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Discretization of a boundary value problem of type p-Bilaplacian

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To my parents Rachid and Nassima and to my brothers Sami, Meriem and Sara.

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Introduction and preliminaries

0.1 Introduction

Let Ω be an open and bounded subset of $\mathbb{R}^d$. For a given real valued function $u : \Omega \to \mathbb{R}$ we denote the gradient and Hessian of u as $Du : \Omega \to \mathbb{R}^d$, $D^2u : \Omega \to \mathbb{R}^d \times \mathbb{R}^d$ respectively and the Laplacian as $\Delta u : \Omega \to \mathbb{R}$. The $p-$Bilaplacian

$$\Delta_p^2 u := \Delta(|\Delta u|^{p-2}\Delta u) = 0 \tag{1}$$

is a nonlinear fourth order elliptic partial differential equation (PDE) which is a nonlinear generalisation of the linear Bilaplacian $\Delta^2 u := \Delta(\Delta u) = 0$. This type of problems typically arise from areas of elasticity, in particular, the nonlinear case for example can be used as a modelisation for travelling waves in suspension bridges [10, 11]. The formal limit of the $p-$Bilaplacian 1 as $p \to \infty$ is the $\infty-$Bilaplacian

$$\Delta_\infty^2 u := (\Delta u)^3 |D(\Delta u)|^2 = 0, \tag{2}$$

obtained in [1]. Solutions to this problem called $\infty-$Biharmonic functions. The method proposed in this work is based on C^0-mixed finite elements. Galerkin approach used to build a scheme convergent to the weak solution of the $p-$Bilaplacian, in part three of [4] it has been proven that the resulting approximations from the approach converge to the $\infty-$Biharmonic function as $p \to \infty$ and $h \to 0$ where h is the mesh parameter.

This work (after preliminaries) has divided into two chapters. In the first, we pose the problem and write it in its weak form, using techniques inspired from the proofs of Theorems 2 and 3 in [12, Section 8.2] we prove the existence and uniqueness of a weak solution. After that we prove some addtional results. In the second chapter we perform the discretisation for fixed p and prove a certain stability bound. Finally, we conclude our work with a certain error estimation.

0.2 Preliminaries

We give some proven results that will be used in our work, beginning by introducing the Sobolev spaces

$$
\begin{aligned}
L^p(\Omega) &= \left\{ \phi \ measurable : \int\limits_{\Omega} |\phi|^p dx < \infty \right\} \text{ for } p \in [1, \infty) \text{ and} \\
L^\infty(\Omega) &= \left\{ \phi \ measurable : ess \sup_{\Omega} |\phi| < \infty \right\}, \\
W^{l,p}(\Omega) &= \{ \phi \in L^p(\Omega) : \ D^\alpha \phi \in L^p(\Omega), \text{ for } |\alpha| \leq l \} \text{ and } H^l(\Omega) := W^{l,2}(\Omega),
\end{aligned}
$$

which are equipped with the following norms and semi-norms:

$$
\begin{aligned}
||v||^p_{L^p(\Omega)} \ &: = \int_\Omega |v|^p dx \text{ for } p \in [1, \infty) \text{ and } ||v||_{L^\infty(\Omega)} := ess \sup_{\Omega} |v| \\
||v||^p_{W^{l,p}(\Omega)} \ &: = \sum_{|\alpha| \leq l} ||D^\alpha v||^p_{L^p(\Omega)} \\
|v|^p_{W^{l,p}(\Omega)} \ &: = \sum_{|\alpha| = l} ||D^\alpha v||^p_{L^p(\Omega)}
\end{aligned}
$$

where $\alpha = \{\alpha_1, ..., \alpha_d\}$ is a multi-index ($\alpha_i \in \mathbb{N}$, $i = \overline{1, d}$), $|\alpha| =^d_{i=1} \alpha_i$ and derivatives D^α are understood in the weak sense.

$$
\begin{aligned}
W_0^{l,p}(\Omega) := \overline{D(\Omega)}^{W^{l,p}(\Omega)} \quad &= \left\{ \phi \in W^{l,p}(\Omega) : \exists (u_n)_{n \in \mathbb{N}} \subset D(\Omega), ||u_n - u||_{W^{l,p}(\Omega)} \to 0 \ \right\} \\
&= \left\{ \phi \in W^{l,p}(\Omega) : \phi|_{\partial\Omega} = D\phi|_{\partial\Omega} = 0 \ \right\}
\end{aligned}
$$

where $D(\Omega) := C_c^\infty(\Omega)$ i.e. the space of infinitely differentiable functions with compact support in Ω.

$$C^1(\overline{\Omega}) = \left\{\phi|_{\overline{\Omega}} : \phi \in C^1(\mathbb{R}^d) \text{ } (\Omega \text{ is bounded in } \mathbb{R}^d)\right\}$$

For a given function $u : \Omega \to \mathbb{R}$, the gradient of u is $Du = \left(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_d}\right)$ and the Laplacian is $\Delta u = \sum_{i=1}^{d} \frac{\partial u}{\partial x_i}$.

In this work we are interested with the case where $l = 2$ and define

$$W_g^{2,p}(\Omega) := g + W_0^{2,p}(\Omega) = \{\phi \in W^{2,p}(\Omega) : \phi|_{\partial\Omega} = g \text{ and } D\phi|_{\partial\Omega} = Dg\},$$

for a prescribed function $g \in W^{2,\infty}(\Omega)$. We have $W_0^{2,p}(\Omega) \subseteq C^1(\overline{\Omega})$ and $||v||_{W_0^{2,p}(\Omega)} \sim ||\Delta v||_{L^p(\Omega)}$ for $v \in W_0^{2,p}(\Omega)$. We define a larger space of $W_g^{2,\infty}(\Omega)$ needed to find a solution of the $\infty-$Bilaplacian as

$$\hat{W}_g^{2,\infty}(\Omega) = \{u \in \bigcap_{p\in(1,\infty)} W_g^{2,p}(\Omega) : \Delta u \in L^\infty(\Omega)\}.$$

For the $p-$Bilaplacian, the action functional is given as

$$L[u;p] = \int_\Omega |\Delta u|^p \, dx.$$

We then then look to find the minimizer over the space $W_g^{2,p}(\Omega)$ by L,that is, to find $u \in W_g^{2,p}(\Omega)$ such that

$$L[u;p] = \min_{v \in W_g^{2,p}(\Omega)} L[u;p].$$

Proposition 0.2.1 *(Weak lower semicontinuity of L [4, Cor 2.3]). The action functional L is weakly lower semi-continous over $W_g^{2,p}(\Omega)$. That is, given a sequence of functions $\{u_j\}_{j\in\mathbb{N}}$ which has a weak limit $u \in W_g^{2,p}(\Omega)$, we have*

$$L[u;p] \leq \liminf_{j\to\infty} L[u_j;p].$$

Remark 0.2.2 *For a given function $v \in L^p(\Omega)$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$|| \mid v \mid^{p-1} ||_{L^q(\Omega)} = ||v||^{p-1}_{L^p(\Omega)}$$

Proposition 0.2.3 *(Poincaré inequality). Let Ω be a bounded domain of $\mathbb{R}^d$. For any $p \in [1, \infty]$, there exists a positive constant $C = C(\Omega, p)$ such that*

$$||u||_{L^p(\Omega)} \leq C(\Omega, p)||Du||_{L^p(\Omega)},$$

for all $u \in W^{1,p}_0(\Omega)$.

