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THÈSE

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Filière: Mathématique

Présentée par

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Intitulée

Analytical and numerical study for an integro-differential nonlinear Volterra equation

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Abstract

The objective of this thesis is to study the existence and uniqueness of the solution of a nonlinear integro-differential equation of Volterra of second kind when the derivative of the solution appears under the sign of integration in a non-linear way, which has been defined without a singular kernel. Then, using Matlab and C ++ we get the approximate solution and error estimates. During this study, we deemed it necessary to study a problem of Fractional Boundary value in the presence of this derivative, it leads to the study of a linear integral Volterra-Fredholm equation of the second type and this by defining a new fractional integral as inverse of the conformable fractional derivative of Caputo.

Key words:

Nonlinear integro-differential equation of Volterra, Fractional derivative without singular kernel, Convergence, Analytical solution, Existence and uniqueness of solution, Fractional Boundary value problem, Green's function, fractional integral.

Mathematics Subject Classification (2010):

26A33, 34A08, 34K37, 45B05, 34A12, 47G20, 47Gxx.

Résumé

L'objectif de cette thèse est d'étudier l'existence et l'unicité de la solution d'une équation intégro-différentielle non linéaire de Volterra de second type lorsque la dérivée de la solution apparaît sous le signe de l'intégration de manière non linéaire, qui a été définie sans noyau singulier. Ensuite, en utilisant Matlab et C ++, nous obtenons la solution approximative et les estimations d'erreur. Au cours de cette étude, nous avons jugé nécessaire d'étudier un problème fractionnaire aux limites en présence de cette dérivée, ce dernier conduit à l'étude d'une équation Volterra-Fredholm intégrale linéaire du second type et ce en définissant une nouvelle intégrale fractionnaire comme inverse de la dérivée fractionnaire conformable de Caputo.

Mots clés:

Équation intégro-différentielle non linéaire de Volterra, Dérivée fractionnaire sans noyau singulier, Convergence, Solution analytique, Existence et unicité de la solution, Problèmes aux limites fractionnaire, fonction de Green, intégrale fractionnaire.

الملــــــــــخص

الهدف من هذه الرسالة هو دراسة وجود وتفرد حل معادلة فولتيرا التكاملية غير الخطية من النوع الثاني عندما يظهر مشتق الحل تحت اشارة التكامل بطريقة غير خطية، والتي تم تعريفه بدون نواة شاذة. ومن ثم استخدمنا $MATLAB$ و $C++$ للحصول على الحل التقريبي وتقدير الخطأ المرتكب. وخلال هاته الدراسة وجدنا أنه من الضروري دراسة المسائل الحدية في وجود هذا المشتق حيث تؤول إلى دراسة معادلة تكاملية خطية لفولتيرا - فريدهولم من النوع الثاني ومن أجل ذلك قمنا بتعريف التكامل الكسري الجديد والذي يعتبر مقلوب مشتق كابوتو الكسري الجديد.

الكلمات المفتاحية:

معادلة فولتيرا التكاملية غير الخطية، مشتق كابوتو الكسري، التكامل الكسري، دراسة المسائل الحدية، الحل التحليلي، التقارب، وجود وتفرد الحل، تابع قرين.

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- **Prof Fateh Ellaggoune**
- **Dr. Noureddine Benrabia**

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Introduction

This thesis is the outcome of my research during my Ph.D. study at Guelma University. The principal materials in the thesis are based on the following articles from this period :

Paper I: Moumen Bekkouche M., Guebbai H. & Kurulay M. *Analytical and numerical study of a nonlinear Volterra integro-differential equations with conformable fractional derivation of Caputo*. Annals of the University of Craiova - Mathematics and Computer Science Series(2020). https://doi.org/.

Paper II: Moumen Bekkouche M., Guebbai H. & Kurulay M.*On the solvability fractional of a boundary value problem with new fractional integral*. J. Appl. Math. Comput. (2020). https://doi.org/10.1007/s12190-020-01368-x.

Paper III: Moumen Bekkouche M., Guebbai H. *Analytical and Numerical Study for an Fractional Boundary Value Problem with conformable fractional derivative of Caputo and its Fractional Integral*. J. Appl. Math. Comput. Mech., JAMCM(2020). https://doi.org/

Paper IV: Moumen Bekkouche M., Guebbai H. & Kurulay M., S. Benmahmoud*A new fractional integral associated with the Caputo–Fabrizio fractional derivative*. Rendiconti del Circolo Matematico di Palermo Series 2 (2020). https://doi.org/10.1007/s12215-020-00557-8.

The organization of this thesis is as follows : The thesis begins in Chapter 1 that contains a brief history about the appearance of the concept of fractional calculus and a presentaion of some fractional derivative. In the sequel, we describe some basic concepts and give some information about direct and inverse problems. Afterward, we collect the results obtained from three articles above and present them respectively in Chapters 2, 3 and 4.

The next three chapters are structured on fractional calculus as follows: We start in Chapter 2, by studying an integro-differential equation of Volterra when the derivative of the solution appears under the sign of integration in a non-linear way. The derivation is conformable fractional of the Caputo type, we prove two theorems of existence and uniqueness for the solutions of this problem in a Banach space specific to the derivative of Caputo. The analytical study is followed by a complete numerical study.

In Chapter 3, we introduce a new definition of fractional integral as an inverse of the conformable fractional derivative of Caputo. By using these definitions, we obtain the basic properties of those fractional derivative and its fractional integral. Finally, we solve a class of fractional boundary value problems as a theoretical application, and we use Matlab to solve this class of fractional boundary value problems. In Chapter 4, we consider a class of a fractional boundary value problem with conformable fractional derivation of the Caputo type. We obtain existence and uniqueness results for this problems based on the new definition of fractional integral. Therefore, the proofs are based upon the reduction of the problem to a equivalent linear Volterra-Fredholm integral equations of the second kind and this analytical study is followed also by a complete numerical study.

Chapter 1

Introduction and Preliminaries of Fractional Calculus

Summary

1.1 History, Definitions and Applications 11

1.1.1 Grünwald–Letnikov derivative 15

1.1.2 Riemann-Liouville fractional derivative 15

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For the convenience of the reader, in this chapter we present here the necessary definitions and fundamental facts of the fractional calculus theory in this chapter. These definitions can and fundamental facts be found in the recent literature.

1.1 Fractional Calculus: History, Definitions and Applications

This part is aimed at the reader who wishes to learn about Fractional Calculus and its possible applications in his/her field(s) of study. The intent is to first expose the reader to the concepts, applicable definitions, and execution of fractional calculus (including a discussion of notation, operators, and fractional order differential equations), and second to show how these may be used to solve several modern problems.

The traditional integral and derivative are, to say the least, a staple for the technology professional, essential as a means of understanding and working with natural and artificial systems. Fractional Calculus is a field of mathematics study that grows out of the traditional definitions of the calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integer value. Consider the physical meaning of the exponent. According to our primary school teachers exponents provide a short notation for what is essentially a repeated multiplication of a numerical value. This concept in itself is easy to grasp and straight forward. However, this physical definition can clearly become confused when considering exponents of non integer value. While almost anyone can verify that $x^3 = x \cdot x \cdot x$, how might one describe the physical meaning of $x^{3.4}$, or moreover the transcendental exponent $x^{1/4}$. One cannot conceive what it might be like to multiply a number or quantity by itself 3.4 times, or $\frac{1}{4}$ times, and yet these expressions have a definite value for any value x, verifiable by infinite series expansion, or more practically, by calculator. Now, in the same way consider the integral and derivative. Although they are indeed concepts of a higher complexity by nature, it is still fairly easy to physically represent their meaning. Once mastered, the idea of completing numerous of these operations, integrations or differentiations follows naturally. Given the satisfaction of a very few restrictions (e.g. function continuity), completing n integrations can become as methodical as multiplication. But the curious mind can not be restrained from asking the question what if n were not restricted to an integer value? Again, at first glance, the physical meaning can become convoluted (pun intended), but as this report will show, fractional calculus flows quite naturally from our traditional definitions. And just as fractional exponents such as the square root may find their way into innumerable equations and applications, it will become apparent that integrations of order 1/2 and beyond can find practical use in many modern problems.

The concept of fractional calculus is popularly believed to have stemmed from a question raised in the year 1695 by Marquis de L'Hopital (1661-1704) to Gottfried Wilhelm Leibniz (1646-1716), which sought the meaning of Leibniz's (currently popular) notation $\frac{d^n y}{dx^n}$ for the derivative of order $n \in \mathbb{N}_0 := \{0, 1, 2, 3, \cdots\}$ when $n = 1/2$ (What if $n = 1/2$?). In his reply, dated 30 September 1695, Leibniz wrote to L'Hopital as follows: "... This is an apparent paradox from which, one day, useful consequences will be drawn. ..."

Subsequent mention of fractional derivatives was made, in some context or the other, by (for example) Euler in 1730, Lagrange in 1772, Laplace in 1812, Lacroix in 1819, Fourier in

1822, Liouville in 1832, Riemann in 1847, Greer in 1859, Holmgren in 1865, Griinwald in 1867, Letnikov in 1868, Sonin in 1869, Laurent in 1884, Nekrassov in 1888, Krug in 1890, and Weyl in 1917. In fact, in his 700-page textbook, entitled "Traite du Calcul Differentiel et du Calcul Integral" (Second edition; Courcier, Paris, 1819), S. F. Lacroix devoted two pages (pp. 409-410) to fractional calculus, showing eventually that

$$\frac{d^{\frac{1}{2}}}{dv^{\frac{1}{2}}}v = \frac{2\sqrt{v}}{\sqrt{\pi}}$$

Leibniz's response, based on studies over the intervening 300 years, has proven at least half right. It is clear that within the 20th century especially numerous applications and physical manifestations of fractional calculus have been found. However, these applications and the mathematical background surrounding fractional calculus are far from paradoxical. While the physical meaning is difficult (arguably impossible) to grasp, the definitions themselves are no more rigorous than those of their integer order counterparts. Understanding of definitions and use of fractional calculus will be made more clear by quickly discussing some necessary but relatively simple mathematical definitions that will arise in the study of these concepts. These are The Gamma Function, The Beta Function, The Laplace Transform, and the Mittag-Leffler Function.

Definition 1.1 (The Gamma Function). *the gamma function (represented by* Γ *, the capital letter gamma from the Greek alphabet) is one commonly used extension of the factorial function to complex numbers. The gamma function is defined for all complex numbers except the non-positive integers. For any positive integer* n*,*

$$\Gamma(n) = (n-1)!.$$

Derived by Daniel Bernoulli, for complex numbers with a positive real part the gamma function is defined via a convergent improper integral:

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx, \quad \textit{for all } z \in \mathbb{R}$$

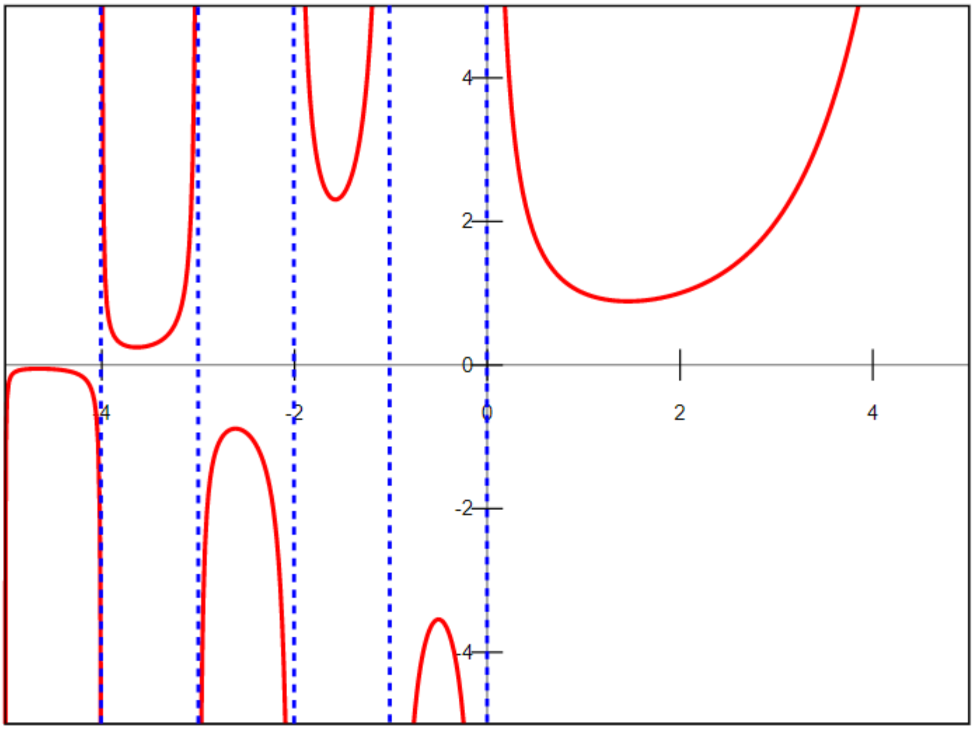

Figure 1.1: The gamma function along part of the real axis

The gamma function then is defined as the analytic continuation of this integral function to a meromorphic function that is holomorphic in the whole complex plane except the non-positive integers, where the function has simple poles. The gamma function has no zeroes, so the reciprocal gamma function $1/\Gamma$ *is an entire function.*

The first mention of a derivative of arbitrary order appears in a text. S. F. Lacroix [1819, pp. 409 - 410] devoted less than two pages of his 700-page text to this topic. He developed a mere mathematical exercise generalizing from a case of integer order. Strating with $y = x^m$, m a positive integer, Lacroix easily develops the $n-$th derivative

$$\frac{d^n y}{dx^n} = \frac{m!}{(m-n)!} x^{m-n}, \quad m \geqq n$$

Using Legendre's symbol for the generalized factorial (the gamma function), he gets

$$\frac{d^n y}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}$$

He then gives the example for $y = x$ and $n = \frac{1}{2}$, and obtains

$$\frac{d^{1/2} y}{dx^{1/2}} = \frac{2\sqrt{x}}{\sqrt{\pi}}$$

It is interesting to note that the result obtained by Lacroix, in the manner typical of the classical formalists of this period, is the same as that yielded by the present-day Riemann-Liouville definition of a fractional derivative. Lacroix's method offered no clue as to a possible application for a derivative of arbitrary order. Now we have several definitions of the fractional derivative, we present:

1.1.1 Grünwald–Letnikov derivative

Grünwald–Letnikov derivative is a basic extension of the natural derivative to fractional one, It was introduced by Anton Karl Grünwald in 1867, and then by Aleksey Vasilievich Letnikov in 1868. Hence, it is written as

$$D^{\alpha} f(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{m=0}^{\infty} (-1)^m \begin{pmatrix} \alpha \\ \\ m \end{pmatrix} f(t - mh)$$

where $n \in \mathbb{N}$, and the binomial coefficient is calculated by the help of the Gamma function.

$$\begin{pmatrix} \alpha \\ \\ m \end{pmatrix} = \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-m+1)}{m!}$$

1.1.2 Riemann-Liouville fractional derivative

Riemann-Liouville fractional derivative acquiring by Riemann in 1847 is defined as follows.

$$\begin{aligned} {}_a^{RL}D_t^{\alpha} f(t) &= \left(\frac{d}{dt}\right)^n \left({}_aD_t^{-(n-\alpha)}\right) f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx, (n = [\alpha]+1, t > a) \end{aligned}$$

where $\alpha > 0$ this operator is an extension of Cauchy's integral from the natural number to real one. Based on the fractionalization algorithm. In addition, according to the above relation, if $0 < \alpha < 1$ then the Riemann-Liouville operator reduced to

$${}_a^{RL}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(x)}{(t-x)^{\alpha}} dx$$

1.1.3 Caputo derivative

Since Riemann-Liouville fractional derivatives failed in the description and modeling of some complex phenomena, Caputo derivative was introduced in 1967. The Caputo derivative of fractional order α $(n-1 \leqslant \alpha < n)$ of function $f(t)$ defined as

$$ {}_{a}^{C}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} \frac{D^{n} f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, $$

if $0 < \alpha < 1$ then the Caputo derivative of fractional order α defined as:

$$ {}_{a}^{C}D_{t}^{(\alpha)} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{a}^{t} \frac{f'(\tau)}{(t-\tau)^{\alpha}} d\tau $$

1.1.4 A new fractional derivative of Caputo (The inverse of the new Fractional Derivative without Singular Kernel)

The authors of the article [6] suggest a new definition of fractional derivative, which assumes two different representations for the temporal and spatial variable. The first representation works on time variables, where the real powers appearing in the solutions of the usual fractional derivative will turn into integer powers, with some simplifications in the formulate and computations. In this framework, it is suitable to use the Laplace transform. The second representation is related to the spatial variables, thus for this non-local fractional derivative it is more convenient to work with the Fourier transform. The interest for this new approach is due to the necessity of using a model describing the behavior of classical viscoelastic materials, thermal media, electromagnetic systems, etc. In fact, the original definition of fractional derivative appears to be particularly convenient for those mechanical phenomena, related with plasticity, fatigue, damage and with electromagnetic hysteresis. When these effects are not present it seems more appropriate to use the new fractional derivative. We have also proposed a new non-local fractional derivative able to describe material heterogeneities and structures with different scales, which cannot be well described by classical local theories. So that, we rely that this spatial fractional derivative can play a meaningful role in the study of the macroscopic behaviors of some materials, related with non-local interactions, which are prevalent in determining the properties of the material. This work also contains some applications and simulations related to the behavior of these new derivatives, applied to classical functions such as trigonometric functions. These simulations show some similarities with the corresponding results by usual fractional derivative.

Let us recall the usual Caputo fractional time derivative (UFD$_t$) of order a, given by:

$$ {}_{a}^{C}D_{t}^{(\alpha)}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_{a}^{t}\frac{f'(\tau)}{(t-\tau)^{\alpha}}d\tau $$

with $a \in [0,1]$ and $a \in [-\infty, t)$, $f \in H^1(a,b)$, and $b > a$. By changing the kernel $(t-\tau)^{-\alpha}$ with the function $\exp\left(-\frac{\alpha}{1-\alpha}t\right)$ and $\frac{1}{\Gamma(1-\alpha)}$ with $\frac{M(\alpha)}{1-\alpha}$.

They obtained the following new definition of fractional time derivative NFD$_t$

Definition 1.2. *For $\alpha \in [0,1]$, the fractional time derivative $\mathscr{D}^{(\alpha)}u(x)$ of order (α) is defined as follow:*

$$ \mathscr{D}^{(\alpha)}\,f(x) = \frac{M(\alpha)}{1-\alpha}\int_{a}^{x} f'(s)\exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right]\,\mathrm{d}s \tag{1.1} $$

where $a \in\,]-\infty, x)$, $b > a$, $f \in H^1(a,b)$, and $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$, for example $M(\alpha) = 1 - 0.2\sin(2\pi\alpha)$, as shown in fig. 1.2.

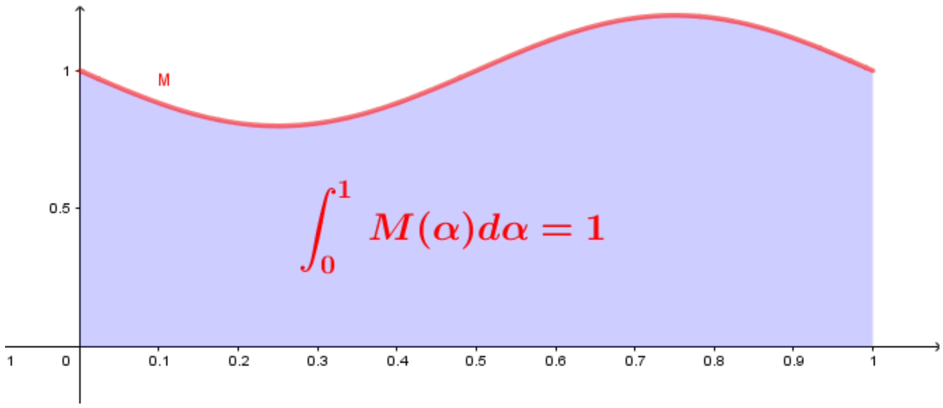

Figure 1.2: An example of a normalization function.

The following Matlab function M has used in our programs:

```
function y=Ma(z)
        r=0.2;
        y=1+r*sin(2*pi*z);
end
```

The NFD$_t$ is zero when $f(t)$ is constant, as in the UFD$_t$, but contrary to the UPD$_t$, the kernel does not have singularity for $t = \tau$. Formulated also NFD$_t$ can also be applied to functions that do not belong to $H^1(a,b)$. Indeed, its can be formulated also for $f \in L^1(-\infty, b)$

and for any $\alpha \in [0,1]$ as

$$\mathscr{D}_t^{(\alpha)} f(t) = \frac{\alpha M(\alpha)}{(1-\alpha)} \int_{-\infty}^{t} (f(t) - f(\tau)) \exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau$$

Now, it is worth to observe that if we put

$$\sigma = \frac{1-\alpha}{\alpha} \in [0,\infty], \alpha = \frac{1}{1+\sigma} \in [0,1]$$

the definition of NFD_t assumes the form

$$\tilde{\mathscr{D}}_t^{(\sigma)} f(t) = \frac{N(\sigma)}{\sigma} \int_a^t f'(\tau) \exp\left[-\frac{(t-\tau)}{\sigma}\right] d\tau$$

this is definition on which our study is based.

