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Study of a Snap problem in the Hilfer - Katugampola sense

Directed by : Dr. Tabouche Nora

Before the jury

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Dedications

The present work is dedicated, with deep reverence and gratitude to my beloved parents,

my father and my mother.

I express my sincerest appreciation for their presence and for being a significant source of support, which contributed to my current state of being.

I hereby convey my genuine sentiments and utmost recognition
towards their remarkable efforts extended towards my welfare
and shall always cherish their invaluable reassurance.

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I express my gratitude to Pr. Chaoui Abderrazak for consenting to evaluate my work.

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الملخص

في هذه المذكرة، المستوحاة من أعمال كل من برحايل، أ.، طابوش، ن.، مطر، م. م.، ساماي م . أ.، كابار، م. ك. أ. ، كزيافونغ وانغ و كزياكونغ يو والتي قاموا فيها بدراسة وجود ووحدانية واستقرار حل مشكلة القيم الحدية سناب من نوع ج - كابوتو. قمنا بالعمل على نموذج حديث لمشكلة سناب ذات ترتيب كسري من نوع هلفر- كاتوكومبولا بشروط تكاملية كسرية (مشكلة جديدة لم تتم دراستها بعد).

أولا، تطرقنا إلى دراسة وجود ووحدانية حل مشكلة سناب ضمن فضاء الدوال المستمرة المزود بثقل، وذلك من خلال تطبيق كل من نظرية النقطة الثابتة لشودر ومبدأ التقليص لبناخ.

بالإضافة إلى ذلك، قمنا بدراسة استقرار الحل وفق معياري أولام هايرز وأولام هايرز المعمم.

في الأخير، قمنا بتقديم مثال لغرض توضيح الدراسة النظرية إلى جانب تبيان النتائج المتحصل عليها.

الكلمات المفتاحية : مشكلة سناب، المشتقة الكسرية من نوع هلفر- كاتوكومبولا، نظرية النقطة الثابتة لشودر، مبدأ التقليص لبناخ، الاستقرار حسب أولام هايرز، الاستقرار حسب أولام هايرز المعمم.

ABSTRACT

In our thesis, inspired by the work of Berhail, A., Tabouche, N., Matar, M.M., Samei, M.E., Kaabar, M.K.A., Xiaofeng Wang and Xiao-Guang Yue, which primarily concerns itself with addressing the underlying issues pertaining to the existence, uniqueness, and stability of the solution to the Snap problem in the $\mathbb{G}$-Caputo sense, we consider a contemporary model of the Snap problem of fractional order in the Hilfer-Katugampola sense with fractional integral conditions (a novel problem that has yet to be studied).

Firstly, we establish the existence and uniqueness of the solution, which is achieved via the concurrent implementation of both Schauder fixed point theorem and Banach contraction principle. Moreover, we explore the stability of the solution to our problem in both Ulam-Hyers and generalized Ulam-Hyers sense. Finally, we provide a numerical simulation in order to understand the theoritical process and illustrate the obtained results.

Key words : Snap problem, Hilfer-Katugampola fractional derivative, Schauder fixed point theorem, Banach contraction principle, Ulam-Hyers stability, Generalized Ulam-Hyers stability.

Résumé

Dans notre mémoire, inspiré par les travaux de Berhail, A., Tabouche, N., Matar, M.M., Samei, M.E., Kaabar, M.K.A., Xiaofeng Wang et Xiao-Guang Yue, qui s'intéressent principalement aux questions relatives à l'existence, l'unicité et la stabilité de la solution du problème Snap au sens de $\mathbb{G}$-Caputo, nous considérons un modèle contemporain du problème Snap d'ordre fractionnaire au sens de Hilfer-Katugampola avec des conditions intégrales fractionnaires (un nouveau problème qui n'a pas encore été étudié).

Tout d'abord, nous établissons l'existence et l'unicité de la solution, ce qui est réalisé par la mise en œuvre simultanée du théorème du point fixe de Schauder et du principe de contraction de Banach. De plus, nous explorons la stabilité de la solution de notre problème au sens d'Ulam-Hyers et d'Ulam-Hyers généralisé. Enfin, nous fournissons une simulation numérique afin de mieux comprendre le processus théorique et d'illustrer les résultats obtenus.

Mots clés : Poblème Snap, Dérivée fractionnaire au sens de Hilfer-Katugampola, Théorème du point fixe de Schauder, Principe de contraction de Banach, Stabilité au sens de Ulam-Hyers, Stabilité au sens de Ulam-Hyers généralisé.

Contents

0.1 *Introduction*

The concept of fractional analysis holds a long-standing historical precedent, dating back as far as the early 1700s, a time when monumental figures such as Newton and Leibniz laid down the fundamental principles of differential and integral calculus. This conception which refers to a function possessing an arbitrary real or complex order has gained substantial prominence within the academic community during the last three decades, and has unlocked new insights into understanding complex and intermittent systems of our contemporary time that classical calculus cannot handle which contributed to numerous advances in various fields of science and engineering. Despite the complexity involved in studying non-integer derivatives, researchers have persistently pursued this area of study, and have made significant progress towards understanding and applying the aforementioned concept.

The origins of fractional calculus can be traced back to a discourse that transpired between prominent mathematicians L'Hopital and Leibniz during the year 1695. L'Hopital made an inquiry regarding the significance of the $d^n f/dt$ derivative when $n = 1/2$, which consequently inspired Leibniz to contemplate the plausibility of derivatives of non integer order. Leibniz denoted the aforementioned concept as a paradox containing potential utilities within practical contexts. However, it was not until 1990 that noteworthy advancements in this domain were realized.

In his article [13] on the Gamma function, a mathematical concept closely connected to the factorial function, Euler presented a quandary that concerned rational numbers. It is possible that this factor may have contributed to the adoption of the descriptor "fractional" in contemporary calculus. In the year 1823, Abel introduced an approach utilizing fractional calculus in the context of a challenge involving differential equations of non-integer order. During the 1830s, Liouville and Riemann, in their own distinct manners, established the concept of fractional derivatives, a method subsequently referred to as the "Riemann-Liouville" approach. Over time, ancillary hypotheses, including the Grunwald-Letnikov, Weyl, and Caputo conjectures, were developed.

Notwithstanding its initial perception as a theoretical concept lacking practical applicability, after the first conference on fractional calculus and its applications taking place at the University of New Haven in 1974 [36]. fractional calculus has since gained considerable recognition for its diverse potential applications across multiple disciplines such as physics, engineering, computer science etc... For instance, in physics, fractional derivatives and integrals are used to model the behavior of systems with memory, such as viscoelastic materials, non-Newtonian fluids, electromagnetic problems that can be modelieed through the utilization of fractional integro-differential equations. In engineering and computer science, fractional calculus also has significant applications in the modelization of complex dynamics, such as those found in aircraft, spacecraft, robots and signal processing, for instance, fractional derivatives and integrals are used to create filters that can separate noise from a signal, extract important features and analyze the time-frequency characteristics of a signal.

Over the years, multiple definitions of non-integer operators have been introduced. However consistent outcomes cannot always be guaranteed. The Riemann-Liouville approach, Caputo's approach, Hadamard's approach [11, 15, 22, 23, 28], and the recently introduced Hilfer-Katugampola derivation. The latter is described in great detail in references [27, 26, 25, 30].

Accordingly, the Snap problem is a mathematical model of acceleration changing rate over time. Specifically, it describes how quickly an object's acceleration changes from "jerk" (rate of change of acceleration) to "snap" (rate of change of jerk). The calculation involves taking derivatives of position with respect to time.
Real-life examples where these concepts apply include automobile safety testing, computer graphics animation, software development and aerospace engineering design.
In ocean engineering phenomena, jerk and snap become even more significant due to their effects on ship motion in waves and the comfort level of passengers on board. Seakeeping studies how jerk and snap influence vessel speed response amplitude operators when traveling through rough seas. Ride comfort explores how these factors impact passenger experiences by measuring vertical accelerations onboard ships during travel. Shock response spectrum is another key consideration for marine structures because it determines how much force is needed to damage them under various conditions such as earthquakes or other types of shocks.

The second derivative of accleration (fourth derivative of position) is a physical quantity called snap or jounce, which can be modeled as :

$$\begin{cases} \frac{dv_1}{dt} = v_2(t), \\ \frac{dv_2}{dt} = v_3(t), \\ \frac{dv_3}{dt} = v_4(t), \\ \frac{dv_4}{dt} = \xi(v_1, v_2, v_3, v_4). \end{cases} \tag{0.1}$$

It is obvious that the model (0.1) can be reduced to the following equation:

$$\frac{d^4 v_1}{dt^4} = \xi\left(v_1, \frac{dv_1}{dt}, \frac{d^2 v_1}{dt^2}, \frac{d^3 v_1}{dt^3}\right). \tag{0.2}$$

In fact, the terms jerk and snap are basically the third and fourth derivatives of our position with regard to time, respectively.

Equation (0.2) contains a 4th-order derivative of the variable v_1, and it describes a 4th-order dynamical vibration model. The corresponding fractional model is achieved by using the fractional derivative (of order less than or equal 1) instead of the standard derivative $\frac{d}{dt}$. In 2017, Elsonbaty et al., by applying the contraction principle, investigated the following jerk system :

$$\begin{cases} \frac{dv_1}{dt} = v_2(t), \qquad \frac{dv_1}{dt} = v_3(t), \\ \frac{dv_3}{dt} = \lambda v_1(t) - \beta v_2(t) - v_3(t) - v_1^3(t), \end{cases}$$

in which derivatives are with respect to time, λ and β denote positive parameters with $\beta \in \mathbb{R}$ [12]. The authors in the recent article [38] considered the $\mathbb{G}$-fractional snap model with a constant initial conditions.

$$\begin{cases} {}^C D_{a^+}^{\alpha,\mathbb{G}} v(t) = v_1(t), & v(a) = u_0, \\ {}^C D_{a^+}^{\beta,\mathbb{G}} v(t) = v_2(t), & v_1(a) = u_1, \\ {}^C D_{a^+}^{\gamma,\mathbb{G}} v(t) = v_3(t), & v_2(a) = u_2, \\ {}^C D_{a^+}^{\delta,\mathbb{G}} v(t) = \mathcal{T}(t, v, v_1, v_2, v_3), & v_3(a) = u_3, \end{cases} \tag{0.3}$$

where the $\mathbb{G}$-Caputo derivatives are illustrated by symbol ${}^C D_{a^+}^{\eta,\mathbb{G}}$, and $\eta \in \{\alpha, \beta, \gamma, \delta\}$ such that $0 < \eta \leq 1$, the increasing function $\mathbb{G} \in C^1[a, b]$ is such that $\mathbb{G}'(t) \neq 0$, $t \in [a, b]$ and continuous function $\mathcal{T} \in C^1([a, b] \times \mathbb{R}^4)$ and $u_0, u_1, u_2, u_3 \in \mathbb{R}$.

In 2022, Berhail et al. [8] studied a nonlinear fractional Snap model with respect to a $\mathbb{G}$-Caputo derivative and subject to non-periodic boundary conditions, which was considered as follows :

$$\begin{cases} {}^C D_{a^+}^{\alpha,\mathbb{G}} v(t) = v_1(t), & v(a) = \lambda_0 v(b), \\ {}^C D_{a^+}^{\beta,\mathbb{G}} v(t) = v_2(t), & v_1(a) = \lambda_1 v_1(b), \\ {}^C D_{a^+}^{\gamma,\mathbb{G}} v(t) = v_3(t), & v_2(a) = \lambda_2 v_2(b), \\ {}^C D_{a^+}^{\delta,\mathbb{G}} v(t) = f(t, v, v_1, v_2, v_3), & v_3(a) = \lambda_3 v_3(b). \end{cases}$$

where the symbol ${}^C D_{a^+}^{\alpha,\alpha'}$, where $\eta \in \alpha, \beta, \gamma, \delta$ is the $\mathbb{G}$-Caputo derivative such that $0 < \eta \leq 1$, the function $\mathbb{G} \in C^1[a, b]$ is increasing such that $\mathbb{G}'(t) \neq 0$, $t \in [a, b]$ $f \in C([a, b] \times \mathbb{R}^4)$ and $\lambda_0, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \backslash \{1\}$. The system was defined as follows:

$$\begin{cases} {}^C D_{a^+}^{\delta,\mathbb{G}} ({}^c D_{a^+}^{\gamma,\mathbb{G}} ({}^c D_{a^+}^{\beta,\mathbb{G}} ({}^c D_{a^+}^{\alpha,\mathbb{G}} v(t))) \\ \qquad = f(t, v(t), {}^c D_{a^+}^{\alpha,\mathbb{G}} v(t), {}^c D_{a^+}^{\beta,\mathbb{G}} ({}^c D_{a^+}^{\alpha,\mathbb{G}} v(t)), {}^c D_{a^+}^{\gamma,\mathbb{G}} ({}^c D_{a^+}^{\beta,\mathbb{G}} ({}^c D_{a^+}^{\alpha,\mathbb{G}} v(t))))), \\ v(a) = \lambda_0 \, v(b), \\ {}^C D_{a^+}^{\alpha,\mathbb{G}} v(a) = \lambda_1 \, {}^C D_{a^+}^{\alpha,\mathbb{G}} v(b), \\ {}^C D_{a^+}^{\beta,\mathbb{G}} ({}^C D_{a^+}^{\alpha,\mathbb{G}} v(a))) = \lambda_2 \, {}^C D_{a^+}^{\beta,\mathbb{G}} ({}^C D_{a^+}^{\alpha,\mathbb{G}} v(b)), \\ {}^C D_{a^+}^{\gamma,\mathbb{G}} ({}^c D_{a^+}^{\beta,\mathbb{G}} ({}^c D_{a^+}^{\alpha,\mathbb{G}} v(a)))) = \lambda_3 \, {}^C D_{a^+}^{\gamma,\mathbb{G}} ({}^C D_{a^+}^{\beta,\mathbb{G}} ({}^C D_{a^+}^{\alpha,\mathbb{G}} v(b))). \end{cases}$$

where $\mathbb{G}'(t) \neq 0$, $t \in [a, b]$ $f \in C([a, b] \times \mathbb{R}^4)$ and $\lambda_0, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \backslash \{1\}$.

However, in our thesis we will study the Snap problem with respect to Hilfer-Katugampola derivative equipped with fractional integral boundary conditions;

$$\begin{cases} {}^{\rho}D_{a^+}^{\alpha,\alpha'} v(t) = v_1(t), \quad t \in [a,b], & {}^{\rho}I_{a^+}^{1-\eta_0} v(a) = c_0, \\ {}^{\rho}D_{a^+}^{\beta,\beta'} v_1(t) = v_2(t), & {}^{\rho}I_{a^+}^{1-\eta_1}(v_1(a)) = c_1, \\ {}^{\rho}D_{a^+}^{\gamma,\gamma'} v_2(t) = v_3(t), & {}^{\rho}I_{a^+}^{1-\eta_2}(v_2(a)) = c_2, \\ {}^{\rho}D_{a^+}^{\delta,\delta'} v_3(t) = f(t, v(t), v_1(t), v_2(t), v_3(t)), & {}^{\rho}I_{a^+}^{1-\eta_3}(v_3(a)) = c_3, \end{cases}$$

where the symbol ${}^{\rho}D_{a^+}^{\zeta,\zeta'}$, with $\zeta \in \{\alpha, \beta, \gamma, \delta\}$ and $\zeta' \in \{\alpha', \beta', \gamma', \delta'\}$ is the Hilfer-Katugampola derivative of order ζ, $(0 < \zeta < 1)$ and type ζ', $(0 \leq \zeta' \leq 1)$.
$c_0, c_1, c_2, c_3 \in \mathbb{R}$, $f \in C_{1-\eta,\rho}^{\eta}([a,b])$ and $\eta_i, i = 0, .., 3$ given by (2.16).
Now, the system can be defined as follows

$$\begin{cases} {}^{\rho}D_{a^+}^{\delta,\delta'}({}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'} v(t)))) \\ \quad = f(t, v(t), {}^{\rho}D_{a^+}^{\alpha,\alpha'} v(t), {}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'} v(t)), {}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'} v(t)))), \qquad t \in [a,b], \\ {}^{\rho}I_{a^+}^{1-\eta_0} v(a) = c_0, \\ {}^{\rho}I_{a^+}^{1-\eta_1}({}^{\rho}D_{a^+}^{\alpha,\alpha'} v(a)) = c_1, \\ {}^{\rho}I_{a^+}^{1-\eta_2}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'} v(a))) = c_2, \\ {}^{\rho}I_{a^+}^{1-\eta_3}({}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'} v(a)))) = c_3, \end{cases} \tag{0.4}$$

such that $f \in C_{1-\eta,\rho}^{\eta}([a,b])$ and $c_0, c_1, c_2, c_3 \in \mathbb{R}$.

This thesis is organized into three distinct chapters. The first Chapter serves as an introduction. Chapter 2 provides an exposition that establishes the existence and uniqueness of the solution the Snap problem (0.4). The third Chapter is devoted to the study of the solution stability to our problem.

- **First chapter** :
  In this chapter, it is pertinent to revisit fundamental definitions of fractional calculus that will prove to be conducive to our study. Specifically, we shall highlight special functions such as the Gamma function and Beta function, as delineated in references [23] and [24], as well as established techniques for fractional derivatives and integrals. Additionaly, we present some fixed point theorems, as documented in reference [16].

- **Second chapter** :
  In this chapter, we establish the existence and uniqueness of the solution to our problem (0.4) by applying Schauder fixed-point theorem and Banach contraction principle.

- **Third chapter** :
  We explore the stability of the solution in the sense of Ulam-Hyers and generalized Ulam-Hyers. Finally, we present an example illustrating the obtained results.

Chapter 1

Preliminaries

The present chapter is primarily intended to furnish a comprehensive exposition of the fundamental precepts underlying fractional calculus. These include the essential characteristics of specific functions, fractional integrals and derivatives, as well as several fixed point theorems that are considered crucial for the advancement of the remaining components of this work.

1.1 The Gamma function and Related Special Functions

1.1.1 The Gamma function

The Gamma function is a crucial function in fractional calculus that extends the concept of factorial n! to non-integer and complex values. This function is essential in various parts of this work, and thus, some important outcomes are discussed in this section.

Definition 1.1 [5, 23, 24] The most basic interpretation of the gamma function is simply a generalization factorial for all real numbers. Its definition is given by:

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt, \tag{1.1}$$

such that : $\Re(Z) > 0$.

Properties The Gamma function Γ satisfies the following properties:

- $\Gamma(z+1) = z\Gamma(z),\ z \in \mathbb{C}.$
- $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)},\ z \in \mathbb{Z}\,, \quad 0 < \Re(Z) < 1.$

- $\Gamma(2z) = \dfrac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\dfrac{1}{2}\right), \ z \in \mathbb{C}.$

- $\Gamma(n) = (n-1)\,!, \ n \in \mathbb{N}^*$

- $\Gamma\left(n+\frac{1}{2}\right) = \dfrac{(2n)\,!}{2^{2n}n\,!}\sqrt{\pi}, \ n \in \mathbb{N}.$

Commonly encountered Gamma function values

- $\Gamma(1) = \Gamma(2) = 1.$

- $\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-t^2}dt = \sqrt{\pi}$ (Gaussian integral).

- $\Gamma\left(\frac{-3}{2}\right) = \frac{4}{3}\sqrt{\pi}, \qquad \Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}.$

- $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}.$

1.1.2 The Beta function

Definition 1.2 [5, 23, 24] The Beta function is an Euler type integral given by the following definition,

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt, \qquad z, w \in \mathbb{C}. \tag{1.2}$$

Such that $\Re(z) > 0, \ \Re(w) > 0.$

Theorem 1.1 *The Beta function B satisfies the following properties:*

- *The Beta function is symmetric, i.e*

$$B(z,w) = B(w,z), \qquad \Re(z) > 0, \ \Re(w) > 0.$$

- $B(z,w) = B(z+1,w) + B(z,w+1), \qquad \Re(z) > 0, \ \Re(w) > 0.$

- $B(z,1) = \dfrac{1}{z}, \ \Re(z) > 0.$

- $B(z,w+1) = \dfrac{z}{w}B(z+1,w), \qquad \Re(z) > 0, \ \Re(w) > 0.$

Remark 1.1 The Beta function is linked to the Gamma function through the following formula,

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}. \tag{1.3}$$

1.1.3 The Mittag-Leffler function

Definition 1.3 [5, 23, 24] The Mittag-Leffler of one parameter is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \quad z \in \mathbb{C}, \quad \alpha > 0. \tag{1.4}$$

Definition 1.4 [5, 23, 24] The Mittag-Leffler of two parameters is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \quad z \in \mathbb{C}, \quad \alpha > 0, \quad \alpha > 0. \tag{1.5}$$

Remark 1.2

$$E_{1,1}(z) = E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$$

1.2 Fractional Integrals and Derivatives

1.2.1 Riemann-Liouville Fractional Integral and Derivative

Some of the original, and most used, forms of fractional operators were developed by Riemann and Liouville. Here we will summarise these definitions and list various useful properties. First we consider the Riemann-Liouville fractional integral. Let f be a continuous function on the interval $[a,b]$, and let us consider the integral

$$(I_a^1 f)(t) = \int_a^t f(s)ds,$$

The second integral of f is then given by

$$(I_a^2 f)(t) = \int_a^t \left(\int_a^s f(\tau)d\tau \right) ds.$$

Exchanging the order of integration we obtain

$$(I_a^2 f)(t) = \int_a^t (t-s)f(s)ds.$$

Comprehensively, the n-th iteration of the operator I can be written as follows

$$I_a^n f(t) = \int_a^t \mathrm{d}t_1 \int_a^{t_1} \mathrm{d}t_2 ... \int_a^{t_{n-1}} f(t_n)\mathrm{d}t_n = \frac{1}{(n-1)\,!} \int_a^t (t-s)^{n-1} f(s)ds, \tag{1.6}$$

Using the well-known property of the Gamma function, $\Gamma(n) = (n-1)\,!$, finally gives

$$I_a^n f(t) = \frac{1}{\Gamma(n)} \int_a^t (t-s)^{n-1} f(s)ds,$$

This formula makes sense even for non-integer values of n, and so we are able to define a fractional version of I, which is termed the Riemann-Liouville integral.

Definition 1.5 [5, 28]
Let f be a continuous function over $[a,b]$ and let $\alpha \in \mathbb{R}_+$, the integral

$$I^{\alpha}_{a+}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds,\ a \in \mathbb{R} \tag{1.7}$$

is called the left Riemann-Liouville fractional intagral of order α and,

$$I^{\alpha}_{b-}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (t-s)^{\alpha-1}f(s)ds,\ b \in \mathbb{R}, \tag{1.8}$$

is called the right Riemann-Liouville fractional integral of order α.

