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Discretization of some parabolic problems

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Discretization of some parabolic problems

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I dedicate this modest work to :

My dearest parents, symbols of kindness, sacrifice and love.

My dear sisters Lamia, Souhila, and my dear brother Chouaib, Soufiane, Hamza, Khaled

My dear husband, and my son Alla eddine

All my family,

All my friends …

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It is with happiness that I consecrate these words as a sign of recognition to all those who have contributed, from near or far, to the realization of this thesis. I would like to express my sincere thanks to all of them. At the beginning, I would like

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ملخص

في هذه الأطروحة نقدم مسألتين مكافئتين جزئيتين. في الحالة الأولى، نتعامل مع معادلة الانتشار الجزئي بشرط حد غير معروف ومعامل غير محلي. باستخدام طريقة روث جنبًا إلى جنب مع طريقة العناصر المحدودة وقياس تكامل إضافي ، نقوم بإعادة بناء حالة ديريكلي المفقودة. يؤدي وجود المعامل غير المحلي إلى استهلاك كبير للوقت عند حل المعادلة عدديًا باستخدام طريقة نيوتن لأن المصفوفة الجاكوبية التي تم الحصول عليها كاملة. من أجل حل هذه المشاكل، قمنا بتطوير أسلوب مستوحى من فكرة جودي. التجربة العددية توضح كفاءة النهج المقترح .

ثانيًا، صحة (وجود وتفرد وبعض نتائج الاستقرار) للمسالة المتعلقة بإعادة بناء حالة ديريكلي الغير المعروفة المعتمدة على الوقت لمعادلة التكامل التفاضلي الكسري، لهذا الغرض، نستخدم قياسًا تكامليًا إضافيًا مع تقدير الوقت الروث. أخيرًا، نقدم بعض الأمثلة العددية لتوضيح النتائج.

الكلمات المفتاحية: معادلات تكاملية تفاضلية كسرية، مشكلة منفصلة، تقدير مسبق، شرط ديريكلي غير معروف.

Abstract

In this thesis we present two fractional parabolic problems. In the first one, we treat the partial diffusion equation with an unknown boundary condition and a non-local coefficient. Using the Rothe method combined with the finite element method and an additional integral measure, we reconstruct the missing Dirichlet state. The presence of the non-local component leads to a large consuming of time when solving the equation numerically using the Newton method because the obtained Jacobian matrix is complete. In order to resolve these problems, we develop a method inspired from Gudi's idea. The numerical experiment demonstrate the efficiency of the proposed approach.

Secondly, the well-posedness(existence, uniqueness and some stability results) of the problem concerning the reconstruction of the unknown time-dependent boundary function for fractional integro-differential equation, for this purpose, we use an additional integral measurement with Rothe time discretization. Finally, we give some numerical examples to illustrate the results.

Key-words: Fractional integro-differential equations, Discrete problem, A priori estimate, Unknown Dirichlet condition .

Résumé

Dans cette thèse nous avons considéré deux problèmes parabolique fractionnaire. Dans le premier, nous avons traité une équation de diffusion partielle avec une condition limite inconnue et un coéfficient non local. En utilisant la méthode de Rothe combinée avec la méthode des éléments fini et une mesure intégrale supplémentaire, nous reconstruirons l'état de Dirichlet manquant. La présence de la composante non locale entraîne des pertes de temps lors de la résolution numérique de l'équation à l'aide de la méthode de Newton car la matrice jacobienne obtenue est complète. Afin de résoudre ces problèmes, nous développons une méthode inspirée de l'idée de Gudi, le test numérique démontre l'efficacité de l'approche proposée.

Deuxièment, Nous prouvons le caractère bien-posé (existence, unicité et stabilité) du problème concernant la reconstruction de la fonction inconnue dépendant du temps sur la frontière pour une équation fractionnaire integro-différentielle, pour cela, on utilise une mesure intégrale supplémentaire avec la discrétisation temporelle de Rothe. Des expériences numériques sont données pour illustrer les résultats.

Mots-Clés: Équations fractionnaires integro-différentielles, Problème discret, Estimation a priori, Condition de Dirichlet inconnue.

Contents

Introduction

Partial differential equations are an indisputable factor in applied mathematics and engineering. The solution of these equations occupies an important place in these fields. Recently, some researchers, from all over the world, work on simulation between the physical model and the previously conducted experiments, which has led to the development and expansion of new theories such as fuzzy sets theory, fractional calculus and integration...etc.

This latter theory became the focus of interest of researchers what generated a wonderful scientific experiments involving several generations of mathematicians and physicists like the celebrated name Leibniz who made some observations about the meaning and possibility of the fractional derivative at the end of the 17th century. This subject represents a branch of derivation and integration theory that unifies and generalizes the concepts of correct differentiation and repeated integration. Leibniz clarified this concept in his famous letter to the hospital when he wanted to initiate the preservation of a possible theory of the incomplete derivation of function. With this theory, models are located on fewer variables and they describe the non-local and memory properties, which let to a great progress in modeling several applications, including signal processing, molecular biology, psychological analysis, etc. We refer the reader to [2], [6], [10], [20], [30], [47], [49], [53], [55], [57].

The analytical resolution of problems related to partial differential equations is often difficult, for this reasons, many numerical methods have been developed, among the most familiar we mention the finite differences, finite volume, finite element and Rothe method, see for instance [4], [20], [21], [29].

In the following, we will discuss the two most common methods of numerical approximation, the first one is the finite element method, which was introduced in the $1950's$, thenceforth, it has since undergone many development and it became one of the most used in many numerical simulation software. This method is based on the variational formulation (i.e. on weak solutions) and consists on approaching the infinite dimensional solutions space by, generally, a finite dimensional subspace.
Any engineer, working in an environment where numerical simulation is an important tool, will be confronted with the said method, therefore, he must know its fundamental principles, and even the latest refinements.

In the framework of numerical approximation, we also discuss the Roth method where discretization in time, or derivatives with respect to a variable are replaced by difference quotients which leads finally to differential equations system. This method was developed by Rothe in 1930 for second order one-dimensional linear parabolic equations, afterwards, it has been adopted by Ladyzhenskaja, for linear parabolic and one-dimensional second order parabolic problems, second order quasi-linear and higher order linear equations. The method of discretization in time is considered as an efficient theoretical and numerical tool in the study of evolution problems.
The discretization scheme of the Rothe method is as follows:
1)The time interval is divided into n sub-intervals (t_{i-1}, t_i), $i = 1, ..., n$ with $t_i = ih$ and $h = \frac{T}{n}$.

2)We replace the derivative $\frac{\partial u}{\partial t}$ by the quotient $\delta u = \frac{u_i - u_{i-1}}{h}$ with $u_i = u_i(x, ih)$ for all $i = 1, ..., n$.

3)We extract a system form of n equations in x where $u_i(x)$ is the unknown, so, at any point t_i, we approximate the continuous problem by a new discrete problem which determine the Roth function u^n defined as the approximation of the solution u by a polynomial of degree 1 on each sub-interval (t_{i-1}, t_i) [38]i.e:

$$u^n(t) = u_{i-1} + (t - t_{i-1})\delta u_i, \quad t \in [t_{i-1}, t_i], \quad i = 1, ..., n$$

With the step functions

$$\overline{u}^n(t) = \begin{cases} u_i & t \in (t_{i-1}, t_i] \\ u_0 & t = 0 \end{cases}, \; i = 1, ..., n.$$

This thesis is devoted, firstly, to the study of a fractional parabolic equation with an unknown boundary condition and a non-local coefficient. Using Rothe's method combined with the finite element method and an additional integral measure, we reconstruct the missing Dirichlet's condition.

We consider an open bounded domain Ω of $\mathbb{R}^n$ with Lipschitz continuous boundary Γ which is divided into two parts Γ_D, Γ_N, that do not overlap i.e $\Gamma_D \cap \Gamma_N = \varnothing$ with $\Gamma = \overline{\Gamma_D} \cup \overline{\Gamma_N}$. Denote by ν the outward normal vector at each point of Γ and T the final time with $I = [0, T]$.

This work aims to the study the following inverse problem of identifying the missing Dirichlet data $\gamma(t)$ on the part Γ_D of the boundry from an additional integral measure .

$$D^{\alpha}_{RL} u(t,x) - a(l(u))\triangle u(t,x) = f(t,x) \qquad \text{in } I \times \Omega \tag{P}$$

$$u(0,x) = u_0(x) \text{ in } \Omega \tag{P_a}$$

$$\nabla u.\nu = g, \text{ on } I \times \Gamma_N \tag{P_b}$$

$$u = \gamma(t) \text{ on } I \times \Gamma_D, \tag{P_c}$$

The complementary measure is of type

$$\int_\Omega I^{1-\alpha}(u(t,x))\,dx = \theta(t), \tag{P_d}$$

where $\alpha \in]0,1[$, $I^{1-\alpha}$ and D_{RL}^{α} are fractional Riemann-Liouville integral and derivative respectively and a is a continuous function.

In the above equation (P), Δ is the usual Laplace operator, l is defined on $L^2(I)$ by

$$l(u) = \int_\Omega u(x,t)\,dx.$$

The problems of non-local terms attract a lot of attention, as they contribute to the modeling of many physical and biological phenomena, where $u(x,t)$ can describe the population density of bacteria susceptible to spread in x at the time t, f is the density of bacteria supplied from outside and the nonlocal dependence of $a(l(u))$ is the diffusion rate of the population which depends on the entire population as shown in [22], [23]. It should be noted that partial differential equations apears in several areas of investigation, for example in science and engineering such as: rheology, fluid flows, electrical networks, viscoelasticity, chemical physics, biosciences, signal prossecing, systems control theory, electrochemistry, mechanics and diffusion processes. For general motivations, relevant theory and its application we refer the reader to [2], [10], [30],

[41], [47]-[50], [53], [55], [57], [60] and references cited therein.

Because of the impossibility of getting exact solutions in the majority of cases, research has turned to methods that give approximate solutions, so, in recent decades, numerical resolution has become of considerable importance for linear and non-linear partial differential equations. One of the most popular methods is Rothe's method, which is commonly used in the temporal discretization of evolution equations where the derivatives with respect to a variable are replaced by difference quotients that eventually lead to systems of differential equations for the functions of the remaining variables. Rothe's method as an approximate approach is well suited not only for proving existence results, but also for various applications. Nowadays, a vast documentation exists on the subject, we can refer to [20], [29], [31], [32], [42].

Our main goal is to construct the missing Dirichlet boundary data $\gamma(t)$ in (Pc) with the help of the auxiliary measurement (P_d). To this purpose, we use Rothe time discretization to approximate the time dependent problem by sequence of elliptic problems combined with finite element method for the spatial variable. But the presence of the non-local coefficient $a(l(u))$ causes a major difficulties. In order to overcome this obstacle we develop a method based on the idea given in [35] parallelly with our problem to ensure the sparsity of the Jacobian matrix.

The second problem concerns the study of fractional integro-differential equation with unknown flux on the Dirichlet boundary by the use of Rothe time discretization. Thus, the objective is identifying the messing Dirichlet condition $\gamma(t)$, also by using the measurement (P_d), in the following problem :

$$D_{RL}^{\alpha} u(t,x) - \triangle u(t,x) = \int_0^t k(s,u(s,x))\,ds + f(t,x) \text{ in } I \times \Omega \tag{1}$$

$$u(0,x) = u_0(x) \quad \text{in} \quad \Omega \tag{2}$$

$$\nabla u.\nu = g, \text{ on } I \times \Gamma_N \tag{3}$$

$$u = \gamma(t) \text{ on } I \times \Gamma_D, \tag{4}$$

from an additional measurement of type

$$\int_{\Omega} I^{1-\alpha}(u(t,x))\,dx = \theta(t), \tag{5}$$

Where k is globally Lipschitz continuous function.

One of our motivation for studying fractional diffusion equations comes from applications in diffusion on fractals area, more precisely, it can describe efficiently anomalous diffusion on some amorphous semiconductors or strongly porous materials [9]. Other interesting applications are related to the study of fractional random walk [51]. We start by stating a related results. Let us mention the works of Oldham et *all* [57], where the authors considered the relation between usual diffusion equation and a fractional diffusion equation. In [56], the model of waves diffusion in viscoelastic medium based on fractional calculus was established. The work [28] treated the following problem

$$\begin{cases} y \text{ is continuous on } [0,T], y(t) \in D(A) \text{ for each } t \in [0,T], \\ D_C^{\alpha} y \text{ exists and is continuous on } [0,T], \\ D_C^{\alpha} y(t) = Ay(t) + f(t), t \in [0,T], \\ y(0) = y^0, \end{cases}$$

where D_C^{α} fractional Caputo derivative, A is generator of an analytic semi-group. The authors he proved the uniqueness of the solution under convenable assumptions on f.