Definition 1.3. *If $n \geq 1$, and $\alpha \in [0,1]$ the fractional time derivative $\mathscr{D}_t^{(\alpha+n)} f(t)$ of order $(n+\alpha)$ is defined by*

$$\mathscr{D}_t^{(\alpha+n)} f(t) := D_t^{(\alpha)} \left(\mathscr{D}_t^{(n)} f(t)\right).$$

Theorem 1.1. *For NFD_t, if the function $f(t)$ is such that*

$$f^{(s)}(a) = 0, s = 1, 2, .., n$$

then, we have

$$\mathscr{D}_t^{(n)} \left(\mathscr{D}_t^{(\alpha)} f(t)\right) = \mathscr{D}_t^{(\alpha)} \left(\mathscr{D}_t^{(n)} f(t)\right)$$

Proof. [6]. We begin considering $n = 1$, then from definition (2.8) of $\mathscr{D}_t^{(\alpha+1)} f(t)$, we obtain

$$\mathscr{D}_t^{(\alpha)} \left(\mathscr{D}_t^{(1)} f(t)\right) = \frac{M(\alpha)}{1-\alpha} \int_a^t \ddot{f}(\tau) \exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau$$

Hence, after an integration by parts and assuming $f'(a) = 0$, we have

$$\begin{aligned}
\mathscr{D}_t^{(\alpha)} \left(\mathscr{D}_t^{(1)} f(t)\right) =& \frac{M(\alpha)}{(1-\alpha)} \int_a^t \left(\frac{d}{d\tau} f'(\tau)\right) \exp - \frac{\alpha(t-\tau)}{1-\alpha} d\tau = \\
& \frac{M(\alpha)}{(1-\alpha)} \left[\int_a^t \frac{d}{d\tau} \left(f'(\tau) \exp - \frac{\alpha(t-\tau)}{1-\alpha} d\tau \right.\right. \\
& \left. - \frac{\alpha}{1-\alpha} \int_a^t f'(\tau) \exp - \frac{\alpha(t-\tau)}{1-\alpha} d\tau \right] \\
=& \frac{M(\alpha)}{(1-\alpha)} \left[f'(t) - \frac{\alpha}{1-\alpha} \int_a^t f'(\tau) \exp - \frac{\alpha(t-\tau)}{1-\alpha} d\tau \right]
\end{aligned}$$

otherwise

$$\mathscr{D}_t^{(1)}\left(\mathscr{D}_t^{(\alpha)}f(t)\right) = \frac{d}{dt}\left(\frac{M(\alpha)}{1-\alpha}\int_a^t f'(\tau)\exp -\frac{\alpha(t-\tau)}{1-\alpha}d\tau\right)$$
$$= \frac{M(\alpha)}{1-\alpha}\left[f'(t) - \frac{\alpha}{1-\alpha}\int_a^t f'(\tau)\exp -\frac{\alpha(t-\tau)}{1-\alpha}d\tau\right].$$

It is easy to generalize the proof for any $n > 1$. □

Definition 1.4. *Let $n \geq 1$, and $\alpha \in [0,1]$ the fractional derivative $\mathscr{D}^{(\alpha+n)}f$ of order $(n+\alpha)$ is defined by*

$$\mathscr{D}^{(\alpha+n)}f(t) := \mathscr{D}_t^{(\alpha)}\left(\mathscr{D}^{(n)}f(t)\right)$$
$$= \frac{M(\alpha)}{1-\alpha}\int_a^t f^{(n+1)}(s)\exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right]\mathrm{d}s$$

Such that

$$\mathscr{D}^{(\alpha+n)}f(t) = \frac{M(\alpha)}{1-\alpha}\int_a^t f^{(n+1)}(s)\exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right]\mathrm{d}s \tag{1.2}$$

The following MATLAB code calculates the new Caputo derivative for any C^{n+1}-continuous function.

```
% The goal of this program is calculate
%            the new Caputo derivative
   clc; close; clear all;
   syms t s;
   a=0;  b=7;
   alpha=1.5;
   f(t)=t^1.5;
  K(t)=Da(f,alpha,a)
```

```
function y=Da(f,alpha,a)
    syms t s;     m=floor(alpha);
    y=int(diff(f(s),m+1)*exp((alpha-m)*(s-t)/(m+1-alpha))
                     ,s,a,t)*Ma(alpha-m)/(m+1-alpha);
end
```

Lemma 1.1. *Let $n \geqslant 1$, and $\gamma \in (n, n+1)$. If we assume $u \in C^{n+1}[a,b]$, then the fractional deferential equation*

$$\mathscr{D}^{(\gamma)} u(x) = 0, \ \forall x \in [a,b]$$

has $u(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0;\ a_i \in \mathbb{R}, i = 0, 1, \cdots, n,$ as unique solutions.

Proof. 0. Let $u(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0;\ a_i \in \mathbb{R}, i = 0, 1, \cdots, n,$ we have $u^{(n+1)}(x) = 0$ such that $\mathscr{D}^{(\gamma)} u(x) = 0, \ \forall x \in [a,b]$.

0. Let $\gamma \in]n, n+1[$, it can be written of the form: $\gamma = n + \alpha$ where $\alpha \in]0,1[$, and $n = [\gamma]$, we suppose $\mathscr{D}^{(\gamma)} u(x) = 0, \ \forall x \in [a,b]$, we have

$$\mathscr{D}^{(\gamma)} u(x) = \frac{M(\alpha)}{1-\alpha} \int_a^x u^{(n+1)}(s) \exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right] \mathrm{d}s$$

and the Leibniz integral rule gives the formula

$$\begin{aligned} \frac{d}{dx}\left(\mathscr{D}^{(\gamma)} u(x)\right) &= \frac{M(\alpha)}{1-\alpha} u^{(n+1)}(t) \\ &\quad - \frac{\alpha}{1-\alpha}\frac{M(\alpha)}{1-\alpha} \int_a^x u^{(n+1)}(s) \exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right] \mathrm{d}s \\ &= \frac{M(\alpha)}{1-\alpha} u^{(n+1)}(t) - \frac{\alpha}{1-\alpha} \mathscr{D}^{(\gamma)} u(x), \end{aligned}$$

and we have $\mathscr{D}^{(\gamma)} u(x) = 0$, consequently $u^{(n+1)}(t) = 0$ therefor,

$u(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0;\ a_i \in \mathbb{R}, i = 0, 1, \cdots, n.$

The proof is complete. □

Lemma 1.2. *Let $\gamma \in (n, n+1), n = [\gamma] \geqslant 0$. Assume that $f \in \mathcal{C}^n[a,b]$, the fractional derivative of order (γ) belongs to $C[a,b]$.*

Proof. 0. Let $x \in (a,b)$ and $s \in (a,x]$, we use part integration to prove this relation

$$\begin{aligned} \mathscr{D}^{(\gamma)} f(x) &= \frac{M(\alpha)}{1-\alpha}\left(f^{(n)}(x) - f^{(n)}(a) \cdot \exp\left[-\frac{\alpha(x-a)}{1-\alpha}\right]\right) \\ &\quad - \frac{\alpha \cdot M(\alpha)}{(1-\alpha)^2} \int_a^x f^{(n)}(s) \exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right] \mathrm{d}s. \end{aligned}$$

Simply to say that a function $\mathscr{D}^{(\gamma)} f$ is continuous (a,b).

0. Then, let $\{x_n\}_{n\in\mathbb{N}}$ real sequence define in $[a,b]$, which this sequence converges to $x \in [a,b]$.

$$\begin{aligned}
\lim_{n\longrightarrow\infty} \mathscr{D}^{(\gamma)} f(x_n) &= \lim_{n\longrightarrow\infty} \frac{M(\alpha)}{1-\alpha}\left(f^{(n)}(x_n) - f^{(n)}(a)\cdot \exp\left[-\frac{\alpha(x_n-a)}{1-\alpha}\right]\right) \\
&- \lim_{n\longrightarrow\infty} \frac{\alpha\cdot M(\alpha)}{(1-\alpha)^2}\int_a^{x_n} f^{(n)}(s)\exp\left[-\frac{\alpha(x_n-s)}{1-\alpha}\right]\,\mathrm{d}s, \\
&= \frac{M(\alpha)}{1-\alpha} f^{(n)}\left(\lim_{n\longrightarrow\infty} x_n\right) \\
&- f^{(n)}(a)\exp\left[-\frac{\alpha\left(\lim_{n\longrightarrow\infty} x_n - a\right)}{1-\alpha}\right] \\
&- \frac{\alpha\cdot M(\alpha)}{(1-\alpha)^2}\int_a^{\lim_{n\longrightarrow\infty} x_n} f^{(n)}(s)\exp\left[-\frac{\alpha\left(\lim_{n\longrightarrow\infty} x_n - s\right)}{1-\alpha}\right]\,\mathrm{d}s, \\
&= \frac{M(\alpha)}{1-\alpha}\left(f^{(n)}(x) - f^{(n)}(a)\cdot\exp\left[-\frac{\alpha(x-a)}{1-\alpha}\right]\right) \\
&- \frac{\alpha\cdot M(\alpha)}{(1-\alpha)^2}\int_a^{x} f^{(n)}(s)\exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right]\,\mathrm{d}s, \\
&= \mathscr{D}^{(\gamma)} f(x).
\end{aligned}$$

Therefor, $\mathscr{D}^{(\gamma)} f$ is continuous $[a,b]$.

□

1.2 Differentiation under the Integral Sign

Suppose a function ϕ is given by the formula

$$\phi(x) = \int_c^d f(x,t)dt, \quad a \le x \le b$$

where c and d are constants. If the integration can be performed explicitly, then $\phi'(x)$ can be found by a computation. However, even when the evaluation of the integral is impossible, it sometimes happens that $\phi'(x)$ can be found. The basic formula is given in the next theorem, known as **Leibniz' Rule**.

Theorem 1.2. *Suppose that ϕ is defined by*

$$\phi(x) = \int_c^d f(x,t)dt, \quad a \le x \le b$$

where c and d are constants. If fand f_x are continuous in the rectangle

$$R = \{(x,t) : a \leq x \leq b, \quad c \leq t \leq d\}$$

then

$$\phi'(x) = \int_c^d f_x(x,t)dt, \quad a < x < b$$

That is, the derivative may be found by differentiating under the integral sign.

We consider a function defined by

$$\phi(x) = \int_{u_0(x)}^{u_1(x)} f(x,t)\,\mathrm{d}t \tag{1.3}$$

where $u_0(x)$ and $u_1(x)$ are continuously differentiable functions for $a \leq x \leq b$ Furthermore, the ranges of u_0 and u_1 are assumed to lie between c and d. To obtain a formula for the derivative $\phi'(x)$, where ϕ is given by an integral such as eq. (1.3), it is simpler to consider a new integral which is more general than eq. (1.3). We define

$$F(x,y,z) = \int_y^z f(x,t)\,\mathrm{d}t, \tag{1.4}$$

and obtain the following corollary of Leibniz' Rule.

Theorem 1.3. *Suppose that f satisfies the conditions of Theorem (1.2) and that F is defined by (1.4) with $c < y$, $z < d$. Then*

$$\frac{\partial F}{\partial x} = \int_y^z f_1(x,t)dt \tag{1.5}$$

$$\frac{\partial F}{\partial y} = -f(x,y) \tag{1.6}$$

$$\frac{\partial F}{\partial z} = f(x,z) \tag{1.7}$$

Proof. Formula (1.5) is Theorem (1.2). Formulas (1.6) and (1.7) are precisely the Fundamental Theorem of Calculus, since taking the partial derivative of F with respect to one variable, say y, implies that x and z are kept fixed. □

Theorem 1.4 (General Rule for Differentiation under the Integral Sign)**.** *Suppose that f and $\partial f/\partial x$ are continuous in the rectangle*

$$R = \{(x,t) : a \leq x \leq b, \quad c \leq t \leq d\}$$

and suppose that $u_0(x), u_1(x)$ *are continuously differentiable for* $a \leq x \leq b$*, with the range of* u_0 *and* u_1 *in* (c, d)*. If* ϕ *is given by*

$$\phi(x) = \int_{u_0(x)}^{u_1(x)} f(x,t)\, \mathrm{d}t$$

then

$$\phi'(x) = f\left[x, u_1(x)\right] u_1'(x) - f\left[x, u_0(x)\right] \cdot u_0'(x) + \int_{u_0(x)}^{u_1(x)} f_x(x,t)\, \mathrm{d}t. \tag{1.8}$$

Proof. We observe that

$$F\left(x, u_0(x), u_1(x)\right) = \phi(x),$$

in Theorem 1.5. Applying the Chain Rule, we get

$$\phi'(x) = F_x + F_y u_0'(x) + F_z u_1'(x)$$

Inserting the values of F_x, F_y, and F_z from eqs. (1.5) to (1.7), we obtain the desired result eq. (1.8). □

Example 1.1. *Find* $\phi'(x)$*, given that*

$$\phi(x) = \int_0^{x^2} \arctan \frac{t}{x^2} dt$$

Solution. We have

$$\frac{\partial}{\partial x}\left(\arctan \frac{t}{x^2}\right) = \frac{-2t/x^3}{1+(t^2/x^4)} = -\frac{2tx}{t^2+x^4}$$

We use formula (1.8) and find

$$\phi'(x) = (\arctan 1) \cdot (2x) - \int_0^{x^2} \frac{2txdt}{t^2+x^4}$$

Setting $t = x^2 u$ in the integral on the right, we obtain

$$\phi'(x) = \frac{\pi x}{2} - \int_0^1 \frac{2x^3 u \cdot x^2 du}{x^4u^2+x^4} = \frac{\pi x}{2} - x\int_0^1 \frac{2udu}{u^2+1} = x\left(\frac{\pi}{2} - \log 2\right).$$

□

Theorem 1.5 (The Cauchy Formula For Repeated Integration)**.** *The Cauchy formula for repeated integration, named after Augustin Louis Cauchy, allows one to compress n anti-differentiation of a function into a single integral. Let* f *be a continuous function on the* *real line. Then the n-th repeated integral of f based at a,*

$$f^{(-n)}(x)=\int_a^x\int_a^{\sigma_1}\cdots\int_a^{\sigma_{n-1}} f\left(\sigma_n\right) \mathrm{d}\sigma_n\cdots\mathrm{d}\sigma_2\mathrm{d}\sigma_1$$

is given by single integration

$$f^{(-n)}(x)=\frac{1}{(n-1)!}\int_a^x(x-t)^{n-1}f(t)\mathrm{d}t$$

Proof. A proof is given by induction. since f is continuous, the base case follows from the Fundamental theorem of calculus

$$\frac{d}{dx}f^{(-1)}(x)=\frac{d}{dx}\int_a^x f(t)dt=f(x)$$

where

$$f^{(-1)}(a)=\int_a^a f(t)dt=0$$

Now, suppose this is true for n, and let us prove it for $n+1$. Apply the induction hypothesis and switching the order of integration,

$$\begin{aligned}
f^{-(n+1)}(x)&=\int_a^x\int_a^{\sigma_1}\cdots\int_a^{\sigma_n} f\left(\sigma_{n+1}\right)\mathrm{d}\sigma_{n+1}\cdots\mathrm{d}\sigma_2\mathrm{d}\sigma_1\\
&=\frac{1}{(n-1)!}\int_{a_x}^{t^\sigma}\int_{a_x}^{\sigma_1}\left(\sigma_1-t\right)^{n-1}f(t)\mathrm{d}t\mathrm{d}\sigma_1\\
&=\frac{1}{(n-1)!}\int_a^x\left(\sigma_1-t\right)^{n-1}f(t)\mathrm{d}\sigma_1\mathrm{d}t\\
&=\frac{1}{n!}\int_a^x(x-t)^n f(t)\mathrm{d}t
\end{aligned}$$

□

Chapter 2

Analytical and numerical study of a nonlinear Volterra integro-differential equations with conformable fractional derivation of Caputo

Summary

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In this chapter, we interest concerns an integro-differential equation of Volterra when the derivative of the solution appears under the sign of integration in a non-linear way. The derivation is conformable fractional of Caputo type, which increases the interest of this study.

We built Lipchitz conditions to obtain the existence and uniqueness of the solution in a Banach space specific to the derivative of Caputo. The analytical study is followed by a complete numerical study.

It should be noted that most of the integro-differential equations which have already been studied in case of the fractional derivative of the unknown u is out integral sign as shown in the references [6, 21], or in case the derivative of the first degree of the unknown under the integral like in the articles [45, 17, 46]. The authors of [45] studied the problem from the following type:

$$u(x) = f(x) + \int_a^x K\left(x, s, u(s), u'(s)\right) \mathrm{d}s, \qquad \forall x \in [a, b].$$

where, $f \in \mathcal{C}^1([a, b])$ and u is unknown, to be found in the same space, and in the paper [17], the authors studied for an integro-differential nonlinear volterra equation with weakly singular kernel as fellow

$$u(t) = \int_a^t p(t-s) K\left(t, s, u(s), u'(s)\right) \mathrm{d}s + f(t), \qquad \forall t \in [a, b]..$$

Therefore, the object of this chapter is to study a nonlinear integro-differential equation of Volterra of the type:

$$f(x) = g(x) + \int_a^x K\left(x, s, f(s), \mathscr{D}^{(\alpha)} f(s)\right) \mathrm{d}s, \qquad \forall x \in [a, b]. \tag{2.1}$$

Where $\mathscr{D}^{(\alpha)} f$ is the new of fractional derivative of Caputo (NFD), without singular kernel (see [6]), and this derivative has many properties as in the reference [27].

The first work, we built Lipchitz conditions to obtain the existence and uniqueness of the solution in a Banach space specific to the derivative of Caputo, then we search a method to approximate this solution and estimate error. In the last part, we give three numerical examples to illustrate the above methods for solve this type integro-differential equations. The exact solution is known and used to show that the numerical solution obtained with our methods is correct. We used MATLAB and C++ to solve these examples.

Definition 2.1. *We consider the space of continuously differentiable functions*

$$F\mathscr{D}^{(\alpha)}(a, b) = \left\{ f \in \mathcal{C}(a, b), \quad \| f \|_\alpha < \infty \right\}, \tag{2.2}$$

$F\mathscr{D}^{(\alpha)}(a,b)$ is a Banach space specific to the new Caputo derivative admits the norm

$$\| f \|_\alpha = \max_{x\in(a,b)} |f(x)| + \max_{x\in(a,b)} |\mathscr{D}^{(\alpha)} f(x)| = \| f \|_\infty + \| \mathscr{D}^{(\alpha)} f \|_\infty . \tag{2.3}$$

Proposition 2.1. *For $\alpha \in (0,1)$, if f is continuous on $[a,b]$, then the function $h(x) = \mathscr{D}^{(\alpha)} f(x)$ is continuous on the interval $[a,b]$.*

Proof. 0. Let $x \in (a,b)$ and $s \in (a,x]$, we have that

$$\frac{d}{ds}\left(f(s) \exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right]\right) = f'(s)\exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right] + \frac{\alpha}{1-\alpha} f(s)\exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right],$$

and hence

$$f'(s)\exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right] = \frac{d}{ds}\left(f(s) \exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right]\right) - \frac{\alpha}{1-\alpha} f(s)\exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right].$$

Then

$$\begin{aligned}\int_a^x f'(s)\exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right] \mathrm{d}s &= \int_a^x \frac{d}{ds}\left(f(s) \exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right]\right) \mathrm{d}s \\ &\quad - \frac{\alpha}{1-\alpha}\int_a^x f(s)\exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right] \mathrm{d}s \\ &= f(x) - f(a)\cdot\exp\left[-\frac{\alpha(x-a)}{1-\alpha}\right] \\ &\quad - \frac{\alpha}{1-\alpha}\int_a^x f(s)\exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right] \mathrm{d}s,\end{aligned}$$

as a result

$$\begin{aligned}\mathscr{D}^{(\alpha)} f(x) &= \frac{M(\alpha)}{1-\alpha}\left(f(x) - f(a)\cdot\exp\left[-\frac{\alpha(x-a)}{1-\alpha}\right]\right) \\ &\quad - \frac{\alpha\cdot M(\alpha)}{(1-\alpha)^2}\int_a^x f(s)\exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right] \mathrm{d}s.\end{aligned} \tag{2.4}$$

Simply to say that a function $\mathscr{D}^{(\alpha)} f$ is continuous (a,b).

0. Then, let $\{x_n\}_{n\in\mathbb{N}}$ real sequence define in $[a,b]$, which this sequence converges to $x \in [a,b]$.

$$\begin{aligned}
\lim_{n\longrightarrow\infty} \mathscr{D}^{(\alpha)} f(x_n) &= \lim_{n\longrightarrow\infty} \frac{M(\alpha)}{1-\alpha}\left(f(x_n) - f(a)\cdot \exp\left[-\frac{\alpha(x_n-a)}{1-\alpha}\right]\right) \\
&- \lim_{n\longrightarrow\infty} \frac{\alpha\cdot M(\alpha)}{(1-\alpha)^2}\int_a^{x_n} f(s)\exp\left[-\frac{\alpha(x_n-s)}{1-\alpha}\right] \mathrm{d}s, \\
&= \frac{M(\alpha)}{1-\alpha}\left(f\left(\lim_{n\longrightarrow\infty} x_n\right) - f(a)\cdot \exp\left[-\frac{\alpha\left(\lim_{n\longrightarrow\infty} x_n - a\right)}{1-\alpha}\right]\right) \\
&- \frac{\alpha\cdot M(\alpha)}{(1-\alpha)^2}\int_a^{\lim_{n\longrightarrow\infty} x_n} f(s)\exp\left[-\frac{\alpha\left(\lim_{n\longrightarrow\infty} x_n - s\right)}{1-\alpha}\right] \mathrm{d}s, \\
&= \frac{M(\alpha)}{1-\alpha}\left(f(x) - f(a)\cdot \exp\left[-\frac{\alpha(x-a)}{1-\alpha}\right]\right) \\
&- \frac{\alpha\cdot M(\alpha)}{(1-\alpha)^2}\int_a^{x} f(s)\exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right] \mathrm{d}s, \\
&= \mathscr{D}^{(\alpha)} f(x).
\end{aligned}$$

finally $\mathscr{D}^{(\alpha)} f$ is continuous on $[a,b]$. □

Proposition 2.2. *$F\mathscr{D}^{(\alpha)}(a,b)$ is a Banach space.*

Proof. It is obvious that $\|\cdot\|_\alpha$ is a semi-norm, and since $\|\cdot\|_\infty$ is strictly positive, and $\|\cdot\|_\alpha \geq \|\cdot\|_\infty$, we see that $\|\cdot\|_\alpha$ is strictly positive too. Now, suppose that $(f_n)_{n\in\mathbb{N}} \subseteq F\mathscr{D}^{(\alpha)}(a,b)$ is a Cauchy sequence.