Proposition 0.2.4 *(Calderon-Zygmund estimate [3, Cor 9.10]). Let Ω be a bounded domain of $\mathbb{R}^d$. Then for any $p \in (1, \infty)$, there exists a positive constant $C = C(d, p) > 0$ such that*

$$||D^2u||_{L^p(\Omega)} \leq C(d, p)||\Delta u||_{L^p(\Omega)},$$

for all $u \in W^{2,p}_0(\Omega)$.

Theorem 0.2.5 *(Hölder's inequality). Let $(\Omega, \sum, \mu)$ be a measure space and let $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for any two measurable functions $f, g : \Omega \to \mathbb{k}$ where $\mathbb{k}$ is the real or the complex field*

$$||fg||_{L^1(\Omega)} \leq ||f||_{L^p(\Omega)}||g||_{L^q(\Omega)}.$$

Theorem 0.2.6 *(ϵ-Young's inequality). For a, b, ϵ positive real numbers we have*

$$ab \leq \frac{1}{p}(\epsilon a)^p + \frac{1}{q}(\frac{b}{\epsilon})^q.$$

Corollary 0.2.7 *[13, Cor 6.12]. In a reflexive Banach space X, every bounded sequence has a weakly convergent subsequence.*

Definition 0.2.8 *Let $P : X \to X$ be a linear operator on a vector space X. P is called* *a projection operator if* $P^2 = P$. *When* X *is a Hilbert space with an inner product* $\langle .,. \rangle$, *a projection* P *is called an orthogonal projection if it satisfies* $\langle Px, y\rangle = \langle x, Py\rangle$ *for all* $x, y \in X$.

Definition 0.2.9 *Let* X *be a real normed linear space. We say a sequence* $\{x_n\}_{n=1}^{\infty} \subset X$ *converges weakly to an element* $x \in X$, *written* $x_n \overset{weakly}{\rightarrow} x$, *if*

$$f(x_n) \rightarrow f(x)$$

for every bounded linear functional $f \in X^*$.

We need the followings in proving the existence and uniqueness of a weak solution to the problem 1.1.

Let $(V, ||.||)$ be a normed vector space, a continuous bilinear form $a(.,.) : V \times V \rightarrow \mathbb{R}$, an element $f \in V^*$ and a non empty subset U of V. with this data we associate an abstarct minimization problem:

$$\begin{cases} \text{Find } u \in U \text{ such that} \\ \quad J(u) = \inf_{v \in U} J(v) \end{cases} \tag{3}$$

where the functional $J : V \rightarrow \mathbb{R}$ is defined by

$$\begin{aligned} J: \quad V &\rightarrow \mathbb{R} \\ v &\rightarrow J(v) = \tfrac{1}{2} a(v, v) - f(v). \end{aligned}$$

Theorem 0.2.10 *[15, The 1.1.2]. An element u is the solution to the abstract minimization problem 3 if and only if it satisfies the relations*

$$\begin{cases} \qquad\qquad u \in U \ \textit{such that} \\ a(u, v - u) \geq f(v - u) \qquad \forall v \in U \end{cases}$$

in the general case, or

$$\begin{cases} u \in U \text{ such that} \\ a(u,v) \geq f(v) \text{ and } a(u,u) = f(u) \qquad \forall v \in U \end{cases}$$

if U is a closed convex cone with vertex 0, or

$$\begin{cases} u \in U \text{ such that} \\ a(u,v) = f(v), \qquad \forall v \in U \end{cases}$$

if U is a closed subspace.

Definition 0.2.11 *[14, Def 10.5]. Let $f : X \to \mathbb{R} \cup \{+\infty\}$. Then f is uniformaly convex with modulus[1] $\phi : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ if ϕ is increasing, ϕ vanishes only at zero, and $\forall x, y \in dom(f)$, $\alpha \in]0,1[$*

$$f(\alpha x + (1-\alpha)y) + \alpha(1-\alpha)\phi(||x-y||) \leq \alpha f(x) + (1-\alpha)f(y).$$

If there exists $\beta > 0$ such that $\phi = \frac{\beta}{2}|.|^2$, then f is strongly convex with constant β.

Proposition 0.2.12 *[4, Prop 2.2] (coercivity of the action functional L). Suppose that $u \in W_0^{2,p}(\Omega)$ and $f \in L^q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$. We have that the action functional $L[.;p]$ is coercive over $W_0^{2,p}(\Omega)$, that is,*

$$L[u;p] \geq C|v|_{2,p}^p - \gamma,$$

for some C≥ 0 and $\gamma \geq 0$. Equivalently, we have that there exists a constant C>0 such that

$$L[v;p] \geq C|v|_{2,p}^p, \qquad \forall v \in W_0^{2,p}(\Omega).$$

[1] A modulus function is a function which gives the absolute value of a number or variable.

Chapter 1

Approximation via the p-Bilaplacian

1.1 Posing the problem and Weak formulation

The Dirichlet problem for the $p-$Bilaplacian is, given $g \in W^{2,\infty}(\Omega)$, to find u such that

$$\begin{cases} \Delta_p^2 u := \Delta(|\Delta u|^{p-2}\Delta u) = 0, & \text{in } \Omega, \\ \qquad u = g, & \text{on } \partial\Omega, \\ \qquad Du = Dg & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

Note that, for $p = 2$, the PDE reduces to the Bilaplacian $\Delta^2 u = 0$. Multiplying the equation by an element v lying in an appropriate space $W_0^{2,p}(\Omega)$, and using integration by parts formula of multivariable calculus and Green's first identity we get

$$\int_\Omega (|\Delta u|^{p-2}\Delta u)\Delta v \mathrm{d}x = 0$$

where $A(u,v) := \int_\Omega (|\Delta u|^{p-2}\Delta u)\Delta v \mathrm{d}x$ is the associated semilinear form of the weak formulation. Therefore, the problem 1.1 can be written in a weak form as: Find $u \in W_g^{2,p}(\Omega)$ that satisfies

$$A(u,v) = 0, \qquad \forall v \in W_0^{2,p}(\Omega).$$

Definition 1.1.1 *(Weak solution). The problem 1.1 has a weak solution if there exists*

$u \in W_g^{2,p}(\Omega)$ such that

$$A(u,v) = \int_\Omega (|\Delta u|^{p-2}\Delta u)\Delta v dx = 0, \qquad \forall v \in W_0^{2,p}(\Omega).$$

u is called a weak solution of 1.1.

1.2 Existence and uniqueness of the solution

Our first result is the next theorem which assure the existence and uniqueness of a weak solution for the problem 1.1. The proof techniques used to prove the theorem are a direct extension of the proofs of Theorems 2 and 3 in [12, Section 8.2].

Theorem 1.2.1 (Existence and uniqueness of a weak solution) *Let $p > 1$[1]. The problem 1.1 has a unique weak solution. That is, there exists a unique element $u \in W_g^{2,p}(\Omega)$ such that*

$$\int_\Omega \left(|\Delta u|^{p-2}\Delta u\right) \Delta v dx = 0, \qquad \forall v \in W_0^{2,p}(\Omega).$$

Proof. In view of Theorem 0.2.10, the problem 1.1 has a unique weak solution is equivalent to there exists a unique minimizer to the action functional L over $W_g^{2,p}(\Omega)$, that is, there exists a unique element $u \in W_g^{2,p}(\Omega)$ such that

$$L[u,p] = \min_{v \in W_g^{2,p}(\Omega)} L[v,p].$$

Existence of a minimizer: Let $b = \inf_{v\in W_g^{2,p}(\Omega)} L[v,p]$. We will prove that there exists an element $u \in W_g^{2,p}(\Omega)$ which satisfies $L[u,p] \leq b$. Since this in turn implies $L[u,p] = b$. So as $u \in W_g^{2,p}(\Omega)$ we have $L[u,p] = b = \inf_{v\in W_g^{2,p}(\Omega)} L[v,p] = \min_{v\in W_g^{2,p}(\Omega)} L[v,p]$.