In [52] Mophou et *all* considered the problem

$$\begin{cases} y \in C\left((0,T],D(A)\right), I^{1-\alpha}y \in C\left([0,T],X\right) \cap C^{1}\left((0,T],X\right), \\ D^{\alpha}_{RL}y(t) = Ay(t) + f(t), t \in [0,T], \\ I^{1-\alpha}y\left(0^{+}\right) = y^{0}, \end{cases}$$

where $I^{1-\alpha}$, D^{α}_{RL} are Riemann-Liouville integral and derivative respectively. They investigated the existence and uniqueness of the solution under suitable conditions on the densely defined linear operator A. In [13], Bahuguna et *all* studied a similar fractional differential equation and showed the existence of the strong solution using Rothe method. Recently, Chaoui et *all* [21] Looked into the following pseudo-parabolic fractional problem

$$\begin{cases} D^{\alpha}_{RL}u(t,x) - \Delta u - \Delta u_{t} = f(t,x), (t,x) \in I \times \Omega, \\ u(0,x) = u_{0}(x), x \in \Omega, \\ u = 0 \text{ on } I \times \partial\Omega, \\ I^{1-\alpha}u\left(0^{+}\right) = U_{1}(x), \end{cases}$$

They proved the well posedness of the problem by the use of Rothe time discretization method. Here, Rothe's method and some weak compactness criterion are used to exhibit the existence of unique solution for the fractional integro-differential problem (3.1)-(3.5) and reconstruct the unknown Dirichlet boundary condition $\gamma(t)$.

This thesis consists of an introduction, three chapters, perspectives and conclusion. The first chapter contain basic notions and elementary definitions of functional and fractional analysis that will be useful in the following chapters.

In the second chapter, we present the original results published in (Dynamics of Continuous, Discrete and Impulsive Systems, journal) concerning the study of fractional parabolic equation with an unknown boundary condition and a non-local coefficient. So,as explained above, using the Rothe method combined with the finite element method and an additional integral measure, we reconstruct the missing Dirichlet condition. We conclude this chapter by some numerical tests to prove the validity of the proposed approach.

The third chapter concerns the full content of the main article were the subject of the publication: Solution to fractional integro-differential equation with unknown flux on the Dirichlet boundary, in which, we prove the existence, uniqueness and some stability results.

Perspectives and future works are given a conclusion.

1

Preliminaries

1.1 Introduction

In this chapter, we mainly describe the general properties of topological vector spaces and the basic definitions of functional analysis, and then we give some definitions and properties of fractional calculi that will be needed in the rest of the thesis (for more information see [15], [16], [17], [37], [38], [40], [41], [53], [55], [59], [62]).

In the following Ω denotes an open of R^n with Lebesgue measure dx.

1.2 Functional spaces

1.2.1 Lebesgue spaces

Definition 1.2.1. *we call Lebesgue space, the vector space of numerical functions u of Ω in (C), and we note it $L^p(\Omega)$. With p an element of $[1;+\infty]$ and Ω an open of R^n, Lebesgue measurable verifying :*

1. If $1 \leq p < +\infty$,

$$\int_{\Omega} |u(x)|^p dx < +\infty,$$

2. If $p = +\infty$,

$$\sup_{x\in\Omega} |u(x)| < +\infty.$$

Where:

$$\sup_{x\in\Omega} |u(x)| = inf M|u(x)| \leq M \text{ p.p.}$$

Definition 1.2.2. *Any complete normalized vector space is called a Banach space.*

Some properties

a) The application of $L^p(\Omega)$ in R^+:

$$u \rightarrow \begin{cases} \|u\|_p = (\int_{\Omega} |u(x)|^p dx)^{\frac{1}{p}}, & 1 \leq p < +\infty \\ \|u\|_{\infty} = sup_{x\in\Omega}|u(x)|, & \text{p}=\infty. \end{cases}$$

Defines a norm on $L^p(\Omega)$, norm by which $L^p(\Omega)$ is a space of Banach.

b) For any real p in $[1,+\infty[$, the dual of $L_p(\Omega)$ is algebraically and topologically isomorphic at $L_q(\Omega)$, with $\frac{1}{p}+\frac{1}{q}=1$, the application of duality is defined by:

$$\mathrm{L}^p(\Omega) \times L^p(\Omega) \longrightarrow C^n$$

$$(u,v) \longrightarrow \int_\Omega u(x)v(x)dx.$$

For any real p in $[1;+\infty[$, the vidual of $L_p(\Omega)$, identifies algebraically and topologically to $L_p(\Omega)$. We say that the space $L_p(\Omega)$ is reflexive.

1.2.2 Hilbert spaces

A Hilbert space is a normalized vector space, and complete for the induced norm, is therefore a special case of Banach space (the norm is defined from a scalar product)

Definition 1.2.3. ***(The** $L^2(\Omega)$ **space)***

Let Ω is an open of R^n we note by L^2 the space of the functions of sommable square on Ω

$$\|f\|_{L^2(\Omega)} = (\int_\Omega |f(x)|^2 dx)^{\frac{1}{2}},$$

with the scalar product :

$$< f,g >= \int_\Omega f(x)g(x)dx.$$

1.2.3 Boschner spaces

Let $\Omega \subset R^N$ and $I=[0;T]$ be an interval of R.

We define the following spaces

1- $C(I,L^2(\Omega))=\{f:I\rightarrow L^2(\Omega)$ which associates to $t,\ f(t)\in L^2(\Omega)$ continues$\}$ with the norm

$$\|f\|_{C(I,L^2(\Omega))}=\max_I\|f(t)\|_{L^2(\Omega)}. \tag{1.1}$$

2- $L^2\left(I,L^2(\Omega)\right)=\{f:I\rightarrow L^2(\Omega)$ with integrable square $\}$ with the norm

$$\|f\|^2_{L^2(I,L^2(\Omega))}=\int_I\|f\|^2_{L^2(\Omega))}dt. \tag{1.2}$$

3-$L^\infty\left(I,H^1(\Omega)\right)=\{f:I\rightarrow H^1(\Omega)$ essensticially bounded$\}$ with the norm

$$\|f\|_{L^\infty(I,H^1(\Omega))}=\sup_I\|f\|_{H^1(\Omega)}\text{ p.p.} \tag{1.3}$$

1.3 Fundamental inequalities

1.3.1 Trace inequality

Let $\Omega\subset R^n$, we have

$$\| Z \|^2_\Gamma\leq\varepsilon\parallel\nabla Z\parallel^2+C_\varepsilon\parallel Z\parallel^2,\forall Z\in H^1(\Omega),0\leq\varepsilon<\varepsilon_0.$$

1.3.2 Cauchy-schwarz inequality.

Let $\Omega\subset R^N,\forall f,g\in L^2(\Omega)$:

$$\left|\int_\Omega f(x)g(x)\,dx\right|\leq\left(\int_\Omega|f(x)|^2dx\right)^{\frac{1}{2}}\left(\int_\Omega|g(x)|^2dx\right)^{\frac{1}{2}}, \tag{1.4}$$

where
$$\left|\int_\Omega \sum_{i=1}^{n} f_i(x) g_i(x) dx\right| \leq \left(\int_\Omega \sum_{i=1}^{N} f_i(x) dx\right)^{\frac{1}{2}} \left(\int_\Omega \sum_{i=1}^{N} g_i(x) dx\right)^{\frac{1}{2}}. \tag{1.5}$$

1.3.3 Hölder inequality

It is a generalization of Cauchy's inequalities. For $1 \leq a, b < \infty$, $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$ such that $\frac{1}{a} + \frac{1}{b} = 1$, we have

• **Continuous form**

$$\Big|\int_\Omega u(x) v(x) dx\Big| \leq \Big(\int_\Omega |u(x)|^a dx\Big)^{\frac{1}{a}} \Big(\int_\Omega |v(x)|^b dx\Big)^{\frac{1}{b}}. \tag{1.6}$$

• **Discrete form**

$$\Big|\int_\Omega \sum_{i=1}^{N} u_i(x) v_i(x) dx\Big| \leq \Big(\int_\Omega \sum_{i=1}^{N} |u_i(x)|^a dx\Big)^{\frac{1}{a}} \Big(\int_\Omega \sum_{i=1}^{N} |v_i(x)|^b dx\Big)^{\frac{1}{b}}. \tag{1.7}$$

1.3.4 Triangular inequality

Let $\Omega \subset R^N$, $\forall u, v \in L^p(\Omega)$, we have

$$\Big(\int_\Omega \big(u(x) + v(x)\big)^2 dx\Big)^{\frac{1}{2}} \leq \Big(\int_\Omega (u(x))^2 dx\Big)^{\frac{1}{2}} + \Big(\int_\Omega (v(x))^2 dx\Big)^{\frac{1}{2}}. \tag{1.8}$$

1.3.5 Young's inequality

Let $a, b > 0$, and $p, q \in]1, +\infty[$, $\frac{1}{p} + \frac{1}{q} = 1$ so

$$a.b \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}. \tag{1.9}$$

1.3.6 ε-inequality

$$\left|x.y\right| \leq \frac{\varepsilon}{2}x^2 + \frac{1}{2\varepsilon}y^2, \forall \varepsilon > 0, \forall x, y.$$

1.3.7 Poincaré's inequality

Let Ω a bounded open of R^n, then there is a constant $C > 0$ such that:

$$\| u \|_{L^p(\Omega)} = \left[\left(\int_\Omega |u|^p \, dx\right)\right]^{\frac{1}{p}} \leq C \left[\left(\int_\Omega |\nabla u|^p \, dx\right)\right]^{\frac{1}{p}}, \forall u \in W^{1,p} \; 1 \leq p \leq \infty. \tag{1.10}$$

In particular the expression $\|\nabla u\|_{L^p(\Omega)}$ is a norm on $W_0^{1,p}(\Omega)$ that is equivalent to the norm $\| u \|_W^{1,p}$ on $H_0^1(\Omega)$, the expression $\int \nabla u \nabla v \, dx$ is a scalar product that induces the $\|\nabla u\|_{L^p(\Omega)}$ equivalent to the norm $\| u \|_{H^1}$.

1.3.8 Trace inequality

Let $\Omega \subset R$, we have

$\|Z\|_\Gamma^2 \leq \epsilon \|\nabla Z\|^2 + C_\epsilon \|Z\|^2, \; \forall Z \in H^1(\Omega), 0 \leq \epsilon \leq \epsilon_0.$

1.4 Useful theorems

1.4.1 Minty-browder theorem.

Theorem 1.4.1. *Let* $d : \; Q_t = (0, T) \times \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ *be monotone in the last variable, i.e.* $(d(t, x, z_1) - d(t, x, z_2))(z_1 - z_2) \geq 0$ *for* $z_1, \; z_2 \in \mathbb{R}^N$,

and

$u_n \rightharpoonup u \; in \; L^p(Q_t)^N$

$d(t,x,u_n) \rightharpoonup v \; in \; L^q(Q_t)^N$

$\lim_{n\to\infty} \sup \int_\Omega d(t,x,u_n)u_n dx \leq \int_\Omega vudx$

Then $v = d(t,x,u)$.

1.4.2 Green's formula

Let be Ω a regular bounded open of class C^2, and $n(x)$ its normal outward. If $u \in H^2(\Omega)$ and $v \in H^2(\Omega)$, we have

$$\int_\Omega \Delta u(x)v(x)dx = -\int_\Omega \nabla u \nabla v dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} v d\sigma. \tag{1.11}$$

1.4.3 Kolmogorov's theorem

Theorem 1.4.2. *Let Ω be a bounded domain in R^N. A set M of functions $f \in L^p(\Omega)$ is precompact iff M is bounded and equicontinuous i.e.*

$$\forall \varepsilon > 0, \exists \delta > 0 \; such \; as \; \forall \; f \in M \int_\Omega \left|f\left(x+y\right) - f\left(x\right)\right|^p dx < \varepsilon \; for \; |y| < \delta.$$

1.4.4 Gronwall's lemma (continuous form)

Lemma 1.4.3. *Let α, β and γ be real valued functions on $I = [a, \infty)$. Assume that $\beta$and γ are continuous. If β nonnegative, α is nondecreasing and if γ satisfies the integral inequality*

$$\gamma(t) \leq \alpha(t) + \int_a^t \beta(s)\gamma(s)ds \; \forall t \in I, \tag{1.12}$$

then

$$\gamma(t) \leq \alpha(t)\, exp\,(\int_a^t \beta(s)\, ds). \tag{1.13}$$

1.4.5 Gronwall's lemma (discrete form)

Lemma 1.4.4. *If*

$$\gamma_n \geq 0, \alpha_n \geq \alpha_{n-1}, \beta_j \geq 0\ et\ \gamma_n \leq \alpha_n + \sum_{j=1}^{n} \beta_j \gamma_j,\ n \geq 0, \tag{1.14}$$

then

$$\gamma_n \leq \alpha_n exp(\sum_{j=1}^{n} \beta_j). \tag{1.15}$$

1.4.6 Abel's lemma

$\sum_{i=1}^{j} z_i(w_i - w_{i-1}) = z_j w_j - z_0 w_0 - \sum_{i=1}^{j}(z_i - z_{i-1}) w_{i-1}, \forall z_i, w_i \in R.$

1.5 Weak convergence

Let E be a Banach space.