Proposition 1.1 *[28] Let $f : [a,b] \to \mathbb{R}$ be a continuous function, the Riemann-Liouville fractional integral has the following property*

$$I^{\alpha}_a[I^{\beta}_a f(t)] = I^{\alpha+\beta}_a f(t),\ \alpha, \beta > 0.$$

Moreover, we have

$$\frac{d}{dt}(I^{\alpha}_a f)(t) = I^{\alpha-1}_a f(t),\ \alpha > 0.$$

Definition 1.6 [5, 28] Let $\alpha > 0$ and $n \in \mathbb{N}^*$ such that $n-1 < \alpha < n$, the left Riemann-Liouville fractional derivative of order α of a function $f : [a,b] \mapsto \mathbb{R}$ is defined as follows

$${}^{RL}D^{\alpha}_{a+}f(t) = \left(\frac{d}{dt}\right)^n I^{n-\alpha}_{a+}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1}f(s)\mathrm{d}s. \tag{1.9}$$

the left Riemann-Liouville fractional derivative of order α of f is defined by

$${}^{RL}D^{\alpha}_{b-}f(t) = \left(-\frac{d}{dt}\right)^n I^{n-\alpha}_{b-}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dt}\right)^n \int_t^b (s-t)^{n-\alpha-1}f(s)\mathrm{d}s. \tag{1.10}$$

Theorem 1.2 *The Riemann-Liouville fractional derivative satisfies the following properties*

1. ***Linearity***

$${}^{RL}D^{\alpha}_{a+}(\lambda f(t) + \mu g(t)) = \lambda\ {}^{RL}D^{\alpha}_{a+}f(t) + \mu\ {}^{RL}D^{\alpha}_{a+}g(t). \tag{1.11}$$

typically, we have

$${}^{RL}D^{\alpha}_{a+}(\ {}^{RL}D^{\beta}_{a+}f)(t) \neq\ {}^{RL}D^{\beta}_{a+}(\ {}^{RL}D^{\alpha}_{a+}f)(t) \neq\ {}^{RL}D^{\alpha+\beta}_{a+}f(t). \tag{1.12}$$

2. ***Composition formulas***
Let $m-1 \leq \alpha < m$ and $n-1 \leq \beta < n$,

$${}^{RL}D^{\alpha}_{a+}(\ {}^{RL}D^{\beta}_{a+}f)(t) =\ {}^{RL}D^{\alpha+\beta}_{a+}f(t) - \sum_{j=1}^{n}[\ {}^{RL}D^{\beta-j}_{a+}f(t)]_{t=a}\frac{(t-a)^{-\alpha-j}}{\Gamma(-\alpha-j+1)}. \tag{1.13}$$

$${}^{RL}D^{\beta}_{a+}(\ {}^{RL}D^{\alpha}_{a+}f)(t) =\ {}^{RL}D^{\alpha+\beta}_{a+}f(t) - \sum_{j=1}^{m}[\ {}^{RL}D^{\alpha-j}_{a+}f(t)]_{t=a}\frac{(t-a)^{-\beta-j}}{\Gamma(-\beta-j+1)}. \tag{1.14}$$

3. *The Riemann-Liouville fractional derivative of a constant c is given by*

$$^{RL}D^{\alpha}_{a+}c = \frac{c}{\Gamma(1-\alpha)}(t-a)^{-\alpha}, \qquad t > a. \tag{1.15}$$

4. *The Riemann-Liouville fractional derivative of a power function $(t-a)^{\nu}$ for $\nu > -1$ is given by,*

$$^{RL}D^{\alpha}_{a+}(t-a)^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)}(t-a)^{\nu-\alpha}, \tag{1.16}$$

and

$$^{RL}D^{\alpha}_{a+}(t-a)^{\alpha-j} = 0, \qquad j = 1, 2, \cdots, \alpha+1. \tag{1.17}$$

1.2.2 Hadamard Fractional Integral and Derivative

Definition 1.7 [28] Let (a, b), $(0 \leq a \leq b \leq \infty)$ be a finite or an infinte interval. The Hadmard fractional integral of order α of a function g is defined by

$$I^{\alpha}_{a}g(t) = \frac{1}{\Gamma(\alpha)}\int_{a}^{t}\left(log\frac{t}{s}\right)^{\alpha-1}\frac{g(s)}{s}ds, \quad a \leq x \leq b. \tag{1.18}$$

Definition 1.8 [28] Let (a, b) ,$(0 \leq a \leq b \leq \infty)$ be a finite or an infinite interval. The Hadamard fractional derivative of order α of a function g is defined as follows,

$$D^{\alpha}_{a}g(t) = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^{n}\int_{a}^{t}\left(log\frac{t}{s}\right)^{n-\alpha-1}\frac{g(s)}{s}\,ds, \quad n = [\alpha]+1, a \leq x \leq b. \tag{1.19}$$

Lemma 1.1 *if $a,\ \alpha,\ \beta > 0$ then*

$$\left(D^{\alpha}_{a}\left(\log\frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log\frac{x}{a}\right)^{\beta-\alpha-1},$$

and

$$\left(I^{\alpha}_{a}\left(\log\frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(\log\frac{x}{a}\right)^{\beta+\alpha-1}.$$

1.2.3 Caputo Fractional Derivative

Definition 1.9 [5, 28] The Caputo fractional derivative of order $\alpha \in \mathbb{R}_{+}$ of a function $f \in C^{n}([a, b])$ is defined by

$$^{C}D^{\alpha}_{a+}f(t) = I^{n-\alpha}_{a^+}f^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-s)^{n-\alpha-1}f^{(n)}(s)ds, \quad t > a, \tag{1.20}$$

with $n-1 < \alpha < n$.

Theorem 1.3 *The Caputo fractional derivative satisfies the following properties*

1. ***Linearity***
   Let $n-1<\alpha<n,\ m,\ n\in\mathbb{N},\ \lambda,\mu\in\mathbb{C}$, *the Caputo fractional derivative is a linear operator*
   $$^{C}D^{\alpha}_{a^+}(\lambda f(t)+\mu g(t))=\lambda\ ^{C}D^{\alpha}_{a^+}f(t)+\mu\ ^{C}D^{\alpha}_{a^+}g(t).$$

2. ***Link with the Riemann-Liouville derivative***
   Let $\alpha>0$ *such that* $n-1<\alpha<n,(n\in\mathbb{N}^*)$, *we suppose that* f *is a function such that* $^{C}D^{\alpha}_{a^+}f(t)$ *and* $^{RL}D^{\alpha}_{a^+}f(t)$ *exist, then*
   $$^{C}D^{\alpha}_{a^+}f(t)=\ ^{RL}D^{\alpha}_{a^+}f(t)-\sum_{k=0}^{n-1}\frac{f^k(a)(t-a)^{k-a}}{\Gamma(k-\alpha+1)}.$$
   We deduce that if $f^k(a)=0$ *for* $k=0,1,2,...,n-1$, *we get* $^{C}D^{\alpha}_{a^+}f(t)=\ ^{RL}D^{\alpha}_{a^+}f(t)$.

3. ***Composition with the fractional integral operator***
   If f *is continuous function, we have*
   $$^{C}D^{\alpha}_{a^+}(I^{\alpha}_{a^+}f)=f\ \ \texttt{et}\ \ I^{\alpha}_{a^+}(\ ^{C}D^{\alpha}_{a^+}f(t))=f(t)-\sum_{k=0}^{n-1}\frac{f^k(a)(t-a)^k}{k\,!},$$

4. *The derivative of a constant funtion is null in the Caputo sense*
   $$^{C}D^{\alpha}_{a^+}\,c=0.$$

1.2.4 Hilfer Fractional Integral and Derivative

The first attempt towards the generalization of the fractional derivative operator was made by Hilfer [19] in the year 2000, who introduced the fractional derivative of two parameters.

Definition 1.10 [17, 45] The left-sided Hilfer fractional derivative of order $0<\gamma<1$ and $0\leq\beta\leq1$ of a function g is defined as,

$$D^{\beta,\gamma}_{a^+}g(t)=(I^{\beta(1-\gamma)}_{a^+}D(I^{(1-\beta)(1-\gamma)}g))(t), \tag{1.21}$$

where $D=\frac{d}{dt}$.

Remark 1.3 [45, 46]

- When $\beta=0$, $0<\gamma<1$ and $a=0$, the Hilfer fractional derivative corresponds to the classical Riemann-Liouville fractional derivative
  $$D^{\gamma,0}_{0^+}g(t)=\frac{d}{dt}I^{1-\gamma}_{0^+}g(t)=\ ^{RL}D^{\gamma}_{a}g(t).$$

- When $\beta=1$, $0<\gamma<1$ and $a=0$, the Hilfer fractional derivative corresponds to the classical Riemann-Liouville fractional derivative
  $$D^{\gamma,1}_{0^+}g(t)=\frac{d}{dt}I^{1-\gamma}_{0^+}g(t)=\ ^{C}D^{\gamma}_{a}g(t).$$

Completely continuous operators

Definition 1.11 [33] Let X, Y be Banach spaces and $\mathcal{F} : D \subset X \to Y$.

(a) The operator $\mathcal{F}$ is said to be bounded if it maps any bounded subset of D into a bounded subset of Y.

(b) The operator $\mathcal{F}$ is said to be completely continuous if it is continuous and maps any bounded subset of D into a relatively compact subset of Y.

It is clear that a continuous operator $\mathcal{F} : D \subset X \to Y$ is completely continuous if and only if for every bounded sequence (u_k) with $u_k \in D$, the sequence $(\mathcal{F}(u_k))$ has a convergent subsequence.

Notice that any completely continuous operator is a bounded operator.

1.3 Theorems

In the subsequent section, we shall proffer certain theorems that will prove advantageous in the ensuing content.

1.3.1 Arzela-Ascoli Theorem

Theorem 1.4 *[16] Let $Y = C([a, b])$ equipped with the norm*

$$\|u\| = \max_{t\in[a,b]} \mid u(t) \mid .$$

If M is a subset of Y, such that

(i) M est uniformely bounded, i.e $\exists r > 0, \|u\| \leq r, \forall u \in M$.

(ii) M is equicontinuous, that is

$$\forall \varepsilon > 0, \exists \delta > 0, \forall t_1, t_2 \in [a, b] \ \ \textit{tel que} \mid t_1 - t_2 \mid < \delta \textit{ et } u \in M \Longrightarrow \mid u(t_1) - u(t_2) \mid < \varepsilon.$$

Therefore, M is relatively compact.

1.3.2 Banach fixed point theorem

Theorem 1.5 *[16]*
Let E be a Banach space and $\mathcal{F} : E \to E$ a contraction operator. Then, there is a unique $u \in E$ such that $\mathcal{F}(u) = u$.

1.3.3 Schauder fixed point theorem

Theorem 1.6 *[16]*
Let X be a nonvoid closed subset of a Banach space E and $\mathcal{F} : X \to X$ a continuous mapping such that $\mathcal{F}(X) \subset E$ is relatively compact. Then $\mathcal{F}$ has at least fixed point in X.

1.3.4 Leray-Schauder fixed point theorem

Theorem 1.7 *[16] Let E be a Banach space and $\mathcal{F} : E \to E$ a completely continuous operator. Let*

$$Y = \{x \in E := \zeta \mathcal{F}(x) \quad for \quad a \quad certain \quad 0 < \zeta < 1\}.$$

Then, either the set Y is non bounded, or $\mathcal{F}$ has at least a fixed point.

Chapter 2

Study of a Snap problem in the Hilfer-Katugampola sense

In this chapter, we introduce the Hilfer-Katugampola fractional derivative [31]. This new formulation is a Hilfer-type fractional differentiation operator, particularly, it is an integer order derivative performing between generalized fractional integrals according to Katugampola [27]. This new fractional derivative interpolates the Hilfer, Hilfer-Hadamard, Riemann-Liouville, Hadamard, Caputo, Caputo-Hadamard and generalized Caputo-type fractional derivatives, as well as the Weyl and Liouville fractional derivatives for particular cases of integration extremes [14, 18, 20].
The objective of this chapter is to examine the Snap boundary problem in the Hilfer-Katugampola sense. This study aims to provide evidence that a unique solution to the problem at hand can be found within the confines of the continuous functions with weighted spaces. In order to accomplish this, it is necessary to utilize two fundamental mathematical principles, namely Shauder's fixed-point theorem and Banach's contraction principle.

2.1 Preliminaries

In this section, fractional calculus in the Hilfer-Katugampola sense will be extensively examined with an extensive and deep analysis in order to gain a great understanding and appreciation of the aforementioned concept principles.

2.1.1 Function Spaces

In order to introduce the Hilfer-Katugampola fractional derivatives, in this section, we propose the function spaces and some results involving these spaces that are adequate for such definition.

Definition 2.1 [25, 30] Let $[a, b]$ be a finite interval on the half-axis $\mathbb{R}^+$ and the parameters $\rho > 0$ and $0 \leq \gamma < 1$.
(1) The weighted space $C_{\gamma,\rho}[a, b]$ of functions g on $(a, b]$ is defined by

$$C_{\gamma,\rho}[a,b] = \left\{ g : [a,b] \to \mathbb{R} : \left(\frac{t^\rho - a^\rho}{\rho}\right)^\gamma g(t) \in C[a,b] \right\},$$

where $0 \leq \gamma < 1$ and with the norm

$$\|g\|_{C_{\gamma,\rho}} = \left\| \left(\frac{t^\rho - a^\rho}{\rho}\right)^\gamma g(t) \right\|_C = \max_{t\in[a,b]} \left| \left(\frac{t^\rho - a^\rho}{\rho}\right)^\gamma g(t) \right|,$$

where $C_{0,\rho}[a, b] = C[a, b]$.

(2) Let $\delta_\rho = (t^\rho \frac{d}{dt})$. For $n \in \mathbb{N}$ we denote by $C^n_{\delta_\rho,\gamma}[a, b]$ the Banach space of functions g which are continuously differentiable on $[a, b]$, with operator δ_ρ, up to order $(n - 1)$ and which have the derivative $\delta^n_\rho g$ of order n on $(a, b]$ such that $\delta^n_\rho g \in C_{\gamma,\rho}[a, b]$, that is,

$$C^n_{\delta_\rho,\gamma}[a,b] = \left\{ g : (a,b] \to \mathbb{R} : \delta^k_\rho \in C[a,b], k = 0, 1 \ldots, n-1, \delta^n_\rho g \in C_{\gamma,\rho}[a,b] \right\}$$

where $n \in \mathbb{N}$, with the norms

$$\|g\|_{C^n_{\delta_\rho,\gamma}} = \sum_{k=0}^{n-1} \left\|\delta^k_\rho g\right\|_C + \left\|\delta^n_\rho g\right\|_{C_{\gamma,\rho}}, \quad \|g\|_{C^n_{\delta_\rho}} = \sum_{k=0}^{n} \max_{x\in[a,b]} \left|\delta^k_\rho g(t)\right|.$$

For $n = 0$, we have

$$C^0_{\delta_\rho,\gamma}[a,b] = C_{\gamma,\rho}[a,b].$$

Lemma 2.1 *[30] Let $0 \leq \gamma < 1$, $a < c < b$, $g \in C_{\gamma,\rho}[a, c]$, $g \in C[c, b]$ and g continuous at c. Then, $g \in C_{\gamma,\rho}[a, c]$.*

2.1.2 Hilfer-Katugampola Fractional Integral and Derivative

Definition 2.2 [27, 26, 30] Let $\alpha, \rho, c \in \mathbb{R}$, such that $\alpha > 0$ and $\rho > 0$. The generalized left-sided fractional integral of $g \in X^\rho_c(a, b)$ of order $\alpha \in \mathbb{C}(Re(\alpha) > 0)$ is defined by

$$^\rho I^\alpha_{a^+} g(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} g(s) ds, \qquad t > a,\ \rho > 0, \tag{2.1}$$

Similarly, the right-sided fractional integral $^\rho I^\alpha_{b^-} g(.)$ is defined by

$$^\rho I^\alpha_{b^-} g(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_t^b (s^\rho - t^\rho)^{\alpha-1} s^{\rho-1} g(s) ds, \qquad t < b,\ \rho > 0, \tag{2.2}$$

Next, we introduce the generalized fractional derivatives corresponding to the fractional integrals, (2.1) and (2.2), respectively.

Definition 2.3 [27, 26, 30] Let $\alpha \in \mathbb{C}$, $Re(\alpha) \geq 0$, $n = [Re(\alpha)] + 1$ and $\rho > 0$. The generalized fractional derivatives, corresponding to the generalized fractional integrals (2.1) and (2.2), are defined for $0 \leq a < t < b < \infty$ by :

$$({}^{\rho}D_{a^+}^{\alpha}g)(t) = \frac{\rho^{\alpha-n-1}}{\Gamma(n-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t (t^{\rho}-s^{\rho})^{n-\alpha+1}s^{\rho-1}g(s)ds, \tag{2.3}$$

and

$$({}^{\rho}D_{b^-}^{\alpha}g)(t) = \frac{\rho^{\alpha-n-1}}{\Gamma(n-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^n \int_t^b (t^{\rho}-s^{\rho})^{n-\alpha+1}s^{\rho-1}g(s)ds, \tag{2.4}$$

if the integrals exist.

Lemma 2.2 *[30] For $\alpha > 0$, ${}^{\rho}I_{a^+}^{\alpha}$ maps $C[a,b]$ into $C[a,b]$.*

Lemma 2.3 *[30] For $\alpha > 0$ and $0 \leq \gamma < 1$, then ${}^{\rho}I_{a^+}^{\alpha}$ is bounded from $C_{\gamma,\rho}[a,b]$ into $C_{\gamma,\rho}[a,b]$.*

Lemma 2.4 *[30] For $\alpha > 0$ and $0 \leq \gamma < 1$, then ${}^{\rho}I_{a^+}^{\alpha}$ is bounded from $C_{\gamma,\rho}[a,b]$ into $C[a,b]$.*

Lemma 2.5 *[4, 30] Let $t > a$, ${}^{\rho}I_{a^+}^{\alpha}$ and ${}^{\rho}D_{a^+}^{\alpha}$ as defined in (2.1) and (2.3) respectively. For $t > a$ then, for $\alpha \geq 0$ and $\zeta > 0$*

$$\left[{}^{\rho}I_{a^+}^{\alpha}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\zeta-1}\right](x) = \frac{\Gamma(\zeta)}{\Gamma(\alpha+\zeta)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha+\zeta-1} \tag{2.5}$$

$$\left[{}^{\rho}D_{a^+}^{\alpha}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\zeta-1}\right](x) = 0, \qquad 0 < \alpha < 1. \tag{2.6}$$

Proof See [4]. ■

Lemma 2.6 *[27, 30] Let $\alpha > 0$, $\beta > 0$, $p \geq 1$, $[a,b] \subset \mathbb{R}_+^*$ and $c \in \mathbb{R}$. Then, for $g \in X_c^p(a,b)$, $\rho > 0$*

$${}^{\rho}I_{a^+}^{\alpha}({}^{\rho}I_{a^+}^{\beta}g) = {}^{\rho}I_{a^+}^{\alpha+\beta}g \quad \text{and} \quad {}^{\rho}D_{a^+}^{\alpha}({}^{\rho}D_{a^+}^{\beta}g) = {}^{\rho}D_{a^+}^{\alpha+\beta}g.$$

Proof See [27, 26, 30] ■

Lemma 2.7 *[30] Let $0 < \alpha < 1$, $0 \leq \gamma < 1$. If $g \in C_{\gamma}[a,b]$ and ${}^{\rho}I_{a^+}^{1-\alpha}g \in C_{\gamma}^1[a,b]$, then :*

$$({}^{\rho}I_{a^+}^{\alpha}\,{}^{\rho}D_{a^+}^{\alpha}g)(t) = g(t) - \frac{({}^{\rho}I_{a^+}^{1-\alpha}g)(a)}{\Gamma(\alpha)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha-1}, \tag{2.7}$$

for all $t \in [a,b]$.

Proof The proof uses integration by parts, with $u = (t^\rho - s^\rho)^{\alpha-1}$ and $dv = \frac{d}{dt}({}^\rho I_{a^+}^{1-\alpha} g)(t)dt$. ■

Lemma 2.8 *[27, 30] Let $\alpha > 0$, $0 \leq \gamma < 1$ and $g \in C_\gamma[a,b]$, then*

$$({}^\rho D_{a^+}^\alpha {}^\rho I_{a^+}^\alpha g)(t) = g(t), \quad \forall t \in [a,b]$$

Proof See [27]. ■

Lemma 2.9 *[30] Let $0 < a < b < \infty$, $\alpha > 0$, $0 \leq \gamma < 1$ and $g \in C_{\gamma,\rho}([a,b])$. If $\alpha > \gamma$ then ${}^\rho I_{a^+}^\alpha(g)$ is continuous on [a,b] and we have,*

$$({}^\rho I_{a^+}^\alpha g)(a) = \lim_{t \to a^+} ({}^\rho I_{a^+}^\alpha) g(t) = 0.$$

Proof Since $g \in C_{\gamma,\rho}([a,b])$ then $\left(\frac{t^\rho - a^\rho}{\rho}\right) g(t)$ is continuous on $[a,b]$ and

$$\left| \left(\frac{t^\rho - a^\rho}{\rho}\right) g(t) \right| \leq M, \qquad t \in [a,b],$$

For some positive constant M. Consequently,

$$|({}^\rho I_{a^+}^\alpha g)(t)| \leq M \left[{}^\rho I_{a^+}^\alpha \left(\frac{t^\rho - a^\rho}{\rho}\right)^{-\gamma} \right](x), \qquad t \in [a,b],$$

and by Lemma 2.5, we can write

$$|({}^\rho I_{a^+}^\alpha g)(t)| \leq M \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-\gamma+1)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-\gamma}. \tag{2.8}$$

As $\alpha > \gamma$, the right side of (2.8) goes to zero when $t \to a^+$. ■

Definition 2.4 [30] Let the orders α, β satisfy $0 < \alpha \leq 1$ and $0 \leq \beta \leq 1$.
The Hilfer-Katugampola fractional derivative (left-sided/right-sided), with respect to t, $\rho > 0$ of a function $g \in C_{1-\gamma,\rho}$ is defined by

$$\begin{aligned} \left({}^\rho D_{a^\pm}^{\alpha,\beta} g\right)(t) &= \left(\pm {}^\rho I_{a^\pm}^{\beta(1-\alpha)} \left(t^{\rho-1} \frac{d}{dt}\right) {}^\rho I_{a^\pm}^{(1-\beta)(1-\alpha)} g\right)(t). \\ &= \left(\pm {}^\rho I_{a^\pm}^{\beta(1-\alpha)} \delta_\rho \, {}^\rho I_{a^\pm}^{(1-\beta)(1-\alpha)} g\right)(t) \end{aligned} \tag{2.9}$$

where I is the generalized fractional integral given in Definition 2.2.
Properties [27, 30] We present some properties of the operator ${}^\rho D_{a^+}^{\alpha,\beta}$.