Suppose b is finite and let $\{v_k\}_{k=1}^\infty \subset W_g^{2,p}(\Omega)$ be such that

$$L[v_k,p] \to b, \tag{1.2}$$

[1]Since for $p > 1$ the space $W^{l,p}(\Omega)$ is a reflexive Banach space and this fact will be needed in the proof.

$\{v_k\}_{k=1}^{\infty}$ is called a minimizing sequence. It is clear that $g \in W_g^{2,p}(\Omega)$ since $g \in W^{2,\infty}(\Omega)$. So for a certain $N \in \mathbb{N}$ we have

$$||\Delta v_k||_{L^p(\Omega)}^p = L[v_k, p] \leq L[g, p] = ||\Delta g||_{L^p(\Omega)}^p, \qquad \forall n \geq N \tag{1.3}$$

and so

$$||\Delta v_k||_{L^p(\Omega)} \leq ||\Delta g||_{L^p(\Omega)}, \qquad \forall n \geq N \tag{1.4}$$

therefore

$$\sup_k ||\Delta v_k||_{L^p(\Omega)} \leq C. \tag{1.5}$$

The coercivity of L gives

$$L[v_k, p] \geq C|v_k|_{2,p}, \tag{1.6}$$

Since b is finite, 1.2 and 1.6 imply

$$\sup_k ||Dv_k||_{L^p(\Omega)} < \infty. \tag{1.7}$$

Now let f be a fixed function in $W_g^{2,p}(\Omega)$. We have $v_k - f \in W_0^{2,p}(\Omega)$ since they are identical on the boundary of Ω. Using Poincaré's inequality and 1.7 we see

$$\begin{aligned} ||v_k||_{L^p(\Omega)} = ||v_k - f + f||_{L^p(\Omega)} &\leq ||v_k - f||_{L^p(\Omega)} + ||f||_{L^p(\Omega)} \\ &\leq C||Dv_k - Df||_{L^p(\Omega)} + ||f||_{L^p(\Omega)} \\ &\leq C||Dv_k||_{L^p(\Omega)} + ||Df||_{L^p(\Omega)} + ||f||_{L^p(\Omega)} \\ &\leq C. \end{aligned} \tag{1.8}$$

hence

$$\sup_k ||v_k||_{L^p(\Omega)} < \infty. \tag{1.9}$$

This estimate , 1.5 and 1.7 imply that $\{v_k\}_{k=1}^{\infty}$ is bounded in $W^{2,p}(\Omega)$. Consequently, in view of Corollory 0.2.7 there exists a subsequence $\{v_{k_j}\}_{j=1}^{\infty} \subset \{v_k\}_{k=1}^{\infty}$ and a function

$u \in W^{2,p}(\Omega)$ such that

$$v_{k_j} \overset{weakly}{\rightarrow} u \qquad \text{in } W^{2,p}(\Omega).$$

Now we prove $u \in W^{2,p}_g(\Omega)$. We have $W^{2,p}_0(\Omega)$ is a weakly closed subspace of $W^{2,p}(\Omega)$ (see [12, Appendix D.4.]). For a fixed $f \in W^{2,p}_g(\Omega)$ as above, $v_k - f \in W^{2,p}_0(\Omega)$. Hence $v - f \in W^{2,p}_0(\Omega)$. Therefore $u \in W^{2,p}_g(\Omega)$ since $f = g$, $Df = Dg$ on $\partial\Omega$. Thus $v_{k_j} \overset{weakly}{\rightarrow} u$ in $W^{2,p}_g(\Omega)$.

Note that $L[v_{k_j}, p] \to b$, L is weakly lower semicontinuous (Proposition 0.2.1). Then,

$$L[u,p] \leq \lim_{j\to\infty} \inf v_{k_j} = b,$$

and so as we said above $L[u,p] = b = \inf_{v\in W^{2,p}_g(\Omega)} L[v,p] = \min_{v\in W^{2,p}_g(\Omega)} L[v,p]$. Hence there exists a minimizer to the action functional L over $W^{2,p}_g(\Omega)$.

Uniqueness of the minimizer: Suppose u_1, u_2 are two minimizers of L so we have, $u_1, u_2 \in W^{2,p}_g(\Omega)$ and $L[u_1,p] = L[u_2,p] = \min_{v\in W^{2,p}_g(\Omega)} L[v,p]$. Let $k := \frac{u_1+u_2}{2}$ (clearly $k \in W^{2,p}_g(\Omega)$ since $u_1 = u_2$, $Du_1 = Du_2$ on $\partial\Omega$) and $\widetilde{L}$ to denote the Lagrangian of L i.e.

$$\begin{aligned} \widetilde{L}: \ \mathbb{R}^d \times \Omega \ &\to \mathbb{R} \\ (c,x) \ &\to \widetilde{L}(c,x) = |div(c)|^p \end{aligned}$$

clearly we have $L[u,p] = \int_\Omega \widetilde{L}(Du(x),x)dx$. From the uniform convexity of $\widetilde{L}$ with respect to the first variable we have the following (see [1, proof page 471])

$$\widetilde{L}(b,x) + D_c\widetilde{L}(b,x).(a-b) + \frac{\theta}{2}|a-b|^2 \leq \widetilde{L}(a,x) \qquad (\theta > 0,\ x \in \Omega,\ a,b \in \mathbb{R}^d). \quad (1.10)$$

By putting $b = \frac{Du_1+Du_2}{2}$, $a = Du_1$ and integrate over Ω, we obtain

$$\int_\Omega \widetilde{L}(\frac{Du_1+Du_2}{2},x) + D_c\widetilde{L}(\frac{Du_1+Du_2}{2},x).(\frac{Du_1-Du_2}{2}) + \frac{\theta}{2}|\frac{Du_1-Du_2}{2}|^2 dx \leq \int_\Omega \widetilde{L}(Du_1,x)dx,$$

so

$$L[k,p]+\int_\Omega D_c\widetilde{L}(\frac{Du_1+Du_2}{2},x).(\frac{Du_1-Du_2}{2})dx+\frac{\theta}{8}\int_\Omega |Du_1-Du_2|^2dx \leq L[u_1,p] \quad (1.11)$$

Similarly, set $b=\frac{Du_1+Du_2}{2}$, $a=Du_2$, we obtain

$$\int_\Omega \widetilde{L}(\frac{Du_1+Du_2}{2},x)+D_c\widetilde{L}(\frac{Du_1+Du_2}{2},x).(\frac{Du_2-Du_1}{2})+\frac{\theta}{2}|\frac{Du_2-Du_1}{2}|^2dx \leq \int_\Omega \widetilde{L}(Du_2,x)dx,$$

so

$$L[k,p]-\int_\Omega D_c\widetilde{L}(\frac{Du_1+Du_2}{2},x).(\frac{Du_1-Du_2}{2})dx+\frac{\theta}{8}\int_\Omega |Du_1-Du_2|^2dx \leq L[u_2,p]. \quad (1.12)$$

Now adding 1.11 to 1.12 and devide by 2, we get

$$L[k,p]+\frac{\theta}{8}\int_\Omega |Du_1-Du_2|^2dx \leq \frac{L[u_1,p]+L[u_2,p]}{2}=L[u_1,p]. \quad (1.13)$$

Now in view of this, as $L[u_1,p]=L[u_2,p]=\min_{v\in W_g^{2,p}(\Omega)} L[v,p]\leq L[k,p]$ we have $L[k,p]=L[u_1,p]$. From 1.13 we deduce $Du_1=Du_2$ on Ω. Hence it follows $u_1=u_2$ since $u_1=u_2=g$ on $\partial\Omega$. Therefore L has a unique minimizer over $W_g^{2,p}(\Omega)$.