Then,

$$\|f_n - f_m\|_\alpha \to 0 \quad \text{as } n,m \to \infty$$

and hence

$$\|f_n - f_m\|_\infty \to 0, \quad \left\|\mathscr{D}^{(\alpha)} f_n - \mathscr{D}^{(\alpha)} f_m\right\|_\infty \to 0, \quad \text{as } n,m\to\infty$$

Thus, the sequences $(f_n)_{n\in\mathbb{N}}$ and $\left(\mathscr{D}^{(\alpha)} f_n\right)_{n\in\mathbb{N}}$ are Cauchy sequences in $(C([a,b],\mathbb{R}),\|\cdot\|_\infty)$ Since this space is complete, both sequences have limits in this space, we name these limits f and h, i.e.

$$\|f_n - f\|_\infty \to 0 \text{ and } \left\|\mathscr{D}^{(\alpha)} f_n - h\right\|_\infty \to 0, \text{ as } n\to\infty.$$

The proof will be complete if we can show that $f \in F\mathscr{D}^{(\alpha)}(a,b)$ and $\lim\limits_{n\to+\infty} \|f_n - f\|_\alpha = 0$.

From eq. (2.4) we have

$$f(x)=\frac{1-\alpha}{M(\alpha)}\mathscr{D}^{(\alpha)}f(x)+f(a)\cdot\exp\left[-\frac{\alpha(x-a)}{1-\alpha}\right]+\frac{\alpha}{1-\alpha}\int_a^x f(s)\exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right]\mathrm{d}s,$$

we note that for every $n\in\mathbb{N}$,

$$\begin{cases} f_n(x) & =\dfrac{1-\alpha}{M(\alpha)}\mathscr{D}^{(\alpha)}f_n(x)+f_n(a)\cdot\exp\left[-\dfrac{\alpha(x-a)}{1-\alpha}\right] \\ & +\dfrac{\alpha}{1-\alpha}\displaystyle\int_a^x f_n(s)\exp\left[-\dfrac{\alpha(x-s)}{1-\alpha}\right]\mathrm{d}s, \\ & \\ \mathscr{D}^{(\alpha)}f_n(x) & =\dfrac{M(\alpha)}{1-\alpha}\left(f_n(x)-f_n(a)\cdot\exp\left[-\dfrac{\alpha(x-a)}{1-\alpha}\right]\right) \\ & -\dfrac{M(\alpha)}{1-\alpha}\left(\dfrac{\alpha}{1-\alpha}\displaystyle\int_a^x f_n(s)\exp\left[-\dfrac{\alpha(x-s)}{1-\alpha}\right]\mathrm{d}s\right) \end{cases}$$

We define

$$\begin{cases} \tilde{f}(x) & =\dfrac{1-\alpha}{M(\alpha)}h(x)+f(a)\cdot\exp\left[-\dfrac{\alpha(x-a)}{1-\alpha}\right]+\dfrac{\alpha}{1-\alpha}\displaystyle\int_a^x f(s)\exp\left[-\dfrac{\alpha(x-s)}{1-\alpha}\right]\mathrm{d}s, \\ & \\ \tilde{h}(x) & =\dfrac{M(\alpha)}{1-\alpha}\left(f(x)-f(a)\cdot\exp\left[-\dfrac{\alpha(x-a)}{1-\alpha}\right]\right) \\ & -\dfrac{M(\alpha)}{1-\alpha}\left(\dfrac{\alpha}{1-\alpha}\displaystyle\int_a^x f(s)\exp\left[-\dfrac{\alpha(x-s)}{1-\alpha}\right]\mathrm{d}s\right) \end{cases}$$

Then

$$\begin{aligned} \left\|f_n-\tilde{f}\right\|_\infty &=\sup_{x\in[a,b]}\left|f_n(x)-\tilde{f}(x)\right| \\ &\leqslant\frac{1-\alpha}{M(\alpha)}\sup_{x\in[a,b]}\left|\mathscr{D}^{(\alpha)}f_n(x)-h(x)\right| \\ &+\sup_{x\in[a,b]}\left|(f_n(a)-f(a))\exp\left[-\frac{\alpha(x-a)}{1-\alpha}\right]\right| \\ &+\sup_{x\in[a,b]}\left|\frac{\alpha}{1-\alpha}\int_a^x(f_n(s)-f(s))\exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right]\mathrm{d}s\right| \\ &\leqslant C_1\|\mathscr{D}^{(\alpha)}f_n-h\|_\infty+C_2\|f_n-f\|_\infty+C_3\|f_n-f\|_\infty\underset{n\to+\infty}{\longrightarrow}0, \end{aligned}$$

and

$$\begin{aligned}
\left\|\mathscr{D}^{(\alpha)} f_n - \tilde{h}\right\|_\infty &= \sup_{x\in[a,b]}\left|\mathscr{D}^{(\alpha)} f_n(x) - \tilde{h}(x)\right| \\
&\leqslant \frac{M(\alpha)}{1-\alpha}\sup_{x\in[a,b]}|f_n(x)-f(x)| \\
&+ \frac{M(\alpha)}{1-\alpha}\sup_{x\in[a,b]}\left|(f_n(a)-f(a))\exp\left[-\frac{\alpha(x-a)}{1-\alpha}\right]\right| \\
&+ \sup_{x\in[a,b]}\left|\frac{\alpha M(\alpha)}{(1-\alpha)^2}\int_a^x (f_n(s)-f(s))\exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right]\,\mathrm{d}s\right| \\
&\leqslant C_1'\|f_n-f\|_\infty + C_2'\|f_n-f\|_\infty + C_3'\|f_n-f\|_\infty \underset{n\to+\infty}{\longrightarrow} 0,
\end{aligned}$$

i.e., f_n converges to $\tilde{f}$ and $\mathscr{D}^{(\alpha)} f_n$ converges to $\tilde{h}$ in the uniform norm. Since the limit of a sequence in normed space is unique, we deduce that $\tilde{f}=f$ and $\tilde{h}=h$. Then

$$\begin{aligned}
\|f_n-f\|_\alpha &= \|f_n-f\|_\infty + \left\|\mathscr{D}^{(\alpha)} f_n - \mathscr{D}^{(\alpha)} f\right\|_\infty \\
&= \|f_n-f\|_\infty + \left\|\mathscr{D}^{(\alpha)} f_n - h\right\|_\infty \underset{n\to+\infty}{\longrightarrow} 0.
\end{aligned}$$

Finally $F\mathscr{D}^{(\alpha)}(a,b)$ is a Banach space. $\square$

2.1 Hypotheses

We consider the following nonlinear integro-differential equation of Volterra:

$$f(x) = g(x) + \int_a^x K\left(x,s,f(s),\mathscr{D}^{(\alpha)} f(s)\right)\,\mathrm{d}s, \qquad \forall x\in[a,b]. \tag{2.5}$$

where $f\in F\mathscr{D}^{(\alpha)}(a,b)$ and f is unknown, and the function

$$\begin{aligned}
K:\ [a,b]^2 \quad \mathbb{R}^2 &\to \mathbb{R} \\
(x,s,\quad u,v) &\to K(x,s,u,v).
\end{aligned}$$

is satisfy the following assumptions:

$$(\mathcal{H}_1)\quad \left\| \begin{array}{l} a)\dfrac{\partial K}{\partial x}\in\mathcal{C}\left([a,b]^2\mathbb{R}^2\right) \\ b)\exists\, m>0,\forall\, x,s\in[a,b],\forall u,v\in\mathbb{R},\max\left(\left|\dfrac{\partial K}{\partial x}(x,s,u,v)\right|,|K(x,s,u,v)|\right)\leqslant m. \end{array}\right.$$

eq. (2.5) bearing several information for the solution f. If both sides of the equation are derived (using Leibniz's rule for differentiation under the integral sign) we obtain $\forall x \in [a,b]$:

$$\mathscr{D}^{(\alpha)} f(x) = \mathscr{D}^{(\alpha)} g(x) + \frac{M(\alpha)}{1-\alpha} \int_a^x \frac{\partial}{\partial s} \left[\int_a^s K\left(s, f(\tau), \mathscr{D}^{(\alpha)} f(\tau)\right) \mathrm{d}\tau \right] \exp\left[-\frac{\alpha(x-s)}{1-\alpha} \right] \mathrm{d}s,$$

$$= \mathscr{D}^{(\alpha)} g(x) + \frac{M(\alpha)}{1-\alpha} \int_a^x K\left(s, s, f(s), \mathscr{D}^{(\alpha)} f(s)\right) \exp\left[-\frac{\alpha(x-s)}{1-\alpha} \right] \mathrm{d}s$$

$$+ \int_a^s \frac{\partial K}{\partial s}(s, \tau, f(\tau), \mathscr{D}^{(\alpha)} f(\tau))\, \mathrm{d}\tau \exp\left[-\frac{\alpha(x-s)}{1-\alpha} \right] \mathrm{d}s$$

$$= \mathscr{D}^{(\alpha)} g(x) + \frac{M(\alpha)}{1-\alpha} \int_a^x \left(K\left(s, s, f(s), \mathscr{D}^{(\alpha)} f(s)\right) \exp\left[-\frac{\alpha(x-s)}{1-\alpha} \right] \right) \mathrm{d}s$$

$$+ \frac{M(\alpha)}{1-\alpha} \int_a^x \left(\int_a^s \frac{\partial K}{\partial s} \left(s, \tau, f(\tau), \mathscr{D}^{(\alpha)} f(\tau)\right) \mathrm{d}\tau \right) \exp\left[-\frac{\alpha(x-s)}{1-\alpha} \right] \mathrm{d}s,$$

So that

$$\mathscr{D}^{(\alpha)} f(x) = \mathscr{D}^{(\alpha)} g(x) + \frac{M(\alpha)}{1-\alpha} \int_a^x \left(K\left(s, s, f(s), \mathscr{D}^{(\alpha)} f(s)\right) \exp\left[-\frac{\alpha(x-s)}{1-\alpha} \right] \right) \mathrm{d}s$$

$$+ \frac{M(\alpha)}{1-\alpha} \int_a^x \left(\int_a^s \frac{\partial K}{\partial s} \left(s, \tau, f(\tau), \mathscr{D}^{(\alpha)} f(\tau)\right) \mathrm{d}\tau \right) \exp\left[-\frac{\alpha(x-s)}{1-\alpha} \right] \mathrm{d}s, \tag{2.6}$$

finally, we obtain

$$\begin{aligned} \mathscr{D}^{(\alpha)} f(x) &= \mathscr{D}^{(\alpha)} g(x) + \frac{M(\alpha)}{1-\alpha} \int_a^x K\left(s, s, f(s), \mathscr{D}^{(\alpha)} f(s)\right) \exp\left[-\frac{\alpha(x-s)}{1-\alpha} \right] \mathrm{d}s \\ &+ \frac{M(\alpha)}{1-\alpha} \int_a^x (x-s) \frac{\partial K}{\partial s} \left(s, s, f(s), \mathscr{D}^{(\alpha)} f(s)\right) \exp\left[-\frac{\alpha(x-s)}{1-\alpha} \right] \mathrm{d}s. \end{aligned} \tag{2.7}$$

Definition 2.2. *For $g \in F\mathscr{D}^{(\alpha)}(a,b)$, we define the functional Ψ_g by:*

$$\begin{array}{rccc} \Psi_g : & F\mathscr{D}^{(\alpha)}(a,b) & \longrightarrow & F\mathscr{D}^{(\alpha)}(a,b) \\ & \varphi & \longrightarrow & \Psi_g(\varphi) \end{array}$$

where

$$\begin{aligned} \Psi_g(\varphi): \quad [a,b] \quad &\longrightarrow \mathbb{R} \\ x \quad &\longrightarrow \Psi_g(\varphi)(x) = g(x) + \int_a^x K\left(x, s, \varphi(s), \mathscr{D}^{(\alpha)}\varphi(s)\right) \mathrm{d}s. \end{aligned}$$

Proposition 2.3. *For $g \in F\mathscr{D}^{(\alpha)}(a,b)$, the functional Ψ_g is continuous from $F\mathscr{D}^{(\alpha)}(a,b)$ to itself.*

Proof. Let $g \in F\mathscr{D}^{(\alpha)}(a,b)$. it is obvious that $\Psi_g(\varphi)$ is continuous on (a,b).
For all $x \in (a,b)$,

$$\begin{aligned} \mathscr{D}^{(\alpha)}\Psi_g(\varphi)(x) = \mathscr{D}^{(\alpha)}g(x) + \frac{M(\alpha)}{1-\alpha}\int_a^x \left(K\left(s,s,\varphi(s),\mathscr{D}^{(\alpha)}\varphi(s)\right)\exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right]\right)\mathrm{d}s \\ + \frac{M(\alpha)}{1-\alpha}\int_a^x \left(\int_a^s \frac{\partial K}{\partial s}\left(s,\tau,\varphi(\tau),\mathscr{D}^{(\alpha)}\varphi(\tau)\right)\mathrm{d}\tau\right)\exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right]\mathrm{d}s, \end{aligned}$$

$\mathscr{D}^{(\alpha)}\Psi_g(\varphi)$ is a continuous function on (a,b) and bounded by the constant $\|g\|_\alpha + \lambda\beta m(b-a)(b-a+1)$, where

$$\lambda = \max_{\alpha\in(0,1)}\frac{M(\alpha)}{1-\alpha} \quad \text{and} \quad \beta = \max_{a\leqslant s\leqslant x\leqslant b}\left(\exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right]\right),$$

therefore $\Psi_g(\varphi) \in F\mathscr{D}^{(\alpha)}(a,b)$.

Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued functions of the space $F\mathscr{D}^{(\alpha)}(a,b)$ which converge to a function $\varphi \in F\mathscr{D}^{(\alpha)}(a,b)$,

$$\begin{aligned} \lim_{n\to+\infty}\Psi_g(\varphi_n)(x) &= \lim_{n\to+\infty}\left(\int_a^x K\left(t,s,\varphi_n(s),\mathscr{D}^{(\alpha)}\varphi_n(s)\right)\mathrm{d}s + g(x)\right) \\ &= \lim_{n\to+\infty}\int_a^x K\left(x,s,\varphi_n(s),\mathscr{D}^{(\alpha)}\varphi_n(s)\right)\mathrm{d}s + g(x) \\ &= \int_a^x \lim_{n\to+\infty} K\left(x,s,\varphi_n(s),\mathscr{D}^{(\alpha)}\varphi_n(s)\right)\mathrm{d}s + g(x) \\ &= \int_a^x K\left(x,s,\lim_{n\to+\infty}\varphi_n(s),\lim_{n\to+\infty}\mathscr{D}^{(\alpha)}\varphi_n(s)\right)\mathrm{d}s + g(x) \\ &= \int_a^x K\left(x,s,\varphi(s),\mathscr{D}^{(\alpha)}\varphi(s)\right)\mathrm{d}s + g(x) \\ &= \Psi_g(\varphi)(x). \end{aligned}$$

□

In this article, the purpose is to get the existence and uniqueness of the solution of eq. (2.1), but use the minimum conditions to ensure its.

2.2 Analysis study

Theorem 2.1 (Existence)**.** *The equation (2.1) has a solution in* $F\mathscr{D}^{(\alpha)}(a,b)$.

Proof. We define the following set:

$$E = \left\{ \varphi \in F\mathscr{D}^{(\alpha)}(a,b), \left| \begin{array}{l} \varphi(a) = g(a), \\ |\Psi_g(\varphi)(x) - f(x)| \leqslant (b-a)m, \\ \left|\mathscr{D}^{(\alpha)}\Psi_g(\varphi)(x) - \mathscr{D}^{(\alpha)} f\right| \leqslant \lambda\beta m(b-a)(b-a+1) \end{array} \right. \right\}.$$

It is obvious that, E is convex and closed set. For all $\varphi \in E$ and all $x \in [a,b]$; $\Psi_g(\varphi)(a) = g(a)$, and

$$|\Psi_g(\varphi)(x) - g(x)| = \left| \int_a^x K\left(t, s, \varphi(s), \mathscr{D}^{(\alpha)}\varphi(s)\right) \mathrm{d}s \right|$$

$$\leqslant (b-a)m$$

$$\begin{aligned} \left|\mathscr{D}^{(\alpha)}\Psi_g(\varphi)(x) - \mathscr{D}^{(\alpha)} g\right| \leqslant & \left| \frac{M(\alpha)}{1-\alpha} \int_a^x \left(K\left(s, s, f(s), \mathscr{D}^{(\alpha)} f(s)\right) \exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right] \right) \mathrm{d}s \right| \\ & + \left| \frac{M(\alpha)}{1-\alpha} \int_a^x \left(\int_a^s \frac{\partial K}{\partial s}\left(s, \tau, f(\tau), \mathscr{D}^{(\alpha)} f(\tau)\right) \mathrm{d}\tau \right) e^{\left[-\frac{\alpha(x-s)}{1-\alpha}\right]} \mathrm{d}s \right| \end{aligned}$$

$$\leqslant \lambda\beta m(b-a)(b-a+1).$$

Then, $\Psi_g(E) \subset E$. But, Ψ_g is continuous on $F\mathscr{D}^{(\alpha)}(a,b)$ to itself, and for all $x,\ x' \in [a,b]$ ($x \leqslant x'$)

$$\begin{aligned}
|\Psi_g(\varphi)(x) - \Psi_g(\varphi)(x')| \leqslant & \left| \int_a^x K\left(x, s, \varphi(s), \mathscr{D}^{(\alpha)}\varphi(s)\right) \mathrm{d}s \right. \\
& \left. - \int_a^{x'} K\left(x', s, \varphi(s), \mathscr{D}^{(\alpha)}\varphi(s)\right) \mathrm{d}s + g(x) - g(x') \right| \\
& \leqslant \left| \int_a^x \left[K\left(x, s, \varphi(s), \mathscr{D}^{(\alpha)}\varphi(s)\right) - K\left(x', s, \varphi(s), \mathscr{D}^{(\alpha)}\varphi(s)\right)\right] \mathrm{d}s \right| \\
& + \left| \int_x^{x'} K\left(x', s, \varphi(s), \mathscr{D}^{(\alpha)}\varphi(s)\right) \mathrm{d}s \right| + |g(x) - g(x')| \\
& \leqslant \left((b-a+1)m + \max_{s\in[a,b]} |g'(s)| \right) |x - x'|
\end{aligned}$$

We use the theorem Schauder for we proved that the functional Ψ_g has a fixed point in $F\mathscr{D}^{(\alpha)}(a,b)$. Which result that eq. (2.1) has solutions in $F\mathscr{D}^{(\alpha)}(a,b)$. □

2.3 Uniqueness

Lemma 2.1 (see [45]). *If the function h is continuous and positive on (a,b), and satisfies*

$$\exists m > 0, \ h(x) \leqslant m \int_a^x h(s)\,\mathrm{d}s. \tag{2.8}$$

Then $\forall x \in [a,b], \ h(x) = 0.$

Proof: *(see [45]).* Since $h(x)$ is continuous function on $[a,b]$, $\exists\ \mu > 0$ realizes the following:

$$h(x) \leqslant \mu, \forall t \in [a,b].$$

Then,

$$h(x) \leqslant m\mu \int_a^x \mathrm{d}s = m\mu(x-a).$$

we integer

$$\begin{aligned}
h(x) &\leqslant m \int_a^x h(s)\,\mathrm{d}s \\
&\leqslant m^2\mu \int_a^x (s-a)\,\mathrm{d}s = m^2\mu \frac{(s-a)^2}{2}
\end{aligned}$$

If we repeat this operation n times, we get:

$$h(x) \leqslant m^n \mu \frac{(x-a)^n}{n} \xrightarrow[n \longrightarrow \infty]{} 0.$$

$\square$

In order to prove the uniqueness of solution of eq. (2.1), we use the hypothesis:

$$(\mathcal{H}_2) \quad \left\| \begin{array}{l} (c) \quad \exists M_1, M_2, \overline{M}_1, \overline{M}_2 \in \mathbb{R}_+, \forall u, v, \overline{u}, \overline{v} \in \mathbb{R}, \forall x, s \in [0, U], \\ \\ \qquad |K(x,s,u,v) - K(x,s,\overline{u},\overline{v})| \leqslant M_1|u-\overline{u}| + M_2|v-\overline{v}| \\ \\ \qquad \left| \dfrac{\partial K}{\partial x}(x,s,u,v) - \dfrac{\partial K}{\partial x}(x,s,\overline{u},\overline{v}) \right| \leqslant \overline{M}_1|u-\overline{u}| + \overline{M}_2|v-\overline{v}| \end{array} \right.$$