Definition 1.5.1. (f_n) *converges weakly in* E *to* f *if we have :*

$$lim_{n\to+\infty} \langle f^{\grave{}}, f_n\rangle = \langle f^{\grave{}}, f\rangle \Longleftrightarrow lim_{n\to+\infty} \langle f^{\grave{}}, f_n - f\rangle = 0, \forall f^{\grave{}} \in É,$$

for É *the dual space of E.*

Notation:

1.We note by $f_n \rightharpoonup f$ the weak convergence in E.

2.We note by $f_n \to f$ the strong convergence in E (i.e. convergence in norm).

Remark 1.5.2. *-If $f_n \to f$ strongly $\|f_n - f\|_E \Longrightarrow 0 \Rightarrow f_n \rightharpoonup f$ because :*

$$\forall f^{\backprime} \in \acute{E} : \langle f^{\backprime}, f_n - f\rangle \leq \|f^{\backprime}\| \|f_n - f\| \longrightarrow 0.$$

Definition 1.5.3. *Let H and K be two spaces of Banach. and G be a continuous linear application of H in K and let x_n be a sequence of H such that $x_n \to^H x$ then $G(x_n) \to^K G(x)$.*

1.6 Semi-continuous and hemi-continuous operator

Definition 1.6.1. *It is said that $A : V \to V^*$ a semi-continuous operator in V.*
If $v_n \in V, n = 1,2,...$ and $v_n \to v$ when $n \to +\infty$ then $A(v_n) \rightharpoonup A(v)$ in V^*.

Definition 1.6.2. *It is said that $A : V \to V^*$ a hemi-continuous operator in V.*
If $u \in V, t_n = 1,2,...$ and $v + t_n u \in v$ then $A(v + t_n u) \rightharpoonup A(v)$ in V^*.

1.7 Fractional calculus.

1.7.1 Gamma function

The Gamma function is simply the general function of the factorial function and is one of the basic tools of fractional calculation.

Definition 1.7.1. *The Gamma function is defined by the integral :*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, z > 0,$$

where $t^{z-1} = exp(z-1)ln(t)$.

proposition

For all $z \in R\{0,-1,-2,...\}$ and $n \in N$ we have:

.$\Gamma(z+1) = z\Gamma(z)$

.$\Gamma(z+n) = z(z+1)..(z+n-1)\Gamma(z)$

.$\Gamma(n+1) = n!$

1.7.2 Riemann-liouville fractional integrals

Definition 1.7.2. *Let* $f \in C\left(\mathbb{R}^+, \mathbb{R}\right)$ *and* $\alpha > 0$, *then the formula*

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds.$$

is called the Riemann-Liouville fractional integral.

1.7.3 Riemann-liouville fractional derivative

Definition 1.7.3. *If* $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$. *Then the following expression*

$$D_{RL}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{-n+\alpha+1}} ds, \ t > 0,$$

is called The Riemann-Liouville fractional derivative where $\alpha \in (n-1, n)$, $n \in \mathbb{N}$.

Lemma 1.7.4. *Let* $f \in C^n([0,T])$, $T > 0$, $\alpha \in (n-1,n)$, $n \in \mathbb{N}$. *Then for* $t \in [0,T]$, *the following properties hold*

$$D_{RL}^{\alpha} f(t) = \frac{d}{dt} I^{1-\alpha} f(t), \; n = 1. \tag{1.16}$$

$$D_{RL}^{\alpha} I^{\alpha} f(t) = f(t). \tag{1.17}$$

$$I^{\alpha} D_{RL}^{\alpha} f(t) = f - \sum_{k=1}^{n} \frac{t^{\alpha-k}}{\Gamma(\alpha-k+k)} \left(I^{k-\alpha} f\right)^{(n-k)}(0). \tag{1.18}$$

$$I^{\alpha} D_{RL}^{\alpha} f(t) = f - \frac{t^{\alpha-1}}{\Gamma(\alpha)} \left(I^{1-\alpha} f\right)(0), \; if\, n = 1. \tag{1.19}$$

2

Discretization scheme of fractional parabolic equation with nonlocal coefficient and unknown flux on the Dirichlet boundary

2.1 Introduction

In this chapter, we are interested in the application of the Rothe method combined with the finite element method on a fractional parabolic problem with an unknown Dirichlet boundary condition and a non local coefficient. This type of problem can be encountered in the modeling of many physical and biological phenomena. This chapter then consists in solving this nonlinear equation, in which we will try to reconstruct the missing Dirichlet condition through a new integral condition, we will show the existence and uniqueness of the approximate solution, as well as some a priori estimates, and give some numerical results .

2.2 Position of the problem

We consider the fractional parabolic equation of the following form .

$$D_{RL}^{\alpha} u(t,x) - a(l(u)) \triangle u(t,x) = f(t,x) \qquad \text{in } I \times \Omega \tag{P}$$

$$u(0,x) = u_0(x) \ \text{ in } \ \Omega \tag{P_a}$$

$$\nabla u.\nu = g, \text{ on } I \times \Gamma_N \tag{P_b}$$

$$u = \gamma(t) \ \text{on } I \times \Gamma_D, \tag{P_c}$$

the complementary measure of type

$$\int_{\Omega} I^{1-\alpha}\left(u\left(t,x\right)\right)dx=\theta\left(t\right), \tag{P_d}$$

where $\alpha\in\left]0,1\right[$. $I^{1-\alpha}$ and D_{RL}^{α} are fractional Riemann-Liouville integral and derivative respectively.

In the above equation (P), Δ is the usual Laplace operator, Denote by a a function from $\mathbb{R}$ into $\mathbb{R}$ such that a is continuous, l is defined on $L^2\left(\Omega\right)$ by

$$l(u)=\int_{\Omega} u(x,t)dx.$$

2.3 Variational formulation and discretization scheme

We define Lebesgue space

$$L^2\left(\Omega\right)=\left\{u:\Omega\longrightarrow\mathbb{R};\ u \text{ is measurable with } \int_{\Omega}\left|u\left(x\right)\right|^2dx\prec\infty\right\},$$

equipped with usual inner product and norm $(\,,)$ and $\|\,.\|$, respectively. We denote by $(,)_{\Gamma}$ and $\|.\|_{\Gamma}$ the inner product and the norm in $L^2\left(\Gamma\right)$. We define also Sobolev space

$$H^1\left(\Omega\right)=\left\{u\in L^2\left(\Omega\right):\ D^{\alpha}u\in L^2\left(\Omega\right)\ \text{for}\ |\alpha|\leq 1\right\},$$

endowed with norm $\|u\|^2_{H^1(\Omega)} = \|u\|^2 + \|\nabla u\|^2$. Due to the Dirichlet condition (P_c) , assume that the Dirichlet trace of the solution is fixed in the Hilbert space

$$V = \left\{u \in H^1(\Omega) \text{ such that } u_{\backslash\Gamma_D} \text{ is constant}\right\},$$

with induced norm from $H^1(\Omega)$.

The following hypotheses are made along this work

(H_1) $u_0 \in V$ and $g{:}I \times \Gamma_N$ is continuous function.

(H_2) $f(t) \in L^2(\Omega)$ and $\left\|f(t) - f\left(t'\right)\right\| \leq l\left|t - t'\right|$.

(H_3) a is Lipschitz continuous and satisfied $a : R \to R^+$is such that

$$0 \prec m \leq a(s) \leq M \prec \infty, \forall s \in R.$$

We will avoid also the degenerate case assuming that $\left|D^\alpha_{RL}u\right| \succ p \succ 0$.

Definition 2.3.1. *The pair (u,γ) weakly solves problem (P) if*

$i)$ $u \in L^2([0,T],V)$ with $I^{1-\alpha}(u) \in C([0,T],V^*)$ and $\partial_t I^{1-\alpha}(u) \in L^2([0,T],V^*)$.

$ii)$ $u_{\backslash\Gamma_D} = \gamma(t)$.

$iii)$ For any $\phi \in V$, we have

$$\int_0^T \left(\partial_t I^{1-\alpha}(u),\phi\right)dt + \int_0^T a(l(u))\left(\nabla u,\nabla\phi\right)dt + \int_0^T a(l(u))\left(g,\phi\right)_{\Gamma_N}dt - \int_0^T \left(f,\phi\right)dt$$

$$= \int_0^T \phi_{\backslash\Gamma_D}\left[\theta' + a(l(u))\left(g,1\right)_{\Gamma_N} - \left(f,1\right)\right]dt.$$

The weak formulation of the problem (P) is defined by

$$\left(\partial_t I^{1-\alpha}(u),\phi\right)+a(l(u))(\left(\nabla u,\nabla\phi\right)+\left(g,\phi\right)_{\Gamma_N}-\phi_{\backslash\Gamma_D}\left(\nabla u.\eta,1\right)_{\Gamma_D})=\left(f,\phi\right),\ \forall\phi\in V\,. \tag{2.1}$$

Hence, testing the identity (2.1) with $\phi=1$ and taking (P_d) into account , we obtain

$$a(l(u))\left(\nabla u.\eta,1\right)_{\Gamma_D}=\theta'+a(l(u))\left(g,1\right)_{\Gamma_N}-\left(f,1\right), \tag{2.2}$$

replacing in (2.1), we get

$$\begin{aligned}\left(\partial_t I^{1-\alpha}(u),\phi\right)+a(l(u))(\left(\nabla u,\nabla\phi\right)+\left(g,\phi\right)_{\Gamma_N})&=\left(f,\phi\right)\\+\phi_{\backslash\Gamma_D}\left[\theta'+a(l(u))\left(g,1\right)_{\Gamma_N}-\left(f,1\right)\right]&.\end{aligned} \tag{2.3}$$

We subdivide the time interval I by points $t_i=i\tau$, $\tau=\frac{T}{n}$, $i=1,...,n$, and denote $u_i=u(t_i,x)$, $\delta u_i=\frac{u_i-u_{i-1}}{\tau}$. After discretization,the recurrent approximation scheme for $i=1,...,n$ became :

$$\begin{aligned}\left(I^{1-\alpha}(u_i)-I^{1-\alpha}(u_{i-1}),\phi\right)+\tau a(l(u))\left(\nabla u_i,\nabla\phi\right)+\tau a(l(u))\left(g_i,\phi\right)_{\Gamma_N}&=\tau\left(f_i,\phi\right)\\+\tau\phi_{\backslash\Gamma_D}\left[\theta_i'+a(l(u))\left(g_i,1\right)_{\Gamma_N}-\left(f_i,1\right)\right]&.\end{aligned} \tag{2.4}$$

Due to the monotony and coercivity of the operator T, then there is a solution u_i at each time step, where T is given by :

$$\left(Tu,\phi\right):=\left(I^{1-\alpha}(u),\phi\right)+\tau a(l(u))\left(\nabla u,\nabla\phi\right)\,.$$

2.4 A priori estimates

Lemma 2.4.1. *The following estimates are made uniformly for n, i, j and τ.*

$$\sum_{i=1}^{l} \tau \|\delta u_i\|^2 \leq C, \ \|\nabla u_i\|^2 \leq C, \ \sum_{i=1}^{l} \|\nabla u_i - \nabla u_{i-1}\|^2 \leq C, \ \|u_i\| \leq C. \tag{2.5}$$

proof. We take $\phi = u_i - u_{i-1}$ as a test function in (2.4) and summing over $i = 1, ..., l$, this leads to :

$$\begin{aligned}&\sum_{i=1}^{l} \tau \left(\delta I^{1-\alpha}(u_i), \delta u_i\right) + \sum_{i=1}^{l} \tau a(l(u_i)) (\nabla u_i, \nabla u_i - \nabla u_{i-1}) = \sum_{i=1}^{l} \tau \left(f_i, \delta u_i\right) \\ &- \sum_{i=1}^{l} \tau a(l(u_i)) \left(g_i, \delta u_i\right)_{\Gamma_N} + \tau \sum_{i=1}^{l} \phi_{\setminus\Gamma_D} \left[\theta'_i + a(l(u_i)) \left(g_i, 1\right)_{\Gamma_N} - \left(f_i, 1\right)\right].\end{aligned} \tag{2.6}$$