We have shown the existence and uniqueness of a minimizer to the action functional L over $W_g^{2,p}(\Omega)$. Hence the problem 1 has a unique weak solution. ■

1.3 Convergence of a $p-$solutions to the $\infty-$ Biharmonic function

Theorem 1.3.1 *(The limit as $p\to\infty$) Let $(u_p)_1^\infty$ be a sequence of weak solutions $u_p\in W_g^{2,p}(\Omega)$ to the $p-$Bilaplacian. Then, there exists a subsequence $(u_{p_j})_{p_j}$ of $(u_p)_1^\infty$ converging uniformly toghther with their derivatives to the $\infty-$Biharmonic function[2] $u_\infty\in$*

[2] We mean by $\infty-$Biharmonic function a solution of the equation $\Delta_\infty^2 u=0$, in fact this solution is unique see [1, 2].

$\hat{W}_g^{2,\infty}(\Omega)$. *That is,*

$$u_{p_j} \to u_\infty \text{ in } C^1(\overline{\Omega}) \text{ as } j \to \infty.$$

Proof. Let $u_p \in W_g^{2,p}(\Omega)$ denote the weak solution of 1.1. u_p minimizes the energy functional $L[., p]$. That is, $\forall p \in [1, \infty)$ we have

$$\begin{aligned} L[u_p, p] &= \int_\Omega |\Delta u_p|^p \mathrm{d}x \\ &\leq L[u, p] = \int_\Omega |\Delta u|^p \mathrm{d}x. \end{aligned}$$

$\forall u \in W_g^{2,p}(e)$. As $g \in W^{2,\infty}(\Omega)$, (the given data appearing in 1.1) so it is in $W_g^{2,p}(\Omega)$ $\forall p \in [1, \infty)$. In particular we have

$$\begin{aligned} L[u_p, p] &= \int_\Omega |\Delta u_p|^p \mathrm{d}x \\ &\leq L[g, p] = \int_\Omega |\Delta g|^p \mathrm{d}x, \qquad \forall p \in [1, \infty), \end{aligned}$$

so

$$||\Delta u_p||^p_{L^p(\Omega)} \leq ||\Delta g||^p_{L^p(\Omega)}, \qquad \forall p \in [1, \infty),$$

therefore

$$||\Delta u_p||_{L^p(\Omega)} \leq ||\Delta g||_{L^p(\Omega)}, \qquad \forall p \in [1, \infty), \tag{1.14}$$

Now fix $k > d$ and take $p \geq k$. Then, by taking $r = \frac{p}{k}$ and $q = \frac{r}{r-1}$ such that $\frac{1}{r} + \frac{1}{q} = 1$ and using Holder's inequlity, we have

$$\begin{aligned} ||\Delta u_p||^k_{L^k(\Omega)} &= \int_\Omega |\Delta u_p|^k dx \\ &\leq (\int_\Omega 1^q \mathrm{d}x)^{1/q} (\int_\Omega |\Delta u_p|^p \mathrm{d}x)^{1/r} \\ &\leq |\Omega|^{\frac{r-1}{r}} ||\Delta u_p||^k_{L^p(\Omega)} = |\Omega|^{1-\frac{k}{p}} ||\Delta u_p||^k_{L^p(\Omega)} \end{aligned}$$

and so

$$||\Delta u_p||_{L^k(\Omega)} \leq |\Omega|^{\frac{1}{k} - \frac{1}{p}} ||\Delta u_p||_{L^p(\Omega)}. \tag{1.15}$$

By using the triangle inequality, a double application of the Poincar inquality taking in consider that $u_p - g \in W_0^{2,k}(\Omega)$ (since $u_p, g \in W_g^{2,p}(\Omega) \subset W_g^{2,k}(\Omega)$) and the Calderon-Zygmund estimate, we obtain

$$\begin{aligned}
||u_p||_{L^k(\Omega)} &\leq ||u_p - g||_{L^k(\Omega)} + ||g||_{L^k(\Omega)} \\
&\leq C||D^2(u_p - g)||_{L^k(\Omega)} + ||g||_{L^k(\Omega)} \\
&\leq C||\Delta u_p - \Delta g||_{L^k(\Omega)} + ||g||_{L^k(\Omega)} \\
&\leq C(||\Delta u_p||_{L^k(\Omega)} + ||\Delta g||_{L^k(\Omega)} + ||g||_{L^k(\Omega)}) \\
&\leq C(||\Delta u_p||_{L^k(\Omega)} + \sum_{i=1}^{d} ||\tfrac{\partial^2 g}{\partial x_i^2}||_{L^k(\Omega)} + ||g||_{L^k(\Omega)}) \\
&\leq C(||\Delta u_p||_{L^k(\Omega)} + ||g||_{W^{2,k}(\Omega)})
\end{aligned}$$

using 1.15, we have

$$||u_p||_{L^k(\Omega)} \leq C(|\Omega|^{\frac{1}{k}-\frac{1}{p}}||\Delta u_p||_{L^p(\Omega)} + ||g||_{W^{2,k}(\Omega)}) \tag{1.16}$$

With the same way we show that

$$||Du_p||_{L^k(\Omega)} \leq C(|\Omega|^{\frac{1}{k}-\frac{1}{p}}||\Delta u_p||_{L^p(\Omega)} + ||g||_{W^{2,k}(\Omega)})$$

In view of 1.14 we infer that

$$||u_p||_{W^{2,k}(\Omega)} \leq C||g||_{W^{2,k}(\Omega)}.$$

Hence

$$\sup_{p>k} ||u_p||_{W^{2,k}(\Omega)} \leq C.$$

This means that the sequence $(u_p)_1^\infty$ is bounded in $W^{2,k}(\Omega)$, so in view of Corrolory 0.2.7 we can extract a sub-sequence $(u_{p_j})_{p_j}$ from $(u_p)_1^\infty$ and a function $u_\infty \in W^{2,k}(\Omega)$ such that

$$u_{p_j} \overset{weakly}{\rightarrow} u_\infty \text{ in } W^{2,k}(\Omega) \text{ as } j \to \infty$$

and

$$
\begin{aligned}
||u_\infty||_{W^{2,k}(\Omega)} &\leq \liminf_{j\to\infty} ||u_{p_j}||_{W^{2,k}(\Omega)} \\
&\leq \liminf_{j\to\infty} C||g||_{W^{2,k}(\Omega)}
\end{aligned}
$$

by the weak lower semi-continuity of the $||.||_{L^k(\Omega)}$. Since this true for any fixed $k \geq d$, it is clear that $u_\infty \in \cap_{k\in(d,\infty)} W^{2,k}(\Omega)$. Further, by the weak lower semi-continuity of the $||.||_{L^k(\Omega)}$ and 1.15 we have $\Delta u_\infty \in L^\infty(\Omega)$ and hence $u_\infty \in \hat{W}^{2,\infty}_g(\Omega)$, therefore concluding the proof. ■

1.4 Mixed formulation of the $p-$Bilaplacian

The mixed formulation we propose is based on the fact that if $\phi(t) = |t|^{p-2}t$, the inverse is defined as $\phi^{-1}(t) = sign(t)|t|^{1/(p-1)} = |t|^{q-2}t$. By putting an auxiliary variable $w = -|\Delta u|^{p-2}\Delta u$, we have

$$
\begin{aligned}
\Delta_p^2 u &= \Delta(|\Delta u|^{p-2}\Delta u) \\
&= -\Delta(-|\Delta u|^{p-2}\Delta u) \\
&= -\Delta w,
\end{aligned}
$$

and

$$
\begin{aligned}
w &= -|\Delta u|^{p-2}\Delta u = -\phi(\Delta u) \\
\phi^{-1}(w) &= \phi^{-1}(-\phi(\Delta u)) = -\phi^{-1}(\phi(\Delta u) \\
|w|^{q-2}w &= -\Delta u.
\end{aligned}
$$