Theorem 2.2 (Uniqueness)**.** *The solution of eq.* (2.1) *is unique.*

Proof.
Let $f, h \in F\mathscr{D}^{(\alpha)}(a,b)$ two different solutions of eq. (2.1). we have:

$$f(x) = g(x) + \int_a^x K\left(x, s, f(s), \mathscr{D}^{(\alpha)} f(s)\right) \mathrm{d}s,$$

$$h(x) = g(x) + \int_a^x K\left(x, s, h(s), \mathscr{D}^{(\alpha)} h(s)\right) \mathrm{d}s,$$

and

$$\begin{aligned} \mathscr{D}^{(\alpha)} f(x) = \mathscr{D}^{(\alpha)} g(x) &+ \frac{M(\alpha)}{1-\alpha} \int_a^x \left(K\left(s, s, f(s), \mathscr{D}^{(\alpha)} f(s)\right) \exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right]\right) \mathrm{d}s \\ &+ \frac{M(\alpha)}{1-\alpha} \int_a^x \left(\int_a^s \frac{\partial K}{\partial s}\left(s, \tau, f(\tau), \mathscr{D}^{(\alpha)} f(\tau)\right) \mathrm{d}\tau \right) \exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right] \mathrm{d}s. \end{aligned}$$

$$\begin{aligned} \mathscr{D}^{(\alpha)} h(x) = \mathscr{D}^{(\alpha)} g(x) &+ \frac{M(\alpha)}{1-\alpha} \int_a^x \left(K\left(s, s, h(s), \mathscr{D}^{(\alpha)} h(s)\right) \exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right]\right) \mathrm{d}s \\ &+ \frac{M(\alpha)}{1-\alpha} \int_a^x \left(\int_a^s \frac{\partial K}{\partial s}\left(s, \tau, h(\tau), \mathscr{D}^{(\alpha)} h(\tau)\right) \mathrm{d}\tau \right) \exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right] \mathrm{d}s. \end{aligned}$$

we define

$$R(x) = |f(x) - h(x)| + \left|\mathscr{D}^{(\alpha)} f(x) - \mathscr{D}^{(\alpha)} h(x)\right|$$

$$\begin{aligned}
|f(x) - h(x)| &\leqslant \int_a^x \left|K\left(x, s, f(s), \mathscr{D}^{(\alpha)} f(s)\right) - K\left(x, s, h(s), \mathscr{D}^{(\alpha)} h(s)\right)\right| \mathrm{d}s \\
&\leqslant \int_a^x \left(M_1 |f(x) - h(x)| + M_2 \left|\mathscr{D}^{(\alpha)} f(x) - \mathscr{D}^{(\alpha)} h(x)\right|\right) \mathrm{d}s \\
&\leqslant \int_a^x \max(M_1, M_2)\left(|f(x) - h(x)| + \left|\mathscr{D}^{(\alpha)} f(x) - \mathscr{D}^{(\alpha)} h(x)\right|\right) \mathrm{d}s \\
&\leqslant \int_a^x \max(M_1, M_2) R(s)\, \mathrm{d}s \\
&\leqslant C_1 \cdot \int_a^x R(s)\, \mathrm{d}s.
\end{aligned}$$

we use the hypothesis ($\mathcal{H}_2$)

$$\begin{aligned}
\left|\mathscr{D}^{(\alpha)} f(x) - \mathscr{D}^{(\alpha)} h(x)\right| &\leqslant \frac{M(\alpha)}{1-\alpha} \Bigg| \int_a^x K\left(s, s, f(s), \mathscr{D}^{(\alpha)} f(s)\right) \\
&\qquad - K\left(s, s, h(s), \mathscr{D}^{(\alpha)} h(s)\right)\Bigg| \exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right] \mathrm{d}s \\
&\quad + \frac{M(\alpha)}{1-\alpha} \int_a^x \int_a^s \Bigg| \frac{\partial K}{\partial s}\left(s, \tau, f(\tau), \mathscr{D}^{(\alpha)} f(\tau)\right) \\
&\qquad - \frac{\partial K}{\partial s}\left(s, \tau, f(\tau), \mathscr{D}^{(\alpha)} f(\tau)\right)\Bigg| \mathrm{d}\tau \cdot \exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right] \mathrm{d}s \\
&\leqslant \lambda \int_a^x \max(M_1, M_2) R(s)\, \mathrm{d}s + \lambda\beta(b-a) \int_a^x \max(\overline{M}_1, \overline{M}_2) R(s)\, \mathrm{d}s \\
&\leqslant C_2 \cdot \int_a^x R(s)\, \mathrm{d}s.
\end{aligned}$$

Consequently

$$R(x) \leqslant \max(C_1, C_2) \int_a^x R(s)\, \mathrm{d}s$$

Using the lemma 2.1

$$R(x) \leqslant m \int_a^x R(s)\, \mathrm{d}s \xRightarrow{\textit{lemma } 2.1} R(x) = 0, \ \forall x \in [a, b]$$

we get: $f(x) = h(x), \ \forall x \in [a, b]$.

$\square$

2.4 Numerical study

Under the assumptions ($\mathcal{H}_1$) and ($\mathcal{H}_2$), we have prove that eq. (2.1) has a unique solution $F\mathscr{D}^{(\alpha)}(a,b)$, we introduce algorithm for finding numerical solution of eq. (2.1). Here, the interval $[a,b]$ is divided in to N-equal sub-intervals, where $x_j = a + j \cdot l$ for all $j \in \{0 \cdots N\}$ and $l = (b-a)/N$. Also, using the following numerical integration formula:

$$\int_a^b f(s)\,\mathrm{d}s = l \sum_{j=1}^{N} w_j f(x_j)$$

where w_j are the weights of the quadrature rule such that $\exists\ W > 0,\ \max\limits_{0 \leqslant j \leqslant N} |w_j| \leqslant W$, on eqs. (2.1) and (2.7) we get the system:

$$f_0 = g(a) \tag{2.9}$$

$$h_0 = \mathscr{D}^{(\alpha)} g(a) = 0 \tag{2.10}$$

$$\begin{aligned} h_n = \mathscr{D}^{(\alpha)} g(x_n) &+ \frac{M(\alpha)}{1-\alpha} l \sum_{j=1}^{n} w_j K(x_j, x_j, f_j, h_j) \exp\left[-\frac{\alpha(x_n - x_j)}{1-\alpha}\right] \\ &+ \frac{M(\alpha)}{1-\alpha} l \sum_{j=1}^{n} w_j (x_n - x_j) \frac{\partial K}{\partial x}(x_j, x_j, f_j, h_j) \exp\left[-\frac{\alpha(x_n - x_j)}{1-\alpha}\right] \end{aligned} \tag{2.11}$$

$$f_n = g(x_n) + l \sum_{j=1}^{n} w_j K(x_n, x_j, f_j, h_j)\,. \tag{2.12}$$

Where f_n approaches $f(x_n)$, and h_n approaches $\mathscr{D}^{(\alpha)} f(x_n)$.

2.5 System study

In order to prove the existence and uniqueness of the solution of the system eqs. (2.9) to (2.12), we use hypotheses ($\mathcal{H}_1$) and ($\mathcal{H}_2$):

Theorem 2.3. *For l is sufficiently small, the system eqs. (2.9) to (2.12) has a unique solution.*

Proof. We supposing that the space $\mathbb{R}^2$ has the first norm:

$$\forall \begin{pmatrix} U \\ V \end{pmatrix} \in \mathbb{R}^2, \quad \left\| \begin{matrix} U \\ V \end{matrix} \right\|_1 = |U| + |V|$$

Now, we define:

$$\forall n \geqslant 1 \quad \Psi_n \begin{pmatrix} U \\ \\ V \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \\ \rho_2 \end{pmatrix},$$

where

$$\begin{aligned}
\rho_1 &= g(x_n) + l\ W_n K\left(x_n, x_n, U, V\right) + l \sum_{j=0}^{n-1} w_j K\left(x_n, x_j, f_j, h_j\right), \\
\rho_2 &= \mathscr{D}^{(\alpha)}\, g(x_n) + \frac{M(\alpha)}{1-\alpha} l\ W_n K\left(x_n, x_n, U, V\right) \\
&\quad + \frac{M(\alpha)}{1-\alpha} l \sum_{j=1}^{n-1} w_j \left(K\left(x_j, x_j, f_j, h_j\right) + (x_n - x_j) \frac{\partial K}{\partial x}\left(x_j, x_j, f_j, h_j\right) \right) \exp\left[-\frac{\alpha(x_n - x_j)}{1-\alpha} \right].
\end{aligned}$$

We have:

$$\left\| \Psi_n \begin{pmatrix} U \\ \\ V \end{pmatrix} - \Psi_n \begin{pmatrix} U' \\ \\ V' \end{pmatrix} \right\|_1 = \left\| \begin{pmatrix} \beta_1 \\ \\ \beta_2 \end{pmatrix} \right\|_1$$

as result

$$\begin{aligned}
\beta_1 &= l\ W_n \left(K\left(x_n, x_n, U, V\right) - K\left(x_n, x_n, U', V'\right) \right) \\
\beta_2 &= \frac{M(\alpha)}{1-\alpha} l\ W_n \left(K\left(x_n, x_n, U, V\right) - K\left(x_n, x_n, U', V'\right) \right).
\end{aligned}$$

however

$$\begin{aligned}
|\beta_1| &\leqslant l\ W \left(M_1 |U - U'| + M_2 |V - V'| \right) \\
|\beta_2| &\leqslant \lambda l\ W \left(M_1 |U - U'| + M_2 |V - V'| \right)
\end{aligned}$$

Then

$$\begin{aligned}
\left\| \Psi_n \begin{pmatrix} U \\ \\ V \end{pmatrix} - \Psi_n \begin{pmatrix} U' \\ \\ V' \end{pmatrix} \right\|_1 &= |\beta_1| + |\beta_2| \\
&\leqslant l\ W \left(M_1 |U - U'| + M_2 |V - V'| \right) \\
&+ \lambda l\ W \left(M_1 |U - U'| + M_2 |V - V'| \right) \\
&\leqslant \max \left(W M_1 l, W M_2 l, W M_1 \lambda l, W M_2 \lambda l \right) \left\| \begin{pmatrix} U \\ \\ V \end{pmatrix} - \begin{pmatrix} U' \\ \\ V' \end{pmatrix} \right\|_1
\end{aligned}$$

Using the theorem of Banach to get a unique solution of the previous system eqs. (2.9) to (2.12)
. □

2.6 Error analysis

In this part, we want to show that the numerical method constructed in the previous section, converges to the exact solution of the eq. (2.1), we define:

$$e_n = |f_n - f(x_n)| + \left| h_n - \mathscr{D}^{(\alpha)} f(x_n) \right|$$

We say that the method is convergent if

$$\lim_{h \longrightarrow 0} \left(\max_{0 \leqslant n \leqslant N} \{e_n\} \right) = 0$$

For $n > 0$ and $\varphi \in F\mathscr{D}^{(\alpha)}(a, b)$, we define the local consistency error by:

$$\begin{aligned}
\delta(h, x_n) = & \left| \int_a^x K\left(t, s, \varphi(s), \mathscr{D}^{(\alpha)} \varphi(s)\right) \mathrm{d}s - l \sum_{j=0}^{n} w_j K\left(x_n, x_j, \varphi_j, h_j\right) \right| \\
& + \frac{M(\alpha)}{1-\alpha} \left| \int_a^x K\left(s, s, \varphi(s), \mathscr{D}^{(\alpha)} \varphi(s)\right) \exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right] \mathrm{d}s \right. \\
& \qquad \left. - l \sum_{j=0}^{n} w_j K\left(x_j, x_j, f_j, h_j\right) \exp\left[-\frac{\alpha(x_n - x_j)}{1-\alpha}\right] \right| \\
& + \frac{M(\alpha)}{1-\alpha} \left| \int_a^x (x-s) \frac{\partial K}{\partial s}\left(s, s, \varphi(s), \mathscr{D}^{(\alpha)} \varphi(s)\right) \exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right] \mathrm{d}s \right. \\
& \qquad \left. - l \sum_{j=0}^{n} w_j (x_n - x_j) \frac{\partial K}{\partial x}\left(x_j, x_j, f_j, h_j\right) \exp\left[-\frac{\alpha(x_n - x_j)}{1-\alpha}\right] \right| .
\end{aligned}$$

We say that the approximation method of the system eqs. (2.9) to (2.12) is consistent with eq. (2.1), if

$$\forall \varphi \in F\mathscr{D}^{(\alpha)}(a, b), \ \lim_{h \longrightarrow 0} \left(\max_{0 \leqslant n \leqslant N} \{\delta(h, x_n)\} \right) = 0$$

Theorem 2.4. *If the approximation method eqs. (2.9) to (2.12) is consistent with eq. (2.1), then*

$$\lim_{h \longrightarrow 0} \left(\max_{0 \leqslant n \leqslant N} \{e_n\} \right) = 0$$

Proof. For $n \geqslant 1$, we have

$$\begin{aligned}
f(x_n) &= g(x_n) + \int_a^{x_n} K\left(x_n, s, f(s), \mathscr{D}^{(\alpha)} f(s)\right) \mathrm{d}s \\
&\simeq g(x_n) + l \sum_{j=0}^{n} w_j K\left(x_n, x_j, f(x_j), \mathscr{D}^{(\alpha)} f(x_j)\right)
\end{aligned}$$

and

$$f_n = g(x_n) + l \sum_{j=0}^{n} w_j K\left(x_n, x_j, f_j, v_j\right).$$

So that

$$\begin{aligned}
|f(x_n) - f_n| &= \left| l \sum_{j=0}^{n} w_j \left(K\left(x_n, x_j, f(x_j), \mathscr{D}^{(\alpha)} f(x_j)\right) - K\left(x_n, x_j, f_j, h_j\right) \right) \right| \\
&\leqslant l\, W \sum_{j=0}^{n} \left| K\left(x_n, x_j, f(x_j), \mathscr{D}^{(\alpha)} f(x_j)\right) - K\left(x_n, x_j, f_j, h_j\right) \right| \\
&\leqslant l\, W \sum_{j=0}^{n} \left(M_1 \left| f(x_j) - f_j \right| + M_2 \left| \mathscr{D}^{(\alpha)} f(x_j) - h_j \right| \right). \qquad (2.13)
\end{aligned}$$

And

$$h_n = \mathscr{D}^{(\alpha)}\, g(x_n) + \frac{M(\alpha)}{1-\alpha} l \sum_{j=1}^{n} w_j K\left(x_j, x_j, f_j, h_j\right) \exp\left[-\frac{\alpha(x_n - x_j)}{1-\alpha}\right]$$
$$+ \frac{M(\alpha)}{1-\alpha} l \sum_{j=1}^{n} w_j (x_n - x_j) \frac{\partial K}{\partial x}\left(x_j, x_j, f_j, h_j\right) \exp\left[-\frac{\alpha(x_n - x_j)}{1-\alpha}\right].$$

$$\mathscr{D}^{(\alpha)}\, f(x) = \mathscr{D}^{(\alpha)}\, g(x) + \frac{M(\alpha)}{1-\alpha} \int_a^x K\left(s, s, f(s), \mathscr{D}^{(\alpha)}\, f(s)\right) \exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right]\, \mathrm{d}s$$
$$+ \frac{M(\alpha)}{1-\alpha} \int_a^x (x-s) \frac{\partial K}{\partial s}\left(s, s, f(s), \mathscr{D}^{(\alpha)}\, f(s)\right) \exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right]\, \mathrm{d}s$$

$$\mathscr{D}^{(\alpha)}\, f(x_n) = \mathscr{D}^{(\alpha)}\, g(x_n) + \frac{M(\alpha)}{1-\alpha} \int_a^{x_n} K\left(s, s, f(s), \mathscr{D}^{(\alpha)}\, f(s)\right) \exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right]\, \mathrm{d}s$$

$$+ \frac{M(\alpha)}{1-\alpha} \int_a^x (x_n - s) \frac{\partial K}{\partial x}\left(s, s, f(s), \mathscr{D}^{(\alpha)}\, f(s)\right) \exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right]\, \mathrm{d}s$$

$$\simeq \mathscr{D}^{(\alpha)}\, g(x_n) + \frac{M(\alpha)}{1-\alpha} l \sum_{j=0}^{n} w_j K\left(x_j, x_j, f(x_j), \mathscr{D}^{(\alpha)}\, f(x_j)\right) \exp\left[-\frac{\alpha(x_n - x_j)}{1-\alpha}\right]$$

$$+ \frac{M(\alpha)}{1-\alpha} l \sum_{j=0}^{n} w_j (x_n - x_j) \frac{\partial K}{\partial x}\left(x_j, x_j, f(x_j), \mathscr{D}^{(\alpha)}\, f(x_j)\right) \exp\left[-\frac{\alpha(x_n - x_j)}{1-\alpha}\right].$$

Thus

$$\begin{aligned}\left|\mathscr{D}^{(\alpha)} f(x_n) - h_n\right| \simeq \frac{M(\alpha)}{1-\alpha} l \Bigg| & \sum_{j=0}^{n} w_j K\left(x_j, x_j, f(x_j), \mathscr{D}^{(\alpha)} f(x_j)\right) \exp\left[-\frac{\alpha(x_n - x_j)}{1-\alpha}\right] \\ & - \sum_{j=0}^{n} w_j K\left(x_j, x_j, f_j, h_j\right) \exp\left[-\frac{\alpha(x_n - x_j)}{1-\alpha}\right] \\ & + \sum_{j=1}^{n} w_j (x_n - x_j) \frac{\partial K}{\partial x}\left(x_j, x_j, f(x_j), \mathscr{D}^{(\alpha)} f(x_j)\right) \exp\left[-\frac{\alpha(x_n - x_j)}{1-\alpha}\right] \\ & - \sum_{j=1}^{n} w_j (x_n - x_j) \frac{\partial K}{\partial x}\left(x_j, x_j, f_j, h_j\right) \exp\left[-\frac{\alpha(x_n - x_j)}{1-\alpha}\right] \Bigg| \\ \leqslant \lambda l\ W & \sum_{j=0}^{n}\left(M_1 \left|f(x_j) - f_j\right| + M_2 \left|\mathscr{D}^{(\alpha)} f(x_j) - h_j\right|\right) \\ & + (b-a)\lambda l\ W \sum_{j=0}^{n}\left(\overline{M}_1 \left|f(x_j) - f_j\right| + \overline{M}_2 \left|\mathscr{D}^{(\alpha)} f(x_j) - h_j\right|\right)\end{aligned}$$

finally

$$\left|\mathscr{D}^{(\alpha)} f(x_n) - h_n\right| \leqslant \lambda l\ W \sum_{j=0}^{n} (M_1 + (b-a)\overline{M}_1) \left|f(x_j) - f_j\right| \tag{2.14}$$

$$\lambda l\ W \sum_{j=0}^{n} +(M_2 + (b-a)\overline{M}_2) \left|\mathscr{D}^{(\alpha)} f(x_j) - h_j\right|. \tag{2.15}$$

From eqs. (2.13) and (2.14) we get:

$$\lim_{h \longrightarrow 0}\left(\max_{0 \leqslant n \leqslant N}\{e_n\}\right) = 0$$

□

2.7 Numerical result

Since the approximation method of eqs. (2.9) to (2.12) converges to the solution of eqs. (2.1) and (2.7), we use the trapezoidal integration method to get that the terms f_n and h_n will can not calculated exactly. but we use the iteration method of Banach for be approached its.

Example 2.1. *Let's take the following nonlinear Volterra integro-differential nonlinear equation as an example:*

$$\forall x \in [0, \pi]; \quad f(x) = g(x) + \int_0^x K\left(x, s, f(s), \mathscr{D}^{(\alpha)} f(s)\right) \mathrm{d}s, \tag{2.16}$$

where $\alpha = 0.5$ *and the kernel*

$$k(x,s,u,v) = \frac{x + e^{-s} + \sin^2(s) + \left(e^{-s} - \sqrt{2}\sin(s+\frac{\pi}{4})\right)^2}{x + e^{-s} + u^2 + v^2}$$

If we take: $g(x) = \sin(x) - x$, *we get* $f(x) = \sin(x)$.

