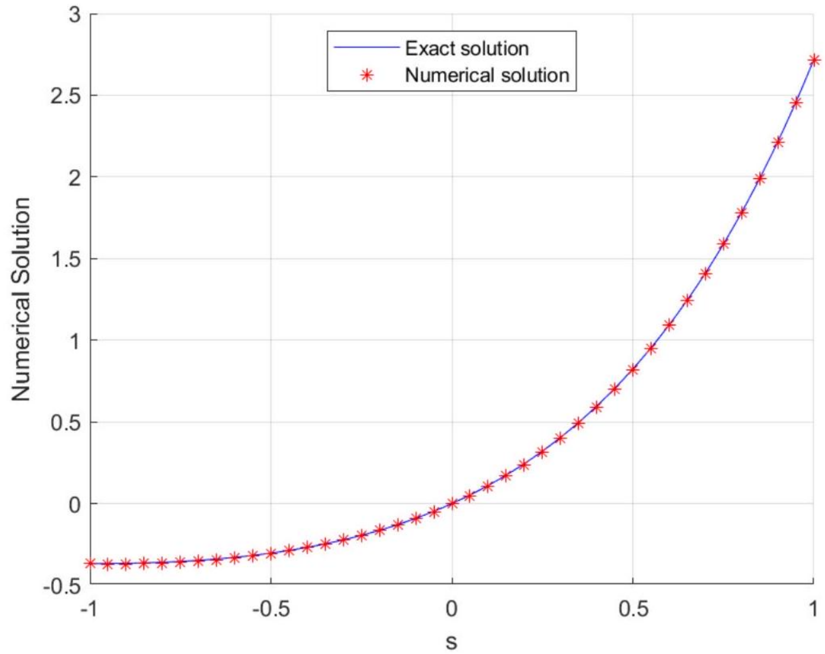

Figure 3.1: Exact and numerical solutions (Example 1) with $n = 2$ and $k = 2$.

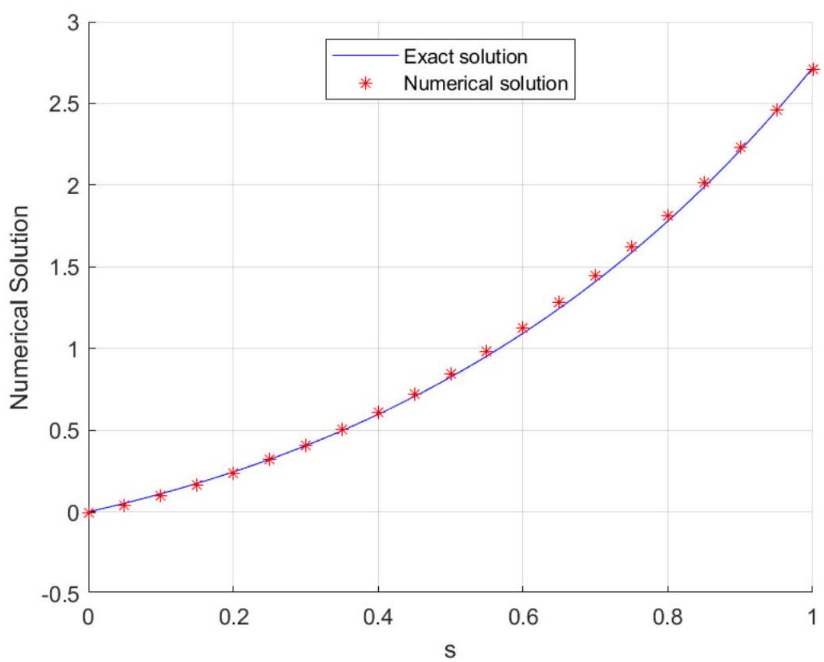

Figure 3.2: Exact and numerical solutions (Example 1) with $n = 3$ and $k = 2$.

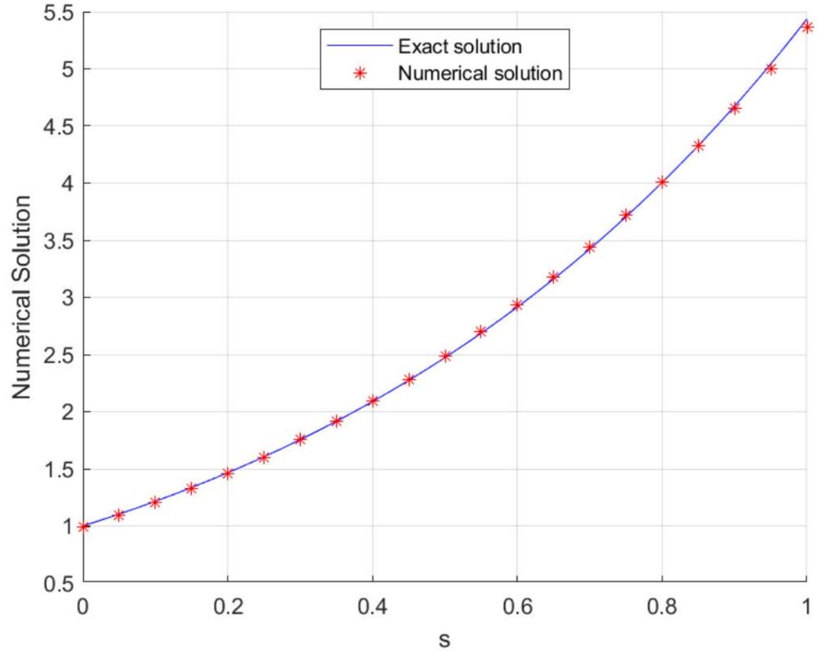

Figure 3.3: The derivative of exact and numerical solutions (Example 1) with $n = 4$ and $k = 1$.