- (P_1) : The operator ${}^\rho D_{a^+}^{\alpha,\beta}$ can be written as

$${}^\rho D_{a^+}^{\alpha,\beta} = {}^\rho I_{a^+}^{\beta(1-\alpha)} \delta_\rho \, {}^\rho I_{a^+}^{1-\gamma} = {}^\rho I_{a^+}^{\beta(1-\alpha)} \, {}^\rho D_{a^+}^\gamma, \quad \gamma = \alpha + \beta(1-\alpha).$$

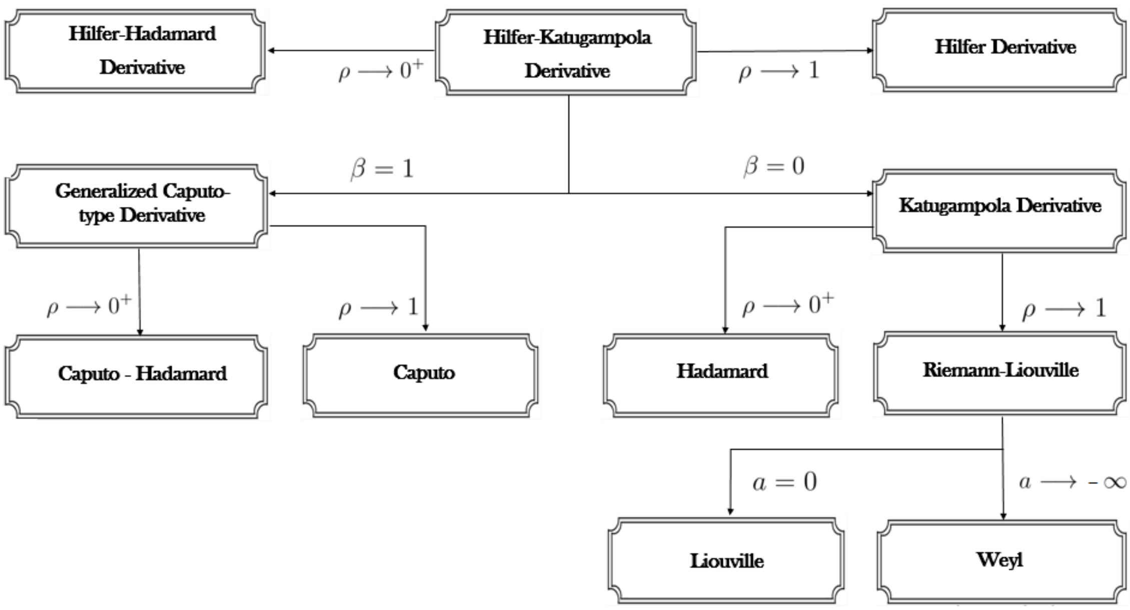
Proof From definition 2.3, we have

$$\begin{aligned}({}^{\rho}D^{\alpha,\beta}_{a^+}g)(t) &= {}^{\rho}I^{\beta(1-\alpha)}_{a^+}\left(t^{1-\rho}\frac{d}{dt}\right)\left(\frac{\rho^{1-(1-\beta)(1-\alpha)}}{\Gamma[(1-\beta)(1-\alpha)]}\int_a^t \frac{s^{\rho}}{(t^{\rho}-s^{\rho})^{1-(1-\beta)(1-\alpha)}}g(s)ds\right)\\ &= {}^{\rho}I^{\beta(1-\alpha)}_{a^+}\left[\frac{\rho^{1-(1-\beta)(1-\alpha)}}{\Gamma[(1-\beta)(1-\alpha)-1]}\int_a^t \frac{s^{\rho}}{(t^{\rho}-s^{\rho})^{1+\alpha+\beta-\alpha\beta}}g(s)ds\right](t)\\ &= ({}^{\rho}I^{\beta(1-\alpha)}_{a^+}\,{}^{\rho}D^{\gamma}_{a^+}g)(t).\end{aligned}$$

■

- (P_2) : The fractional derivative ${}^{\rho}D^{\alpha,\beta}_{a^+}$ is an interpolator of the following fractional derivatives
  - Hilfer fractional derivative when $(\rho \to 1)$ [18].
  - Hilfer-Hadamard fractional derivative when $(\rho \to 0)$ [24],
  - Generalized fractional derivative when $(\beta = 0)$ [26],
  - Caputo-type fractional derivative when $(\beta = 1)$ [32],
  - Riemann-Liouville fractional derivative when $(\beta = 0, \rho \to 1)$ [28],
  - Hadamard fractional derivative when $(\beta = 0, \rho \to 0)$ [28],
  - Caputo fractional derivative when $(\beta = 1, \rho \to 1)$ [28],
  - Caputo-Hadamard fractional derivative when $(\beta = 1, \rho \to 0)$ [15],
  - Liouville fractional derivative when $(\beta = 0, \rho \to 1, a = 0)$ [28],
  - Weyl fractional derivative $(\beta = 0, \rho \to, a = -\infty)$ [20].

The diagram presented below serves as a visual representation that supports the aforementioned facts.



- (P_3) : We consider the following parameters $\alpha, \beta, \gamma, \rho$ satisfying

$$\gamma = \alpha + \beta(1-\alpha), \qquad 0 < \alpha, \beta, \gamma < 1, \qquad 0 \leq \rho < 1.$$

Thus, we define the spaces

$$C^{\alpha,\beta}_{1-\gamma,\rho}[a,b] = \{g \in C_{1-\gamma,\rho}[a,b], \quad {}^{\rho}D^{\alpha,\beta}_{a^+} g \in C_{\mu,\rho}[a,b]\}$$

$$C^{\gamma}_{1-\gamma,\mu}[a,b] = \{g \in C_{1-\gamma,\mu}[a,b], \quad {}^{\rho}D^{\gamma}_{a+} g \in C_{1-\gamma,\rho}[a,b]\}$$

where $C_{\mu,\rho}[a,b]$ and $C_{1-\gamma,\rho}[a,b]$ are weighted spaces of continuous functions on $(a,b]$ defined by item (2) in Definition 2.1.
Since ${}^{\rho}D^{\alpha,\beta}_{a^+} = {}^{\rho} I^{\gamma(1-\alpha)}_{a^+} {}^{\rho}D^{\gamma}_{a^+} g$, it follows from Lemma (2.3)

$$C^{\gamma}_{1-\gamma,\mu}[a,b] \subset C^{\alpha,\beta}_{1-\gamma,\mu}[a,b]$$

Lemma 2.10 *[30] Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta(1-\alpha)$. If $g \in C^{\gamma}_{1-\gamma}[a,b]$, then*

$${}^{\rho}I^{\gamma}_{a^+} {}^{\rho}D^{\gamma}_{a^+} g = {}^{\rho}I^{\alpha}_{a^+} {}^{\rho}D^{\alpha,\beta}_{a^+} g \tag{2.10}$$

and

$${}^{\rho}D^{\gamma}_{a^+} {}^{\rho}I^{\alpha}_{a^+} g = {}^{\rho}D^{\beta(1-\alpha)}_{a^+} g. \tag{2.11}$$

Proof First, we prove (2.10). By using both Lemma 2.5 and Property (P_1), we get

$${}^{\rho}I^{\gamma}_{a^+} {}^{\rho}D^{\gamma}_{a^+} = {}^{\rho}I^{\gamma}_{a^+} {}^{\rho}I^{-\beta(1-\alpha)}_{a^+} {}^{\rho}D^{\alpha,\beta}_{a^+} g = {}^{\rho}I^{\alpha+\beta-\alpha\beta}_{a^+} {}^{\rho}I^{-\beta+\alpha\beta}_{a^+} {}^{\rho}D^{\alpha,\beta}_{a^+} g = {}^{\rho}I^{\alpha}_{a^+} {}^{\rho}D^{\alpha,\beta}_{a^+} g.$$

In order to prove (2.11), we use Definition 2.4 and Lemma 2.5 to obtain

$${}^{\rho}D^{\gamma}_{a^+} {}^{\rho}I^{\alpha}_{a^+} g = \delta_{\rho} {}^{\rho}I^{1-\gamma}_{a^+} {}^{\rho}I^{\alpha}_{a^+} g = \delta_{\rho} {}^{\rho}I^{1-\beta+\alpha\beta}_{a^+} g = \delta_{\rho} {}^{\rho}I^{1-\beta(1-\alpha)}_{a^+} g = {}^{\rho}D^{\beta(1-\alpha)}_{a^+} g.$$

■

Lemma 2.11 *Let $g \in L(a,b)$. If ${}^{\rho}D^{\beta(1-\alpha)}_{a^+} g$ exists on $L(a,b)$, then*

$${}^{\rho}D^{\alpha,\beta}_{a^+} {}^{\rho}I^{\alpha}_{a^+} g = {}^{\rho}I^{\beta(1-\alpha)}_{a^+} {}^{\rho}D^{\beta(1-\alpha)}_{a^+} g. \tag{2.12}$$

Proof From (2.6), Definition 2.3 and Definition 2.4, we get

$${}^{\rho}D^{\alpha,\beta}_{a^+} {}^{\rho}I^{\alpha}_{a^+} g = {}^{\rho}I^{\beta(1-\alpha)}_{a^+} {}^{\rho}D^{\gamma}_{a^+} {}^{\rho}I^{\alpha}_{a^+} g = {}^{\rho}I^{\beta(1-\alpha)}_{a^+} \delta_{\rho} {}^{\rho}I^{1-\gamma}_{a^+} {}^{\rho}I^{\alpha}_{a^+} g = {}^{\rho}I^{\beta(1-\alpha)}_{a^+} \delta_{\rho} {}^{\rho}I^{1-\beta(1-\alpha)}_{a^+} g.$$

Thus,

$${}^{\rho}D^{\alpha,\beta}_{a^+} {}^{\rho}I^{\alpha}_{a^+} g = {}^{\rho}I^{\beta(1-\alpha)}_{a^+} {}^{\rho}D^{\beta(1-\alpha)}_{a^+} g.$$

■

Lemma 2.12 *Let* $0<\alpha<1$, $0\leq\beta\leq 1$ *and* $\gamma=\alpha+\beta(1-\alpha)$. *If* $g\in C_{1-\gamma}[a,b]$ *and* ${}^{\rho}I_{a^+}^{1-\beta(1-\alpha)}\in C_{1-\gamma}^{1}[a,b]$, *then* ${}^{\rho}D_{a^+}^{\alpha,\beta}{}^{\rho}I_{a^+}^{\alpha}g$ *exists on* $(a,b])$ *and*

$$ {}^{\rho}D_{a^+}^{\alpha,\beta}{}^{\rho}I_{a^+}^{\alpha}g=g, \qquad t\in[a,b]. \tag{2.13}$$

Proof Using Lemma (2.7), Lemma (2.3) and Lemma (2.11), we obtain

$$\begin{aligned}({}^{\rho}D_{a^+}^{\alpha,\beta}{}^{\rho}I_{a^+}^{\alpha}g)(t) &= ({}^{\rho}I_{a^+}^{\beta(1-\alpha)}{}^{\rho}D_{a^+}^{\beta(1-\alpha)}g)(t)\\ &= g(t)-\frac{{}^{\rho}I_{a^+}^{1-\beta(1-\alpha)}g(a)}{\Gamma(\alpha+\beta)}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\beta(1-\alpha)-1}\\ &= g(t), \qquad t\in(a,b].\end{aligned} \tag{2.14}$$

■

2.2 Problem definition

We consider the following fractional differential problem,

$$\begin{cases} {}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t) = v_1(t), \quad t\in[a,b], & {}^{\rho}I_{a^+}^{1-\eta_0}v(a)=c_0,\\ {}^{\rho}D_{a^+}^{\beta,\beta'}v_1(t)=v_2(t), & {}^{\rho}I_{a^+}^{1-\eta_1}(v_1(a))=c_1,\\ {}^{\rho}D_{a^+}^{\gamma,\gamma'}v_2(t)=v_3(t), & {}^{\rho}I_{a^+}^{1-\eta_2}(v_2(a))=c_2,\\ {}^{\rho}D_{a^+}^{\delta,\delta'}v_3(t) = f(t,v(t),v_1(t),v_2(t),v_3(t)), & {}^{\rho}I_{a^+}^{1-\eta_3}(v_3(a))=c_3,\end{cases}$$

where the symbol ${}^{\rho}D_{a^+}^{\zeta,\zeta'}$, with $\zeta\in\{\alpha,\beta,\gamma,\delta\}$ and $\zeta'\in\{\alpha',\beta',\gamma',\delta'\}$ is the Hilfer-Katugampola derivative of order ζ, $(0<\zeta<1)$ and type ζ', $(0\leq\zeta'\leq 1)$.
$c_0,c_1,c_2,c_3\in\mathbb{R}$, $f\in C_{1-\eta,\rho}^{\eta}([a,b])$ and $\eta_i, i=0,..,3$ given by (2.16).
Now, the system can be defined as follows

$$\begin{cases} {}^{\rho}D_{a^+}^{\delta,\delta'}({}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))\\ \qquad = f(t,v(t),{}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t),{}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)),{}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))), \qquad t\in[a,b],\\ {}^{\rho}I_{a^+}^{1-\eta_0}v(a)=c_0,\\ {}^{\rho}I_{a^+}^{1-\eta_1}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(a))=c_1,\\ {}^{\rho}I_{a^+}^{1-\eta_2}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(a)))=c_2,\\ {}^{\rho}I_{a^+}^{1-\eta_3}({}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(a))))=c_3,\end{cases} \tag{2.15}$$

such that $f\in C_{1-\eta,\rho}^{\eta}([a,b])$ and $c_0,c_1,c_2,c_3\in\mathbb{R}$.

In the following, we consider the parameters $\alpha,\beta,\gamma,\delta,\alpha',\beta',\gamma',\delta'$ and μ satisfying,

$$\eta_0=\alpha+\alpha'(1-\alpha), \qquad \eta_1=\beta+\beta'(1-\beta), \qquad \eta_2=\gamma+\gamma'(1-\gamma), \qquad \eta_3=\delta+\delta'(1-\delta). \tag{2.16}$$

Such that,

$$\eta_0, \eta_1, \eta_2, \eta_3 \in [0,1), \qquad \alpha, \beta, \gamma, \delta \in (0,1), \qquad \alpha^{'}, \beta^{'}, \gamma^{'}, \delta^{'} \in [0,1]$$

and

$$\mu \in [0,1).$$

We put : $\eta = \eta_0 + \eta_1 + \eta_2 + \eta_3$, such that $\eta \in [0,1)$

$$C_{1-\eta}^{\alpha+\beta+\gamma+\delta,\alpha^{'}+\beta^{'}+\gamma^{'}+\delta^{'}}([a,b]) = \{f \in C_{1-\eta,\rho}([a,b]), \quad {}^{\rho}D_{a^+}^{\alpha+\beta+\gamma+\delta,\alpha^{'}+\beta^{'}+\gamma^{'}+\delta^{'}} f \in C_{\mu,\rho}([a,b])\}$$

and

$$C_{1-\eta}^{\eta}([a,b]) = \{f \in C_{1-\eta,\rho}([a,b]), \quad {}^{\rho}D_{a^+}^{\eta} f \in C_{1-\eta,\rho}([a,b])\}.$$

It is obvious that,

$$C_{1-\eta}^{\eta}([a,b]) \subset C_{1-\eta}^{\alpha+\beta+\gamma+\delta,\alpha^{'}+\beta^{'}+\gamma^{'}+\delta^{'}}([a,b])$$

In order to prove the main theorems of existence and uniqueness of the solution of the fractional Snap problem (2.15), we present the following key lemma, which describes the corresponding integral equation.

Lemma 2.13 *Let* $\eta = (\alpha+\beta+\gamma+\delta) + (\alpha^{'}+\beta^{'}+\gamma^{'}+\delta^{'}) - (\alpha\alpha^{'}+\beta\beta^{'}+\gamma\gamma^{'}+\delta\delta^{'})$ *where* $0 < (\alpha+\beta+\gamma+\delta) < 1$ *and* $0 \leq \alpha^{'}+\beta^{'}+\gamma^{'}+\delta^{'} \leq 1$. *If* $f : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ *is a function such that*

$$f(\cdot, v(\cdot), {}^{\rho}D_{a^+}^{\alpha,\alpha^{'}} v(\cdot), {}^{\rho}D_{a^+}^{\beta,\beta^{'}}({}^{\rho}D_{a^+}^{\alpha,\alpha^{'}} v(\cdot)), {}^{\rho}D_{a^+}^{\gamma,\gamma^{'}}({}^{\rho}D_{a^+}^{\beta,\beta^{'}}({}^{\rho}D_{a^+}^{\alpha,\alpha^{'}} v(\cdot)))) \in C_{1-\eta,\rho}[a,b]$$

for all $v \in C_{1-\eta,\rho}[a,b]$. *A function* $v \in C_{1-\eta}^{\eta}([a,b])$ *is the solution of the Snap problem* (2.15) *if and only if* v *satisfies the following integral equation*

$$\begin{aligned} v(t) = {} & \frac{c_0}{\Gamma(\eta_0)} \left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\eta_0-1} + \frac{c_1}{\Gamma(\eta_1)} {}^{\rho}I_{a^+}^{\alpha} \left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\eta_1-1} \\ & + \frac{c_2}{\Gamma(\eta_2)} {}^{\rho}I_{a^+}^{\alpha+\beta} \left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\eta_2-1} + \frac{c_3}{\Gamma(\eta_3)} {}^{\rho}I_{a^+}^{\alpha+\beta+\gamma} \left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\eta_3-1} \\ & + {}^{\rho}I_{a^+}^{\alpha+\beta+\gamma+\delta} f(t, v(t), {}^{\rho}D_{a^+}^{\alpha,\alpha^{'}} v(t), {}^{\rho}D_{a^+}^{\beta,\beta^{'}}({}^{\rho}D_{a^+}^{\alpha,\alpha^{'}} v(t)), {}^{\rho}D_{a^+}^{\gamma,\gamma^{'}}({}^{\rho}D_{a^+}^{\beta,\beta^{'}}({}^{\rho}D_{a^+}^{\alpha,\alpha^{'}} v(t)))). \end{aligned} \tag{2.17}$$

Proof

We start by showing the implication in this sense ($\Longrightarrow$).

Consider v(t) satisfying the snap problem. Hence, by applying the δ-th integral ${}^{\rho}I_{a^+}^{\delta}$ to both sides of the first equation of problem (2.15) and by using Lemma 2.10, we obtain

$$\begin{aligned} & {}^{\rho}I_{a^+}^{\delta}({}^{\rho}D_{a^+}^{\delta,\delta^{'}}({}^{\rho}D_{a^+}^{\gamma,\gamma^{'}}({}^{\rho}D_{a^+}^{\beta,\beta^{'}}({}^{\rho}D_{a^+}^{\alpha,\alpha^{'}} v(t))))) \\ & \qquad = {}^{\rho}I_{a^+}^{\delta} f(t, v(t), {}^{\rho}D_{a^+}^{\alpha,\alpha^{'}} v(t), {}^{\rho}D_{a^+}^{\beta,\beta^{'}}({}^{\rho}D_{a^+}^{\alpha,\alpha^{'}} v(t)), {}^{\rho}D_{a^+}^{\gamma,\gamma^{'}}({}^{\rho}D_{a^+}^{\beta,\beta^{'}}({}^{\rho}D_{a^+}^{\alpha,\alpha^{'}} v(t)))) \end{aligned}$$

$$
\begin{aligned}
&{}^{\rho}I^{\eta_3}_{a^+}({}^{\rho}D^{\eta_3}_{a^+}({}^{\rho}D^{\gamma,\gamma'}_{a^+}({}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)))))\\
&\qquad= {}^{\rho}I^{\delta}_{a^+}f(t,v(t),{}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t),{}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)),{}^{\rho}D^{\gamma,\gamma'}_{a^+}({}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)))).
\end{aligned}
$$

Therefore, according to Lemma 2.7 we have

$$
\begin{aligned}
{}^{\rho}D^{\gamma,\gamma'}_{a^+}({}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)))) &= \frac{{}^{\rho}I^{1-\eta_3}_{a_+}({}^{\rho}D^{\gamma,\gamma'}_{a^+}({}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v)))(a)}{\Gamma(\eta_3)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\eta_3-1}\\
&+ {}^{\rho}I^{\delta}_{a^+}f(t,v(t),{}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t),{}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)),{}^{\rho}D^{\gamma,\gamma'}_{a^+}({}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)))).
\end{aligned}
$$

Such that,

$$\eta_3 = \delta + \delta'(1-\delta).$$

By using the fourth boundary condition of the problem (2.15), we get

$${}^{\rho}I^{1-\eta_3}_{a_+}({}^{\rho}D^{\gamma,\gamma'}_{a^+}({}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v)))(a) = c_3.$$

Thus, we have

$$
\begin{aligned}
{}^{\rho}D^{\gamma,\gamma'}_{a^+}({}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)))) &= \frac{c_3}{\Gamma(\eta_3)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\eta_3-1}\\
&+ {}^{\rho}I^{\delta}_{a^+}f(t,v(t),{}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t),{}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)),{}^{\rho}D^{\gamma,\gamma'}_{a^+}({}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)))). \quad (2.18)
\end{aligned}
$$

Next, we apply the γ-th integral to (2.18) using Lemma 2.6 and Lemma 2.10

$$
\begin{aligned}
&{}^{\rho}I^{\gamma}_{a^+}({}^{\rho}D^{\gamma,\gamma'}_{a^+}({}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t))))\\
&\qquad= {}^{\rho}I^{\gamma+\delta}_{a^+}f(t,v(t),{}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t),{}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)),{}^{\rho}D^{\gamma,\gamma'}_{a^+}({}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)))).
\end{aligned}
$$

$$
\begin{aligned}
&{}^{\rho}I^{1-\eta_2}_{a^+}({}^{\rho}D^{1-\eta_2}_{a^+}({}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t))))\\
&\qquad= {}^{\rho}I^{\gamma+\delta}_{a^+}f(t,v(t),{}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t),{}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)),{}^{\rho}D^{\gamma,\gamma'}_{a^+}({}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)))).
\end{aligned}
$$

By using Lemma 2.7, we get

$$
\begin{aligned}
{}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)) &= \frac{{}^{\rho}I^{1-\eta_2}_{a_+}({}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v))(a)}{\Gamma(\eta_2)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\eta_2-1} + \frac{c_3}{\Gamma(\eta_3)}{}^{\rho}I^{\gamma}_{a_+}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\eta_3-1}\\
&+ {}^{\rho}I^{\gamma+\delta}_{a^+}f(t,v(t),{}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t),{}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)),{}^{\rho}D^{\gamma,\gamma'}_{a^+}({}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)))).
\end{aligned}
$$

Where,

$$\eta_2 = \gamma + \gamma'(1-\gamma).$$

It follows from the third boundary condition of (2.15)

$${}^{\rho}I^{1-\eta_2}_{a_+}({}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v))(a) = c_2,$$

and

$$
\begin{aligned}
{}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)) &= \frac{c_2}{\Gamma(\eta_2)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_2-1} + \frac{c_3}{\Gamma(\eta_3)}{}^{\rho}I^{\gamma}_{a_+}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_3-1} \\
&+ {}^{\rho}I^{\gamma+\delta}_{a^+}f(t,v(t),{}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t),{}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)),{}^{\rho}D^{\gamma,\gamma'}_{a^+}({}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)))). \quad (2.19)
\end{aligned}
$$