The following MATLAB with C++ codes solve the equation (2.16) numerically, and plot the absolute errors and the approximate solution of this equation with the exact solution.

```
% The goal of this program is calculate the approximate
%            solution of the following problem
%      u(t)=f(t)+\int(a,t)k(t,s,u,Du)ds      t in [a,b]
clc ; clear all;
%%%%%%%%%%%% Variables %%%%%%%%%%%%%%%%%%
alpha=0.5;
a=0;
b=pi;
%%%%%%%%%%%%% functions %%%%%%%%%%%%%%%%%
syms t s x y;
S(t)=t*sin(t);
DS(t)=Da(S,alpha,a)
k(t,s,x,y)= sin(t)*(t+exp(-s)+(s*sin(s))^2
                  +(exp(-s) - cos(s)+ s*cos(s)
                  + s*sin(s))^2)/(t+exp(-s)+x^2+y^2)
K(t,s,x,y)= diff(k(t,s,x,y),t)
```

```
#include "moumen.h"
double f(double),Ma(double),S(double),DS(double),
Df(double),KK(double ,double ,double,double);
double K(double ,double ,double,double),
               k(double ,double ,double,double);
main ()
{
int N=64,n,j;
```

```
double  alpha,a,b,t,h,beta,u[N+1],
                    v[N+1],X[N+1],l[N+1],U,DU;
alpha=0.5;
a=0.;
b=2.;
h=(b-a)/N;
beta=h*Ma(alpha)/(1-alpha)  ;
for (n=0;n<=N;n++)
      X[n]=a+h*n;
u[0]=f(a);
v[0]=Df(a);
l[0]=fabs(S(X[0])-u[0])+fabs(DS(X[0])-v[0]);
for (n=1;n<=N;n++)
{
DU=KK(X[0],X[0],u[0],v[0])*exp(alpha*(X[0]-X[n])
                /(1-alpha))/2.;
U=k(X[n],X[0],u[0],v[0])/2.  ;
     for (j=1;j<n;j++)
   {
           DU=DU+KK(X[j],X[j],u[j],v[j])*exp(alpha*(X[j]
                            -X[n])/(1-alpha));
           U=U+k(X[n],X[j],u[j],v[j])  ;
   }
           DU=DU+KK(X[n],X[n],f(X[n])+h*U,Df(X[n])
                          +beta*DU)/2.;
           U=U+ k(X[n],X[n],f(X[n])+h*U,Df(X[n])
                          +beta*DU)/2.  ;
     v[n]=Df(X[n])+beta*DU;
     u[n]=f(X[n])+h*U;
     l[n]=fabs(S(X[n])-u[n])+fabs(DS(X[n])-v[n]);
}

FILE  *f1,*f2,*f3;
```

```
f1=fopen("Approximate.txt","w");
for(n=0;n<=N;n++)
fprintf(f1,"%lf   \t %lf   \n", n*h , u[n] );
fclose(f1);
plot("plot  'Approximate.txt' pointtype 5,
                x*x+x  linewidth 2 ;
                pause mouse  \n ");
endplot();
f2=fopen("Error.txt","w");
for(n=0;n<=N;n++)
fprintf(f2,"%lf   \t %lf   \n", n*h , l[n] );
fclose(f2);
plot1("plot  'Error.txt' pointtype 7;
                     pause mouse  \n");
endplot1();
}
double S(double t)
{
return t*t+t;
}
double DS(double t)
{
return 4.*t+2.*exp(-t)-2.;
}
double f(double t)
{
 return  t*t;
}
double Df(double t)
{
 return 4.*t+4.*exp(-t)-4.;
}
double k(double t,double s,double x,double y)
```

```
{
return (t+exp(-2.*s) -4.*s+4.*s*exp(-s)+8.*s*s
          +8.*s*s*s + 4.*s*s*s*s + 1.)/(t
          +2.*exp(-s)+4.*x*x+y*y/4.);
}
double K(double t,double s,double x,double y)
{
return 1./(t+2.*exp(-s)+4.*x*x+y*y/4)
            -(t + exp(-2.*s) -4.*s + 4.*s*exp(-s)
            +8.*s*s+8.*s*s*s+4.*s*s*s*s+1.)
            /((t+2.*exp(-s)+4.*x*x+y*y/4.)*(t
            +8.*exp(-s)+4.*x*x+y*y/4.));
}
double KK(double t,double s,double x,double y)
{
return  k(t,s,x,y)+(t-s)*K(t,s,x,y);
}
double Ma(double t)
{
double r=0.2;
return 1+r*sin(2*pi*t);
}
```

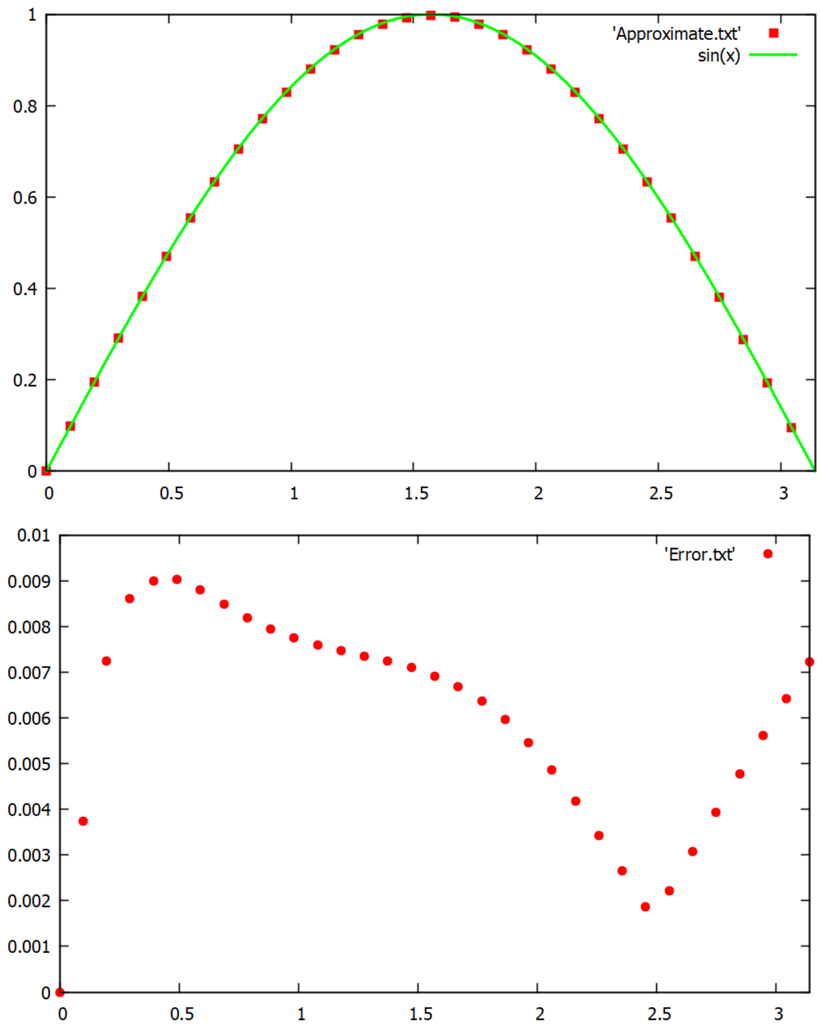

Figure 2.1: Numerical solution of example 2.1 for ($N = 32$) with the exact solution, and error obtained.

N	$\max\limits_{0\leqslant i\leqslant N}\left(\lvert f_i - f(x_i)\rvert + \lvert h_i - \mathscr{D}^{(\alpha)} f(x_i)\rvert\right)$
32	0.01
64	2.5E-3
128	6E-4
256	1.4E-4
512	4E-5
1024	8E-6
2048	2E-6
4096	1E-6

Table 2.1: The numerical error obtained for the example 2.1

Example 2.2. *Let's take the following nonlinear Volterra integro-differential nonlinear equation as an example:*

$$\forall x \in [0, \pi]; \quad f(x) = g(x) + \int_0^x K\left(x, s, f(s), \mathscr{D}^{(\alpha)} f(s)\right) \mathrm{d}s, \tag{2.17}$$

where $\alpha = 0.5$ *and the kernel*

$$k(x, s, u, v) = \frac{2x + e^{-2s} + 4\sinh^2(s) + 2(e^s - 1)^2}{2x + e^{-s} + 2u^2 + v^2}$$

If we take: $g(x) = e^x - 1 - x$*, we get:* $f(x) = e^x - 1$.

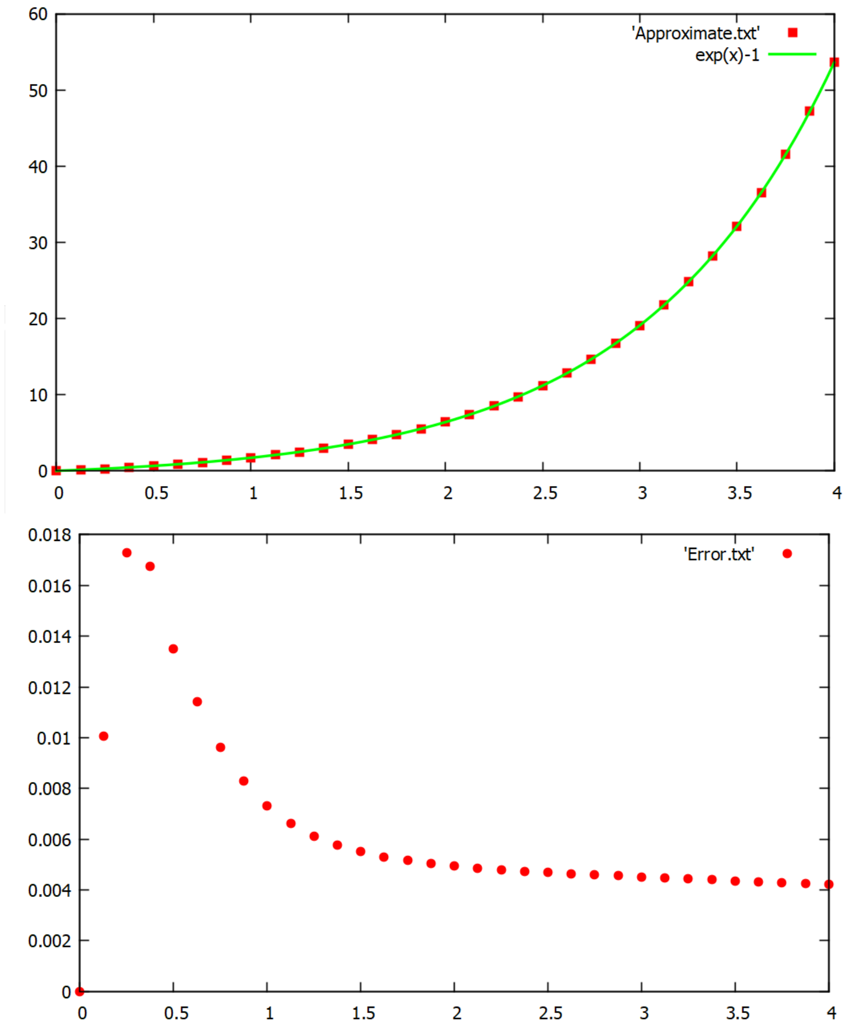

Figure 2.2: Numerical solution of example 2.2 for $(N = 32)$ with the exact solution, and error obtained.

Example 2.3. *Let's take the following nonlinear Volterra integro-differential nonlinear equation*

N	$\max\limits_{0\leqslant i\leqslant N}\left(\lvert f_i - f(x_i)\rvert + \lvert h_i - \mathscr{D}^{(\alpha)} f(x_i)\rvert\right)$
32	1.8E-2
64	4.5E-3
128	1.2E-3
256	3E-4
512	7E-5
1024	2E-5
2048	4E-6
4096	1E-6

Table 2.2: Numerical error obtained for example 2.2

as an example:

$$\forall x \in [0,\pi]; \quad f(x) = t^2 + \int_0^x \frac{e^{-2s} - 2e^{-s} - 4s + 4se^{-s} + 8s^2 + 8s^3 + 4s^4 + 1 + x}{x + 2e^{-s} + (2f(s))^2 + \left(\mathscr{D}^{(\alpha)} f(s)/2\right)^2}\, \mathrm{d}s, \tag{2.18}$$

where $\alpha = 0.5$, *we get:* $f(x) = t^2 + t$.

N	$\max\limits_{0\leqslant i\leqslant N}\left(\lvert f_i - f(x_i)\rvert + \lvert h_i - \mathscr{D}^{(\alpha)} f(x_i)\rvert\right)$
32	4.5E-3
64	1.2E-3
128	1.2E-3
256	2.5E-4
512	2E-5
1024	4E-6
2048	1E-6

Table 2.3: Numerical error obtained for example 2.3

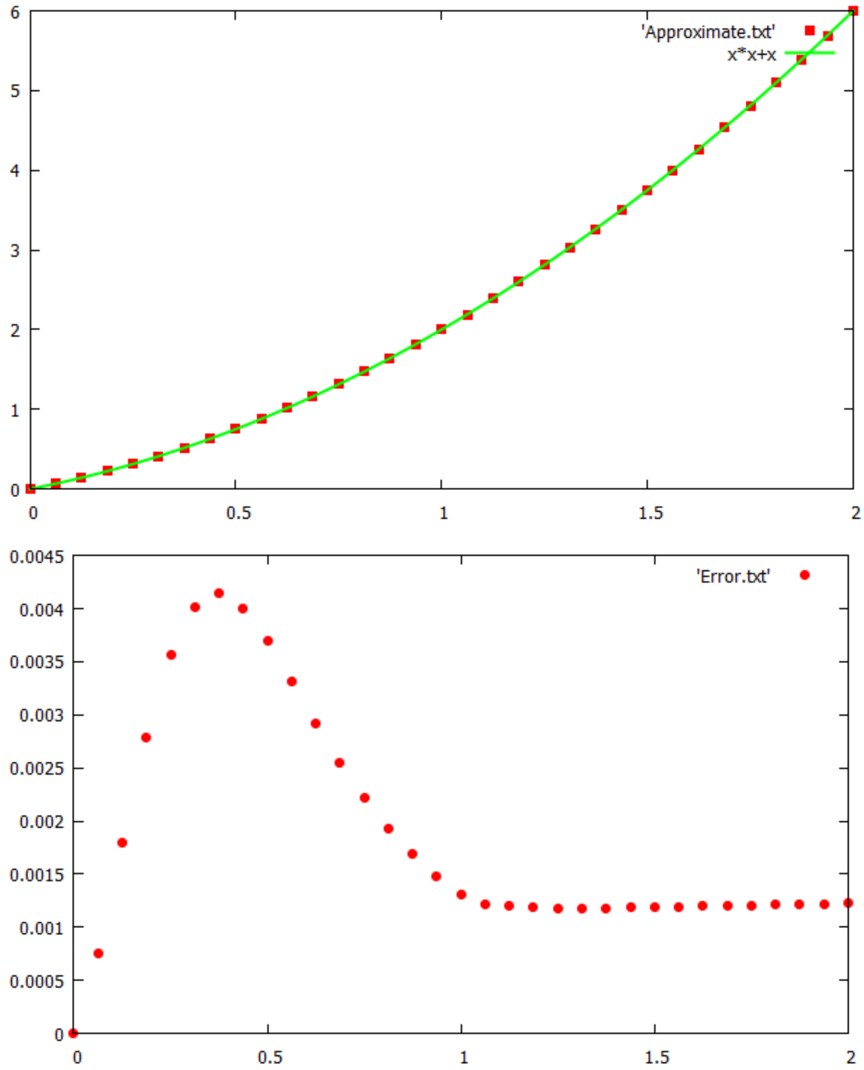

Figure 2.3: Numerical solution of example 2.3 for ($N = 32$) with the exact solution, and error obtained.

In this chapter we prove the existence and uniqueness of nonlinear integro-differential equation of Volterra with the use of the minimum of hypotheses that ensure this, and then we solved the problem numerically using numerical methods and programming the problem using the Matlab which is characterised by slowness, that needs the division of the program into two parts. A part using the Matlab and the second part using C++ which gave us satisfying results and in a short time.

Chapter 3

A New Definition of Fractional Integral and its Properties with a Theoretical Application

Summary

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In this chapter, we introduce a new definition of fractional integral as an inverse of the conformable fractional derivative of Caputo. By using these definitions, we obtain the basic properties of those fractional derivative and its fractional integral. Finally, we solve a class of fractional boundary value problems as a theoretical application, and we use Matlab to solve this class of fractional boundary value problems.

The usual integral and derivative are (to say the least) a staple for the new technology, essential as a means of understanding and working with natural and artificial systems. Recently, many authors have participated in the development of the fractional calculus (differentiation and integration of arbitrary order). The applications of fractional calculus often appear in the fields such as generalized voltage dividers[50], electric conductance of biological systems[14], capacitor theory[36], engineering[10], electrode-electrolyte interface models[27], feedback amplifiers, medical[14, 39], fractional order models of neurons[4], analysis of special functions[24], and fitting experimental data[5].

In this chapter, we launch a new fractional Integral operator, we investigate some properties of the new fractional Integral operator. As concerns the properties of the fractional derivative operator, we are interested in recalling some extended functions

As a theoretical application a class of fractional boundary value problems is solved. To the best of our knowledge, this is the first work that solve problem with the new concept of fractional derivative recently introduced by Caputo and Fabrizio in paper [6], which has many properties mentioned in article [27].

Lemma 3.1. *Let $\alpha \in]0,1[$ and $f \in \mathcal{C}^1[a,b]$ a non constant function. The equations*

$$\mathscr{D}^{(\alpha)}\, u(x) = f(x), \quad in\ [a,b] \tag{3.1}$$

$$u'(x) = \frac{1}{M(\alpha)}\left[(1-\alpha)f'(x) + \alpha f(x)\right], \quad in\ [a,b] \tag{3.2}$$

have the same solution.

Proof. If u is a solution of eq. (3.1), then

$$\frac{M(\alpha)}{1-\alpha}\int_a^x u'(s)\exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right] \mathrm{d}s = f(x)$$

and the Leibniz integral rule gives the formula

$$\frac{M(\alpha)}{1-\alpha}u'(x) - \frac{\alpha}{1-\alpha}\frac{M(\alpha)}{1-\alpha}\int_a^x u'(s)\exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right] \mathrm{d}s = f'(x)$$

$$\frac{M(\alpha)}{1-\alpha}u'(x) = \frac{\alpha}{1-\alpha}\frac{M(\alpha)}{1-\alpha}\int_a^x u'(s)\exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right] \mathrm{d}s + f'(x),$$

consequently

$$\frac{M(\alpha)}{1-\alpha}u'(x) = \frac{\alpha}{1-\alpha}f(x) + f'(x).$$

So that

$$u'(x) = \frac{1}{M(\alpha)}\left(\alpha f(x) + (1-\alpha)f'(x)\right)$$

□

Example 3.1. *Let* $\alpha = 0.5$, $M(\alpha) = 1$, *and* $f(x) = x$, *the solution of the equation*

$$\mathscr{D}^{(\alpha)}\, u(x) = x$$

is the same solution of the equation

$$\begin{aligned} u'(x) &= \frac{1}{M(\alpha)}\left(\alpha f(x) + (1-\alpha) f'(x)\right) \\ &= \frac{1}{2}x + \frac{1}{2}. \end{aligned}$$

Therefore

$$u(x) = \frac{1}{4}x^2 + \frac{1}{2}x + c;\; c \in \mathbb{R}.$$

Lemma 3.2. *Let* $\gamma \in]1,2]$ *and* $f \in \mathcal{C}^1[a,b]$ *a non constant function, then : The equations*

$$\mathscr{D}^{(\gamma)}\, u(x) = f(x), \quad \textit{in } [a,b] \tag{3.3}$$

$$u''(x) = \frac{1}{M(\gamma-1)}[(2-\gamma)f'(x) + (\gamma-1)f(x)], \quad \textit{in } [a,b] \tag{3.4}$$

have the same solution.

Proof. Let u be a solution of eq. (3.3), then, it satisfies :

$$\frac{M(\gamma-1)}{2-\gamma}\int_a^x u''(\tau)\exp\left[-\frac{(\gamma-1)(x-\tau)}{2-\gamma}\right]d\tau = f(x)$$

and the Leibniz integral rule gives the formula

$$\frac{M(\gamma-1)}{2-\gamma}u''(x) - \frac{(\gamma-1)}{2-\gamma}\frac{M(\gamma-1)}{2-\gamma}\int_a^x u''(\tau)\exp\left[-\frac{(\gamma-1)(x-\tau)}{2-\gamma}\right]d\tau = f'(x)$$

$$\frac{M(\gamma-1)}{2-\gamma}u''(x) = \frac{(\gamma-1)}{2-\gamma}\mathscr{D}^{(\gamma)}\, u(x) + f'(x)$$

So that

$$u''(x) = \frac{1}{M(\gamma-1)}\left((2-\gamma)f'(x) + (\gamma-1)f(x)\right)$$

□

Example 3.2. *For $\gamma = 1.5$ find the Green's function for the following boundary value problem*

$$\begin{cases} \mathscr{D}^{(\gamma)}\, u(x) = x, \ 0 \le x \le 1 \\ u(0) = 0, u(1) = 0 \end{cases} \tag{3.5}$$

Solution: From Lemma 3.2 we have:

$$\mathscr{D}^{(\gamma)}\, u(x) = x, \ 0 \le x \le 1 \Leftrightarrow u''(x) = \frac{1}{2}(x+1), \ 0 \le x \le 1$$

We get a boundary value problem equivalent to boundary value problem (3.5)

$$\begin{cases} u''(x) = \frac{1}{2}(x+1), \ 0 \le x \le 1 \\ u(0) = 0, u(1) = 0 \end{cases} \tag{3.6}$$

The homogeneous equation $u'' = 0$ has the fundamental solutions $u_1(x) = x$ and $u_2(x) = x - 1$ to satisfy the boundary conditions $B_0[u] = u(0) = 0$ and $B_1[u] = u(1) = 0$ respectively.