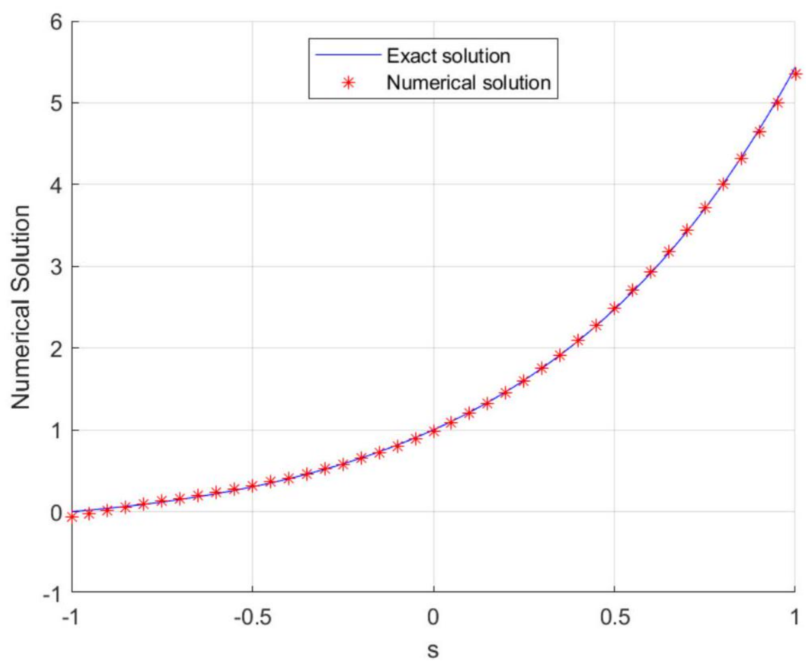

Figure 3.4: The derivative of exact and numerical solutions (Example 1) with $n = 4$ and k = 3.

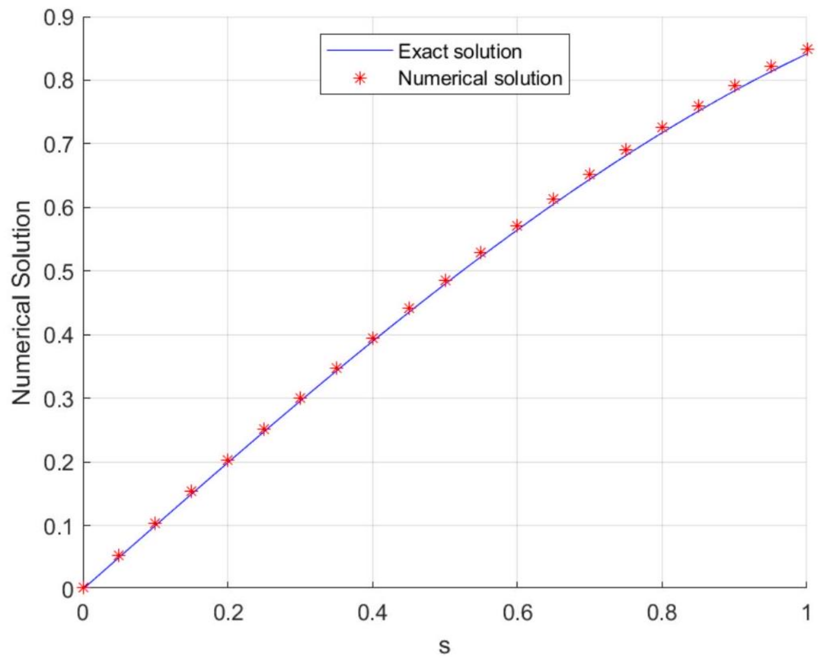

Figure 3.5: Exact and numerical solutions (Example 2) with $n = 3$ and $k = 1$.

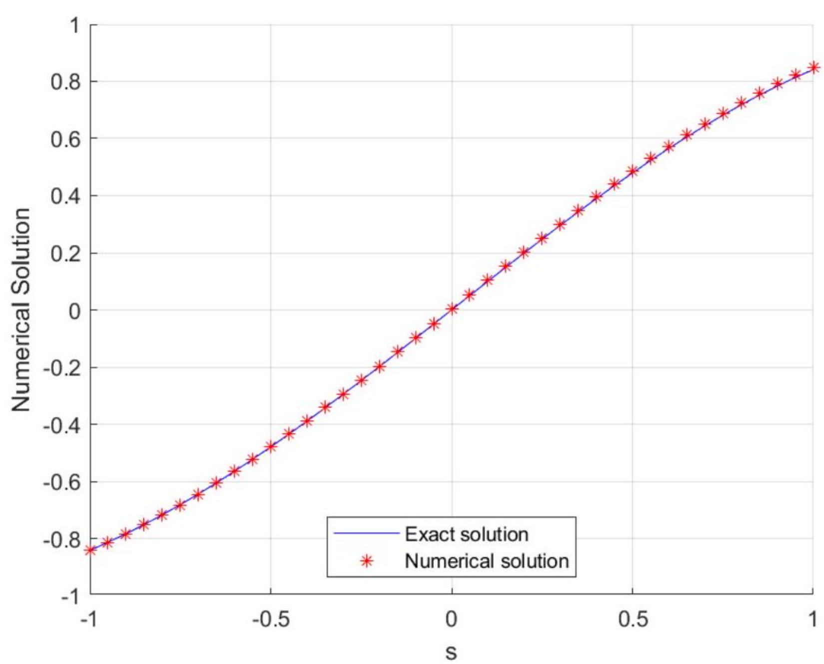

Figure 3.6: Exact and numerical solutions (Example 2) with $n = 4$ and $k = 1$.

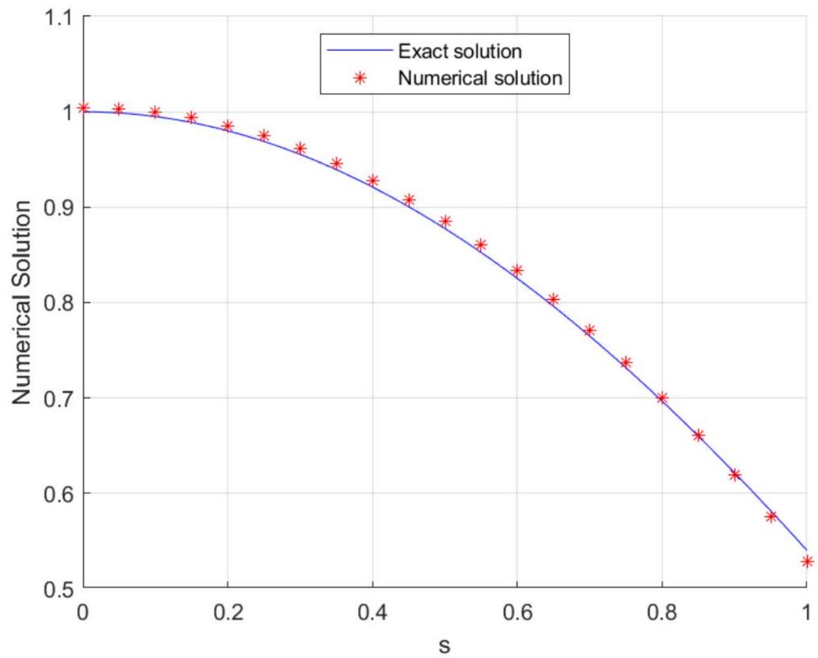

Figure 3.7: The derivative of exact and numerical solutions (Example 2) with $n = 4$ and $k = 1$.