Next, we apply the β-th integral to (2.19) using Lemma 2.6 and Lemma 2.10

$$
\begin{aligned}
&{}^{\rho}I^{\beta}_{a^+}({}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t))) \\
&\quad = {}^{\rho}I^{\beta+\gamma+\delta}_{a^+}f(t,v(t),{}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t),{}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)),{}^{\rho}D^{\gamma,\gamma'}_{a^+}({}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)))).
\end{aligned}
$$

$$
\begin{aligned}
&{}^{\rho}I^{1-\eta_1}_{a^+}({}^{\rho}D^{1-\eta_1}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t))) = \\
&\qquad {}^{\rho}I^{\beta+\gamma+\delta}_{a^+}f(t,v(t),{}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t),{}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)),{}^{\rho}D^{\gamma,\gamma'}_{a^+}({}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)))).
\end{aligned}
$$

According to Lemma 2.7, we have

$$
\begin{aligned}
{}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t) &= \frac{{}^{\rho}I^{1-\eta_1}_{a_+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v)(a)}{\Gamma(\eta_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_1-1} + \frac{c_2}{\Gamma(\eta_2)}{}^{\rho}I^{\beta}_{a_+}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_2-1} \\
&+ \frac{c_3}{\Gamma(\eta_3)}{}^{\rho}I^{\beta+\gamma}_{a_+}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_3-1} \\
&+ {}^{\rho}I^{\beta+\gamma+\delta}_{a^+}f(t,v(t),{}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t),{}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)),{}^{\rho}D^{\gamma,\gamma'}_{a^+}({}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)))).
\end{aligned}
$$

Such that

$$\eta_1 = \beta + \beta'(1-\beta).$$

The second boundary condition of (2.15) implies

$${}^{\rho}I^{1-\eta_1}_{a_+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v)(a) = c_1.$$

Thus,

$$
\begin{aligned}
{}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t) &= \frac{c_1}{\Gamma(\eta_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_1-1} + \frac{c_2}{\Gamma(\eta_2)}{}^{\rho}I^{\beta}_{a_+}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_2-1} + \frac{c_3}{\Gamma(\eta_3)}{}^{\rho}I^{\beta+\gamma}_{a_+}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_3-1} \\
&+ {}^{\rho}I^{\beta+\gamma+\delta}_{a^+}f(t,v(t),{}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t),{}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)),{}^{\rho}D^{\gamma,\gamma'}_{a^+}({}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)))). \quad (2.20)
\end{aligned}
$$

Finally, we apply the integral operator ${}^{\rho}I^{\alpha}_{a^+}$ to (2.20), by using Lemma 2.6 and Lemma 2.10 we obtain

$${}^{\rho}I^{\alpha}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v)(t) = {}^{\rho}I^{\alpha+\beta+\gamma+\delta}_{a^+}f(t,v(t),{}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t),{}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)),{}^{\rho}D^{\gamma,\gamma'}_{a^+}({}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)))).$$

$${}^{\rho}I^{1-\eta_0}_{a^+}({}^{\rho}D^{1-\eta_0}_{a^+}v)(t) = {}^{\rho}I^{\alpha+\beta+\gamma+\delta}_{a^+}f(t,v(t),{}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t),{}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)),{}^{\rho}D^{\gamma,\gamma'}_{a^+}({}^{\rho}D^{\beta,\beta'}_{a^+}({}^{\rho}D^{\alpha,\alpha'}_{a^+}v(t)))).$$

According to Lemma 2.7, we have

$$v(t) = \frac{{}^\rho I_{a^+}^{1-\eta_0} v(a)}{\Gamma(\eta_0)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\eta_0 - 1} + \frac{c_1}{\Gamma(\eta_1)} {}^\rho I_{a+}^{\alpha} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\eta_1 - 1}$$
$$+ \frac{c_2}{\Gamma(\eta_2)} {}^\rho I_{a+}^{\alpha+\beta} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\eta_2 - 1} + \frac{c_3}{\Gamma(\eta_3)} {}^\rho I_{a+}^{\alpha+\beta+\gamma} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\eta_3 - 1}$$
$$+ {}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta} f(t, v(t), {}^\rho D_{a^+}^{\alpha,\alpha'} v(t), {}^\rho D_{a^+}^{\beta,\beta'} ({}^\rho D_{a^+}^{\alpha,\alpha'} v(t)), {}^\rho D_{a^+}^{\gamma,\gamma'} ({}^\rho D_{a^+}^{\beta,\beta'} ({}^\rho D_{a^+}^{\alpha,\alpha'} v(t)))).$$

with,

$$\eta_0 = \alpha + \alpha^{'}(1 - \alpha).$$

The first boundary condition of (2.15) leads to ${}^\rho I_{a^+}^{1-\eta_0} v(a) = c_0$.
Hence,

$$v(t) = \frac{c_0}{\Gamma(\eta_0)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\eta_0 - 1} + \frac{c_1}{\Gamma(\eta_1)} {}^\rho I_{a+}^{\alpha} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\eta_1 - 1}$$
$$+ \frac{c_2}{\Gamma(\eta_2)} {}^\rho I_{a+}^{\alpha+\beta} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\eta_2 - 1} + \frac{c_3}{\Gamma(\eta_3)} {}^\rho I_{a+}^{\alpha+\beta+\gamma} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\eta_3 - 1}$$
$$+ {}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta} f(t, v(t), {}^\rho D_{a^+}^{\alpha,\alpha'} v(t), {}^\rho D_{a^+}^{\beta,\beta'} ({}^\rho D_{a^+}^{\alpha,\alpha'} v(t)), {}^\rho D_{a^+}^{\gamma,\gamma'} ({}^\rho D_{a^+}^{\beta,\beta'} ({}^\rho D_{a^+}^{\alpha,\alpha'} v(t)))). \quad (2.21)$$

Therefore, we see that $v(t)$ fulfills , and the proof in ($\Longrightarrow$) is ended.

($\Longleftarrow$) Now, we prove the converse

By applying ${}^{\rho}D_{a^+}^{\alpha,\alpha'}$ on both sides of (2.17) and using Lemma 2.8, we obtain

$$\begin{aligned}{}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t) &= \frac{c_0}{\Gamma(\eta_0)}{}^{\rho}D_{a^+}^{\alpha,\alpha'}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_0-1}+\frac{c_1}{\Gamma(\eta_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_1-1}\\ &+\frac{c_2}{\Gamma(\eta_2)}{}^{\rho}I_{a_+}^{\beta}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_2-1}+\frac{c_3}{\Gamma(\eta_3)}{}^{\rho}I_{a_+}^{\beta+\gamma}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_3-1}\\ &+{}^{\rho}I_{a^+}^{\beta+\gamma+\delta}f(t,v(t),{}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t),{}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)),{}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))).\end{aligned}$$

By using Lemma 2.5, we get ${}^{\rho}D_{a^+}^{\alpha,\alpha'}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_0-1}=0$. Therefore

$$\begin{aligned}{}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t) &= \frac{c_1}{\Gamma(\eta_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_1-1}+\frac{c_2}{\Gamma(\eta_2)}{}^{\rho}I_{a_+}^{\beta}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_2-1}+\frac{c_3}{\Gamma(\eta_3)}{}^{\rho}I_{a_+}^{\beta+\gamma}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_3-1}\\ &+{}^{\rho}I_{a^+}^{\beta+\gamma+\delta}f(t,v(t),{}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t),{}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)),{}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))).\end{aligned} \tag{2.22}$$

By applying ${}^{\rho}D_{a^+}^{\beta,\beta'}$ to (2.22) and using Lemma 2.8, we obtain

$$\begin{aligned}{}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)) &= \frac{c_1}{\Gamma(\eta_1)}{}^{\rho}D_{a^+}^{\beta,\beta'}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_1-1}+\frac{c_2}{\Gamma(\eta_2)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_2-1}\\ &+\frac{c_3}{\Gamma(\eta_3)}{}^{\rho}I_{a_+}^{\gamma}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_3-1}\\ &+{}^{\rho}I_{a^+}^{\gamma+\delta}f(t,v(t),{}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t),{}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)),{}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))).\end{aligned}$$

By using Lemma 2.5, we get ${}^{\rho}D_{a^+}^{\beta,\beta'}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_1-1}=0$. Hence

$$\begin{aligned}{}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)) &= \frac{c_2}{\Gamma(\eta_2)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_2-1}+\frac{c_3}{\Gamma(\eta_3)}{}^{\rho}I_{a_+}^{\gamma}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_3-1}\\ &+{}^{\rho}I_{a^+}^{\gamma+\delta}f(t,v(t),{}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t),{}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)),{}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))).\end{aligned} \tag{2.23}$$

By applying ${}^{\rho}D_{a^+}^{\gamma,\gamma'}$ to (2.23), we obtain

$$\begin{aligned}{}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t))) &= \frac{c_2}{\Gamma(\eta_2)}{}^{\rho}D_{a^+}^{\gamma,\gamma'}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_2-1}+\frac{c_3}{\Gamma(\eta_3)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_3-1}\\ &+{}^{\rho}I_{a^+}^{\delta}f(t,v(t),{}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t),{}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)),{}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))).\end{aligned}$$

By using Lemma 2.5, we have ${}^{\rho}D_{a^+}^{\gamma,\gamma'}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_2-1}=0$, and

$$\begin{aligned}{}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t))) &= \frac{c_3}{\Gamma(\eta_3)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_3-1}\\ &+{}^{\rho}I_{a^+}^{\delta}f(t,v(t),{}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t),{}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)),{}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))).\end{aligned} \tag{2.24}$$

By applying ${}^{\rho}D_{a^+}^{\delta,\delta'}$ to (2.24), we get

$$
{}^{\rho}D_{a^+}^{\delta,\delta'}({}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))) = \frac{c_3}{\Gamma(\eta_3)}\,{}^{\rho}D_{a^+}^{\gamma,\gamma'}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\eta_3-1}
$$

$$
+ f(t, v(t), {}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t), {}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)), {}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))).
$$

Lemma 2.5 gives ${}^{\rho}D_{a^+}^{\delta,\delta'}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\eta_3-1} = 0.$

Therefore, we obtain the first equation of problem (2.15)

$$
{}^{\rho}D_{a^+}^{\delta,\delta'}({}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))) = f(t, v(t), {}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t), {}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)), {}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))).
$$

Now, in order to prove the boundary conditions of the problem (2.15) we rewrite the following integral equation (2.17)

$$
\begin{aligned}
v(t) &= \frac{c_0}{\Gamma(\eta_0)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\eta_0-1} + \frac{c_1}{\Gamma(\eta_1)}\,{}^{\rho}I_{a_+}^{\alpha}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\eta_1-1} \\
&+ \frac{c_2}{\Gamma(\eta_2)}\,{}^{\rho}I_{a_+}^{\alpha+\beta}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\eta_2-1} + \frac{c_3}{\Gamma(\eta_3)}\,{}^{\rho}I_{a_+}^{\alpha+\beta+\gamma}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\eta_3-1} \\
&+ {}^{\rho}I_{a^+}^{\alpha+\beta+\gamma+\delta} f(\cdot, v(\cdot), {}^{\rho}D_{a^+}^{\alpha,\alpha'}v(\cdot), {}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(\cdot)), {}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(\cdot)))).
\end{aligned}
$$

By using Lemma 2.5

$$
\begin{aligned}
v(t) &= \frac{c_0}{\Gamma(\eta_0)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\eta_0-1} + \frac{c_1}{\Gamma(\eta_1)}\frac{\Gamma(\eta_1)}{\Gamma(\alpha+\eta_1)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha+\eta_1-1} \\
&+ \frac{c_2}{\Gamma(\eta_2)}\frac{\Gamma(\eta_2)}{\Gamma(\alpha+\beta+\eta_2)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha+\beta+\eta_2-1} + \frac{c_3}{\Gamma(\eta_3)}\frac{\Gamma(\eta_3)}{\Gamma(\alpha+\beta+\gamma+\eta_3)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1} \\
&+ {}^{\rho}I_{a^+}^{\alpha+\beta+\gamma+\delta} f(t, v(t), {}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t), {}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)), {}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
v(t) &= \frac{c_0}{\Gamma(\eta_0)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\eta_0-1} + \frac{c_1}{\Gamma(\alpha+\eta_1)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha+\eta_1-1} \\
&+ \frac{c_2}{\Gamma(\alpha+\beta+\eta_2)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha+\beta+\eta_2-1} + \frac{c_3}{\Gamma(\alpha+\beta+\gamma+\eta_3)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1} \\
&+ {}^{\rho}I_{a^+}^{\alpha+\beta+\gamma+\delta} f(t, v(t), {}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t), {}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)), {}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))).
\end{aligned}
\tag{2.25}
$$

In order to prove the first boundary condition of (2.15), we apply ${}^{\rho}I_{a^+}^{1-\eta_0}$ to (3.3),

$$
\begin{aligned}
{}^{\rho}I_{a^+}^{1-\eta_0}v(t) &= \frac{c_0}{\Gamma(\eta_0)}{}^{\rho}I_{a^+}^{1-\eta_0}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\eta_0-1} + \frac{c_1}{\Gamma(\alpha+\eta_1)}{}^{\rho}I_{a^+}^{1-\eta_0}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\eta_1-1}\\
&+\frac{c_2}{\Gamma(\alpha+\beta+\eta_2)}{}^{\rho}I_{a^+}^{1-\eta_0}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-1}+\frac{c_3}{\Gamma(\alpha+\beta+\gamma+\eta_3)}{}^{\rho}I_{a^+}^{1-\eta_0}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1}\\
&+ {}^{\rho}I_{a^+}^{1-\eta_0}\,{}^{\rho}I_{a^+}^{\alpha+\beta+\gamma+\delta}f(t,v(t),{}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t),{}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)),{}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))).
\end{aligned}
$$

According to Lemma 2.5, we have

$$
\begin{aligned}
{}^{\rho}I_{a^+}^{1-\eta_0}v(t) &= \frac{c_0}{\Gamma(\eta_0)}\frac{\Gamma(\eta_0)}{\Gamma(1-\eta_0+\eta_0)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\eta_0+\eta_0-1}\\
&+\frac{c_1}{\Gamma(\alpha+\eta_1)}\frac{\Gamma(\alpha+\eta_1)}{\Gamma(1-\eta_0+\alpha+\eta_1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\eta_0+\alpha+\eta_1-1}\\
&+\frac{c_2}{\Gamma(\alpha+\beta+\eta_2)}\frac{\Gamma(\alpha+\beta+\eta_2)}{\Gamma(1-\eta_0+\alpha+\beta+\eta_2)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\eta_0+\alpha+\beta+\eta_2-1}\\
&+\frac{c_3}{\Gamma(\alpha+\beta+\gamma+\eta_3)}\frac{\Gamma(\alpha+\beta+\gamma+\eta_3)}{\Gamma(1-\eta_0+\alpha+\beta++\eta_3)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\eta_0+\alpha+\beta+\gamma+\eta_3-1}\\
&+{}^{\rho}I_{a^+}^{1-\eta_0+\alpha+\beta+\gamma+\delta}f(t,v(t),{}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t),{}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)),{}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
{}^{\rho}I_{a^+}^{1-\eta_0}v(t) &= c_0+\frac{c_1}{\Gamma(1-\eta_0+\alpha+\eta_1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{-\eta_0+\alpha+\eta_1}\\
&+\frac{c_2}{\Gamma(1-\eta_0+\alpha+\beta+\eta_2)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{-\eta_0+\alpha+\beta+\eta_2}\\
&+\frac{c_3}{\Gamma(1-\eta_0+\alpha+\beta+\gamma+\eta_3)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{-\eta_0+\alpha+\beta+\gamma+\eta_3}\\
&+{}^{\rho}I_{a^+}^{1-\eta_0+\alpha+\beta+\gamma+\delta}f(t,v(t),{}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t),{}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)),{}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))).
\end{aligned}
$$

For $t = a$,

$$
{}^{\rho}I_{a^+}^{1-\eta_0}v(a) = c_0+{}^{\rho}I_{a^+}^{1-\eta_0+\alpha+\beta+\gamma+\delta}f(a,v(a),{}^{\rho}D_{a^+}^{\alpha,\alpha'}v(a),{}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(a)),{}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(a)))).
$$

According to Lemma 2.9, we have

$$
{}^{\rho}I_{a^+}^{1-\eta_0}v(a) = c_0.
$$

Now, we prove the second boundary condition of the problem (2.15), we have by (2.22)

$$
\begin{aligned}
{}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t) &= \frac{c_1}{\Gamma(\eta_1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\eta_1-1}+\frac{c_2}{\Gamma(\eta_2)}{}^{\rho}I_{a_+}^{\beta}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\eta_2-1}+\frac{c_3}{\Gamma(\eta_3)}{}^{\rho}I_{a_+}^{\beta+\gamma}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\eta_3-1}\\
&+{}^{\rho}I_{a^+}^{\beta+\gamma+\delta}f(t,v(t),{}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t),{}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)),{}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))).
\end{aligned}
$$

By using Lemma 2.5, we obtain

$$
\begin{aligned}
{}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t) &= \frac{c_1}{\Gamma(\eta_1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\eta_1-1} + \frac{c_2}{\Gamma(\eta_2)}\frac{\Gamma(\eta_2)}{\Gamma(\beta+\eta_2)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta+\eta_2-1} \\
&+ \frac{c_3}{\Gamma(\eta_3)}\frac{\Gamma(\eta_3)}{\Gamma(\beta+\gamma+\eta_3)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta+\gamma+\eta_3-1} \\
&+ {}^{\rho}I_{a^+}^{\beta+\gamma+\delta} f(t, v(t), {}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t), {}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)), {}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
{}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t) &= \frac{c_1}{\Gamma(\eta_1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\eta_1-1} + \frac{c_2}{\Gamma(\beta+\eta_2)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta+\eta_2-1} \\
&+ \frac{c_3}{\Gamma(\beta+\gamma+\eta_3)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta+\gamma+\eta_3-1} \\
&+ {}^{\rho}I_{a^+}^{\beta+\gamma+\delta} f(t, v(t), {}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t), {}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)), {}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))).
\end{aligned}
\tag{2.26}
$$

By applying ${}^{\rho}I_{a^+}^{1-\eta_1}$ to (2.26), we obtain

$$
\begin{aligned}
{}^{\rho}I_{a^+}^{1-\eta_1}{}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t) &= \frac{c_1}{\Gamma(\eta_1)}{}^{\rho}I_{a^+}^{1-\eta_1}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\eta_1-1} + \frac{c_2}{\Gamma(\beta+\eta_2)}{}^{\rho}I_{a^+}^{1-\eta_1}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta+\eta_2-1} \\
&+ \frac{c_3}{\Gamma(\beta+\gamma+\eta_3)}{}^{\rho}I_{a^+}^{1-\eta_1}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta+\gamma+\eta_3-1} \\
&+ {}^{\rho}I_{a^+}^{1-\eta_1}\,{}^{\rho}I_{a^+}^{\beta+\gamma+\delta} f(t, v(t), {}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t), {}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)), {}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))).
\end{aligned}
$$

According to Lemma 2.5 and Lemma 2.6, we get

$$
\begin{aligned}
{}^{\rho}I_{a^+}^{1-\eta_1}{}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t) &= \frac{c_1}{\Gamma(\eta_1)}\frac{\Gamma(\eta_1)}{\Gamma(1-\eta_1+\eta_1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\eta_1+\eta_1-1} \\
&+ \frac{c_2}{\Gamma(\beta+\eta_2)}\frac{\Gamma(\beta+\eta_2)}{\Gamma(1-\eta_1+\beta+\eta_2)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\eta_1+\beta+\eta_2-1} \\
&+ \frac{c_3}{\Gamma(\beta+\gamma+\eta_3)}\frac{\Gamma(\beta+\gamma+\eta_3)}{\Gamma(1-\eta_1+\beta+\gamma+\eta_3)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\eta_1+\beta+\gamma+\eta_3-1} \\
&+ {}^{\rho}I_{a^+}^{1-\eta_1+\beta+\gamma+\delta} f(t, v(t), {}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t), {}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)), {}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
{}^{\rho}I_{a^+}^{1-\eta_1}{}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t) &= c_1 + \frac{c_2}{\Gamma(1-\eta_1+\beta+\eta_2)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{-\eta_1+\beta+\eta_2} \\
&+ \frac{c_3}{\Gamma(1-\eta_1+\beta+\gamma+\eta_3)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{-\eta_1+\beta+\gamma+\eta_3} \\
&+ {}^{\rho}I_{a^+}^{1-\eta_1+\beta+\gamma+\delta} f(t, v(t), {}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t), {}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)), {}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))).
\end{aligned}
$$

For $t = a$,

$$ {}^{\rho}I_{a^+}^{1-\eta_1}\, {}^{\rho}D_{a^+}^{\alpha,\alpha'} v(a) = c_1 + {}^{\rho}I_{a^+}^{1-\eta_1+\beta+\gamma+\delta} f(a, v(a), {}^{\rho}D_{a^+}^{\alpha,\alpha'} v(a), {}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'} v(a)), {}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'} v(a)))). $$

According to Lemma 2.9, we have

$$ {}^{\rho}I_{a^+}^{1-\eta_1}\, {}^{\rho}D_{a^+}^{\alpha,\alpha'} v(a) = c_1. $$

Next, we prove the third boundary condition of the problem (2.15), we have by (2.19)

$$ {}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'} v(t)) = \frac{c_2}{\Gamma(\eta_2)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\eta_2-1} + \frac{c_3}{\Gamma(\eta_3)}\, {}^{\rho}I_{a_+}^{\gamma}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\eta_3-1} + {}^{\rho}I_{a^+}^{\gamma+\delta} f(t, v(t), {}^{\rho}D_{a^+}^{\alpha,\alpha'} v(t), {}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'} v(t)), {}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'} v(t)))). $$

By using Lemma 2.5, we get

$$ {}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'} v(t)) = \frac{c_2}{\Gamma(\eta_2)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\eta_2-1} + \frac{c_3}{\Gamma(\eta_3)}\frac{\Gamma(\eta_3)}{\Gamma(\gamma+\eta_3)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma+\eta_3-1} + {}^{\rho}I_{a^+}^{\gamma+\delta} f(t, v(t), {}^{\rho}D_{a^+}^{\alpha,\alpha'} v(t), {}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'} v(t)), {}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'} v(t)))). $$

Thus

$$ {}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'} v(t)) = \frac{c_2}{\Gamma(\eta_2)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\eta_2-1} + \frac{c_3}{\Gamma(\gamma+\eta_3)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma+\eta_3-1} + {}^{\rho}I_{a^+}^{\gamma+\delta} f(t, v(t), {}^{\rho}D_{a^+}^{\alpha,\alpha'} v(t), {}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'} v(t)), {}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'} v(t)))). \tag{2.27} $$