Then $W(u_1, u_2)(x) = 1$ and therefore

$$G(x,s) = \begin{cases} s(x-1) & \text{if} \quad 0 \le s \le x \\ x(s-1) & \text{if} \quad x \le s \le 1 \end{cases}$$

Thus solve eq. (3.6) with

$$\begin{aligned} u(x) &= \int_0^x sf(s)ds(x-1) + \int_x^1 (s-1)f(s)dsx \\ &= (x-1)\int_0^x s \cdot \frac{1}{2}(s+1)ds + x\int_x^1 (s-1)\cdot\frac{1}{2}(s+1)ds \\ &= \frac{1}{2}(x-1)\int_0^x (s^2+s)ds + \frac{1}{2}x\int_x^1 (s^2-1)ds \\ &= \frac{1}{2}(x-1)\left(\frac{1}{3}s^3 + \frac{1}{2}s^2\right) + \frac{1}{2}x\left(\frac{1}{3} - 1 - \frac{1}{3}x^3 + x\right) \\ &= \frac{1}{6}x^4 - \frac{1}{6}x^3 + \frac{1}{4}x^3 - \frac{1}{4}x^2 - \frac{1}{3}x - \frac{1}{6}x^4 + \frac{1}{2}x^2 \\ &= \frac{1}{12}x^3 + \frac{1}{4}x^2 - \frac{1}{3}x. \end{aligned}$$

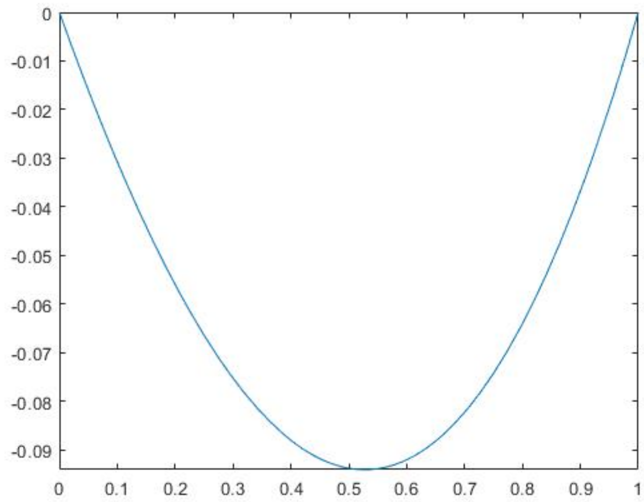


Figure 3.1: Eexact solution of boundary value problem (3.5)

□

3.1 The First Definition of New Fractional Integral

In this section, we present our new definition as a theorem

Theorem 3.1. *Let $n \geq 1$, $\alpha \in [0,1]$. The formula :*

$$I_a^{n+\alpha} f(t) = \frac{1}{M(\alpha) \cdot n!} \int_a^t (t-s)^n [\alpha f(s) + (1-\alpha) f'(s)] \, \mathrm{d}s$$

where $f \in \mathcal{C}^1[a,b]$, and $M(\alpha)$, is a normalization function such that

$M(0) = M(1) = 1$ is a new fractional integral of order $(n+\alpha)$, and it's as an inverse of the conformable fractional derivative of Caputo of order $(n+\alpha)$.

Proof. From Definitions 1.2 and 1.4, we obtain

$$\begin{aligned}
\mathscr{D}^{(\alpha+n)} f(t) &= \mathscr{D}_t^{(\alpha)} \left(\mathscr{D}^{(n)} f(t) \right) \\
&= \frac{M(\alpha)}{1-\alpha} \int_a^t f^{(n+1)}(s) \exp\left[-\frac{\alpha(t-s)}{1-\alpha} \right] \mathrm{d}s,
\end{aligned}$$

using the following Leibniz integral rule

$$\frac{d}{dt} \left(\int_{a(t)}^{b(t)} g(t,s) ds \right) = g(t, b(t)) \cdot \frac{d}{dt} b(t) - g(t, a(t)) \cdot \frac{d}{dt} a(t) + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} g(t,s) ds$$

and considering $a(t) = a$, $b(t) = t$, and $g(t,s) = f^{(n+1)}(s)\exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right]$,
$\Rightarrow \frac{d}{dt}b(t) = 1$, $\frac{d}{dt}a(t) = 0$, and $\frac{\partial}{\partial t}g(t,s) = -\frac{\alpha}{1-\alpha}f^{(n+1)}(s)\exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right]$ we obtain

$$\frac{d}{dt}\left(\mathscr{D}^{(\alpha+n)}f(t)\right) = \frac{M(\alpha)}{1-\alpha}f^{(n+1)}(t) - \frac{\alpha}{1-\alpha}\underbrace{\left(\frac{M(\alpha)}{1-\alpha}\int_a^t f^{(n+1)}(s)\exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right]\mathrm{d}s\right)}_{=\mathscr{D}^{(\alpha+n)}f(t)}$$

So that

$$\frac{d}{dt}\left(\mathscr{D}^{(\alpha+n)}f(t)\right) = \frac{M(\alpha)}{1-\alpha}f^{(n+1)}(t) - \frac{\alpha}{1-\alpha}\mathscr{D}^{(\alpha+n)}f(t).$$

We then obtain

$$f^{(n+1)}(t) = \frac{1}{M(\alpha)}\left[(1-\alpha)\frac{d}{dt}\left(\mathscr{D}^{(\alpha+n)}f(t)\right) + \alpha\mathscr{D}^{(\alpha+n)}f(t)\right].$$

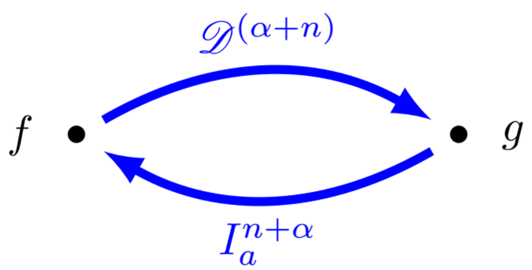
Now, using Cauchy formula for evaluating the $(n+1)^{th}$ integration of the function $f^{(n+1)}(t)$

$$\begin{aligned} f(t) &= \int_0^t\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_n} f^{(n+1)}(t_n)\,dt_n\cdots dt_1 dt \\ &= \frac{1}{n!}\int_0^t (t-s)^n f^{(n+1)}(t)dt \\ &= \frac{1}{n!M(\alpha)}\int_a^t (t-s)^n\left[(1-\alpha)\frac{d}{dt}\left(\mathscr{D}^{(\alpha+n)}f(s)\right) + \alpha\mathscr{D}^{(\alpha+n)}f(s)\right]\mathrm{d}s, \end{aligned}$$

If we consider that $g(x) = \mathscr{D}^{(\alpha+n)}f(x)$

$$\Rightarrow f(t) = \frac{1}{M(\alpha)\cdot n!}\int_a^t (t-s)^n[\alpha g(s) + (1-\alpha)g'(s)]\,\mathrm{d}s$$

Finally, we obtained an expression of the inverse of the conformable fractional derivative of Caputo as shown in the following figure:



$$f(t) = I_a^{n+\alpha}g(t) = \frac{1}{M(\alpha)\cdot n!}\int_a^t (t-s)^n[\alpha g(s) + (1-\alpha)g'(s)]\,\mathrm{d}s$$

The proof is complete. $\square$

Lemma 3.3. *Let $\gamma \in (1,2)$. If we assume $u \in C^1(0,1)$, then the fractional deferential equation*

$$\mathscr{D}^{(\gamma)}\, u(x) = 0$$

has $u(x) = c_1 x + c_2;\ c_1, c_2 \in \mathbb{R}$ as unique solutions.

Proof. In order to prove the previous Lemma, it is sufficient to prove the following equivalence:

$$\forall x \in [a,b],\ \int_a^x f(s)\,\mathrm{d}s = 0 \Leftrightarrow \forall x \in [a,b], f(x) = 0 \tag{3.7}$$

Suppose F is an anti-derivative of f, with f continuous on $[a,b]$. We have that

$$\begin{aligned} F(x) - F(a) = \int_a^x f(s)\,\mathrm{d}s = 0 &\Leftrightarrow \forall x \in [a,b], F(x) = F(a), \\ &\Leftrightarrow \forall x \in [a,b], F'(x) = 0, \\ &\Leftrightarrow \forall x \in [a,b], f(x) = 0, \end{aligned}$$

Now, using (3.7) to prove Lemma 3.3, $\forall x \in [0,1]$ we have that

$$\begin{aligned} \mathscr{D}^{(\gamma)}\, u(x) = 0 \Leftrightarrow \frac{M(\gamma-1)}{2-\gamma}\int_a^x u''(\tau)\exp\left[-\frac{(\gamma-1)(x-\tau)}{2-\gamma}\right] d\tau = 0, \\ \Leftrightarrow \int_a^x u''(\tau)\exp\left[-\frac{(\gamma-1)(x-\tau)}{2-\gamma}\right] d\tau = 0, \\ \Leftrightarrow \forall x \in [a,b], u''(x) = 0 \\ \Leftrightarrow \forall x \in [a,b], u(x) = c_1 x + c_2;\ c_1, c_2 \in \mathbb{R}. \end{aligned}$$

$\square$

Lemma 3.4. *Assume that $u \in \mathcal{C}^1(0,1)$ with a fractional derivative of order $\gamma \in (1,2)$, that belongs to $C(0,1)$. Then, those statements holds*

0. *if $u(a) = 0$, then $\mathscr{D}^{(\gamma)}\, I_a^{\gamma} u(x) = u(x)$.*

0. *$I_a^{\gamma}\mathscr{D}^{(\gamma)}\, u(x) = u(x) + ax + b,\ a, b \in \mathbb{R}$.*

Proof. 0. Let $\gamma \in]1,2[$, we have that

$$I_a^{\gamma} u(x) = \frac{1}{M(\gamma-1)}\int_a^x (x-s)[(\gamma-1)u(s) + (2-\gamma)u'(s)]\,\mathrm{d}s$$

$$\mathscr{D}^{(\gamma)}u(x)=\frac{M(\gamma-1)}{2-\gamma}\int_a^x u''(\tau)\exp\left[-\frac{(\gamma-1)(x-\tau)}{2-\gamma}\right]d\tau.$$

Before outlining the method needed, we wish to recall the useful transformation formula

$$\int_0^x\int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_{n-1}} f(x_n)\,dx_n\cdots dx_1=\frac{1}{(n-1)!}\int_0^x(x-t)^{n-1}f(t)dt \tag{3.8}$$

For practical consideration

$$\int_0^x\int_0^x f(t)dtdt=\int_0^x(x-t)f(t)dt \tag{3.9}$$

Using the formula (3.9) we obtain

$$\begin{aligned}
\mathscr{D}^{(\gamma)}\left(I_a^{\gamma}u(x)\right)&=\frac{\gamma-1}{2-\gamma}\int_a^x\frac{d^2}{d\tau^2}\left[\int_a^{\tau}(\tau-s)u(s)\,\mathrm{d}s\right]\exp\left[-\frac{(\gamma-1)(x-\tau)}{2-\gamma}\right]d\tau\\
&+\int_a^x\frac{d^2}{d\tau^2}\left[\int_a^{\tau}(\tau-s)u'(s)\,\mathrm{d}s\right]\exp\left[-\frac{(\gamma-1)(x-\tau)}{2-\gamma}\right]d\tau\\
&=\frac{\gamma-1}{2-\gamma}\int_a^x u(\tau)\exp\left[-\frac{(\gamma-1)(x-\tau)}{2-\gamma}\right]d\tau\\
&+\int_a^x u'(\tau)\exp\left[-\frac{(\gamma-1)(x-\tau)}{2-\gamma}\right]d\tau\\
&=\frac{\gamma-1}{2-\gamma}\int_a^x u(\tau)\exp\left[-\frac{(\gamma-1)(x-\tau)}{2-\gamma}\right]d\tau\\
&+u(\tau)\exp\left[-\frac{(\gamma-1)(x-\tau)}{2-\gamma}\right]\Bigg|_a^x\\
&-\frac{\gamma-1}{2-\gamma}\int_a^x u(\tau)\exp\left[-\frac{(\gamma-1)(x-\tau)}{2-\gamma}\right]d\tau\\
&=u(x)-u(a)\exp\left[-\frac{(\gamma-1)(x-a)}{2-\gamma}\right]\\
&=u(x).
\end{aligned}$$

0. Firstly, we have that

$$\mathscr{D}^{(\gamma)}u(x)=\frac{M(\gamma-1)}{2-\gamma}\int_a^x u''(\tau)\exp\left[-\frac{(\gamma-1)(x-\tau)}{2-\gamma}\right]d\tau.$$

We derive both sides of the equation, we get that

$$\begin{aligned}\frac{d}{dx}\left(\mathscr{D}^{(\gamma)}\,u(x)\right) &= \frac{M(\gamma-1)}{2-\gamma}u''(x) \\ &\quad - \frac{\gamma-1}{2-\gamma}\frac{M(\gamma-1)}{2-\gamma}\int_a^x u''(\tau)\exp\left[-\frac{(\gamma-1)(x-\tau)}{2-\gamma}\right]d\tau \\ &= \frac{M(\gamma-1)}{2-\gamma}u''(x) - \frac{\gamma-1}{2-\gamma}\mathscr{D}^{(\gamma)}\,u(x)\end{aligned}$$

using the last formula

$$\begin{aligned}I_a^\gamma\left(\mathscr{D}^{(\gamma)}\,u(x)\right) &= \frac{1}{M(\gamma-1)}\int_a^x (x-s)\left[(\gamma-1)\mathscr{D}^{(\gamma)}\,u(s) + (2-\gamma)\frac{d}{ds}\left(\mathscr{D}^{(\gamma)}\,u(s)\right)\right]\mathrm{d}s \\ &= \frac{\gamma-1}{M(\gamma-1)}\int_a^x (x-s)\mathscr{D}^{(\gamma)}\,u(s)\,\mathrm{d}s + \int_a^x (x-s)u''(s)\,\mathrm{d}s \\ &\quad - \frac{\gamma-1}{M(\gamma-1)}\int_a^x (x-s)\mathscr{D}^{(\gamma)}\,u(s)\,\mathrm{d}s \\ &= \int_a^x (x-s)u''(s)\,\mathrm{d}s \\ &= (x-a)u'(a) + \int_a^x u'(s)\,\mathrm{d}s \\ &= u(x) + (x-a)u'(a) + c \\ &= u(x) + c_1x + c_2.\end{aligned}$$

□

3.2 Theoretical Application

On the other hand, we will study the following fractional boundary value problem:

$$\begin{cases}\mathscr{D}^{(\gamma)}\,u(x) = f(x), \;\; 0 \le x \le 1 \\ u(0) = 0, u(1) = 0\end{cases} \tag{3.10}$$

In the following, we present the Green's function of fractional differential equation boundary value problem.

Lemma 3.5. *Consider a non constant function $f \in \mathcal{C}^1(0,1)$, u is a solution of problem (3.10) if, and only if, it satisfies the following integral equation*

$$u(x) = \int_0^1 G_1(x,s)f(s)\,\mathrm{d}s + \int_0^1 G_2(x,s)f'(s)\,\mathrm{d}s \tag{3.11}$$

where

$$G_1(x,s)=\frac{\gamma-1}{M(\gamma-1)}\begin{cases}s(x-1), & 0\le x\le s\le 1\\ x(s-1), & 0\le s\le x\le 1.\end{cases}$$

and

$$G_2(x,s)=\frac{2-\gamma}{M(\gamma-1)}\begin{cases}s(x-1), & 0\le x\le s\le 1\\ x(s-1), & 0\le s\le x\le 1.\end{cases}$$

Proof. We may apply Lemma 3.4 to reduce first equation of problem (3.10) to an equivalent integral equation

$$I_a^\gamma \mathscr{D}^{(\gamma)}u(x)=I_a^\gamma f(x)+ax+b,$$

for some $a,b\in\mathbb{R}$. Consequently, the general solution of problem (3.10) is

$$u(x)=\frac{1}{M(\gamma-1)}\int_0^x (x-s)[(\gamma-1)f(s)+(2-\gamma)f'(s)]\,\mathrm{d}s+ax+b$$

By second equation of problem (3.10), we get $b=0$ and

$$a=-\frac{1}{M(\gamma-1)}\int_0^1 (1-s)[(\gamma-1)f(s)+(2-\gamma)f'(s)]\,\mathrm{d}s$$

Therefore, the unique solution of problem (3.10)

$$\begin{aligned}u(x)&=\frac{1}{M(\gamma-1)}\int_0^x (x-s)[(\gamma-1)f(s)+(2-\gamma)f'(s)]\,\mathrm{d}s\\&\quad-\frac{1}{M(\gamma-1)}\int_0^1 x(1-s)[(\gamma-1)f(s)+(2-\gamma)f'(s)]\,\mathrm{d}s\\&=\frac{1}{M(\gamma-1)}\int_0^x [(x-s)-x(1-s)][(\gamma-1)f(s)+(2-\gamma)f'(s)]\,\mathrm{d}s\\&\quad-\frac{1}{M(\gamma-1)}\int_x^1 x(1-s)[(\gamma-1)f(s)+(2-\gamma)f'(s)]\,\mathrm{d}s\\&=\frac{1}{M(\gamma-1)}\int_0^x s(x-1)[(\gamma-1)f(s)+(2-\gamma)f'(s)]\,\mathrm{d}s\\&\quad-\frac{1}{M(\gamma-1)}\int_x^1 x(1-s)[(\gamma-1)f(s)+(2-\gamma)f'(s)]\,\mathrm{d}s\\&=\int_0^1 G_1(x,s)f(s)\,\mathrm{d}s+\int_0^1 G_2(x,s)f'(s)\,\mathrm{d}s.\end{aligned}$$

The proof is complete. □

Example 3.3. *For $\gamma=1.75$ and $M(\alpha)=1-0.2\sin(2\pi\alpha)$, let find the Green's function for the* *following boundary value problem*

$$\begin{cases} \mathscr{D}^{(\gamma)}\, u(x) = x^2 + x, \;\; 0 \le x \le 1 \\ u(0) = 0, u(1) = 0 \end{cases} \tag{3.12}$$

Solution: From Lemma 3.5 we have:

$$G_1(x,s) = \frac{15}{16} \begin{cases} s(x-1), \;\; 0 \le x \le s \le 1 \\ x(s-1), \;\; 0 \le s \le x \le 1. \end{cases}$$

and

$$G_2(x,s) = \frac{5}{16} \begin{cases} s(x-1), \;\; 0 \le x \le s \le 1 \\ x(s-1), \;\; 0 \le s \le x \le 1. \end{cases}$$

Thus solve eq. (3.12) with

$$\begin{aligned} u(x) &= \int_0^1 G_1(x,s) f(s)\,\mathrm{d}s + \int_0^1 G_2(x,s) f'(s)\,\mathrm{d}s \\ &= \frac{15}{16}(x-1)\int_0^x s(s^2+s)\,\mathrm{d}s + \frac{15}{16}x\int_x^1 (s-1)(s^2+s)\,\mathrm{d}s \\ &\quad + \frac{5}{16}(x-1)\int_0^x s(2s+1)\,\mathrm{d}s + \frac{5}{16}x\int_x^1 (s-1)(2s+1)\,\mathrm{d}s \\ &= \frac{5}{64}x^4 + \frac{25}{96}x^3 + \frac{5}{32}x^2 - \frac{95}{192}x. \end{aligned}$$

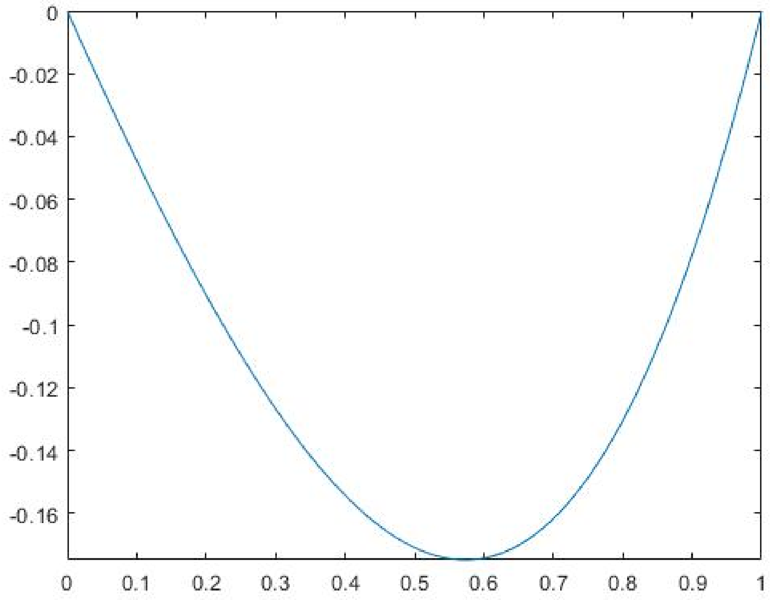


Figure 3.2: Eexact solution of boundary value problem (3.12)

□

Example 3.4. *For* $\gamma = 1.75$, *let find the Green's function for the following boundary value problem*

$$\begin{cases} \mathscr{D}^{(\gamma)}\, u(x) = x^2 + x, \;\; 0 \le x \le 1 \\ u(0) = 0, u(1) = 0 \end{cases} \tag{3.13}$$