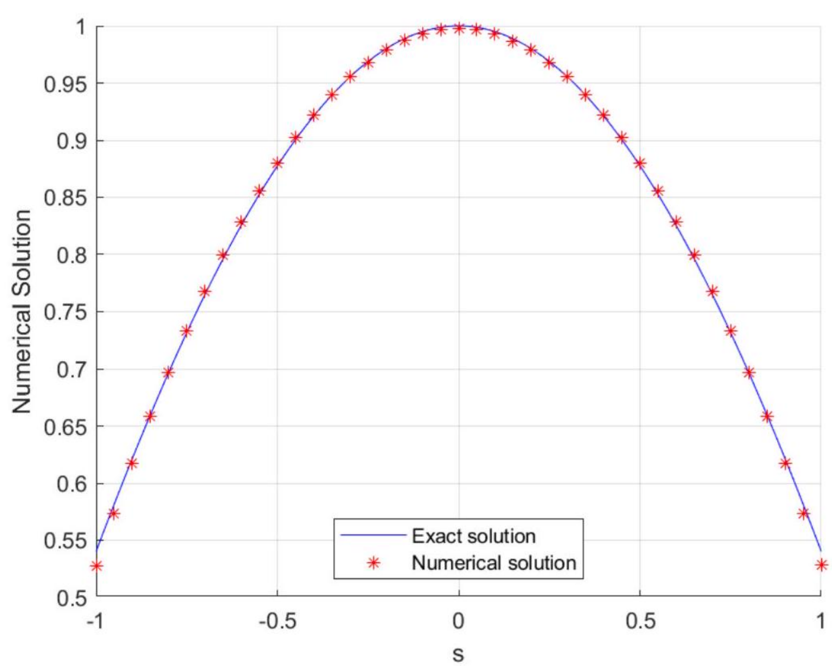

Figure 3.8: The derivative of exact and numerical solutions (Example 2) with $n = 4$ and $k = 2$.

n	$E(n,2)$	$E(n,3)$	$E(n,4)$	$E(n,5)$
3	3.3368e − 03	3.2190e − 03	3.2189e − 03	3.2188e − 03
5	5.4483e − 04	2.9483e − 05	2.6844e − 05	2.6746e − 05
7	5.4679e − 04	1.1965e − 05	8.6072e − 07	1.3264e − 07
9	5.4680e − 04	1.1966e − 05	8.5800e − 07	7.9817e − 08

Table 3.2: Numerical results for Example 2.

3.2 Collocation method

In this section, we present another method to solve the proposed equation, that is the collocation method. In this method, we seek a numerical solution represented by Laguerre polynomials.

3.2.1 Operational integration matrix

The operational matrix of integration for $n = 3$ is obtained as follows:

$$\begin{aligned}
\int_{-1}^{z} \varphi_0(t)dt &= \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \end{bmatrix} P(z),\\
\int_{-1}^{z} \varphi_1(t)dt &= \begin{bmatrix} \frac{3}{2} & -1 & 1 & 0 & 0 \end{bmatrix} P(z),\\
\int_{-1}^{z} \varphi_2(t)dt &= \begin{bmatrix} \frac{13}{6} & 0 & 1 & -1 & 0 \end{bmatrix} P(z),\\
\int_{-1}^{z} \varphi_3(t)dt &= \begin{bmatrix} \frac{73}{24} & 0 & 0 & 1 & -1 \end{bmatrix} P(z).
\end{aligned}$$

Therefore,

$$\int_{-1}^{z} C^T P(t)dt = C^T M_{4\times 5} Q(z),$$

where

$$M_{4\times 5} = \begin{pmatrix} \frac{3}{2} & -1 & 1 & 0 & 0 \\ \frac{3}{2} & -1 & 1 & 0 & 0 \\ \frac{13}{6} & 0 & 1 & -1 & 0 \\ \frac{73}{24} & 0 & 0 & 1 & -1 \end{pmatrix}.$$

Similarly, we can find the operational matrix of integration M for an arbitrary value n.

Let $N \in \mathbb{N}^*$ and consider the following grid points in the interval $[-1,1]$:

$$\Delta_N = \left\{ z_i = -1 + ih, h = \frac{2}{N+1}, i = 0, 1, \ldots, N \right\}.$$

Lemma 3.1. *The matrix*

$$W_n = \begin{pmatrix} \varphi_0(z_0) & \varphi_1(z_0) & \cdots & \varphi_n(z_0) \\ \varphi_0(z_1) & \varphi_1(z_1) & \cdots & \varphi_n(z_1) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_0(z_n) & \varphi_1(z_n) & \cdots & \varphi_n(z_n) \end{pmatrix}$$

is invertible for any n.

Proof. Let C_n be the matrix that contains the coefficients for the Laguerre polynomials of degree less than or equals n. We have

$$C_n = \begin{pmatrix} 1 & 1 & \cdots & \frac{1}{n!} \\ 0 & -1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{(-1)^n}{n!} \end{pmatrix},$$

and let V_n be the Vandermonde matrix

$$V_n = \begin{pmatrix} 1 & z_0 & z_0^2 & \cdots & z_0^n \\ 1 & z_1 & z_1^2 & \cdots & z_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_n & z_n^2 & \cdots & z_n^n \end{pmatrix}.$$

It is well known that the matrices C_n and V_n are invertible. We have

$$W_n = V_n C_n,$$

Hence, the matrix W_n is invertible as well.

Now, we approximate the function $\beta(z)$ as

$$\beta(z) \approx C^T P(z), \tag{3.13}$$

such that

$$\beta\left(z_i\right) = C^T P\left(z_i\right), \quad \forall \quad i = 0, \cdots n.$$

□

3.2.2 Method description

Consider the following integro-differential equation:

$$\begin{cases} \beta(z) = f(z) + \int_{-1}^{1} p(|z-s|)\Psi\left(z, s, \beta(s), \beta'(s)\right) ds, z \in [-1, 1], \\ \beta(-1) = \beta_0. \end{cases} \tag{3.14}$$

By taking the derivatives of (3.14), we get

$$\begin{aligned} \beta'(z) = f'(z) &+ \int_{-1}^{1} \operatorname{sign}(z-s) p'(|z-s|)\Psi\left(z, s, \beta(s), \beta'(s)\right) ds \\ &+ \int_{-1}^{1} p(|z-s|)\frac{\partial \Psi}{\partial z}\left(z, s, \beta(s), \beta'(s)\right) ds, \end{aligned} \tag{3.15}$$

where

$$\operatorname{sign}(z-s) = \begin{cases} +1, & z > s, \\ 0, & z = s, \\ -1, & z < s. \end{cases}$$