By applying ${}^{\rho}I_{a^+}^{1-\eta_2}$ to (2.27), we obtain

$$ {}^{\rho}I_{a^+}^{1-\eta_2}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'} v(t))) = \frac{c_2}{\Gamma(\eta_2)}\, {}^{\rho}I_{a^+}^{1-\eta_2}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\eta_2-1} + \frac{c_3}{\Gamma(\gamma+\eta_3)}\, {}^{\rho}I_{a^+}^{1-\eta_2}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma+\eta_3-1} + {}^{\rho}I_{a^+}^{1-\eta_2}\, {}^{\rho}I_{a^+}^{\gamma+\delta} f(t, v(t), {}^{\rho}D_{a^+}^{\alpha,\alpha'} v(t), {}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'} v(t)), {}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'} v(t)))). $$

According to Lemma 2.5 and Lemma 2.6, we have

$$ {}^{\rho}I_{a^+}^{1-\eta_2}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'} v(t))) = \frac{c_2}{\Gamma(\eta_2)}\frac{\Gamma(\eta_2)}{\Gamma(1-\eta_2+\eta_2)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{1-\eta_2+\eta_2-1} $$

$$ + \frac{c_3}{\Gamma(\gamma+\eta_3)}\frac{\Gamma(\gamma+\eta_3)}{\Gamma(1-\eta_2+\gamma+\eta_3)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{1-\eta_2+\gamma+\eta_3-1} $$

$$ + {}^{\rho}I_{a^+}^{1-\eta_2+\gamma+\delta} f(t, v(t), {}^{\rho}D_{a^+}^{\alpha,\alpha'} v(t), {}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'} v(t)), {}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'} v(t)))). $$

Thus,

$$ {}^{\rho}I_{a^+}^{1-\eta_2}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t))) = c_2 + \frac{c_3}{\Gamma(1-\eta_2+\gamma+\eta_3)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{-\eta_2+\gamma+\eta_3} $$

$$ + \ {}^{\rho}I_{a^+}^{1-\eta_2+\gamma+\delta}f(t,v(t),{}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t),{}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)),{}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))). $$

For $t = a$,

$$ {}^{\rho}I_{a^+}^{1-\eta_2}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(a))) = c_2 + {}^{\rho}I_{a^+}^{1-\eta_2+\gamma+\delta}g(a) $$

By using Lemma 2.9, we get

$$ {}^{\rho}I_{a^+}^{1-\eta_2}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(a))) = c_2. $$

Finally, we prove the last boundary condition of the problem (2.15), we have by (2.18)

$$ {}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v))(t) = \frac{c_3}{\Gamma(\eta_3)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_3-1} $$
$$ +{}^{\rho}I_{a^+}^{\delta}f(t,v(t),{}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t),{}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)),{}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))). \quad (2.28) $$

Now, we apply ${}^{\rho}I_{a^+}^{1-\eta_3}$ to (2.28)

$$ {}^{\rho}I_{a^+}^{1-\eta_3}({}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t))) = \frac{c_3}{\Gamma(\eta_3)}{}^{\rho}I_{a^+}^{1-\eta_3}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_3-1} $$
$$ + \ {}^{\rho}I_{a^+}^{1-\eta_3}\,{}^{\rho}I_{a^+}^{\delta}f(t,v(t),{}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t),{}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)),{}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))). $$

According to Lemma 2.5 and Lemma 2.6, we get

$$ {}^{\rho}I_{a^+}^{1-\eta_3}({}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t))) = \frac{c_3}{\Gamma(\eta_3)}\frac{\Gamma(\eta_3)}{\Gamma(1-\eta_3+\eta_3)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta_3+\eta_3-1} $$
$$ + \ {}^{\rho}I_{a^+}^{1-\eta_3+\delta}f(t,v(t),{}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t),{}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)),{}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))). $$

Thus,

$$ {}^{\rho}I_{a^+}^{1-\eta_3}({}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t))) = c_3 $$
$$ + \ {}^{\rho}I_{a^+}^{1-\eta_3+\delta}f(t,v(t),{}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t),{}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)),{}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))). $$

For $t = a$,

$$ {}^{\rho}I_{a^+}^{1-\eta_3}({}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(a))) = c_3 $$
$$ + \ {}^{\rho}I_{a^+}^{1-\eta_3+\delta}f(a,v(a),{}^{\rho}D_{a^+}^{\alpha,\alpha'}v(a),{}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(a)),{}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(a))))..$$

By using Lemma 2.9, we obtain

$$^{\rho}I_{a^+}^{1-\eta_3}({}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v)))(a)=c_3.$$

The proof is fulfilled. ■

2.3 Existence and uniquness

In all the sequel, we consider the operator $\mathcal{F}:C_{1-\eta,\rho}([a,b])\longrightarrow C_{1-\eta,\rho}([a,b])$, defined and given by

$$\begin{aligned}\mathcal{F}v(t)&=\frac{c_0}{\Gamma(\eta_0)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_0-1}+\frac{c_1}{\Gamma(\eta_1)}\,{}^{\rho}I_{a^+}^{\alpha}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_1-1}\\&+\frac{c_2}{\Gamma(\eta_2)}\,{}^{\rho}I_{a^+}^{\alpha+\beta}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_2-1}+\frac{c_3}{\Gamma(\eta_3)}\,{}^{\rho}I_{a^+}^{\alpha+\beta+\gamma}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_3-1}\\&+{}^{\rho}I_{a^+}^{\alpha+\beta+\gamma+\delta}f(t,v(t),{}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t),{}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)),{}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))).\end{aligned}$$

Which can be written as,

$$\begin{aligned}\mathcal{F}v(t)&=\frac{c_0}{\Gamma(\eta_0)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_0-1}+\frac{c_1}{\Gamma(\alpha+\eta_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha+\eta_1-1}\\&+\frac{c_2}{\Gamma(\alpha+\beta+\eta_2)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-1}+\frac{c_3}{\Gamma(\alpha+\beta+\gamma+\eta_3)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1}\\&+{}^{\rho}I_{a^+}^{\alpha+\beta+\gamma+\delta}f(t,v(t),{}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t),{}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)),{}^{\rho}D_{a^+}^{\gamma,\gamma'}({}^{\rho}D_{a^+}^{\beta,\beta'}({}^{\rho}D_{a^+}^{\alpha,\alpha'}v(t)))).\end{aligned}\tag{2.29}$$

Notations

We denote by K_1,K_2,K_3 the following constants

$$\begin{aligned}K_1=&\frac{\rho^{\eta-2-\alpha-\beta-\gamma-\delta}}{\Gamma(\alpha+\beta+\gamma+\delta+1)}(b^\rho-a^\rho)^{1-\eta+\alpha+\beta+\gamma+\delta}+\frac{\rho^{\eta-2-\beta-\gamma-\delta}}{\Gamma(\beta+\gamma+\delta+1)}(b^\rho-a^\rho)^{1-\eta+\beta+\gamma+\delta}\\&+\frac{\rho^{\eta-2-\alpha-\gamma-\delta}}{\Gamma(\alpha+\gamma+\delta+1)}(b^\rho-a^\rho)^{1-\eta+\alpha+\gamma+\delta}+\frac{\rho^{\eta-2-\alpha-\beta-\delta}}{\Gamma(\alpha+\beta+\delta+1)}(b^\rho-a^\rho)^{1-\eta+\alpha+\beta+\delta},\end{aligned}$$

$$\begin{aligned}\mathrm{K}_2=&\left|\frac{c_0}{\Gamma(\eta_0)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\eta_0-\eta}+\frac{c_1}{\Gamma(\alpha+\eta_1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha+\eta_1-\eta}\right.\\&\left.+\frac{c_2}{\Gamma(\alpha+\beta+\eta_2)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-\eta}+\frac{c_3}{\Gamma(\alpha+\beta+\gamma+\eta_3)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-\eta}\right|,\end{aligned}$$

$$K_3=\frac{\rho^{\eta-2-\delta-\gamma-\beta-\alpha}}{\Gamma(\delta+\gamma+\beta+\alpha+1)}\,(b^\rho-a^\rho)^{1-\eta+\alpha+\beta+\gamma+\delta}.$$

Set $\tilde{f}(t)=f(t,0,0,0,0),\qquad t\in[a,b].\quad$ And $\;\Omega=\;K_3\,\|\tilde{f}\|_{C_{1-\eta,\rho}}$.

Hypotheses

Before stating and proving the main results, we introduce the following hypotheses

(H_1) $\exists L > 0$ such that $x, y, z, w, x^{'}, y^{'}, z^{'}, w^{'} \in C^{\eta}_{1-\eta,\rho}([a,b])$,

$|f(t,x,y,z,w) - f(t,x^{'},y^{'},z^{'},w^{'})| \leq L(|x-x^{'}| + |y-y^{'}| + |z-z^{'}| + |w-w^{'}|), t \in [a,b]$.

(H_2) For all $x, y, z, w \in \mathbb{R}$, there exists a constant $M > 0$ such that

$$|f(t,x,y,z,w)| \leq M, \qquad t \in [a,b].$$

(H_3) The constant $LK_1 < 1$.

2.3.1 Existence

Within this section, we will prove the existence of the problem (2.15) solution which is based on Schauder fixed point theorem.

Theorem 2.1 *Let* (H_1) *be held, the problem* (2.15) *has at least one solution in* $C^{\eta}_{1-\eta,\rho}([a,b])$.

Proof

In order to prove the existence result, we transform the problem (2.15) into a fixed point problem.
In fact, since the problem (2.15) is equivalent to an integral equation (2.17), the fixed points of $\mathcal{F}$ are the solutions of the problem (2.15), we will establish then, the hypotheses of Schauder fixed point theorem. Hence, the proof consists of several steps.

First Step

We prove that $\mathcal{F}$ is a continuous operator. We consider the bounded set $\mathcal{B}_r \subset C^{\eta}_{1-\eta,\rho}([a,b])$, such that

$$\mathcal{B}_r = \{\ v \in C^{\eta}_{1-\eta,\rho}([a,b]) : \|v\|_{C_{1-\eta,\rho}} \leq r\ \}, \qquad r > 0,$$

with

$$r \geq \frac{K_2 + L\Omega}{1 - LK_1}.$$

Let $(v_n)_{n\in\mathbb{N}} \in \mathcal{B}_r$ a sequcence of real numbers, such that $\lim_{n\to\infty} \|v_n - v\|_{C_{1-\eta,\rho}} = 0$. Then, $\forall t \in [a,b]$, we have

$$\left|(\mathcal{F}v_n(t)-\mathcal{F}v(t))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right|$$

$$=\Bigg|\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\Bigg(\frac{c_0}{\Gamma(\eta_0)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_0-1}+\frac{c_1}{\Gamma(\alpha+\eta_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha+\eta_1-1}$$

$$+\frac{c_2}{\Gamma(\alpha+\beta+\eta_2)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-1}+\frac{c_3}{\Gamma(\alpha+\beta+\gamma+\eta_3)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1}$$

$$+\,{}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}f(t,v_n(t),{}^\rho D_{a^+}^{\alpha,\alpha'}v_n(t),{}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'}v_n(t)),{}^\rho D_{a^+}^{\gamma,\gamma'}({}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'}v_n(t))))$$

$$-\frac{c_0}{\Gamma(\eta_0)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_0-1}-\frac{c_1}{\Gamma(\alpha+\eta_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha+\eta_1-1}-\frac{c_2}{\Gamma(\alpha+\beta+\eta_2)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-1}$$

$$-\frac{c_3}{\Gamma(\alpha+\beta+\gamma+\eta_3)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1}$$

$$-\,{}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}f(t,v(t),{}^\rho D_{a^+}^{\alpha,\alpha'}v(t),{}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'}v(t)),{}^\rho D_{a^+}^{\gamma,\gamma'}({}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'}v(t))))\Bigg|$$

$$\leq\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}$$

$$\times\Bigg({}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}\Big|f(t,v_n(t),{}^\rho D_{a^+}^{\alpha,\alpha'}v_n(t),{}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'}v_n(t)),{}^\rho D_{a^+}^{\gamma,\gamma'}({}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'}v_n(t))))$$

$$-f(t,v(t),{}^\rho D_{a^+}^{\alpha,\alpha'}v(t),{}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'}v(t)),{}^\rho D_{a^+}^{\gamma,\gamma'}({}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'}v(t))))\Big|\Bigg)$$

According to hypothesis (H_1), we have

$$\left|(\mathcal{F}v_n(t)-\mathcal{F}v(t))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right|$$

$$\leq L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}{}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}\Big(|v_n(t)-v(t)|+\left|{}^\rho D_{a^+}^{\alpha,\alpha'}v_n(t)-{}^\rho D_{a^+}^{\alpha,\alpha'}v(t)\right|$$

$$+\left|{}^\rho D_{a^+}^{\beta,\beta'}\left({}^\rho D_{a^+}^{\alpha,\alpha'}v_n(t)\right)-{}^\rho D_{a^+}^{\beta,\beta'}\left({}^\rho D_{a^+}^{\alpha,\alpha'}v(t)\right)\right|$$

$$+\left|{}^\rho D_{a^+}^{\gamma,\gamma'}\left({}^\rho D_{a^+}^{\beta,\beta'}\left({}^\rho D_{a^+}^{\alpha,\alpha'}v_n(t)\right)-{}^\rho D_{a^+}^{\gamma,\gamma'}\left({}^\rho D_{a^+}^{\beta,\beta'}\left({}^\rho D_{a^+}^{\alpha,\alpha'}v(t)\right)\right)\right|\Big)$$

By using Property (P_1), we obtain

$$\left|(\mathcal{F}v_n(t)-\mathcal{F}v(t))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right|$$
$$\leq L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\ {}^\rho I_{a_+}^{\alpha+\beta+\gamma+\delta}\Big(|v_n(t)-v(t)|+\left|{}^\rho I_{a^+}^{\alpha'(1-\alpha)}\ {}^\rho D_{a^+}^{\eta_0}v_n(t)-\ {}^\rho I_{a^+}^{\alpha'(1-\alpha)}\ {}^\rho D_{a^+}^{\eta_0}v(t)\right|$$
$$+\left|{}^\rho I_{a^+}^{\alpha'(1-\alpha)+\beta'(1-\beta)}\ {}^\rho D_{a^+}^{\eta_0+\eta_1}v_n(t)-\ {}^\rho I_{a^+}^{\alpha'(1-\alpha)+\beta'(1-\beta)}\ {}^\rho D_{a^+}^{\eta_0+\eta_1}v(t)\right|$$
$$+\left|{}^\rho I_{a^+}^{\alpha'(1-\alpha)+\beta'(1-\beta)+\gamma'(1-\gamma)}\ {}^\rho D_{a^+}^{\eta_0+\eta_1+\eta_2}v_n(t)-\ {}^\rho I_{a^+}^{\alpha'(1-\alpha)+\beta'(1-\beta)+\gamma'(1-\gamma)}\ {}^\rho D_{a^+}^{\eta_0+\eta_1+\eta_2}v(t)\right|\Big)$$
$$\leq L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\ {}^\rho I_{a_+}^{\alpha+\beta+\gamma+\delta}\Big(|v_n(t)-v(t)|+\left|\ {}^\rho D_{a^+}^{\alpha,\alpha'}|v_n(t)-v(t)|\right|+\left|\ {}^\rho D_{a^+}^{\alpha+\beta,\alpha'+\beta'}|v_n(t)-v(t)|\right|$$
$$+\left|\ {}^\rho D_{a^+}^{\alpha+\beta+\gamma,\alpha'+\beta'+\gamma'}|v_n(t)-v(t)|\right|\Big)$$

Then,

$$\left|(\mathcal{F}v_n(t)-\mathcal{F}v(t))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right|$$
$$\leq L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\ \Big({}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}|v_n(t)-v(t)|+\left|{}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}\ {}^\rho D_{a^+}^{\alpha,\alpha'}|v_n(t)-v(t)|\right|$$
$$+\left|{}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}\ {}^\rho D_{a^+}^{\alpha+\beta,\alpha'+\beta'}|v_n(t)-v(t)|\right|+\left|{}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}\ {}^\rho D_{a^+}^{\alpha+\beta+\gamma,\alpha'+\beta'+\gamma'}|v_n(t)-v(t)|\right|\Big)$$
$$\leq L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\ \Big({}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}|v_n(t)-v(t)|+\left|{}^\rho I_{a^+}^{\beta+\gamma+\delta}({}^\rho I_{a_+}^{\alpha}\ {}^\rho D_{a^+}^{\alpha,\alpha'})|v_n(t)-v(t)|\right|$$
$$+\left|{}^\rho I_{a^+}^{\alpha+\gamma+\delta}({}^\rho I_{a^+}^{\beta}\ {}^\rho D_{a^+}^{\beta,\beta'})|v_n(t)-v(t)|\right|+\left|{}^\rho I_{a^+}^{\alpha+\beta+\delta}({}^\rho I_{a^+}^{\gamma}\ {}^\rho D_{a^+}^{\gamma,\gamma'})|v_n(t)-v(t)|\right|\Big)$$

According to Lemma 2.10, we obtain

$$\left|(\mathcal{F}v_n(t)-\mathcal{F}v(t))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right|$$
$$\leq L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\ \Big({}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}|v_n(t)-v(t)|+\left|{}^\rho I_{a^+}^{\beta+\gamma+\delta}({}^\rho I_{a^+}^{\eta_0}\ {}^\rho D_{a^+}^{\eta_0})|v_n(t)-v(t)|\right|$$
$$+\left|{}^\rho I_{a^+}^{\alpha+\gamma+\delta}({}^\rho I_{a^+}^{\eta_1}\ {}^\rho D_{a^+}^{\eta_1})|v_n(t)-v(t)|\right|+\left|{}^\rho I_{a_+}^{\alpha+\beta+\delta}({}^\rho I_{a^+}^{\eta_2}\ {}^\rho D_{a^+}^{\eta_2})|v_n(t)-v(t)|\right|\Big)$$

As stated in Lemma 2.7

$$\left|(\mathcal{F}v_n(t)-\mathcal{F}v(t))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right| \leq L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\left[\left|{}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}|v_n(t)-v(t)|\right|\right.$$

$$+\left|{}^\rho I_{a^+}^{\beta+\gamma+\delta}\left(|v_n(t)-v(t)|-\frac{{}^\rho I_{a^+}^{\eta_0}|v_n-v|(a)}{\Gamma(\eta_0)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_0-1}\right)\right|$$

$$+\left|{}^\rho I_{a^+}^{\alpha+\gamma+\delta}\left(|v_n(t)-v(t)|-\frac{{}^\rho I_{a^+}^{\eta_1}|v_n-v|(a)}{\Gamma(\eta_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_1-1}\right)\right|$$

$$\left.+\left|{}^\rho I_{a^+}^{\alpha+\beta+\delta}\left(|v_n(t)-v(t)|-\frac{{}^\rho I_{a^+}^{\eta_2}|v_n-v|(a)}{\Gamma(\eta_2)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_2-1}\right)\right|\right]$$

By using Lemma 2.9, we get

$$\left|(\mathcal{F}v_n(t)-\mathcal{F}v(t))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right|$$

$$\leq L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\left[\left|{}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}\|v_n-v\|_{C_{1-\eta,\rho}}\right|+\left|{}^\rho I_{a^+}^{\beta+\gamma+\delta}\|v_n-v\|_{C_{1-\eta,\rho}}\right|\right.$$

$$\left.+\left|{}^\rho I_{a^+}^{\alpha+\gamma+\delta}\|v_n-v\|_{C_{1-\eta,\rho}}\right|+\left|{}^\rho I_{a^+}^{\alpha+\beta+\delta}\|v_n-v\|_{C_{1-\eta,\rho}}\right|\right]$$

$$\leq L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\left[\frac{\rho^{-1-\alpha-\beta-\gamma-\delta}}{(\alpha+\beta+\gamma+\delta)\Gamma(\alpha+\beta+\gamma+\delta)}(t^\rho-a^\rho)^{\alpha+\beta+\gamma+\delta}\|v_n-v\|_{C_{1-\eta,\rho}}\right.$$

$$+\frac{\rho^{-1-\beta-\gamma-\delta}}{(\beta+\gamma+\delta)\Gamma(\beta+\gamma+\delta)}(t^\rho-a^\rho)^{\beta+\gamma+\delta}\|v_n-v\|_{C_{1-\eta,\rho}}$$

$$+\frac{\rho^{-1-\alpha-\gamma-\delta}}{(\alpha+\gamma+\delta)\Gamma(\alpha+\gamma+\delta)}(t^\rho-a^\rho)^{\alpha+\gamma+\delta}\|v_n-v\|_{C_{1-\eta,\rho}}$$

$$\left.+\frac{\rho^{-1-\alpha-\beta-\delta}}{(\alpha+\beta+\delta)\Gamma(\alpha+\beta+\delta)}(t^\rho-a^\rho)^{\alpha+\beta+\delta}\|v_n-v\|_{C_{1-\eta,\rho}}\right]$$

$$\leq L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\|v_n-v\|_{C_{1-\eta,\rho}}\left[\frac{\rho^{-1-\alpha-\beta-\gamma-\delta}}{\Gamma(\alpha+\beta+\gamma+\delta+1)}(b^\rho-a^\rho)^{\alpha+\beta+\gamma+\delta}\right.$$

$$+\frac{\rho^{-1-\beta-\gamma-\delta}}{\Gamma(\beta+\gamma+\delta+1)}(b^\rho-a^\rho)^{\beta+\gamma+\delta}+\frac{\rho^{-1-\alpha-\gamma-\delta}}{\Gamma(\alpha+\gamma+\delta+1)}(b^\rho-a^\rho)^{\alpha+\gamma+\delta}$$

$$\left.+\frac{\rho^{-1-\alpha-\beta-\delta}}{\Gamma(\alpha+\beta+\delta+1)}(b^\rho-a^\rho)^{\alpha+\beta+\delta}\right]$$

Finally,

$$\left|(\mathcal{F}v_n(t) - \mathcal{F}v(t))\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\eta}\right|$$

$$\leq L\left[\frac{\rho^{\eta-2-\alpha-\beta-\gamma-\delta}}{\Gamma(\alpha+\beta+\gamma+\delta+1)}(b^\rho - a^\rho)^{1-\eta+\alpha+\beta+\gamma+\delta} + \frac{\rho^{\eta-2-\beta-\gamma-\delta}}{\Gamma(\beta+\gamma+\delta+1)}(b^\rho - a^\rho)^{1-\eta+\beta+\gamma+\delta}\right.$$

$$\left. + \frac{\rho^{\eta-2-\alpha-\gamma-\delta}}{\Gamma(\alpha+\gamma+\delta+1)}(b^\rho - a^\rho)^{1-\eta+\alpha+\gamma+\delta} + \frac{\rho^{\eta-2-\alpha-\beta-\delta}}{\Gamma(\alpha+\beta+\delta+1)}(b^\rho - a^\rho)^{1-\eta+\alpha+\beta+\delta}\right]\|v_n - v\|_{C_{1-\eta,\rho}}$$

$$\leq LK_1\|v_n - v\|_{C_{1-\eta,\rho}}.$$

Hence,

$$\left|(\mathcal{F}v_n(t) - \mathcal{F}v(t))\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\eta}\right| \leq LK_1\|v_n - v\|_{C_{1-\eta,\rho}}.$$

Therefore,

$$\|(\mathcal{F}v_n)(t) - (\mathcal{F}v)(t)\|_{C_{1-\eta,\rho}} \longrightarrow 0, \qquad when \quad n \to \infty.$$

Consequently, $\mathcal{F}$ is continuous.