From the hypothesis $(H3)$ and the mean value theorem we get

$$\begin{aligned}&C \sum_{i=1}^{l} \tau \|\delta u_i\|^2 + \frac{1}{2} m \left[\|\nabla u_l\|^2 - \|\nabla u_0\|^2 + \sum_{i=1}^{l} \tau \|\nabla u_i - \nabla u_{i-1}\|^2\right] \\ \leq \ &\sum_{i=1}^{l} \tau \left(\delta I^{1-\alpha}(u_i), \delta u_i\right) + \sum_{i=1}^{l} \tau M (\nabla u_i, \nabla u_i - \nabla u_{i-1}) .\end{aligned} \tag{2.7}$$

With Cauchy-Schwarz and Young inequalities we obtain

$$\sum_{i=1}^{l} \tau \left(f_i, \delta u_i\right) \leq C \left(\frac{1}{\varepsilon} + \varepsilon \sum_{i=1}^{l} \tau \|\nabla \delta u_i\|^2\right) . \tag{2.8}$$

It follows from the trace inequality that

$$\left|\sum_{i=1}^{l} \tau a(l(u_i)) \left(g_i, \delta u_i\right)_{\Gamma_N}\right| \leq \varepsilon M \|\nabla u_l\|^2 + C_\varepsilon . \tag{2.9}$$

Similarly, we can estimate the last term of (2.6) as follows

$$\begin{aligned}
& \tau \sum_{i=1}^{l} \delta u_{i\backslash\Gamma_D} \left[\theta_i' + a(l(u_i)) \left(g_i, 1\right)_{\Gamma_N} - \left(f_i, 1\right)\right] \\
\leq \;& C \|u_l\|_{\Gamma_D} \left(\left|\theta_l'\right| + \|f_l\| + M \|g_l\|_{\Gamma_N} + 1\right) + C \\
& + C\tau \sum_{i=1}^{l} \|u_{i-1}\|_{\Gamma_D} \left(\left|\delta\theta_i'\right| + \|\delta f_l\| + M \|\delta g_i\|_{\Gamma_N} + 1\right) \\
\leq \;& \varepsilon \|\nabla u_l\|^2 + C_\varepsilon . \qquad (2.10)
\end{aligned}$$

By summarizing all the previous considerations cited above, we fixed ε sufficiently small. Gronwall's discrete lemma concludes the proof.

to show the last part of the estimate (2.5), we substitute ϕ by u_i in (2.4) and with the same previous way we find the result.

We define Roth functions as follows

$$u^n(t) = u_{i-1} + (t - t_{i-1}) \delta u_i, \quad t \in [t_{i-1}, t_i], \quad i = 1, ..., n \qquad (2.11)$$

$$I_n\left(\overline{u}^n(t)\right) = I^{1-\alpha}(u_{i-1}) + (t - t_{i-1}) \delta I^{1-\alpha}(u_i), \quad t_{i-1} \leq t \leq t_i, \text{ and } i \in \{1, ..., n\}. \qquad (2.12)$$

With the step functions

$$\overline{u}^n(t) = \begin{cases} u_i & t \in (t_{i-1}, t_i] \\ u_0 & t = 0 \end{cases}, \; i = 1, ..., n. \qquad (2.13)$$

$$\overline{I_n}\left(\overline{u}^n(t)\right) = \begin{cases} I^{1-\alpha}(u_i), & t_{i-1} < t \leq t_i \\ I^{1-\alpha} u\left(0^+\right) = U_1 & t = 0 \end{cases}, \; i \in \{1, ..., n\}. \qquad (2.14)$$

$$f^n(t) = \begin{cases} f_i & t \in [t_{i-1}, t_i] \\ f_0 & t = 0 \end{cases} \qquad i = 1, ..., n\,. \tag{2.15}$$

Likewise, we define g_n, θ_n and θ'_n.

Remark 2.4.2. *A direct consequence of lemma* 2.3.1 *is the next a priori estimate.*

Lemma 2.4.3. *There is an independent positive constant C at i, j, n, and τ, such that*

$$\int_I \|\partial_t I_n\|^2_{V^*} \leq C. \tag{2.16}$$

2.5 Existence results

It results from the application of the lemma 2.3.3

$$\max_I \left\| \overline{I_n} \right\| + \|\partial_t I_n\|_{L^2(I,V^*)} \leq C.$$

Indeed, there exist $w \in C(I, V^*) \cap L^\infty\left(I, L^2(\Omega)\right.$ with $\partial_t w \in L^2(I, V^*)$ and subsequence I_{n_k} ,(see [38], lemma.1. (3.13)) in which

$$I_{n_k} \longrightarrow w \text{ in } C\left(I, V^*\right), \ \overline{I_{n_k}}(t) \rightharpoonup w(t) \text{ in } L^2(\Omega)\,. \tag{2.17}$$

$$I_{n_k}(t) \rightharpoonup w(t) \text{ in } L^2(\Omega), \ \partial_t I_{n_k} \rightharpoonup \partial_t w \text{ in } L^2\left(I, V^*\right)\,. \tag{2.18}$$

It can be deduced from this, based on the lemma (2.3.1), that$\{\overline{u}^n\}_n$ and $\{\partial_t I_n(\overline{u}^n)\}$ are uniformly bounded in $L^2(I, V)$, so a subsequence $\{\overline{u}^{n_k}\}_{k\in\mathbb{N}}$ can be extracted

$$\overline{u}^{n_k} \underset{k\longrightarrow\infty}{\rightharpoonup} u \text{ in } L^2(I, V)\,.$$

$$\partial_t I_{nk}\left(\overline{u}^{nk}\right) \underset{k\longrightarrow\infty}{\rightharpoonup} \partial_t w \text{ in } L^2(I,V) . \tag{2.19}$$

Therefore, with the aid of Kolmogorov's compactness theorem

$$\overline{u}^{n_k} \underset{k\longrightarrow\infty}{\longrightarrow} u \text{ in } L^2(I,V).$$

$$\partial_t I_{nk}\left(\overline{u}^{nk}\right) \underset{k\longrightarrow\infty}{\longrightarrow} \partial_t w \text{ in } L^2(I,V) . \tag{2.20}$$

By assembling all the obtained results, we can demonstrate our existence theorem.

Theorem 2.5.1. *From the definition* 2.2.1, *there exists* $u \in L^2(I,V)$ *and* $\gamma \in L^2(0,T)$ *such that* $\{u,\gamma\}$ *solves* (P).

proof. The demonstration of this theorem is in two steps

Step1: In the context of (2.18)-(2.19)and Minty-Browder theorem [9] , we have

$$w = I^{1-\alpha}(u) \text{ and } \partial_t w = \partial_t I^{1-\alpha}(u). \tag{2.21}$$

Observe that these results can also be derived as follows

First, using the Lebesgue dominated theorem, we conclude that $w = I^{1-\alpha}(u)$.

Afterwards, from the equality

$$\left(I_{n_k}\left(\overline{u}^{n_k}(t)\right) - U_1, \phi\right) = \int_0^t \left(\partial_t I_{n_k}\left(\overline{u}^{n_k}(s)\right), \phi\right) ds, \tag{2.22}$$

as $k \longrightarrow \infty$, we get

$$\left(I^{1-\alpha}(u) - U_1, \phi\right) = \int_0^t \left(\partial_t w(s), \phi\right) ds . \tag{2.23}$$

Hence

$$\left(I^{1-\alpha}(u) - U_1 - \int_0^t \partial_t w(s)\,ds, \phi\right) = 0. \tag{2.24}$$

Therefore, we can write

$$\partial_t I^{1-\alpha}(u) = \partial_t w.$$

Now, through the relationships (2.12), we can reorganize the variational equation (2.4) as follows

$$\int_I (\partial_t I_n(t), \phi) + a(l(\overline{u}^{n_k})) \int_I (\nabla \overline{u}^{n_k}, \nabla \phi) = \int_I (f^n, \phi) - a(l(\overline{u}^{n_k})) \int_I (\overline{g}_n, \phi)_{\Gamma_N}$$

$$+ \int_I \phi_{\backslash \Gamma_D} \left[\overline{\theta'_n} + a(l(\overline{u}^{n_k}))\left(\overline{g_n}, 1\right)_{\Gamma_N} - (f_n, 1)\right]. \tag{2.25}$$

From the lipschicity of f , we can deduce

$$\|f^n(t) - f\|_{L^2(I, L^2(\Omega))} \leq \frac{C}{n},$$

and therefore

$$f^n(t) \underset{n \longrightarrow \infty}{\longrightarrow} f \text{ in } L^2\left(I, L^2(\Omega)\right). \tag{2.26}$$

When n goes to infinity in (2.25) , we conclude that u satifies (iii) of the definition 2.2.1.

Step2: (Existence of γ**)** Our objective in this step is to prove the existence of γ that satisfied (ii). For this purpose, we will present the following function

$$\gamma_n = u_{n \backslash \Gamma_D} \text{ and } \gamma = u_{\backslash \Gamma_D}.$$

Across relationships (2.11), (2.12), it follows that

$$\left\| u^n - \overline{u}^n \right\|_{L^2(I,V)} = (\tau - (t - t_{i-1})) \left\| \delta u_i \right\|_{L^2(I,V)} \leq \frac{C}{n}.$$

This implies that (u^n) and $\left(\overline{u}^n\right)$ have the same limit u. Taking into account the inclusion of $H^1(\Omega) \hookrightarrow L^2(\Gamma)$, this allows to get

$$\begin{aligned} \int_0^T \left\| \gamma_n - \gamma \right\|_{\Gamma_D}^2 &= \left\| u^n - u \right\|_{L^2((0,T),L^2(\Gamma_D))}^2 \leq \left\| u^n - u \right\|_{L^2((0,T),L^2(\Gamma))}^2 \\ &\leq C \left\| u^n - u \right\|_{L^2((0,T),V)}^2 \underset{n \longrightarrow \infty}{\longrightarrow} 0. \end{aligned}$$

Thus

$$\gamma_n \longrightarrow \gamma = u_{\backslash \Gamma_D} in \;\; L^2\left(I, L^2(\Gamma_D)\right).$$

2.6 Full discretization

At each time t_i, $i = 1...n$, we consider a triangulation Υ_h^i composed of triangles T^i, such that no vertices of any triangle within the inner side of another triangle. Let V_h^i is the discrete space of V which defined by

$$V_h^i = \{\phi_h \in H^1(\Omega) \;\; \text{such that} \;\; \phi_{h\backslash \Gamma_D} = C \;\; \text{and} \; \phi_h|_{T^i} \; \text{is polynomial of degree one} \; \forall T^i \in \Upsilon_h^i\}.$$

Let $\{p_j\}_1^N$ the inner vertices of Υ_h^i and$\{\phi_j(x)\}_{j=1}^N$ are the basic function in V_h^i, so that any function is the pyramidal function takes the value 1 at each inner vertex but it disappears at the other vertices. The solution u_h^i can be written as follow

$$u_h^i(t) = \sum_{j=1}^{N} \alpha_j^i \phi_j(x) \in V_h^i.$$

Therfore, the full discrete formulation consists to search a solution $u_h^i \in V_h$, such that

$u_h(0) = u_h^0$, and $\forall v_h \in V_h^i$

$$\left(\partial_t I^{1-\alpha} u_h^i, v_h\right) + \tau a(l(u_h^i)) \left(\nabla u_h^i, \nabla v_h\right) + \tau a(l(u_h^i)) \left(g^i, v_h\right)_{\Gamma_N} = \tau \left(f^i, v_h\right)$$

$$+ \tau v_{h\setminus\Gamma_D} \left[\theta_i' + a(l(u_h^i)) \left(g^i, 1\right)_{\Gamma_N} - \left(f^i, 1\right)\right] \tag{2.27}$$

$$+ v_{h\setminus\Gamma_D} \left[\theta_i' + a(l(u_h^i)) \left(g^i, 1\right)_{\Gamma_N} - \left(f^i, 1\right)\right].$$

We approximate $\partial_t I^{1-\alpha}(u_h^i)$ by $\frac{u_h^i - \alpha u_h^{i-1}}{\tau^\alpha}$ (See[2]).