This enable us to reformulate the problem in the following mixed system:

$$\begin{cases} -\Delta u &= |w|^{q-2}w, \\ -\Delta w &= 0. \end{cases} \tag{1.17}$$

The mixed formulation can be written in a weak form as: Find a pair $(u, w) \in W_g^{2,p}(\Omega) \times L^q(\Omega)$ such that

$$\begin{cases} a(w,\psi) + b(u,\psi) = & 0, \\ b(\phi, w) = & 0, \end{cases} \qquad \forall(\psi,\phi) \in L^q(\Omega) \times W_0^{2,p}(\Omega), \tag{1.18}$$

where the semilinear form $a(.,.)$ and the bilinear form $b(.,.)$ are given by

$$\begin{cases} a(w,\psi) = \int_\Omega |w|^{q-2}w\psi \mathrm{d}x \\ b(u,\psi) = \int_\Omega \Delta u\psi \mathrm{d}x \end{cases}. \tag{1.19}$$

1.4.1 Existence and uniqueness of the solution

We have just seen that the problem 1 has been reformulated to the mixed form 1.17. Althought we already know that the problem has a unique solution as a consequence of Theorem 1.2.1, we will show that the solution of the mixed formulation satisfies the following estimation $||\Delta u||_{L^p(\Omega)} + ||w||_{L^q(\Omega)}^{q-1} \leq C||\Delta g||_{L^p(\Omega)}$. Since the result will be useful hencefoth. We begin by showing the following result.

Proposition 1.4.1 *(Inf-sup stability of $b(.,.)$ over $W_0^{2,p}(\Omega)$). For any $u_0 \in W_0^{2,p}(\Omega)$, the bilinear form $b(.,.)$ satisfies the following inf-sup property:*

$$||\Delta u_0||_{L^p(\Omega)} \leq C \sup_{0 \neq v \in L^q(\Omega)} \frac{b(u_0, v)}{||v||_{L^q(\Omega)}}.$$

Proof. Let $u_0 \in W_0^{2,p}(\Omega)$. Then, we have $|\Delta u_0|^{p-2}\Delta u_0 \in L^q(\Omega)$. Therefore, by taking

$v = |\Delta u_0|^{p-2} \Delta u_0$ we have

$$b(u_0, v) = ||\Delta u_0||^p_{L^p(\Omega)}$$

and that

$$\begin{aligned} ||v||_{L^q(\Omega)} &= || \, | \Delta u_0 |^{p-1} \, ||_{L^q(\Omega)} \\ &= ||\Delta u_0||^{p-1}_{L^p(\Omega)}, \end{aligned}$$

in view of the property given in Remark 0.2.2. Hence we have

$$\begin{aligned} b(u_0, v) &= ||\Delta u_0||^p_{L^p(\Omega)} \\ &= ||\Delta u_0||_{L^p(\Omega)} ||v||_{L^q(\Omega)} \end{aligned}$$

and

$$||\Delta u_0||_{L^p(\Omega)} = \frac{b(u_0, v)}{||v||_{L^q(\Omega)}}$$

so

$$||\Delta u_0||_{L^p(\Omega)} \leq \sup_{0 \neq v \in L^q(\Omega)} \frac{b(u_0, v)}{||v||_{L^q(\Omega)}}$$

which implies the desired result. ■

Theorem 1.4.2 *(The mixed formulation is well posed). For every $g \in W^{2,\infty}(\Omega)$, there exists a unique pair (u, w) solving 1.18 that satisfies*

$$||\Delta u||_{L^p(\Omega)} + ||w||^{q-1}_{L^q(\Omega)} \leq C ||\Delta g||_{L^p(\Omega)}.$$

Proof. The existence and uniqueness of the solution is a direct consequence of Theorem 1.2.1 since w is just an auxiliary variable depending on u. Now let $u_0 := u - g \in$

$W_0^{2,p}(\Omega)$, using Remark 0.2.2 we obtain

$$\begin{aligned}||\Delta u_0||_{L^p(\Omega)} &\leq ||\Delta u||_{L^p(\Omega)} + ||\Delta g||_{L^p(\Omega)} \\ &\leq ||\mid w \mid^{q-2} w||_{L^p(\Omega)} + ||\Delta g||_{L^p(\Omega)} \\ &\leq ||\mid w \mid^{q-1} ||_{L^p(\Omega)} + ||\Delta g||_{L^p(\Omega)} \\ &\leq ||w||^{q-1}_{L^q(\Omega)} + ||\Delta g||_{L^p(\Omega)}.\end{aligned} \tag{1.20}$$

Now by taking $\psi = w$ and $\phi = u_0$ in 1.18. Then,

$$\begin{aligned}a(w,w) + b(u,w) &= 0 \\ b(u_0,w) &= 0\end{aligned}$$

and in particular

$$a(w,w) + b(g,w) = 0.$$

This in turn implies

$$\int_\Omega |w|^q \mathrm{d}x = -\int_\Omega \Delta g w \mathrm{d}x.$$

Hence

$$\begin{aligned}||w||^q_{L^q(\Omega)} &= -\int_\Omega \Delta g w \mathrm{d}x \\ &\leq \int_\Omega |\Delta g w| \mathrm{d}x \\ &\leq ||\Delta g||_{L^p(\Omega)} ||w||_{L^q(\Omega)}\end{aligned}$$

by Holder's inquality, so

$$||w||^{q-1}_{L^q(\Omega)} \leq ||\Delta g||_{L^p(\Omega)}, \tag{1.21}$$

Using the fact that $||\Delta u||_{L^p(\Omega)} \leq ||\Delta u_0||_{L^p(\Omega)} + ||\Delta g||_{L^p(\Omega)}$ and 1.20, 1.21, we have

$$\begin{aligned}
||\Delta u||_{L^p(\Omega)} + ||w||_{L^q(\Omega)}^{q-1} &\leq ||\Delta u_0||_{L^p(\Omega)} + ||\Delta g||_{L^p(\Omega)} + ||\Delta g||_{L^p(\Omega)} \\
&\leq ||w||_{L^q(\Omega)}^{q-1} + ||\Delta g||_{L^p(\Omega)} + 2||\Delta g||_{L^p(\Omega)} \\
&\leq 4||\Delta g||_{L^p(\Omega)}
\end{aligned}$$

which implies the desired result. ∎

Chapter 2

Discretisation of the p-Bilaplacian

2.1 Introduction

In this chapter we exibit a finite element discretisation to the mixed formulation of the p-Bilaplacian using Galerkin approach. Let $\mathcal{F}$ be a conforming triangulation of Ω, that is, $\mathcal{F}$ is a finite collection satisfying the followings

(1) The elements of $\mathcal{F}$ are open simplexes i.e. segment for $d = 1$, triangle for $d = 2$, tetrahedron for $d = 3$ ...etc.

(2) Given $A, B \in \mathcal{F}$ we have that $\overline{A} \cap \overline{B}$ is a full lower-dimentional simplex i.e. it is either $\emptyset$, a vertex, an edge, a face, or the whole of $\overline{A}$ and $\overline{B}$ (this happen when $A = B$).

(3) $\cup_{A \in \mathcal{F}} \overline{A} = \overline{\Omega}$.