Second Step

We prove that $\mathcal{F}(\mathcal{B}_r) \subset \mathcal{B}_r$. By hypothesis (H_1), for all $t \in [a,b]$, $v \in \mathcal{B}_r$, we have

$$\begin{aligned}
&\left|\mathcal{F}v(t)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\eta}\right| \\
&= \Bigg|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\eta}\Bigg(\frac{c_0}{\Gamma(\eta_0)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\eta_0-1} + \frac{c_1}{\Gamma(\alpha+\eta_1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\eta_1-1} \\
&+ \frac{c_2}{\Gamma(\alpha+\beta+\eta_2)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-1} + \frac{c_3}{\Gamma(\alpha+\beta+\gamma+\eta_3)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1} \\
&+ {}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta} f(t, v(t), {}^\rho D_{a^+}^{\alpha,\alpha'} v(t), {}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} v(t)), {}^\rho D_{a^+}^{\gamma,\gamma'}({}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} v(t))))\Bigg|. \\
&\leq \Bigg|\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\eta}\Bigg(\frac{c_0}{\Gamma(\eta_0)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\eta_0-1} + \frac{c_1}{\Gamma(\alpha+\eta_1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha+\eta_1-1} \\
&+ \frac{c_2}{\Gamma(\alpha+\beta+\eta_2)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-1} + \frac{c_3}{\Gamma(\alpha+\beta+\gamma+\eta_3)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1}\Bigg)\Bigg| \\
&+ \left|\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\eta} {}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta} f(t, v(t), {}^\rho D_{a^+}^{\alpha,\alpha'} v(t), {}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} v(t)), {}^\rho D_{a^+}^{\gamma,\gamma'}({}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} v(t))))\right|.
\end{aligned}$$

Then,

$$\begin{aligned}
&\left|\mathcal{F}v(t)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\eta}\right| \\
&\leq \Bigg|\frac{c_0}{\Gamma(\eta_0)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\eta_0-\eta} + \frac{c_1}{\Gamma(\alpha+\eta_1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha+\eta_1-\eta} \\
&+ \frac{c_2}{\Gamma(\alpha+\beta+\eta_2)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-\eta} + \frac{c_3}{\Gamma(\alpha+\beta+\gamma+\eta_3)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-\eta}\Bigg| \\
&+ \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\eta} {}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}\Big| f(t, v(t), {}^\rho D_{a^+}^{\alpha,\alpha'} v(t), {}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} v(t)), {}^\rho D_{a^+}^{\gamma,\gamma'}({}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} v(t)))) \\
&- f(t,0,0,0,0) + f(t,0,0,0,0)\Big|
\end{aligned}$$

According to hypothesis (H_1),

$$\left|\mathcal{F}v(t)\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right|$$

$$\leq K_2+L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\,{}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}\Bigg(|v(t)|+\left|{}^\rho D_{a^+}^{\alpha,\alpha'}v(t)\right|+\left|{}^\rho D_{a^+}^{\beta,\beta'}\left({}^\rho D_{a^+}^{\alpha,\alpha'}v(t)\right)\right|$$

$$+\left|{}^\rho D_{a^+}^{\gamma,\gamma'}\left({}^\rho D_{a^+}^{\beta,\beta'}\left({}^\rho D_{a^+}^{\alpha,\alpha'}v(t)\right)\right)\right|\Bigg)+\;L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\,{}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}\|\tilde{f}\|_{C_{1-\eta,\rho}}$$

By using Property (P_1), we obtain

$$\left|(\mathcal{F}v(t))\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\right|$$

$$\leq K_2+L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\,{}^\rho I_{a_+}^{\alpha+\beta+\gamma+\delta}\left(|v(t)|+\left|{}^\rho I_{a^+}^{\alpha'(1-\alpha)}\,{}^\rho D_{a^+}^{\eta_0}v(t)\right|+\left|{}^\rho I_{a^+}^{\alpha'(1-\alpha)+\beta'(1-\beta)}\,{}^\rho D_{a^+}^{\eta_0+\eta_1}v(t)\right|\right.$$

$$\left.+\left|{}^\rho I_{a^+}^{\alpha'(1-\alpha)+\beta'(1-\beta)+\gamma'(1-\gamma)}\,{}^\rho D_{a^+}^{\eta_0+\eta_1+\eta_2}v(t)\right|\right)$$

$$+\;L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\frac{\rho^{-1-\alpha-\beta-\gamma-\delta}}{(\alpha+\beta+\gamma+\delta)\Gamma(\alpha+\beta+\gamma+\delta)}(t^\rho-a^\rho)^{\alpha+\beta+\gamma+\delta}\;\|\tilde{f}\|_{C_{1-\eta,\rho}}$$

$$\leq K_2+L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\,{}^\rho I_{a_+}^{\alpha+\beta+\gamma+\delta}\left(|v(t)|+\left|{}^\rho D_{a^+}^{\alpha,\alpha'}|v(t)|\right|+\left|{}^\rho D_{a^+}^{\alpha+\beta,\alpha'+\beta'}|v(t)|\right|\right.$$

$$\left.+\left|{}^\rho D_{a^+}^{\alpha+\beta+\gamma,\alpha'+\beta'+\gamma'}|v(t)|\right|\right)+\;L\frac{\rho^{\eta-2-\alpha-\beta-\gamma-\delta}}{\Gamma(\alpha+\beta+\gamma+\delta+1)}(b^\rho-a^\rho)^{1-\eta+\alpha+\beta+\gamma+\delta}\;\|\tilde{f}\|_{C_{1-\eta,\rho}}$$

Then,

$$\left|(\mathcal{F}v_n(t)-\mathcal{F}v(t))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right|$$

$$\leq K_2+L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\left({}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}|v(t)|+\left|{}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}\,{}^\rho D_{a^+}^{\alpha,\alpha'}|v(t)|\right|\right.$$

$$\left.+\left|{}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}\,{}^\rho D_{a^+}^{\alpha+\beta,\alpha'+\beta'}|v(t)|\right|+\left|{}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}\,{}^\rho D_{a^+}^{\alpha+\beta+\gamma,\alpha'+\beta'+\gamma'}|v(t)|\right|\right)+\;L\,K_3\;\|\tilde{f}\|_{C_{1-\eta,\rho}}$$

$$\leq K_2+L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\left({}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}|v(t)|+\left|{}^\rho I_{a^+}^{\beta+\gamma+\delta}({}^\rho I_{a_+}^{\alpha}\,{}^\rho D_{a^+}^{\alpha,\alpha'})|v(t)|\right|\right.$$

$$\left.+\left|{}^\rho I_{a^+}^{\alpha+\gamma+\delta}({}^\rho I_{a^+}^{\beta}\,{}^\rho D_{a^+}^{\beta,\beta'})|v(t)|\right|+\left|{}^\rho I_{a^+}^{\alpha+\beta+\delta}({}^\rho I_{a^+}^{\gamma}\,{}^\rho D_{a^+}^{\gamma,\gamma'})|v(t)|\right|\right)\;+\;L\,K_3\;\|\tilde{f}\|_{C_{1-\eta,\rho}}$$

According to Lemma 2.10, we obtain

$$\left|(\mathcal{F}v_n(t) - \mathcal{F}v(t))\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\eta}\right|$$
$$\leq K_2 + L\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\eta}\Big({}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}|v(t)| + \left|{}^\rho I_{a^+}^{\beta+\gamma+\delta}({}^\rho I_{a^+}^{\eta_0}\,{}^\rho D_{a^+}^{\eta_0})|v(t)|\right|$$
$$+\left|{}^\rho I_{a^+}^{\alpha+\gamma+\delta}({}^\rho I_{a^+}^{\eta_1}\,{}^\rho D_{a^+}^{\eta_1})|v(t)|\right| + \left|{}^\rho I_{a_+}^{\alpha+\beta+\delta}({}^\rho I_{a^+}^{\eta_2}\,{}^\rho D_{a^+}^{\eta_2})|v(t)|\right|\Big) + L\,K_3\|\tilde{f}\|_{C_{1-\eta,\rho}}$$

As stated in Lemma 2.7

$$\left|(\mathcal{F}v(t))\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\eta}\right|$$
$$\leq K_2 + L\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\eta}\Bigg[\left|{}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}|v(t)|\right| + \left|{}^\rho I_{a^+}^{\beta+\gamma+\delta}\left(|v(t)| - \frac{{}^\rho I_{a^+}^{\eta_0}|v|(a)}{\Gamma(\eta_0)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\eta_0-1}\right)\right|$$
$$+\left|{}^\rho I_{a^+}^{\alpha+\gamma+\delta}\left(|v(t)| - \frac{{}^\rho I_{a^+}^{\eta_1}|v|(a)}{\Gamma(\eta_1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\eta_1-1}\right)\right|$$
$$+\left|{}^\rho I_{a^+}^{\alpha+\beta+\delta}\left(|v(t)| - \frac{{}^\rho I_{a^+}^{\eta_2}|v|(a)}{\Gamma(\eta_2)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\eta_2-1}\right)\right|\Bigg] + L\,\Omega$$

By using Lemma 2.9, we get

$$\left|(\mathcal{F}v(t))\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\eta}\right|$$
$$\leq K_2 + L\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\eta}\Bigg[\left|{}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}\|v\|_{C_{1-\eta,\rho}}\right| + \left|{}^\rho I_{a^+}^{\beta+\gamma+\delta}\|v\|_{C_{1-\eta,\rho}}\right| + \left|{}^\rho I_{a^+}^{\alpha+\gamma+\delta}\|v\|_{C_{1-\eta,\rho}}\right|$$
$$+\left|{}^\rho I_{a^+}^{\alpha+\beta+\delta}\|v\|_{C_{1-\eta,\rho}}\right|\Bigg] + L\,\Omega$$

$$\leq K_2 + L\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\eta}\Bigg[\frac{\rho^{-1-\alpha-\beta-\gamma-\delta}}{(\alpha+\beta+\gamma+\delta)\Gamma(\alpha+\beta+\gamma+\delta)}(t^\rho - a^\rho)^{\alpha+\beta+\gamma+\delta}\|v\|_{C_{1-\eta,\rho}}$$
$$+\frac{\rho^{-1-\beta-\gamma-\delta}}{(\beta+\gamma+\delta)\Gamma(\beta+\gamma+\delta)}(t^\rho - a^\rho)^{\beta+\gamma+\delta}\|v\|_{C_{1-\eta,\rho}}$$
$$+\frac{\rho^{-1-\alpha-\gamma-\delta}}{(\alpha+\gamma+\delta)\Gamma(\alpha+\gamma+\delta)}(t^\rho - a^\rho)^{\alpha+\gamma+\delta}\|v\|_{C_{1-\eta,\rho}}$$
$$+\frac{\rho^{-1-\alpha-\beta-\delta}}{(\alpha+\beta+\delta)\Gamma(\alpha+\beta+\delta)}(t^\rho - a^\rho)^{\alpha+\beta+\delta}\|v\|_{C_{1-\eta,\rho}}\Bigg] + L\,\Omega$$

Thus,

$$\left|(\mathcal{F}v(t))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right|$$

$$\leq K_2+L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\|v\|_{C_{1-\eta,\rho}}\left[\frac{\rho^{-1-\alpha-\beta-\gamma-\delta}}{\Gamma(\alpha+\beta+\gamma+\delta+1)}(b^\rho-a^\rho)^{\alpha+\beta+\gamma+\delta}\right.$$

$$+\frac{\rho^{-1-\beta-\gamma-\delta}}{\Gamma(\beta+\gamma+\delta+1)}(b^\rho-a^\rho)^{\beta+\gamma+\delta}+\frac{\rho^{-1-\alpha-\gamma-\delta}}{\Gamma(\alpha+\gamma+\delta+1)}(b^\rho-a^\rho)^{\alpha+\gamma+\delta}$$

$$\left.+\frac{\rho^{-1-\alpha-\beta-\delta}}{\Gamma(\alpha+\beta+\delta+1)}(b^\rho-a^\rho)^{\alpha+\beta+\delta}\right]+L\,\Omega$$

Finally,

$$\left|(\mathcal{F}v(t))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right|$$

$$\leq K_2+L\left[\frac{\rho^{\eta-2-\alpha-\beta-\gamma-\delta}}{\Gamma(\alpha+\beta+\gamma+\delta+1)}(b^\rho-a^\rho)^{1-\eta+\alpha+\beta+\gamma+\delta}+\frac{\rho^{\eta-2-\beta-\gamma-\delta}}{\Gamma(\beta+\gamma+\delta+1)}(b^\rho-a^\rho)^{1-\eta+\beta+\gamma+\delta}\right.$$

$$\left.+\frac{\rho^{\eta-2-\alpha-\gamma-\delta}}{\Gamma(\alpha+\gamma+\delta+1)}(b^\rho-a^\rho)^{1-\eta+\alpha+\gamma+\delta}+\frac{\rho^{\eta-2-\alpha-\beta-\delta}}{\Gamma(\alpha+\beta+\delta+1)}(b^\rho-a^\rho)^{1-\eta+\alpha+\beta+\delta}\right]\|v\|_{C_{1-\eta,\rho}}+L\,\Omega$$

$$\leq K_2+LK_1\|v\|_{C_{1-\eta,\rho}}+L\,\Omega$$

$$\leq (K_2+L\Omega)+LK_1r$$

$$\leq (K_2+L\Omega)\left(\frac{1-LK_1}{1-LK_1}\right)+LK_1r$$

$$\leq \left(\frac{K_2+L\Omega}{1-LK_1}\right)-\frac{LK_1(K_2+L\Omega)}{1-LK_1}+LK_1r$$

$$\leq r-LK_1r+LK_1r.$$

Therefore,

$$\left|(\mathcal{F}v(t))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right|\leq r$$

We obtain $\|\mathcal{F}v\|_{C_{1-\eta,\rho}}\leq r$. It follows that $\mathcal{F}(\mathcal{B}_r)\subset\mathcal{B}_r$

Third Step

We prove that $\mathcal{F}$ is uniformly bounded.

By hypothesis (H_2), for all $t \in [a,b]$, $v \in \mathcal{B}_r$, we have

$$\begin{aligned}
&\left|\mathcal{F}v(t)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\eta}\right| \\
&= \Bigg|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\eta}\Bigg(\frac{c_0}{\Gamma(\eta_0)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\eta_0-1} + \frac{c_1}{\Gamma(\alpha+\eta_1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\eta_1-1} \\
&+ \frac{c_2}{\Gamma(\alpha+\beta+\eta_2)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-1} + \frac{c_3}{\Gamma(\alpha+\beta+\gamma+\eta_3)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1} \\
&+ {}^\rho I_{a+}^{\alpha+\beta+\gamma+\delta} f(t, v(t), {}^\rho D_{a^+}^{\alpha,\alpha'} v(t), {}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} v(t)), {}^\rho D_{a^+}^{\gamma,\gamma'}({}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} v(t))))\Bigg)\Bigg| \\
&\leq \Bigg|\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\eta}\Bigg(\frac{c_0}{\Gamma(\eta_0)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\eta_0-1} + \frac{c_1}{\Gamma(\alpha+\eta_1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha+\eta_1-1} \\
&+ \frac{c_2}{\Gamma(\alpha+\beta+\eta_2)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-1} + \frac{c_3}{\Gamma(\alpha+\beta+\gamma+\eta_3)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1} + {}^\rho I_{a_+}^{\alpha+\beta+\gamma+\delta} M\Bigg)\Bigg| \\
&\leq \Bigg|\frac{c_0}{\Gamma(\eta_0)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\eta_0-\eta} + \frac{c_1}{\Gamma(\alpha+\eta_1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha+\eta_1-\eta} + \frac{c_2}{\Gamma(\alpha+\beta+\eta_2)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-\eta} \\
&+ \frac{c_3}{\Gamma(\alpha+\beta+\gamma+\eta_3)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-\eta}\Bigg| \\
&+ {}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}\left|f(t, v(t), {}^\rho D_{a^+}^{\alpha,\alpha'} v(t), {}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} v(t)), {}^\rho D_{a^+}^{\gamma,\gamma'}({}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} v(t))))\right| \\
&\leq K_2 + \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\eta}\frac{\rho^{-1-\alpha-\beta-\gamma-\delta}}{(\alpha+\beta+\gamma+\delta)\Gamma(\alpha+\beta+\gamma+\delta)}(t^\rho - a^\rho)^{\alpha+\beta+\gamma+\delta} M \\
&\leq K_2 + \frac{\rho^{\eta-2-\alpha-\beta-\gamma-\delta}}{\Gamma(\alpha+\beta+\gamma+\delta+1)}(b^\rho - a^\rho)^{1-\eta+\alpha+\beta+\gamma+\delta} M
\end{aligned}$$

Therefore,

$$\left|\mathcal{F}v(t)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\eta}\right| \leq K_2 + MK_3$$

Hence, $\|\mathcal{F}v\|_{C_{1-\eta,\rho}} \leq K_2 + MK_3$.

It follows that $\mathcal{F}$ is uniformly bounded.

Fourth Step

At this point, we will prove that $\mathcal{F}$ is equicontinuous.

Let $v \in \mathcal{B}_r$ and $t_1, t_2 \in [a,b]$ such that $t_1 < t_2$, we have :

$$\left|(\mathcal{F}v(t_2) - \mathcal{F}v(t_1))\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\eta}\right|$$

$$= \left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\eta}\left(\frac{c_0}{\Gamma(\eta_0)}\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{\eta_0-1} + \frac{c_1}{\Gamma(\alpha+\eta_1)}\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{\alpha+\eta_1-1}\right.\right.$$

$$+ \frac{c_2}{\Gamma(\alpha+\beta+\eta_2)}\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-1} + \frac{c_3}{\Gamma(\alpha+\beta+\gamma+\eta_3)}\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1}$$

$$+ {}^\rho I_{a+}^{\alpha+\beta+\gamma+\delta} f(t_2, v(t_2), {}^\rho D_{a^+}^{\alpha,\alpha'} v(t_1), {}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} v(t_2)), {}^\rho D_{a^+}^{\gamma,\gamma'}({}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} v(t_2))))$$

$$- \frac{c_0}{\Gamma(\eta_0)}\left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{\eta_0-1} - \frac{c_1}{\Gamma(\alpha+\eta_1)}\left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{\alpha+\eta_1-1} - \frac{c_2}{\Gamma(\alpha+\beta+\eta_2)}\left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-1}$$

$$- \frac{c_3}{\Gamma(\alpha+\beta+\gamma+\eta_3)}\left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1}$$

$$\left.- {}^\rho I_{a+}^{\alpha+\beta+\gamma+\delta} f(t_1, v(t_1), {}^\rho D_{a^+}^{\alpha,\alpha'} v(t_1), {}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} v(t_1)), {}^\rho D_{a^+}^{\gamma,\gamma'}({}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} v(t_1))))\right|$$

$$\leq \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\eta}\left(\left(\frac{|c_0|}{\Gamma(\eta_0)}\left(\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{\eta_0-1} - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{\eta_0-1}\right)\right.\right.$$

$$+ \frac{|c_1|}{\Gamma(\alpha+\eta_1)}\left(\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{\alpha+\eta_1-1} - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{\alpha+\eta_1-1}\right)$$

$$+ \frac{|c_2|}{\Gamma(\alpha+\beta+\eta_2)}\left(\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-1} - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-1}\right)$$

$$+ \frac{|c_3|}{\Gamma(\alpha+\beta+\gamma+\eta_3)}\left(\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1} - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1}\right)$$

$$+ {}^\rho I_{a+}^{\alpha+\beta+\gamma+\delta}\left|f(t_2, v(t_2), {}^\rho D_{a^+}^{\alpha,\alpha'} v(t_2), {}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} v(t_2)), {}^\rho D_{a^+}^{\gamma,\gamma'}({}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} v(t_2))))\right.$$

$$\left.- f(t_1, v(t_1), {}^\rho D_{a^+}^{\alpha,\alpha'} v(t_1), {}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} v(t_1)), {}^\rho D_{a^+}^{\gamma,\gamma'}({}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} v(t_1))))\right|$$

By hypothesis (H_2), we have

$$\left|(\mathcal{F}v(t_2)-\mathcal{F}v(t_1))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right|$$

$$\leq\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\left(\frac{|c_0|}{\Gamma(\eta_0)}\left(\left(\frac{t_2^\rho-a^\rho}{\rho}\right)^{\eta_0-1}-\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{\eta_0-1}\right)\right.$$

$$+\frac{|c_1|}{\Gamma(\alpha+\eta_1)}\left(\left(\frac{t_2^\rho-a^\rho}{\rho}\right)^{\alpha+\eta_1-1}-\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{\alpha+\eta_1-1}\right)$$

$$+\frac{|c_2|}{\Gamma(\alpha+\beta+\eta_2)}\left(\left(\frac{t_2^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-1}-\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-1}\right)$$

$$+\frac{|c_3|}{\Gamma(\alpha+\beta+\gamma+\eta_3)}\left(\left(\frac{t_2^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1}-\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1}\right)$$

$$+\frac{M\rho^{1-\alpha-\beta-\gamma-\delta}}{\Gamma(\alpha+\beta+\gamma+\delta)}\left(\int_a^{t_2}(t_2^\rho-s^\rho)^{\alpha+\beta+\gamma+\delta-1}s^{\rho-1}ds-\int_a^{t_1}(t_1^\rho-s^\rho)^{\alpha+\beta+\gamma+\delta-1}s^{\rho-1}ds.\right)$$

Hence,

$$\left|(\mathcal{F}v(t_2)-\mathcal{F}v(t_1))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right|$$

$$\leq\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\left(\frac{|c_0|}{\Gamma(\eta_0)}\left(\left(\frac{t_2^\rho-a^\rho}{\rho}\right)^{\eta_0-1}-\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{\eta_0-1}\right)\right.$$

$$+\frac{|c_1|}{\Gamma(\alpha+\eta_1)}\left(\left(\frac{t_2^\rho-a^\rho}{\rho}\right)^{\alpha+\eta_1-1}-\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{\alpha+\eta_1-1}\right)$$

$$+\frac{|c_2|}{\Gamma(\alpha+\beta+\eta_2)}\left(\left(\frac{t_2^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-1}-\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-1}\right)$$

$$+\frac{|c_3|}{\Gamma(\alpha+\beta+\gamma+\eta_3)}\left(\left(\frac{t_2^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1}-\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1}\right)$$

$$+\frac{M}{\Gamma(\delta+\gamma+\beta+\alpha)}\left(\int_a^{t_2}\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\delta-1}s^{\rho-1}ds-\int_a^{t_1}\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\delta-1}s^{\rho-1}ds.\right)$$