So

$$\left(u_h^i, v_h\right) + \tau^\alpha a(l(u_h^i)) \left(\nabla u_h^i, \nabla v_h\right) + \tau^\alpha a(l(u_h^i)) \left(g^i, v_h\right)_{\Gamma_N} = \tau^\alpha f^i, v_h) + \alpha \left(u_h^{i-1}, v_h\right)$$

$$+ \tau^\alpha v_{h\setminus\Gamma_D} \left[\theta_i' + a(l(u_h^i)) \left(g^i, 1\right)_{\Gamma_N} - \left(f^i, 1\right)\right]. \tag{2.28}$$

The finite element method (2.28) provides a system of nonlinear algebraic equations

$$F_j(\bar{\alpha}^i) = F_j(u_h^i) = 0 \quad 1 \leq j \leq N, \tag{2.29}$$

where $\bar{\alpha}^i = [\alpha_1^i, \alpha_2^i, ..., \alpha_N^i]$, and

$$F_j(u_h^i) = \left(u_h^i, v_h\right) + \tau^\alpha a(l(u_h^i))\left(\nabla u_h^i, \nabla v_h\right) + \tau^\alpha a(l(u_h^i))\left(g^i, v_h\right)_{\Gamma_N} - \tau^\alpha f^i, v_h) - \alpha\left(u_h^{i-1}, v_h\right)$$

$$-\tau^\alpha v_{h \setminus \Gamma_D}\left[\theta_i' + a(l(u_h^i))\left(g^i, 1\right)_{\Gamma_N} - \left(f^i, 1\right)\right]. \tag{2.30}$$

In order to solve nonlinear algebraic equations, we have seen that the Newton-Raphson iterative method is appropriate to solve this type of equation because it is a very rapid convergence. But in our present situation, the non-local term poses the problem that it destroys the rarity of the Jacobian matrix of the Newton-Raphson method, in fact the Jacobian is a complete matrix.

Thus, to compute the solution $\bar{\alpha}^i$ with the Newton-Raphson iterative method, we first compute the Jacobian matrix such: any element of the Jacobian takes the form

$$\frac{\partial F_j}{\partial \alpha_j^i} = (\phi_j, \phi_l) + \tau^\alpha a'(l(u_h^i))(\int_\Omega \phi_j)\left(\nabla u_h^i, \nabla \phi_l\right) + \tau^\alpha a(l(u_h^i))\left(\nabla \phi_j, \nabla \phi_l\right)$$

$$+\tau^\alpha a'(l(u_h^i))(\int_\Omega \phi_j)\left(g^i, \phi_l\right)_{\Gamma_N} - \tau^\alpha \phi_{l \setminus \Gamma_D} a'(l(u_h^i))(\int_\Omega \phi_j)\left(g^i, 1\right)_{\Gamma_N}. \tag{2.31}$$

Hence, the resolution process using the Newton-Raphson method needs equipment with a large memory and also includes a lot of operations. To overcome this, we have modified the scheme (2.28) according to the technic used by Gudi in [35]. After that, problem (2.28) can be rewritten like this:

Find $d \in \mathbb{R}$, and $u_h^i \in V_h^i$ such that

$$l(u_h^i) - d = 0, \tag{2.32}$$

$$\left(u_h^i, v_h\right) + \tau^\alpha a(d)\left(\nabla u_h^i, \nabla v_h\right) + \tau^\alpha a(d)\left(g^i, v_h\right)_\Gamma - \tau^\alpha (f^i, v_h) - \alpha\left(u_h^{i-1}, v_h\right)$$

$$-\tau^\alpha v_{h\setminus\Gamma_D}\left[\theta_i' + a(d)\left(g^i, 1\right)_{\Gamma_N} - \left(f^i, 1\right)\right] = 0. \tag{2.33}$$

Take $v_h = \phi_j$, and we reformulate the equations (2.32)-(2.33) as follows :

$$F_j(u_h^i, d) = \left(u_h^i, \phi_j\right) + \tau^\alpha a(d)\left(\nabla u_h^i, \nabla \phi_j\right) + \tau^\alpha a(d)\left(g^i, \phi_j\right)_{\Gamma_N} - \tau^\alpha (f^i, \phi_j)$$

$$-\left(u_h^{i-1}, \phi_j\right) - \tau^\alpha \phi_{j\setminus\Gamma_D}\left[\theta_i' + a(d)\left(g^i, 1\right)_{\Gamma_N} - \left(f^i, 1\right)\right], \tag{2.34}$$

$$F_{N+1}^i = l(u_h^i) - d. \tag{2.35}$$

This implies

$$J\begin{bmatrix}\bar{\alpha}\\ \beta\end{bmatrix}=\begin{bmatrix}A & b\\ c & \delta_{11}\end{bmatrix}\begin{bmatrix}\bar{\alpha}\\ \beta\end{bmatrix}=\begin{bmatrix}\bar{F}\\ F_{N+1},\end{bmatrix}, \tag{2.36}$$

where $A = A_{N\times N}$, $b = b_{N\times 1}$ and $c = c_{1\times N}$ take the form

$$\begin{aligned}
A_{jl} &= (\phi_l,\phi_j)+\tau^{\alpha}a(d)(\nabla\phi_l,\nabla\phi_j),\ 1\le l,\ j\le N\\
b_{j1} &= \tau^{\alpha}a'(d)(\nabla u_h^i,\nabla\phi_j)+\tau^{\alpha}a'(d)\left(g^i,\phi_j\right)_{\Gamma_N}-\tau^{\alpha}a'(d)\left(g^i,1\right)_{\Gamma_N},\ 1\le j\le N\\
c_{1l} &= (\int_{\Omega}\phi_l),\ 1\le l\le N\\
\delta_{11} &= -1,
\end{aligned}$$

and $\bar{\alpha}^i = [\alpha_1^i,\alpha_2^i,...,\alpha_N^i]^T$, $\bar{F}^i = [F_1^i,F_2^i,...,F_N^i]^T$.

The matrix system (2.36) can be solved by using the Sherman-Morrison Woodbury formula and block elimination with one-refinement algorithm see more in [24], [35], [36].

2.7 Numerical results

Now to illustrate the results of a simple numerical experimen, we use Roth's approximation in time discretization and the finite element method for spatial discretization . We consider Ω as an interval, so we take for example $\Omega = (0,1)$,

$I=(0,1)$, and the test solution $u(t,x)=xt(1-\frac{1}{2}x)(1-\frac{1}{2}t)$.

We take $\alpha=1/2$, and $a(l(u))=1+cos(d)$ with

$$f(t,x) = \frac{1}{\sqrt{\Pi}}(x-x^2)(\frac{-4}{3}t^{\frac{3}{2}}+2t^{\frac{1}{2}})-a(d)(-t+\frac{1}{2}t^2),$$

and

$$\theta^{'}(t) = \frac{1}{\sqrt{\Pi}}(\frac{4}{9}t^{\frac{3}{2}}+\frac{2}{3}t^{\frac{1}{2}}).$$

For Newton's iteration, we consider initial guess u^0 as follows

$$u^0=\begin{cases} 1, & \text{at interior node} \\ 0, & \text{at boundary node} \end{cases}$$

The numerical error is calculated first on u_γ in $L^2(0,T)$ and therefore we take N large enough in the following $N=10^3$ and we assign the time step $\tau=\frac{1}{10k}$, $k=1,2...6$. The table 1 gives the numerical errors and Figure 2.1 shows the results of error in loglog-plot

Table 1.

τ	$\Vert u(t,1)-u^{(N)}\Vert_{L^2([0,T])}$
$\frac{1}{10}$	$4.2866e-003$
$\frac{1}{20}$	$2.2108e-003$
$\frac{1}{30}$	$1.4928e-003$
$\frac{1}{40}$	$1.0800e-003$
$\frac{1}{50}$	$8.9679e-004$
$\frac{1}{60}$	$7.4682e-004$

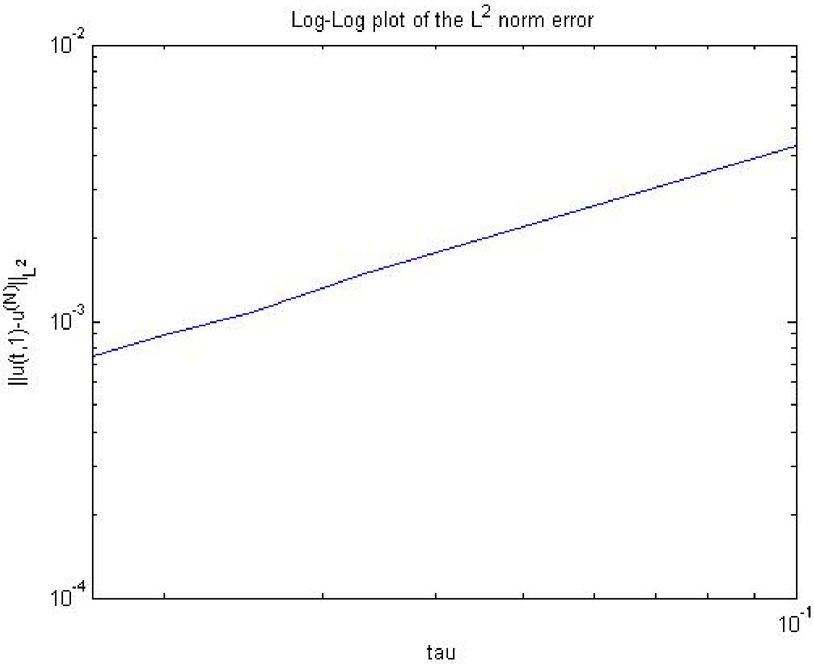

Figure 2.1:

Second numerically error are computed at $T = 1$ with $\tau = 10^{-4}$ and varying $N = 10k,\ k = 1,2....6$, where $h = \frac{1}{N}$, The Table 2.2 below gives the numerical errors and Figure 2.2 shows the results of error in loglog-plot

Table 2.

h	$\|u(x,T) - u^{(j)}\|_{H^1}$
$\frac{1}{10}$	$1.1423e-002$
$\frac{1}{20}$	$2.7195e-003$
$\frac{1}{30}$	$1.1787e-003$
$\frac{1}{40}$	$6.8134e-004$
$\frac{1}{50}$	$4.6090e-004$
$\frac{1}{60}$	$3.4135e-004$

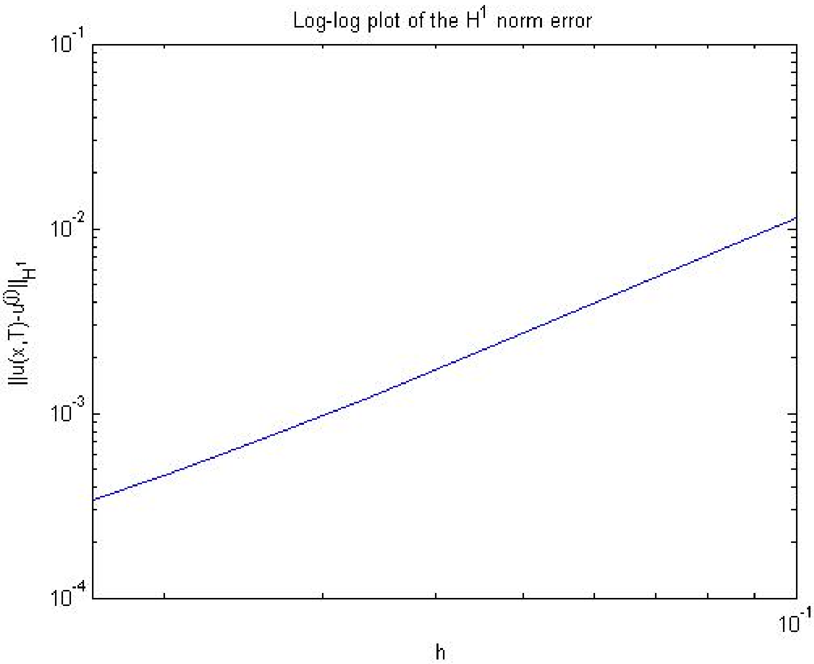

Figure 2.2:

3

Solution to fractional integro-differential equation with unknown flux on the Dirichlet boundary

3.1 Introduction

In this chapter, we are interested in the application of the Rothe discretization scheme in time, to find an approximate solution of a parabolic partial differential diffusion equation with unknown Dirichlet conditions. We will prove the existence of the weak solution, then we show its uniqueness and some numerical results demonstrate the potential of this method.

3.2 Position of the problem

Consider the parabolic integro-differential equation of the following form

$$D_{RL}^{\alpha} u(t,x) - \triangle u(t,x) = \int_0^t k(s,u(s,x))\,ds + f(t,x) \text{in } I \times \Omega \tag{3.1}$$

$$u(0,x) = u_0(x) \ \text{ in } \ \Omega \tag{3.2}$$

$$\nabla u.\nu = g, \text{ on } I \times \Gamma_N \tag{3.3}$$

$$u = \gamma(t) \text{ on } I \times \Gamma_D, \tag{3.4}$$

from an additional measurement of type

$$\int_{\Omega} I^{1-\alpha}(u(t,x))\,dx = \theta(t), \tag{3.5}$$

where $\alpha \in]0,1[$, the function k introduced in (Hyp_3). $I^{1-\alpha}$ is the fractional integral and the derivative D_{RL}^{α} are in Riemann-Liouville sense.