Let ξ be the collection of common interfaces of the triangulation $\mathcal{F}$ and say $c \in \xi$ if $c \subset int(\Omega)$ and $c \in \partial\xi$ if $c \subset \partial\Omega$. The shape regularity constant of $\mathcal{F}$ is defined as follows

$$\mu(S) := \inf_{A \in \mathcal{F}} \frac{\rho_A}{h_A},$$

where ρ_A is the radius of the largest ball included in A and h_A is the diameter of A. An indexed family of triangulations $\{\mathcal{F}^n\}_n$ is called *shape regular* if

$$\mu := \inf_n \mu(\mathcal{F}^n) > 0.$$

Now let $P^k(\mathcal{F})$ to be the space of piecewise polynomials of degree $k \geq 2$ over the triangulation $\mathcal{F}$, that is,

$$P^k(\mathcal{F}) = \{\phi : \phi|_A \in P^k(A)\ \forall A \in \mathcal{F}\},$$

and we define the *finite element space*

$$V := P^k(\mathcal{F}) \cap C^0(\Omega),$$

to be the space of continuous piecwise polynomial functions of degree k. For an arbitrary scalar function v and vector function $\mathbf{v}$ we define jump operators over an edge $c = A_1 \cap A_2$ as $[v] = v|_{A_1}\mathbf{n}_{A_1} + v|_{A_2}\mathbf{n}_{A_2}$, $[\mathbf{v}] = \mathbf{v}|_{A_1}.\mathbf{n}_{A_1} + \mathbf{v}|_{A_2}.\mathbf{n}_{A_2}$. When $c \in \partial\xi$ we understand $[v] = v|_A.\mathbf{n}_{\partial\Omega}$ and $[\mathbf{v}] = \mathbf{v}|_A.\mathbf{n}_{\partial\Omega}$.

Moreover, we define the *meshsize function* h of $\mathcal{F}$ as follows

$$h : \Omega \to \mathbb{R}$$
$$x \to h(x) := \max\nolimits_{\overline{A}\ni x} h_A.$$

If there exists a constant $C \geq 0$ such that $\max_{x\in\Omega} h \leq C \min_{x\in\Omega} h$ the mesh is called a quasi-uniform mesh. In what follows we assume that all triangulations are **shape-regular** and **quasi-uniform**.

Definition 2.1.1 *(Ritz projection operators). We define the Ritz projection operators R and Neumann Ritz projection $\overline{R}$ through requiring*

$$\int_\Omega D(Rv).D\phi dx = \int_\Omega Dv.D\phi dx \qquad \forall\phi \in V \cap H^1_0(\Omega),$$

$$\int_\Omega D(\overline{R}w) \cdot D\psi dx = \int_\Omega Dw \cdot D\psi dx \qquad \forall\psi \in V$$

$$\int_\Omega \overline{R}w\,dx = \int_\Omega w\,dx.$$

where Rv coincides with an appropriate interpolant of v on the boundary. These operators satisfing the following approximation propertie for a quasi-uniform meshe (see [5]): given $v \in W^{k+1,q}(\Omega)$, and $k \geq 2$ we have

$$||v-Rv||_{L^q(\Omega)}+||h(Dv-D(Rv))||_{L^q(\Omega)}+(\sum_{K\in S}||h^2(\Delta v-\Delta(Rv))||^q_{L^q(K)})^{1/q} \leq Ch^{k+1}|v|_{k+1,q}.$$

$$||v-\overline{R}v||_{L^q(\Omega)}+||h(Dv-D(\overline{R}v))||_{L^q(\Omega)}+(\sum_{K\in S}||h^2(\Delta v-\Delta(\overline{R}v))||^q_{L^q(K)})^{1/q} \leq Ch^{k+1}|v|_{k+1,q}.$$

Definition 2.1.2 *(Mesh-dependent norms). We introduce the mesh-dependent $L^p-$and $W^{2,p}-$norms to be*

$$\begin{aligned} ||w_h||^p_{L^p_h(\Omega)} &:= ||w_h||^p_{L^p(\Omega)} + ||h^{1/p}w_h||^p_{L^p(\xi)} \\ ||w_h||^p_{W^{2,p}_h(\Omega)} &:= ||\Delta_h w_h||^p_{L^p(\Omega)} + ||h^{1/p-1}[Dw_h]||^p_{L^p(\xi)}, \end{aligned}$$

where Δ_h denotes an elementwise Laplace operator.

2.2 Galerkin discretisation

The Galerkin discretisation of 1 is, to find $(u_h, w_h) \in V_g \times V$ such that

$$\begin{aligned} a(w_h,\psi) + b_h(u_h,\psi) &= 0 \\ b_h(\phi, w_h) &= 0, \qquad \forall(\psi,\phi) \in V \times V_0, \end{aligned} \tag{2.1}$$

where $V_g := \{\phi \in V : \phi|_{\partial\Omega} = Rg\}$, the bilinear form $a(.,.)$ is that of 1.19 and $b_h(.,.)$ is a consistent discretisation of $b(.,.)$ defined as follows

$$b_h(u_h,\psi) = -\sum_{K\in S}\int_K \Delta u_h \psi \mathrm{d}x + \int_\xi [Du_h]\psi \mathrm{d}s.$$

Notice that the method is equivalent to finding $(u_h, w_h) \in V_g \times V$ such that

$$\int_\Omega \mid w_h \mid^{q-2} w_h \psi + Du_h \cdot D\psi \mathrm{d}x = \int_{\partial\Omega} Dg \cdot \mathbf{n}\psi \mathrm{d}s$$
$$\int_\Omega Dw_h \cdot D\phi \mathrm{d}x = 0, \qquad \forall(\psi, \phi) \in V \times V_0.$$

Hence the Ritz projection operator from Definition 2.1.1 is the b_h-orthogonal projection onto V_g, that is, $R : H^1_g(\Omega) \to V_g$ such that, for $v \in H^1_g(\Omega)$

$$b_h(Rv - v, \phi) = 0 \qquad \forall \phi \in V_0.$$

Remark 2.2.1 *We defined the mesh-dependent norms as above to ensure the boundedness property*

$$|b_h(u_h, v_h)| \leq ||u_h||_{W^{2,p}_h(\Omega)} ||v_h||_{L^q_h(\Omega)}.$$

and to have $||.||_{L^p_h(\Omega)} \sim ||.||_{L^p(\Omega)}$, $||.||_{W^{2,p}_h(\Omega)} \sim ||.||_{W^{2,p}(\Omega)}$[1].

2.3 Existence and uniqueness of the solution

Now we will prove an important estimation for u_h and v_h that will be used in proving a certain error estimation in the next section. We need the following Lemma to prove the result

Lemma 2.3.1 [2] *The bilinear form b_h satisfies the following inf-sup property:*

$$||\Phi||_{W^{2,p}_h(\Omega)} \leq C \sup_{0 \neq v_h \in V_0} \frac{b_h(\Phi, v_h)}{||v_h||_{L^q_h(\Omega)}}, \qquad \forall \Phi \in V_0.$$

Proof. *See [4, page 10].* ■

Theorem 2.3.2 *(existence and uniqueness of solution to 2.1). There exists a unique pair*

[1] See [4, Remark 3.4].

[2] Here the mesh is assumed to be **quasi-uniform** as we said in the beginning of this chapter. Otherwise, we do not now if the result still true.

$(u_h, w_h) \in V_g \times V$ solving 2.1. They satisfy the stability bound

$$||u_h||_{W_h^{2,p}(\Omega)} + ||w_h||_{L^q(\Omega)}^{q-1} \leq C||\Delta g||_{L^p(\Omega)},$$

the right hand side is finite since $g \in W^{2,\infty}(\Omega)$.