Let us approximate the unknown function $\beta'(z)$ by using (3.13). Then,

$$\beta'(z) = C^T P(z). \tag{3.16}$$

By integrating (3.16) with respect to z from 0 to z, we get

$$\beta(z) = \beta_0 + C^T M Q(z). \tag{3.17}$$

Now, substituting (3.17) and (3.16) into (3.15) yields the following equation:

$$\begin{aligned} C^T P(z) = f'(z) &+ \int_{-1}^{1} \operatorname{sign}(z-s) p'(|z-s|)\Psi\left(z, s, \beta_0 + C^T M P(s), C^T P(s)\right) ds \\ &+ \int_{-1}^{1} p(|z-s|)\frac{\partial \Psi}{\partial z}\left(z, s, \beta_0 + C^T M P(s), C^T P(s)\right) ds. \end{aligned} \tag{3.18}$$

Collocating the equation (3.18) by the grids points Δ_N leads to the nonlinear algebraic system:

$$
\begin{aligned}
C^T P(z_i) = f'(z_i) &+ \int_{-1}^{1} \operatorname{sign}(z_i - s)\, p'(z_i - s\,|)\, \Psi\left(z_i, s, \beta_0 + C^T M P(s), C^T P(s)\right) ds \\
&+ \int_{-1}^{1} p(|z_i - s|)\, \frac{\partial \Psi}{\partial z}\left(z_i, s, \beta_0 + C^T M P(s), C^T P(s)\right) ds,
\end{aligned} \tag{3.19}
$$

that can be written as

$$
C^T W_n = D^T = \left[d_0\left(C^T\right), d_1\left(C^T\right), \cdots, d_n\left(C^T\right)\right],
$$

where W_n is described in the lemma above and for $i = 1, \ldots, n$,

$$
\begin{aligned}
d_i\left(C^T\right) = f'(z_i) &+ \int_{-1}^{1} \operatorname{sign}(z_i - s)\, p'(z_i - s\,|)\, \Psi\left(z_i, s, \beta_0 + C^T M P(s), C^T P(s)\right) ds \\
&+ \int_{-1}^{1} p(|z_i - s|)\, \frac{\partial \Psi}{\partial z}\left(z_i, s, \beta_0 + C^T M P(s), C^T P(s)\right) ds.
\end{aligned}
$$

Therefore, by lemma 3.1,

$$
C^T = D^T W_n^{-1}.
$$

The nonlinear algebraic system (3.19)is solved by an iterative process. Let us consider the following system:

$$
\begin{cases}
C_{k+1}^T = \left[d_0\left(C_k^T\right), d_1\left(C_k^T\right), \cdots, d_n\left(C_k^T\right)\right] W_n^{-1}, \\
C_0^T = \left(f'(z_0), f'(z_1), \cdots, f'(z_n)\right), \quad k \in \mathbb{N}.
\end{cases} \tag{3.20}
$$

By solving the system (3.20) for a suitable value of k, we can obtain the coefficients C^T. Finally, the desired approximation for $\beta'(z)$ is computed by (3.13) .

3.2.3 Numerical examples

In order to demonstrate the effectiveness of the method proposed in this section, we provide several numerical examples that highlight its accuracy. These examples serve as a way to test and validate our method, and compare its results with other existing methods. Through these examples, we aim to show that our method can accurately approximate solutions for integro-differential equations with a weakly singular kernel.

n	$E(n,4)$	CPUtime
2	1.1023e − 01	0.045137
3	1.5531e − 02	0.031160
4	3.3139e − 03	0.035663
5	3.1440e − 04	0.367585
6	4.6722e − 05	0.045970

Table 3.3: Numerical results (Example 1)

n	$E(n,3)$	CPUtime
2	4.8956e − 03	0.0058
3	3.4910e − 03	0.1127
4	6.7750e − 05	0.1054
5	4.8455e − 05	0.1216
6	5.4069e − 07	0.1506

Table 3.4: Numerical results (Example 2)

Example 1

Consider the following equation:

$$\begin{cases} \beta(z) = f(z) + \displaystyle\int_{-1}^{1} \sqrt{|z-s|}\cos\left[z+\beta(s)+e^s-\beta'(s)\right] ds \quad \forall z \in [-1,1], \\ u(-1) = -e^{-1}, \end{cases}$$

with $f(z) = e^z - \dfrac{2}{3}\left(\sqrt{(1-z)^3} + \sqrt{(1+z)^3}\right)\cos(z)$ so that the exact solution is $\beta(z) = ze^z$.

Example 2

Consider the following equation:

$$\begin{cases} \beta(z) = \sin(z) + \dfrac{8}{7}\left((1+z)^{\frac{7}{4}} + (1-z)^{\frac{7}{4}}\right) + \displaystyle\int_{-1}^{1} |z-s|^{\frac{3}{4}}\sqrt{3+\beta(s)^2+\beta'(s)^2} ds, \\ u(-1) = -\sin(-1), \quad \forall z \in [-1,1], \end{cases}$$

such that the exact solution is $\beta(z) = \sin(z)$.