3.3 Variational formulation and discretization scheme

Due to the Dirichlet condition (3.4), it will be necessary to look for a solution in the following Hilbert space

$$V = \left\{ u \in H^1(\Omega) \text{ such that } u_{\backslash \Gamma_D} \text{ is constant} \right\}.$$

with induced norm from $H^1(\Omega)$. Along this work we shall always assume the following assumptions

1. Hyp_1: $u_0 \in V$ and $g : I \times \Gamma_N$ is continuous function

2. Hyp_2: $f(t) \in L^2(\Omega)$ and $\| f(t) - f(t') \| \leq l \left| t - t' \right|$

3. Hyp_3: k is globally Lipschitz continuous function.

We will avoid also the degenerate case assuming that $\left| D^{\alpha}_{RL} u \right| > p > 0$.

Definition 3.3.1. *The couple (u, γ) is said weak solution of problem (3.1)-(3.5) if*

1. $u \in L^2(I, V)$ *with* $I^{1-\alpha}(u) \in C(I, V^*)$ *and* $\partial_t I^{1-\alpha}(u) \in L^2(I, V^*)$.

2. $u_{\backslash \Gamma_D} = \gamma(t)$.

3. *For any $\phi \in V$, we have*

$$\int_I \left(\partial_t I^{1-\alpha}(u), \phi\right) dt + \int_I \left(\nabla u, \nabla \phi\right) dt + \int_I (g,\phi)_{\Gamma_N} dt = \int_I (f,\phi)\, dt + \int_I \left(\int_0^t k(s,u(s))\, ds, \phi\right) dt + \int_I \phi_{\backslash \Gamma_D} \left[\theta' + (g,1)_{\Gamma_N} - (f,1) - \left(\int_0^t k(s,u(s))\, ds, 1\right)\right] dt. \tag{3.6}$$

The weak formulation of the problem (3.1)-(3.5) is defined by

$$\left(\partial_t I^{1-\alpha}(u), \phi\right) + \left(\nabla u, \nabla \phi\right) + (g,\phi)_{\Gamma_N} - \phi_{\backslash \Gamma_D} \left(\nabla u.\eta, 1\right)_{\Gamma_D} = (f,\phi) + \left(\int_0^t k(s,u(s))\, ds, \phi\right), \forall \phi \in V. \tag{3.7}$$

Hence, testing the identity (3.7) with $\phi = 1$ and taking (3.5) into account , we obtain

$$\left(\nabla u.\eta, 1\right)_{\Gamma_D} = \theta' + (g,1)_{\Gamma_N} - (f,1) - \left(\int_0^t k(s,u(s))\, ds, 1\right), \tag{3.8}$$

replacing in (3.7), we get

$$\left(\partial_t I^{1-\alpha}(u), \phi\right) + \left(\nabla u, \nabla \phi\right) + (g,\phi)_{\Gamma_N} = (f,\phi) + \left(\int_0^t k(s,u(s))\, ds, \phi\right) + \phi_{\backslash \Gamma_D} \left[\theta' + (g,1)_{\Gamma_N} - (f,1) - \left(\int_0^t k(s,u(s))\, ds, 1\right)\right]. \tag{3.9}$$

Let us dividing the interval I into n sub intervals of length $h = \frac{T}{n}$ and denote $u_i = u(t_i, x)$, $t_i = ih, \delta u_i = \frac{u_i - u_{i-1}}{h}, i = 1, ..., n$. After discretization, the recurrent approximation scheme for $i = 1, ..., n$ becomes

$$\left(I^{1-\alpha}(u_i) - I^{1-\alpha}(u_{i-1}), \phi\right) + h\left(\nabla u_i, \nabla \phi\right) + h\left(g_i, \phi\right)_{\Gamma_N} = h\left(f_i, \phi\right) + h\sum_{j=1}^{i-1}\left(k\left(t_j, u_j\right)h, \phi\right) h\phi_{\backslash \Gamma_D}\left[\theta'_i + \left(g_i, 1\right)_{\Gamma_N} - \left(f_i, 1\right) - \sum_{j=1}^{i-1}\left(k\left(t_j, u_j\right)h, 1\right)\right]. \tag{3.10}$$

In the following Lemma, we ensure the existence of a weak solution at each time step

Lemma 3.3.2. *Suppose that* $g(t) \in L^2(\Gamma_N)$ *and* $\theta'(t) \in \mathbb{R}, \forall t \in [0,T]$. *Then, at each time step* t_i, *there exists weak solution* u_i *of the problem (3.10).*

Proof. Let's define the operator $T : V \longrightarrow V'$ by

$$\left\langle T(u), \phi \right\rangle = \left(I^{1-\alpha}(u), \phi\right) + h\left(\nabla u, \nabla \phi\right).$$

According to the theory of monotone operators, it suffices to show that $T(u)$ is bounded, semi-continuous, coercive and monotone operator.
First, we show that T is bounded. To that aim, we have

$$\begin{aligned}\left\langle T(u_i), \phi \right\rangle &= \left(I^{1-\alpha}(u_{i-1}), \phi\right) - h\left(g_i, \phi\right)_{\Gamma_N} + h\left(f_i, \phi\right) + h\sum_{j=1}^{i-1}\left(k\left(t_j, u_j\right)h, \phi\right) \\ &\quad + h\phi_{\setminus\Gamma_D}\left[\theta'_i + \left(g_i, 1\right)_{\Gamma_N} - \left(f_i, 1\right) - \sum_{j=1}^{i-1}\left(k\left(t_j, u_j\right)h, 1\right)\right].\end{aligned}$$

Cauchy-Schwarz inequality implies

$$\begin{aligned}\left|\left\langle T(u_i), \phi \right\rangle\right| &= \left\|I^{1-\alpha}(u_{i-1})\right\|\left\|\phi\right\| + h\left\|g\right\|_{\Gamma_N}\left\|\phi\right\| + h\left\|f_i\right\|\left\|\phi\right\| + h^2\sum_{j=1}^{i-1}\left\|k\left(t_j, u_j\right)\right\|\left\|\phi\right\| \\ &\quad + h\left|\phi_{\setminus\Gamma_D}\right|\left[\left|\theta'_i\right| + \left\|g_i\right\|_{\Gamma_N}\|1\| + \left\|f_i\right\|\|1\| + h\sum_{j=1}^{i-1}\left(\left\|k\left(t_j, u_j\right)\right\|\|1\|\right)\right].\end{aligned}$$

Applying standard arguments, we get

$$\left\langle T(u_i), \phi \right\rangle \le C(i)\left\|\phi\right\|_{H^1},$$

which proves that $T(u_i)$ is bounded.

Next, from the mean value Theorem and the the hypothesis (Hyp$_3$), we have

$$\begin{aligned}\langle T(u)-T(v), u-v\rangle &= \left(I^{1-\alpha}(u)-I^{1-\alpha}(v), u-v\right)+h\|\nabla(u-v)\|^2 \\ &\geq \left(I^{1-\alpha}\right)'(\xi)\|u-v\|^2+h\|\nabla(u-v)\|^2 \\ &\geq p\|u-v\|^2+h\|\nabla(u-v)\|^2 \geq m(i)\|u-v\|^2_{H^1(\Omega)},\end{aligned}$$

and so T is monotone. We can similarly check that

$$\langle T(u), u\rangle \geq C(h)\|u\|^2_{H^1(\Omega)},$$

this shows the coercivity of T.

Finally, for $v_n \longrightarrow v$ in V, we have

$$\begin{aligned}\left|\left\langle T(v_n)-T(v), \phi\right\rangle\right| &= \left|\left(I^{1-\alpha}(v_n)-I^{1-\alpha}(v), \phi\right)+h\left(\nabla(v_n-v), \phi\right)\right| \\ &\leq C\left\|I^{1-\alpha}(v_n)-I^{1-\alpha}(v)\right\|\left\|\phi\right\|+\|v_n-v\|_{H^1(\Omega)}\left\|\phi\right\|.\end{aligned}$$

Since $v_n \longrightarrow v$ in $H^1(\Omega)$, it follows that

$$\left|\left\langle T(v_n)-T(v), \phi\right\rangle\right| \longrightarrow 0, \forall \phi \in V.$$

Consequently, T is semi-continuous. This completes the proof.

3.4 Necessary a priori estimates

To show the existence result, we derive some a priori estimates

Lemma 3.4.1. *The estimates*

$$\sum_{i=1}^{l} h\|\delta u_i\|^2 \leq C, \|\nabla u_i\|^2 \leq C, \sum_{i=1}^{l} \|\nabla u_i - \nabla u_{i-1}\|^2 \leq C, \|u_i\| \leq C, \tag{3.11}$$

take place uniformly in n, i, j and h.

proof. Choosing $\phi = u_i - u_{i-1}$ as a test function in (3.10) and summing over $i = 1, ..., l$, yields

$$\begin{aligned}\sum_{i=1}^{l} h\left(\delta I^{1-\alpha}(u_i), \delta u_i\right) + \sum_{i=1}^{l} (\nabla u, \nabla u_i - \nabla u_{i-1}) = \sum_{i=1}^{l} h\left(f_i, \delta u_i\right) - \sum_{i=1}^{l} h\left(g_i, \delta u_i\right)_{\Gamma_N} \\ + h\sum_{i=1}^{l}\sum_{j=1}^{i-1}\left(k\left(t_j, u_j\right)h, \delta u_i\right) + h\sum_{i=1}^{l}\phi_{\backslash\Gamma_D}\left[\theta_i' + \left(g_i, 1\right)_{\Gamma_N} - \left(f_i, 1\right) - \sum_{j=1}^{i-1} h\left(k\left(t_j, u_j\right)h, 1\right)\right].\end{aligned} \tag{3.12}$$

Thanks to Abel's summing formula, it follows that

$$\begin{aligned}& C\sum_{i=1}^{l} h\|\delta u_i\|^2 + \frac{1}{2}\left[\|\nabla u_l\|^2 - \|\nabla u_0\|^2 + \sum_{i=1}^{l}\|\nabla u_i - \nabla u_{i-1}\|^2\right] \\ \leq & \sum_{i=1}^{l} h\left(\delta I^{1-\alpha}(u_i), \delta u_i\right) + \sum_{i=1}^{l}(\nabla u_i, \nabla u_i - \nabla u_{i-1}).\end{aligned} \tag{3.13}$$

By Cauchy-Schwarz and Young inequalities we may write

$$\sum_{i=1}^{l} h\left(f_i, \delta u_i\right) \leq C\left(\frac{1}{\varepsilon} + \varepsilon\sum_{i=1}^{l} h\|\nabla\delta u_i\|^2\right). \tag{3.14}$$

Now, due to the assumption on k, we get

$$h^2\sum_{i=1}^{l}\sum_{j=1}^{i-1}\left(k\left(t_j, u_j\right)h, \delta u_i\right) \leq C\left(\frac{1}{\varepsilon} + \varepsilon\sum_{i=1}^{l} h\|\nabla\delta u_i\|^2\right). \tag{3.15}$$

It follows from the trace inequality and Abel's summing formula that

$$\left|\sum_{i=1}^{l} h\left(g_i, \delta u_i\right)_{\Gamma_N}\right| \leq \varepsilon \left\|\nabla u_l\right\|^2 + C_\varepsilon. \tag{3.16}$$

Similarly, we can estimate the last term in (3.12) as follows

$$\begin{aligned}
& h\sum_{i=1}^{l} \delta u_{i\setminus\Gamma_D}\left[\theta_i' + \left(g_i, 1\right)_{\Gamma_N} - \left(f_i, 1\right) - \sum_{j=1}^{i-1} h\left(k\left(t_j, u_j\right)h, 1\right)\right] \\
\leq\ & C\left\|u_l\right\|_{\Gamma_D}\left(\left|\theta_l'\right| + \left\|f_l\right\| + \left\|g_l\right\|_{\Gamma_N} + 1\right) + C \\
& + Ch\sum_{i=1}^{l}\left\|u_{i-1}\right\|_{\Gamma_D}\left(\left|\delta\theta_i'\right| + \left\|\delta f_l\right\| + \left\|\delta g_i\right\|_{\Gamma_N} + 1\right) \\
\leq\ & \varepsilon\left\|\nabla u_l\right\|^2 + C_\varepsilon.
\end{aligned} \tag{3.17}$$

Summarizing all considerations above, fixing ε sufficiently small, then the discrete Gronwall Lemma conclude the proof. To show the last estimate in (3.11), we test by $\phi = u_i$ in (3.10) and in the same way as we did before, we get the result.