Proof. existence and uniqueness the solution is that of Theorem 1.4.2. Now we begin by noting that for $\psi = w_h$ and $V_0 \ni \phi = u_{h,0} := u_h - Rg$ in 2.1 we have

$$\begin{aligned} a(w_h, w_h) + b_h(u_h, w_h) &= 0, \\ b_h(u_h - Rg, w_h) &= 0. \end{aligned}$$

by substraction, we see

$$a(w_h, w_h) + b_h(Rg, w_h) = 0.$$

Now, by definition, we have

$$\begin{aligned} ||w_h||_{L^q(\Omega)}^q &= |b_h(Rg, w_h)| \\ &\leq ||Rg||_{W_h^{2,p}(\Omega)} ||w_h||_{L_h^q(\Omega)}, \\ &\leq C||\Delta g||_{L^p(\Omega)} ||w_h||_{L^q(\Omega)}. \end{aligned}$$

using Remark 2.2.1 and Lemma 2.2.1 and so

$$||w_h||_{L^q(\Omega)}^{q-1} \leq C||\Delta g||_{L^p(\Omega)}. \tag{2.2}$$

We also have

$$\begin{aligned}
||u_{h,0}||_{W_h^{2,p}(\Omega)} &\le C \sup_{0\neq v_h\in V_0} \tfrac{b_h(u_{h,0},v_h)}{||v_h||_{L_h^q(\Omega)}} \\
&\le C\left(\sup_{0\neq v_h\in V_0} \tfrac{b_h(u_h,v_h)}{||v_h||_{L_h^q(\Omega)}} + \sup_{0\neq v_h\in V_0} \tfrac{b_h(Rg,-v_h)}{||v_h||_{L_h^q(\Omega)}}\right) \\
&\le C\left(\sup_{0\neq v_h\in V_0} \tfrac{b_h(u_h,v_h)}{||v_h||_{L_h^q(\Omega)}} + \sup_{0\neq v_h\in V_0} \tfrac{b_h(Rg,v_h)}{||v_h||_{L_h^q(\Omega)}}\right) \\
&\le C\left(\sup_{0\neq v_h\in V_0} \tfrac{-a(u_h,v_h)}{||v_h||_{L_h^q(\Omega)}} + \sup_{0\neq v_h\in V_0} \tfrac{b_h(g,v_h)}{||v_h||_{L_h^q(\Omega)}}\right) \\
&\le C(|| \mid w_h \mid^{q-1} ||_{L^p(\Omega)} + ||\Delta g||_{L^p(\Omega)}) \\
&\le C(||w_h||^{q-1}_{L^q(\Omega)} + ||\Delta g||_{L^p(\Omega)}) \\
&\le C||\Delta g||_{L^p(\Omega)}
\end{aligned} \tag{2.3}$$

by the discrete inf-sup condition in Lemma 2.3.1 and the same argument as in the proof of Theorem 1.4.2 and 2.2.

Since

$$\begin{aligned}
||u_h||_{W_h^{2,p}(\Omega)} &\le ||u_{h,0}||_{W_h^{2,p}(\Omega)} + ||Rg||_{W_h^{2,p}(\Omega)} \\
&\le C(||u_{h,0}||_{W_h^{2,p}(\Omega)} + ||\Delta g||_{L^p(\Omega)}) \\
&\le C||\Delta g||_{L^p(\Omega)}
\end{aligned} \tag{2.4}$$

by 2.3. Now 2.2, 2.4 imply the claimed result. ■

2.4 Error estimation (Main result)

Finally, we arrived to our main result which is proving some error estimation. First, we state some technical properties that will be used in the theorem that follows.

Lemma 2.4.1 *Let $w \in L^p(\Omega)$ and $w_h, v_h \in V$, for any $p \geq 2$, there exist constants*

(1) $C_1 > 0$ such that

$$C_1 \frac{||w-w_h||^2_{L^q(\Omega)}}{||w||^{2-q}_{L^q(\Omega)} + ||w_h||^{2-q}_{L^q(\Omega)}} \le a(w, w-w_h) - a(w_h, w-w_h). \tag{2.5}$$

(2) $C_2 > 0$ such that

$$C_2 \int_\Omega \left| \mid w \mid^{q-2} w - \mid w_h \mid^{q-2} w_h \right| |w - w_h| dx \le a(w, w - w_h) - a(w_h, w - w_h). \quad (2.6)$$

(3) $C_3 > 0$ such that

$$a(w, w - v_h) - a(w_h, w - v_h) \le C_3 \left(\int_\Omega \left| |w|^{q-2} w - |w_h|^{q-2} w_h \right| |w - w_h| dx \right)^{1/p} ||w - v_h||_{L^q(\Omega)}. \quad (2.7)$$

Theorem 2.4.2 *(Error estimation). Let $(u, w) \in W_g^{k+1,p}(\Omega) \times W^{k+1,q}(\Omega)$ be the unique solution of 1.18 and $(u_h, w_h) \in V_g \times V$ be the finite element approximation satisfying 2.1. Then, we have the following estimate*

$$||w - w_h||_{L^q(\Omega)} + ||u - u_h||^{p-1}_{W_h^{2,p}(\Omega)} \le C(h^{\frac{q}{2}(k+1)} |w|^{q/2}_{W^{k+1,q}(\Omega)} + h^{k+1} |w|_{W^{k+1,q}(\Omega)} + h^{k-1} |u|_{W^{k+1,p}(\Omega)}). \quad (2.8)$$

Proof. In view of 1.18 and 2.1 we have the following Galerkin orthogonality results

$$\begin{aligned} b_h(\phi, w - w_h) &= 0 \qquad \forall \phi \in V_0, \quad (2.9) \\ a(w, \psi) - a(w_h, \psi) + b_h(u - u_h, \psi) &= 0 \qquad \forall \phi \in V, \end{aligned}$$

Now from the previous lemma we see

$$\frac{C_1 ||w - w_h||^2_{L^q(\Omega)}}{2(||w||^{2-q}_{L^q(\Omega)} + ||w_h||^{2-q}_{L^q(\Omega)})} + \frac{C_2}{2} \int_\Omega \left| \mid w \mid^{q-2} w - \mid w_h \mid^{q-2} w_h \right| \mid w - w_h \mid dx \le a(w, w - w_h) - a(w_h, w - w_h). \quad (2.10)$$

Now for $k \in V$ some approximation of w (k will be specified later) and using the second equality of 2.9, we have

$$a(w, w - w_h) - a(w_h, w - w_h) = a(w, w - k) - a(w_h, w - k) + a(w, k - w_h) - a(w_h, k - w_h)$$
$$= \underbrace{a(w, w - k) - a(w_h, w - k)}_{=:I} + \underbrace{b_h(u - u_h, w_h - k)}_{=:II}. \tag{2.11}$$

We proceed to bound these terms (I, II) separately, starting with I.

Using the previous lemma and $\epsilon-$Young's inequality, we have

$$\begin{aligned} a(w, w - k) - a(w_h, w - k) &\leq C_3 \left(\int_\Omega \left| \mid w \mid^{q-2} w - \mid w_h \mid^{q-2} w_h \right| \mid w - w_h \mid \mathrm{d}x\right)^{1/p} ||w - k||_{L^q(\Omega)}, \\ &\leq \tfrac{\epsilon^p}{p} \int_\Omega \left| \mid w \mid^{q-2} w - \mid w_h \mid^{q-2} w_h \right| \mid w - w_h \mid \mathrm{d}x + \tfrac{C_3^q}{q\epsilon^q} ||w - k||^q_{L^q(\Omega)}. \end{aligned}$$

By choosing $\epsilon = \left(\frac{C_2 p}{2}\right)^{1/p}$, we obtain

$$I = a(w, w-k) - a(w_h, w-k) \leq \frac{C_2}{2} \int_\Omega \left| \mid w \mid^{q-2} w - \mid w_h \mid^{q-2} w_h \right| \mid w - w_h \mid \mathrm{d}x + C(q)||w - k||^q_{L^q(\Omega)}. \tag{2.12}$$

ϵ picked in a way where the first term of the right hand side of 2.12 will be simplified with the second term of the left hand side of 2.10.