Solution: From Lemma 3.5 we have:

$$G_1(x,s) = \frac{15}{16} \begin{cases} s(x-1), \;\; 0 \le x \le s \le 1 \\ x(s-1), \;\; 0 \le s \le x \le 1. \end{cases}$$

and

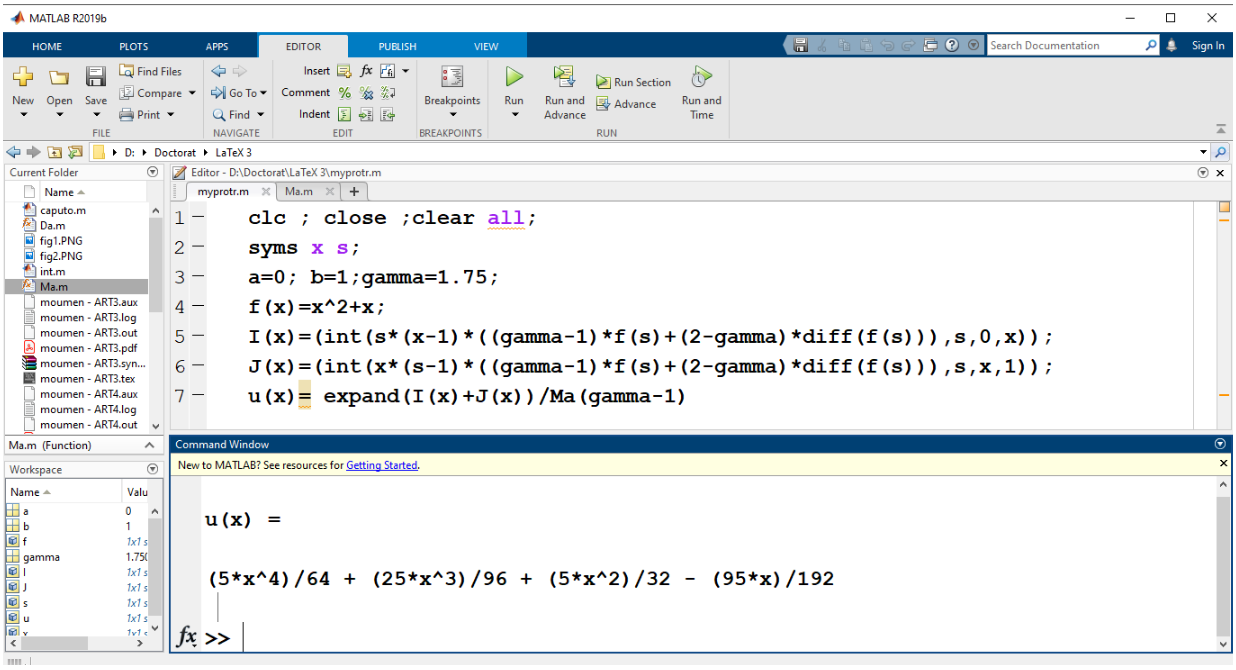
$$G_2(x,s) = \frac{5}{16} \begin{cases} s(x-1), \;\; 0 \le x \le s \le 1 \\ x(s-1), \;\; 0 \le s \le x \le 1. \end{cases}$$

Thus solve eq. (3.13) with

$$
\begin{aligned}
u(x) &= \int_0^1 G_1(x,s)f(s)\,\mathrm{d}s + \int_0^1 G_2(x,s)f'(s)\,\mathrm{d}s \\
&= \frac{15}{16}(x-1)\int_0^x s(s^2+s)\,\mathrm{d}s + \frac{15}{16}x\int_x^1 (s-1)(s^2+s)\,\mathrm{d}s \\
&\quad + \frac{5}{16}(x-1)\int_0^x s(2s+1)\,\mathrm{d}s + \frac{5}{16}x\int_x^1 (s-1)(2s+1)\,\mathrm{d}s \\
&= \frac{5}{64}x^4 + \frac{25}{96}x^3 + \frac{5}{32}x^2 - \frac{95}{192}x.
\end{aligned}
$$

Finally, we program the problem (3.13) using Matlab to get the solution $u(x)$, as follows:

```
% The goal of this program is calculate
%         the new Caputo derivative
 clc ; close ;clear all;
 syms x s;
 a=0; b=1; gamma=1.75;
 f(x)=x^2+1;
 I(x)=(int(s*(x-1)*((gamma-1)*f(s)
        +(2-gamma)*diff(f(s))),s,0,x));
 J(x)=(int(x*(s-1)*((gamma-1)*f(s)
        +(2-gamma)*diff(f(s))),s,x,1));
 u(x)= expand(I(x)+J(x))/Ma(gamma-1)
```



□

3.3 The Second Definition of New Fractional Integral

In this section, we introduce a new definition of fractional integral as a theorem:

Theorem 3.2. *Let $n \geq 1$, $\alpha \in [0,1]$, and $f \in \mathcal{C}^1[a,b]$. The formula :*

$$I_a^{n+\alpha} f(t) = \frac{1}{M(\alpha) \cdot n!} \int_a^t (t-s)^{n-1} [\alpha(t-s) + n(1-\alpha)] f(s)\, \mathrm{d}s$$

where $f \in \mathcal{C}^1[a,b]$, and $M(\alpha)$, is a normalization function such that $M(0) = M(1) = 1$ is a new fractional integral of order $(n+\alpha)$.

Proof. From Definitions 1.2 and 1.4, we obtain

$$\mathscr{D}^{(\alpha+n)} f(x) = \frac{M(\alpha)}{1-\alpha} \int_a^t f^{(n+1)}(s) \exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right] \mathrm{d}s$$

and Leibniz integral rule gives the formula

$$\frac{d}{dt}\left(\mathscr{D}^{(\alpha+n)} f(t)\right) = \frac{M(\alpha)}{1-\alpha} f^{(n+1)}(t) - \frac{\alpha}{1-\alpha} \frac{M(\alpha)}{1-\alpha} \int_a^t f^{(n+1)}(s) \exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right] \mathrm{d}s$$

So that

$$\frac{d}{dt}\left(\mathscr{D}^{(\alpha+n)}f(t)\right)=\frac{M(\alpha)}{1-\alpha}f^{(n+1)}(t)-\frac{\alpha}{1-\alpha}\mathscr{D}^{(\alpha+n)}f(t).$$

We then obtain

$$f^{(n+1)}(t)=\frac{1}{M(\alpha)}\left[(1-\alpha)\frac{d}{dt}\left(\mathscr{D}^{(\alpha+n)}f(t)\right)+\alpha\mathscr{D}^{(\alpha+n)}f(t)\right].$$

Now, we use the Cauchy formula for evaluating the $(n+1)^{th}$ integration of the function $f^{(n+1)}(t)$

$$f(t)=\frac{1}{n!M(\alpha)}\int_a^t (t-s)^n\left[(1-\alpha)\frac{d}{ds}\left(\mathscr{D}^{(\alpha+n)}f(s)\right)+\alpha\mathscr{D}^{(\alpha+n)}f(s)\right]\,\mathrm{d}s.$$

Finally, we get

$$\begin{aligned}I_a^{n+\alpha}f(t)&=\frac{1}{M(\alpha)\cdot n!}\int_a^t (t-s)^n[\alpha f(s)+(1-\alpha)f'(s)]\,\mathrm{d}s\\&=\frac{1}{M(\alpha)\cdot n!}\int_a^t (t-s)^{n-1}[\alpha(t-s)+n(1-\alpha)]f(s)\,\mathrm{d}s.\end{aligned}$$

□

Lemma 3.6. *Let $\gamma\in(n,n+1), n=[\gamma]\geqslant 0$. Assume that $u\in\mathcal{C}^n[a,b]$, then those statements holds:*

0. $I_a^\gamma\left(\mathscr{D}^{(\gamma)}u(t)\right)=u(t)+\sum_{i=0}^{n}a_it^i,\ a_i\in\mathbb{R}\ i=0,1,\ldots,n.$
0. *if $u(a)=0$, then $\mathscr{D}^{(\gamma)}\left(I_a^\gamma u(t)\right)=u(t)$.*

Proof. 0. Let $\gamma\in]n,n+1[$, it can be written in the form: $\gamma=n+\alpha$ where $\alpha\in]0,1[$, and $n=[\gamma]$, we have

$$\begin{aligned}I_a^\gamma\left(\mathscr{D}^{(\gamma)}u(t)\right)&=\frac{1}{M(\alpha)\cdot n!}\int_a^t (t-s)^{n-1}[\alpha(t-s)+n(1-\alpha)]\mathscr{D}^{(\gamma)}u(s)\,\mathrm{d}s.\\&=\frac{\alpha}{M(\alpha)\cdot n!}\int_a^t (t-s)^n\mathscr{D}^{(\gamma)}u(s)\,\mathrm{d}s+\frac{(1-\alpha)}{M(\alpha)\cdot n!}\int_a^t (t-s)^n\frac{d}{ds}\left(\mathscr{D}^{(\gamma)}u(s)\right)\,\mathrm{d}s\\&=\frac{\alpha}{M(\alpha)\cdot n!}\int_a^t (t-s)^n\mathscr{D}^{(\gamma)}u(s)\,\mathrm{d}s\\&+\frac{(1-\alpha)}{M(\alpha)\cdot n!}\int_a^t (t-s)^n\left(\frac{M(\alpha)}{1-\alpha}u^{(n+1)}(s)-\frac{\alpha}{1-\alpha}\mathscr{D}^{(\gamma)}u(s)\right)\,\mathrm{d}s\\&=\frac{1}{n!}\int_a^t (t-s)^n u^{(n+1)}(s)\,\mathrm{d}s\\&=u(t)+\sum_{i=0}^{n}a_it^i,\ a_i\in\mathbb{R}\ i=0,1,\ldots,n\end{aligned}$$

0. Let $\gamma \in]n, n+1[$, it can be written in the form: $\gamma = n + \alpha$ where $\alpha \in]0,1[$, and $n = [\gamma]$, we have

$$
\begin{aligned}
\mathscr{D}^{(\gamma)}\left(I_a^\gamma u(t)\right) &= \frac{\alpha}{(1-\alpha)} \int_a^t \frac{d^{(n+1)}}{ds^{(n+1)}} \left[\frac{1}{n!} \int_a^s (s-x)^n u(x) \mathrm{d}x\right] \exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right] \mathrm{d}s \\
&+ \int_a^t \frac{d^{(n+1)}}{ds^{(n+1)}} \left[\frac{1}{n!} \int_a^s (s-x)^n u'(x) \mathrm{d}x\right] \exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right] \mathrm{d}s \\
&= \frac{\alpha}{(1-\alpha)} \int_a^t u(s) \exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right] \mathrm{d}s + \int_a^t u'(s) \exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right] \mathrm{d}s \\
&= \frac{\alpha}{(1-\alpha)} \int_a^t u(s) \exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right] \mathrm{d}s \\
&+ u(s) \exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right]\Bigg|_a^t - \frac{\alpha}{(1-\alpha)} \int_a^t u(s) \exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right] \mathrm{d}s \\
&= u(t) - u(a) \exp\left[-\frac{\alpha(t-a)}{1-\alpha}\right] \\
&= u(t).
\end{aligned}
$$

□

Chapter 4

Analytical and Numerical Study for a Fractional Boundary Value Problem with a conformable fractional derivative of Caputo and its Fractional Integral

Summary

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We study the existence and uniqueness of the solution of a fractional boundary value problem with conformable fractional derivation of the Caputo type, which increases the interest of this study. In order to study this problem we have introduced a new definition of fractional integral as an inverse of the conformable fractional derivative of Caputo, therefore, the proofs are based upon the reduction of the problem to a equivalent linear Volterra-Fredholm integral equations of the second kind, and we have built the minimum conditions to obtain the existence and uniqueness of this solution. The analytical study is followed by a complete numerical study.

Recently, papers have been published that deal with the existence and multiplicity of the solution of nonlinear initial fractional differential equation by the use of techniques of nonlinear analysis, see [36, 26, 39, 54]. However, most of the papers offer the problem using the standard Riemann-Liouville differentiation. However, Our aim is to study the existence and the uniqueness of the solution for a class of fractional boundary value problems. To the best of our knowledge, this is the first work that solves problem with the conformable fractional derivative by Caputo and Fabrizio in paper [6], which has many properties mentioned in the article [27]. The interest for in this new approach is due to the necessity of using a model to describe the behavior of classical viscoelastic materials, electromagnetic systems, thermal media, etc. In fact, the original definition of Caputo's fractional derivative appears to be particularly convenient for those mechanical phenomena, related to damage and with electromagnetic hysteresis, fatigue and plasticity. When these effects are not present it seems more appropriate to use the new fractional derivative [6].

In this chapter, We study the existence and uniqueness of the solution of the fractional differential equation boundary value problem, as follows:

$$\begin{cases} \mathscr{D}^{(\gamma)} u(x) + q(x)u(x) = f(x), \quad 0 \leqslant x \leqslant 1 \\ u(0) = u(1) = 0 \end{cases} \tag{4.1}$$

where $1 < \gamma < 2$ is a real number, q is the potential function, and $f : [0,1] \to \mathbb{R}$ is continuous. and $\mathscr{D}^{(\gamma)}$ is the new fractional derivative, and we introduce a new definition of its fractional integral with some properties, using this fractional integral upon problem (4.1) to obtain an equivalent linear Volterra-Fredholm integral equations of second kind. Finally, by the means of some theorems, the existence and uniqueness of solutions are obtained, and we introduce an algorithm for finding a numerical solution of this problem class.

4.1 Analytic Study

In the following, we suppose the function $M(\alpha) = 1$.

Lemma 4.1. *Given $q \in \mathcal{C}[0,1]$, and $1 < \gamma < 2$, the solution of*

$$\begin{aligned} &\mathscr{D}^{(\gamma)}\, u(x) + q(x)u(x) = f(x), \quad 0 \leqslant x \leqslant 1 \\ &u(0) = u(1) = 0 \end{aligned} \tag{4.2}$$

satisfies the following linear Volterra-Fredholm integral equations of the second kind

$$u(x) + \int_0^x G(x,s)u(s)\,\mathrm{d}s + \int_0^1 K(x,s)u(s)\,\mathrm{d}s = g(x) \tag{4.3}$$

where $g(x) = \displaystyle\int_0^x (x-1)(\alpha s - 1 + \alpha) f(s)\,\mathrm{d}s + \int_0^1 x(\alpha s - 1) f(s)\,\mathrm{d}s,$

$$G(x,s) = (x-1)(\alpha s - 1 + \alpha)q(s) \text{ and } K(x,s) = x(\alpha s - 1)q(s).$$

Proof. We may apply Lemma 3.6 to reduce Eq. (4.2) to an equivalent integral equation

$$\begin{aligned} I_0^\gamma \left(\mathscr{D}^{(\gamma)}\, u(x)\right) &= I_0^\gamma \left(f(x) - q(x)u(x)\right) \\ \Rightarrow u(x) + cx + d &= \int_0^x \left[\alpha(x-s) + (1-\alpha)\right]\left(f(s) - q(s)u(s)\right)\,\mathrm{d}s \end{aligned}$$

Using boundary conditions $u(0) = u(1) = 0$, we have $d = 0$, and

$$c = \int_0^1 (1 - \alpha s)\left(f(s) - q(s)u(s)\right)\,\mathrm{d}s.$$

Therefore, the unique solution of problem (4.2) is

$$\begin{aligned} u(x) &= \int_0^x \left[\alpha(x-s) + (1-\alpha)\right]\left(f(s) - q(s)u(s)\right)\,\mathrm{d}s \\ &+ \int_0^1 x(\alpha s - 1)\left(f(s) - q(s)u(s)\right)\,\mathrm{d}s \\ &= \int_0^x [\alpha(x-s) + (1-\alpha)] f(s)\,\mathrm{d}s + \int_0^1 x(\alpha s - 1) f(s)\,\mathrm{d}s \\ &- \int_0^x [\alpha(x-s) + (1-\alpha)] q(s)u(s)\,\mathrm{d}s - \int_0^1 x(\alpha s - 1) q(s)u(s)\,\mathrm{d}s \\ &= g(x) - \int_0^x G(x,s)u(s)\,\mathrm{d}s - \int_0^1 K(x,s)u(s)\,\mathrm{d}s. \end{aligned}$$

The proof is complete. □

4.2 Existence and uniqueness of the solution

The classical approach to proving the existence and uniqueness of the solution of (4.3) is the Picard method. This consists of the simple iteration for $n = 1, 2, \ldots$

$$u_n(x) = g(x) + \int_0^x G(x,s)u_{n-1}(s)\,\mathrm{d}s + \int_0^1 K(x,s)u_{n-1}(s)\,\mathrm{d}s \tag{4.4}$$

with $u_0(x) = g(x)$. For ease of manipulation, it is convenient to introduce

$$v_n(x) = v_n(x) - v_{n-1}(x), \quad n = 1, 2, \ldots \tag{4.5}$$

with $v_0(x) = g(x)$. On subtracting from (4.4), the same equation with n replaced by $n-1$, an we see that

$$v_n(x) = \int_0^x k(x,s)v_{n-1}(s)ds, \quad n = 1, 2, \ldots$$

Also, from (4.5)

$$u_n(x) = \sum_{i=0}^{n} v_i(x) \tag{4.6}$$

The following theorem uses this iteration to prove the existence and uniqueness of the solution under quite restrictive conditions, namely that $G(x,s),\ K(x,s)$ and $g(x)$ are continuous.

Theorem 4.1. *If $g(x)$ is continuous in $0 \leqslant x \leqslant 1$, and the function $K(x,s)$, $G(x,s)$ are continuous in $0 \leqslant s \leqslant x \leqslant 1$, and $\max\limits_{0 \leqslant s \leqslant x \leqslant 1} |K(x,s)| < 1$, then the integral equation (4.3) possesses a unique continuous solution for $0 \leqslant x \leqslant 1$.*

Proof. Choose M_1, M_2 and M_3 such that

$$|g(x)| \leqslant M_1, \quad 0 \leqslant x \leqslant 1$$

$$|G(x,s)| \leqslant M_2, \quad 0 \leqslant s \leqslant x \leqslant 1$$

$$|K(x,s)| \leqslant M_3, \quad 0 \leqslant s \leqslant x \leqslant 1 \text{ where } M_3 < 1.$$

We first prove by induction that

$$|v_n(x)| \leqslant \frac{M_1(M_2x)^n}{n!} + M_1M_3^n, \quad 0 \leqslant x \leqslant 1, \quad n = 0, 1, \ldots \tag{4.7}$$

this bound makes it obvious that the sequence $u_n(x)$ in (4.6) converges, and we can write

$$u(x) = \sum_{i=0}^{\infty} v_i(x) \tag{4.8}$$

We now show that this $u(x)$ satisfies equation (4.3). The series (4.8) is uniformly convergent since the terms $v_i(x)$ are dominated by $M_1(M_2x)^i/i! + M_1M_3^i$. Consequently, we can interchange the order of integration and summation in the following expression to obtain

$$\begin{aligned}\int_0^x G(x,s)\sum_{i=0}^{\infty} v_i(s)\,\mathrm{d}s + \int_0^1 K(x,s)\sum_{i=0}^{\infty} v_i(s)\,\mathrm{d}s &= \sum_{i=0}^{\infty}\int_0^x G(x,s)v_i(s)\,\mathrm{d}s \\ &\quad+ \sum_{i=0}^{\infty}\int_0^1 K(x,s)v_i(s)\,\mathrm{d}s \\ &= \sum_{i=0}^{\infty} v_{i+1}(s) \\ &= \sum_{i=0}^{\infty} v_i(s) - g(x).\end{aligned}$$

Each of the $v_i(x)$ is clearly continuous. Therefore $u(x)$ is continuous, since it is the limit of a uniformly convergent sequence of continuous functions.

To show that $u(x)$ is the only continuous solution, suppose there exists another continuous solution $\tilde{u}(x)$ of (4.3) Then

$$u(x) - \tilde{u}(x) = \int_0^x G(x,s)(u(s) - \tilde{u}(s))\,\mathrm{d}s + \int_0^1 K(x,s)(u(s) - \tilde{u}(s))\,\mathrm{d}s \tag{4.9}$$

since $f(x)$ and $\tilde{f}(x)$ are both continuous, there exists a constant C such that

$$|u(x) - \tilde{u}(x)| \leqslant C, \quad 0 \leqslant x \leqslant 1$$

Substituting this into (4.9)

$$|u(x) - \tilde{u}(x)| \leqslant C(M_2x + M_3), \quad 0 \leqslant x \leqslant 1$$

and repeating the step shows that

$$|u(x) - \tilde{u}(x)| \leqslant C\left(\frac{(M_2x)^n}{n!} + M_3^n\right), \quad 0 \leqslant x \leqslant 1, \quad \text{for any } n.$$

For a large enough n, the right-hand side is arbitrarily small, therefore, we must have

$$u(x) - \tilde{u}(x), \quad 0 \leqslant x \leqslant 1$$

$\square$

Theorem 4.2. *If $f(x),\ q(x)$ are continuous in $[0,1]$, and $\max\limits_{0\leqslant x\leqslant 1} |q(x)| < 1$, then the fractional boundary value problem (4.1) possesses a unique continuous solution for $0 \leqslant x \leqslant 1$.*

Proof. If $f(x),\ q(x)$ are continuous in $[0,1]$, then it is clear that the following functions

$$\begin{aligned}
g(x) &= \int_0^x (x-1)(\alpha s - 1 + \alpha) f(s)\,\mathrm{d}s + \int_0^1 x(\alpha s - 1) f(s)\,\mathrm{d}s,\\
G(x,s) &= (x-1)(\alpha s - 1 + \alpha) q(s),\\
K(x,s) &= x(\alpha s - 1) q(s),
\end{aligned}$$

are continuous, and $|K(x,s)| = |x(\alpha s - 1)q(s)| \leqslant |q(s)| < 1, \quad \forall x, s \in [0,1]$, which means that integral equation (4.3) possesses a unique continuous solution for $0 \leqslant x \leqslant 1$. Therefore, there is a unique continuous solution of the fractional boundary value problem (4.1) for $0 \leqslant x \leqslant 1$. $\square$

4.3 Numerical study

In this section, we introduce an algorithm for finding a numerical solution of linear Volterra-Fredholm integral equations of the second kind, the methods based upon trapezoidal rule. For all $N \in \mathbb{N}$, Here the interval $[0,1]$ in to N equal sub-intervals, where $h = (b-a)/N$, and $x_i = a + i \cdot h$ for all $i \in \{0 \cdots N\}$.