Let us define Rothe functions by the piecewise linear interpolation with respect to the time t

$$u^n(t) = u_{i-1} + \left(t - t_{i-1}\right)\delta u_i, \quad t \in \left[t_{i-1}, t_i\right], \quad i = 1, \ldots, n \tag{3.18}$$

$$I_n\left(\overline{u}^n(t)\right) = I^{1-\alpha}\left(u_{i-1}\right) + \left(t - t_{i-1}\right)\delta I^{1-\alpha}\left(u_i\right), \quad t_{i-1} \leq t \leq t_i, \text{ and } \ i \in \{1, \ldots, n\} \tag{3.19}$$

With the step functions

$$\overline{u}^n(t) = \begin{cases} u_i & t \in \left(t_{i-1}, t_i\right] \\ u_0 & t = 0 \end{cases}, \ i = 1, \ldots, n. \tag{3.20}$$

$$\overline{I_n}\left(\overline{u}^n(t)\right)=\begin{cases} I^{1-\alpha}(u_i), & t_{i-1}<t\leq t_i \\ I^{1-\alpha}u\left(0^+\right)=U_1 & t=0 \end{cases}, \; i\in\{1,...,n\}\,. \tag{3.21}$$

We denote by f^n and K^n the functions

$$f^n(t)=\begin{cases} f_i & t\in[t_{i-1},t_i] \\ f_0 & t=0 \end{cases} \quad i=1,...,n\,. \tag{3.22}$$

$$K^n(t)=\begin{cases} K_i & t\in[t_{i-1},t_i] \\ K_0 & t=0 \end{cases} \quad i=1,...,n\,. \tag{3.23}$$

Similarly, we define g_n, θ_n and θ'_n.

Lemma 3.4.2. *There exists a positive constant C independent to i, j, and h such that*

$$\left\|\delta I^{1-\alpha}(u_i)\right\|_{V^*}\leq C. \tag{3.24}$$

proof. From (3.10) we get

$$\begin{aligned}\left(\delta I^{1-\alpha}(u_i),\phi\right)&=-\left(\nabla u_i,\nabla\phi\right)-\left(g_i,\phi\right)_{\Gamma_N}+\left(f_i,\phi\right)+\sum_{j=1}^{i-1}\left(k\left(t_j,u_j\right)h,\phi\right)\\ \phi_{\backslash\Gamma_D}&\left[\theta'_i+\left(g_i,1\right)_{\Gamma_N}-\left(f_i,1\right)-\sum_{j=1}^{i-1}\left(k\left(t_j,u_j\right)h,1\right)\right].\end{aligned} \tag{3.25}$$

To estimate the term on the RHS of (3.25), we use standard estimates and take Lemma 3.3.1 into account to obtain

$$\left|\left(\delta I^{1-\alpha}(u_i),\phi\right)\right|\leq C\left\|\phi\right\|_{H^1(\Omega)}.$$

Then

$$\left\|\delta I^{1-\alpha}(u_i)\right\|_{V^*}=\sup_{\|\phi\|_V\leq 1}\left|\left(\delta I^{1-\alpha}(u_i),\phi\right)\right|\leq C.$$

This completes the proof.

Remark 3.4.3. *A direct consequence of Lemma 3.4.2 is the next a priori estimate:*

$$\int_I \|\partial_t I_n\|^2_{V^*} \leq C,$$

where C is a positive constant independent to i, j, n, and h.

3.5 Existence results

It follows from Lemma 3.3.1 and 3.3.2 that

$$\max_I \left\|\overline{I_n}\right\| + \|\partial_t I_n\|_{L^2(I,V^*)} \leq C.$$

Then, as in ([38] Lemma.1.), there exist $w \in C(I, V^*) \cap L^\infty\left(I, L^2(\Omega)\right)$ with $\partial_t w \in L^2(I, V^*)$ and subsequence I_{n_k} for which

$$\begin{aligned} I_{n_k} &\longrightarrow w \text{ in } C\left(I, V^*\right), \overline{I_{n_k}}(t) \rightharpoonup w(t) \text{ in } L^2(\Omega), && (3.26)\\ I_{n_k}(t) &\rightharpoonup w(t) \text{ in } L^2(\Omega), \partial_t I_{n_k} \rightharpoonup \partial_t w \text{ in } L^2\left(I, V^*\right). && (3.27)\end{aligned}$$

We deduce, in view of Lemma 3.3.1, that $\{\overline{u}^n\}_n$ and $\{\partial_t I_n(\overline{u}^n)\}$ are uniformly bounded in $L^2(I, V)$, therefore we can extract subsequence $\{\overline{u}^{n_k}\}_{k\in\mathbb{N}}$

$$\begin{aligned} &\overline{u}^{n_k} \underset{k\longrightarrow\infty}{\rightharpoonup} u \text{ in } L^2(I, V)\\ &\{\partial_t I_n(\overline{u}^n)\} \underset{k\longrightarrow\infty}{\rightharpoonup} \partial_t w \text{ in } L^2(I, V), && (3.28)\end{aligned}$$

hence, thanks to Kolomogorov compactness Theorem

$$\begin{aligned}
&\overline{u}^{n_k} \underset{k\longrightarrow\infty}{\longrightarrow} u \quad \text{in } L^2(I,V)\\
&\partial_t I_{nk}\left(\overline{u}^{nk}\right) \underset{k\longrightarrow\infty}{\longrightarrow} \partial_t w \text{ in } L^2(I,V)\ .
\end{aligned} \tag{3.29}$$

Collecting all the obtained results, we can state our existence Theorem

Theorem 3.5.1. *There exists $u \in L^2(I,V)$ and $\gamma \in L^2(0,T)$ such that $\{u,\gamma\}$ solves (3.1)-(3.5) in the sense of definition* 3.2.1.

proof. The proof of this Theorem is divided into tow steps.

Step1: In light of (3.27)-(3.28) and the Minty-Browder Theorem [9], we have

$$w = I^{1-\alpha}(u) \text{ and } \partial_t w = \partial_t I^{1-\alpha}(u). \tag{3.30}$$

Note that this results can also be obtained as follows:

Firstly, by virtue of the Lebesgue dominated Theorem, we conclude that $w = I^{1-\alpha}(u)$.

Secondly, from the equality

$$\left(I_{n_k}\left(\overline{u}^{n_k}(t)\right) - U_1, \phi\right) = \int_0^t \left(\partial_t I_{n_k}\left(\overline{u}^{n_k}(s)\right), \phi\right) ds. \tag{3.31}$$

We get as $k \longrightarrow \infty$

$$\left(I^{1-\alpha}(u) - U_1, \phi\right) = \int_0^t \left(\partial_t w(s), \phi\right) ds. \tag{3.32}$$

Hence

$$\left(I^{1-\alpha}(u) - U_1 - \int_0^t \partial_t w(s)\, ds, \phi\right) = 0. \tag{3.33}$$

Therefore we can write

$$\partial_t I^{1-\alpha}(u) = \partial_t w.$$

Now, using relations (3.18) -(3.23), we can rearrange the variational equation (3.10) as follows

$$\begin{aligned}&\int_I \left(\partial_t I_n(t), \phi\right) + \int_I \left(\nabla \overline{u}^{n_k}, \nabla \phi\right) = \int_I \left(f^n, \phi\right) + \int_I \left(K^n(t), \phi\right) - \int_I \left(\overline{g}_n, \phi\right)_{\Gamma_N} \\ &\int_I \phi_{\backslash \Gamma_D}\left[\overline{\theta'_n} + \left(\overline{g_n}, 1\right)_{\Gamma_N} - \left(f_n, 1\right) - \left(K^n(t), 1\right)\right]\end{aligned} \tag{3.34}$$

From the Lipschicity of f we deduce that

$$\left\| f^n(t) - f \right\|_{L^2(I, L^2(\Omega))} \leq \frac{C}{n},$$

and so

$$f^n(t) \underset{n \longrightarrow \infty}{\longrightarrow} f \text{ in } L^2\left(I, L^2(\Omega)\right). \tag{3.35}$$

Now using the same arguments than in [13], we have

$$K^{n_k} \underset{k \longrightarrow \infty}{\rightharpoonup} k(s, u(s)) \quad \text{in } L^2\left(I, L^2(\Omega)\right). \tag{3.36}$$

Passing to the limit for $n \longrightarrow \infty$ in (3.34), we conclude that u satisfies (3) of the Definition 3.2.1

Step2: (Existence of γ) our goal in this step is to prove the existence of γ satisfying (2). To this purpose, let us introduce the following function

$$\gamma_n = u_{n \backslash \Gamma_D} \text{ and } \gamma = u_{\backslash \Gamma_D}.$$

It follows from Lemma 3.3.1 that

$$\left\| u^{n}-\overline{u}^{n}\right\|_{L^{2}(I,V)}=(h-(t-t_{i-1}))\left\|\delta u_{i}\right\|_{L^{2}(I,V)}\leq\frac{C}{n}.$$

This implies that (u^{n}) and $\left(\overline{u}^{n}\right)$ have the same limit u. Taking the imbedding $H^{1}(\Omega)\hookrightarrow L^{2}(\Gamma)$ into account, we obtain

$$\begin{aligned}\int_{0}^{T}\left\|\gamma_{n}-\gamma\right\|_{\Gamma_{D}}^{2} &= \int_{0}^{T}\left\|u^{n}-u\right\|_{\Gamma_{D}}^{2}\leq\int_{0}^{T}\left\|u^{n}-u\right\|_{\Gamma}^{2}\\ &\leq C\int_{0}^{T}\left\|u^{n}-u\right\|_{V}^{2}\underset{n\longrightarrow+\infty}{\longrightarrow}0.\end{aligned}$$

Thus

$$\gamma_{n}\longrightarrow\gamma=u_{\backslash\Gamma_{D}}\text{ in }L^{2}\left(I,L^{2}(\Gamma_{D})\right).$$

3.6 Uniqueness of weak solution

The uniqueness result we are going to prove, implies that the convergence obtained in Theorem 3.5.1 are still true for the whole sequence (u_{n}).

Theorem 3.6.1. *Under the hypothesis $(Hyp_{1})-(Hyp_{3})$, the problem (3.9) admits a unique weak solution u for $t\in[0,T]$.*

proof. We suppose that the problem (3.9) admits two weak solutions u_{1}, u_{2}. Subtracting (3.9) written for u_{2} from the same identity written for u_{1} and substituting $\phi = u_1 - u_2 = u$, then by integration from 0 to $\tau, \tau \in [0, T]$ we get

$$\begin{aligned}
& \left(I^{1-\alpha} u_1(\tau) - I^{1-\alpha} u_2(\tau), u_1(\tau) - u_2(\tau)\right) dt + \left(\int_0^\tau \nabla(u_1 - u_2), \nabla(u_1 - u_2)\right) dt \\
= & \ (u_1 - u_2)_{\backslash\Gamma_D} \left(\int_0^\tau \int_0^t (k(s, u_1(s)) ds - k(s, u_2(s))) ds dt, 1\right) \\
& + \left(\int_0^\tau \int_0^t (k(s, u_1(s)) ds - k(s, u_2(s))) ds dt, u_1(\tau) - u_2(\tau)\right). \qquad (3.37)
\end{aligned}$$

It follows from the mean value Theorem that

$$\begin{aligned}
& p \int_0^\eta \|u_1(\tau) - u_2(\tau)\|^2 d\tau + \frac{1}{2} \left\|\nabla \int_0^\eta u_1(\tau) - u_2(\tau) d\tau\right\|^2 \qquad (3.38) \\
\leq & \ \left(I^{1-\alpha} u_1(\tau) - I^{1-\alpha} u_2(\tau), u_1(\tau) - u_2(\tau)\right) + \left(\int_0^\tau \nabla(u_1 - u_2), \nabla(u_1 - u_2)\right) dt.
\end{aligned}$$