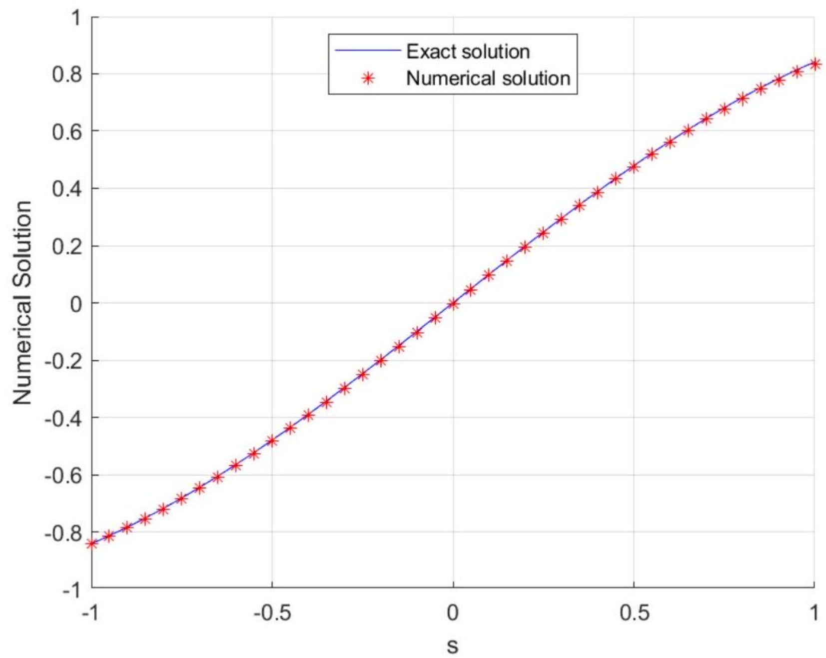

Figure 3.9: Exact and numerical solutions (Example 1), with $n = 2$

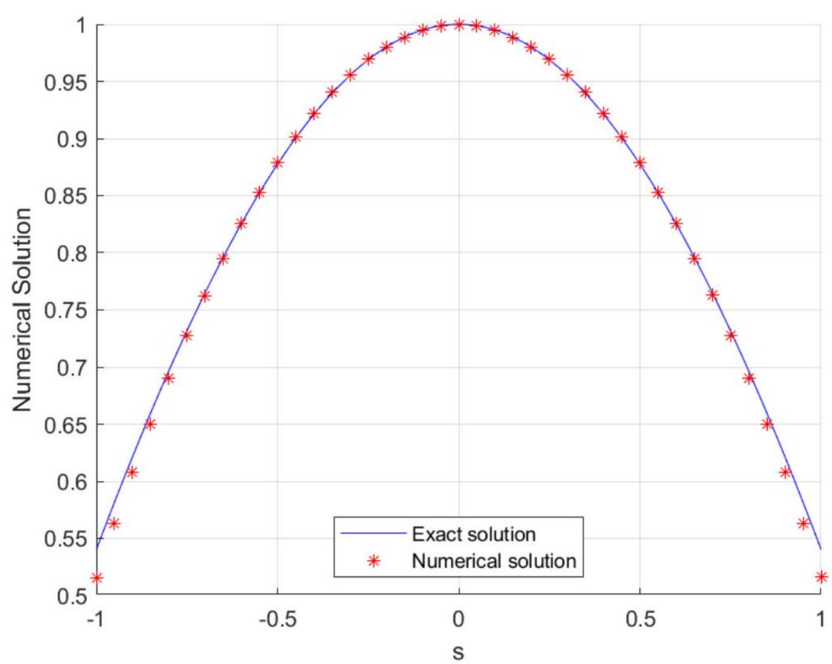

Figure 3.10: Exact and numerical solutions (Example 1), with $n = 3$

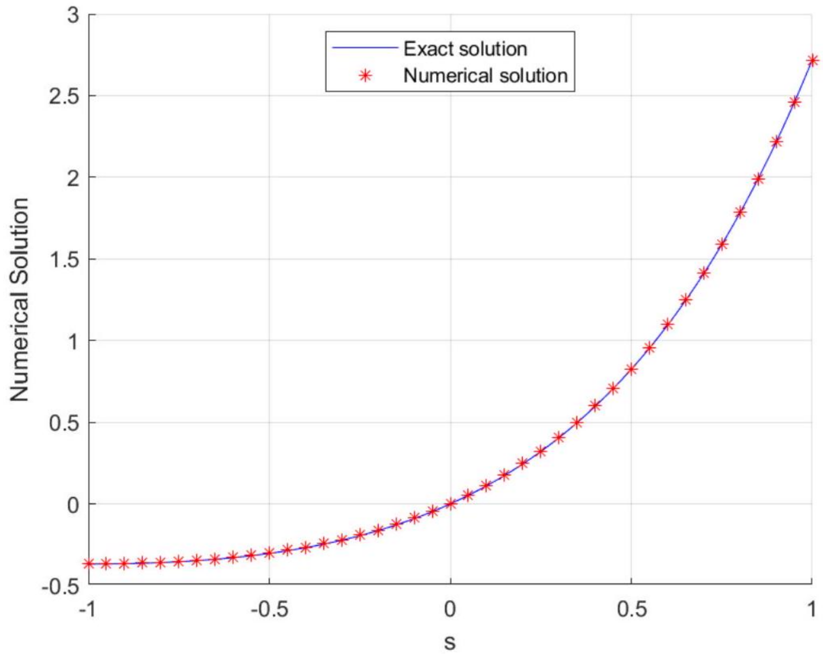

Figure 3.11: Exact and numerical solutions (Example 2), with $n = 2$

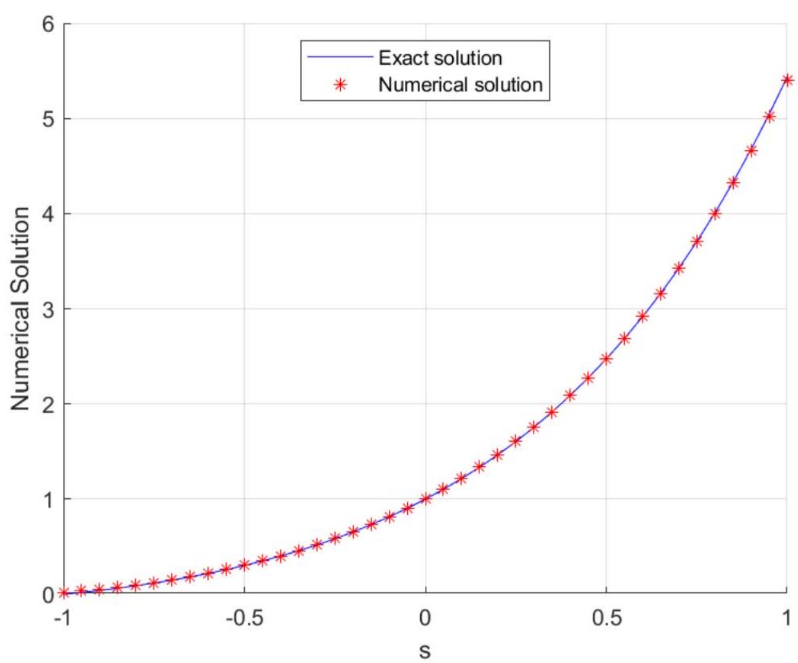

Figure 3.12: Exact and numerical solutions (Example 2), with $n = 3$

Discussion

In this chapter, we have presented two different methods for solving integro-differential equations with weakly singular kernels. The efficiency of both methods is demonstrated through several examples. Moreover, we can compare advantages and disadvantages of each method. The Galerkin method is more accurate but it is quite complex and requires more computation. On the other hand, the collocation method is less accurate than the Galerkin method but it is simpler and requires less computational complexity.