To estimate II we choose k such that

$$b_h(\phi, k) = 0 \qquad \forall \phi \in V_0, \tag{2.13}$$

[3]. Now the definition of w_h from 2.1 and 2.13 imply

$$b_h(\phi, w_h - k) = 0 \qquad \forall \phi \in V_0,$$

[3] The Neumann Ritz projection operator $\overline{R}w$ given in Definition 3.1 satisfying this requirement.

and so

$$
\begin{aligned}
b_h(u-u_h, w_h-k) &= b_h(u-u_h, w_h-k) - b_h(R(u-u_h), w_h-k) \\
&= b_h(u-u_h-R(u-u_h), w_h-k) \\
&= b_h(u-Ru, w_h-k) + b_h(Ru_h-u_h, w_h-k) \\
&= b_h(u-Ru, w_h-k).
\end{aligned}
$$

Using this, and the boundedness of b_h from Remark 2.2.1, we obtain

$$
\begin{aligned}
II = b_h(u-u_h, w_h-k) = b_h(u-Ru, w_h-k) \\
&\leq ||u-Ru||_{W_h^{2,p}(\Omega)}||w_h-k||_{L_h^q(\Omega)} \\
&\leq C||u-Ru||_{W_h^{2,p}(\Omega)}||w_h-k||_{L^q(\Omega)} \\
&\leq \frac{C}{4\epsilon}||u-Ru||^2_{W_h^{2,p}(\Omega)} + \epsilon||w_h-k||^2_{L^q(\Omega)} \qquad (2.14)
\end{aligned}
$$

by Remark 2.2.1 $||.||_{L_h^p(\Omega)} \sim ||.||_{L^p(\Omega)}$ and applying Young's inequality after multiplication by $\frac{C}{\sqrt{2\epsilon}}\frac{\sqrt{2\epsilon}}{C} = 1$. Now applying triangular and Young's inequalies to the second term of the right hand side of 2.14, we see

$$
\begin{aligned}
\epsilon||w_h-k||^2_{L^q(\Omega)} &\leq \epsilon(||w-w_h||_{L^q(\Omega)} + ||w-k||_{L^q(\Omega)})^2 \\
&\leq \epsilon(||w-w_h||^2_{L^q(\Omega)} + ||w-k||^2_{L^q(\Omega)} + 2||w-w_h||_{L^q(\Omega)}||w-k||_{L^q(\Omega)}) \\
&\leq \epsilon(||w-w_h||^2_{L^q(\Omega)} + ||w-k||^2_{L^q(\Omega)} + ||w-w_h||^2_{L^q(\Omega)} + ||w-k||^2_{L^q(\Omega)}) \\
&\leq 2\epsilon(||w-w_h||^2_{L^q(\Omega)} + ||w-k||^2_{L^q(\Omega)}).
\end{aligned} \qquad (2.15)
$$

2.14 and 2.15 imply

$$
II = b_h(u-u_h, w_h-k) \leq \frac{C}{4\epsilon}||u-Ru||^2_{W_h^{2,p}(\Omega)} + 2\epsilon(||w-w_h||^2_{L^q(\Omega)} + ||w-k||^2_{L^q(\Omega)}). \qquad (2.16)
$$

Substituting 2.12 and 2.16 into 2.10 bearing in mind 2.11 and taking ϵ small enough we obtain

$$||w-w_h||^2_{L^q(\Omega)} \leq C(||w-k||^q_{L^q(\Omega)} + ||u-Ru||^2_{W_h^{2,p}(\Omega)} + ||w-k||^2_{L^q(\Omega)}), \tag{2.17}$$

this allows to us to use the approximability of R and $\overline{R}$ and concluding the proof for the auxiliary variable.

Now we show a bound concerning the primal variable i.e. we bound the second term appearing in the left hand side of 2.8. Beginning with the following: In view of the definition of R and Galerkin orthogonality we note that

$$\begin{aligned} 0 &= a(w,\phi) - a(w_h,\phi) + b_h(u-u_h,\phi) \\ &= a(w,\phi) - a(w_h,\phi) + b_h(Ru-u_h,\phi) \end{aligned}$$

$\forall \phi \in V_0$. Using this, inf-sup property from Lemma 2.3.1 and the equivalence of the L^q-norm and its discrete counterpart. we obtain

$$\begin{aligned} ||Ru-u_h||_{W_h^{2,p}(\Omega)} &\leq \sup_{0\neq\phi\in V_0} \tfrac{b_h(Ru-u_h,\phi)}{||\phi||_{L_h^q(\Omega)}} \\ &= \sup_{0\neq\phi\in V_0} \tfrac{a(w_h,\phi)-a(w,\phi)}{||\phi||_{L_h^q(\Omega)}} \\ &\leq C_3 \sup_{0\neq\phi\in V_0} \tfrac{\left(\int_\Omega \left||w|^{q-2}w-|w_h|^{q-2}w_h\right||w-w_h|\mathrm{d}x\right)^{1/p}||\phi||_{L^q(\Omega)}}{||\phi||_{L_h^q(\Omega)}} \\ &\leq C_3 C \left(\int_\Omega \left||w|^{q-2}w-|w_h|^{q-2}w_h\right||w-w_h|\mathrm{d}x\right)^{1/p}, \end{aligned} \tag{2.18}$$

Now Lemma 2.4.1 and $\epsilon-$Young's inequality giving us

$$\begin{aligned} C_2\int_\Omega \left| |w|^{q-2} w - |w_h|^{q-2} w_h\right| |w-w_h|\mathrm{d}x &\leq a(w,w-w_h) - a(w_h,w-w_h) \\ &\leq C_3 \left(\int_\Omega \left||w|^{q-2}w-|w_h|^{q-2}w_h\right||w-w_h|\mathrm{d}x\right)^{1/p} \\ &\quad \times ||w-w_h||_{L^q(\Omega)} \\ &\leq \frac{\epsilon^p}{p}\int_\Omega \left||w|^{q-2}w-|w_h|^{q-2}w_h\right||w-w_h|\mathrm{d}x \\ &\quad + \frac{C_3^q}{q\epsilon^q}||w-w_h||^q_{L^q(\Omega)}. \end{aligned} \tag{2.19}$$

let $\epsilon = (\frac{pC_2}{2})^{1/p}$, so

$$\int_\Omega \big| \mid w \mid^{q-2} w - \mid w_h \mid^{q-2} w_h \big| \, |w - w_h| \mathrm{d}x \leq C||w - w_h||^q_{L^q(\Omega)}. \tag{2.20}$$

By substituting 2.20 into 2.18 we get

$$||Ru - u_h||_{W_h^{2,p}(\Omega)} \leq C||w - w_h||^{q/p}_{L^q(\Omega)}.$$

The result follows from the fact

$$\begin{aligned} ||u - u_h||_{W_h^{2,p}(\Omega)} &= ||u - Ru + Ru - u_h||_{W_h^{2,p}(\Omega)} \\ &\leq ||Ru - u||_{W_h^{2,p}(\Omega)} + ||Ru - u_h||_{W_h^{2,p}(\Omega)} \end{aligned}$$

and using the approximation properties of the Ritz projection, concluding the proof. ■

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