For the first term in the RHS of (3.37) we have

$$\begin{aligned}
& \int_0^\eta \left(\int_0^\tau \int_0^t (k(s, u_2(s)) ds - k(s, u_1(s))) ds dt, 1\right) (u_1 - u_2)_{\backslash\Gamma_D} d\tau \\
= & \int_0^\eta \left(\int_0^t (k(s, u_2(s)) ds - k(s, u_1(s))), 1\right) \int_0^\tau (u_1 - u_2)_{\backslash\Gamma_D} ds d\tau \\
& - \int_0^\eta \left[\left(\int_0^\tau (k(s, u_2(s)) ds - k(s, u_1(s))), 1\right) \int_0^\tau (u_1 - u_2)_{\backslash\Gamma_D} ds\right] d\tau \\
\leq & \left|\left(\int_0^\eta \int_0^t (k(s, u_2(s)) ds - k(s, u_1(s))) ds dt, 1\right) \int_0^\eta (u_1 - u_2)_{\backslash\Gamma_D} d\tau\right| \qquad (3.39) \\
& + \left|\int_0^\eta \left[\left(\int_0^\tau (k(s, u_2(s)) ds - k(s, u_1(s))), 1\right) \int_0^\tau (u_1 - u_2)_{\backslash\Gamma_D} ds\right] d\tau\right|.
\end{aligned}$$

Next, using Cauchy-Schwarz inequality, then the first term in the right hand side of (3.39) can be estimated as follows

$$\begin{aligned}
& \left|\left(\int_0^\eta \int_0^t (k(s, u_1(s)) ds - k(s, u_2(s))) ds dt, 1\right) \int_0^\eta (u_1 - u_2)_{\backslash\Gamma_D} d\tau\right| \\
\leq & \left\|\int_0^\eta \int_0^t (k(s, u_2(s)) ds - k(s, u_1(s))) ds dt\right\| \left\|\int_0^\eta (u_1(\tau) - u_2(\tau)) d\tau\right\|_{\Gamma_D} \qquad (3.40)
\end{aligned}$$

Now, trace inequality together with Lipschitz continuity of k and Young inequality imply

$$
\begin{aligned}
& \left\| \int_0^\eta \int_0^t (k(s,u_2(s))\,ds - k(s,u_1(s)))\,dsdt \right\| \left\| \int_0^\eta (u_1(\tau)-u_2(\tau))\,d\tau \right\|_{\Gamma_D} \\
\leq\ & C \int_0^\eta \int_0^t \|u_2(s)-u_1(s)\|\,dsdt \left(\left\| \nabla \int_0^\eta (u_1(\tau)-u_2(\tau))\,d\tau \right\| + \left\| \int_0^\eta (u_1(\tau)-u_2(\tau))\,d\tau \right\| \right) \qquad (3.41) \\
\leq\ & C_\varepsilon \int_0^\eta \int_0^t \|u_2(s)-u_1(s)\|^2\,dsdt + \varepsilon \left(\left\| \nabla \int_0^\eta (u_1(\tau)-u_2(\tau))\,d\tau \right\|^2 + \int_0^\eta \|(u_1(\tau)-u_2(\tau))\|^2\,d\tau \right).
\end{aligned}
$$

Using similar techniques, we get for the remaining terms in (3.37) and (3.39)

$$
\begin{aligned}
& \left| \int_0^\eta \left[\left(\int_0^\tau (k(s,u_2(s))\,ds - k(s,u_1(s))), 1 \right) \int_0^\tau (u_1-u_2)_{\backslash\Gamma_D}\,ds \right] d\tau \right| \\
\leq\ & C \int_0^\eta \int_0^t \|u_2(s)-u_1(s)\|^2\,ds + C\int_0^\eta \left\| \nabla \int_0^\tau (u_1(s)-u_2(s))\,ds \right\|^2 d\tau. \qquad (3.42)
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_0^\eta \left(\int_0^\tau \int_0^t (k(s,u_1(s))\,ds - k(s,u_2(s)))\,dsdt, u_1(\tau)-u_2(\tau) \right) d\tau \\
\leq\ & \varepsilon \int_0^\eta \|u_1(\tau)-u_2(\tau)\|^2\,d\tau + C_\varepsilon \int_0^\eta \int_0^t \|u_1(s)-u_2(s)\|^2\,dsdt. \qquad (3.43)
\end{aligned}
$$

Collecting (3.38) -(3.43), choosing ε sufficiently small, we obtain

$$
\begin{aligned}
& \int_0^\eta \|u_1(\tau)-u_2(\tau)\|^2\,d\tau + \left\| \nabla \int_0^\eta (u_1(\tau)-u_2(\tau))\,d\tau \right\|^2 \\
\leq\ & C\int_0^\eta \left(\int_0^t \|u_1(s)-u_2(s)\|^2\,ds + \left\| \nabla \int_0^t (u_1(s)-u_2(s))\,ds \right\|^2 \right) dt. \qquad (3.44)
\end{aligned}
$$

The Theorem 3.6.1 is proved.

3.7 Numerical results

In order to illustrate our theoretical prediction, we shall present the results of a numerical experiment with (3.1)-(3.5) on simple case. Let $\Omega = (0,1)$, the time interval $(0,1)$ i.e. $T = 1$, the function $K(t,s) = \exp(t-s)$, and the test solution $u(t,x) = (x-\frac{1}{2}x^2)(t-\frac{1}{2}t^2)$. For Riemann-Liouville derivative we take $\alpha = \frac{1}{2}$, Next, we calculate the other data to get

$$f(t,x) = \frac{1}{\sqrt{\Pi}}(x-\frac{1}{2}x^2)(\frac{-4}{3}t^{\frac{3}{2}}+2t^{\frac{1}{2}})-(-t+\frac{1}{2}t^2)+\frac{1}{4}(-2+x)xt^2 \text{ and } \theta^{'}(t) = \frac{1}{\sqrt{\Pi}}(\frac{4}{9}t^{\frac{3}{2}}+\frac{2}{3}t^{\frac{1}{2}}).$$

To solve our problem numerically, we combine Roth's time discretization with finite element method for spatial variable, using first order Lagrange polynomials. We approximate $\partial_t I^{1-\alpha}(u_h^i)$ by $\frac{u_h^i-\alpha u_h^{i-1}}{\tau^\alpha}$.

The numerical error is calculated first on u_γ in $L^2(0,T)$ and therefore we take N large enough in the following $N = 2^5$ and we assign the time step $\tau = \frac{1}{10k}$, $k = 1,2...4$. The table 1. gives the numerical errors and Fig-1. shows the results of error in loglog-plot.

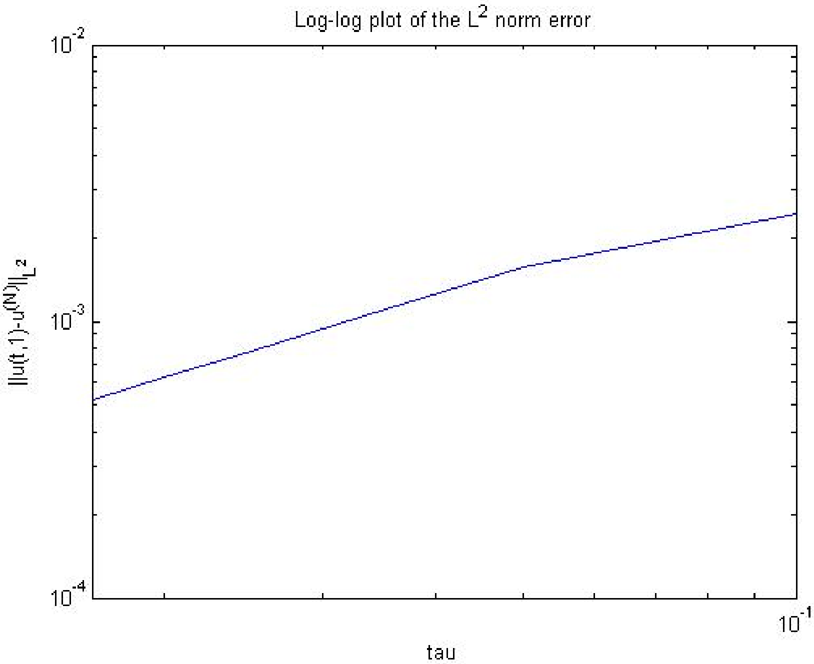

Figure 3.1:

Table 1.

τ	$\Vert u(t,1)-u^{(N)}\Vert_{L^2([0,T])}$
$\frac{1}{10}$	$2.4630e-003$
$\frac{1}{20}$	$1.5709e-003$
$\frac{1}{30}$	$1.0484e-003$
$\frac{1}{40}$	$7.8232e-004$
$\frac{1}{50}$	$6.2511e-004$
$\frac{1}{60}$	$5.2084e-004$

Second numerically error are calculated at final time level $T = 1$ in $H^1(0,1)$, with $\tau = 2^{-7}$ and varying $N = 10k,\ k = 1,2....4$, where $h = \frac{1}{N}$, The Table 2. below gives the numerical errors and Fig-2 shows the results of error in loglog-plot. All the computations were performed by using MATLAB.

Table 2.

h	$\Vert u(x,T) - u^{(j)} \Vert_{H^1}$
$\frac{1}{10}$	$1.5484e-002$
$\frac{1}{20}$	$5.5341e-003$
$\frac{1}{30}$	$3.1671e-003$
$\frac{1}{40}$	$2.1759e-003$
$\frac{1}{50}$	$1.6440e-003$
$\frac{1}{60}$	$1.3160e-003$

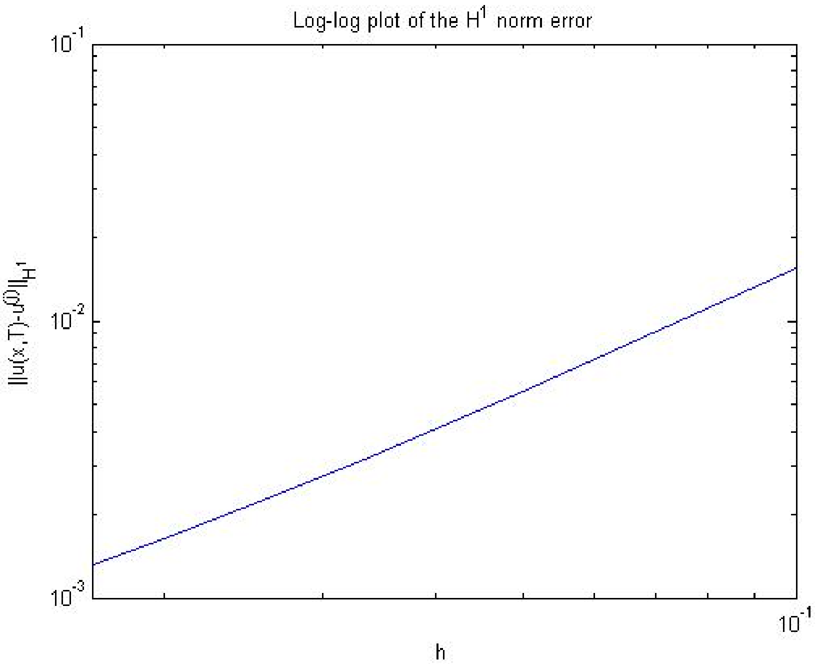

Figure 3.2:

Conclusion and perspectives

In this thesis, we proved the existence and uniqueness of the solution of two nonlinear fractional parabolic problems. In the first work the nonlocal term poses difficulties. In order to remedy these difficulties, we develop an idea given by Gudi and a numerical experiment is given to demonstrate the effectiveness of the proposed approach.

In the second work, we focus on the solution of a fractional diffusion of an integro-differential equation. We proved that the unknown Dirichlet condition can be obtained by using the prescribed measure, were proved the existence and uniqueness of the solution, by applying the Rothe method and the theory of monotone operators. The numerical example given in this work has demonstrated the theoretical results.

To conclude, we cite a research topic, not in the immediate area of the work presented in this thesis, but to which our attention can be directed. Thus, we can study a degenerated partial differential equation taking care to extract the estimates a posteriori.

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Research activities

INTERNATIONAL PUBLICATIONS

- A. Labadla and A. Chaoui : Discretization scheme of fractional parabolic equation with nonlocal coefficient and unknown flux on the Dirichlet boundary, *Dynamics of Continuous, Discrete and Impulsive Systems http:monotone.uwaterloo.ca/j̃ournal.*

- A. Labadla, A. Chaoui, M. Djaghout :Solution to fractional integro-differential equation with unknown flux on the Dirichlet boundary *Discontinuity, Nonlinearity, and Complexity* .

INTERNATIONAL COMMUNICATIONS

- *Approximation solution for some parabolic equation* , International Conference on Advances in Applied Mathematics ICAAM-2018 December 17-20, 2018, Sousse-Tunisia

NATIONAL COMMUNICATIONS

- *Approximate solution for fractional diffusion equation with unnknown boundary condition*, Doctoral days on Informatics and Applied Mathematics, Université 8 Mai 1945 Guelma, 3-4 October 2018.

- *Full discritization for some parabolic equations*, " JNMA'19". Université Larbi Ben Mhidi - Oum El Bouaghi -, 27 Juin 2019.