Chapter 4

Fractional Integro-Differential Equation with the Caputo-Fabrizio Sense

In this chapter, we demonstrate the applicability of our technique in solving fractional integro-differential equations with the Caputo-Fabrizio derivative. Specifically, we concentrate on the following equation:

$$\begin{cases} u(z) = g(z) + \int_0^1 K\left(z, s, u(s), \mathcal{D}^\alpha u(s)\right) ds, \\ u(0) = 0, \end{cases} \tag{4.1}$$

where $\partial_x K, K \in \mathcal{C}\left([0,1]^2 \times \mathbb{R}^2\right)$, $0 < \alpha < 1, u(z), g(z) \in H^1[0,1]$, and $\mathcal{D}^\alpha$ denotes the Caputo-Fabrizio derivative of order α.

We explain the construction of the fractional operational matrix of integration utilizing Hermite wavelets. We then apply the collocation technique followed by the iterative method to approximate the numerical solution. To verify the accuracy of our process, we present several computational examples.

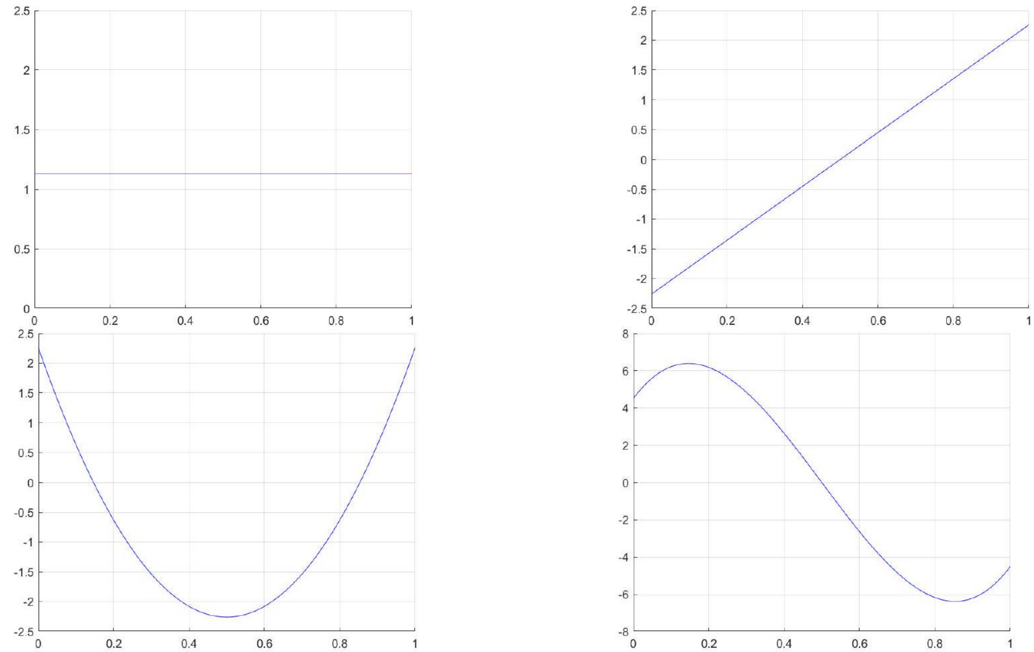

Figure 4.1: Hermite wavelets for $n = 0, 1, 2, 3$.

4.1 Hermite wavelets

The Hermite wavelets are given as follows :

$$\psi_{i,j}(z) = \begin{cases} \dfrac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} H_j\left(2^k z - 2i + 1\right), & \dfrac{i-1}{2^{k-1}} \leqslant z \leqslant \dfrac{i}{2^{k-1}}, \\ 0, & \text{otherwise} , \end{cases}$$

where $i = 1, 2, \ldots, 2^{k-1}.k > 0$ is an integer number, $j = 0, 1, 2, \ldots, n-1$, H_j is the Hermite polynomial of degree j.

Any function $u(z)$ in $L^2_w(\mathbb{R})$ can be written in the following form:

$$u(z) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} c_{i,j} \psi_{i,j}(z), \tag{4.2}$$

where $c_{i,j} = \langle u, \psi_{i,j} \rangle$ with $\langle ., . \rangle$ being the scalar product in the Hilbert space $L^2(\mathbb{R})$. Then, we get the approximate function for $u(z)$ by truncating the series (4.2) as follows:

$$u_n(z) = \sum_{i=1}^{2^{k-1}} \sum_{j=0}^{n-1} c_{i,j} \psi_{i,j}(z) = C^T P(z), \tag{4.3}$$

where $P(z)$ and C^T are $2^{k-1}n \times 1$ matrices:

$$P(z) = \left[\psi_{1,0}, \psi_{1,1}, \ldots, \psi_{1,n-1}, \psi_{2,0}, \psi_{2,1}, \ldots, \psi_{2,n-1}, \ldots, \psi_{2^{k-1},0}, \psi_{2^{k-1},1} \ldots, \psi_{2^{k-1},n-1}\right]^T,$$

and

$$C^T = \left[c_{1,0}, c_{1,1}, \ldots, c_{1,n-1}, c_{1,0}, c_{2,1}, \ldots, c_{2,n-1}, \ldots, c_{2^{k-1},0}, \ldots, c_{2^{k-1},n-1}\right].$$

4.2 Fractional operational integration matrix

If $k = 1$, then both $P(z)$ and C^T would be:

$$C^T = [\alpha_0, \alpha_1, \ldots, \alpha_{n-1}],$$
$$P(z) = [\psi_0(z), \psi_1(z), \ldots, \psi_{n-1}(z)].$$

W_n denotes a matrix comprising the coefficients associated with the Hermite wavelets:

$$W_n = \frac{1}{\sqrt{\pi}} \begin{pmatrix} 2 & -4 & 4 & \cdots & H_{n-1}(-1) \\ 0 & 8 & -32 & \cdots & \vdots \\ 0 & 0 & 32 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2^{2n-1} \end{pmatrix},$$

and

$$Z_n(z) = \left(1, z, z^2, \ldots, z^{n-1}\right) \quad , \quad P_n(z) = [\psi_0(z), \psi_1(z), \ldots, \psi_{n-1}(z)].$$

We have:

$$P_n(z) = Z_n(z) W_n.$$

Consider an integral matrix denoted by N within the canonical polynomial basis:

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1/2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1/3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1/n \end{pmatrix}.$$

Subsequently, the operational integration matrix in the Hermite wavelets basis is given as follows:

$$\int_0^z C^T P_n(y) dy = C^T M P_{n+1}(z),$$

where

$$M = W_n^{-1} L W_{n+1}.$$

Furthermore, employing the previously defined notation in (1.1), we can express the fractional operational integration matrix in the following manner:

$$\mathcal{I}^{\alpha}\left(C^T P_n(z)\right) = C^T[\alpha M + (1-\alpha)F]P_{n+1}(z) = C^T Q_n(z), \tag{4.4}$$

where $Q_n(z) = [(1-\alpha)F + \alpha M]P_{n+1}(z)$ and $\quad F = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$