Thus,

$$\left|(\mathcal{F}v(t_2)-\mathcal{F}v(t_1))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right|$$
$$\leq\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\left(\frac{|c_0|}{\Gamma(\eta_0)}\left(\left(\frac{t_2^\rho-a^\rho}{\rho}\right)^{\eta_0-1}-\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{\eta_0-1}\right)\right.$$
$$+\frac{|c_1|}{\Gamma(\alpha+\eta_1)}\left(\left(\frac{t_2^\rho-a^\rho}{\rho}\right)^{\alpha+\eta_1-1}-\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{\alpha+\eta_1-1}\right)$$
$$+\frac{|c_2|}{\Gamma(\alpha+\beta+\eta_2)}\left(\left(\frac{t_2^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-1}-\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-1}\right)$$
$$+\frac{|c_3|}{\Gamma(\alpha+\beta+\gamma+\eta_3)}\left(\left(\frac{t_2^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1}-\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1}\right)$$
$$+\frac{M}{\Gamma(\delta+\gamma+\beta+\alpha)}\left(\int_a^{t_1}\left(\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\delta-1}-\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\delta-1}\right)s^{\rho-1}ds\right.$$
$$+\int_{t_1}^{t_2}\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\delta-1}s^{\rho-1}ds$$

Therefore,

$$\left|(\mathcal{F}v(t_2)-\mathcal{F}v(t_1))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right|$$
$$\leq\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\left(\frac{|c_0|}{\Gamma(\eta_0)}\left(\left(\frac{t_2^\rho-a^\rho}{\rho}\right)^{\eta_0-1}-\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{\eta_0-1}\right)\right.$$
$$+\frac{|c_1|}{\Gamma(\alpha+\eta_1)}\left(\left(\frac{t_2^\rho-a^\rho}{\rho}\right)^{\alpha+\eta_1-1}-\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{\alpha+\eta_1-1}\right)$$
$$+\frac{|c_2|}{\Gamma(\alpha+\beta+\eta_2)}\left(\left(\frac{t_2^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-1}-\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-1}\right)$$
$$+\frac{|c_3|}{\Gamma(\alpha+\beta+\gamma+\eta_3)}\left(\left(\frac{t_2^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1}-\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1}\right)$$
$$+\frac{M}{(\delta+\gamma+\beta+\alpha)\Gamma(\delta+\gamma+\beta+\alpha)}$$
$$\times\left(\left(\left(\frac{t_2^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\delta}-\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\delta}+\int_a^{t_2}\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\delta-1}s^{\rho-1}ds\right.\right.$$
$$-\int_a^{t_1}\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\delta-1}s^{\rho-1}\,ds$$

Finally,

$$\left|(\mathcal{F}v(t_2)-\mathcal{F}v(t_1))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right|$$

$$\leq \left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\left(\frac{|c_0|}{\Gamma(\eta_0)}\left(\left(\frac{t_2^\rho-a^\rho}{\rho}\right)^{\eta_0-1}-\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{\eta_0-1}\right)\right.$$

$$+\frac{|c_1|}{\Gamma(\alpha+\eta_1)}\left(\left(\frac{t_2^\rho-a^\rho}{\rho}\right)^{\alpha+\eta_1-1}-\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{\alpha+\eta_1-1}\right)$$

$$+\frac{|c_2|}{\Gamma(\alpha+\beta+\eta_2)}\left(\left(\frac{t_2^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-1}-\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-1}\right)$$

$$+\frac{|c_3|}{\Gamma(\alpha+\beta+\gamma+\eta_3)}\left(\left(\frac{t_2^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1}-\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1}\right)$$

$$+\frac{M}{\Gamma(\alpha+\beta+\gamma+\delta+1)}\left(\left(\frac{t_2^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\delta}-\left(\frac{t_1^\rho-a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\delta}\right)$$

$$\left.-\frac{M}{\Gamma(\alpha+\beta+\gamma+\delta+1)}\left(\frac{t_2^\rho-t_1^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\delta}\right..$$

Consequently, we have

$$\|(\mathcal{F}v)(t_2)-(\mathcal{F}v)(t_1)\|_{C_{1-\eta_i,\rho}}\longrightarrow 0,\ when\ t_1\to t_2.$$

which implies that $\mathcal{F}$ is equicontinuous.

Thus, by Ascoli-Arzela theorem, the operator $\mathcal{F}$ is completely continuous and by Schauder fixed point theorem the operator $\mathcal{F}$ has a fixed point $v\in\mathcal{B}_r$.

2.3.2 Uniqueness

Theorem 2.2 *We suppose that hypotheses* (H_1) *and* (H_3) *are satisfied, then the problem* (2.15) *has a unique solution in* $C_{1-\eta,\rho}([a,b])$.

Proof For all $u, v \in C_{1-\eta,\rho}([a,b])$ and for $t \in [a,b]$, we have

$$\left|(\mathcal{F}v(t) - \mathcal{F}u(t))\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\eta}\right|$$

$$= \left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\eta}\left(\frac{c_0}{\Gamma(\eta_0)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\eta_0-1} + \frac{c_1}{\Gamma(\alpha+\eta_1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\eta_1-1}\right.\right.$$

$$+ \frac{c_2}{\Gamma(\alpha+\beta+\eta_2)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-1} + \frac{c_3}{\Gamma(\alpha+\beta+\gamma+\eta_3)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1}$$

$$+ {}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta} f(t, v(t), {}^\rho D_{a^+}^{\alpha,\alpha'} v(t), {}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} v(t)), {}^\rho D_{a^+}^{\gamma,\gamma'}({}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} v(t))))$$

$$- \frac{c_0}{\Gamma(\eta_0)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\eta_0-1} - \frac{c_1}{\Gamma(\alpha+\eta_1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\eta_1-1} - \frac{c_2}{\Gamma(\alpha+\beta+\eta_2)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-1}$$

$$- \frac{c_3}{\Gamma(\alpha+\beta+\gamma+\eta_3)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1}$$

$$\left. - {}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta} f(t, u(t), {}^\rho D_{a^+}^{\alpha,\alpha'} u(t), {}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} u(t)), {}^\rho D_{a^+}^{\gamma,\gamma'}({}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} u(t))))\right|$$

$$\leq \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\eta}$$

$$\times \left({}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}\left| f(t, v(t), {}^\rho D_{a^+}^{\alpha,\alpha'} v(t), {}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} v(t)), {}^\rho D_{a^+}^{\gamma,\gamma'}({}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} v(t))))\right.\right.$$

$$\left.\left. - f(t, u(t), {}^\rho D_{a^+}^{\alpha,\alpha'} u(t), {}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} u(t)), {}^\rho D_{a^+}^{\gamma,\gamma'}({}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} u(t))))\right|\right)$$

According to hypothesis (H_1), we have

$$\left|(\mathcal{F}v_n(t) - \mathcal{F}v(t))\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\eta}\right|$$

$$\leq L\left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\eta} {}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}\left(|v(t) - u(t)| + \left|{}^\rho D_{a^+}^{\alpha,\alpha'} v(t) - {}^\rho D_{a^+}^{\alpha,\alpha'} u(t)\right|\right.$$

$$+ \left|{}^\rho D_{a^+}^{\beta,\beta'}\left({}^\rho D_{a^+}^{\alpha,\alpha'} v(t)\right) - {}^\rho D_{a^+}^{\beta,\beta'}\left({}^\rho D_{a^+}^{\alpha,\alpha'} u(t)\right)\right|$$

$$\left. + \left|{}^\rho D_{a^+}^{\gamma,\gamma'}\left({}^\rho D_{a^+}^{\beta,\beta'}\left({}^\rho D_{a^+}^{\alpha,\alpha'} v(t)\right) - {}^\rho D_{a^+}^{\gamma,\gamma'}\left({}^\rho D_{a^+}^{\beta,\beta'}\left({}^\rho D_{a^+}^{\alpha,\alpha'} u(t)\right)\right)\right|\right)$$

By using Property (P_1), we obtain

$$\left|(\mathcal{F}v(t)-\mathcal{F}u(t))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right|$$

$$\leq L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\ {}^\rho I_{a_+}^{\alpha+\beta+\gamma+\delta}\Big(|v(t)-u(t)|+\left|{}^\rho I_{a^+}^{\alpha'(1-\alpha)}\,{}^\rho D_{a^+}^{\eta_0}v(t)-\,{}^\rho I_{a^+}^{\alpha'(1-\alpha)}\,{}^\rho D_{a^+}^{\eta_0}u(t)\right|$$

$$+\left|{}^\rho I_{a^+}^{\alpha'(1-\alpha)+\beta'(1-\beta)}\,{}^\rho D_{a^+}^{\eta_0+\eta_1}v(t)-\,{}^\rho I_{a^+}^{\alpha'(1-\alpha)+\beta'(1-\beta)}\,{}^\rho D_{a^+}^{\eta_0+\eta_1}u(t)\right|$$

$$+\left|{}^\rho I_{a^+}^{\alpha'(1-\alpha)+\beta'(1-\beta)+\gamma'(1-\gamma)}\,{}^\rho D_{a^+}^{\eta_0+\eta_1+\eta_2}v(t)-\,{}^\rho I_{a^+}^{\alpha'(1-\alpha)+\beta'(1-\beta)+\gamma'(1-\gamma)}\,{}^\rho D_{a^+}^{\eta_0+\eta_1+\eta_2}u(t)\right|\Big)$$

$$\leq L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\ {}^\rho I_{a_+}^{\alpha+\beta+\gamma+\delta}\Big(|v(t)-u(t)|+\left|{}^\rho D_{a^+}^{\alpha,\alpha'}|v(t)-u(t)|\right|+\left|{}^\rho D_{a^+}^{\alpha+\beta,\alpha'+\beta'}|v(t)-u(t)|\right|$$

$$+\left|{}^\rho D_{a^+}^{\alpha+\beta+\gamma,\alpha'+\beta'+\gamma'}|v(t)-u(t)|\right|\Big)$$

Then,

$$\left|(\mathcal{F}v(t)-\mathcal{F}u(t))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right|$$

$$\leq L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\Big({}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}|v(t)-u(t)|+\left|{}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}\,{}^\rho D_{a^+}^{\alpha,\alpha'}|v(t)-u(t)|\right|$$

$$+\left|{}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}\,{}^\rho D_{a^+}^{\alpha+\beta,\alpha'+\beta'}|v(t)-u(t)|\right|+\left|{}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}\,{}^\rho D_{a^+}^{\alpha+\beta+\gamma,\alpha'+\beta'+\gamma'}|v(t)-u(t)|\right|\Big)$$

$$\leq L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\Big({}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}|v(t)-u(t)|+\left|{}^\rho I_{a^+}^{\beta+\gamma+\delta}({}^\rho I_{a_+}^{\alpha}\,{}^\rho D_{a^+}^{\alpha,\alpha'})|v(t)-u(t)|\right|$$

$$+\left|{}^\rho I_{a^+}^{\alpha+\gamma+\delta}({}^\rho I_{a^+}^{\beta}\,{}^\rho D_{a^+}^{\beta,\beta'})|v(t)-u(t)|\right|+\left|{}^\rho I_{a^+}^{\alpha+\beta+\delta}({}^\rho I_{a^+}^{\gamma}\,{}^\rho D_{a^+}^{\gamma,\gamma'})|v(t)-u(t)|\right|\Big)$$

According to Lemma 2.10, we obtain

$$\left|(\mathcal{F}v(t)-\mathcal{F}u(t))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right|$$

$$\leq L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\Big({}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}|v(t)-u(t)|+\left|{}^\rho I_{a^+}^{\beta+\gamma+\delta}({}^\rho I_{a^+}^{\eta_0}\,{}^\rho D_{a^+}^{\eta_0})|v(t)-u(t)|\right|$$

$$+\left|{}^\rho I_{a^+}^{\alpha+\gamma+\delta}({}^\rho I_{a^+}^{\eta_1}\,{}^\rho D_{a^+}^{\eta_1})|v(t)-u(t)|\right|+\left|{}^\rho I_{a_+}^{\alpha+\beta+\delta}({}^\rho I_{a^+}^{\eta_2}\,{}^\rho D_{a^+}^{\eta_2})|v(t)-u(t)|\right|\Big)$$

As stated in Lemma 2.7

$$\left|(\mathcal{F}v(t)-\mathcal{F}u(t))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right| \leq L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\Bigg[\left|{}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}|v(t)-u(t)|\right|$$

$$+\left|{}^\rho I_{a^+}^{\beta+\gamma+\delta}\left(|v(t)-u(t)|-\frac{{}^\rho I_{a^+}^{\eta_0}|v-u|(a)}{\Gamma(\eta_0)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_0-1}\right)\right|$$

$$+\left|{}^\rho I_{a^+}^{\alpha+\gamma+\delta}\left(|v(t)-u(t)|-\frac{{}^\rho I_{a^+}^{\eta_1}|v-u|(a)}{\Gamma(\eta_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_1-1}\right)\right|$$

$$+\left|{}^\rho I_{a^+}^{\alpha+\beta+\delta}\left(|v(t)-u(t)|-\frac{{}^\rho I_{a^+}^{\eta_2}|v-u|(a)}{\Gamma(\eta_2)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_2-1}\right)\right|\Bigg]$$

By using Lemma 2.9, we get

$$\left|(\mathcal{F}v(t)-\mathcal{F}u(t))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right|$$

$$\leq L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\Bigg[\left|{}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}\|v-u\|_{C_{1-\eta,\rho}}\right|+\left|{}^\rho I_{a^+}^{\beta+\gamma+\delta}\|v-u\|_{C_{1-\eta,\rho}}\right|$$

$$+\left|{}^\rho I_{a^+}^{\alpha+\gamma+\delta}\|v-u\|_{C_{1-\eta,\rho}}\right|+\left|{}^\rho I_{a^+}^{\alpha+\beta+\delta}\|v-u\|_{C_{1-\eta,\rho}}\right|\Bigg]$$

$$\leq L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\Bigg[\frac{\rho^{-1-\alpha-\beta-\gamma-\delta}}{(\alpha+\beta+\gamma+\delta)\Gamma(\alpha+\beta+\gamma+\delta)}(t^\rho-a^\rho)^{\alpha+\beta+\gamma+\delta}\|v-u\|_{C_{1-\eta,\rho}}$$

$$+\frac{\rho^{-1-\beta-\gamma-\delta}}{(\beta+\gamma+\delta)\Gamma(\beta+\gamma+\delta)}(t^\rho-a^\rho)^{\beta+\gamma+\delta}\|v-u\|_{C_{1-\eta,\rho}}$$

$$+\frac{\rho^{-1-\alpha-\gamma-\delta}}{(\alpha+\gamma+\delta)\Gamma(\alpha+\gamma+\delta)}(t^\rho-a^\rho)^{\alpha+\gamma+\delta}\|v-u\|_{C_{1-\eta,\rho}}$$

$$+\frac{\rho^{-1-\alpha-\beta-\delta}}{(\alpha+\beta+\delta)\Gamma(\alpha+\beta+\delta)}(t^\rho-a^\rho)^{\alpha+\beta+\delta}\|v-u\|_{C_{1-\eta,\rho}}\Bigg]$$

$$\leq L\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\|v-u\|_{C_{1-\eta,\rho}}\Bigg[\frac{\rho^{-1-\alpha-\beta-\gamma-\delta}}{\Gamma(\alpha+\beta+\gamma+\delta+1)}(b^\rho-a^\rho)^{\alpha+\beta+\gamma+\delta}$$

$$+\frac{\rho^{-1-\beta-\gamma-\delta}}{\Gamma(\beta+\gamma+\delta+1)}(b^\rho-a^\rho)^{\beta+\gamma+\delta}+\frac{\rho^{-1-\alpha-\gamma-\delta}}{\Gamma(\alpha+\gamma+\delta+1)}(b^\rho-a^\rho)^{\alpha+\gamma+\delta}$$

$$+\frac{\rho^{-1-\alpha-\beta-\delta}}{\Gamma(\alpha+\beta+\delta+1)}(b^\rho-a^\rho)^{\alpha+\beta+\delta}\Bigg]$$

Finally,

$$\begin{aligned}
&\left|(\mathcal{F}v(t)-\mathcal{F}u(t))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right| \\
&\leq L\left[\frac{\rho^{-1-\alpha-\beta-\gamma-\delta}}{\Gamma(\alpha+\beta+\gamma+\delta+1)}(b^\rho-a^\rho)^{\alpha+\beta+\gamma+\delta}+\frac{\rho^{-1-\beta-\gamma-\delta}}{\Gamma(\beta+\gamma+\delta+1)}(b^\rho-a^\rho)^{\beta+\gamma+\delta}\right. \\
&\left.+\frac{\rho^{-1-\alpha-\gamma-\delta}}{\Gamma(\alpha+\gamma+\delta+1)}(b^\rho-a^\rho)^{\alpha+\gamma+\delta}+\frac{\rho^{-1-\alpha-\beta-\delta}}{\Gamma(\alpha+\beta+\delta+1)}(b^\rho-a^\rho)^{\alpha+\beta+\delta}\right]\|v-u\|_{C_{1-\eta,\rho}} \\
&\leq LK_1\|v-u\|_{C_{1-\eta,\rho}}
\end{aligned}$$

Hence,

$$\left|(\mathcal{F}v(t)-\mathcal{F}u(t))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right|\leq LK_1\|v-u\|_{C_{1-\eta,\rho}}$$

Therefore,

$$\|\mathcal{F}v(t)-\mathcal{F}u(t)\|_{C_{1-\eta,\rho}}\leq LK_1\|v-u\|_{C_{1-\eta,\rho}}$$

Since $LK_1<1$, the operator $\mathcal{F}$ is a contraction. Therefore, according to the Banach contraction principle, the problem (2.15) has a unique solution. ■

Chapter 3

Stability in the sense of Ulam-Hyers

3.1 Introduction

The stability of functional equations has garnered significant attention due to its critical relevance in the field of science and engineering. Numerous notions on stability have been explored, each with distinct definitions that are not equivalent. However, authors in their work on this subject employ the identical denomination "stability", which encompasses (Lyapounov stability, Von Neumann stability, asymptotic stability, among other types of stability ...)

During a public lecture held at the University of Wisconsin in 1940, Ulam presented a theoretical problem that pertained to the stability of functional equations, specifically those related to group homomorphisms [42]. The inquiry pertains to the circumstances in which an additive application is considered close enough to an approximated additive mapping.

Let $\mathcal{G}_1$ be a group and $\mathcal{G}_2$ be a metric group equipped with the distance $d(.,.)$.

For all $\varepsilon > 0$, there exists $\delta > 0$ such that if the function $f : \mathcal{G}_1 \to \mathcal{G}_2$ satisfies the following inequality

$$d(f(xy), f(x)f(y)) \leq \delta, \qquad \forall x, y \in \mathcal{G}_1,$$

then is there a homomorphism $F : \mathcal{G}_1 \to \mathcal{G}_2$ such that

$$d(f(x), F(x)) < \varepsilon, \qquad \forall x \in \mathcal{G}_1?$$

Later in 1941, Hyers [21] was able to partly answer the question. Then between 1982 and 1998, Rassias [34, 35] established stability in the Hyers-Ulam sense.

Many researchers have been interested in the stability problem in the Ulam-Hyers sense. Moreover a large number of monographs and articles have been published with the aim of generalising Hyers' results [1, 3, 6, 9, 21, 37, 40, 41, 43, 44]. However, In this chapter, we aim to define crucial criteria for evaluating the stability of the solution to the Snap problem (2.15) using both Ulam-Hyers and generalized Ulam-Hyers principles.

3.2 Definitions

Definition 3.1 [8] The fractional Snap problem (2.15) is Ulam-Hyers stable if there exists a real number $\mathcal{C} > 0$ such that for each $\varepsilon > 0$ and for each solution $v \in C_{1-\eta,\rho}([a,b])$ of the inequality

$$|\mathcal{F}v(t)| \leq \varepsilon, \qquad t \in [a,b]. \tag{3.1}$$

There exists a solution $u \in C_{1-\eta,\rho}([a,b])$ of the equation of problem (2.15) with

$$\|v-u\|_{C_{1-\eta,\rho}} < \mathcal{C}\varepsilon, \qquad t \in [a,b].$$

Definition 3.2 [8] The fractional Snap problem (2.15) is generalized Ulam-Hyers stable if there exists a function $\varphi \in C_{1-\eta,\rho}([a,b])$, $\varphi(0) = 0$ such that for each solution $v \in C_{1-\eta,\rho}([a,b])$ of the inequality

$$|\mathcal{F}v(t)| \leq \varepsilon, \qquad t \in [a,b]. \tag{3.2}$$

There exists a solution $u \in C_{1-\eta,\rho}([a,b])$ of the equation of problem (2.15) with

$$\|v-u\|_{C_{1-\eta,\rho}} < \varphi(\varepsilon), \qquad t \in [a,b].$$

Remark 3.1 [8] A function $v \in \mathcal{B}_r$ is a solution of the inequality (3.2) if and only if there exists a function $h \in C_{1-\eta,\rho}([a,b])$ such that

1. $|h(t)| \leq \varepsilon, \quad t \in [a,b]$.
2. For all $t \in [a,b]$,

$$\begin{aligned}&{}^{\rho}D_{a+}^{\delta,\delta'}({}^{\rho}D_{a+}^{\gamma,\gamma'}({}^{\rho}D_{a+}^{\beta,\beta'}\,{}^{\rho}D_{a+}^{\alpha,\alpha'}v(t))))\\ &= f(t,v(t),{}^{\rho}D_{a+}^{\alpha,\alpha'}v(t),{}^{\rho}D_{a+}^{\beta,\beta'}\,{}^{\rho}D_{a+}^{\alpha,\alpha'}v(t)),{}^{\rho}D_{a+}^{\gamma,\gamma'}({}^{\rho}D_{a+}^{\beta,\beta'}\,{}^{\rho}D_{a+}^{\alpha,\alpha'}v(t)))) + h(t).\end{aligned}$$

3.3 Stability Study

Theorem 3.1 *It is assumed that the following hypotheses are satisfied*

(H_1) $\exists L > 0$ *such that* $x, y, z, w, x', y', z', w' \in C^{\eta}_{1-\eta,\rho}([a,b])$,

$$|f(t,x,y,z,w) - f(t,x',y',z',w')| \leq L(|x-x'| + |y-y'| + |z-z'| + |w-w'|) \quad , t \in [a,b].$$

(H_2) $LK_1 < 1$,

Then the solution of problem (2.15) is stable in the sense of Ulam-Hyers (UH) and generalized Ulam-Hyers sense (GUH).