The formula of the numerical integration is:

$$\int_a^b f(s)\,\mathrm{d}s \approx \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{N} f(x_j) + f(b)\right]$$

we apply this formula in eq. (4.3), and we obtain:

$$\begin{aligned}
g(x_i) = u(x_i) &+ \frac{h}{2}\left[G(x_i,x_0)u(x_0) + 2\sum_{j=1}^{i-1} G(x_i,x_j)u(x_j) + G(x_i,x_i)u(x_i)\right]\\
&+ \frac{h}{2}\left[K(x_i,x_0)u(x_0) + 2\sum_{j=1}^{N-1} K(x_i,x_j)u(x_j) + K(x_i,x_N)u(x_N)\right]
\end{aligned}$$

$$\Rightarrow \forall i = 0, \ldots, N,\ g_i = u_i + \frac{h}{2}\left[G_{i0}u_0 + 2\sum_{j=1}^{i-1} G_{ij}u_j + G_{ii}u_i\right] + \frac{h}{2}\left[K_{i0}u_0 + 2\sum_{j=1}^{N-1} K_{ij}u_j + K_{iN}u_N\right]$$

This leads to

$$\frac{h}{2}(G_{i0} + K_{i0})u_0 + h\sum_{j=1}^{i-1}(G_{ij} + K_{ij})u_j + \frac{h}{2}\left(\frac{2}{h} + G_{ii} + 2K_{ii}\right)u_i + h\sum_{j=i+1}^{N-1} K_{ij}u_j + \frac{h}{2}K_{iN}u_N = g_i$$

Finally, we get a system of $N + 1$ equations, which is:

$$AU = B \tag{4.10}$$

when $B = (g_0, g_1, \ldots, g_N),\ U = (u_0, u_1, \ldots, u_N)$, and $A = (a_{ij})_{i,j=0,\ldots,N}$;

$$a_{ij} = \begin{cases} h \cdot K_{00}/2 + 1 & \text{if } i = j = 0, \\ h \cdot K_{0j} & \text{if } j = 1, \ldots, N-1, \\ h \cdot K_{0j}/2 & \text{if } j = N, \\ h \cdot (G_{i0} + K_{i0})/2 & \text{if } i = 1, \ldots, N, \\ h \cdot (G_{ii} + 2K_{ii})/2 + 1 & \text{if } i = j = 1, \ldots, N-1, \\ h \cdot (G_{ii} + K_{ii})/2 + 1 & \text{if } i = j = N, \\ h \cdot K_{iN}/2 & \text{if } i = 1, \ldots, N-1, \\ h \cdot (G_{ij} + K_{ij}) & \text{if } i = 2, \ldots, N, j = 1, \ldots, i-1, \\ h \cdot K_{ij} & \text{if } i = 1, \ldots, N-1, j = i+1, \ldots, N-1, \end{cases}$$

$$A = h \cdot \begin{pmatrix} \frac{K_{00}}{2} + \frac{1}{h} & K_{01} & K_{02} & \cdots & K_{0N-1} & \frac{K_{0N}}{2} \\ \frac{K_{10}+G_{10}}{2} & \frac{K_{11}+G_{11}}{2} + \frac{1}{h} & K_{12} & \cdots & K_{1N-1} & \frac{K_{1N}}{2} \\ \frac{K_{20}+G_{20}}{2} & \frac{K_{21}+G_{21}}{2} & \frac{K_{22}+G_{22}}{2} + \frac{1}{h} & \cdots & \vdots & \frac{K_{1N}}{2} \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \cdots & \ddots & \frac{K_{N-1N-1}+G_{N-1N-1}}{2} + \frac{1}{h} & \frac{K_{N-1N}}{2} \\ \frac{K_{N0}+G_{N0}}{2} & \frac{K_{N1}+G_{N1}}{2} & \cdots & \cdots & \frac{K_{NN-1}+G_{NN-1}}{2} & \frac{K_{NN}+G_{NN}}{2} + \frac{1}{h} \end{pmatrix}$$

We have chosen to write our system in its general matrix form without taking into account the fact that $u_0 = u_N = 0$. However, we can see that

$$g_0 = K_{0j} = 0, \ \forall j \in \{0 \dots N\} \Rightarrow u_0 = 0.$$

In same way, we get $u_N = 0$.

4.4 Numerical result

In this section, we give three numerical examples to illustrate the above methods for solve the linear Volterra-Fredholm integral equations of the second kind. The exact solution is known and used to show that the numerical solution obtained with our methods is correct. We used MATLAB to solve these examples.

Example 4.1. *Consider the following Fractional Boundary value problem:*

$$\begin{aligned} &\mathscr{D}^{(\gamma)} u(x) + q(x)u(x) = f(x), \quad 0 \leqslant x \leqslant 1 \\ &u(0) = u(1) = 0 \end{aligned} \tag{4.11}$$

where $\gamma = 1.5, \ q(x) = 1,$ *and*

$$f(x) = \frac{39xe^x - 8e^{-x} - 9e^x - 2xe^{-x} - 17x + 17}{20},$$

with the exact solution $u(x) = (e^x - 1)(x - 1)$.

The following MATLAB code solves the equation (4.11) numerically, and plot the absolute errors and the approximate solution of this equation with the exact solution.

```
% ce programe qui calcul la solution approchee de la ... probleme
% ... u(t)=f(t)+\int(a,t)k1(t,s,u(s))ds+\int(a,b)k2(t,s,u(s))ds ... t in [a,b]
clc ; clear;
%%%%%%%%%%%% variables %%%%%%%%%%%%%%%%%%
gamma=1.5;
alpha=gamma-floor(gamma);
a=0;
b=1;
N=16;
h=(b-a)/N;
tic
%%%%%%%%%%%%% functions %%%%%%%%%%%%%%%%%
syms t s ;
u(t)=(t-1)*(exp(t)-1);
q(t)=1+0*t;
f(t)=sinh(t) + (exp(t) - 1)*(t - 1) + t*exp(t);
V(t,s)=(alpha*(t-s)+1-alpha)*q(s);
F(t,s)=t*(alpha*s-1)*q(s);
g(t)=int(V(t,s)*f(s),s,0,t)+ int(F(t,s)*f(s),s,0,1);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
for n=1:N+1
      X(n)=a+h*(n-1);
      B(n)=g(X(n));
end
for i=1:N+1
    for j=1:N+1
     v1(i,j)=V(X(i),X(j));
     v2(i,j)=F(X(i),X(j));
    end
end
```

```

A=zeros(N+1,N+1);
for i=1:N+1
    for j=1:N+1
     v1(i,j)=V(X(i),X(j));
     v2(i,j)=F(X(i),X(j));
    end
end
A(1,1)=1+h*v2(1,1)/2;
for j=2:N
     A(1,j)=h*v2(1,j);
end
A(1,N+1)=h*v2(1,N+1)/2;
for i=2:N
     A(i,1)=h*(v1(i,1)+v2(i,1))/2;
     A(i,i)=1+h*(v1(i,i)+2*v2(i,i))/2;
     A(i,N+1)=h*v2(i,N+1)/2;
end
A(N+1,1)=h*(v1(N+1,1)+v2(N+1,1))/2;
A(N+1,N+1)=1+h*(v1(i,i)+v2(i,i))/2;
for i=3:N+1
    for j=2:i-1
     A(i,j)=h*(v1(i,j)+v2(i,j));
    end
end
for i=2:N
    for j=i+1:N
     A(i,j)=h*v2(i,j);
    end
end
U=vpa(inv(A))*B';
for i=1:N+1
    r(i)=vpa(abs(U(i)-u(X(i))));
```

```
end
toc
plot(X,U);
grid on;
plot(X,r);
grid on;
```

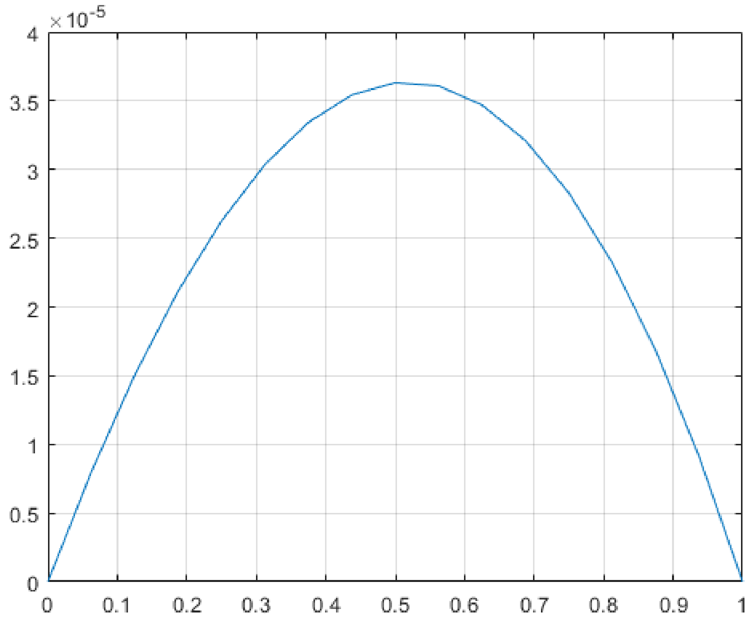

Figure 4.1: The Absolute Error of Test Example (4.1) with $N = 16$.

Example 4.2. *Consider the following Fractional Boundary value problem:*

$$\begin{aligned} &\mathscr{D}^{(\gamma)}\, u(x) + q(x)u(x) = f(x), \quad 0 \leqslant x \leqslant 1 \\ &u(0) = u(1) = 0 \end{aligned} \tag{4.12}$$

where $\gamma = 1.35,\ q(x) = 1,$ *and*

$$f(x) = \frac{3(169\pi^2 + 49)\sin(\pi x) + 780\pi^3(\cos(\pi x) - e^{-7x/13}) - 420\pi^2\sin(\pi x)}{169\pi^2 + 49}$$

with the exact solution $u(x) = 3\sin(\pi x)$.

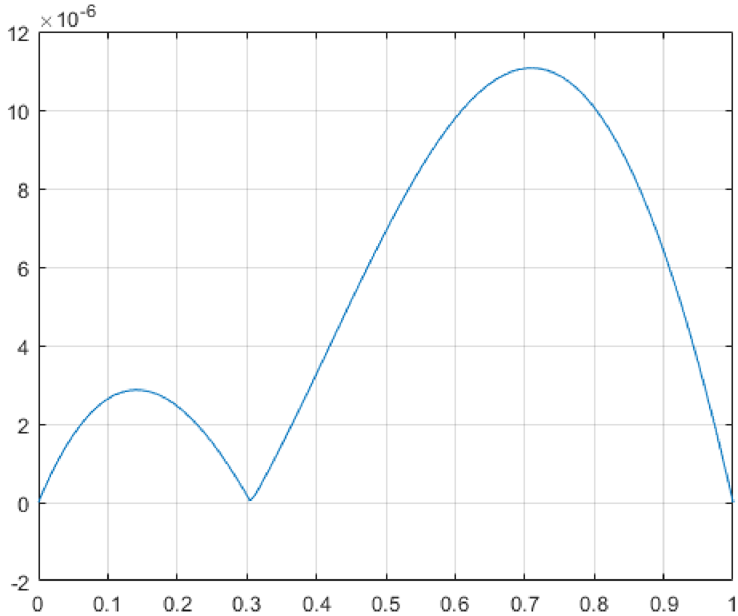

Figure 4.2: The Absolute Error of Test Example (4.2) with $N = 128$.

Example 4.3. *Consider the following Fractional Boundary value problem:*

$$\begin{aligned} &\mathscr{D}^{(\gamma)} u(x) + q(x)u(x) = f(x), \quad 0 \leqslant x \leqslant 1 \\ &u(0) = u(1) = 0 \end{aligned} \tag{4.13}$$

where $\gamma = 1.75$, $q(x) = \dfrac{19 + 2e^{-x}}{20}$, *and*

$$\begin{aligned} f(x) = {} & \frac{x(x+\cos(\pi x))(2e^{-x}+19)}{20} - (\pi^2 - 9)\frac{4\pi^2(\cos(\pi x) + e^{-3x}}{(\pi^2+9)^2} \\ & + \frac{8}{3}(1 - e^{-3x}) + \frac{(3x + 8\pi^2)\cos(\pi x) + (x - 30)\pi \sin(pix) + \pi e^{-3x}}{(\pi^2+9)^2} \end{aligned}$$

with the exact solution $u(x) = x(\cos(\pi x) + x)$.

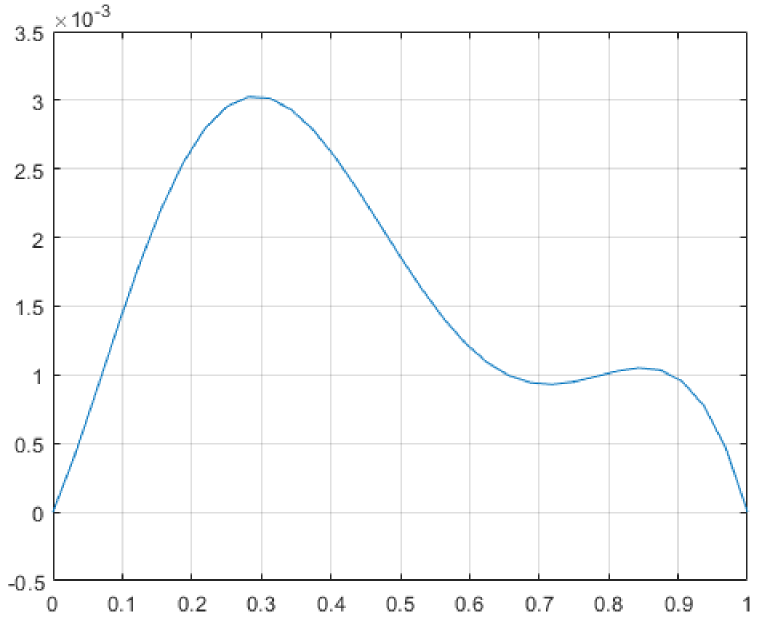

Figure 4.3: The Absolute Error of Test Example (4.3) with $N = 16$.

Example 4.4. *Consider the following Fractional Boundary value problem:*

$$\begin{aligned} &\mathscr{D}^{(\gamma)} u(x) + q(x)u(x) = f(x), \quad 0 \leqslant x \leqslant 1 \\ &u(0) = u(1) = 0 \end{aligned} \tag{4.14}$$

where $\gamma = 1.9$, $q(x) = t - 1$, *and* $f(x) = 10x$. *In this case, we don't know the exact solution.*

The following MATLAB code solves the equation (4.14) numerically, and plot the approximate solution of this equation.

```
% ce programe qui calcul la solution approchee de la ...
    probleme
% ...
    u(t)=f(t)+\int(a,t)k1(t,s,u(s))ds+\int(a,b)k2(t,s,u(s))ds ...
        t in [a,b]
clc ; clear;
%%%%%%%%%%%% variables %%%%%%%%%%%%%%%%%%
gamma=1.9;
alpha=gamma-floor(gamma);
a=0;
```

```
b=1;
N=16;
h=(b-a)/N;
%%%%%%%%%%%%% functions %%%%%%%%%%%%%%%%%
syms t s ;
q(t)=t-1;
f(t)=10*t;
V(t,s)=(alpha*(t-s)+1-alpha)*q(s);
F(t,s)=t*(alpha*s-1)*q(s);
g(t)=int(V(t,s)*f(s),s,0,t)+ int(F(t,s)*f(s),s,0,1);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
for n=1:N+1
      X(n)=a+h*(n-1);
      B(n)=g(X(n));
end
for i=1:N+1
    for j=1:N+1
      v1(i,j)=V(X(i),X(j));
      v2(i,j)=F(X(i),X(j));
    end
end

A=zeros(N+1,N+1);
for i=1:N+1
    for j=1:N+1
      v1(i,j)=V(X(i),X(j));
      v2(i,j)=F(X(i),X(j));
    end
end
A(1,1)=1+h*v2(1,1)/2;
for j=2:N
      A(1,j)=h*v2(1,j);
end
```

```
A(1,N+1)=h*v2(1,N+1)/2;
for i=2:N
      A(i,1)=h*(v1(i,1)+v2(i,1))/2;
      A(i,i)=1+h*(v1(i,i)+2*v2(i,i))/2;
      A(i,N+1)=h*v2(i,N+1)/2;
end
A(N+1,1)=h*(v1(N+1,1)+v2(N+1,1))/2;
A(N+1,N+1)=1+h*(v1(i,i)+v2(i,i))/2;
for i=3:N+1
    for j=2:i-1
      A(i,j)=h*(v1(i,j)+v2(i,j));
    end
end
for i=2:N
    for j=i+1:N
      A(i,j)=h*v2(i,j);
    end
end
U=vpa(inv(A))*B';
plot(X,U);
grid on;
```

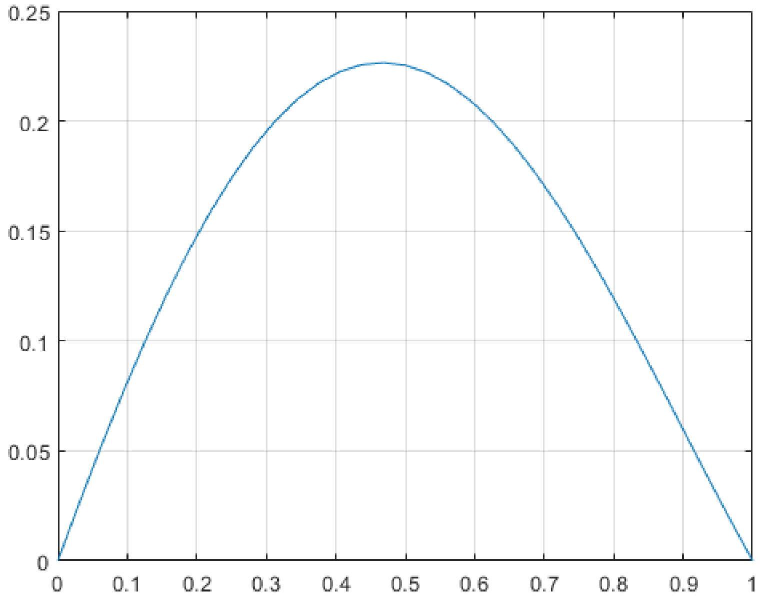

Figure 4.4: The approximate solution of test Example (4.4) with $N = 32$.

Conclusion

In this thesis, we first dealt with some Volterra integro-differential problems in fractional calculus by using the new derivative of Caputo, and we proved the existence and uniqueness of each solution of this problems. Convergences of the obtained solutions are also justified in order to establish that the formal solutions are analitic solutions. Afterwards, we have examined with a conformable fractional derivative of Caputo and its Fractional Integral. Considered problem reduced to the equivalent linear Volterra-Fredholm integral equations of the second kind.

As perspectives, we will try to study the generalized Fractional Boundary Value Problem with with a conformable fractional derivative of Caputo and its Fractional Integral. This generalized version represents a challenge from the analytical point of view, i.e. the existence and uniqueness of the solution. However, the numerical side remains the same.

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Annex

MATLAB Operators and Special Characters

Some common commands are listed in this section a full specification of each can be obtained using the help system.

$+$	Addition.
-	Subtraction.
*	Multiplication.
/	Division.
abs	Absolute value.
sqrt	Square root function.
clc	Clear command window.
clear	Clear variables and functions from memory.
$\wedge$	Exponentiation.
pi	the mathematical constant π.
$'$	Transpose.
%	To comment out one line in a multiline command.
clc	clean command window. After this function, all previous command written on window will clean.
close all	closes all figures, and window.
clear all	this build in function clear all variable created in work space of matlab.

syms Create symbolic variables and functions.

diff(f) differentiates f with respect to x, and there are several forms

$$
\begin{aligned}
\textbf{diff(f(x))} &= f'(x)\\
\textbf{diff(f(x),n)} &= f^{(n)}(x)\\
\textbf{diff(f(x,y),n,y)} &= \frac{d^{(n)} f(x,y)}{dy^n}\\
\textbf{diff(f,x1,...,xN)} &= \frac{d}{dx_n}\cdots\frac{d}{dx_1} f(x_1, x_2, \cdots, x_n).
\end{aligned}
$$

int Integrate, and there are several forms

$$\textbf{int(f,s,a,t)} = \int_a^t f(s)\,\mathrm{d}s, \quad \textbf{int(f,omega)} = \int_\Omega f(s)\,\mathrm{d}s.$$

Da(f,alpha,a) the new fractional derivative of Caputo $\mathscr{D}^{(\alpha)} f(t)$

floor the nearest integer in the direction of negative.

Ma M the function used in section 1.1.4.

A(i,j) A_{ij} Element of the matrix A.

V(i) V_i Element of the vector V.

== Equal to.

∼= Not equal to.

> Greater than.

>= Greater than or equal to.

< Less than.

<= Less than or equal to.

& Logical AND.

— Logical OR .

∽ Logical NOT.

Compare Matlab to C/C++

Pros Matlab:

- is a higher level of programming language.
- is easier to start.
- provides convenient tools and built-in functions.
- provides user-friendly graphical interface.
- is good for post data analysis and visualization.
- Matlab is better for numerical treatments.
- Matlab is more popular in engineering.
- Symbolic Math Toolbox™ provides functions for solving, plotting, and manipulating symbolic math equations. You can create, run, and share symbolic math code using the MATLAB® Live Editor. The toolbox provides functions in common mathematical areas such as calculus, linear algebra, algebraic and ordinary differential equations, equation simplification, and equation manipulation.

Cons Matlab:

- in many cases, the computational speed of Matlab is slower than that of C/C++.