4.3 Method description

Consider the following Fredholm integro-differential equation:

$$u(z) = f(z) + \int_0^1 K\left(z, s, u(s), \mathcal{D}^{\alpha} u(s)\right) dy, \quad u(0) = 0. \tag{4.5}$$

By taking the derivative of both sides of equation (4.5) and using the Caputo-Fabrizio derivative of order α, we obtain

$$\mathcal{D}^{\alpha} u(z) = \mathcal{D}^{\alpha} g(z) + \int_0^1 \mathcal{D}_x^{\alpha} K\left(z, s, u(s), \mathcal{D}^{\alpha} u(s)\right) ds. \tag{4.6}$$

We approach the unknown function $\mathcal{D}^{\alpha} u(z)$ by using the formula (4.3)

$$\mathcal{D}^{\alpha} u(z) \approx C^T P_n(z). \tag{4.7}$$

To approximate the unknown function $u(z)$, we perform an integration of (4.7) with the help of the operational matrix of fractional integration mentioned earlier (4.4). This process yields:

$$\begin{aligned} u_n(z) &= \mathcal{I}^{\alpha}\left(C^T P_n(z)\right) \\ &= C^T Q_n(z). \end{aligned} \tag{4.8}$$

Now, substitute (4.7) and (4.8) into (4.6) to obtain:

$$C^T P_n(z) = \mathcal{D}^{\alpha} g(z) + \int_0^1 \mathcal{D}_x^{\alpha} K\left(z, s, C^T Q_n(s), C^T P_n(s)\right) ds. \tag{4.9}$$

Using the grid points $z_i = \dfrac{2i+1}{2(n+1)}$, where $i \in \mathbb{N}$ and $i \leq n-1$, we apply collocation to equation (4.9). This results in the formation of the subsequent nonlinear algebraic system:

$$C^T A = D^T, \tag{4.10}$$

such that

$$D^T = [d_0, d_1, \cdots, d_{n-1}],$$

where

$$d_i = \mathcal{D}^\alpha g(z_i) + \int_0^1 \mathcal{D}_x^\alpha K\left(z_i, s, C^T Q_n(s), C^T P_n(s)\right) ds, \text{ for } i = 0, \cdots, n-1,$$

and

$$A = \begin{pmatrix} \psi_0(z_0) & \psi_0(z_1) & \cdots & \psi_0(z_{n-1}) \\ \psi_1(z_0) & \psi_1(z_1) & \cdots & \psi_1(z_{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{n-1}(z_0) & \psi_{n-1}(z_1) & \cdots & \psi_{n-1}(z_{n-1}) \end{pmatrix}.$$

We apply the iterative method to solve the system (4.10). To do this, we introduce the following system:

$$C_{k+1}^T A = D_k^T,$$

where

$$D_k^T = \left[d_0^k, d_1^k \cdots, d_n^k\right],$$

and

$$d_i^k = \mathcal{D}^\alpha g(z_i) + \int_0^1 \mathcal{D}_x^\alpha K\left(z_i, s, C_k^T Q_n(s), C_k^T P_n(s)\right) ds.$$

For a suitable value of k, we find the vector C_k^T, then substitute the coefficients of C_k^T into (4.8) to compute the approximate solution to equation (4.5).

4.4 Numerical experiments

Here, some illustrative experiments are included to demonstrate the efficiency of our method. We introduce the error as follows:

$$E_n = \max_{i=\overline{0,n-1}} |u_n(z_i) - u(z_i)|,$$

where $u(z)$ represents the true solution, $u_n(z)$ stands for the approximate solution, n signifies the degree of Hermite wavelets.

First experiment

Let be given the fractional Fredholm integro-differential equation:

$$\begin{cases} \forall z \in [0,1], u(z) = g(z) + \displaystyle\int_0^1 \ln\left[\frac{6}{5}\left(\cos(s) - ze^{-3s}\right) + \frac{2}{5}u(s) - \mathcal{D}^\alpha u(s)\right] ds, \\ u(0) = 0, \end{cases}$$

where

$$g(z) = \cos(z) + \ln\left[\frac{6}{5}(1+z)\right] - \frac{3}{2}.$$

The exact solution to this equation takes the form of $u(z) = \sin(z)$, when the fractional order of differentiation is $\alpha = 0.5$.

Second experiment

Consider the following equation:

$$\begin{cases} u(z) = g(z) - \displaystyle\int_0^1 \frac{\sin(z+s)}{1 + \mathcal{D}^\alpha u(s) + 2su(s)} ds, \quad \forall z \in [0,1], \\ u(0) = 0. \end{cases}$$

In this context, we have $g(z) = e^{-z} - 1 + \cos(1+z) - \cos(z)$, and the exact solution is expressed as $u(z) = e^{-z} - 1$ if the fractional order of differentiation is $\alpha = 0.75$.

Third experiment

We have the following equation:

$$\begin{cases} \forall z \in [0,1], u(z) = -\dfrac{5}{2}z^2 + \displaystyle\int_0^1 z^2\sqrt{12 - 3e^{-s} + 2\mathcal{D}^\alpha u(s) + u(s)} ds, \\ u(0) = 0, \end{cases}$$

such that the order of derivation is $\alpha = \dfrac{2}{3}$, and the exact solution of the equation is $u(z) = z^2$.

Fourth experiment

Let the following equation:

$$\begin{cases} \forall z \in [0,1], u(z) = ze^{-\frac{z}{3}} - \cos(z) + \displaystyle\int_0^1 \sin\left(z - s + \frac{2}{9}s^2 - \frac{4}{3}x + e^{\frac{s}{3}}u(s) + e^{\frac{s}{3}}\mathcal{D}^\alpha u(s)\right) ds, \\ u(0) = 0. \end{cases}$$

The order for this example is $\alpha = \dfrac{1}{4}$, and the exact solution is $u(z) = ze^{-\frac{z}{3}} - \cos(z)$.

n	3	4	5	6	7
E_n	6.988E − 05	1.433E − 06	4.934E − 08	6.723e − 10	5.444E − 10

Table 4.1: Numerical results (First experiment).

n	3	4	5	6	7
E_n	7.949E − 04	2.759E − 05	7.617E − 07	1.741E − 08	8.503E − 09

Table 4.2: Numerical results (Second experiment)

n	3	4	5	6	7
E_n	4.402E − 04	5.399E − 05	5.730E − 06	4.661E − 07	4.355E − 08

Table 4.3: Numerical results (Third experiment) .

n	3	4	5	6	7
E_n	1.301E − 05	1.762E − 07	1.065E − 08	7.066E − 11	4.737E − 11

Table 4.4: Numerical results (Fourth experiment).