Proof Let $\varepsilon \in \mathbb{R}^+$, $v \in \mathcal{B}_r$ a particular solution of the inequality (3.2) then by Remark (3.1), we have

$$\begin{aligned} v(t) &= \frac{c_0}{\Gamma(\eta_0)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\eta_0-1} + \frac{c_1}{\Gamma(\alpha+\eta_1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\eta_1-1} \\ &+ \frac{c_2}{\Gamma(\alpha+\beta+\eta_2)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-1} + \frac{c_3}{\Gamma(\alpha+\beta+\gamma+\eta_3)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1} \\ &+ {}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}(f(t, v(t), {}^\rho D_{a^+}^{\alpha,\alpha'} v(t), {}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} v(t)), {}^\rho D_{a^+}^{\gamma,\gamma'}({}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} v(t))) + h(t)). \end{aligned} \tag{3.3}$$

Let $u \in \mathcal{B}_r$ be the unique solution of problem (2.15) then by Lemma (2.13) the integral equation of problem (2.15) is given by

$$\begin{aligned} u(t) &= \frac{c_0}{\Gamma(\eta_0)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\eta_0-1} + \frac{c_1}{\Gamma(\alpha+\eta_1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\eta_1-1} \\ &+ \frac{c_2}{\Gamma(\alpha+\beta+\eta_2)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-1} + \frac{c_3}{\Gamma(\alpha+\beta+\gamma+\eta_3)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1} \\ &+ {}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta} f(t, u(t), {}^\rho D_{a^+}^{\alpha,\alpha'} u(t), {}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} u(t)), {}^\rho D_{a^+}^{\gamma,\gamma'}({}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} u(t)))). \end{aligned} \tag{3.4}$$

On the other hand, according to Remark 3.1, for all $t \in [a, b]$, we have

$$\left|(\mathcal{F}v(t) - \mathcal{F}u(t))\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\eta}\right|$$

$$\begin{aligned} &= \Bigg|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\eta}\Bigg(\frac{c_0}{\Gamma(\eta_0)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\eta_0-1} + \frac{c_1}{\Gamma(\alpha+\eta_1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\eta_1-1} \\ &+ \frac{c_2}{\Gamma(\alpha+\beta+\eta_2)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-1} + \frac{c_3}{\Gamma(\alpha+\beta+\gamma+\eta_3)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1} \\ &+ {}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}(f(t, v(t), {}^\rho D_{a^+}^{\alpha,\alpha'} v(t), {}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} v(t)), {}^\rho D_{a^+}^{\gamma,\gamma'}({}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} v(t)))) + h(t)) \\ &- \frac{c_0}{\Gamma(\eta_0)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\eta_0-1} - \frac{c_1}{\Gamma(\alpha+\eta_1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\eta_1-1} - \frac{c_2}{\Gamma(\alpha+\beta+\eta_2)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\eta_2-1} \\ &- \frac{c_3}{\Gamma(\alpha+\beta+\gamma+\eta_3)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\beta+\gamma+\eta_3-1} \\ &- {}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta} f(t, u(t), {}^\rho D_{a^+}^{\alpha,\alpha'} u(t), {}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} u(t)), {}^\rho D_{a^+}^{\gamma,\gamma'}({}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'} u(t))))\Bigg| \end{aligned}$$

Hence,

$$\left|(\mathcal{F}v(t)-\mathcal{F}u(t))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right|$$

$$\leq\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\left({}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}\Big|f(t,v(t),{}^\rho D_{a^+}^{\alpha,\alpha'}v(t),{}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'}v(t)),{}^\rho D_{a^+}^{\gamma,\gamma'}({}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'}v(t))))\right.$$

$$\left.-f(t,u(t),{}^\rho D_{a^+}^{\alpha,\alpha'}u(t),{}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'}u(t)),{}^\rho D_{a^+}^{\gamma,\gamma'}({}^\rho D_{a^+}^{\beta,\beta'}({}^\rho D_{a^+}^{\alpha,\alpha'}u(t))))\Big|\right)$$

$$+\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}{}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}|h(t)|$$

According to hypothesis (H_1), we have

$$\left|(\mathcal{F}v(t)-\mathcal{F}u(t))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right|$$

$$\leq L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}{}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}\left(|v(t)-u(t)|+\left|{}^\rho D_{a^+}^{\alpha,\alpha'}v(t)-{}^\rho D_{a^+}^{\alpha,\alpha'}u(t)\right|\right.$$

$$+\left|{}^\rho D_{a^+}^{\beta,\beta'}\left({}^\rho D_{a^+}^{\alpha,\alpha'}v(t)\right)-{}^\rho D_{a^+}^{\beta,\beta'}\left({}^\rho D_{a^+}^{\alpha,\alpha'}u(t)\right)\right|$$

$$\left.+\left|{}^\rho D_{a^+}^{\gamma,\gamma'}\left({}^\rho D_{a^+}^{\beta,\beta'}\left({}^\rho D_{a^+}^{\alpha,\alpha'}v(t)\right)-{}^\rho D_{a^+}^{\gamma,\gamma'}\left({}^\rho D_{a^+}^{\beta,\beta'}\left({}^\rho D_{a^+}^{\alpha,\alpha'}u(t)\right)\right)\right|\right)+\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}{}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}\varepsilon$$

By using Property (P_1), we obtain

$$\left|(\mathcal{F}v(t)-\mathcal{F}u(t))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right|$$

$$\leq L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}{}^\rho I_{a_+}^{\alpha+\beta+\gamma+\delta}\left(|v(t)-u(t)|+\left|{}^\rho I_{a^+}^{\alpha'(1-\alpha)}\,{}^\rho D_{a^+}^{\eta_0}v(t)-{}^\rho I_{a^+}^{\alpha'(1-\alpha)}\,{}^\rho D_{a^+}^{\eta_0}u(t)\right|\right.$$

$$+\left|{}^\rho I_{a^+}^{\alpha'(1-\alpha)+\beta'(1-\beta)}\,{}^\rho D_{a^+}^{\eta_0+\eta_1}v(t)-{}^\rho I_{a^+}^{\alpha'(1-\alpha)+\beta'(1-\beta)}\,{}^\rho D_{a^+}^{\eta_0+\eta_1}u(t)\right|$$

$$\left.+\left|{}^\rho I_{a^+}^{\alpha'(1-\alpha)+\beta'(1-\beta)+\gamma'(1-\gamma)}\,{}^\rho D_{a^+}^{\eta_0+\eta_1+\eta_2}v(t)-{}^\rho I_{a^+}^{\alpha'(1-\alpha)+\beta'(1-\beta)+\gamma'(1-\gamma)}\,{}^\rho D_{a^+}^{\eta_0+\eta_1+\eta_2}u(t)\right|\right)$$

$$+\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\frac{\rho^{-1-\alpha-\beta-\gamma-\delta}}{(\alpha+\beta+\gamma+\delta)\Gamma(\alpha+\beta+\gamma+\delta)}(t^\rho-a^\rho)^{\alpha+\beta+\gamma+\delta}\varepsilon$$

$$\leq L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}{}^\rho I_{a_+}^{\alpha+\beta+\gamma+\delta}\left(|v(t)-u(t)|+\left|{}^\rho D_{a^+}^{\alpha,\alpha'}|v(t)-u(t)|\right|+\left|{}^\rho D_{a^+}^{\alpha+\beta,\alpha'+\beta'}|v(t)-u(t)|\right|\right.$$

$$\left.+\left|{}^\rho D_{a^+}^{\alpha+\beta+\gamma,\alpha'+\beta'+\gamma'}|v(t)-u(t)|\right|\right)+\frac{\rho^{\eta-2-\alpha-\beta-\gamma-\delta}}{\Gamma(\alpha+\beta+\gamma+\delta+1)}(b^\rho-a^\rho)^{1-\eta+\alpha+\beta+\gamma+\delta}\varepsilon$$

Then,

$$\left|(\mathcal{F}v(t)-\mathcal{F}u(t))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right|$$

$$\leq L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\Big({}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}|v(t)-u(t)|+\left|{}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}\,{}^\rho D_{a^+}^{\alpha,\alpha'}|v(t)-u(t)|\right|$$

$$+\left|{}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}\,{}^\rho D_{a^+}^{\alpha+\beta,\alpha'+\beta'}|v(t)-u(t)|\right|+\left|{}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}\,{}^\rho D_{a^+}^{\alpha+\beta+\gamma,\alpha'+\beta'+\gamma'}|v(t)-u(t)|\right|\Big)+\,K_3\varepsilon$$

$$\leq L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\Big({}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}|v(t)-u(t)|+\left|{}^\rho I_{a^+}^{\beta+\gamma+\delta}({}^\rho I_{a_+}^{\alpha}\,{}^\rho D_{a^+}^{\alpha,\alpha'})|v(t)-u(t)|\right|$$

$$+\left|{}^\rho I_{a^+}^{\alpha+\gamma+\delta}({}^\rho I_{a^+}^{\beta}\,{}^\rho D_{a^+}^{\beta,\beta'})|v(t)-u(t)|\right|+\left|{}^\rho I_{a^+}^{\alpha+\beta+\delta}({}^\rho I_{a^+}^{\gamma}\,{}^\rho D_{a^+}^{\gamma,\gamma'})|v(t)-u(t)|\right|\Big)+\,K_3\varepsilon$$

According to Lemma 2.10, we have

$$\left|(\mathcal{F}v(t)-\mathcal{F}u(t))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right|$$

$$\leq L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\Big({}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}|v(t)-u(t)|+\left|{}^\rho I_{a^+}^{\beta+\gamma+\delta}({}^\rho I_{a^+}^{\eta_0}\,{}^\rho D_{a^+}^{\eta_0})|v(t)-u(t)|\right|$$

$$+\left|{}^\rho I_{a^+}^{\alpha+\gamma+\delta}({}^\rho I_{a^+}^{\eta_1}\,{}^\rho D_{a^+}^{\eta_1})|v(t)-u(t)|\right|+\left|{}^\rho I_{a_+}^{\alpha+\beta+\delta}({}^\rho I_{a^+}^{\eta_2}\,{}^\rho D_{a^+}^{\eta_2})|v(t)-u(t)|\right|\Big)+\,K_3\varepsilon$$

As stated in Lemma 2.7

$$\left|(\mathcal{F}v(t)-\mathcal{F}u(t))\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\right|$$

$$\leq L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\left[\left|{}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}|v(t)-v(t)|\right|\right.$$

$$+\left|{}^\rho I_{a^+}^{\beta+\gamma+\delta}\left(|v(t)-u(t)|-\frac{{}^\rho I_{a^+}^{\eta_0}|v-u|(a)}{\Gamma(\eta_0)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_0-1}\right)\right|$$

$$+\left|{}^\rho I_{a^+}^{\alpha+\gamma+\delta}\left(|v(t)-u(t)|-\frac{{}^\rho I_{a^+}^{\eta_1}|v-u|(a)}{\Gamma(\eta_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_1-1}\right)\right|$$

$$\left.+\left|{}^\rho I_{a^+}^{\alpha+\beta+\delta}\left(|v(t)-u(t)|-\frac{{}^\rho I_{a^+}^{\eta_2}|v-u|(a)}{\Gamma(\eta_2)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\eta_2-1}\right)\right|\right]+\,K_3\,\varepsilon$$

By using Lemma 2.9, we obtain

$$
\begin{aligned}
&\left|(\mathcal{F}v(t)-\mathcal{F}u(t))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right| \\
&\leq L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\left[\left|{}^\rho I_{a^+}^{\alpha+\beta+\gamma+\delta}\|v-u\|_{C_{1-\eta,\rho}}\right|+\left|{}^\rho I_{a^+}^{\beta+\gamma+\delta}\|v-u\|_{C_{1-\eta,\rho}}\right|\right. \\
&\left.+\left|{}^\rho I_{a^+}^{\alpha+\gamma+\delta}\|v-u\|_{C_{1-\eta,\rho}}\right|+\left|{}^\rho I_{a^+}^{\alpha+\beta+\delta}\|v-u\|_{C_{1-\eta,\rho}}\right|\right]+K_3\,\varepsilon \\
&\leq L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\left[\frac{\rho^{-1-\alpha-\beta-\gamma-\delta}}{(\alpha+\beta+\gamma+\delta)\Gamma(\alpha+\beta+\gamma+\delta)}(t^\rho-a^\rho)^{\alpha+\beta+\gamma+\delta}\|v-u\|_{C_{1-\eta,\rho}}\right. \\
&+\frac{\rho^{-1-\beta-\gamma-\delta}}{(\beta+\gamma+\delta)\Gamma(\beta+\gamma+\delta)}(t^\rho-a^\rho)^{\beta+\gamma+\delta}\|v-u\|_{C_{1-\eta,\rho}} \\
&+\frac{\rho^{-1-\alpha-\gamma-\delta}}{(\alpha+\gamma+\delta)\Gamma(\alpha+\gamma+\delta)}(t^\rho-a^\rho)^{\alpha+\gamma+\delta}\|v-u\|_{C_{1-\eta,\rho}} \\
&\left.+\frac{\rho^{-1-\alpha-\beta-\delta}}{(\alpha+\beta+\delta)\Gamma(\alpha+\beta+\delta)}(t^\rho-a^\rho)^{\alpha+\beta+\delta}\|v-u\|_{C_{1-\eta,\rho}}\right]+K_3\,\varepsilon \\
&\leq L\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\eta}\|v-u\|_{C_{1-\eta,\rho}}\left[\frac{\rho^{-1-\alpha-\beta-\gamma-\delta}}{\Gamma(\alpha+\beta+\gamma+\delta+1)}(b^\rho-a^\rho)^{\alpha+\beta+\gamma+\delta}\right. \\
&+\frac{\rho^{-1-\beta-\gamma-\delta}}{\Gamma(\beta+\gamma+\delta+1)}(b^\rho-a^\rho)^{\beta+\gamma+\delta}+\frac{\rho^{-1-\alpha-\gamma-\delta}}{\Gamma(\alpha+\gamma+\delta+1)}(b^\rho-a^\rho)^{\alpha+\gamma+\delta} \\
&\left.+\frac{\rho^{-1-\alpha-\beta-\delta}}{\Gamma(\alpha+\beta+\delta+1)}(b^\rho-a^\rho)^{\alpha+\beta+\delta}\right]+K_3\,\varepsilon
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\left|(\mathcal{F}v(t)-\mathcal{F}u(t))\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\eta}\right| \\
&\leq L\left[\frac{\rho^{\eta-2-\alpha-\beta-\gamma-\delta}}{\Gamma(\alpha+\beta+\gamma+\delta+1)}(b^\rho-a^\rho)^{1-\eta+\alpha+\beta+\gamma+\delta}+\frac{\rho^{\eta-2-\beta-\gamma-\delta}}{\Gamma(\beta+\gamma+\delta+1)}(t^\rho-a^\rho)^{1-\eta+\beta+\gamma+\delta}\right. \\
&\left.+\frac{\rho^{\eta-2-\alpha-\gamma-\delta}}{\Gamma(\alpha+\gamma+\delta+1)}(b^\rho-a^\rho)^{1-\eta+\alpha+\gamma+\delta}+\frac{\rho^{\eta-2-\alpha-\beta-\delta}}{\Gamma(\alpha+\beta+\delta+1)}(b^\rho-a^\rho)^{1-\eta+\alpha+\beta+\delta}\right]\|v-u\|_{C_{1-\eta,\rho}} \\
&+K_3\,\varepsilon \\
&\leq LK_1\|v-u\|_{C_{1-\eta,\rho}}+K_3\,\varepsilon
\end{aligned}
$$

Consequently,

$$\|v-u\|_{C_{1-\eta,\rho}} \leq \frac{K_3}{1-LK_1}\,\varepsilon.$$

This proves the existence of a positive real number $\mathcal{C}=\frac{K_3}{1-LK_1}$. And so, by Definition 3.1, the solution of problem (2.15) is Ulam-Hyers stable. By posing $\varphi(\varepsilon)=\mathcal{C}\varepsilon$, the problem (2.15) is therefore stable in the generalized Ulam-Hyers sense. ■

3.4 Numerical Simulation

Within this particular segment, it is our intention to provide a comprehensive example that validates the theoretical process outlined in both Chapters 2 and 3.

Example

We consider the following fractional Snap problem :

$$\begin{cases} {}^{0.4}D_{0^+}^{0.01,0.02}v(t) = v_1(t), \\ {}^{0.4}D_{0^+}^{0.03,0.04}v_1(t) = v_2(t), \\ {}^{0.4}D_{0^+}^{0.05,0.06}v_2(t) = v_3(t), \\ {}^{0.4}D_{0^+}^{0.07,0.08}v_3(t) = \frac{t}{1+\sqrt{2}} + \frac{1}{100}\cos(v) + \frac{\cos(v_1)}{100(1+\cos^2(v_1))} + \frac{3}{100}\big(cos(v_2)+cos(v_3)\big), \end{cases} \tag{3.5}$$

for $t\in[0,1]$ and,

$${}^{0.4}I_{0^+}^{0.9702}v(0)=-1, \quad {}^{0.4}I_{0^+}^{0.9312}v_1(0)=2, \quad {}^{0.4}I_{0^+}^{0.893}v_2(0)=0, \quad {}^{0.4}I_{0^+}^{0.856}v_3(0)=1.$$

Clearly,

$$\alpha=0.01,\alpha^{'}=0.02 \qquad \beta=0.04,\beta^{'}=0.03, \qquad \gamma=0.06,\gamma^{'}=0.05 \qquad \delta=0.08,\delta^{'}=0.07$$

$$\eta_0=0.0298, \qquad \eta_1=0.0688, \qquad \eta_2=0.107, \qquad \eta_3=0.1444, \qquad \eta=0.35.$$

And

$$f(t,v,v_1,v_2,v_3)=\frac{t}{1+\sqrt{2}}+\frac{1}{100}\cos(v)+\frac{\cos(v_1)}{100(1+cos^2(v_1))}+\frac{3}{100}\Big(cos(v_2)+cos(v_3)\Big).$$

Thus, we can rewrite the above system as follows

$$\begin{cases} {}^{0.4}D_{0^+}^{0.07,0.08}({}^{0.4}D_{0^+}^{0.05,0.06}({}^{0.4}D_{0^+}^{0.03,0.04}({}^{0.4}D_{0^+}^{0.01,0.02}v(t)))\\ = \frac{t}{1+\sqrt{2}} + \frac{1}{100}\cos(v) + \frac{\cos(v_1)}{100(1+cos^2(v_1))} + \frac{3}{100}\big(cos(v_2)+cos(v_3)\big), \qquad t\in[0,1]\\ {}^{0.4}I_{0^+}^{0.9702}v(0) = -1.\\ {}^{0.4}I_{0^+}^{0.9312}({}^{0.4}D_{0^+}^{0.01,0.02}v(0)) = 2.\\ {}^{0.4}I_{0^+}^{0.893}({}^{0.4}D_{0^+}^{0.03,0.04}({}^{0.4}D_{0^+}^{0.01,0.02}v(0))) = 0.\\ {}^{0.4}I_{0^+}^{0.856}({}^{0.4}D_{0^+}^{0.05,0.06}({}^{0.4}D_{0^+}^{0.03,0.04}({}^{0.4}D_{0^+}^{0.01,0.02}v(0)))) = 1. \end{cases} \tag{3.6}$$

Using all the data provided above in our problem (3.6), we obtain

$$\begin{aligned} &\Big|f(t,v,v_1,v_2,v_3)) - f(t,v',v_1',v_2',v_3')\Big|\\ &\leq \Bigg|\frac{t}{1+\sqrt{2}} + \frac{1}{100}\cos(v) + \frac{\cos(v_1)}{100(1+cos^2(v_1))} + \frac{1}{100}\Bigg(cos(v_2)+cos(v_3)\Bigg)\\ &- \Bigg(\frac{t}{1+\sqrt{2}} + \frac{1}{100}\cos(v') + \frac{\cos(v_1')}{100(1+cos^2(v_1'))} + \frac{1}{100}\Bigg(\cos(v_2')+\cos(v_3')\Bigg)\Bigg|\\ &\leq \frac{1}{100}\Bigg|\cos(v)-\cos(v')\Bigg| + \frac{1}{100}\Bigg|\frac{\cos(v_1)}{(1+cos^2(v_1))} - \frac{\cos(v_1')}{(1+cos^2(v_1'))}\Bigg|\\ &+ \frac{1}{100}\Bigg|\Bigg(\cos(v_2)-\cos(v_2')\Bigg) + \Bigg(\cos(v_3)-cos(v_3')\Bigg)\Bigg|\\ &\leq \frac{3}{100}\Bigg(|v-v'| + |v_1-v_1'| + |v_2-v_2'| + |v_3-v_3'|\Bigg) \end{aligned}$$

$$\big|f(t,v,v_1,v_2,v_3)\big| \leq \frac{3}{100}.$$

Consequently, both hypotheses (H_1) and (H_2) are satisfied with

$$\begin{aligned} K_1 = &\frac{(0.4)^{0.35-2-0.01-0.04-0.06-0.08}}{\Gamma(0.01+0.04+0.06+0.08+1)} + \frac{(0.4)^{0.35-2-0.04-0.06-0.08}}{\Gamma(0.04+0.06+0.08+1)}\\ &+ \frac{(0.4)^{0.35-2-0.01-0.06-0.08}}{\Gamma(0.01+0.06+0.08+1)} + \frac{(0.4)^{0.35-2-0.01-0.04-0.08}}{\Gamma(0.01+0.04+0.08+1)}, \end{aligned}$$

$$\begin{aligned} \mathrm{K}_2 = &\Bigg|\frac{-1}{\Gamma(0.0298)}\left(\frac{1}{0.4}\right)^{0.0298-0.35} + \frac{2}{\Gamma(0.01+0.0688)}\left(\frac{1}{0.4}\right)^{0.01+0.0688-0.35}\\ &+ \frac{1}{\Gamma(0.01+0.04+0.06+0.1444)}\left(\frac{1}{0.4}\right)^{0.01+0.04+0.06+0.1444-0.35}\Bigg|, \end{aligned}$$

$$K_3 = \frac{(0.4)^{0.35-2-0.08-0.06-0.04-0.01}}{\Gamma(0.08+0.06+0.04+0.01+1)}.$$

Thus,

$$\begin{cases} K_1 = 22.6638896801, \\ K_2 = 0.38515175692, \\ K_3 = 5.86143855961. \end{cases}$$

We obtain,

$$LK_1 = \frac{22.6638896801 \times 3}{100} = 0.6799166904$$

Therefore

$$LK_1 < 1.$$

Hence, the hypotheses of Theorem 2.1 are satisfied. Then, the problem (3.6) has a unique solution in $C^{0.35}_{0.65,0.4}([0,1])$.

On the other hand, the hypotheses of Theorem 3.1 are satisfied, which ensures the stability in both Ulam-Hyers and generalized Ulam-Hyers sense.

Conclusion

Our work offers a significant contribution, primarily encompassing the subsequent aspects

A new extension to the study of the Snap problem in the Hilfer-Katugampola sense with fractional integral conditions. Through rigorous analysis, we have successfully established an integral representation of the problem at hand. This has allowed us to effectively convert it into a fixed-point problem. Thereby, Schauder fixed-point theorem was the key to our analysis to establish the existence of the solution to the Snap problem. Nevertheless, through the incorporation of an added condition, we were able to attain the uniqueness of the solution by means of Banach contraction principle.

Furthermore, using only two conditions of the existence and uniqueness of the solution to our problem, we were able to explore and establish the stability of the solution in both Ulam-Hyers and generalized Ulam-Hyers sense.

At last, we provided a numerical simulation in order to illustrate the theoretical results.

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