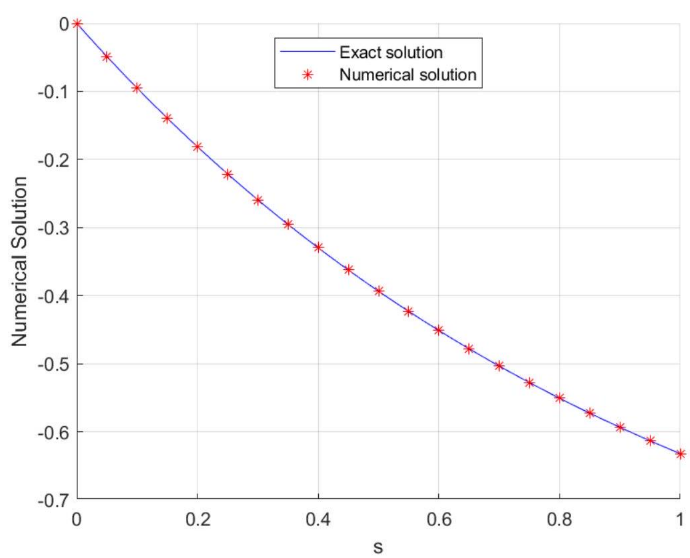

Figure 4.2: Exact and approximate solutions (First experiment), $n = 7$.

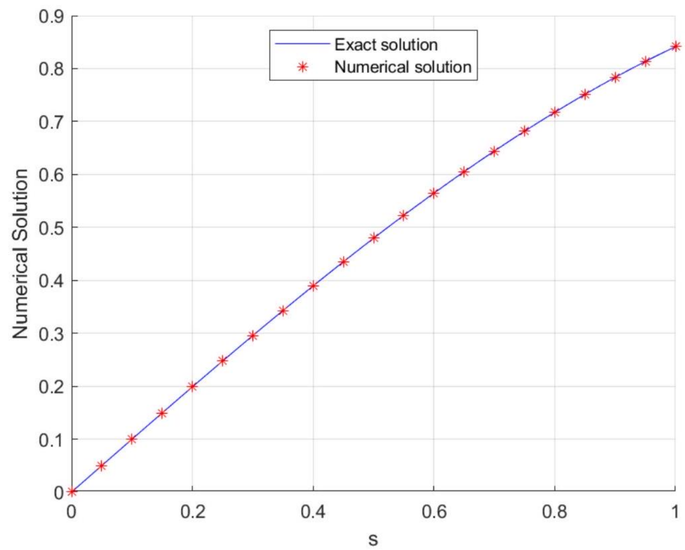

Figure 4.3: Exact and approximate solutions (Second experiment), $n = 7$.

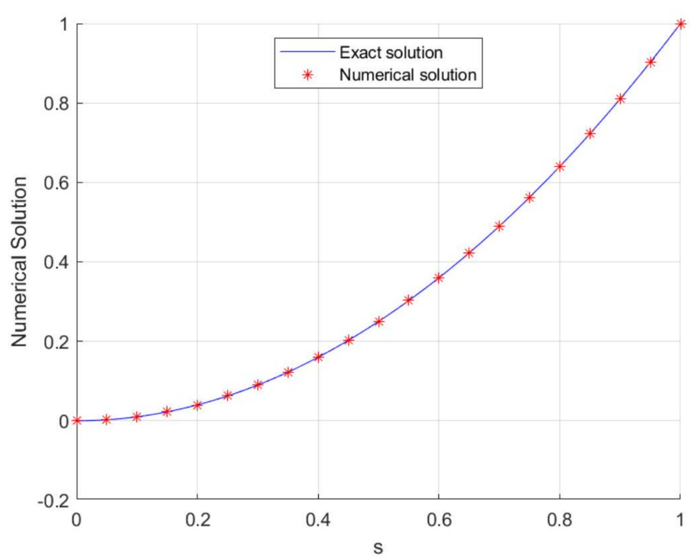

Figure 4.4: Exact and approximate solutions (Third experiment) , $n = 7$.

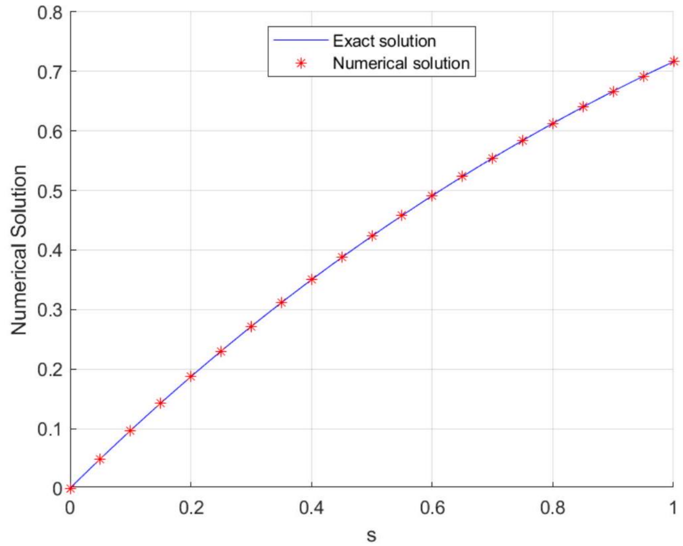

Figure 4.5: Exact and approximate solutions (Fourth experiment) $n = 7$.

Results interpretation

Tables 4.1, 4.2, 4.3, and 4.4 display the error E_n for various degrees n, illustrating that the method's performance improves as n increases. Additionally, Figures 4.2,4.3, 4.4, and 4.5 present graphs depicting both the exact and approximate solutions, showcasing their remarkable similarity. Thus, the aforementioned examples serve as compelling evidence of the efficiency and validity of our numerical approach.

Conclusion

This thesis has been focused on an in-depth investigation of various types of integrodifferential equations. Specifically, we have studied equations with regular kernels, weakly singular kernels, and those in the fractional case. To solve these equations, we have employed the popular projection method along with classical orthogonal polynomials. By applying this approach, we are able to transform in each case the main equations into a nonlinear algebraic system, that can then be solved using the Picard successive approximations. To validate the effectiveness and accuracy of our proposed methods, we have also developed algorithms using the Matlab platform. This allowed us to implement our methods and present numerical examples that demonstrate their applicability and performance. Through our comprehensive analysis and numerical examples, we have shown that our proposed approach can accurately and efficiently solve a wide range of integro-differential equations.

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